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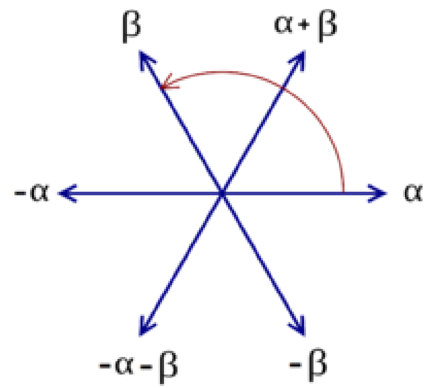
Root system

In mathematics, a **root system** is a configuration of vectors in a Euclidean space satisfying certain geometrical properties. The concept is fundamental in the theory of Lie groups and Lie algebras, especially the classification and representations theory of semisimple Lie algebras. Since Lie groups (and some analogues such as algebraic groups) and Lie algebras have become important in many parts of mathematics during the twentieth century, the apparently special nature of root systems belies the number of areas in which they are applied. Further, the classification scheme for root systems, by Dynkin diagrams, occurs in parts of mathematics with no overt connection to Lie theory (such as singularity theory). Finally, root systems are important for their own sake, as in spectral graph theory.[1]

Contents

Definitions and examples



The six vectors of the root system A_2.

As a first example, consider the six vectors in 2-dimensional Euclidean space, $\mathbf{R}^2$, as shown in the image at the right; call them **roots**. These vectors span the whole space. If you consider the line perpendicular to any root, say β, then the reflection of $\mathbf{R}^2$ in that line sends any other root, say α, to another root. Moreover, the root to which it is sent equals α + nβ, where n is an integer (in this case, n equals 1). These six vectors satisfy the following definition, and therefore they form a root system; this one is known as A_2.

Definition

Let V be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by $(\cdot,\cdot)$. A **root system** Φ in V is a finite set of non-zero vectors (called **roots**) that satisfy the following conditions:[2][3]

1. The roots span V.
2. The only scalar multiples of a root $\alpha \in \Phi$ that belong to Φ are α itself and $-\alpha$.
3. For every root $\alpha \in \Phi$, the set Φ is closed under reflection through the hyperplane perpendicular to α.
4. (**Integrality**) If α and β are roots in Φ, then the projection of β onto the line through α is an *integer or half-integer* multiple of α.

An equivalent way of writing conditions 3 and 4 is as follows:

3. For any two roots $\alpha, \beta \in \Phi$, the set Φ contains the element $\sigma_\alpha(\beta) := \beta - 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}\alpha \in \Phi$.
4. For any two roots $\alpha, \beta \in \Phi$, the number $\langle \beta, \alpha \rangle := 2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

Some authors only include conditions 1–3 in the definition of a root system.[4] In this context, a root system that also satisfies the integrality condition is known as a **crystallographic root system**.[5] Other authors omit condition 2; then they call root systems satisfying condition 2 **reduced**.[6] In this article, all root systems are assumed to be reduced and crystallographic.

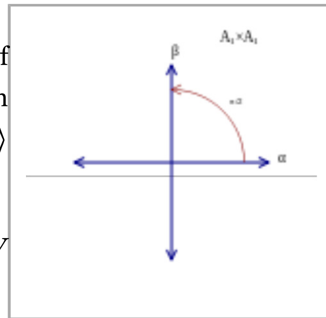
In view of property 3, the integrality condition is equivalent to stating that β and its reflection $\sigma_\alpha(\beta)$ differ by an integer multiple of α. Note that the operator

$$\langle \cdot, \cdot \rangle \colon \Phi \times \Phi \to \mathbb{Z}$$

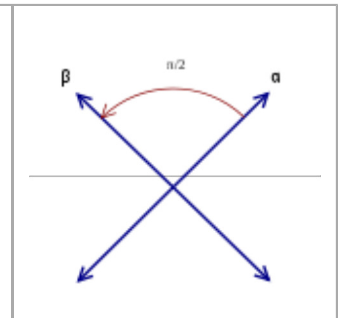
defined by property 4 is not an inner product. It is not necessarily symmetric and is linear only in the first argument.

The **rank** of a root system Φ is the dimension of V. Two root systems may be combined by regarding the Euclidean spaces they span as mutually orthogonal subspaces of a common Euclidean space. A root system which does not arise from such a combination, such as the systems A_2, B_2, and G_2 pictured to the right, is said to be **irreducible**.

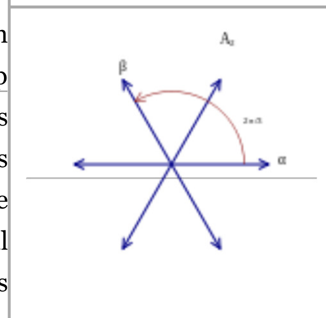
Rank-2 root systems



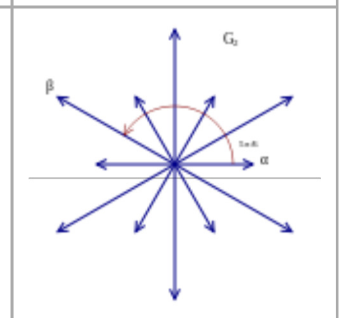
Root system $A_1 \times A_1$



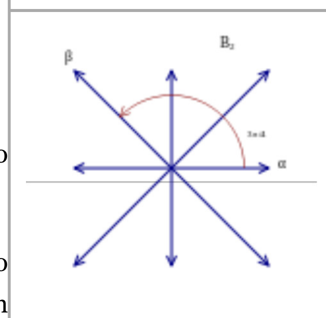
Root system D_2



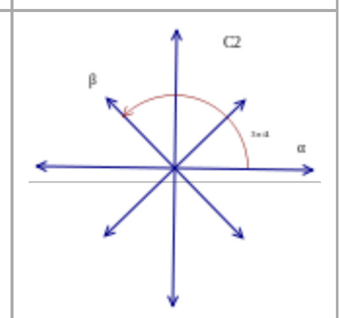
Root system A_2



Root system G_2



Root system B_2



Root system C_2

Two root systems (E_1, Φ_1) and (E_2, Φ_2) are called **isomorphic** if there is an invertible linear transformation $E_1 \to E_2$ which sends Φ_1 to Φ_2 such that for each pair of roots, the number $\langle x, y \rangle$ is preserved.[7]

The **root lattice** of a root system Φ is the **Z**-submodule of V generated by Φ. It is a lattice in V.

