Complex and p-adic L functions on classical groups. Admissible measures, special values.

("Fonctions L p-adiques et complexes sur les groupes classiques : mesures admissibles, valeurs spéciales").

Alexei PANTCHICHKINE
Institut Fourier, University Grenoble-Alpes

Summer School on the Theory of Motives and the Theory of Numbers (L'École d'été à LAMA)

Fonctions zetas, polyzetas, séries arithmétiques : applications aux motifs et à la théorie des nombres.

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This Summer School on Theory of Motives and Number Theory

at the crossroad of automorphic L functions (complex and p-adic), zeta functions, polyzeta functions and dynamical zeta function, was conceived as a continuation of a seris of Conferences "Zeta-functions I-VI", held in J.-V.Poncelet Laboratory UMI 2615 du CNRS, The Higher School of Economics, and Independent University in Moscow, and mainly organized by Alexey ZYKIN, professor of the French Polynesia University in Tahiti, who tragically dissapeared in April 2017 together with his wife Tatiana MAKAROVA.

The Summer Schools «Algebra and Geometry» (Yaroslavl, RUSSIA, July 2012-2016), were also largely organized by Alexey ZYKIN together with Fyodor BOGOMOLOV and Courant Institute (New York, USA), see the video and pdf of lectures at
`http://bogomolov-lab.ru/SHKOLA2012/talks/panchishkin.html`

Alexey ZYKIN near Grenoble on June 22, 2012

Figure: Climbing the mountain Chamchaude with Siegfried BOECHERER

Contents : Mini-course of Alexei Pantchichkine (6h) :

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Lecture N°3. Distributions, measures, Kummer congruences. Kubota-Leopoldt p-adic zeta function and Iwasawa algebra.
The analytic structure of $\mathcal{Y}_S$ and p-adic Mellin transform
Lecture N°4. p-adic L-functions on classical groups. Ordinary case. Admissible measures, special values.

(L'intervention d'Alexei Pantchichkine:
1) Groupes classiques, le cas $\mathrm{GL}(n)$, le cas symplectique et unitaires. Formes modulaires et formes automorphes, exemples.
2) Formes modulaies hermitiennes. Fonctions L complexes sur les groupes classiques. Algèbres de Hecke. Methode de Rankin-Selberg
3) Distributions, mesures, congruences de Kummer.Fonction zêta p-aique de Kubota-Leopoldt et l'algèbre d'Iwasawa.
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Lecture N°1. Classical groups, the case of $\mathrm{GL}(n)$

The sympectic and unitary cases. Modular forms and automorphic forms. ("Groupes classiques, le cas $\mathrm{GL}(n)$ le cas symplectique et unitaires. Formes modulaires et formes automorphes, exemples").

- Linear Algebraic Groups. §1-6 of [Bor66]
- Radical. Parabolic subgroups. Reductive groups.
- Structure theorems for reductive groups.
  §6.5 of[MaPa], Automorphic Forms and The Langlands Program
- §6.5.1 A Relation Between Classical Modular Forms and Representation Theory
- §I-II of [Bor79]: I. Definition of the L-group
- II. Quasi-split groups

Reductive groups

Recall: an algebraic group is irreducible if and only if it is connected. The connected component of the identity of G will be denoted by G^0. The index of G^0 in G is finite.

Definition

Let G be an algebraic group over a field k. The radical $R(G)$ of G is the greatest connected normal subgroup of G; the unipotent radical $R_u(G)$ is the greatest connected unipotent normal subgroup of G. The group G is semisimple (resp. reductive) if $R(G) = \{e\}$ (resp. $R_u(G) = \{e\}$).

The definitions of $R(G)$ and $R_u(G)$ make sense, because if H, H' are connected normal and solvable (resp. unipotent) subgroups, then so is $H \cdot H'$. Both radicals are k-closed if G is a k-group. Clearly, $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$. The quotient G/R(G) is semisimple, and $G/R_u(G)$ is reductive. In characteristic zero, the unipotent radical has a complement; more precisely: Let G be defined over k. There exists a maximal reductive k-subgroup H of G such that $G = H \cdot R_u(G)$, the product being a semidirect product of algebraic groups. If H' is a reductive subgroup of G defined over k, then H' is conjugate over k to a subgroup of H of G such that $G = H \cdot R_u(G)$, the product being a semidirect product of algebraic groups. If H' is a reductive subgroup of G defined over k, then H' is conjugate over k to a subgroup of H.

Theorem (5.2 of [Bor66])

Let G be an algebraic group. The following conditions are equivalent :
(1) G^0 is reductive,
(2) $G^0 = S \cdot G'$, where S is a central torus and G' is semisimple,

Theorem (5.3 of [Bor66])

Let G be a connected algebraic group.
(1) All maximal tori of G are conjugate. Every semisimple element is contained in a torus. The centralizer of any subtorus is connected.
(2) All maximal connected solvable subgroups are conjugate. Every element of G belongs to one such group.
(3) If P is a closed subgroup of G, then G/P is a projective variety if and only if P contains a maximal connected solvable subgroup.

Characters and roots.

A character of G is a rational representation of degree 1; $\chi : G \to \mathrm{GL}_1$. The set of characters of G is a commutative group, denoted by $X(G)$ or $\hat{G}$. The group $\hat{G}$ is finitely generated; it is free if G is connected [Bor66], p.6. If one wants to write the composition-law in $\hat{G}$ multiplicatively, the value at $g \in G$ of $\chi \in \hat{G}$ should be noted $\chi(g)$. But since one is accustomed to add roots of Lie algebras, it is also natural to write the composition in G additively. The value of χ at g will then be denoted by g^{χ}. To see the similarity between roots and characters take $\Omega = \mathbb{C}$; if $X \in \mathfrak{g}$, the Lie algebra of G, $(e^{x})^{\chi} = ed_{\chi}(X)$, where d_{χ} is the differential at e; d_{χ} is a linear form over $\mathfrak{g}$. In the sequel, we not make any notational distinction between a character and its differential at e.

Let $g \in \mathrm{GL}(n, \Omega)$, g can be written uniquely as the product $g = g_s \cdot g_n$, where g_s is a semisimple matrix (i.e., g. can be made diagonal) and g_n is a unipotent matrix (i.e., the only eigenvalue of g_n is 1, or equivalently $g_n - I$ is nilpotent) and $g_s \cdot g_n = g_n \cdot g_s$.

Example: the case of $\mathrm{GL}(n)$

The rank of G is the common dimension of the maximal tori, (notation $\mathrm{rk}(G)$). A closed subgroup P of G is called parabolic, if G/P is a projective variety. A maximal connected closed solvable subgroup is called Borel subgroup.

Exemple. $G = \mathrm{GL}_n$. A flag $\mathcal{F}$ in a vector space V is a properly increasing sequence of subspaces

$$\mathcal{F} : 0 \neq V_1 \subset \cdots \subset V_t \subset V_{t+1} = V.$$

The sequence (d_i) $(d_i = \dim v_i, i = 1, \cdots t))$ describes the type of the flag. If $d_i = i$ and $t = \dim V - 1$, we speak of a full flag. A parabolic subgroup of GL_n is the stability group of a flag $\mathcal{F}$ in Ω^n. G/P is the manifold of flags of the same type as F, and is well known to be a projective variety. A Borel subgroup is the stability group of a full flag. In a suitable basis, it is the group of all upper triangular matrices.

The case of orthogonal group $G = SO(F)$

of a nondegenerate quadratic form F on a vector space V_k (where, to be safe, one takes char $k \neq 2$). In a suitable basis

$$F(x_1, \cdots, x_n) = x_1 x_n + x_2 x_{n-1} + \cdots + x_q x_{n-q+1} + F_0(x_{q+1}, \cdots, x_{n-q})$$

where F_0 does not represent zero rationally. The index of F, the dimension of the maximal isotropic subspaces in V, is equal to q. A maximal k-split torus S is given by the set of following diagonal matrices:

$$\begin{pmatrix} s^{\lambda_1} & & & & & & & & & & & \\ & s^{\lambda_2} & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & s^{\lambda_q} & & & & & & & & \\ & & & & 1 & & & & & & & \\ & & & & & \ddots & & & & & & \\ & & & & & & 1 & & & & & \\ & & & & & & & s^{-\lambda_q} & & & & \\ & & 0 & & & & & & s^{-\lambda_{q-1}} & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & s^{-\lambda_1} \end{pmatrix}$$

Let $SO(F_0)$ denote the proper orthogonal group

of the quadratic form F_0 , imbedded in $SO(F)$ by acting trivially on $x_1, \cdots, x_q, x_{n-q+1}, ..., x_n$. Then $Z(S) = S \times SO(F_0)$. The minimal parabolic k-subgroups are the stability groups of the full isotropic flags. For the above choice of S, and ordering of the coordinates, the standard full isotropic flag is

$$[e_1] \subset [e_1, e_2] \subset \cdots \subset [e_1, \cdots e_q]$$

The corresponding minimal parabolic k-subgroup takes then the form

$$P = \left\{ \begin{pmatrix} A_0 & A_1 & A_2 \\ 0 & B & A_3 \\ 0 & 0 & A_4 \end{pmatrix} \right\}$$

where A_0 and A_4 are upper triangular $q \times q$ matrices, $B \in SO(F_0)$, with additional relations that insure that $P \subset SO(F)$. The unipotent radical U of P is the set of matrices in P, where $B = I, A_0, A_4$ are unipotent, and

$$A_4 = {}^{\sigma}A_0^{-1}; \;\; Q \cdot A_3 + {}^{t}A_1 \cdot J \cdot A_4 = 0,$$
$${}^{t}A_4 \cdot J \cdot A_2 + {}^{t}A_3 \cdot Q \cdot A_3 + {}^{t}A_2 \cdot J \cdot A_4 = 0$$

where Q is the matrix of the quadratic form F_0, J is the $q \times q$ matrix with one's in the nonprincipal diagonal and zeros elsewhere, and σ is the transposition with respect to the same diagonal, $({}^{\sigma}M = J^{t}MJ)$.

