**Fonctions L p-adiques et complexes sur les groupes classiques : mesures admissibles, valeurs spéciales**

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1) Groupes classiques, le cas GL(n), le cas symplectique et unitaires. Formes modulaires et formes automorphes, exemples.

2) Formes modulaies hermitiennes. Fonctions L complexes sur les groupes classiques. Algèbres de Hecke. Methode de Rankin-Selberg

3) Distributions, mesures, congruences de Kummer.Fonction zêta p-aique de Kubota-Leopoldt et l’algèbre d'Iwasawa.

4) Fonctions L p-adiques sur les groupes classiques : mesures admissibles, valeurs spéciales
**Linear Algebraic Groups.**
5. Radical. Parabolic subgroups. Reductive groups.

6. Structure theorems for reductive groups. P.11-17

7. Representations in characteristic zero (3).

AUTOMORPHIC L-FUNCTIONS
I. L-groups .................................................................................... 28

1. Classification.

2. Definition of the L-group ......................................................... 29
3. Parabolic subgroups .................................................................. 31
4. Remarks on induced groups ......................................................... 33

5. Restriction of scalars ............................................................... 34
II. Quasi-split groups; the unramified case (p.35)
III. Weil groups and representations. Local factors ................ 39

8. Definition of \Phi(G) ..................................................................... 39

9. The correspondence for tori ..................................................... . 41

10. Desiderata ............................................................................. . 43

11. Outline of the construction over **R, C** ............................................ 46

12. Local factors. . ....................................................................... 48

IV. The L-function of an automorphic representation ....................... . 49

13. The L-function of an irreducible admissible representation of G\_A49
14. The L-function of an automorphic representation ......... ............... 52
V. Lifting problems ................................................................ ........... 54

15. L-homomorphisms of L-groups ................................................... 54

16. Local lifting .......................................................................... . 55

17. Global lifting .......................................................................... . 56

18. Relations with other types of L-functions ..................................... 5.5
6.5.Automorphic Forms and The Langlands Program. . . . . . . . . . .

6.5.1 A Relation Between Classical Modular Forms and Representation Theory . . . . 332

6.5.2 Automorphic L-Functions . . . . . . . . . . . . . . . . . . . . . . . . . . 335

Further analytic properties of automorphic L-functions . 338

6.5.3 The Langlands Functoriality Principle . . . . . . . . . . . . . . . 338

6.5.4 Automorphic Forms and Langlands Conjectures . . . . . . . 339

5. *Radical. Parabolic subgroups. Reductive groups.*
RAPPELS. 5.1. DEFINITIONS. Let G be an algebraic group. The radical R(G) of G is the greatest connected normal subgroup of G; the unipotent radical Ru(G) is the greatest connected unipotent normal subgroup of G. The group G is semisimple (resp. reductive) if R(G) = {e} (resp. Ru(G) = {e}).The definitions of R(G) and Ru(G) make sense, because if H, H' are connected normal and solvable (resp. unipotent) subgroups, then so is H · H'. Both radicals are k-closed if G is a k-group. Clearly, R(G) = R(G0) and Ru(G) = Ru(G0). The quotient G/R(G) is semisimple, and G/Ru(G) is reductive. In characteristic zero, the unipotent radical has a complement; more precisely: Let G be defined

over k. There exists a maximal reductive k-subgroup H of G such that

G = H · Ru(G), the product being a semidirect product of algebraic groups. If H' is a reductive

subgroup of G defined over k, then H' is conjugate over k to a subgroup of H.

5.3. THEOREM [ll Let G be a connected algebraic group.

(1) All maximal tori of Gare conjugate. Every semisimple element is contained in a torus. The centralizer of any subtorus is connected.

(2) All maximal connected solvable subgroups are conjugate. Every element of G belongs to one such group.

(3) If P is a closed subgroup of G, then G/P is a projective variety if and only if P contains a maximal connected solvable subgroup.

6. Structure theorems for reductive groups.
6.1. *Root systems.* Let V be a finite dimensional real vector space endowed with a positive nondegenerate scalar product. A subset \Phi of V is a *root system* when

(1) \Phi consists of a finite number of nonzero vectors that generate V, and is

symmetric (\Phi = -\Phi).

