Lie algebra, semi-simple

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A Lie algebra that has no non-zero solvable ideals (see Lie algebra, solvable). Henceforth finite-dimensional semi-simple Lie algebras over a field k of characteristic 0 are considered (for semi-simple Lie algebras over a field of non-zero characteristic see Lie algebra).

The fact that a finite-dimensional Lie algebra $\mathfrak g$ is semi-simple is equivalent to any of the following conditions:

1) $\mathfrak g$ does not contain non-zero Abelian ideals;

2) the Killing form of $\mathfrak g$ is non-singular (Cartan's criterion);

3) $\mathfrak g$ splits into the direct sum of non-Abelian simple ideals;

4) every finite-dimensional linear representation of $\mathfrak g$ is completely reducible (in other words: every finite-dimensional $\mathfrak g$-module is semi-simple);

5) the one-dimensional cohomology of $\mathfrak g$ with values in an arbitrary finite-dimensional $\mathfrak g$-module is trivial.

Any ideal and any quotient algebra of a semi-simple Lie algebra is also semi-simple. The decomposition of a semi-simple Lie algebra mentioned in condition 3) is unique. A special case of condition 5) is the following assertion: All derivations of a semi-simple Lie algebra are inner. The property of a Lie algebra of being semi-simple is preserved by both extensions and restrictions of the ground field.

Let $\mathfrak g$ be a semi-simple Lie algebra over an algebraically closed field k. The adjoint representation maps $\mathfrak g$ isomorphically onto the linear Lie algebra $\mathrm{ad}\ \mathfrak g$, which is the Lie algebra of the algebraic group $\mathrm{Aut}\ \mathfrak g$ of all automorphisms of $\mathfrak g$ and is therefore an algebraic Lie algebra (cf. Lie algebra, algebraic). An element $X \in \mathfrak g$ is said to be semi-simple (nilpotent) if $\mathrm{ad}\ X$ is semi-simple (respectively, nilpotent). This property of an element X is preserved by any homomorphism of $\mathfrak g$ into another semi-simple Lie algebra. The identity component $(\mathrm{Aut}\ \mathfrak g)^0$ coincides with the group of inner automorphisms of $\mathfrak g$, that is, it is generated by the automorphisms of the form $\exp(\mathrm{ad}\ X)$, $X \in \mathfrak g$.

In the study of semi-simple Lie algebras over an algebraically closed field k an important role is played by the roots of a semi-simple Lie algebra, which are defined as follows. Let $\mathfrak h$ be a Cartan subalgebra of $\mathfrak g$. For a non-zero linear function $\alpha \in \mathfrak h^*$, let $\mathfrak g_\alpha$ denote the linear subspace of $\mathfrak g$ given by the condition

$$\mathfrak g_\alpha = \{X \in \mathfrak g :\ [H, X] = \alpha(H)X, H \in \mathfrak h\} .$$

If $\mathfrak g_\alpha \neq 0$, then α is called a root of $\mathfrak g$ with respect to $\mathfrak h$. The set Σ of all non-zero roots is called the root system, or system of roots, of $\mathfrak g$. One has the root decomposition

$$\mathfrak g = \mathfrak h + \sum_{\alpha \in \Sigma} \mathfrak g_\alpha .$$

The root system and the root decomposition of a semi-simple Lie algebra have the following properties:

a) Σ generates $\mathfrak h^*$ and is a reduced root system in the abstract sense (in the linear hull of Σ over the field of the real numbers). The system Σ is irreducible if and only if $\mathfrak g$ is simple.

b) For any $\alpha \in \Sigma$,

$$\dim \mathfrak g_\alpha = \dim [\mathfrak g_\alpha, \mathfrak g_{-\alpha}] = 1 .$$

There is a unique element $H_\alpha \in [\mathfrak g_\alpha, \mathfrak g_{-\alpha}]$ such that $\alpha(H_\alpha) = 2$.

c) For every non-zero $X_\alpha \in \mathfrak g_\alpha$ there is a unique $Y_\alpha \in \mathfrak g_{-\alpha}$ such that $[X_\alpha, Y_\alpha] = H_\alpha$, and

$$[H_\alpha, X_\alpha] = 2X_\alpha \quad \textrm{and} \quad [H_\alpha, Y_\alpha] = -2Y_\alpha .$$

Moreover,

$$\beta(H_\alpha) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}, \quad \alpha, \beta \in \Sigma,$$

where $(,)$ is the scalar product induced by the Killing form.

d) If $\alpha, \beta \in \Sigma$ and $\alpha + \beta \neq 0$, then $\mathfrak g_\alpha$ and $\mathfrak g_\beta$ are orthogonal with respect to the Killing form and $[\mathfrak g_\alpha, \mathfrak g_\beta] = \mathfrak g_{\alpha+\beta}$.