Weyl group

The group of isometries of V generated by reflections through hyperplanes associated to the roots of Φ is called the Weyl group of Φ. As it acts faithfully on the finite set Φ, the Weyl group is always finite. In the A_2 case, the "hyperplanes" are the lines perpendicular to the roots, indicated by dashed lines in the figure. The Weyl group is the symmetry group of an equilateral triangle, which has six elements. In this case, the Weyl group is not the full symmetry group of the root system (e.g., a 60-degree rotation is a symmetry of the root system but not an element of the Weyl group).

Rank two examples

There is only one root system of rank 1, consisting of two nonzero vectors $\{\alpha, -\alpha\}$. This root system is called A_1.

In rank 2 there are four possibilities, corresponding to $\sigma_\alpha(\beta) = \beta + n\alpha$, where $n = 0, 1, 2, 3$.[8] Note that a root system is not determined by the lattice that it generates: $A_1 \times A_1$ and B_2 both generate a square lattice while A_2 and G_2 generate a hexagonal lattice, only two of the five possible types of lattices in two dimensions.

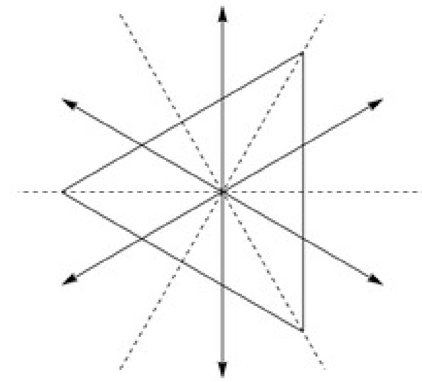
Whenever Φ is a root system in V, and U is a subspace of V spanned by $\Psi = \Phi \cap U$, then Ψ is a root system in U. Thus, the exhaustive list of four root systems of rank 2 shows the geometric possibilities for any two roots chosen from a root system of arbitrary rank. In particular, two such roots must meet at an angle of 0, 30, 45, 60, 90, 120, 135, 150, or 180 degrees.

Root systems arising from semisimple Lie algebras

If $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, we can construct a root system as follows. We say that $\alpha \in \mathfrak{h}^*$ is a **root** of $\mathfrak{g}$ relative to $\mathfrak{h}$ if $\alpha \neq 0$ and there exists some $X \neq 0 \in \mathfrak{g}$ such that

$$[H, X] = \alpha(H)X$$

for all $H \in \mathfrak{h}$. One can show[9] that there is an inner product for which the set of roots forms a root system. The root system of $\mathfrak{g}$ is a fundamental tool for analyzing the structure of $\mathfrak{g}$ and classifying its representations. (See the section below on Root systems and Lie theory.)



The Weyl group of the A_2 root system is the symmetry group of an equilateral triangle

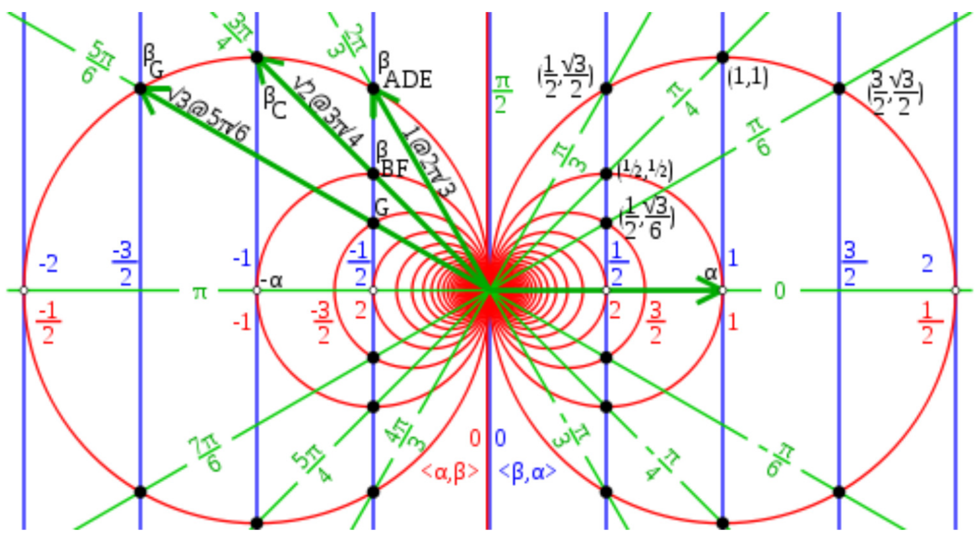
History

The concept of a root system was originally introduced by Wilhelm Killing around 1889 (in German, *Wurzelsystem*[10]).[11] He used them in his attempt to classify all simple Lie algebras over the field of complex numbers. Killing originally made a mistake in the classification, listing two exceptional rank 4 root systems, when in fact there is only one, now known as F_4. Cartan later corrected this mistake, by showing Killing's two root systems were isomorphic.[12]

Killing investigated the structure of a Lie algebra L, by considering (what is now called) a Cartan subalgebra $\mathfrak{h}$. Then he studied the roots of the characteristic polynomial $\det(\mathrm{ad}_L x - t)$, where $x \in \mathfrak{h}$. Here a *root* is considered as a function of $\mathfrak{h}$, or indeed as an element of the dual vector space $\mathfrak{h}^*$. This set of roots form a root system inside $\mathfrak{h}^*$, as defined above, where the inner product is the Killing form.[13]

Elementary consequences of the root system axioms

The cosine of the angle between two roots is constrained to be a half-integral multiple of a square root of an integer. This is because $\langle \beta, \alpha \rangle$ and $\langle \alpha, \beta \rangle$ are both integers, by assumption, and



The integrality condition for $\langle \beta, \alpha \rangle$ is fulfilled only for β on one of the vertical lines, while the integrality condition for $\langle \alpha, \beta \rangle$ is fulfilled only for β on one of the red circles. Any β perpendicular to *α* (on the *Y* axis) trivially fulfills both with 0, but does not define an irreducible root system.
Modulo reflection, for a given *α* there are only 5 nontrivial possibilities for *β*, and 3 possible angles between *α* and *β* in a set of simple roots. Subscript letters correspond to the series of root systems for which the given *β* can serve as the first root and α as the second root (or in F_4 as the middle 2 roots).

