### 6.2.7 Zeta Functions, $L$-Functions and Motives

(cf. [Man68], [Del79]). As we have seen with the example of the Dedekind zeta function $\zeta_K(s)$, the zeta function $\zeta(X,s)$ of an arithmetic scheme $X$ can often be expressed in terms of $L$–functions of certain Galois representations. This link seems to be universal in the following sense.

Let $X \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic scheme over the maximal order $\mathcal{O}_K$ of a number field $K$ such that the generic fiber $X_K = X \otimes_{\mathcal{O}_K} K$ is a smooth projective variety of dimension $d$, and let

$$\zeta(X,s) = \prod_{\mathfrak{p}} \zeta(X(\mathfrak{p}),s)$$

be its zeta function, where $X(\mathfrak{p}) = X \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{p})$ is the reduction of $X$ modulo a maximal ideal $\mathfrak{p} \subset \mathcal{O}_K$. The shape of the function $\zeta(X(\mathfrak{p}),s)$ is described by the Weil conjecture (W4). If we assume that all $X(\mathfrak{p})$ are smooth projective varieties over $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_q$ then we obtain the following expressions for $\zeta(X,s)$:

$$\zeta(X,s) = \prod_{i=0}^{2d} L_i(X,s)^{(-1)^{i+1}}, \tag{6.2.56}$$

where

$$L_i(X,s) = \prod_{\mathfrak{p}} P_{i,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

and $P_{i,\mathfrak{p}}(X,t) \in \overline{\mathbb{Q}}[t]$ denote polynomials from the decomposition of the zeta function

$$\zeta(X(\mathfrak{p}),s) = \prod_{i=0}^{2d} P_{i,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{(-1)^{i+1}}.$$

In order to prove the conjecture (W4) ("the Riemann Hypothesis over a finite field"), Deligne identified the functions $L_i(X,s)$ with the $L$–functions of certain rational $l$–adic Galois representations

$$\rho_{X,i} : G_K \to \mathrm{Aut}\ H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l); \quad L_i(X,s) = L(\rho_{X,i}, s)$$

defined by a natural action of the Galois group $G_K$ on the $l$–adic cohomology groups $H^*_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l)$ using the transfer of structure

$$\begin{array}{lcl} X_{\overline{K}} & = & X_K \otimes \overline{K} \\ \downarrow & & \\ \mathrm{Spec}\ \overline{K} & \xrightarrow{\sigma} & \mathrm{Spec}\ \overline{K} \ \ (\sigma \in \mathrm{Aut}\ \overline{K}). \end{array}$$

If $X_K$ is an algebraic curve then there are $G_K$–module isomorphisms

$$H^1_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) \cong V_l(J) = T_l(X) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

(the Tate module of the Jacobian of $X$),

$$H^0_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) = \mathbb{Q}_l, \quad H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) \cong V_l(\mu)$$

($V_l(\mu) = T_l(\mu) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ the Tate module of $l$–power roots of unity). This implies the following explicit expressions for the $L$–functions

$$L_0(X,s) = \zeta_K(s), \qquad L_2(X,s) = \zeta_K(s-1),$$

and the zeta function

$$L_1(X,s) = L(X,s) = \prod_{\mathfrak{p}} P_{1,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

(where $\deg P_{1,\mathfrak{p}}(X,t) = 2g$, $g$ is the genus of the curve $X_K$) is often called the $L$–function of the curve $X$.

For topological varieties cohomology classes can be represented using cycles (by Poincaré duality), or using cells if the variety is a CW–complex. Grothendieck has conjectured that an analogue of CW–decomposition must exist for algebraic varieties over $K$. In view of this decomposition the factorization of the zeta function (6.2.56) should correspond to the decomposition of the variety into "generalized cells", which are no longer algebraic varieties but *motives*, elements of a certain larger category $\mathcal{M}_K$. This category is constructed in several steps, starting from the category $\mathcal{V}_K$ of smooth projective varieties over $K$.

*Step 1).* One constructs first an additive category $\mathcal{M}'_K$ in which $\mathrm{Hom}(M, N)$ are $\mathbb{Q}$–linear vector spaces, and one constructs a contravariant functor $H^*$ from $\mathcal{V}_K$ to $\mathcal{M}'_K$, which is bijective on objects (i.e. with objects $H^*(X)$ one for each $X \in \mathcal{O}b(\mathcal{V}_K)$). This category is endowed with the following additional structures:

a) a tensor product $\otimes$ satisfying the standard commutativity, associativity and distributivity constraints;
b) the functor $H^*$ takes disjoint unions of varieties into direct sums and products into tensor products (by means of a natural transformation compatible with the commutativity and associativity).