(2) for every \alpha \in \Phi, s\_\alpha( \Phi) =\Phi, where s\_\alpha denotes reflection with respect to the hyperplane perpendicular to \alpha.
(3) if \alpha, \beta \in \Phi, then 2(\alpha, \beta)/( \alpha, \alpha) \in \Z. The group generated by the symmetries

s\_\alpha (\alpha \in \Phi) is called the Weyl group of \Phi (notation W(\Phi) is finite. The integers 2(\alpha, \beta)/( \alpha, \alpha)

 are called the Cartan integers of \Phi. Condition (3) means that for

every \alpha and \beta of \Phi, (s\_\alpha (\beta) - \beta > is an integral multiple of \alpha, since

s\_\alpha (\beta) = \beta - 2(\alpha, \beta)/( \alpha, \alpha).

For the theory of reductive groups we shall have to enlarge slightly the notion of root system: if M is a subspace of V, we say that \Phi) is a root system in (N, M) if it generates a subspace P supplementary to M, and is a root system in P.

The Weyl group W(\Phi)) is then understood to act trivially on M.

A root system \Phi in Vis the direct sum of \Phi ' \subset V' and \Phi " \subset V", if V = V' \oplus V", and \Phi = \Phi ' \cup \Phi ". The root system is called *irreducible* if it is not the direct sum of two subsystems.

6.2. *Properties of root systems*.

(1) Every root system is direct sum of irreducible root systems.

(2) If \alpha and \lambda\alpha \in \Phi, then \lambda = ± 1, ±(1/2), or ±2.

The root system \Phi, is called *reduced* when for every \alpha \in \Phi,, the only multiples of \alpha belonging to \Phi are ±\alpha

To every root system \Phi, there belongs two natural reduced systems by removing for every \alpha \in \Phi, the longer (or the shorter) multiple

of \alpha :

(3) The only reduced irreducible root systems are the usual ones:

A\_n (n \ge 1), B\_n (n \ge 2), C\_n (n \ge 3), D\_n (n \ge 4),

G\_2, F\_4 , E\_6 , E\_7 , E\_8

(4) For each dimension n, there exists one irreducible nonreduced system, denoted by BC\_n (see below).

EXAMPLES : B\_n: Take R^n with the standard metric and basis

{x\_1, · · ·, x\_n}·

B\_n ={±(x\_i± x\_i) (i <j)\_i and ±x\_i; (1 \le i \le n)}.
W(B\_n) = { s \in GL(n, R)ls a product of a permutation matrix

with a symmetry with respect to a coordinate subspace}

C\_n = {±(X; ± Xj) (i < j) and ±2x\_i; (1 \le i \le n)},

W(C\_n) = W(B\_n),

 BC = { + (x. + x ·) n - I - J (i < j), ±x\_i and ±2x\_i (1 \le i \le n)},

W(BC\_n) = W(B\_n).

DEFINITION. A hyperplane of Vis called singular if it is orthogonal to a root \alpha \in \Phi. A Weyl-chamber C^0 is a connected component of the complement of the union of the singular hyperplanes.

To a Weyl-chamber, is associated an ordering of the roots defined by:

\alpha > 0, if (\alpha, v) > 0 for every v in C^0.

6.3. Roots of a reductive group, with reference to a torus. Let G be a reductive group, and S a torus of G. It operates on the Lie-algebra \fg of G by the adjoint representation. Since S consists of semisimple elements, Ad\_\fg S is diagonalizable

 \fg = \fg\_0^{(S)}\oplus \coprod\_\alpha \fg\_\alpha^{(S)}

The set \Phi(G, S) of roots of G relative to the torus S is the set of nontrivial characters of S appearing in the above decomposition of the adjoint representation.
If T \subset S, every root of G relative to T that is not trivial on S defines a root relative to S. If Tis maximal \Phi (G, T) = \Phi (G) is the set of roots of G in the usual sense.

6.6. EXAMPLES. (1) G = GL(n),

S =group of diagonal mattices={\diag{s^\lambda\_1, s^\lambda\_2,.., s^\lambda\_n }}
where \lambda\_i \in ^S is such that s^\lambda\_i = s\_{ii}. S is obviously a split torus and is maximal.