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \cdot 2\frac{(\alpha,\beta)}{(\beta,\beta)} = 4\frac{(\alpha,\beta)^2}{|\alpha|^2|\beta|^2} = 4\cos^2(\theta) = (2\cos(\theta))^2 \in \mathbb{Z}.$$

Since $2\cos(\theta) \in [-2, 2]$, the only possible values for $\cos(\theta)$ are $0, \pm\frac{1}{2}, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{3}}{2}$ and $\pm\frac{\sqrt{4}}{2} = \pm 1$, corresponding to angles of 90°, 60° or 120°, 45° or 135°, 30° or 150°, and 0° or 180°. Condition 2 says that no scalar multiples of α other than 1 and -1 can be roots, so 0 or 180°, which would correspond to 2α or -2α, are out. The diagram at right shows that an angle of 60° or 120° corresponds to roots of equal length, while an angle of 45° or 135° corresponds to a length ratio of $\sqrt{2}$ and an angle of 30° or 150° corresponds to a length ratio of $\sqrt{3}$.

In summary, here are the only possibilities for each pair of roots.[14]

- Angle of 90 degrees; in that case, the length ratio is unrestricted.
- Angle of 60 or 120 degrees, with a length ratio of 1.
- Angle of 45 or 135 degrees, with a length ratio of $\sqrt{2}$.
- Angle of 30 or 150 degrees, with a length ratio of $\sqrt{3}$.

Positive roots and simple roots

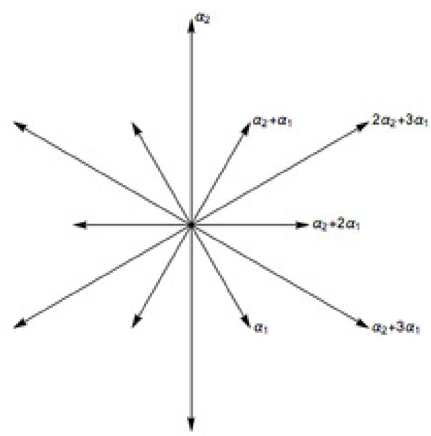
Given a root system Φ we can always choose (in many ways) a set of **positive roots**. This is a subset Φ^+ of Φ such that

- For each root $\alpha \in \Phi$ exactly one of the roots α, $-\alpha$ is contained in Φ^+.
- For any two distinct $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$.

If a set of positive roots Φ^+ is chosen, elements of $-\Phi^+$ are called **negative roots**.

An element of Φ^+ is called a **simple root** if it cannot be written as the sum of two elements of Φ^+. (The set of simple roots is also referred to as a **base** for Φ.) The set Δ of simple roots is a basis of V with the following additional special properties:[15]

- Every root $\alpha \in \Phi$ is linear combination of elements of Δ with *integer* coefficients.
- For each $\alpha \in \Phi$, the coefficients in the previous point are either all non-negative or all non-positive.



The labeled roots are a set of positive roots for the G_2 root system, with α_1 and α_2 being the simple roots

For each root system Φ there are many different choices of the set of positive roots—or, equivalently, of the simple roots—but any two sets of positive roots differ by the action of the Weyl group.[16]

Dual root system and coroots

If Φ is a root system in *V*, the **coroot** $\alpha^\vee$ of a root α is defined by

$$\alpha^\vee = \frac{2}{(\alpha, \alpha)}\,\alpha.$$

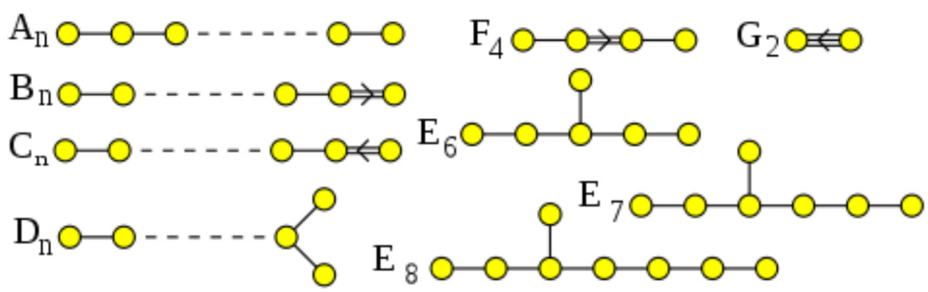
The set of coroots also forms a root system $\Phi^\vee$ in *V*, called the **dual root system** (or sometimes *inverse root system*). By definition, $\alpha^{\vee\vee} = \alpha$, so that Φ is the dual root system of $\Phi^\vee$. The lattice in *V* spanned by $\Phi^\vee$ is called the *coroot lattice*. Both Φ and $\Phi^\vee$ have the same Weyl group *W* and, for *s* in *W*,

$$(s\alpha)^\vee = s(\alpha^\vee).$$

If Δ is a set of simple roots for Φ, then $\Delta^\vee$ is a set of simple roots for $\Phi^\vee$.[17]

In the classification described below, the root systems of type A_n and D_n along with the exceptional root systems E_6, E_7, E_8, F_4, G_2 are all self-dual, meaning that the dual root system is isomorphic to the original root system. By contrast, the B_n and C_n root systems are dual to one another, but not isomorphic (except when $n = 2$).

Classification of root systems by Dynkin diagrams



Pictures of all the connected Dynkin diagrams

A root system is irreducible if it can not be partitioned into the union of two proper subsets $\Phi = \Phi_1 \cup \Phi_2$, such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$.

Irreducible root systems correspond to certain graphs, the **Dynkin diagrams** named after Eugene Dynkin. The classification of these graphs is a simple matter of combinatorics, and induces a classification of irreducible root systems.

Constructing the Dynkin diagram

Given a root system, select a set Δ of simple roots as in the preceding section. The vertices of the associated Dynkin diagram correspond to the roots in Δ. Edges are drawn between vectors as follows, according to the angles. (Note that the angle between simple roots is always at least 90 degrees.)

- No edge if the vectors are orthogonal,
- An undirected single edge if they make an angle of 120 degrees,
- A directed double edge if they make an angle of 135 degrees, and
- A directed triple edge if they make an angle of 150 degrees.

