## Lecture N°3. The abstract Kummer congruences and the $p$-adic Mellin transform

A useful criterion for the existence of a measure with given properties is:

**Proposition (The abstract Kummer congruences)**

*(see [Kat]). Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$ in the ring $\mathcal{C}(Y, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group $Y$ with values in the ring of integers $\mathcal{O}_p$ of $\mathbb{C}_p$ such that $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(Y, \mathbb{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure $\mu$ on $Y$ with the property*

$$\int_Y f_i d\mu = a_i$$

*is equivalent to the following congruences, for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all of which vanish*

$$\sum_i b_i f_i(y) \in p^n \mathcal{O}_p \text{ for all } y \in Y \text{ implies } \sum_i b_i a_i \in p^n \mathcal{O}_p. \qquad (4.11)$$

**Remark**

*Since $\mathbb{C}_p$-measures are characterised as bounded $\mathbb{C}_p$-valued distributions, every $\mathbb{C}_p$-measures on $Y$ becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.*



*Proof of proposition 4.1.* The necessity is obvious since

$$\begin{aligned}\sum_i b_i a_i &= \int_Y (p^n \mathcal{O}_p - \text{valued function}) d\mu = \\ &= p^n \int_Y (\mathcal{O}_p - \text{valued function}) d\mu \in p^n \mathcal{O}_p.\end{aligned}$$

In order to prove the sufficiency we need to construct a measure $\mu$ from the numbers $a_i$. For a function $f \in \mathcal{C}_(Y, \mathcal{O}_p)$ and a positive integer $n$ there exist elements $b_i \in \mathbb{C}_p$ such that only a finite number of $b_i$ does not vanish, and

$$f - \sum_i b_i f_i \in p^n \mathcal{C}(Y, \mathcal{O}_p),$$

according to the density of the $\mathbb{C}_p$-span of $\{f_i\}$ in $\mathcal{C}(Y, \mathbb{C}_p)$. By the assumption (4.11) the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo $p^n$ (i.e. does not depend on the choice of $b_i$). Following N.M. Katz ([Kat]), we denote this value by "$\int_Y f d\mu \bmod p^n$". Then we have that the limit procedure

$$\int_Y f d\mu = \lim_{n\to\infty} \text{“}\int_Y f d\mu \bmod p^n\text{”} \in \varprojlim_n \mathcal{O}_p / p^n \mathcal{O}_p = \mathcal{O}_p,$$

gives the measure $\mu$.

## Mazur's measure

Let $c > 1$ be a positive integer coprime to

$$M_0 = \prod_{q \in S} q$$

with $S$ being a fixed set of prime numbers. Using the criterion of the proposition 4.1 we show that the $\mathbb{Q}$ -valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c), \qquad f_c(x) = f(cx), \tag{4.12}$$

turns out to be a measure where $E_k(f)$ are defined by (4.8), $f \in \mathrm{Step}(Y, \mathbb{Q}_p)$ and the field $\mathbb{Q}$ is viewed as a subfield of $\mathbb{C}_p$.
Define the generelized Bernoulli polynomials $B_{k,f}^{(M)}(X)$ as

$$\sum_{k=0}^{\infty} B_{k,f}^{(M)}(X) \frac{t^k}{k!} = \sum_{a=0}^{M-1} f(a) \frac{te^{(a+X)t}}{e^{Mt} - 1}, \tag{4.13}$$

and the generalized sums of powers

$$S_{k,f}(M) = \sum_{a=0}^{M-1} f(a) a^k.$$



Then the definition (4.13) formally implies that

$$\frac{1}{k}[B_{k,f}^{(M)}(M) - B_{k,f}^{(M)}(0)] = S_{k-1,f}(M), \tag{4.14}$$

and also we see that

$$B_{k,f}^{(M)}(X) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f} X^{k-i} = B_{k,f} + kB_{k-1,f} X + \cdots + B_{0,f} X^k. \tag{4.15}$$

The last identity can be rewritten symbolically as

$$B_{k,f}(X) = (B_f + X)^k.$$

The equality (4.14) enables us to calculate the (generalized) sums of powers in terms of the (generalized) Bernoulli numbers. In particular this equality implies that the Bernoulli numbers $B_{k,f}$ can be obtained by the following $p$-adic limit procedure (see [La76]):

$$B_{k,f} = \lim_{n \to \infty} \frac{1}{Mp^n} S_{k,f}(Mp^n) \quad (\text{a } p\text{-adic limit}), \tag{4.16}$$

where $f$ is a $\mathbb{C}_p$-valued function on $Y = \mathbb{Z}_S$. Indeed, if we replace $M$ in (4.14) by $Mp^n$ with growing $n$ and let $D$ be the common denominator of all coefficients of the polynomial $B_{k,f}^{(M)}(X)$. Then we have from (4.15) that

$$\frac{1}{k}\left[B_{k,f}^{(Mp^n)}(M) - B_{k,f}^{(M)}(0)\right] \equiv B_{k-1,f}(Mp^n) \pmod{\frac{1}{kD} p^2 n}. \tag{4.17}$$

The proof of (4.16) is accomplished by division of (4.17) by $Mp^n$ and by application of the formula (4.14).