A minimal parabolic k-subgroup P is given by the upper triangular matrices, which is in this case a Borel subgroup. The unipotent radical U of P is given by the group of upper triangular matrices with ones in the diagonal. If e\_{ij} is the matrix having all components zero except that with index (i,j) equal .to 1, Ad\_G s(e\_{ij}) = (s^\lambda\_i/ s^\lambda\_j) e\_{ij}. So the positive roots are \lambda\_i - \lambda\_j (i < j) since the Lie algebra of U is generated by e\_{ij} (i < j). The simple roots are

(\lambda\_1 - \lambda\_2, \lambda\_2 - \lambda\_3, …,

\lambda\_{n-1} - \lambda\_n). The Weyl group is generated by s\_\alpha, where \alpha is a positive root; since for

\alpha = \lambda\_i - \lambda\_j, s\_\alpha permutes the i and j axis, 1 W\_k =\fS\_n, the group of permutations of the basis elements. The parabolic subgroups are the stability groups of flags.

(2) G "splits over k" (i.e., G has a maximal torus which splits over k). Example (1) enters in this category. The k-roots are just the usual roots. A minimal parabolic k-subgroup is a maximal connected solvable subgroup. If k is algebraically closed G always splits over k and this gives just the usual properties of semisimple or reductive linear groups

(voir aussi pp.90-105 de mon cours « Algèbre-2 », « Groupes classiques, algèbre géométrique et applications » à l'Institut Fourier 2015).

7. *Representations in characteristic zero* (3). We assume here the ground field to be of characteristic zero, and G to be semisimple, connected. Let P = Z(S) · U be a minimal parabolic k-group, where U = Ru(P), and S is a maximal k-split torus. We put on X(S) an ordering such that u is the sum of the positive k-root spaces.

*Definition of the L-group*1.1.There is a canonical bijection between isomorphism classes of connected reductive k-groups and isomorphism classes of root systems. It is defined by associating to G the root datum \psi(G) = (X\*(T), \phi, X\*(T), \phi^v) where T is a maximal torus of G, X\*(T) (X\*(T)) the group of characters (1-parameter subgroups) of T and(\Phi) (\Phi^v) the set of roots (coroots) of G with respect to T.

1.2. The choice of a Borel subgroup B \supset Tis equivalent to that of a basis \Delta of \Phi(G, T). The previous bijection yields one between isomorphism classes of triples (G, B, T) and isomorphism classes of based root data

Classification of Algebraic Semisimple Groups

by J. TITS p.33-63

**1. Algebraically closed fields, Dynkin diagrams.**

1.1. Dynkin diagrams ([3], [S], [7], [11], [13], [29], [47]).

1.1.1. Notations.

1.1.2. Ordinary Dynkin diagram.

The main purpose of §1 is to indicate how important data relative to a group G

can be read- on its Dynkin diagrams.

1.2. Classification- up to isogeny ((7}, [11]). p.34

1.2.2. The main theorem.

THEOREM 1. p.34

The field K being given, a semisimple group G is characterizeed up to strict isogeny by its Dynkin diagram. It is almost simple if and only if the diagram is connected. Any semisimple group G is strictly isagenous to a direct product of simple- groups whose Dynkin diagrams are the connected components of the diagramof G. The complete list of Dynkin diagrams of almost simple groups is given in Table I ; each diagram of that table determines a strict isogeny class of almost simple groups over any given field K.

1.3. Weyl groups ([S], [7], [29], [34], [47]), p.35

1.3.1.

1.3.2. Generators and relations.

1.3.3. Affine Weyl group.

1.4. Coefficients of the dominant root; dimension ([3], [5], [14], [24], [33]).p.35

1.4.l.

1.5. Classification up to isomorphism; automorphism group and center ([31 [5],

[7], [12), [14), [29]).p.36

1.5.l. Opposition involution.

1.5.2. The cocenter C\*.

1.5.3. Automorphism groups of Dynkin diagrams.

1.5.4. Classification up to isomorphism; simply connected and adjoint groups.

1.5.5. Center.

1.5.6. Automorphism group.

1.6. Parabolic subgroups.p.38

**2. Non algebraically closed field. Index and anisotropic kernels.** p.38

2.1. Introduction; notations **([2], [4]).**

2.2. Anisotropic kernels.p.39

2.3. Index **([4],** [26], [27], [38], **[40],** [42]).p.39

2.4. An example; orthogonal groups.p.39

2.5. How to deduce the relative root systemfrom the index ([4], [40]).p.35

2.5.1. Two elements of Li which do not belong to Li0 have the same restriction

to S if and only if they belong to the same orbit of r.