The term "directed edge" means that double and triple edges are marked with an arrow pointing toward the shorter vector. (Thinking of the arrow as a "greater than" sign makes it clear which way the arrow is supposed to point.)

Note that by the elementary properties of roots noted above, the rules for creating the Dynkin diagram can also be described as follows. No edge if the roots are orthogonal; for nonorthogonal roots, a single, double, or triple edge according to whether the length ratio of the longer to shorter is 1, $\sqrt{2}$, $\sqrt{3}$. In the case of the G_2 root system for example, there are two simple roots at an angle of 150 degrees (with a length ratio of $\sqrt{3}$). Thus, the Dynkin diagram has two vertices joined by a triple edge, with an arrow pointing from the vertex associated to the longer root to the other vertex. (In this case, the arrow is a bit redundant, since the diagram is equivalent whichever way the arrow goes.)

Classifying root systems

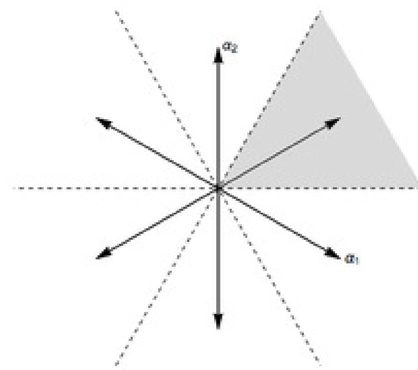
Although a given root system has more than one possible set of simple roots, the Weyl group acts transitively on such choices.[18] Consequently, the Dynkin diagram is independent of the choice of simple roots; it is determined by the root system itself. Conversely, given two root systems with the same Dynkin diagram, one can match up roots, starting with the roots in the base, and show that the systems are in fact the same.[19]

Thus the problem of classifying root systems reduces to the problem of classifying possible Dynkin diagrams. A root systems is irreducible if and only if its Dynkin diagrams is connected.[20] Dynkin diagrams encode the inner product on E in terms of the basis Δ, and the condition that this inner product must be positive definite turns out to be all that is needed to get the desired classification.

The actual connected diagrams are as indicated in the figure. The subscripts indicate the number of vertices in the diagram (and hence the rank of the corresponding irreducible root system).

Weyl chambers and the Weyl group

If $\Phi \subset V$ is a root system, we may consider the hyperplane perpendicular to each root α. Recall that σ_α denotes the reflection about the hyperplane and that the Weyl group is the group of transformations of V generated by all the σ_α's. The complement of the set of hyperplanes is disconnected, and each connected component is called a **Weyl chamber**. If we have fixed a particular set Δ of simple roots, we may define the **fundamental Weyl chamber** associated to Δ as the set of points $v \in V$ such that $(\alpha, v) > 0$ for all $\alpha \in \Delta$.



The shaded region is the fundamental Weyl chamber for the base $\{\alpha_1, \alpha_2\}$

Since the reflections $\sigma_\alpha,\ \alpha \in \Phi$ preserve Φ, they also preserve the set of hyperplanes perpendicular to the roots. Thus, each Weyl group element permutes the Weyl chambers.

The figure illustrates the case of the A_2 root system. The "hyperplanes" (in this case, one dimensional) orthogonal to the roots are indicated by dashed lines. The six 60-degree sectors are the Weyl chambers and the shaded region is the fundamental Weyl chamber associated to the indicated base.

A basic general theorem about Weyl chambers is this:[21]

> **Theorem**: The Weyl group acts freely and transitively on the Weyl chambers. Thus, the order of the Weyl group is equal to the number of Weyl chambers.

In the A_2 case, for example, the Weyl group has six elements and there are six Weyl chambers.

A related result is this one:[22]

> **Theorem**: Fix a Weyl chamber C. Then for all $v \in V$, the Weyl-orbit of v contains exactly one point in the closure $\bar{C}$ of C.

Root systems and Lie theory

Irreducible root systems classify a number of related objects in Lie theory, notably the following:

- simple complex Lie algebras (see the discussion above on root systems arising from semisimple Lie algebras),
- simply connected complex Lie groups which are simple modulo centers, and
- simply connected compact Lie groups which are simple modulo centers.

In each case, the roots are non-zero weights of the adjoint representation.

We now give a brief indication of how irreducible root systems classify simple Lie algebras over $\mathbb{C}$, following the arguments in Humphreys.[23] A preliminary result says that a semisimple Lie algebra is simple if and only if the associated root system is irreducible.[24] We thus restrict attention to irreducible root systems and simple Lie algebras.

- First, we must establish that for each simple algebra $\mathfrak{g}$ there is only one root system. This assertion follows from the result that the Cartan subalgebra of $\mathfrak{g}$ is unique up to automorphism,[25] from which it follows that any two Cartan subalgebras give isomorphic root systems.
- Next, we need to show that for each irreducible root system, there can be at most one Lie algebra, that is, that the root system determines the Lie algebra up to isomorphism.[26]
- Finally, we must show that for each irreducible root system, there is an associated simple Lie algebra. This claim is obvious for the root systems of type A, B, C, and D, for which the associated Lie algebras are the classical algebras. It is then possible to analyze the exceptional algebras in a case-by-case fashion. Alternatively, one can develop a systematic procedure for building a Lie algebra from a root system, using Serre's relations.[27]

For connections between the exceptional root systems and their Lie groups and Lie algebras see E_8, E_7, E_6, F_4, and G_2.

Properties of the irreducible root systems

Φ	$\|\Phi\|$	$\|\Phi^<\|$	*I*	*D*	$\|W\|$
A_n ($n \geq 1$)	$n(n+1)$			$n+1$	$(n+1)!$
B_n ($n \geq 2$)	$2n^2$	$2n$	2	2	$2^n\, n!$
C_n ($n \geq 3$)	$2n^2$	$2n(n-1)$	2^{n-1}	2	$2^n\, n!$
D_n ($n \geq 4$)	$2n(n-1)$			4	$2^{n-1}\, n!$
E_6	72			3	51840
E_7	126			2	2903040
E_8	240			1	696729600
F_4	48	24	4	1	1152
G_2	12	6	3	1	12

Irreducible root systems are named according to their corresponding connected Dynkin diagrams. There are four infinite families (A_n, B_n, C_n, and D_n, called the **classical root systems**) and five exceptional cases (the **exceptional root systems**). The subscript indicates the rank of the root system.

