6.5 Automorphic Forms and The Langlands Program

6.5.1 A Relation Between Classical Modular Forms and Representation Theory

(cf. [Bor79], [PSh79]). The domain of definition of the classical modular forms (the upper half plane) is a homogeneous space $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ of the reductive group $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$:

$$\mathbb{H} = \mathrm{GL}_2(\mathbb{R})/\mathcal{O}(2) \cdot Z,$$

where $Z = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \mid x \in \mathbb{R}^\times\}$ is the center of $G(\mathbb{R})$ and $\mathcal{O}(2)$ is the orthogonal group, see (6.3.1). Therefore each modular form

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in \mathcal{M}_k(N, \psi) \subset \mathcal{M}_k(\Gamma_N) \qquad (6.5.1)$$

can be lifted to a function $\tilde{f}$ on the group $\mathrm{GL}_2(\mathbb{R})$ with the invariance condition

$$\tilde{f}(\gamma g) = \tilde{f}(g) \quad \text{for all} \quad \gamma \in \Gamma_N \subset \mathrm{GL}_2(\mathbb{R}).$$

In order to do this let us consider the function

$$\tilde{f}(g) = \begin{cases} f(g(i))j(g,i)^{-k} & \text{if } \det g > 0, \\ f(g(-i))j(g,-i)^{-k} & \text{if } \det g < 0, \end{cases} \qquad (6.5.2)$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $j(g,i) = |\det g|^{-1/2}(cz+d)$ is the factor of automorphy.

One has $\tilde{f}(xg) = \exp(-ik\theta)\tilde{f}(g)$ if $x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation through the angle θ.

Consider the group $\mathrm{GL}_2(\mathbb{A})$ of non-degenerate matrices with coefficients in the adele ring $\mathbb{A}$ and its subgroup

$$U(N) \qquad (6.5.3)$$
$$= \left\{ g = 1 \times \prod_p g_p \in \mathrm{GL}_2(\mathbb{A}) \mid g_p \in \mathrm{GL}_2(\mathbb{Z}_p), g_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\mathbb{Z}_p \right\}.$$

From the *chinese remainder theorem* (the *approximation theorem*) one obtains the following coset decomposition:

$$\Gamma_N \backslash \mathrm{GL}_2(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})/U(N), \qquad (6.5.4)$$

using which we may consider $\tilde{f}$ as a function on the homogeneous space (6.5.4), or even on the adele group $\mathrm{GL}_2(\mathbb{A})$.

The action of $\mathrm{GL}_2(\mathbb{A})$ on $\tilde{f}$ by group shifts defines a representation $\pi = \pi_f$ of the group $\mathrm{GL}_2(\mathbb{A})$ in the space of smooth complex valued functions on $\mathrm{GL}_2(\mathbb{A})$, for which

$$\left(\pi(h)\tilde{f}\right)(g) = \tilde{f}(gh) \quad (g, h \in \mathrm{GL}_2(\mathbb{A})).$$

The condition that the representation π_f be irreducible has a remarkable arithmetical interpretation: it is equivalent to f being an eigenfunction of the Hecke operators for almost all p. If this is the case then one has an infinite tensor product decomposition

$$\pi = \bigotimes_v \pi_v, \tag{6.5.5}$$

where the π_v are representations of the local groups $\mathrm{GL}_2(\mathbb{Q}_v)$ with $v = p$ or ∞.

Jacquet and Langlands chose irreducible representations of groups such as $\mathrm{GL}_2(\mathbb{Q}_v)$ as a starting point for the construction of L–functions (cf. [JL70], [Bor79]). These representations can be classified and explicitly described. Thus for the representations π_v in (6.5.5) one can verify for almost all $v = v_p$ that the representation π_v has the form of an induced representation $\pi_v = \mathrm{Ind}(\mu_1 \otimes \mu_2)$ from a one dimensional representation of the subgroup of diagonal matrices

$$(\mu_1 \otimes \mu_2) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \mu_2(x)\mu_1(y),$$

where $\mu_i : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ are unramified quasicharacters (see §6.2.4). This classification makes it possible to define for almost all p an element $h_p = \begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. From this one can construct the following Euler product (the L–function of the automorphic representation π)

$$L(\pi, s) = \prod_{p \notin S} L(\pi_p, s) = \prod_{p \notin S} \det(1_2 - p^{-s} h_p)^{-1} \tag{6.5.6}$$

in which the product is extended over all but a finite number of primes.