Example: Unitary group

Let us review some background and set up standard notation. Let E be a quadratic imaginary field, embedded in $\mathbb{C}$; $0 \leq m \leq n$ and $\Lambda = \mathcal{O}_E^{n+m}$. Let

$$I_{n,m} = \begin{pmatrix} & & I_m \\ & I_{n-m} & \\ I_m & & \end{pmatrix}$$

where I_ℓ is the unit matrix of size ℓ, and introduce the perfect hermitian pairing

$$(u, v) = {}^t\bar{u} I_{n,m} v$$

on Λ. Let $G = GU(\Lambda, (\,,\,))$ be the group of unitary similitudes of Λ, regarded as a group scheme over $\mathbb{Z}$; and denote by $\nu : G \to \mathbf{G}_m$ the similitude character. For any commutative ring R

$$G(R) = \{g \in GL_{n+m}(\mathcal{O}_E \otimes R) | \forall u, v \in \Lambda \otimes R,\ (gu, gv) = \nu(g)(u, v)\}.$$

Then $G(R) = GU(n, m)$ is the *general unitary group of signature* (n, m), and $G(\mathbb{C}) \cong GL_{n+m}(\mathbb{C}) \otimes \mathbb{C}$.

Automorphic complex L-functions on classical groups.

- §6.5. of [MaPa] Automorphic Forms and The Langlands Program
- Automorphic L-Functions
- Analytic properties of automorphic L-functions
- Hecke algebras.
- Section IV of [Bor79] The L-function of an automorphic representation.

Lecture N°2. Classical and Hermitian modular forms

- Classical modular forms: the case of GL_2
- Geometric algebra (see [Dieudonné], [Garrett])
- Sesquilinear formes, Hermitian and antihermitian forms
- Automorphic complex L-functions on classical groups.
- Hermitian modular forms and L functions.
- Hecke algebras.
- The Rankin-Selberg method.

Why study L-values attached to modular forms?

A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$

$\rightsquigarrow$ Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$

$\rightsquigarrow$ A numbe (solution)

Example : (Hardy-Ramanujan)

$$p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3),$$
$$\lambda_n = \sqrt{n - 1/24},$$

$\uparrow$ Good bases, finite dimensions, many relations and identities

$\uparrow$ Values of L-functions, periods, congruences, .

Other examples: Birch and Swinnerton-Dyer conjecture, ... L-values attached to modular forms

Modular forms, zeta functions, L-functions

Eisenstein series $E_k = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n \in \mathcal{M}_k$, a modular forms for even weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z})$, $q = e^{2\pi i z}$), and $E_2 \in \mathcal{QM}$ a quasimodular form. The ring of quasimodular forms, closed under differential operator $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, used in arithmetic, $\zeta(s)$ is the Riemann zeta function, $\zeta(-1) = -\frac{1}{12}$, $E_2 = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n$ is also a p-adic modular form (due to J.-P.Serre,[Se73], p.211)

Elliptic curves $E : y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, A.Wiles's modular forms $f_E = \sum_{n=1}^{\infty} a_n q^n$ with $a_p = p - CardE(\mathbb{F}_p)$

($p \not| \, 4a^3 + 27b^2$), and the L-function $L(E, s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Zeta-functions or L-functions

They are attached to various mathematical objects as certain Euler products.

- L-functions link such objects to each other (a general form of functoriality);
- Special L-values answer fundamental questions about these objects in the form of a number (complex or p-adic).

Computing these numbers use integration theory of Dirichlet-Hecke characters along p-adic and complex valued measures.
This approach originates in the Dirichlet class number formula using the L-values in order to compute class numbers of algebraic number fields through Dirichlet's L-series $L(s,\chi)$: for an imaginary quadratic field K of discriminant $-D < -4$, $\chi_D(n) = \left(\frac{-D}{n}\right)$

$$h_D = \frac{\sqrt{D}L(1,\chi_D)}{2\pi} = L(0,\chi) = -\frac{1}{D}\sum_{a=1}^{D-1}\chi_D(a)a.$$

(Example: $\mathrm{disc}(\mathbb{Q}(\sqrt{-5}))) = -20$, $h_{20} = 2$; in PARI/GP $\chi_{20}(n) =$ kronecker(-20,n), `gp > -sum(x=1,19,x*kronecker(-20,x))/20`
`% 29 = 2`
Another famous example: the Millenium BSD Conjecture gives the rank of an elliptic curve E as the order of $L(E,s)$ at s=1 (i.e. the residue of its logarithmic derivative, see [MaPa], Ch.6).

A short story of critical values, see [YS]

Euler discovered $\zeta(2)=\frac{\pi^2}{6}$, and $\frac{2\zeta(2n)}{(2\pi i)^{2n}}=-\frac{B_{2n}}{(2n)!}\in\mathbb{Q}, (n\geq 1)$.
These are examples of critical values (in the sense of Deligne): for a more general zeta function $\mathcal{D}(s)$ the critical values are defined using its gamma factor $\Gamma_{\mathcal{D}}(s)$ such that the product $\Gamma_{\mathcal{D}}(s)\mathcal{D}(s)$ satisfies a standard functional equation under the symmetry $s\mapsto v-s$. Then $\mathcal{D}(n)$, $n\in\mathbb{Z}$ is a critical value of $\mathcal{D}(s)$ if both $\Gamma_{\mathcal{D}}(n)$ and $\Gamma_{\mathcal{D}}(v-n)$ are finite.

Hurwitz [Hur1899] showed a striking analogy to Euler's theorem:

$$\frac{\sum'_{\alpha\in\mathbb{Z}[i]}\alpha^{-4m}}{\Omega^{4m}}=\frac{H_m}{(4m)!}\in\mathbb{Q}, \Omega=2\int_0^1\frac{dx}{\sqrt{1-x^4}}=2.6220575542\cdots$$

for $1\leq m\in\mathbb{Z}$, where $\alpha=a+ib$, $a,b\in\mathbb{Z}$ are non-zero Gaussian integers and H_m are Hurwitz numbers (recursively computed, [Sl]):
$H_1,H_2,\cdots=\frac{1}{10},\frac{3}{10},\frac{567}{130},\frac{43659}{170},\frac{392931}{10},\cdots$. Recall the formula:
Let $\wp$ be the Weierstrass $\wp$-function satisfying $\wp'^2=4\wp^3-4\wp$.
Then $\wp(z)=\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{2^{4n}H_nz^{4n-2}}{4n(4n-2)!}$. A rapid computation of these values: take the Fourier expansion of the Eisenstein series at $z=i$, $q=e^{-2\pi}$:

$$G_{4m}(z)=\sum_{a,b}{}'(az+b)^{-4m}=2\zeta(4m)+\frac{2(2\pi)^{4m}}{(4m-1)!}\sum_{d\geq 1}\frac{d^{4m-1}q^d}{(1-q^d)},$$

$\frac{G_{4m}(i)}{\Omega^{4m}}=\frac{H_m}{(4m)!}$, π,Ω – periods of $\zeta(s)$ and of $E: y^2=4x^3-4x$.

Classical modular forms

are introduced as certain holomorphic functions on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$, which can be regarded as a homogeneous space for the group $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$:

$$\mathbb{H} = \mathrm{GL}_2(\mathbb{R})/\mathrm{O}(2) \cdot Z, \tag{1}$$

where $Z = \{\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\ | x \in \mathbb{R}^\times\}$ is the center of $G(\mathbb{R})$ and $\mathrm{O}(2)$ is the orthogonal group. The group $\mathrm{GL}_2^+(\mathbb{R})$ of matrices $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ with positive determinant acts on $\mathbb{H}$ by fractional linear transformations; on cosets this action transforms into the natural action by group shifts. Let Γ be a subgroup of finite index in the modular group $\mathrm{SL}_2(\mathbb{Z})$.

Definition of a modular form

A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of (integral) weight k with respect to Γ iff the conditions a) and b) are satisfied:

- a) *Automorphy condition*

$$f((a_\gamma z + b_\gamma)/(c_\gamma z + d_\gamma)) = (c_\gamma z + d_\gamma)^k f(z) \qquad (2)$$

  for all elements $\gamma \in \Gamma$;
- b) *Regularity at cusps*: f is regular at cusps $z \in \mathbb{Q} \cup i\infty$ (the cusps can be viewed as fixed points of parabolic elements of Γ); this means that for each element $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the function $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ admits a Fourier expansion over non–negative powers of $q^{1/N} = e(z/N)$ for a natural number N. One writes traditionally

$$q = e(z) = \exp(2\pi i z).$$

  A modular form $f(z) = \sum_{n=0}^{\infty} a(n) e(nz/N)$ is called a cusp form if f vanishes at all cusps (i.e. if the above Fourier expansion contains only positive powers of $q^{1/N}$), see [LangMF], [MaPa]

The complex vector space of all modular (resp. cusp) forms of weight k with respect to Γ is denoted by $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$).
A basic fact from the theory of modular forms is that the spaces of modular forms are finite dimensional. Also, one has $\mathcal{M}_k(\Gamma)\mathcal{M}_l(\Gamma) \subset \mathcal{M}_{k+l}(\Gamma)$. The direct sum

$$\mathcal{M}(\Gamma) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma)$$

turns out to be a graded algebra over $\mathbb{C}$ with a finite number of generators.
An example of a modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$ of weight $k \geq 4$ is given by the *Eisenstein series*

$$G_k(z) = \sum_{m_1, m_2 \in \mathbb{Z}}{}' (m_1 + m_2 z)^{-k} \tag{3}$$

(prime denoting $(m_1, m_2) \neq (0,0)$). For these series the automorphy condition (2) can be deduced straight from the definition. One has $G_k(z) \equiv 0$ for odd k and

$$G_k(z) = \frac{2(2\pi i)^k}{(k-1)!} \left[-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) e(nz) \right], \tag{4}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k^{th} *Bernoulli number*.
The graded algebra $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to the polynomial ring of the (independent) variables G_4 and G_6.

Examples

Recall that B_k denote the Bernoullli numbers defined by the development

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}B_k\frac{x^k}{k!},$$

One has for even $k\geq 2$, $2\zeta(k)=-\frac{(2\pi i)^k B_k}{k!}$, $G_k(z)=\frac{2(2\pi i)^k}{(k-1)!}\left[-\frac{B_k}{2k}+\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right]$, $E_k(z)=1-\frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$

$$E_4(z)=1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\in\mathcal{M}_4(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_6(z)=1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\in\mathcal{M}_6(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_8(z)=1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\in\mathcal{M}_8(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{10}(z)=1-264\sum_{n=1}^{\infty}\sigma_9(n)q^n\in\mathcal{M}_{10}(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{12}(z)=1+\frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n\in\mathcal{M}_{12}(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{14}(z)=1-24\sum_{n=1}^{\infty}\sigma_{13}(n)q^n\in\mathcal{M}_{14}(\mathrm{SL}(2,\mathbb{Z})).$$ (Proof see in [Se70]).

```
with PARI/GP:
gp > k=14;Ek=1-(2*k)/bernfrac(k)*sum(d=1,20,d^(k-1)*q^d/(1-q^d)+O(q^4>
% = 1 - 24*q - 196632*q^2 - 38263776*q^3 + O(q^4)
```

Fast computation of the Ramanujan function:

Put $h_k := \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n = \sum_{d=1}^{\infty} \frac{d^{k-1} q^d}{1-q^d}$. The classical fact is that $\Delta = (E_4^3 - E_6^2)/1728$ where $E_4 = 1 + 240 h_4$ and $E_6 = 1 - 504 h_6$.

