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Special unitary group

In mathematics, the **special unitary group** of degree n, denoted $\mathrm{SU}(n)$, is the Lie group of $n\times n$ unitary matrices with determinant 1.

(More general unitary matrices may have complex determinants with absolute value 1, rather than real 1 in the special case.)

The group operation is matrix multiplication. The special unitary group is a subgroup of the unitary group $\mathrm{U}(n)$, consisting of all $n\times n$ unitary matrices. As a compact classical group, $\mathrm{U}(n)$ is the group that preserves the standard inner product on $\mathbf{C}^n$.[nb 1] It is itself a subgroup of the general linear group, $\mathrm{SU}(n) \subset \mathrm{U}(n) \subset \mathrm{GL}(n, \mathbf{C})$.

The $\mathrm{SU}(n)$ groups find wide application in the Standard Model of particle physics, especially SU(2) in the electroweak interaction and SU(3) in quantum chromodynamics.[1]

The simplest case, SU(1), is the trivial group, having only a single element. The group SU(2) is isomorphic to the group of quaternions of norm 1, and is thus diffeomorphic to the 3-sphere. Since unit quaternions can be used to represent rotations in 3-dimensional space (up to sign), there is a surjective homomorphism from SU(2) to the rotation group SO(3) whose kernel is $\{+I, -I\}$.[nb 2] SU(2) is also identical to one of the symmetry groups of spinors, Spin(3), that enables a spinor presentation of rotations.

Contents

Properties

The special unitary group $\mathrm{SU}(n)$ is a real Lie group (though not a complex Lie group). Its dimension as a real manifold is $n^2 - 1$. Topologically, it is compact and simply connected.[2] Algebraically, it is a simple Lie group (meaning its Lie algebra is simple; see below).[3]

The center of $\mathrm{SU}(n)$ is isomorphic to the cyclic group Z_n, and is composed of the diagonal matrices $\zeta\, I$ for ζ an n^{th} root of unity and I the $n \times n$ identity matrix.

Its outer automorphism group, for $n \geq 3$, is Z_2, while the outer automorphism group of $\mathrm{SU}(2)$ is the trivial group.

A maximal torus, of rank $n - 1$, is given by the set of diagonal matrices with determinant 1. The Weyl group is the symmetric group S_n, which is represented by signed permutation matrices (the signs being necessary to ensure the determinant is 1).

The Lie algebra of $\mathrm{SU}(n)$, denoted by $\mathfrak{su}(n)$, can be identified with the set of traceless antihermitian $n \times n$ complex matrices, with the regular commutator as Lie bracket. Particle physicists often use a different, equivalent representation: the set of traceless hermitian $n \times n$ complex matrices with Lie bracket given by $-i$ times the commutator.

Lie algebra

The Lie algebra $\mathrm{su}(n)$ of $\mathrm{SU}(n)$ consists of $n \times n$ skew-Hermitian matrices with trace zero.[4] This (real) Lie algebra has dimension $n^2 - 1$. More information about the structure of this Lie algebra can be found below in the section "Lie algebra structure."

Fundamental representation

In the physics literature, it is common to identify the Lie algebra with the space of trace-zero *Hermitian* (rather than the skew-Hermitian) matrices. That is to say, the physicists' Lie algebra differs by a factor of i from the mathematicians'. With this convention, one can then choose generators T_a that are traceless hermitian complex $n \times n$ matrices, where:

$$T_a T_b = \frac{1}{2n}\delta_{ab} I_n + \frac{1}{2}\sum_{c=1}^{n^2-1}(if_{abc} + d_{abc})T_c$$

where the f are the structure constants and are antisymmetric in all indices, while the d-coefficients are symmetric in all indices.

As a consequence, the anticommutator and commutator are:

$$\{T_a, T_b\} = \frac{1}{n}\delta_{ab} I_n + \sum_{c=1}^{n^2-1} d_{abc} T_c$$
$$[T_a, T_b] = i \sum_{c=1}^{n^2-1} f_{abc} T_c\,.$$

The factor of i in the commutation relations arises from the physics convention and is not present when using the mathematicians' convention.

We may also take

$$\sum_{c,e=1}^{n^2-1} d_{ace} d_{bce} = \frac{n^2-4}{n}\delta_{ab}$$

as a normalization convention.

Adjoint representation

In the $(n^2 - 1)$ -dimensional adjoint representation, the generators are represented by $(n^2 - 1) \times (n^2 - 1)$ matrices, whose elements are defined by the structure constants themselves:

$$(T_a)_{jk} = -i f_{ajk}.$$

The group SU(2)

SU(2) is the following group,[5]

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} :\ \ \alpha, \beta \in \mathbf{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}\ ,$$

where the overline denotes complex conjugation.