It turns out that the function $L(\pi, s)$ coincides essentially with the Mellin transform of the modular form f:

$$L(s, f) = L(\pi_f, s + (k-1)/2).$$

The notion of a primitive form f also takes on a new meaning: the corresponding function $\tilde{f}$ from the representation space of an irreducible representation π must have a maximal stabilizer. The theory of Atkin–Lehner can be reformulated as saying that the representation π_f occurs with multiplicity one in the regular representation of the group $\mathrm{GL}_2(\mathbb{A})$ (the space of all square integrable functions).

More generally, an *automorphic representation* is defined as an irreducible representation of an adele reductive group $G(\mathbb{A})$ in the space of functions on $G(\mathbb{A})$ with some growth and smoothness conditions.

Jacquet and Langlands constructed for irreducible admissible automorphic representations π of the group $\mathrm{GL}_2(\mathbb{A})$ analytic continuations of the corresponding L–functions $L(\pi, s)$, and established functional equations relating $L(\pi, s)$ to $L(\tilde{\pi}, 1-s)$, where $\tilde{\pi}$ is the dual representation. For the functions $L(\pi_f, s)$ this functional equation is exactly Hecke's functional equation (see (6.3.44)).

Note that the notion of an automorphic representation includes as special cases: 1) the classical elliptic modular forms, 2) the real analytic wave modular forms of Maass, 3) Hilbert modular forms, 4) real analytic Eisenstein series of type $\sum' \dfrac{y^s}{|cz+d|^{2s}}$, 5) Hecke L-series with Grössen–characters (or rather their inverse Melin transforms), 6) automorphic forms on *quaternion algebras* etc.

Interesting classes of Euler products are related to finite dimensional complex representations

$$r : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C}).$$

Let us consider the Euler product

$$L(\pi, r, s) = \prod_p L(\pi_p, r, s), \tag{6.5.7}$$

where

$$L(\pi_p, r, s) = \det(1_m - p^{-s} r(h_p))^{-1}.$$

These products converge absolutely for $\mathrm{Re}(s) \gg 0$, and, conjecturally, admit analytic continuations to the entire complex plane and satisfy functional equations (cf. [Bor79], [BoCa79], [L71a], [Del79], [Se68a]).

This conjecture has been proved in some special cases, for example when $r = \mathrm{Sym}^i\mathrm{St}$ is the i^{th} symmetric power of the standard representation $\mathrm{St} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ for $i = 2, 3, 4, 5$ (cf. [Sh88]).

The Ramanujan–Petersson conjecture, proved by Deligne, can be formulated as saying that the absolute values of the eigenvalues of $h_p \in \mathrm{GL}_2(\mathbb{C})$ for a cusp form f are all equal to 1.

As a consequence of the conjectured analytic properties of the functions (6.5.7) one could deduce the following conjecture of Sato and Tate about the distribution of the arguments of the Frobenius elements: let $\alpha(p) = e^{i\varphi_p}$ $(0 \le \varphi_p \le \pi)$ be an eigenvalue of the matrix h_p defined above. Then for cusp forms f without complex multiplication (i.e. the Mellin transform of f is not the L–function of a Hecke Grössencharacter (see §6.2.4) of an imaginary quadratic field) the arguments φ_p are conjecturally uniformly distributed in the segment $[0, \pi]$ with respect to the measure $\dfrac{2}{\pi}\sin^2\varphi\, d\varphi$ (cf. [Se68a]).

In the case of complex multiplication the analytic properties of the L-functions are reduced to the corresponding properties of the L-functions of Hecke Grössencharacters (see §6.2.4), which imply the uniform distribution of the arguments φ_p with respect to the usual Lebesgue measure.