Computing with PARI-GP see [BBBCO], The PARI/GP number theory system), http://pari.math.u-bordeaux.fr $h_k := \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n = \sum_{d=1}^{\infty} \frac{d^{k-1} q^d}{1-q^d} \Longrightarrow$

```
gp > h6=sum(d=1,20,d^5*q^d/(1-q^d)+O(q^20))
gp > h4=sum(d=1,20,d^3*q^d/(1-q^d)+O(q^20)
gp > Delta=((1+240*h4)^3-(1-504*h6)^2)/1728
```

Congruence of Ramanujan $\tau(n) \equiv \sum_{d|n} d^{11} \mod 691$:

```
gp > (Delta-h12)/691
% = -3*q^2 - 256*q^3 - 6075*q^4 - 70656*q^5 - 525300*q^6 + O(q^7)
```

More programs of computing $\tau(n)$ (see [Sl])

```
(MAGMA) M12:=ModularForms(Gamma0(1), 12); t1:=Basis(M12)[2];
PowerSeries(t1[1], 100); Coefficients($1);
(PARI) a(n)=if(n<1, 0, polcoeff(x*eta(x+x*O(x^n))^24, n))
(PARI) {tau(n)=if(n<1, 0, polcoeff(x*(sum(i=1, (sqrtint(8*n-7)+1)\2,
(-1)^i*(2*i-1)*x^((i^2-i)/2), O(x^n)))^8, n));}
gp > tau(6911)
%3 = -615012709514736031488
```

Geometric algebra (see [Dieudonné], [Garrett])

- $\mathrm{GL}(n)$ (geometric study).
- Bilinear and Hermitian forms, classical groups
- Witt theorem and extensions of isometries

This section is based on notions of geometric algebra.
Concerning matrix notation, for a rectangular matrix $A = (a_{ij})$ let ${}^tA$ denote the transpose of A. If the entries of A belong to a ring D with involution muni d'involution σ, let A^σ given by $A^\sigma_{ij} = a^\sigma_{ij}$.
Geometric study of $\mathrm{GL}(n)$ and its subgroups. The group $\mathrm{GL}(n)$ is a basic classical group showing the most interesting phenomena used in many other situations. The general linear group $\mathrm{GL}(n, k)$ is the group of all invertible $n \times n$ matrices with entries in a commutative field k.The special linear group $\mathrm{SL}(n, k)$ is its subgroup of all $n \times n$ of determinant 1.
For an approach less dependant of coordonnates fix a k-vector space V of dimension n and let $\mathrm{GL}_k(V)$ be the group of all k-linear automorphisms of V . Any choice of a base in V gives an isomorphism $\mathrm{GL}_k(V) \to \mathrm{GL}(n, k)$ using the matrix of linear mapping in the chosen base. Let $e_1, \cdots, e_n$ the standard bases of k^n giving the isomorphhism $\mathrm{GL}_k(k^n) \to \mathrm{GL}(n, k)$.

Conjugation of parabolic subgroups.

Let $V = k^n$ and $\mathfrak{F}$ the standard flag of type $(d_1, \cdots, d_m)$, the parabolic subgroup $P_{\mathfrak{F}}$ is represented by blocs

$$\begin{pmatrix} d_1 \times d_1 & * & * & * \\ & (d_2 - d_1) \times (d_2 - d_1) & * & * \\ & & \ddots & * \\ & \cdots & \cdots & \\ 0 & 0 & 0 & (n - d_m) \times (n - d_m) \end{pmatrix}$$

Any $g \in P = P_{\mathfrak{F}}$ induce a natural mapping on the quotients $Vd_i / V_{d_{i-1}}$, where $V_{d_0} = 0$ and $V_{d_{m+1}} = V$).

Then the unipotent radical $R_u P =$
$\{p \in P_{\mathfrak{F}} \mid p = id \text{ on } Vd_i / V_{d_{i-1}} \text{ on } V / V_{d_m}\}$ is represnted by

$$\begin{pmatrix} 1_{d_1} & * & * & * \\ & 1_{d_2 - d_1} & * & * \\ & & \ddots & * \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 1_{n - d_m} \end{pmatrix}.$$

Levi components and conjugation

Choose a complement V'_{n-d_i} of V_{d_i} in V with the property $V'_{n-d_m} \subset \cdots \subset V'_{n-d_1}$ (an opposit flag $\mathcal{F}'$ of $\mathcal{F}$) with the opposit parabolic $P' = P_{\mathcal{F}'}$. Then $M = P \cap P'$ is called a complementary Levi component in $P = M \ltimes R_u P$, a standard semi-direct product. Then the standard Levi component is the group of matrices of the form

$$\begin{pmatrix} d_1 \times d_1 & 0 & 0 & 0 \\ & (d_2 - d_1) \times (d_2 - d_1) & 0 & 0 \\ & & \ddots & 0 \\ & \dots & \dots & \\ 0 & 0 & 0 & (n-d_m) \times (n-d_m) \end{pmatrix}.$$

Proposition

a) All the parabolic subgroups of given type are conjugate in $\mathrm{GL}_k(V)$

b) All the Levi components of parabolic subgroup P *are conjugate by elements of* P

c) All the maximal k*-split tori are conjugate in* $\mathrm{GL}_k(V)$*.*

∎

Extension to modules over a scew field

This section applies unchanged when k is replaced by a scew field (a division ring) D. Without coordinates, define a vector space V of finite dimension over a scew field (a division ring) D as a finitely generated (left or right) module.

If D is not commutative, there is a modification in viewing at D-lineair endomorphisms.The the ring $\mathrm{End}_D(V)$ of all D-lineair endomorphisms does not contain D naturallurally. Then a choice of D-bases for a vector space D of given dimension gives an isomorphism $\mathrm{End}_D(V)$ to $n \times n$ matrices with coeffcients in D^{opp}, where D^{opp} is the opposite ring to D, i.e.with the same additive group D but with the multiplication $*$, given by $x * y = yx$ where yx is the multiplication in D. The linear group $\mathrm{GL}(n, D)$ over D is the group of all the invertible $n \times n$ matrices over D. A version without coordinates is $GL_D(V)$,and a choice of D-bases of V gives an isomorphism $\mathrm{GL}_D(V) \to \mathrm{GL}(n, D^{opp})$. Definitions concerning flags and parabolics are identical to the commutative case. A flag $\mathcal{F}$ in V is a chain $\mathcal{F} = (V_{d_1} \subset V_{d_2} \subset \cdot \subset V_{d_m})$ of subspaces.

Proposition

a) All the parabolic subgroups of a given type are conjugate in $\mathrm{GL}_D(V)$
b) All the Levi components of parabolic subgroup P are conjugate by elements of P.

∎

Bilinear, sesquilinear and Hermitian forms; classical groups

The classical groups are defined as certain isomtries or similitudes of "formes" on the vector spaces. First, orthogonal and symplectic groups are defined. These can be included into more general families.

Bilinear forms, symmetric and symplectic forms

Let $Q(v) = \langle v, v\rangle$ be the quadratic form attached to a symmetric bilinear form $\langle\cdot,\cdot\rangle$ on a k-vector space V.
The associated orthogonal group $O(Q)$ is the group of isometries of Q (or of $\langle\cdot,\cdot\rangle$), defined as

$$O(Q) = O(\langle\cdot,\cdot\rangle) = \{g \in GL_k(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle\},$$

and the group of orthogonal similitudes is $GO(Q) = GO(\langle\cdot,\cdot\rangle)$

$$= \{g \in GL_k(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \nu(g)\langle v_1, v_2\rangle\}$$

If $\forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = -\langle v_1, v_2\rangle$, then the bilinear form $f : V \times V \to k, f(v_1, v_2) = \langle v_1, v_2\rangle$ is said symplectic.The symplectic group attached to f is the group of isometries of the form $f = \langle v_1, v_2\rangle$ defined by
$\mathrm{Sp}(f) = \{g \in GL_k(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle\}$, puis the group of symplectis symilitudes $\mathrm{GSp}(f) =$
$\{g \in GL_k(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \nu(g)\langle v_1, v_2\rangle\}$.

Sesquilinear formes, Hermitian and antihermitian forms

Let K be a quadratic extension of k, its subfield fixed be the involution σ.

Definition

a) A k-bilinear form $f : V \times V \to K, f(v_1, v_2) = \langle v_1, v_2 \rangle$ on a K-vectorspace V of finite dimension is said sesquilinear (with an implicite reference to σ) if $\langle \alpha v_1, \beta v_2 \rangle = \alpha^\sigma \beta \langle v_1, v_2 \rangle$ ($\forall \alpha, \beta \in K$ and $v_1, v_2 \in V$).
b) A sesquilinear form $f(v_1, v_2) = \langle v_1, v_2 \rangle$ on a K-vector space V of finite dimension is said hermitian if $\forall v_1, v_2 \in V, \langle v_2, v_1 \rangle = \langle v_1, v_2 \rangle^\sigma$.