Diffeomorphism with S^3

If we consider α, β as a pair in $\mathbb{C}^2$, then rewriting $|\alpha|^2 + |\beta|^2 = 1$ with the underlying real coordinates becomes the equation

$$x_1^2 + y_1^2 + x_2^2 + y_2^2 = 1$$

hence we have the equation of the sphere. This can also be seen using an embedding: the map

$$\varphi\colon \mathbf{C}^2 \to \mathrm{M}(2, \mathbf{C})$$
$$\varphi(\alpha, \beta) = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

where M(2, **C**) denotes the set of 2 by 2 complex matrices, is an injective real linear map (by considering $\mathbf{C}^2$ diffeomorphic to $\mathbf{R}^4$ and M(2, **C**) diffeomorphic to $\mathbf{R}^8$). Hence, the restriction of φ to the 3-sphere (since modulus is 1), denoted S^3, is an embedding of the 3-sphere onto a compact submanifold of M(2, **C**), namely $\varphi(S^3)$ = SU(2).

Therefore, as a manifold, S^3 is diffeomorphic to SU(2) and so S^3 is a compact, connected Lie group.

Isomorphism with unit quaternions

The complex matrix:

$$\begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix} \qquad (a,b,c,d \in \mathbb{R})$$

can be mapped to the quaternion:

$$a+bi+cj+dk$$

This map is in fact an isomorphism. Additionally, the determinant of the matrix is the norm of the corresponding quaternion. Clearly any matrix in SU(2) is of this form, and since it has determinant 1 it the corresponding quaternion has norm 1. Thus SU(2) is isomorphic to the unit quaternions.[6]

Lie Algebra

The Lie algebra of SU(2) consists of 2×2 skew-Hermitian matrices with trace zero.[7] Explicitly, this means

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ia & -\overline{z} \\ z & -ia \end{pmatrix} :\ a \in \mathbf{R}, z \in \mathbf{C} \right\} .$$

The Lie algebra is then generated by the following matrices,

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

which have the form of the general element specified above.

These satisfy $u_3u_2 = -u_2u_3 = -u_1$ and $u_2u_1 = -u_1u_2 = -u_3$. The commutator bracket is therefore specified by

$$[u_3, u_1] = 2u_2, \qquad [u_1, u_2] = 2u_3, \qquad [u_2, u_3] = 2u_1 \ .$$

The above generators are related to the Pauli matrices by $u_1 = -i\ \sigma_1$, $u_2 = -i\ \sigma_2$ and $u_3 = -i\ \sigma_3$. This representation is routinely used in quantum mechanics to represent the spin of fundamental particles such as electrons. They also serve as unit vectors for the description of our 3 spatial dimensions in loop quantum gravity.

The Lie algebra serves to work out the representations of SU(2).

The group SU(3)

Topology

The group SU(3) is a simply connected compact Lie group.[8] Its topological structure can be understood by noting that SU(3) acts transitively on the unit sphere S^5 in $\mathbb{C}^3 = \mathbb{R}^6$. The stabilizer of an arbitrary point in the sphere is isomorphic to SU(2), which topologically is a 3-sphere. It then follows that SU(3) is a fiber bundle over the base S^5 with fiber S^3. Since the fibers and the base are simply connected, the simple connectedness of SU(3) then follows by means of a standard topological result (the long exact sequence of homotopy groups for fiber bundles).[9]

Representation theory

The representation theory of SU(3) is well understood.[10] Descriptions of these representations, from the point of view of its complexified Lie algebra $\mathbf{sl}(3;\mathbb{C})$, may be found in the articles on Lie algebra representations or the Clebsch–Gordan coefficients for SU(3).

Lie algebra

The generators, T, of the Lie algebra su(3) of SU(3) in the defining representation, are:

$$T_a = \frac{\lambda_a}{2}.$$

where λ, the Gell-Mann matrices, are the $\mathrm{SU}(3)$ analog of the Pauli matrices for $\mathrm{SU}(2)$:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} & & \\
\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned}$$

These $\lambda_{\rm a}$ span all traceless Hermitian matrices H of the Lie algebra, as required. Note that λ_2, λ_5, λ_7 are antisymmetric.

They obey the relations

$$\begin{aligned}
&[T_a, T_b] = i \sum_{c=1}^{8} f_{abc} T_c \\
&\{T_a, T_b\} = \frac{1}{3}\delta_{ab} I_3 + \sum_{c=1}^{8} d_{abc} T_c \, ,
\end{aligned}$$

(or, equivalently, $\{\lambda_a, \lambda_b\} = \frac{4}{3}\delta_{ab} I_3 + 2\sum_{c=1}^{8} d_{abc}\lambda_c$).

