algebraic. Thus it is expressible in terms of characters $z \in \mathbb{C}^\times \to z^m \bar{z}^n$, $m, n \in \mathbb{Z}$.

Only arithmeticic automorphic representations should correspond to motives. Thus the second element of our nexus is to be the collection $\mathfrak{A}$ of automorphic representations π for F, each attached to a group ${}^\lambda H$. Because of functoriality, in the stronger form described, π is no longer bound to any particular group G.

A central problem is to establish a bijective correspondence between the two elements introduced. Major progress was made by Wiles in his proof of the conjecture of Taniyama and Shimura. Since he had – and still would have – only an extremely limited form of functoriality to work with, his arguments do not appear in exactly the form just suggested. Moreover, there are two further extremely important elements in the nexus in which he works to which we have not yet come.

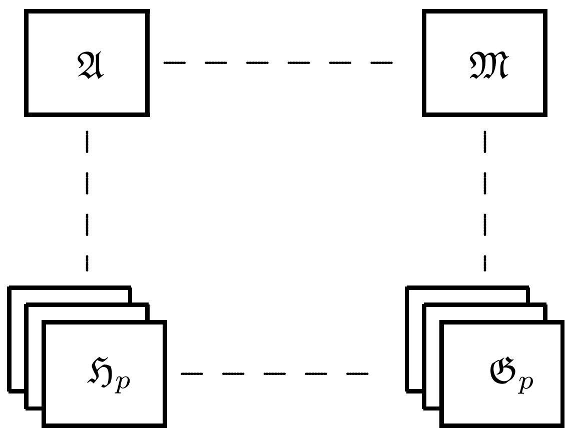
To each motive M and each prime p is attached a p-adic representation of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/F)$ of dimension equal to the rank of the motive. The third element of the nexus is not, however, the collection of p-adic Galois representations – subject to whatever constraints are necessary and appropriate. Rather it is a foliated space, in which the leaves are parametrized by p and in which there are passages from one leaf to another, permitted in so far as each p-adic representation is contained in a compatible family of representations, one for each prime. We are allowed to move from one leaf to another provided we move from one element of a compatible family to another element of the same family. The arguments of Wiles and others, those who preceded and those who followed him, rely on an often very deep analysis of the connectivity properties of the third element, either by p-adic deformation within a fixed leaf, in which often little more is demanded than congruence modulo p, or by passage from one leaf to another in the way described (cf. [Kh]) and their comparison with analogous properties of yet a fourth element whose general definition appears to be somewhat elusive.

For some purposes, but not for all, it can be taken to consist of representations of a suitably defined Hecke algebra. For automorphic representations attached to the group G over F, the Hecke algebra is defined in terms of smooth, compactly supported functions f on $G(\mathbb{A}_F^f)$, $\mathbb{A}_F^f$ being the adèles whose components at infinity are 0. They act by integration on the space of any representation π of $G(\mathbb{A}_F)$, in particular on the space of an automorphic representation or on automorphic forms.

Let $\mathbb{A}_F^\infty$ be the product of F_v at the infinite places. When the Lie group $G(\mathbb{A}_F^\infty)$, defines a bounded symmetric domain – or more precisely when a Shimura variety is attached to the group G – then there are quotients of the symmetric domain that are algebraic varieties defined over number fields. There are vector bundles defined over the same field whose de Rham cohomology groups can be interpreted as spaces of automorphic forms for the group G on which the Hecke operators will then act. The images of the Hecke algebra will be finite-dimensional algebras over some number field L and can often even be given an integral structure and then, by tensoring with the ring $\mathcal{O}_\mathfrak{p}$ of integral elements at a place $\mathfrak{p}$ of L over p, a p-adic structure, imparted of course to its spectrum. In so far as these rings form the fourth element of the nexus, the leaves are clear, as is the passage from one leaf to another. It seems to correspond pretty much to taking two different places $\mathfrak{p}$ and $\mathfrak{q}$ without changing the homomorphism over L.

The four elements form a square, motives at the top left, automorphic representations at the top right, the leaves $\mathfrak{G}_p$ of the p-adic representations at the bottom left, and the fourth as yet only partly defined element $\mathfrak{H}_p$ at the bottom right. The heart of the proof of Fermat's theorem is to deduce from the existence of one couple $\{M, {}^{\mu}G_M\} \in \mathfrak{M}$ and $\{\pi, {}^{\lambda}H_{\pi}\} \in \mathfrak{A}$ of corresponding pairs the existence of others. We pass from $\{M, {}^{\mu}G_M\}$ in $\mathfrak{M}$ to some leaf in the element below, thus to the corresponding p-adic Galois representation $\mathfrak{s}_p \in \mathfrak{G}_p$, and from $\{\pi, {}^{\lambda}H_{\pi}\}$ to an object $\mathfrak{h}_p \in \mathfrak{H}_p$, the fourth element of the nexus. Then the essence of the arguments of Wiles and Taylor-Wiles is to show that movement in $\mathfrak{G}_p$ of the prescribed type is faithfully reflected in permissible movements in $\mathfrak{H}_p$ and that if in $\mathfrak{G}_p$ the movement in leads to an image of a pair in M then the corresponding movement in $\mathfrak{H}_p$ leads to an element of $\mathfrak{A}$. These two pairs will then necessarily correspond in the sense that the associated Frobenius-Hecke classes will be the same.

As a summary of the proof of Fermat's theorem, the preceding paragraph is far too brief, but it places two features in relief. There has to be an initial seeding of couples with one term from $\mathfrak{M}$ and one from $\mathfrak{A}$ that are known for some reason or another to correspond and it has to be possible to compare the local structures of the two spaces $\mathfrak{G}$ and $\mathfrak{H}$.



The easiest seeds arise for G an algebraic torus, for then an automorphic representation π is a character of $T(\mathbb{A}_F)$ and if the character is of type A_0, thus if the representation is arithmetic, the process begun in [W] and continued by the construction of the Taniyama group ([LS]), should construct both the p-adic representations and the motive $\{M, {}^{L}T\}$ corresponding to $\{\pi, {}^{L}T\}$. From them others can be constructed by functoriality, a formality for M.

Although they are somewhat technical, it is useful to say a few words about the correspondence for tori, partly because it serves as a touchstone when trying to understand the general lucubrations, partly because the Taniyama group, the vehicle that establishes the correspondence between arithmetic automorphic forms on tori and motives, is not familiar to everyone. Most of what we need about it is formulated either as a theorem or as a conjecture in one of the papers listed in [LS], but that is clear only on close reading. In particular, it is not stressed in these papers that the correspondence yields objects with equal L-functions