The arithmetical nature of the numbers $e^{i\varphi_p}$ is close to that of the signs of Gauss sums $\alpha(p) = g(\chi)/\sqrt{p}$ where $g(\chi) = \sum_{u=1}^{p-1} \chi(u)e(u/p)$, χ being a primitive Dirichlet character modulo p. Even if χ is a quadratic character, the precise evaluation of the sign $\alpha(p) = \pm 1$ is rather delicate (see [BS85]). If χ is a cubic character, i.e. if $\chi^3 = 1$ then $p = 6t + 1$, and the sums lie inside the 1st, the 3rd or the 5th sextant of the complex plane. Using methods from the theory of automorphic forms S.J.Patterson and D.R.Heath–Brown solved the problem of Kummer on the distribution of the arguments of cubic Gauss sums by means of a cubic analogue of the theta series, which is a certain automorphic form on the threefold covering of the group GL_2 ([Del80a], [HBP79], [Kub69]).

6.5.2 Automorphic L-Functions

The approach of Jacquet–Langlands made it possible to extend the whole series of notions and results concerning L-functions to the general case of automorphic representations of reductive groups over a global field K. Let G be a linear group over K, $G_{\mathbb{A}} = G(\mathbb{A})$ its group of points with coefficients in the adele ring of the field K. Automorphic representations are often defined as representations belonging to the regular smooth representation of the group $G_{\mathbb{A}}$, and one denotes by the symbol $\mathfrak{A}(G/K)$ the set of equivalence classes of irreducible admissible automorphic representations of $G_{\mathbb{A}}$. A representation π from this class admits a decomposition $\pi = \otimes_v \pi_v$ where $v \in \Sigma_K$ runs through the places of K and the π_v are representations of the groups $G_v = G(K_v)$. In order to construct L-functions, the L-group ${}^L G$ of G is introduced. Consider the tuple of root data (cf. [Bor79], [Spr81])

$$\psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^\vee) \tag{6.5.8}$$

of the group G; here T is a maximal torus of G (over a separable closure of the ground field K); $X^*(T)$ is the group of characters of T; $X_*(T)$ the group of one parameter subgroups of T and Δ (resp. $\Delta^\vee$) is a basis of the root system (resp. the dual basis of the system of coroots). The connected component of the Langlands L-group ${}^L G^0$ is defined to be the complex reductive group obtained by inversion $\psi_0 \mapsto \psi_0^\vee$, whose root data is isomorphic to the inverse

$$\psi_0(G)^\vee = (X_*(T), \Delta^\vee, X^*(T), \Delta). \tag{6.5.9}$$

If G is a simple group, then the group ${}^L G(\mathbb{C})$ can be characterized upto a central isogeny by one of the types A_n, B_n, $\dots$, G_2 of the Cartan–Killing classification. It is known that the map $\psi_0 \mapsto \psi_0^\vee$ interchanges the types B_n and C_n, and leaves all other types fixed. Thus if $G = \mathrm{Sp}_n$ (respectively

GSp_n), then ${}^LG^0 = \mathrm{SO}_{2n+1}(\mathbb{C})$ (resp. ${}^LG^0 = \mathrm{Spin}_{2n+1}(\mathbb{C})$). The whole group ${}^LG$ is then defined as the semi–direct product of ${}^LG^0$ with the Galois group $\mathrm{Gal}(K^s/K)$ of an extension K^s of the ground field K over which G splits (i.e. its maximal torus T becomes isomorphic to GL_1^r). This semi-direct product is determined by the action of the Galois group $\Gamma_K = \mathrm{Gal}(K^s/K)$ on the set of maximal tori defined over K^s.

The most important classification result of the Langlands theory states that if

$$\pi = \bigotimes_v \pi_v \in \mathfrak{A}(G/K)$$

then for almost all v the local component π_v corresponds to a unique conjugacy class of an element h_v in the group ${}^LG$.