A basis $\{ \alpha_1, \dots, \alpha_n \}$ of the root system Σ is also called a system of simple roots of the algebra $\mathfrak g$. Let Σ_+ be the system of positive roots with respect to the given basis and let $X_{-\alpha} = Y_\alpha (\alpha \in \Sigma_+)$. Then the elements

$$H_{\alpha_1}, \dots, H_{\alpha_k}, X_\alpha \quad (\alpha \in \Sigma)$$

form a basis of $\mathfrak g$, called a Cartan basis. On the other hand, the elements

$$X_{\alpha_i}, X_{-\alpha_i} \quad (i = 1, \dots, n)$$

form a system of generators of $\mathfrak g$, and the defining relations have the following form:

$$[[X_{\alpha_i}, X_{-\alpha_i}], X_{\alpha_j}] = n(i, j) X_{\alpha_j},$$
$$[[X_{\alpha_i}, X_{-\alpha_i}], X_{-\alpha_j}] = -n(i, j) X_{\alpha_j},$$
$$(\textrm{ad}\, X_{\alpha_i})^{1-n(i,j)} X_{\alpha_j} = 0,$$
$$(\textrm{ad}\, X_{-\alpha_i})^{1-n(i,j)} X_{-\alpha_j} = 0 .$$

Here $i, j = 1, \dots, n$ and

$$n(i, j) = \alpha_j(H_i) = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} .$$

Property d) implies that

$$[X_\alpha, X_\beta] = \begin{cases} N_{\alpha,\beta} X_{\alpha+\beta} & \text{if } \alpha+\beta \in \Sigma, \\ 0 & \text{if } \alpha+\beta \notin \Sigma, \end{cases}$$

where $N_{\alpha,\beta} \in k$. The elements X_α ($\alpha \in \Sigma_+$) can be chosen in such a way that

$$N_{\alpha,\beta} = -N_{-\alpha,-\beta} \quad \text{and} \quad N_{\alpha,\beta} = \pm(p+1),$$

where p is the largest integer such that $\beta - p\alpha \in \Sigma$. The corresponding Cartan basis is called a Chevalley basis. The structure constants of $\mathfrak{g}$ in this basis are integers, which makes it possible to associate with $\mathfrak{g}$ Lie algebras and algebraic groups (see Chevalley group) over fields of arbitrary characteristic. If $k = \mathbf{C}$, then the linear hull over $\mathbf{R}$ of the vectors

$$iH_\alpha, \quad X_\alpha - X_{-\alpha}, \quad i(X_\alpha + X_{-\alpha}) \quad (\alpha \in \Sigma_+)$$

is a compact real form of $\mathfrak{g}$.

A semi-simple Lie algebra is defined up to an isomorphism by its Cartan subalgebra and the corresponding root system. More precisely, if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are semi-simple Lie algebras over k, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are their Cartan subalgebras and Σ_1 and Σ_2 are the corresponding root systems, then every isomorphism $\mathfrak{h}_1 \rightarrow \mathfrak{h}_2$ that induces an isomorphism of the root systems Σ_1 and Σ_2 can be extended to an isomorphism $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$. On the other hand, any reduced root system can be realized as the root system of some semi-simple Lie algebra. Thus, the classification of semi-simple Lie algebras (respectively, simple non-Abelian Lie algebras) over an algebraically closed field k essentially coincides with the classification of reduced root systems (respectively, irreducible reduced root systems).

Simple Lie algebras that correspond to root systems of types A–D are said to be classical and have the following form.

Type A_n, $n \geq 1$. $\mathfrak{g} = \mathfrak{sl}(n+1, k)$, the algebra of linear transformations of the space k^{n+1} with trace 0; $\dim \mathfrak{g} = n(n+2)$.

Type B_n, $n \geq 2$. $\mathfrak{g} = \mathfrak{so}(2n+1, k)$, the algebra of linear transformations of the space k^{2n+1} that are skew-symmetric with respect to a given non-singular symmetric bilinear form; $\dim \mathfrak{g} = n(2n+1)$.

Type C_n, $n \geq 3$. $\mathfrak{g} = \mathfrak{sp}(n, k)$, the algebra of linear transformations of the space k^{2n} that are skew-symmetric with respect to a given non-singular skew-symmetric bilinear form; $\dim \mathfrak{g} = n(2n+1)$.