In this definition the group $\mathrm{Hom}(H^*(X), H^*(Y))$ is defined as a certain group of classes of correspondences between $X$ and $Y$. For a smooth projective variety $X$ over $K$ denote by $Z^i(X)$ the vector space over $\mathbb{Q}$ whose basis is the set of all irreducible closed subschemes of codimension $i$, and denote by $Z^i_R(X)$ its quotient space modulo cohomological equivalence of cycles. Then in Grothendieck's definition, for fields $K$ of characteristic zero one puts

$$\mathrm{Hom}(H^*(Y), H^*(X)) = Z_R^{\dim(Y)}(X \times Y).$$

*Step 2. The category* $\mathcal{M}_{\mathrm{eff},K}$ *of false effective motives.* This is obtained from $\mathcal{M}'_K$ by formally adjoining the images of all projections (i.e. of idempotent morphisms). In this category every projection arises from a direct sum decomposition. Categories with a tensor product and with the latter property are called *caroubien* or *pseudo–Abelian* categories; $\mathcal{M}_{\mathrm{eff},K}$ is the pseudo–Abelian envelope of $\mathcal{M}'_K$, cf. [Del79].

*Step 3. The category* $\overset{\circ}{\mathcal{M}}_K$ *of false motives.* Next we adjoin to $\mathcal{M}_{\mathrm{eff},K}$ all powers of the Tate object $\mathbb{Q}(1) = \underline{\mathrm{Hom}}(L, \mathbb{Q})$, where $L = \mathbb{Q}(-1) = H^2(\mathbb{P}^1)$ is the Lefschetz object and $\underline{\mathrm{Hom}}$ denotes the internal Hom in $\mathcal{M}_{\mathrm{eff},K}$. As a result we get the category $\overset{\circ}{\mathcal{M}}_K$ of "false motives". The category $\overset{\circ}{\mathcal{M}}_K$ can be obtained by a universal construction which converts the functor $M \to M \otimes \mathbb{Q}(-1) = M(-1)$ into an invertible functor. Each object of $\overset{\circ}{\mathcal{M}}_K$ has the form $M(n)$ with some $M$ from $\mathcal{M}_{\mathrm{eff},K}$.
Note that for $X \in \mathcal{O}b(\mathcal{V}_K)$ the objects $H^i(X)$ are defined as the images of appropriate projections and

$$H^*(X) = \bigoplus_{i=0}^{2d} H^i(X).$$

The category $\overset{\circ}{\mathcal{M}}_K$ is a $\mathbb{Q}$–linear rigid Abelian category with the commutativity rule

$$\Psi^{r,s} : H^r(X) \otimes H^s(Y) \cong H^s(Y) \otimes H^r(X), u \otimes v \mapsto (-1)^{rs} v \otimes u,$$

which implies that the rank $\mathrm{rk}(H(X)) = \sum(-1)^r \dim H^r(X)$ could be negative (in fact it coincides with the *Euler characteristic* of $X$).

*Step 4. The category $\mathcal{M}_K$ of true motives* is obtained from $\overset{\circ}{\mathcal{M}}_K$ by a modification of the above commutativity constraint, in which the sign $(-1)^{rs}$ is dropped. This is a $\mathbb{Q}$–linear *Tannakian category*, formed by direct sums of factors of the type $M \subset H^r(X)(m)$, see [Del79].

Tannakian categories are characterized by the property that every such category (endowed with a fiber functor) can be realized as the category of finite dimensional representations of some (pro–) algebraic group.
In particular, the thus obtained category of motives can be regarded as the category of finite dimensional representations of a certain (pro–) algebraic group (the so-called *motivic Galois group*).
Each standard cohomology theory $\mathcal{H}$ on $\mathcal{V}_K$ (a functor from $\mathcal{V}_K$ to an Abelian category with the Künneth formula and with some standard functoriality properties) can be extended to the category $\mathcal{M}_K$. This extension thus defines the $\mathcal{H}$–realizations of motives.

In order to construct $L$–functions of motives one uses the following realizations:

a) *The Betti realization $H_B$*: for a field $K$ embedded in $\mathbb{C}$ and $X \in \mathcal{O}b(\mathcal{V}_K)$ the singular cohomology groups (vector spaces over $\mathbb{Q}$) are defined

$$\mathcal{H} : X \mapsto H^*(X(\mathbb{C}), \mathbb{Q}) = H_B(X).$$

One has a Hodge decomposition of the complex vector spaces

$$H_B(M) \otimes \mathbb{C} = \oplus H_B^{p,q}(M) \quad (h^{p,q} = \dim_{\mathbb{C}} H_B^{p,q}(M)),$$

so that $\overline{H_B^{p,q}(M)} = H_B^{q,p}(M)$. If $K \subset \mathbb{R}$ then the complex conjugation on $X(\mathbb{C})$ defines a canonical involution $F_\infty$ on $H_B(M)$, which may be viewed as the Frobenius element at infinity.

b) *The $l$–adic realizations $H_l$*: if Char $K \neq l$, $X \in \mathcal{O}b(\mathcal{V}_K)$ then the $l$–adic cohomology groups are defined as certain vector spaces over $\mathbb{Q}_l$

$$\mathcal{H} : X \mapsto H^*_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_l) = H_l(X).$$

There is a natural action of the Galois group $G_K$ on $H_l(X)$ by way of which one assigns an $l$–adic representation to a motive $M \in \mathcal{M}_K$

$$\rho_{M,l} : G_K \longrightarrow \mathrm{Aut}\ H_l(M).$$

A non–trivial fact is that these representations are $E$–rational for some $E$, $E \subset \mathbb{C}$ in the sense of §6.2.1.