In an irreducible root system there can be at most two values for the length $(\alpha, \alpha)^{1/2}$, corresponding to **short** and **long** roots. If all roots have the same length they are taken to be long by definition and the root system is said to be **simply laced**; this occurs in the cases A, D and E. Any two roots of the same length lie in the same orbit of the Weyl group. In the non-simply laced cases B, C, G and F, the root lattice is spanned by the short roots and the long roots span a sublattice, invariant under the Weyl group, equal to $r^2/2$ times the coroot lattice, where r is the length of a long root.

In the adjacent table, $|\Phi^<|$ denotes the number of short roots, I denotes the index in the root lattice of the sublattice generated by long roots, D denotes the determinant of the Cartan matrix, and $|W|$ denotes the order of the Weyl group.

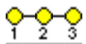
Explicit construction of the irreducible root systems

A_n

Let V be the subspace of $\mathbf{R}^{n+1}$ for which the coordinates sum to 0, and let Φ be the set of vectors in V of length $\sqrt{2}$ and which are *integer vectors,* i.e. have integer coordinates in $\mathbf{R}^{n+1}$. Such a vector must have all but two coordinates equal to 0, one coordinate equal to 1, and one equal to –1, so there are $n^2 + n$ roots in all. One choice of simple roots expressed in the standard basis is: $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $1 \leq i \leq n$.

The reflection σ_i through the hyperplane perpendicular to $\boldsymbol{\alpha}_i$ is the same as permutation of the adjacent i-th and $(i+1)$-th coordinates. Such transpositions generate the full permutation group. For adjacent simple roots, $\sigma_i(\boldsymbol{\alpha}_{i+1}) = \boldsymbol{\alpha}_{i+1} + \boldsymbol{\alpha}_i = \sigma_{i+1}(\boldsymbol{\alpha}_i) = \boldsymbol{\alpha}_i + \boldsymbol{\alpha}_{i+1}$, that is, reflection is equivalent to adding a multiple of 1; but reflection of a simple root perpendicular to a nonadjacent simple root leaves it unchanged, differing by a multiple of 0.

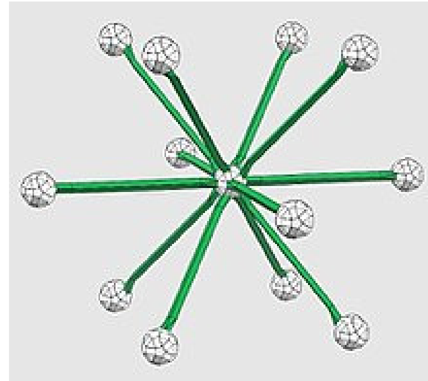
Simple roots in $\mathbf{A_3}$

	e_1	e_2	e_3	e_4
α_1	1	–1	0	0
α_2	0	1	–1	0
α_3	0	0	1	–1
				

The A_n root lattice - that is, the lattice generated by the A_n roots - is most easily described as the set of integer vectors in $\mathbf{R}^{n+1}$ whose components sum to zero.

The A_3 root lattice is known to crystallographers as the **face-centered cubic** (**fcc**) (or **cubic close packed**) lattice.[28]

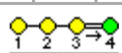
The A_3 root system (as well as the other rank-three root systems) may be modeled in the Zometool Construction set.[29]



Model of the $\boldsymbol{A_3}$ root system in the Zometool system.

B_n

Simple roots in $\mathbf{B_4}$

	e_1	e_2	e_3	e_4
α_1	1	-1	0	0
α_2	0	1	-1	0
α_3	0	0	1	-1
α_4	0	0	0	1
				

Let $V = \mathbf{R}^n$, and let Φ consist of all integer vectors in V of length 1 or $\sqrt{2}$. The total number of roots is $2n^2$. One choice of simple roots is: $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $1 \le i \le n - 1$ (the above choice of simple roots for $\mathbf{A_{n-1}}$), and the shorter root $\boldsymbol{\alpha}_n = \mathbf{e}_n$.

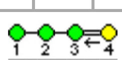
The reflection σ_n through the hyperplane perpendicular to the short root $\boldsymbol{\alpha}_n$ is of course simply negation of the **n**th coordinate. For the long simple root $\boldsymbol{\alpha}_{n-1}$, $\sigma_{n-1}(\boldsymbol{\alpha}_n) = \boldsymbol{\alpha}_n + \boldsymbol{\alpha}_{n-1}$, but for reflection perpendicular to the short root, $\sigma_n(\boldsymbol{\alpha}_{n-1}) = \boldsymbol{\alpha}_{n-1} + 2\boldsymbol{\alpha}_n$, a difference by a multiple of 2 instead of 1.

The B_n root lattice - that is, the lattice generated by the B_n roots - consists of all integer vectors.

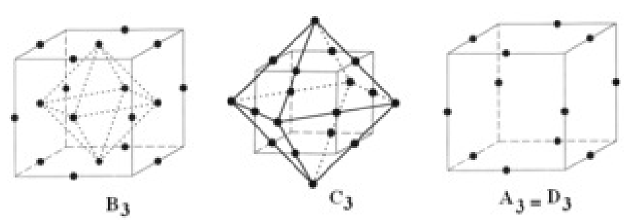
B_1 is isomorphic to A_1 via scaling by $\sqrt{2}$, and is therefore not a distinct root system.

C_n

Simple roots in $\mathbf{C_4}$

	e_1	e_2	e_3	e_4
α_1	1	-1	0	0
α_2	0	1	-1	0
α_3	0	0	1	-1
α_4	0	0	0	2
				

Let $V = \mathbf{R}^n$, and let Φ consist of all integer vectors in V of length $\sqrt{2}$ together with all vectors of the form 2λ, where λ is an integer vector of length 1. The total number of roots is $2n^2$. One choice of simple roots is: $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $1 \le i \le n - 1$ (the above choice of simple roots for $\mathbf{A_{n-1}}$), and the longer root $\boldsymbol{\alpha}_n = 2\mathbf{e}_n$. The reflection $\sigma_n(\boldsymbol{\alpha}_{n-1}) = \boldsymbol{\alpha}_{n-1} + \boldsymbol{\alpha}_n$, but $\sigma_{n-1}(\boldsymbol{\alpha}_n) = \boldsymbol{\alpha}_n + 2\boldsymbol{\alpha}_{n-1}$.