Type D_n, $n \geq 4$. $\mathfrak{g} = \mathfrak{so}(2n, k)$, the algebra of linear transformations of the space k^{2n} that are skew-symmetric with respect to a given non-singular symmetric bilinear form; $\dim \mathfrak{g} = n(2n-1)$.

The simple Lie algebras corresponding to the root systems of types E_6, E_7, E_8, F_4, G_2 are called special, or exceptional (see Lie algebra, exceptional).

The Cartan matrix of a semi-simple Lie algebra over an algebraically closed field also determines this algebra uniquely up to an isomorphism. The Cartan matrices of the simple Lie algebras have the following form:

$$A_n: \quad \left\| \begin{array}{rrrrr} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ . & . & . & \dots & . \\ 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 2 \end{array} \right\|,$$

$$B_n: \quad \left\| \begin{array}{rrrrrr} 2 & -1 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 0 \\ 0 & 0 & 0 & \dots & 2 & -2 \\ 0 & 0 & 0 & \dots & -1 & 2 \end{array} \right\|,$$

$$C_n: \quad \left\| \begin{array}{rrrrrrr} 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{array} \right\|,$$

$$D_n: \quad \left\| \begin{array}{rrrrrrrr} 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 & 0 \\ . & . & . & \dots & . & . & . & . \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & 2 \end{array} \right\|,$$

$$E_6: \quad \left\| \begin{array}{rrrrrr} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{array} \right\|,$$

$$E_7: \quad \left\|\begin{array}{rrrrrrr} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right\|,$$

$$E_8: \quad \left\|\begin{array}{rrrrrrrr} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{array}\right\|,$$

$$F_4: \quad \left\|\begin{array}{rrrr} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{array}\right\|, \qquad G_2: \quad \left\|\begin{array}{rr} 2 & -1 \\ -3 & 2 \end{array}\right\|.$$

The classification of split semi-simple Lie algebras over an arbitrary field k of characteristic zero (a semi-simple Lie algebra $\mathfrak{g}$ is said to be split if it has a Cartan subalgebra $\mathfrak{h}\subset\mathfrak{g}$ such that all characteristic roots of the operators $\mathrm{ad}\, X$, $X\in\mathfrak{h}$, lie in k) goes in the same way as in the case of an algebraically closed field. Namely, to every irreducible reduced root system corresponds a unique split semi-simple Lie algebra. In particular, split semi-simple Lie algebras of types A–D have the form stated above, except that in the cases B and D one must consider non-singular symmetric bilinear forms with Witt index n.

The problem of classifying arbitrary semi-simple Lie algebras over k reduces to the following problem: To list, up to an isomorphism, all k-forms $\mathfrak{g}_0\subset\mathfrak{g}$, that is, all k-subalgebras $\mathfrak{g}_0\subset\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{g}_0\otimes_k K$. Here K is an algebraically closed extension of k and $\mathfrak{g}$ is a given semi-simple Lie algebra over K. The solution of this problem can also be obtained in terms of root systems (see Form of an algebraic group; Form of an (algebraic) structure). When $\mathfrak{g}$ is a classical simple Lie algebra over k (other than D_4), there is another method of classifying k-forms in $\mathfrak{g}$, based on an examination of simple associative algebras (see [3]).

When $k=\mathbf{R}$ the classification of semi-simple Lie algebras goes as follows (see [6], [7]). Every simple non-Abelian Lie algebra over $\mathbf{R}$ is either a simple Lie algebra over $\mathbf{C}$ (regarded as an algebra over $\mathbf{R}$), or the real form of a simple Lie algebra over $\mathbf{C}$. The classification of real forms $\mathfrak{g}_0$ in a simple classical Lie algebra $\mathfrak{g}$ over $\mathbf{C}$ is as follows:

I) Type A_n: $\mathfrak{g}=\mathfrak{sl}(n+1,\mathbf{C})$, $n\geq 1$. A_I: $\mathfrak{g}_0=\mathfrak{sl}(n+1,\mathbf{R})$. A_{II}: $n+1=2m$ is even, $\mathfrak{g}_0=\mathfrak{su}^*(2n)$, the subalgebra of elements of $\mathfrak{sl}(2m,\mathbf{C})$ that preserve a certain quaternion structure. A_{III}: $\mathfrak{g}_0=\mathfrak{su}(p,n+1-p)$, the subalgebra of elements of $\mathfrak{sl}(n+1,\mathbf{C})$ that are skew-symmetric with respect to a non-singular Hermitian form of positive index p, $0\leq p\leq(n+1)/2$.