2.5.2. Relative root system.

2.5.3. Relative Weyl group. p40

2.5.4. Parabolic subgroups.p.412.5.5. An example.

2.6. Isogeny.p.37

2.6.1. Simply connected covering and adjoint group.

2.6.2. Definitions. The groups G and G of the preceding proposition will be

called respectively the simply connected covering and the adjoint group of G.

Two groups will be said (strictly) isogenous over k or k-isogenous if all the

groups and (central) isogenies which occur in the definition of §1.2.1 are defined

over k.

2.6.3. PROPOSITION 3.

2.7. A Witt-type theorem/or the semisimple groups ([27), [38), [40)).p.43

2.7.1. THEOREM 2. p.43

2.7.2. REMARKS. (a) The group G is already determined, up to k-isomorphism,

by its K-isomorphism class, its index and its semisimple anisotropic kernel,

given up to k-isogeny.

(b) The group G is determined up to strict k-isogeny by its strict K-isogeny

class, its index and the k-isogeny class of its semisimple anisotropic kernel.

This is an immediate consequence of the preceding theorem and the Propositions

2 and 3.

(c) The reader will have no difficulty to state for the reductive groups a theorem

analogous to Theorem 2.

(d) There is a trivial but sometimes useful generalization of the Theorem 2,

which we want to mention. Let S' be any k-split torus in G, let A' be the set of

simple roots of G with respect to some maximal torus T' containing S' and some

ordering of X\*(T') compatible with an ordering of X\*(S') and let \Delta’\_0 be the set of simple roots vanishing on S'. Exactly as in §2.3, we can define the \*-action

of \Gamma on \Delta'. Let us call partial index (relative to S') the data consisting of \Delta' (together with the Dynkin diagram), \Delta\_0 and the \*-action of \Delta on \Delta’. Then, in the statement of §2.7.1, one can replace the index and the semisimple anisotropic kernel respectively by the partial index relative to some k-split torus S' and the "corresponding semisimple kernel" \Dc\Zc(S') (which is still defined over k, but is no longer anisotropic in general).

3. **Cl assification.** p.45According to the Theorem 2, the problem of classifying the semisimple algebraic groups over a given field k can be decomposed into two

steps which can roughly be formulated as follows:

(1) Find all admissible indices of semisimple groups over k;

(2) For a given index, find all possible semisimple anisotropic kernels.

These two questions will be discussed here. However, we shall not consider

the problem of classifying all anisotropic groups over k, which theoretically falls

under (2) and is usually by far the most difficult part of the classification problem.

3.1. Preliminary reductions.p.45

3.1.1. Reduction to the simply connected (or to the adjoint) case.

Every semisimple simply connected group defined over k is in a unique way a

direct product of almost k-simple simply connected groups (a group is almost

k-simple if it has no infinite normal subgroup defined over k). If G is almost k-simple and simply connected, there exists a field k' and an (absolutely) almost simple simply connected group H defined over k', such that G ~ k Rk'/k(H).

3.2. Some necessary conditions (independent of the ground field) for the admissibility of indices ([4], [38), [40]).p.46

3.2.1. Self-opposition. The index of a group G is invariant under the opposition

involution i (that is, i commutes with the \*-action of r, and leaves invariant L\0).

3.2.2. An induction process.

3.3. Further admissibility conditions, for various special ground fields. p.46

3.3.1. Finite.fields ([20], [31], [33], [43]).

3.3.2. Real numbers ([17], [18], [26], [40]).

3.3.3. p-adics ([23], [45]). Let k be the field of p-adic numbers (for some p).

Then, a group G which is anisotropic and absolutely almost simple is of inner

type A\_n( §2.3).

3.3.4. Number fields ([I], **[19]).** By means of the "Hasse principle," one can

also use §§3.3.2 and 3.3.3 to exclude certain indices in the case of number fields.

All admissibility conditions stated in §3.3 are necessary conditions.

3.4. Necessary and sufficient conditions on the anisotropic kernel ([30], **[31], [38)).** p.43

3.4.1. Statement of the problem; notations.

3.4.2. Cohomological formulation of the condition

3.4.3. Linear representations: terminology, notations.

3.4.4. Representation-theoretical formulation of the condition.p50

TABLE I: **Dynkin Diagrams**

EXPLANATIONS p53

TABLE II: **Indices**

EXPLANATIONS p.54

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