The f are the structure constants of the Lie algebra, given by:

$$f_{123} = 1$$
$$f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}$$
$$f_{458} = f_{678} = \frac{\sqrt{3}}{2},$$

while all other f_{abc} not related to these by permutation are zero. In general, they vanish, unless they contain an odd number of indices from the set {2,5,7}.[nb 3]

The symmetric coefficients d take the values:

$$d_{118} = d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}}$$
$$d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}$$
$$d_{146} = d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}\ .$$

They vanish if the number of indices from the set {2,5,7} is odd.

A generic SU(3) group element generated by a traceless 3×3 hermitian matrix H, normalized as $\operatorname{tr}(H^2) = 2$, can be expressed as a *second order* matrix polynomial in H[11]:

$$\begin{aligned}
&\exp(i\theta H) \\
&= \left[-\tfrac{1}{3} I \sin(\varphi + 2\pi/3)\sin(\varphi - 2\pi/3) - \tfrac{1}{2\sqrt{3}}\ H \sin(\varphi) - \tfrac{1}{4}\ H^2\right] \frac{\exp\left(\tfrac{2}{\sqrt{3}}\ i\theta \sin\varphi\right)}{\cos\left(\varphi + \tfrac{2\pi}{3}\right)\cos\left(\varphi - \tfrac{2\pi}{3}\right)} \\
&\quad + \left[-\tfrac{1}{3}\ I \sin(\varphi) \sin\left(\varphi - \frac{2\pi}{3}\right) - \tfrac{1}{2\sqrt{3}}\ H \sin\left(\varphi + \frac{2\pi}{3}\right) - \tfrac{1}{4}\ H^2\right] \frac{\exp\left(\tfrac{2}{\sqrt{3}}\ i\theta \sin\left(\varphi + \tfrac{2\pi}{3}\right)\right)}{\cos(\varphi)\cos\left(\varphi - \tfrac{2\pi}{3}\right)} \\
&\quad + \left[-\tfrac{1}{3}\ I \sin(\varphi) \sin\left(\varphi + \frac{2\pi}{3}\right) - \tfrac{1}{2\sqrt{3}}\ H \sin\left(\varphi - \frac{2\pi}{3}\right) - \tfrac{1}{4}\ H^2\right] \frac{\exp\left(\tfrac{2}{\sqrt{3}}\ i\theta \sin\left(\varphi - \tfrac{2\pi}{3}\right)\right)}{\cos(\varphi)\cos\left(\varphi + \tfrac{2\pi}{3}\right)}
\end{aligned}$$

where

$$\varphi \equiv \tfrac{1}{3}\left(\arccos\left(\tfrac{3}{2}\sqrt{3}\det H\right) - \tfrac{\pi}{2}\right).$$

Lie algebra structure

As noted above, the Lie algebra $\mathfrak{su}(n)$ of $\mathrm{SU}(n)$ consists of $n \times n$ skew-Hermitian matrices with trace zero.[12]

The complexification of the Lie algebra $\mathfrak{su}(n)$ is $\mathfrak{sl}(n;\mathbb{C})$, the space of all $n \times n$ complex matrices with trace zero.[13] A Cartan subalgebra then consists of the diagonal matrices with trace zero,[14] which we identify with vectors in $\mathbb{C}^n$ whose entries sum to zero. The roots then consist of all the $n(n-1)$ permutations of $(1, -1, 0, \ldots, 0)$.

A choice of simple roots is

$$\begin{aligned}&(1,-1,0,\dots,0),\\&(0,1,-1,\dots,0),\\&\qquad\vdots\\&(0,0,0,\dots,1,-1).\end{aligned}$$

So, SU(n) is of rank $n - 1$ and its Dynkin diagram is given by A_{n-1}, a chain of $n - 1$ nodes: .[15] Its Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0\\ -1 & 2 & -1 & \dots & 0\\ 0 & -1 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & 2 \end{pmatrix}.$$

Its Weyl group or Coxeter group is the symmetric group S_n, the symmetry group of the $(n - 1)$-simplex.

Generalized special unitary group

For a field F, the **generalized special unitary group over F**, SU(p, q; F), is the group of all linear transformations of determinant 1 of a vector space of rank $n = p + q$ over F which leave invariant a nondegenerate, Hermitian form of signature (p, q). This group is often referred to as the **special unitary group of signature p q over F**. The field F can be replaced by a commutative ring, in which case the vector space is replaced by a free module.