Root system B_3, C_3, and A_3=D_3 as points within a cube and octahedron

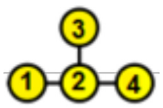
The C_n root lattice - that is, the lattice generated by the C_n roots - consists of all integer vectors whose components sum to an even integer.

C_2 is isomorphic to B_2 via scaling by $\sqrt{2}$ and a 45 degree rotation, and is therefore not a distinct root system.

D_n

Simple roots in $\mathbf{D_4}$

	e_1	e_2	e_3	e_4
α_1	1	-1	0	0
α_2	0	1	-1	0
α_3	0	0	1	-1
α_4	0	0	1	1



Let $V = \mathbf{R}^n$, and let Φ consist of all integer vectors in V of length $\sqrt{2}$. The total number of roots is $2n(n-1)$. One choice of simple roots is: $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$, for $1 \le i < n$ (the above choice of simple roots for $\mathbf{A_{n-1}}$) plus $\boldsymbol{\alpha}_n = \mathbf{e}_n + \mathbf{e}_{n-1}$.

Reflection through the hyperplane perpendicular to $\boldsymbol{\alpha}_n$ is the same as transposing and negating the adjacent $\mathbf{n}$-th and $(\mathbf{n-1})$-th coordinates. Any simple root and its reflection perpendicular to another simple root differ by a multiple of 0 or 1 of the second root, not by any greater multiple.

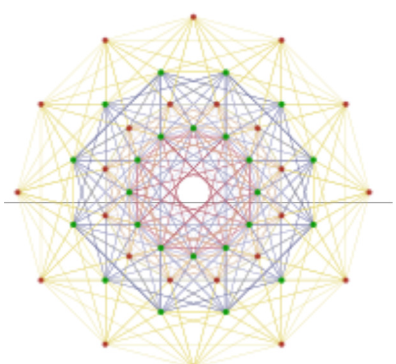
The D_n root lattice - that is, the lattice generated by the D_n roots - consists of all integer vectors whose components sum to an even integer. This is the same as the C_n root lattice.

The D_n roots are expressed as the vertices of a rectified n-orthoplex, Coxeter-Dynkin diagram: . The $2n(n-1)$ vertices exist in the middle of the edges of the n-orthoplex.

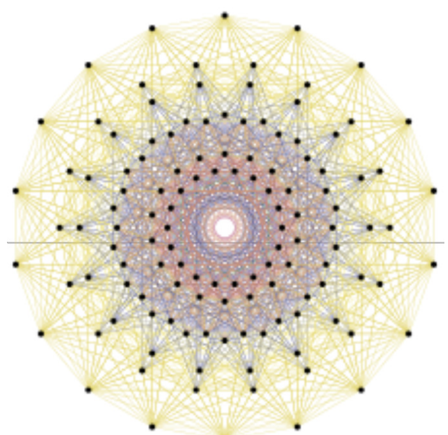
D_3 coincides with A_3, and is therefore not a distinct root system. The 12 D_3 root vectors are expressed as the vertices of , a lower symmetry construction of the cuboctahedron.

D_4 has additional symmetry called triality. The 24 D_4 root vectors are expressed as the vertices of , a lower symmetry construction of the 24-cell.

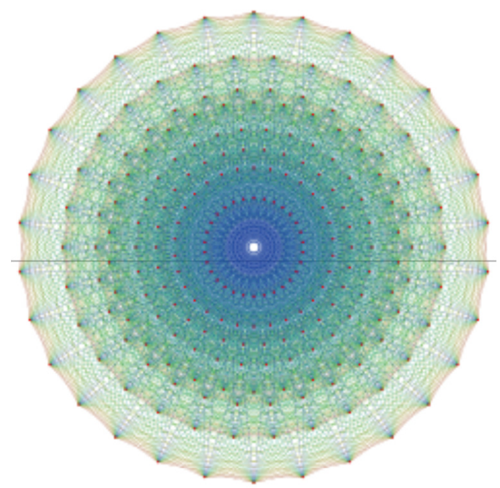
E_6, E_7, E_8



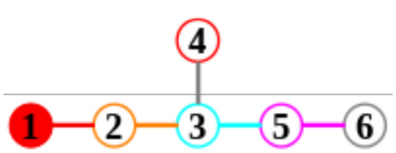
72 vertices of 1_{22} represent the root vectors of E_6
(Green nodes are doubled in this E6 Coxeter plane projection)

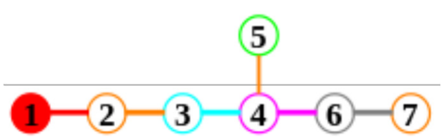


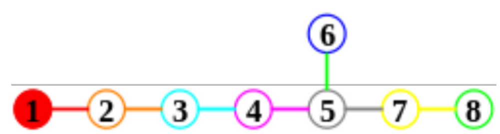
126 vertices of 2_{31} represent the root vectors of E_7



240 vertices of 4_{21} represent the root vectors of E_8







- The E_8 root system is any set of vectors in $\mathbf{R}^8$ that is congruent to the following set:

$$D_8 \cup \{ \tfrac{1}{2}(\textstyle\sum_{i=1}^{8} \varepsilon_i \mathbf{e}_i) : \varepsilon_i = \pm 1,\ \varepsilon_1 \cdots \varepsilon_8 = +1\}.$$

The root system has 240 roots. The set just listed is the set of vectors of length $\sqrt{2}$ in the E8 root lattice, also known simply as the E8 lattice or Γ_8. This is the set of points in $\mathbf{R}^8$ such that:

1. all the coordinates are integers or all the coordinates are half-integers (a mixture of integers and half-integers is not allowed), and
2. the sum of the eight coordinates is an even integer.

Thus,

$$E_8 = \{\boldsymbol{\alpha} \in \mathbf{Z}^8 \cup (\mathbf{Z}+½)^8 : |\boldsymbol{\alpha}|^2 = \sum \boldsymbol{\alpha}_i^2 = 2, \sum \boldsymbol{\alpha}_i \in 2\mathbf{Z}\}.$$

- The root system E_7 is the set of vectors in E_8 that are perpendicular to a fixed root in E_8. The root system E_7 has 126 roots.
- The root system E_6 is not the set of vectors in E_7 that are perpendicular to a fixed root in E_7, indeed, one obtains D_6 that way. However, E_6 is the subsystem of E_8 perpendicular to two suitably chosen roots of E_8. The root system E_6 has 72 roots.