Using the general construction of 6.2.1 one defines the $L$–functions

$$L(M,s) = \prod_v L_v(M,s) \quad (v \text{ finite}),$$

where $L_v(M,s)^{-1} = L_{\mathfrak{p}_v}(M, \mathrm{N}\mathfrak{p}_v^{-s})^{-1}$ are certain polynomials in the variable $t = \mathrm{N}\mathfrak{p}_v^{-s}$ with coefficients in $E$.

For Archimedean places $v$ one chooses a complex embedding $\tau_v : K \to \mathbb{C}$ defining $v$. Then the factors $L_v(M,s)$ are constructed using the Hodge decomposition $H_B(M) \otimes \mathbb{C} = \oplus H_B^{p,q}(M)$ and the action of the involution $F_\infty$ (see the table in 5.3. of [Del79]).

According to a general conjecture the product

$$\Lambda(M,s) = \prod_v L_v(M,s) \quad (v \in \Sigma_K).$$

admits an analytic (meromorphic) continuation to the entire complex plane and satisfies a certain (conjectural) functional equation of the form

$$\Lambda(M,s) = \varepsilon(M,s)\Lambda(M^\vee, 1-s),$$

where $M^\vee$ is the motive dual to $M$ (its realizations are duals of those of $M$), and $\varepsilon(M,s)$ is a certain function of $s$ which is a product of an exponential function and a constant.

One has the following equation

$$\Lambda(M(n),s) = \Lambda(M, s+n).$$

A motive $M$ is called pure of weight $w$ if $h^{p,q} = 0$ for $p+q \neq w$. In this case we put $\mathrm{Re}(M) = -\dfrac{w}{2}$. The Weil conjecture W4) (see section 6.1.3) implies that for a sufficiently large finite set $S$ of places of $K$ the corresponding Dirichlet series (and the Euler product)

$$L_S(M,s) = \prod_{v \notin S} L_v(M,s)$$

converges absolutely for $\mathrm{Re}(M) + \mathrm{Re}(s) > 1$.

For points $s$ on the boundary of absolute convergence (i.e. for $\mathrm{Re}(M) + \mathrm{Re}(s) = 1$ there is the following general conjecture (generalizing the theorem of Hadamard and de la Vallée–Poussin):

a) the function $L_S(M,s)$ does not vanish for $\mathrm{Re}(M)+\mathrm{Re}(s)=1$;
b) the function $L_S(M,s)$ is entire apart from the case when $M$ has even weight $-2n$ and contains as a summand the motive $\mathbb{Q}(n)$; in the last case there is a pole at $s=1-n$.

For example, for the motive $\mathbb{Q}(-1)$ one has

$$H_B(\mathbb{Q}(-1)) = H^2(\mathbb{P}^1(\mathbb{C}),\mathbb{Q}), \quad H_l(\mathbb{Q}(-1)) \cong V_l(\mu) = T_l(\mu)\otimes_{\mathbb{Z}_l}\mathbb{Q}_l,$$

$w=2$, $n=-1$ and the $L$ -function

$$L(\mathbb{Q}(-1),s) = \zeta_K(s-1)$$

has a simple pole at $s=2$.

There are some very general conjectures on the existence of a correspondence between motives and compatible systems of $l$–adic representations. Nowadays these conjectures essentially determine key directions in arithmetical research ([CR01], [Tay02], [BoCa79], [Bor79], [Ta79]). We mention only a remarkable fact that in view of the proof of the theorem of G. Faltings (see §5.5) an Abelian variety is uniquely determined upto isogeny by the corresponding $l$–adic Galois representation on its Tate module.

This important result is cruicial also in Wiles' marvelous proof: in order to show that every semistable elliptic curve $E$ over $\mathbb{Q}$ admits a modular parametrisation (see §7.2), it is enough (due to Faltings) to check that for some prime $p$ the $L$–function of the Galois representation $\rho_{p,E}$ coinsides with the Mellin transform of a modular form of weight two (Wiles has used $p=3$ and $p=5$). In other words, the generating series of such a representation, defined starting from the traces of Frobenius elements, is a modular form of weight two which is proved by counting all possible deformations of the Galois representation in question taken modulo $p$.