Let us consider the Euler product

$$L(\pi, r, s) = \prod_{v \notin S} L(\pi_v, r, s), \tag{6.5.10}$$

where S is a finite set of places of K,

$$L(\pi_v, r, s) = \det(1_m - \mathrm{N}v^{-s} r(h_v))^{-1}.$$

Langlands has shown that if $\pi \in \mathfrak{A}(G/K)$ then the product in (6.5.10) converges absolutely for all s with sufficiently large real part $\mathrm{Re}(s)$ (cf. [L71a]). The product (6.5.10) defines an automorphic L-function only up to a finite number of Euler factors. Although this is sufficient for certain questions related to analytic continuation of these functions, the precise form of these missing factors is very important in the study of the functional equations. A list of standard conjectures on the analytic properties of the L-functions (6.5.10) can be found in A.Borel's paper [Bor79]

We refer to recent introductory texts to the theory of automorphic L-functions and the Langlands program: [BCSGKK3], [Bum97], [Iw97],

For the group $G = \mathrm{GL}_n$ and the standard representation $r = r_n = \mathrm{St} : {}^LG^0 \xrightarrow{\sim} \mathrm{GL}_n(\mathbb{C})$ the main analytic properties of the L-functions (6.5.10) are proved in [JPShS], [GPShR87], [Sh88], [JSh] (see also [Bum97], [BCSGKK3], [CoPSh94]).

Also in the case $G = \mathrm{GL}_n$ the multiplicity one theorem (an analogue of the theorem of Atkin–Lehner) (cf. [AL70], [Mi89], [Li75]) has been extended (cf. [Gel75], [Gel76]). This is closely related to the non-vanishing theorem: for a cuspidal representation π one has $L(\pi, r_n, 1) \neq 0$.

For GL_3 an analogue of Weil's inverse theorem (see §6.3.8) has been proved: if all the L-functions of type $L(\pi \otimes \chi, r_3, s)$ (where χ is a Hecke character and π is an irreducible admissible representation) can be holomorphically continued to the entire complex plane, then the representation π can be realized in the space of cusp forms ([CoPSh94], [JPShS]). More recent results on the case of GL_n, cf. [CoPSh02].

Interesting classes of L-functions attached to Siegel modular forms were introduced and studied in [An74], [An79a], [AK78]. These modular forms and their L-functions have deep arithmetical significance and are closely related to the classical problem on the number of representations of a positive definite integral quadratic form by a given integral quadratic form (as generating functions, or theta–series). These numbers arise in Siegel's general formula considered above (5.3.71). From the point of view of the theory of automorphic representations, Siegel modular forms correspond to automorphic forms on the symplectic group $G = \mathrm{GSp}_n$. In this case the dual Langlands group coincides with the universal covering $\mathrm{Spin}_{2n+1}(\mathbb{C})$ of the orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{C})$. To construct L-functions one uses the following two kinds of representation of the L-group ${}^LG = \mathrm{Spin}_{2n+1} \rtimes \mathrm{Gal}(K^s/K)$: ρ_{2n+1} and r_n, where ρ_{2n+1} is the standard representation of the orthogonal group, and r_n is the spinor representation of dimension 2^n. It is convenient to consider the following matrix realization of the orthogonal group:

$$\mathrm{SO}_{2n+1}(\mathbb{C}) = \left\{g \in \mathrm{SL}_{2n+1}(\mathbb{C}) \mid {}^tgG_ng = G_n\right\},$$

with a quadratic form defined by the matrix

$$G_n = \begin{pmatrix} 0_n & 1_n & 0 \\ \cdots & \cdots & \cdots \\ 1_n & 0_n & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \quad 1_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $\pi = \otimes_v \pi_v \in \mathfrak{A}(\mathrm{GSp}_n/K)$ then for almost all v the representation π_v corresponds to a conjugacy class h_v in ${}^LG$ whose image in the standard representation is given by a diagonal matrix of the type

$$\rho_{2n+1}(h_v) = \left\{\alpha_{1,v}, \cdots, \alpha_{n,v}, \alpha_{1,v}^{-1}, , \alpha_{n,v}^{-1}, 1\right\},$$

and in the spinor representation it becomes

$$r_n(h_v) = \{\beta_{0,v}, \beta_{0,v}\alpha_{1,v}, \cdots, \beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v}\cdots\alpha_{i_m,v}, \cdots\},$$

where for every $m \leq n$ all possible products of the type

$$\beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v}\cdots\alpha_{i_m,v}, \quad 1 \leq i_1 < i_2 < \cdots < i_m \leq n$$

arise.