An alternative description of the E_8 lattice which is sometimes convenient is as the set Γ''_8 of all points in $\mathbf{R}^8$ such that

- all the coordinates are integers and the sum of the coordinates is even, or
- all the coordinates are half-integers and the sum of the coordinates is odd.

The lattices Γ_8 and Γ''_8 are isomorphic; one may pass from one to the other by changing the signs of any odd number of coordinates. The lattice Γ_8 is sometimes called the *even coordinate system* for E_8 while the lattice Γ''_8 is called the *odd coordinate system*.

Simple roots in E_8 even coordinates:

1	-1	0	0	0	0	0	0
0	1	-1	0	0	0	0	0
0	0	1	-1	0	0	0	0
0	0	0	1	-1	0	0	0
0	0	0	0	1	-1	0	0
0	0	0	0	0	1	-1	0
0	0	0	0	0	1	1	0
-½	-½	-½	-½	-½	-½	-½	-½

One choice of simple roots for E_8 in the even coordinate system with rows ordered by node order in the alternate (non-canonical) Dynkin diagrams (above) is:

$$\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \text{ for } 1 \le i \le 6, \text{ and}$$
$$\boldsymbol{\alpha}_7 = \mathbf{e}_7 + \mathbf{e}_6$$

(the above choice of simple roots for D_7) along with

$$\boldsymbol{\alpha}_8 = \boldsymbol{\beta}_0 = -\frac{1}{2}\left(\sum_{i=1}^{8} e_i\right) = (-½,-½,-½,-½,-½,-½,-½,-½).$$

One choice of simple roots for E_8 in the odd coordinate system with rows ordered by node order in alternate (non-canonical) Dynkin diagrams (above) is:

Simple roots in E_8: odd coordinates

1	-1	0	0	0	0	0	0
0	1	-1	0	0	0	0	0
0	0	1	-1	0	0	0	0
0	0	0	1	-1	0	0	0
0	0	0	0	1	-1	0	0
0	0	0	0	0	1	-1	0
0	0	0	0	0	0	1	-1
-½	-½	-½	-½	-½	½	½	½

$$\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}, \text{ for } 1 \le i \le 7$$

(the above choice of simple roots for A_7) along with

$$\boldsymbol{\alpha}_8 = \boldsymbol{\beta}_5, \text{ where}$$
$$\boldsymbol{\beta}_j = \frac{1}{2}\left(-\sum_{i=1}^{j} e_i + \sum_{i=j+1}^{8} e_i\right).$$

(Using $\boldsymbol{\beta}_3$ would give an isomorphic result. Using $\boldsymbol{\beta}_{1,7}$ or $\boldsymbol{\beta}_{2,6}$ would simply give A_8 or D_8. As for $\boldsymbol{\beta}_4$, its coordinates sum to 0, and the same is true for $\boldsymbol{\alpha}_{1\ldots7}$, so they span only the 7-dimensional subspace for which the coordinates sum to 0; in fact $-2\boldsymbol{\beta}_4$ has coordinates (1,2,3,4,3,2,1) in the basis ($\boldsymbol{\alpha}_i$).)

Since perpendicularity to $\boldsymbol{\alpha}_1$ means that the first two coordinates are equal, E_7 is then the subset of E_8 where the first two coordinates are equal, and similarly E_6 is the subset of E_8 where the first three coordinates are equal. This facilitates explicit definitions of E_7 and E_6 as:

$$E_7 = \{\alpha \in \mathbf{Z}^7 \cup (\mathbf{Z}+½)^7 : \sum \alpha_i^2 + \alpha_1^2 = 2, \sum \alpha_i + \alpha_1 \in 2\mathbf{Z}\},$$
$$E_6 = \{\alpha \in \mathbf{Z}^6 \cup (\mathbf{Z}+½)^6 : \sum \alpha_i^2 + 2\alpha_1^2 = 2, \sum \alpha_i + 2\alpha_1 \in 2\mathbf{Z}\}$$

Note that deleting $\boldsymbol{\alpha}_1$ and then $\boldsymbol{\alpha}_2$ gives sets of simple roots for E_7 and E_6. However, these sets of simple roots are in different E_7 and E_6 subspaces of E_8 than the ones written above, since they are not orthogonal to $\boldsymbol{\alpha}_1$ or $\boldsymbol{\alpha}_2$.

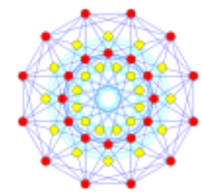
F_4

Simple roots in F_4

	e_1	e_2	e_3	e_4
α_1	1	-1	0	0
α_2	0	1	-1	0
α_3	0	0	1	0
α_4	-½	-½	-½	-½

For F_4, let $V = \mathbf{R}^4$, and let Φ denote the set of vectors α of length 1 or √2 such that the coordinates of 2α are all integers and are either all even or all odd. There are 48 roots in this system. One choice of simple roots is: the choice of simple roots given above for B_3, plus $\boldsymbol{\alpha}_4 = -\frac{1}{2}\sum_{i=1}^4 e_i$.

The F_4 root lattice - that is, the lattice generated by the F_4 root system - is the set of points in $\mathbf{R}^4$ such that either all the coordinates are integers or all the coordinates are half-integers (a mixture of integers and half-integers is not allowed). This lattice is isomorphic to the lattice of Hurwitz quaternions.



48-root vectors of F4, defined by vertices of the 24-cell and its dual, viewed in the Coxeter plane

G_2

Simple roots in G_2

	e_1	e_2	e_3
α_1	1	-1	0
α_2	-1	2	-1

The root system G_2 has 12 roots, which form the vertices of a hexagram. See the picture above.

One choice of simple roots is: $(\boldsymbol{\alpha}_1, \boldsymbol{\beta} = \boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1)$ where $\boldsymbol{\alpha}_i = \mathbf{e}_i - \mathbf{e}_{i+1}$ for i = 1, 2 is the above choice of simple roots for A_2.

The G_2 root lattice - that is, the lattice generated by the G_2 roots - is the same as the A_2 root lattice.