The unitary group $U(f)$ is the group of isometries f (or of $\langle \cdot, \cdot \rangle$), defined as

$$U(f) = U(\langle \cdot, \cdot \rangle) = \{g \in GL_K(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle\},$$

and the group of unitary similitudes is $GU(f) = GU(\langle \cdot, \cdot \rangle)$

$$= \{g \in GL_K(V), \exists \nu(g) \in K^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \nu(g) \langle v_1, v_2 \rangle\}$$

Simultaneuos treatment of general isometries groups

Over a division algebra D with an anti-involution σ. Note that $\sigma : D \to D$ satisfies the properties

$$\forall \alpha, \beta \in D, \alpha^{\sigma\sigma} = \alpha, \ (\alpha+\beta)^\sigma = \alpha^\sigma + \beta^\sigma \text{ and } (\alpha\beta)^\sigma = \beta^\sigma \alpha^\sigma$$

Let Z be the center of D. Suppose that D is of finite dimension over Z, and that $k = \{x \in Z | x^\sigma = x\}$. Let V be a D-vector space of finite dimension, and fix $\varepsilon = \pm 1$. Let $f = \langle \cdot, \cdot \rangle, f : V \times V \to D$ a k-bilinear form with values in D on V such that $\forall \alpha, \beta \in D$, $\forall v_1, v_2 \in V, \langle v_2, v_1 \rangle = \varepsilon \langle v_1, v_2 \rangle^\sigma, \ \langle \alpha v_1, \beta v_2 \rangle = \alpha^\sigma \langle v_1, v_2 \rangle \beta$. Such a form is said ε-hermitian on V, and such space V (endowed with $\langle \cdot, \cdot \rangle$) is called a (D, σ, ε)-space.
The group of isometries $U(f)$ of f (or of $\langle \cdot, \cdot \rangle$), is defined as

$$U(f) = U(\langle \cdot, \cdot \rangle) = \{g \in GL_D(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle\},$$

and the group of isometry similitudes is $GU(f) = GU(\langle \cdot, \cdot \rangle)$

$$= \{g \in GL_D(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \nu(g) \langle v_1, v_2 \rangle\}$$

Orthogonalisation and isotropy vectors

A D - vector subspace U in a (D, ε, σ) - vector space admits an orthogonal complement $U^{\perp} = \{u' \in V \langle u', u \rangle = 0, \forall u \in U\}$. Note that $U \cap U^{\perp} = 0$ is not valid in general. The kernel V is denoted $V^{\perp}$. The form is called non degenerate if $V^{\perp} = 0$. Suppose for simplicity that the space V is non-degenerate.

If V_1, V_2 are two (D, ε, σ) - vector spaces endowed with forms, respectively, $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$, then the direct sum $V_1 \oplus V_2$ of D- vector spaces is a (D, ε, σ) - vector space with the form

$$\langle v_1 + v_2, v_1' + v_2' \rangle = \langle v_1, v_1' \rangle_1 \langle v_2, v_2' \rangle_2$$

called the orthogonal sum. In general, two subspaces V_1, V_2 of a (D, ε, σ) - vector space V are orthogonal if $V_1 \subset V_2^{\perp}$ or equivalently, if $V_2 \subset V_1^{\perp}$.

If $\langle v, v \rangle = 0$ for $v \in V$, then v is called an isotropic vecteur. If $\langle v, v' \rangle = 0$ for all $v, v' \in U$ for a subspace U of V then U is a (totally) isotropic. If there is no isotropic non zero vector in U, then U is said anisotropic.

Proposition

Let V be a (D, ε, σ)- non degenerate vector space with a subspace U. Then U is non degenerate iff $V = U \oplus U^{\perp}$, with $U^{\perp}$ non degenerate.

Orthogonalisation in (D, σ, ϵ)-spaces

This is used for classification of orthogonal and hermitians spaces

Proposition

Let V be a (D, ε, σ)- non degenerate vector space. Suppose that the case where $\varepsilon = -1$, $D = k$, and σ trivial is excluded. If the product $\langle \cdot, \cdot \rangle$ does not vanish identically then there exists $v \in V$ with $\langle v, v \rangle \neq 0$. If V is non degenerate then it has an orthogonal basis.

Proof. Suppose that $\langle v, v \rangle = 0$ for all $v \in V$. Then

$$0 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \varepsilon \langle x, y \rangle^{\sigma}.$$

If $\varepsilon = 1$ and the product $\langle x + y, x + y \rangle$ does not vanish identically, then there exist x, y such that $\langle x, y \rangle = 1$. Contradiction. Suppose that $\varepsilon = -1$ and σ non trivial on D. Then there exists $\alpha \in D$ such that $\alpha \neq \alpha^{\sigma}$, with $\omega = \alpha - \alpha^{\sigma}$, $\omega = -\omega^{\sigma}$. If $\langle x, y \rangle$ does not vanish identically then there existe x, y such that $\langle x, y \rangle = 1$. Then one has $0 = \langle \omega x, y \rangle + \varepsilon \langle \omega x, y \rangle^{\sigma} = \omega^{\sigma} \langle x, y \rangle + \langle x, y \rangle^{\sigma} \omega = -\omega + \varepsilon \omega = -2\omega$, Contradiction. ∎

In order to construct an orthogonal basis, one uses induction on dimension. If the dimension of a non degenerate vector space V is 1, then any non-zero forme admits an orthogonal basis. In general, one finds $v \in V$ such that $\langle v, v \rangle \neq 0$. Then $Dv^{\perp}$ is non degenerate and V is the orthogonal direct sum of Dv and $Dv^{\perp}$, by the previous proposition.

Hermitian modular forms.

Automorphic complex L-functions on classical groups. Hecke algebras. The Rankin-Selberg method.

Hermitian modular group $\Gamma_{n,K}$ *and the standard zeta function* $\mathcal{Z}(s;\mathbf{f})$ *(definitions)*

Let $K=\mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field, $\theta=\theta_K$ its quadratic character, $n\in\mathbb{N}, n'=\left[\frac{n}{2}\right]$. The Hermitian group

$$\Gamma_{n,K}=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{GL}_{2n}(\mathcal{O}_K)|M\eta_n M^*=\eta_n\right\}, \eta_n=\begin{pmatrix}0_n & -I_n\\ I_n & 0_n\end{pmatrix}$$

$$\mathcal{Z}(s,\mathbf{f})=\left(\prod_{i=1}^{2n}L(2s-i+1,\theta^{i-1})\right)\sum_{\mathfrak{a}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

(via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a})=\lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a}\subset\mathcal{O}_K$)

$=\prod_{\mathfrak{q}}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q}\subset\mathcal{O}_K$,

with $\deg\mathcal{Z}_{\mathfrak{q}}(X)=2n$, the Satake parameters $t_{i,\mathfrak{q}}, i=1,\cdots,n$),

$\mathcal{D}(s,\mathbf{f})=\mathcal{Z}(s-\frac{\ell}{2}+\frac{1}{2},\mathbf{f})$ (Motivically normalized standard zeta function

with a functional equation $s\mapsto\ell-s;\mathrm{rk}=4n$)

Examples of Hermitian cusp forms

The Hermitian Ikeda lift, [Ike08]. Assume $n = 2n'$ even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \theta(p)p^{2k-2s})^{-1} \prod_{p | D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \theta(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \theta(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X)$$

For $p|D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_{2n}.$$

The first Hermitian lift (even case)

Theorem 5.1 (Case E) of [Ike08] Assume that $n = 2n'$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

In the case when n is odd, consider a similar lifting for a normalized Hecke eigenform $n = 2n' + 1$ is odd. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N$ $\in \mathcal{S}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p, define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_n.$$

The second Hermitian lift (odd case)

Theorem 5.2 (Case O) of [Ike08]. Assume that $n = 2n' + 1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

The lift $Lift^{(n)}(f)$ of f is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem 13.6).

Theorem 18.1 of [Ike08]. Let n, n', and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(n)}(f) \neq 0$. Let $L(s, Lift^{(n)}(f), st)$ be the L-function of $Lift^{(n)}(\mathbf{f})$ associated to $st : {}^L\mathcal{G} \to \mathrm{GL}_{4n}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(n)}(f), st)$ is equal to

$$\prod_{i=1}^{n} L(s + k + n' - i + \frac{1}{2}, f) L(s + k + n' - i + \frac{1}{2}, f, \theta).$$

Moreover, the $4n$ charcteristic roots of $L(s, Lift^{(n)}(f), st)$ given as follows: for $i = 1, \cdots, n$

$$\alpha_p p^{-k-n'+i-\frac{1}{2}}, \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}$$

Functional equation of the lift (Sho Takemori)

There are two cases [Ike08]: the even case (E) and the odd case (O):

$$\begin{cases} f \in S_{2k+1}(\Gamma_0(D), \theta), F = Lift^{(n)}(f) & (E) \\ \text{(the lift is of even degree } n = 2n' \text{ and of weight } 2k+2n') & \\ f \in S_{2k}(\mathrm{SL}(\mathbb{Z})), F = Lift^{(n)}(f) & (O) \\ \text{(the lift is of odd degree } n = 2n'+1 \text{ and of weight } 2k+2n'). & \end{cases}$$

Then, up to bad Euler factors, the standard L-function of $F = Lift^{(n)}(f)$ is given by

$$\prod_{i=1}^{n} L(s+k+n'-i+\tfrac{1}{2}, f)L(s+k+n'-i+\tfrac{1}{2}, f, \theta)$$

$$= \begin{cases} \prod_{i=1}^{2n'} L(s+k+n'-i+\frac{1}{2}, f)L(s+k+n'-i+\frac{1}{2}, f, \theta) & (E) \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s, 2n'+1-i), f) & \\ L(t(s,i), f, \theta)L(t(s, 2n'+1-i), f, \theta) & \\ \prod_{i=1}^{2n'+1} L(s+k+n'-i+\frac{1}{2}, f) & \\ \times L(s+k+n'-i+\frac{1}{2}, f, \theta) & (O) \\ = L(s+k-\frac{1}{2}, f)L(s+k-\frac{1}{2}, f, \theta) & \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s, 2n'+2-i), f) & \\ L(t(s,i), f, \theta)L(t(s, 2n'+2-i), f, \theta) & \end{cases}$$

where $t(s,i) = s+k+n'-i+\frac{1}{2}$.