The element h_v is uniquely defined upto the action of the Weyl group W_n generated by the substitutions

$$\beta_{0,v} \mapsto \beta_{0,v}\alpha_{i,v}, \quad \alpha_{i,v} \mapsto \alpha_{i,v}^{-1}, \quad \alpha_{j,v} \mapsto \alpha_{j,v} \quad (j \neq i)$$

and by all possible substitutions of the coordinates

$$\alpha_{i_1,v}, \alpha_{i_2,v}, \cdots \alpha_{i_n,v}.$$

A.N.Andrianov has established meromorphic continuations and functional equations for automorphic L-functions of the type $L(\pi_f, r_n, s)$ where π_f is the automorphic representation of $\mathrm{GSp}_n(\mathbb{A})$ over $\mathbb{Q}$ attached to a Siegel modular form f with respect to $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, $n = 2$. He has also studied the holomorphy properties of these *spinor L-functions* for various classes of Siegel modular forms f, cf. [An74], [An79a] . Analytic properties of such functions are related to versions of the theory of new forms in the Siegel modular case for $n = 2$, cf. [AP2000]. A.N. Andrianov and V.L.Kalinin in [AK78] have studied the analytic properties of the *standard L-functions* $L(\pi_f, \rho_{2n+1}, s)$, where π_f is the automorphic representation of $\mathrm{GSp}_n(\mathbb{A})$ over $\mathbb{Q}$ attached to a Siegel modular form f with respect to the congruence subgroup $\Gamma_0^n(N) \subset \mathrm{Sp}_n(\mathbb{Z})$. For $n = 1$ these L-functions coincide with the symmetric squares of Hecke series, previously studied by Shimura.

A general *doubling method* giving explicit constructions of many automorphic L-functions, was developed in [Boe85] and [GPShR87].

Further analytic properties of automorphic L-functions

We refer to Sarnak's plenary lecture [Sar98] to ICI–1998, and to the related papers [IwSa99], [KS99], [KS99a], [LRS99], [KiSha99].

In [IwSa99], four fundamental conjectures were discussed: (A) Grand Riemann hypothesis; (B) Subconvexity problem; (C) Generalized Ramanujan conjecture; (D) Birch and Swinnerton-Dyer conjecture. Another problem which is related to (D) is a special value problem. Namely, the question as to whether an L-function vanishes at a special point on the critical line.

From the classical point of view, analytic and arithmetic properties of new classes of automorphic L-functions where studied in new Shimura's books [Shi2000], [Shi04], using a developed theory of Eisenstein series on reductive groups.

6.5.3 The Langlands Functoriality Principle

(cf. [Bor79], [BoCa79], [Gel75], [Pan84] , and for recent developments, [Lau02], [Hen01], [Car2000], [Li2000], [BCSGKK3], [CKPShSh]). This important principle establishes ties between automorphic representations of different reductive groups H and G. A homomorphism of the L-groups $u : {}^LH \to {}^LG$ attached to G and H is called an L-homomorphism if the restriction of u to ${}^LH^0(\mathbb{C})$ is a complex analytic homomorphism to ${}^LG^0(\mathbb{C})$, and u induces the identity map on the Galois group G_K. The functoriality principle is formulated in terms of the conjugacy classes of the matrices h_v corresponding to the local components π_v of an irreducible admissible representation $\pi = \otimes_v \pi_v$ of the group $H(\mathbb{A}_K)$. It includes the following statements:

1) *locally*: for almost all v there exists an irreducible admissible representation $u_*(\pi_v)$ of the group $G_v = G(K_v)$ which corresponds to the conjugacy class of the element $u(h_v)$ in ${}^LG$;