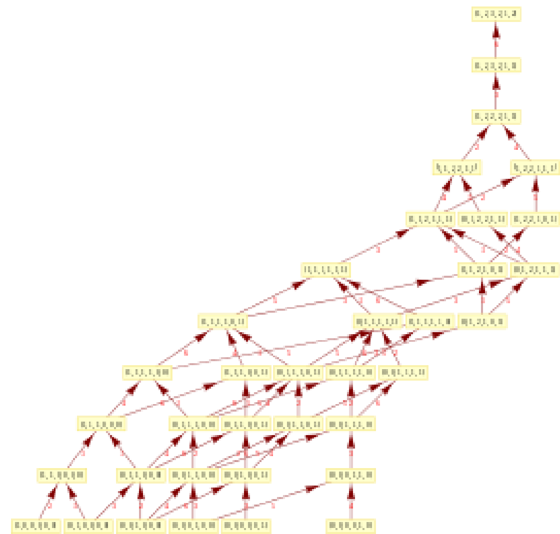
The root poset

The set of positive roots is naturally ordered by saying that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a nonnegative linear combination of simple roots. This poset is graded by $\mathbf{deg}\left(\sum_{\alpha\in\Delta}\lambda_\alpha\alpha\right) = \sum_{\alpha\in\Delta}\lambda_\alpha$, and has many remarkable combinatorial properties, one of them being that one can determine the degrees of the fundamental invariants of the corresponding Weyl group from this poset.[30] The Hasse graph is a visualization of the ordering of the root poset.

See also

- ADE classification
- Affine root system
- Coxeter–Dynkin diagram
- Coxeter group

- Coxeter matrix
- Dynkin diagram
- root datum
- Semisimple Lie algebra
- Weights in the representation theory of semisimple Lie algebras
- Root system of a semi-simple Lie algebra
- Weyl group



Hasse diagram of E6 root poset with edge labels identifying added simple root position

Notes

1. "Graphs with least eigenvalue −2; a historical survey and recent developments in maximal exceptional graphs" (http://www.sciencedirect.com/science/article/pii/S0024379502003774). *Linear Algebra and its Applications*. **356**: 189–210. doi:10.1016/S0024-3795(02)00377-4 (https://doi.org/10.1016%2FS0024-3795%2802%2900377-4).
2. Bourbaki, Ch.VI, Section 1
3. Humphreys (1972), p.42
4. Humphreys (1992), p.6
5. Humphreys (1992), p.39
6. Humphreys (1992), p.41
7. Humphreys (1972), p.43
8. Hall 2015 Proposition 8.8
9. Hall 2015 Section 7.5
10. Killing (1889)
11. Bourbaki (1998), p.270
12. Coleman, p.34
13. Bourbaki (1998), p.270
14. Hall 2015 Proposition 8.6
15. Hall 2015 Theorem 8.16
16. Hall 2015 Proposition 8.28
17. Hall 2015 Proposition 8.18
18. This follows from Hall 2015 Proposition 8.23
19. Hall 2015 Proposition 8.32
20. Hall 2015 Proposition 8.23
21. Hall 2015 Propositions 8.23 and 8.27
22. Hall 2015 Proposition 8.29
23. See various parts of Chapters III, IV, and V of Humphreys 1972, culminating in Section 19 in Chapter V
24. Hall 2015, Theorem 7.35
25. Humphreys 1972, Section 16
26. Humphreys 1972 Part (b) of Theorem 18.4
27. Humphreys 1972 Section 18.3 and Theorem 18.4

28. Conway, John Horton; Sloane, Neil James Alexander; & Bannai, Eiichi. *Sphere packings, lattices, and groups*. Springer, 1999, Section 6.3.
29. Hall 2015 Section 8.9
30. Humphreys (1992), Theorem 3.20

References

- Adams, J.F. (1983), *Lectures on Lie groups*, University of Chicago Press, ISBN 0-226-00530-5
- Bourbaki, Nicolas (2002), *Lie groups and Lie algebras, Chapters 4–6 (translated from the 1968 French original by Andrew Pressley)*, Elements of Mathematics, Springer-Verlag, ISBN 3-540-42650-7. The classic reference for root systems.
- Bourbaki, Nicolas (1998). *Elements of the History of Mathematics*. Springer. ISBN 3540647678.
- A.J. Coleman (Summer 1989), "The greatest mathematical paper of all time", *The Mathematical Intelligencer*, **11** (3): 29–38, doi:10.1007/bf03025189 (https://doi.org/10.1007%2Fbf03025189)
- Hall, Brian C. (2015), *Lie groups, Lie algebras, and representations: An elementary introduction*, Graduate Texts in Mathematics, **222** (2nd ed.), Springer, ISBN 978-3319134666
- Humphreys, James (1992). *Reflection Groups and Coxeter Groups*. Cambridge University Press. ISBN 0521436133.
- Humphreys, James (1972). *Introduction to Lie algebras and Representation Theory*. Springer. ISBN 0387900535.
- Killing, *Die Zusammensetzung der stetigen endlichen Transformationsgruppen* Mathematische Annalen, Part 1 (http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0031&DMDID=DMDLOG_0026&L=1): Volume 31, Number 2 June 1888, Pages 252-290 doi:10.1007/BF01211904 (https://doi.org/10.1007%2FBF01211904); Part 2 (http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0033&DMDID=DMDLOG_0007&L=1): Volume 33, Number 1 March 1888, Pages 1–48 doi:10.1007/BF01444109 (https://doi.org/10.1007%2FBF01444109); Part3 (http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0034&DMDID=DMDLOG_0009&L=1): Volume 34, Number 1 March 1889, Pages 57–122 doi:10.1007/BF01446792 (https://doi.org/10.1007%2FBF01446792); Part 4 (http://gdz.sub.uni-goettingen.de/index.php?id=11&PPN=PPN235181684_0036&DMDID=DMDLOG_0019&L=1): Volume 36, Number 2 June 1890,Pages 161-189 doi:10.1007/BF01207837 (https://doi.org/10.1007%2FBF01207837)
- Kac, Victor G. (1994), *Infinite dimensional Lie algebras*.
- Springer, T.A. (1998). *Linear Algebraic Groups, Second Edition*. Birkhäuser. ISBN 0817640215.

Further reading

- Dynkin, E. B. *The structure of semi-simple algebras.* (in Russian) Uspehi Matem. Nauk (N.S.) 2, (1947). no. 4(20), 59–127.

External links

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