The Gamma factor $\Gamma_{\mathcal{Z}}(s)$ of Ikeda's lift

In the even case since $(2k+1) - t(s,i) = t(1-s, 2n'+1-i)$, using the Hecke functional equation in the symmetric terms of the product, gives the functional equation of the standard L function of the form $s \mapsto 1-s$, and the gamma factor is given by

$$\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+k+n'-i+1/2)^2 = \Gamma_{\mathfrak{D}}(s+n'+\frac{1}{2}).$$

In the odd case when $f \in S_{2k}(SL_2(\mathbb{Z}))$, the lift is of degree $n = 2n'+1$ and of weight $2k+2n'$. By $2k - t(s,i) = t(1-s, 2n+2-i)$, the standard L functions has functional equation of the form $s \mapsto 1-s$ and the gamma factor is the same. Hence the Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f and used as a pattern, extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of even weight ℓ, which equals in the lifted case to $\ell = 2k+2n'$, where $k = (\ell - 2n')/2 = \ell/2 - n' = \ell/2 - n'$, when the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes (see p.37) $\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-n'+n'-i+(1/2))^2 = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-i+(1/2))^2 = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

Eisenstein series and the Rankin-Selberg method

The (Siegel-Hermite) Eisenstein series $E_{2\ell}^{(n)}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined by
$E_{2\ell}^{(n)}(Z) = \sum_{g \in \Gamma_{K,\infty}^{(n)} \backslash \Gamma_K^{(n)}} (\det g)^{\ell} j(g, Z)^{-2\ell}$. The series converges absolutely for $\ell > n$. Define the normalized Eisenstein series $\mathcal{E}_{2\ell}^{(n)}(Z)$ by $\mathcal{E}_{2\ell}^{(n)}(Z) = 2^{-n} \prod_{i=1}^{n} L(i - 2\ell, \theta^{i-1}) \cdot E_{2\ell}^{(n)}(Z)$ If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H, X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in a Rankin-Selberg integral.

Also, we set , for $s \in \mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$,

$$E(Z, s, \ell, \psi) = \sum_{g \in C_\infty \backslash C} \psi(g)(\det g)^{\ell} j(g, Z)^{-2\ell} |(\det g) j(g, Z)|^{-s}.$$

A Rankin-Selberg integral representation: the simplest case

Let us recall a Rankin-Selberg integral representation in the simplest elliptic modular case of GL_2.

An integral representation of Rankin-Selberg type

Include reading in the simplest elliptic modular case of GL_2

The integral representation of Rankin-Selberg type in the Hermitian modular case:

Theorem 4.1 (Shimura, Klosin), see [Bou16], p.13.

Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(C, \psi))$ of scalar weight ℓ, ψ mod $\mathfrak{c}$, such that $\forall \mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s + 3n/2, \mathbf{f}, \chi) =$$
$$\Lambda_\mathfrak{c}(s + 3n/2, \theta\psi\chi) \cdot C_0\langle \mathbf{f}, \theta_\mathcal{T}(\chi)\mathcal{E}(\bar{s} + n, \ell - \ell_\theta, \chi^\rho\psi)\rangle_{C''},$$

where $\mathcal{E}(Z, s, \ell - \ell_\theta, \psi)_{C''}$ is a normalized group theoretic Eisenstein series with components as above of level $\mathfrak{c}''$ divisible by $\mathfrak{c}$, and weight $\ell - \ell_\theta$. Here $\langle\cdot,\cdot\rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level $\mathfrak{c}''$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^\iota(s + h), \Gamma_n^\iota(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s - j),$$

where $h = 0$ or 1, C_0 a subgroup index.

Automorphic complex L-functions on classical groups: Hecke algebras, motives and Galois representations, [La06]

- §6.5. of [MaPa] Automorphic Forms and The Langlands Program
- 6.2.6 of [MaPa]: The Weil Group and its Representations (cf. [Ta79], [Wei74a])
- Algebraic Hecke-Weil characters
- 6.2.7 of [MaPa]: Zeta Functions, L-Functions and Motives (cf. [Man68], [Del79]).
- Automorphic forms and their weights. Complex analytic weight space. Motivic weights, Introduction to [EHLS].

(reading the following materials)

Kubota-Leopoldt p-adic zeta function and Iwasawa algebra.

Zeta values and Bernoulli Numbers A key result in number theory is the Euler product expansion of the Riemann zeta $\zeta(s)$:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \qquad \text{(defined for } \mathrm{Re}(s) > 1).$$

The set of arguments s for which $\zeta(s)$ is defined was extended by Riemann to all $s \in \mathbb{C}$, $s \neq 1$. The special values $\zeta(1-k)$ at negative integers are rational numbers: $\zeta(1-k) = -\frac{B_k}{k}$, satisfying certain Kummer congruences $\bmod p^m$, where B_k are Bernoulli numbers, defined by the

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{te^t}{e^t - 1}; B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = B_5 = \cdots = 0, B_4 = -\frac{1}{30},$$

$$B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = \frac{691}{2730},\ B_{14} = -\frac{7}{6}, \zeta(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

The denominators of B_k are small (Sylvester-Lipschitz):
$\forall c \in \mathbb{Z} \Longrightarrow c^k(c^k - 1)\frac{B_k}{k} \in \mathbb{Z}$ (see in [Mi-St]), Bernoulli polynomials $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i} = "(x+B)^k"$

$$S_k(N) = \sum_{n=1}^{N-1} n^k = \frac{1}{k+1}\left[B_{k+1}(N) - B_{k+1}\right],$$

$B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$

Bernoulli numbers and Kummer congruences

Kubota and Leopoldt constructed [KuLe64] a p-adic interpolation of these special values, explained by Mazur via a p-adic measure μ_c on $\mathbb{Z}_p$ and Kummer congruences for the Bernoulli numbers, see [Ka78] (p is a prime number, $c > 1$ an integer prime to p). Writing the normalized values

$$\zeta_{(p)}^{(c)}(-k) = (1 - p^k)(1 - c^{k+1})\zeta(-k) = \int_{\mathbb{Z}_p^*} x^k d\mu_c(x)$$

produces the Kummer congruences in the form: for any polynomial $h(x) = \sum_{i=0}^n \alpha_i x^i$ over $\mathbb{Z}$,

$$\forall x \in \mathbb{Z}_p, \sum_{i=0}^n \alpha_i x^i \in p^m\mathbb{Z}_p \Longrightarrow \sum_{i=0}^n \alpha_i \zeta_{(p)}^{(c)}(-i) \in p^m\mathbb{Z}_p,$$

Indeed, integrating the above polynomial $h(x)$ over μ_c produces the congruences. The existence of μ_c is deduced from the above formula for the sum of k-th powers $S_k(p^r)$ for $r \to \infty$, restricted to numbers n, prime to p.
In order to define such a measure μ_c it suffices for any continuous function $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$ to define its integral $\int_{\mathbb{Z}_p} \phi(x) d\mu_c$.
Approximating $\phi(x)$ by a polynomial (when the integral is already defined), pass to the limit (which is well defined due to Kummer congruences).

Kubota-Leopoldt p-adic zeta-function

The domain of definition of p-adic zeta functions is the p-adic analytic group $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ of all continuous p-adic characters of the profinite group $\mathbb{Z}_p^\times$, where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$) (over complex numbers $\mathbb{C} = Hom_{cont}(\mathbb{R}_+^*, \mathbb{C}^*)$, y run the characters $t \mapsto t^s$.

Define $\zeta_p : \mathcal{Y}_p \to \mathbb{C}_p$ on the space as the p-adic Mellin transform

$$\zeta_p(y) = \frac{\int_{\mathbb{Z}_p^*} y(x) d\mu_c(x)}{1 - cy(c)} = \frac{\mathcal{L}_{\mu_c}(y)}{1 - cy(c)},$$

with a single simple pole at $y = y_p^{-1} \in \mathcal{Y}_p$, where $y_p(x) = x$ the inclusion character $\mathbb{Z}_p^* \hookrightarrow \mathbb{C}_p^*$ and $y(x) = \chi(x)x^{k-1}$ is a typical arithmetical character ($y = y_p^{-1}$ becomes $k = 0$, $s = 1 - k = 1$).

Explicitly: Mazur's measure is given by $\mu_c(a + p^v\mathbb{Z}_p) = \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v})$, $B_1(x) = x - \frac{1}{2}$, ([LangMF], Ch.XIII), we see the zeta distribution $\mu_s|_{s=0}(a + (N)) = -B_1(\frac{a}{N})$.

Then the binomial formula

$\int_Z (1+t)^z d\mu_c = \sum_{n=0}^{\infty} t^n \int_Z \binom{z}{n} d\mu_c$, gives the analyticity of $\zeta_p(y)$ on $t = y(1+p) - 1$ in the unit disc $\{t \in \mathbb{C}_p \| \ |t|_p < 1\}$.

The abstract Kummer congruences, p-adic Mellin transform and the Iwasawa algebra

A useful criterion for the existence of a measure with given properties is:

Proposition (The abstract Kummer congruences, see [Ka78])

Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(\mathcal{X}_p, O_p)$ in the ring $\mathcal{C}(\mathcal{X}_p, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group $\mathcal{X}_p$ with values in the ring of integers O_p of $\mathbb{C}$ such that $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on $\mathcal{X}_p$ with the property

$$\int_{\mathcal{X}_p} f_i d\mu = a_i$$

is equivalent to the following congruences, for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(x) \in p^n \mathcal{O}_p \text{ for all } x \in \mathcal{X}_p \text{ implies } \sum_i b_i a_i \in p^n \mathcal{O}_p. \quad (5)$$

Remark

Since $\mathbb{C}_p$-measures are characterised as bounded $\mathbb{C}_p$-valued distributions, every $\mathbb{C}_p$-measures on Y becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.

Proof of proposition 8. The necessity is obvious since

$$\sum_i b_i a_i = \int_{\mathcal{X}_p} (p^n \mathcal{O}_p - \text{valued function}) d\mu = \\ = p^n \int_{\mathcal{X}_p} (\mathcal{O}_p - \text{valued function}) d\mu \in p^n \mathcal{O}_p.$$

In order to prove the sufficiency we need to construct a measure μ from the numbers a_i. For a function $f \in \mathcal{C}(\mathcal{X}_p, \mathcal{O}_p)$ and a positive integer n there exist elements $b_i \in \mathbb{C}$ such that only a finite number of b_i does not vanish, and

$$f - \sum_i b_i f_i \in p^n \mathcal{C}(\mathcal{X}_p, \mathcal{O}_p),$$

according to the density of the $\mathbb{C}$-span of $\{f_i\}$ in $\mathcal{C}(\mathcal{X}_p, \mathbb{C})$. By the assumption (5) the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo p^n (i.e. does not depend on the choice of b_i). Following N.M. Katz ([Ka78]), we denote this value by "$\int_{\mathcal{X}_p} f d\mu \bmod p^n$". Then we have that the limit procedure

$$\int_{\mathcal{X}_p} f d\mu = \lim_{n \to \infty} \text{``}\int_{\mathcal{X}_p} f d\mu \bmod p^n\text{''} \in \varprojlim_n \mathcal{O}_p / p^n \mathcal{O}_p = \mathcal{O}_p,$$

gives the measure μ.