II) Type B_n: $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbf{C})$, $n \geq 2$. B_I: $\mathfrak{g}_0 = \mathfrak{so}(p, 2n+1-p)$, the algebra of a linear transformations of the space $\mathbf{R}^{2n+1}$ that are skew-symmetric with respect to a non-singular symmetric bilinear form of positive index p, $0 \leq p \leq n$.

III) Type C_n: $\mathfrak{g} = \mathfrak{sp}(n, \mathbf{C})$, $n \geq 3$. C_I: $\mathfrak{g}_0 = \mathfrak{sp}(n, \mathbf{R})$, the algebra of linear transformations of the space $\mathbf{R}^{2n}$ that are skew-symmetric with respect to a non-singular skew-symmetric bilinear form. C_{II}: $\mathfrak{g}_0 = \mathfrak{sp}(p, n-p)$, $0 \leq p \leq n/2$, the subalgebra of $\mathfrak{su}(2p, 2(n-p))$ consisting of transformations that preserve a certain quaternion structure.

IV) Type D_n: $\mathfrak{g} = \mathfrak{so}(2n, \mathbf{C})$, $n \geq 4$. D_I: $\mathfrak{g}_0 = \mathfrak{so}(p, 2n-p)$, the algebra of linear transformations of the space $\mathbf{R}^{2n}$ that are skew-symmetric with respect to a non-singular bilinear symmetric form of positive index p, $0 \leq p \leq n$. D_{III}: $\mathfrak{g}_0 = \mathfrak{so}^* (2n, \mathbf{C})$, the subalgebra of $\mathfrak{so}(2n, \mathbf{C})$ consisting of transformations that preserve a certain quaternion structure.

Semi-simple Lie algebras over the field $\mathbf{C}$ were first considered in papers by W. Killing , who gave a classification of them, although in his proofs there were gaps, which were filled by E. Cartan [2]. In the papers of Killing and Cartan the roots of a Lie algebra appeared as the characteristic roots of the operator $\text{ad } X$. Cartan also gave a classification of real semi-simple Lie algebras by establishing a deep connection between these algebras and globally symmetric Riemannian spaces (cf. Globally symmetric Riemannian space).

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Comments

The defining relations, mentioned above, $(\text{ad } X_{\alpha_i})^{1-n(i,j)}(X_{\alpha_j}) = 0$, are known as the Serre relations.

It is customary to encode the information contained in the Cartan matrices A_n–G_2 by means of the so-called Dynkin diagrams.'
<tbody> </tbody>

A_n · ***(n nodes)***

B_n · ***(n nodes, $n \geq 2$)***

C_n · ***(n nodes, $n \geq 3$)***

D_n · ***(n nodes, $n \geq 4$)***

E_6 · ***(6 nodes)***

E_7 · ***(7 nodes)***

E_8 · ***(8 nodes)***

F_4 · ***(4 nodes)***

G_2 · ***(2 nodes)***

The rules for recovering the Cartan matrix from the corresponding Dynkin diagram (also called Dynkin graph occasionally) are as follows. Number the vertices, e.g.,

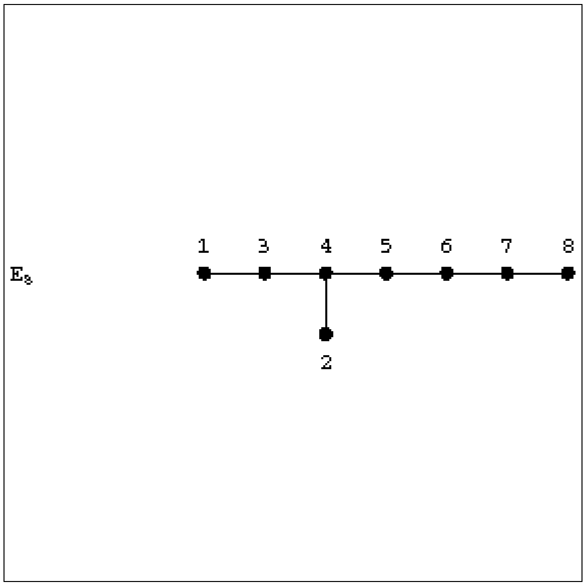

Figure: l058510a

On the diagonal of the Cartan matrix all elements are equal to 2. If nodes i and j are not directly linked, then the matrix entries $a_{ji} = a_{ij} = 0$. If two nodes i, j are directly linked by a single edge, then $a_{ij} = -1 = a_{ji}$. If two nodes i, j are directly linked by a double, respectively triple, edge and the arrow points from i to j, then $a_{ij} = -2$, $a_{ji} = -1$, respectively $a_{ij} = -3$, $a_{ji} = -1$.

References

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