Specifically, fix a Hermitian matrix A of signature p q in GL(n, $\mathbf{R}$), then all

$$M \in \mathrm{SU}(p,q,R)$$

satisfy

$$\begin{aligned}M^*AM &= A\\ \det M &= 1.\end{aligned}$$

Often one will see the notation SU(p, q) without reference to a ring or field; in this case, the ring or field being referred to is $\mathbf{C}$ and this gives one of the classical Lie groups. The standard choice for A when $F = \mathbf{C}$ is

$$A = \begin{bmatrix} 0 & 0 & i\\ 0 & I_{n-2} & 0\\ -i & 0 & 0 \end{bmatrix}.$$

However, there may be better choices for A for certain dimensions which exhibit more behaviour under restriction to subrings of $\mathbf{C}$.

Example

An important example of this type of group is the Picard modular group $\mathrm{SU}(2,\ 1;\ \mathbf{Z}[i])$ which acts (projectively) on complex hyperbolic space of degree two, in the same way that $\mathrm{SL}(2,9;\mathbf{Z})$ acts (projectively) on real hyperbolic space of dimension two. In 2005 Gábor Francsics and Peter Lax computed an explicit fundamental domain for the action of this group on HC^2.[16]

A further example is $\mathrm{SU}(1,\ 1;\ \mathbf{C})$, which is isomorphic to $\mathrm{SL}(2,\mathbf{R})$.

Important subgroups

In physics the special unitary group is used to represent bosonic symmetries. In theories of symmetry breaking it is important to be able to find the subgroups of the special unitary group. Subgroups of $\mathrm{SU}(n)$ that are important in GUT physics are, for $p > 1$, $n - p > 1$,

$$\mathrm{SU}(n) \supset \mathrm{SU}(p) \times \mathrm{SU}(n-p) \times \mathrm{U}(1),$$

where × denotes the direct product and $\mathrm{U}(1)$, known as the circle group, is the multiplicative group of all complex numbers with absolute value 1.

For completeness, there are also the orthogonal and symplectic subgroups,

$$\begin{aligned}\mathrm{SU}(n) &\supset \mathrm{SO}(n),\\ \mathrm{SU}(2n) &\supset \mathrm{Sp}(n).\end{aligned}$$

Since the rank of SU(n) is $n-1$ and of $\mathrm{U}(1)$ is 1, a useful check is that the sum of the ranks of the subgroups is less than or equal to the rank of the original group. $\mathrm{SU}(n)$ is a subgroup of various other Lie groups,

$$\begin{aligned}\mathrm{SO}(2n) &\supset \mathrm{SU}(n)\\ \mathrm{Sp}(n) &\supset \mathrm{SU}(n)\\ \mathrm{Spin}(4) &= \mathrm{SU}(2) \times \mathrm{SU}(2)\\ \mathrm{E}_6 &\supset \mathrm{SU}(6)\\ \mathrm{E}_7 &\supset \mathrm{SU}(8)\\ \mathrm{G}_2 &\supset \mathrm{SU}(3)\end{aligned}$$

See spin group, and simple Lie groups for E_6, E_7, and G_2.

There are also the accidental isomorphisms: $\mathrm{SU}(4) = \mathrm{Spin}(6)$, $\mathrm{SU}(2) = \mathrm{Spin}(3) = \mathrm{Sp}(1)$,[nb 4] and $\mathrm{U}(1) = \mathrm{Spin}(2) = \mathrm{SO}(2)$.

One may finally mention that $\mathrm{SU}(2)$ is the double covering group of $\mathrm{SO}(3)$, a relation that plays an important role in the theory of rotations of 2-spinors in non-relativistic quantum mechanics.

See also

- Unitary group
- Projective special unitary group, $\mathrm{PSU}(n)$

- Orthogonal group
- Generalizations of Pauli matrices
- Representation theory of SU(2)

Remarks

1. For a characterization of $\mathrm{U}(n)$ and hence $\mathrm{SU}(n)$ in terms of preservation of the standard inner product on $\mathbb{C}^n$, see Classical group.
2. For an explicit description of the homomorphism $\mathrm{SU}(2) \to \mathrm{SO}(3)$, see Connection between SO(3) and SU(2).
3. So fewer than 1/6 of all f_{abc}s are non-vanishing.
4. $\mathrm{Sp}(n)$ is the compact real form of $\mathrm{Sp}(2n, \mathbf{C})$. It is sometimes denoted $\mathrm{USp}(2n)$. The dimension of the $\mathrm{Sp}(n)$-matrices is $2n \times 2n$.

Notes

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