Mazur's measure over $\mathfrak{X}_S = \mathbb{Z}_S$

Let $c > 1$ be a positive integer coprime to $M_0 = \prod_{q \in S} q$ with S being a fixed set of primes containing p. Using the criterion of the proposition 8 we show that the $\mathbb{Q}$ -valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c), \quad f_c(x) = f(cx), \tag{6}$$

turns out to be a measure where $E_k(f)$ are defined in [LangMF], $f \in \mathrm{Step}(\mathfrak{X}, \mathbb{Q}_p)$ and the field $\mathbb{Q}$ is viewed as a subfield of $\mathbb{C}_p$.
Define the generalized Bernoulli polynomials $B_{k,f}^{(M)}(X)$ as

$$\sum_{k=0}^{\infty} B_{k,f}^{(M)}(X) \frac{t^k}{k!} = \sum_{a=0}^{M-1} f(a) \frac{te^{(a+X)t}}{e^{Mt} - 1}, \tag{7}$$

and the generalized sums of powers

$$S_{k,f}(M) = \sum_{a=0}^{M-1} f(a) a^k.$$

Then the definition (7) formally implies that

$$\frac{1}{k}[B_{k,f}^{(M)}(M) - B_{k,f}^{(M)}(0)] = S_{k-1,f}(M), \tag{8}$$

and also we see that

$$B_{k,f}^{(M)}(X) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f} X^{k-i} = B_{k,f} + kB_{k-1,f}X + \cdots + B_{0,f}X^k. \tag{9}$$

The last identity can be rewritten symbolically as

$$B_{k,f}(X) = (B_f + X)^k.$$

The equality (8) enables us to calculate the (generalized) sums of powers in terms of the (generalized) Bernoulli numbers. In particular this equality implies that the Bernoulli numbers $B_{k,f}$ can be obtained by the following p-adic limit procedure (see [LangMF]):

$$B_{k,f} = \lim_{n\to\infty} \frac{1}{Mp^n} S_{k,f}(Mp^n) \quad \text{(a } p\text{-adic limit)}, \tag{10}$$

where f is a $\mathbb{C}_p$-valued function on $\mathcal{X}_p = \mathbb{Z}_S$. Indeed, if we replace M in (8) by Mp^n with growing n and let D be the common denominator of all coefficients of the polynomial $B_{k,f}^{(M)}(X)$. Then we have from (9) that

$$\frac{1}{k}\left[B_{k,f}^{(Mp^n)}(M) - B_{k,f}^{(M)}(0)\right] \equiv B_{k-1,f}(Mp^n) \pmod{\frac{1}{kD}p^2 n}. \tag{11}$$

The proof of (10) is accomplished by division of (11) by Mp^n and by application of the formula (8).

Now we can directly show that the distribution E_k^c defined by (6) are in fact bounded measures. If we use (5) and take the functions $\{f_i\}$ to be all of the functions in $\mathrm{Step}(\mathcal{X}_p, O_p)$. Let $\{b_i\}$ be a system of elements $b_i \in \mathbb{C}_p$ such that for all $x \in \mathcal{X}_p$ the congruence

$$\sum_i b_i f_i(x) \equiv 0 \pmod{p^n} \tag{12}$$

holds. Set $f = \sum_i b_i f_i$ and assume (without loss of generality) that the number n is large enough so that for all i with $b_i \neq 0$ the congruence

$$B_{k,f_i} \equiv \frac{1}{Mp^n} S_{k,f_i}(Mp^n) \pmod{p^n} \tag{13}$$

is valid in accordance with (10). Then we see that

$$B_{k,f} \equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}, \tag{14}$$

hence we get by definition (6):

$$\begin{aligned} E_k^c(f) &= B_{k,f} - c^k B_{k,f_c} && (15) \\ &\equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i \left[f_i(a)a^k - f_i(ac)(ac)^k \right] \pmod{p^n}. \end{aligned}$$

Let $a_c \in \{0, 1, \cdots, Mp^n - 1\}$, such that $a_c \equiv ac \pmod{Mp^n}$, then the map $a \longmapsto a_c$ is well defined and acts as a permutation of the set $\{0, 1, \cdots, Mp^n - 1\}$, hence (15) is equivalent to the congruence

$$E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv \sum_i \frac{a_c^k - (ac)^k}{Mp^n} \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}. \tag{16}$$

Now the assumption (11) formally inplies that $E_k^c(f) \equiv 0 \pmod{p^n}$, completing the proof of the abstact congruences and the construction of measures E_k^c.

Remark

The formula (15) *also implies that for all* $f \in \mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$ *the following holds*

$$E_k^c(f) = kE_1^c(x_p^{k-1} f) \tag{17}$$

where $x_p : \mathcal{X}_p \longrightarrow \mathbb{C}_p \in \mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$ *is the composition of the projection* $\mathcal{X}_p \longrightarrow \mathbb{Z}_p$ *and the embedding* $\mathbb{Z}_p \hookrightarrow \mathbb{C}_p$*.*

Indeed if we put $a_c = ac + Mp^n t$ for some $t \in \mathbb{Z}$ then we see that

$$a_c^k - (ac)^k = (ac + Mp^n t)^k - (ac)^k \equiv kMp^n t (ac)^{k-1} \pmod{(Mp^n)^2},$$

and we get that in (16):

$$\frac{a_c^k - (ac)^k}{Mp^n} \equiv k(ac)^{k-1} \frac{a_c - ac}{Mp^n} \pmod{Mp^n}.$$

The last congruence is equivalent to saying that the abstract Kummer congruences (5) are satisfied by all functions of the type $x_p^{k-1} f_i$ for the measure E_1^c with $f_i \in \mathrm{Step}(\mathcal{X}_p, \mathbb{C}_p)$ establishing the identity (17).

The domain of definition of the non-Archimedean zeta functions

In the classical case the set on which zeta functions are defined is the set of complex numbers $\mathbb{C}$ which may be viewed equally as the set of all continuous characters (more precisely, quasicharacters) via the following isomorphism:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}_+^\times, \mathbb{C}^\times) \\ s & \longmapsto & (x \longmapsto x^s) \end{array} \qquad (18)$$

The construction which associates to a function $h(x)$ on $\mathbb{R}_+^\times$ (with certain growth conditions as $x \to \infty$ and $x \to 0$) the following integral

$$L_h(s) = \int_{\mathbb{R}_+^\times} h(x) x^s \frac{dx}{x}$$

(which converges probably not for all values of s) is called the *Mellin transform*.

For example, if $\zeta(s)=\sum_{n\geq 1} n^{-s}$ is the Riemann zeta function, then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(x)=1/(1-e^{-x})$:

$$\zeta(s)\Gamma(s)=\int_0^\infty \frac{1}{1-e^{-x}}x^s\frac{dx}{x}, \tag{19}$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s)>1$. For an arbitrary function of type

$$f(z)=\sum_{n=1}^{\infty} a(n)e^{2i\pi nz}$$

with $z=x+iy\in\mathbb{H}$ in the upper half plane $\mathbb{H}$ and with the growth condition $a(n)=O(n^c)$ $(c>0)$ on its Fourier coefficients, we see that the zeta function

$$L(s,f)=\sum_{n=1}^{\infty} a(n)n^{-s},$$

essentially coincides with the Mellin transform of $f(z)$, that is

$$\frac{\Gamma(s)}{(2\pi)^s}L(s,f)=\int_0^\infty f(iy)y^s\frac{dy}{y}. \tag{20}$$

Both sides of the equality (20) converge absolutely for $\mathrm{Re}(s) > 1 + c$. The identities (19) and (20) are immediately deduced from the well known integral representation for the gamma-function

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}, \qquad (21)$$

where $\frac{dy}{y}$ is a measure on the group $\mathbb{R}_+^\times$ which is invariant under the group translations (Haar measure). The integral (21) is absolutely convergent for $\mathrm{Re}(s) > 0$ and it can be interpreted as the integral of the product of an additive character $y \mapsto e^{-y}$ of the group $\mathbb{R}^{(+)}$ restricted to $\mathbb{R}_+^\times$, and of the multiplicative character $y \mapsto y^s$, integration is taken with respect to the Haar measure $\mathrm{d}y/y$ on the group $\mathbb{R}_+^\times$.

p-adic Mellin transform

In the theory of the non-Archimedean integration one considers the group $\mathbb{Z}_S^\times$ (the group of units of the S-adic completion of the ring of integers $\mathbb{Z}$) instead of the group $\mathbb{R}_+^\times$, and the Tate field $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$) instead of the complex field $\mathbb{C}$. The domain of definition of the p-adic zeta functions is the p-adic analytic group

$$\mathcal{Y}_S = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_S^\times, \mathbb{C}_p^\times) = \mathcal{Y}(\mathbb{Z}_S^\times), \tag{22}$$

where:

$$\mathbb{Z}_S^\times \cong \oplus_{q \in S} \mathbb{Z}_q^\times,$$

and the symbol

$$\mathcal{Y}(G) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_p^\times) \tag{23}$$

denotes the functor of all p-adic characters of a topological group G.

The analytic structure of $\mathcal{Y}_S$

Let us consider in detail the structure of the topological group $\mathcal{Y}_S$. Define

$$U_p = \{x \in \mathbb{Z}_p^\times \quad | \quad x \equiv 1 \pmod{p^\nu}\},$$

where $\nu = 1$ or $\nu = 2$ according as $p > 2$ or $p = 2$. Then we have the natural decomposition

$$\mathcal{Y}_S = \mathcal{Y}\left((\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times\right) \times \mathcal{Y}(U_p). \tag{24}$$

The analytic structure on $\mathcal{Y}(U_p)$ is defined by the following isomorphism (which is equivalent to a special choice of a local parameter):

$$\varphi : \mathcal{Y}(U_p) \xrightarrow{\sim} T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\},$$

where $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topoplogical generator of the multiplicative group $U_p \cong \mathbb{Z}_p$. An arbitrary character $\chi \in \mathcal{Y}_S$ can be uniquely represented in the form $\chi = \chi_0\chi_1$ where χ_0 is trivial on the component U_p, and χ_1 is trivial on the other component

$$(\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times.$$

The character χ_0 is called the *tame component*, and χ_1 the *wild component* of the character χ. We denote by the symbol $\chi_{(t)}$ the (wild) character which is uniquely determined by the condition

$$\chi_{(t)}(1+p^\nu) = t$$

with $t \in \mathbb{C}_p$, $|t|_p < 1$.
In some cases it is convenient to use another local coordinate s which is analogous to the classical argument s of the Dirichlet series:

$$\begin{array}{ccc} \mathcal{O}_p & \longrightarrow & \mathcal{Y}_S \\ s & \longmapsto & \chi^{(s)}, \end{array}$$

where $\chi^{(s)}$ is given by
$\chi^{(s)}((1+p^\nu)^\alpha) = (1+p^\nu)^{\alpha s} = \exp(\alpha s \log(1+p^\nu))$. The character $\chi^{(s)}$ is defined only for such s for which the series exp is p-adically convergent (i.e. for $|s|_p < p^{\nu-1/(p-1)}$). In this domain of values of the argument we have that $t = (1+p^\nu)^s - 1$. But, for example, for $(1+t)^{p^n} = 1$ there is certainly no such value of s (because $t \neq 1$), so that the s-coordonate parametrizes a smaller neighborhood of the trivial character than the t-coordinate (which parametrizes all wild characters) (see [Ma73], [Ma76]).

Recall that an analytic function $F : T \longrightarrow \mathbb{C}_p$ ($T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\}$), is defined as the sum of a series of the type $\sum_{i\geq 0} a_i(t-1)^i$ ($a_i \in \_Cp$), which is assumed to be absolutely convergent for all $t \in T$. The notion of an analytic function is then obviously extended to the whole group $\mathcal{Y}_S$ by shifts. The function

$$F(t) = \sum_{i=0}^{\infty} a_i(t-1)^i$$

is bounded on T iff all its coefficients a_i are universally bounded. This last fact can be easily deduced for example from the basic properties of the Newton polygon of the series $F(t)$ (see [Ko80], [Am-V]). If we apply to these series the Weierstrass preparation theorem (see [Ko80], [Ma73]), we see that in this case the function F has only a finite number of zeroes on T (if it is not identically zero).

p-adic analytic functions on $\mathcal{Y}_S$

Consider the torsion subgroup $\mathcal{Y}_S^{\rm tors} \subset \mathcal{Y}_S$. This subgroup is discrete in $\mathcal{Y}_S$ and its elements $\chi \in \mathcal{Y}_S^{\rm tors}$ can be obviously identified with primitive Dirichlet characters $\chi \mod M$ such that the support $S(\chi) = S(M)$ of the conductor of χ is containded in S. This identification is provided by a fixed embedding denoted

$$i_p : \overline{\mathbb{Q}}^\times \hookrightarrow \mathbb{C}_p^\times$$

if we note that each character $\chi \in \mathcal{Y}_S^{\rm tors}$ can be factored through some finite factor group $(\mathbb{Z}/M\mathbb{Z})^\times$:

$$\chi : \mathbb{Z}_S^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \overset{i_p}{\hookrightarrow} \mathbb{C}_p^\times,$$

and the smallest number M with the above condition is the conductor of $\chi \in \mathcal{Y}_S^{\rm tors}$.

The symbol x_p will denote the composition of the natural projection $\mathbb{Z}_S^\times \to \mathbb{Z}_p^\times$ and of the natural embedding $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, so that $x_p \in \mathcal{Y}_S$ and all integers k can be considered as the characters $x_p^k : y \longmapsto y^k$.

Let us consider a bounded $\mathbb{C}_p$-analytic function F on $\mathcal{Y}_S$. The above statement about zeroes of bounded $\mathbb{C}_p$-analytic functions implies now that the function F is uniquely determined by its values $F(\chi_0\chi)$, where χ_0 is a fixed character and χ runs through all elements $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ with possible exclusion of a finite number of characters in each analyticity component of the decomposition (24). This condition is satisfied, for example, by the set of characters $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ with the S-complete conductor (i.e. such that $S(\chi) = S$), and even for a smaller set of characters, for example for the set obtained by imposing the additional assumption that the character χ^2 is not trivial (see [Ma73]). Let μ be a (bounded) $\mathbb{C}_p$-valued measure on $\mathbb{Z}_S^\times$. Then the *non-Archimedean Mellin transform* of the measure μ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbb{Z}_S^\times} x \mathrm{d}\mu, \qquad (x \in \mathcal{Y}_S), \tag{25}$$

which represents a bounded $\mathbb{C}_p$-analytic function

$$L_\mu : \mathcal{Y}_S \longrightarrow \mathbb{C}_p. \tag{26}$$

Indeed, the boundedness of the function L_μ is obvious since all characters $x \in \mathcal{Y}_S$ take values in O_p and μ also is bounded. The analyticity of this function expresses a general property of the integral (25), namely that it depends analytically on the parameter $x \in \mathcal{Y}_S$. However, we give below a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description will also imply that every bounded $\mathbb{C}_p$-analytic function on $\mathcal{Y}_S$ is the Mellin transform of a certain measure μ.

The Iwasawa algebra

Let $\mathcal{O}$ be a closed subring in $\mathcal{O}_p = \{z \in \mathbb{C}_p \quad | \quad |z|_p \le 1\}$,

$$G = \varprojlim_i G_i, \qquad (i \in I),$$

a profinite group. Then the canonical homomorphism $G_i \xleftarrow{\pi_{ij}} G_j$ induces a homomorphism of the corresponding group rings

$$\mathcal{O}[G_i] \longleftarrow \mathcal{O}[G_j].$$

Then the *completed group ring* $\mathcal{O}[[G]]$ is defined as the projective limit

$$\mathcal{O}[[G]] = \varprojlim_i \mathcal{O}[[G_i]], \qquad (i \in I).$$

Let us consider also the set $\mathrm{Dist}(G, \mathcal{O})$ of all $\mathcal{O}$-valued distributions on G which itself is an $\mathcal{O}$-module and a ring with respect to multiplication given by the *convolution of distributions*, which is defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \longrightarrow \mathcal{O},$$

(see the previous section) as follows:

We noticed above that the theorem 9 would imply a description of $\mathbb{C}_p$-analytic bounded functions on $\mathcal{Y}_S$ in terms of measures. Indeed, these functions are defined on analyticity components of the decomposition (24) as certain power series with p-adically bounded coefficients, that is, power series, whose coefficients belong to $\mathcal{O}_p$ after multiplication by some constant from $\mathbb{C}_p{}^\times$. Formulas for coefficients of these series can be also deduced from the proof of the theorem. However, we give a more direct computation of these coefficients in terms of the corresponding measures. Let us consider the component aU_p of the set $\mathbb{Z}_S^\times$ where

$$a \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q\neq} \mathbb{Z}_q^\times,$$

and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on U_p defined by restriction of μ to the subset $aU_p \subset \mathbb{Z}_S^\times$.

Consider the isomorphism $U_p \cong \mathbb{Z}_p$ given by:

$$y = \gamma^x \quad (x \in \mathbb{Z}_p, y \in U_p),$$

with some choice of the generator γ of U_p (for example, we can take $\gamma = 1 + p^\nu$). Let μ'_a be the corresponding measure on $\mathbb{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbb{Z}_p} \binom{x}{i} d\mu'_a(x) = a_i, \tag{30}$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbb{C}_p$-span of the family

$$\left\{ \binom{x}{i} \right\} \quad (i \in \mathbb{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbb{Z}_p, \mathcal{O}_p)$ according to Mahler's interpolation theorem for continuous functions on $\mathbb{Z}_p$). Indeed, from the basic properties of the interpolation polynomials it follows that

$$\sum_i b_i \binom{x}{i} \equiv 0 \pmod{p^n} \quad (\text{for all } x \in \mathbb{Z}_p) \Longrightarrow b_i \equiv 0 \pmod{p^n}.$$

We can now apply the abstract Kummer congruences (see proposition 8), which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exists a measure with the property (30).

Coefficients of power series and the Iwasawa isomorphism

We state that the Mellin transform L_{μ_a} of the measure μ_a is given by the power series $F_a(t)$ with coefficients as in (30), that is

$$\int_{U_p} \chi_{(t)}(y)\mathrm{d}\mu(ay) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu_a'(x) \right) (t-1)^i \qquad (31)$$

for all wild characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that (31) is valid for all characters of the type $y \longmapsto y^m$, where m is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

which implies that

$$\int_{U_p} y^m \mathrm{d}\mu(ay) = \int_{\mathbb{Z}_p} \gamma^{mx} \mathrm{d}\mu_a'(x) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu_a'(x) \right) (\gamma^m - 1)^i,$$

establishing (31).

Lecture N°4. p-adic L-functions on classical groups.

Ordinary case. Admissible measures, special values. ("Fonctions L p-adiques sur les groupes classiques : cas ordinaire, mesures admissibles, valeurs spéciales").

Admissible measures: Definition.

Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, \cdots, h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform.

Theorem

([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Complex and p-adic L-functions on classical groups.

- Automorphic forms and their weights. Complex analytic weight space. Motivic weights, introduction [EHLS].
- 6.2.6 The Weil Group and its Representations (cf. [Ta79], [Wei74a]).
- 6.2.7 Zeta Functions, L-Functions and Motives (cf. [Man68], [Del79]). [MaPa]
- Algebraic characters
- Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$
- Main Theorem in the Hermitian case

Modular forms as a tool in arithmetic

We view modular forms as:

1) q-power series

$$f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$$ and as

2) holomorphic functions on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$

where $q = \exp(2\pi i z)$, $z \in \mathbb{H}$, and define L-function

$$L(f, s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}$$

for a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ (its Mellin transfor

A famous example: the Ramanujan function $\tau(n)$

The function Δ (of the variable z) is defined by the formal expansion

$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$
$= q \prod_{m=1}^{\infty} (1 - q^m)^{24}$
$= q - 24q^2 + 252q^3 + \cdots$

is a cusp form of weight $k = 12$ for the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$).

$\tau(1) = 1, \tau(2) = -24,$
$\tau(3) = 252, \tau(4) = -1472$
$\tau(m)\tau(n) = \tau(mn)$
for $(n, m) = 1$,
$|\tau(p)| \leq 2p^{11/2}$
for all primes p .

Analytic p-adic theory: zeta values vs. coefficients

It was much developed in the 60th in [Iw], [Se73] and [Wa].

Modular methods are applicable to the p-adic analytic continuation of $\zeta(s)$ itself through the normalized Eisenstein series:

$$\frac{(k-1)!}{2(2\pi i)^k}G_k(z) = \frac{\zeta(1-k)}{2}+\sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n = -\frac{B_k}{2k}+\sum_{d\geq 1}\frac{d^{k-1}q^d}{1-q^d},$$

modular forms of even weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z})$ as follows:

J.-P.Serre noticed [Se73], p.206, that the constant term

$$\frac{\zeta(1-k)}{2}(1-p^{k-1}) \text{ expresses by } \sigma^*_{k-1}(n) = \sum_{d|n} d^{k-1}\ (p \not| d, n \geq 1),$$

the higher coefficients of the normalized Eisenstein series $\bmod p^r$. In this way $\zeta_p^*(1-k)$ can be continually extended to $s \in \mathbb{Z}_p$ with a single simple pole at $s=1$ starting from $s = 1-k$ (see [Se73]).

The Hurwitz numbers naturally appear as the critical values of the Hecke L-function of ideal character $L(s,\psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})N\mathfrak{a}^{-s}$,

$\psi((\alpha)) = \alpha^m, \alpha \equiv 1 \bmod (2+2i)$, also defined for any imaginary quadratic field K, and $g_\psi = \sum_{\mathfrak{a}} \psi(\mathfrak{a})q^{N\mathfrak{a}}$ is a modular form of weight $m+1$. Its p-adic analytic continuation over m and s was constructed by Yu.I.Manin and M.M.Vishik (1974, [Ma-Vi]).

Recall: Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s;\mathbf{f})$

Let $\theta = \theta_K$ be the quadratic character attached to K, $n' = \left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\}, \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$$

$$\mathcal{Z}(s,\mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ (Motivically normalized standard zeta function

with a functional equation $s \mapsto \ell - s; \mathrm{rk} = 4n$)

Main result: p-adic interpolation of all critical values $\mathcal{D}(s,\mathbf{f},\chi)$,
$n \le s \le \ell - n, \chi \bmod p^r$.

Automorphic forms, p-adic theory of weights.

p-adic analytic weight space. Motivic and arithmetical weights, introduction to [EHLS]

[Lan13] , Arithmetic compactifications of PEL-type shimura varieties, London Mathematical Society Monographs, vol. 36, Princeton University Press, 2013.

For the purposes of subsequently defining p-adic modular forms for unitary groups we assume that the PEL data considered also satisfy:

• B has no type D factor;

• $\langle\cdot,\cdot\rangle$: $L \otimes \mathbb{Z}_p \times L \otimes \mathbb{Z}_p \to \mathbb{Z}_p(1)$ is a perfect pairing;

• p $\not|$ Disc($\mathcal{O}_B$), where Disc($\mathcal{O}_B$) is the discriminant of ($\mathcal{O}_B$) over $\mathbb{Z}$ defined in [Lan13, Def. 1.1.1.6]; this condition implies that $(\mathcal{O}_B) \otimes (\mathcal{O}_B)$ is a maximal $(\mathcal{O}_B)$-order in B and that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a product of matrix algebras.

Associate a group scheme $G = G_P$ over $\mathbb{Z}$ with such a PEL datum P: for any $\mathbb{Z}$-algebra R

$$G(R) = \{(g,\nu) \in \mathrm{GL}_{\mathcal{O}_B\otimes R}(L\otimes R)\times R^\times : \langle gx, gy\rangle = \nu\langle x, y\rangle \forall x, y \in L\otimes R\}.$$

Then $G_{/\mathbb{Q}}$ is a reductive group, and by our hypotheses with respect to p, $G_{/\mathbb{Q}}$ is smooth and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact of $G(\mathbb{Q}_p)$.

Definitions of $\mathcal{X}_p$ $\mathcal{Y}_p$ through B, T.

Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$

The Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0, d] \to \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h_+^{j,j}}\Gamma_{\mathbb{R}}(s-j+1)^{h_-^{j,j}}, & \text{if } 2j = w, \end{cases} \text{ where}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s),$$
$$h^{j,j} = h_+^{j,j} + h_-^{j,j}, \sum_j h^{j,w-j} = d.$$

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \dots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$.

Main Theorem (the Hermitian case)

Let $\Omega_{\mathbf{f}} = \langle \mathbf{f}, \mathbf{f} \rangle$ be the period attached to a Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f}_0$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n + 2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s, \chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s, \chi)$ including a finite Euler product, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p\left((\prod_{\mathfrak{q}|p}\prod_{i=1}^n t_{\mathfrak{q},i})p^{-n(n+1)}\right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by:

$$\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} d\mu^*_{\mathcal{D}},\ X = \mathbb{Z}_p^*,\ \text{where } s^* = \ell - n,\ s_* = n.$$

(iii) in the admissible case assume that

$0 < h \le \dfrac{s^* - s_* + 1}{2} = \dfrac{\ell + 1 - 2n}{2}$, where

$h = \mathrm{ord}_p\left((\prod_{\mathfrak{q}|p}\prod_{i=1}^n t_{\mathfrak{q},i})p^{-n(n+1)}\right) > 0$, Then there exist h–admissible measures $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by $i_p\left(A_p(s,\chi)\dfrac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.

(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.

(a) Interpretation of s^*: the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

Proof of the Main Theorem (ii): Kummer congruences

Let us se the notation $\mathcal{D}_p^{alg}(m, \mathbf{f}, \chi) = A_p(s, \chi)\dfrac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}}$

The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m, \chi)$ as Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_{\mathcal{D}}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j, χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi_j) = A_p(s, \chi)\dfrac{\langle \mathbf{f}, h_j\rangle}{\Omega_{\mathbf{f}}}$ with all $h_j = \sum_{\mathcal{T}} b_{j,\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing $\mathbf{f}$ and defined over $\bar{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi) \equiv 0 \mod p^N$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = s^* - s_j, \text{ where } s^* = \ell - n \text{ in our case.}$$

Computing the Petersson products of a given modular form $\mathbf{f}(Z) = \sum_H a_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_H b_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ uses a linear form $\ell_{\mathbf{f}} : h \mapsto \dfrac{\langle \mathbf{f}, h\rangle}{\langle \mathbf{f}, \mathbf{f}\rangle}$ defined over a subfield $k \subset \bar{\mathbb{Q}}$.

Admissible Hermitian case

Let $\mathbf{f} \in \mathcal{S}_k(C;\psi)$ be a Hecke eigenform for the congruence subgroup C of level $\mathfrak{c}$. Let $\mathfrak{p}$ be a prime of K prime to $\mathfrak{c}$, which is inert over F. Then we say that $\mathbf{f}$ is pre-ordinary at $\mathfrak{p}$ if there exists an eigenform $0 \neq \mathbf{f}_0 \in \mathcal{M}_{\{p\}} \subset \mathcal{S}_k(Cp,\psi)$ with Satake parameters $t_{\mathfrak{p},i}$ such that

$$\left\| \left(\prod_{i=1}^{n} t_{\mathfrak{p},i} \right) N(\mathfrak{p})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\|\|_p$ the normalized absolute value at p.

The admissible case corresponds to

$$\left\| \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right\|_p = p^{-h} \text{ for a positive } h > 0.$$

An interpretation of h as the difference $h = P_{N,p}(d/2) - P_H(d/2)$ comes from the above explicit relations.

Existence of h-admissible measures

of Amice-Vélu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p\left(\prod_{\mathfrak{q}|p}\left(\prod_{i=1}^n t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right) = h.$$

We use an easy condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathcal{O}_K \otimes \mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa] and check congruences of the type

$$U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}(a+(p^v)\Big) \in Cp^{vj}\mathcal{O}[[q]]$$

for all $j = 0, 1, \ldots, \varkappa h - 1$. Here $s = j' + s_*$, $\Phi_{j'}(a + (p^v))$ a certain convolution, i.e.

$$\Phi_{j'}(\chi) = \theta(\chi) \cdot \mathcal{E}(s, \chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathcal{E}(s, \chi)$ with any Dirichlet character χ mod p^r. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

Proof of the Main Theorem (iii): (admissible case)

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s,\mathbf{f},\chi)$ and an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on $\mathbf{f}_0$ the Rankin-Selberg integral of $\mathcal{F}_{s,\chi} := \theta(\chi)\cdot\mathcal{E}(s,\chi)$ gives

$$\mathcal{D}^{alg}(s,\mathbf{f},\chi) = \frac{\langle \mathbf{f}_0, \theta(\chi)\cdot\mathcal{E}(s,\chi)\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} \text{ (the Petersson product on } G = GU(\eta_n))$$

$$= \alpha_{\mathbf{f}}^{-v}\frac{\langle \mathbf{f}_0, U(p^v)(\theta(\chi)\cdot\mathcal{E}(s,\chi))\rangle}{\langle \mathbf{f},\mathbf{f}\rangle} = \alpha_{\mathbf{f}}^{-v}\frac{\langle f_0, U(p^v)(\mathcal{F}_{s,\chi})\rangle}{\langle \mathbf{f},\mathbf{f}\rangle}.$$

Modication in the admissible case: instead of Kummer congruences,

to estimate p-adically the integrals of test functions: $M = p^v$:

$$\int_{a+(M)} (x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\int_{a+(M)} x^{j'} d\mathcal{D}^{alg}, \text{ using}$$

the orthogonality of characters and the sequence of zeta distributions

$$\int_{a+(M)} x^j d\mathcal{D}^{alg} = \frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times}\sum_{\chi \bmod M}\chi^{-1}(a)\int_X \chi(x)x^j d\mathcal{D}^{alg},$$

$$\int_X \chi d\mathcal{D}^{alg}_{s_-+j} = \mathcal{D}^{alg}(s^*-j,f,\chi) =: \int_X \chi(x)x^j d\mathcal{D}^{alg}.$$

Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j,\chi} = \sum_\xi v(\xi, s^*-j,\chi)q^\xi$: for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times}\sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\sum_{\chi \bmod M}\chi^{-1}(a)v(p^v\xi, s^*-j',\chi)q^\xi$$

$$\in Cp^{vj}\mathcal{O}[[q]] \quad \text{(This is a quasimodular form if } j' \neq s^*\text{)}$$

The resulting measure $\mu_\mathcal{D}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X, \mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p-adic Mellin transform $\mathcal{L}_{\mu_\mathcal{D}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \mathrm{ord}_p(\alpha)$.

Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients: $\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i} (i = 1, \dots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$.

Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.

The abstract Kummer congruences are checked for a family of test elements.

In the admissible case it suffices to check binomial congruences for the Fourier coefficients as above in place of Kummer congruences.

Thanks for your attention!

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