Now we can directly show that the distribution $E_k^c$ defined by (4.12) are in fact bounded measures. If we use (4.11) and take the functions $\{f_i\}$ to be all of the functions in $\mathrm{Step}(Y, \mathcal{O}_p)$. Let $\{b_i\}$ be a system of elements $b_i \in \mathbb{C}_p$ such that for all $y \in Y$ the congruence

$$\sum_i b_i f_i(y) \equiv 0 \pmod{p^n} \tag{4.18}$$

holds. Set $f = \sum_i b_i f_i$ and assume (without loss of generality) that the number $n$ is large enough so that for all $i$ with $b_i \neq 0$ the congruence

$$B_{k,f_i} \equiv \frac{1}{Mp^n} S_{k,f_i}(Mp^n) \pmod{p^n} \tag{4.19}$$

is valid in accordance with (4.16). Then we see that

$$B_{k,f} \equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}, \tag{4.20}$$

hence we get by definition (4.12):

$$\begin{aligned} E_k^c(f) &= B_{k,f} - c^k B_{k,f_c} \qquad (4.21)\\ &\equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i \left[ f_i(a) a^k - f_i(ac)(ac)^k \right] \pmod{p^n}. \end{aligned}$$



Let $a_c \in \{0, 1, \cdots, Mp^n - 1\}$, such that $a_c \equiv ac \pmod{Mp^n}$, then the map $a \longmapsto a_c$ is well defined and acts as a permutation of the set $\{0, 1, \cdots, Mp^n - 1\}$, hence (4.21) is equivalent to the congruence

$$E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv \sum_i \frac{a_c^k - (ac)^k}{Mp^n} \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}. \tag{4.22}$$

Now the assumption (4.17) formally inplies that $E_k^c(f) \equiv 0 \pmod{p^n}$, completing the proof of the abstact congruences and the construction of measures $E_k^c$.

**Remark**

*The formula (4.21) also implies that for all $f \in \mathcal{C}(Y, \mathbb{C}_p)$ the following holds*

$$E_k^c(f) = k E_1^c(x_p^{k-1} f) \tag{4.23}$$

*where $x_p : Y \longrightarrow \mathbb{C}_p \in \mathcal{C}(Y, \mathbb{C}_p)$ is the composition of the projection $Y \longrightarrow \mathbb{Z}_p$ and the embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}_p$.*

Indeed if we put $a_c = ac + Mp^n t$ for some $t \in \mathbb{Z}$ then we see that

$$a_c^k - (ac)^k = (ac + Mp^n t)^k - (ac)^k \equiv kMp^n t (ac)^{k-1} \pmod{(Mp^n)^2},$$

and we get that in (4.22):

$$\frac{a_c^k - (ac)^k}{Mp^n} \equiv k(ac)^{k-1} \frac{a_c - ac}{Mp^n} \pmod{Mp^n}.$$

The last congruence is equivalent to saying that the abstract Kummer congruences (4.11) are satisfied by all functions of the type $x_p^{k-1} f_i$ for the measure $E_1^c$ with $f_i \in \mathrm{Step}(Y, \mathbb{C}_p)$ establishing the identity (4.23).

## The domain of definition of the non-Archimedean zeta functions

In the classical case the set on which zeta functions are defined is the set of complex numbers $\mathbb{C}$ which may be viewed equally as the set of all continuous characters (more precisely, quasicharacters) via the following isomorphism:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}_+^\times, \mathbb{C}^\times) \\ s & \longmapsto & (y \longmapsto y^s) \end{array} \qquad (4.24)$$

The construction which associates to a function $h(y)$ on $\mathbb{R}_+^\times$ (with certain growth conditions as $y \to \infty$ and $y \to 0$) the following integral

$$L_h(s) = \int_{\mathbb{R}_+^\times} h(y) y^s \frac{dy}{y}$$

(which converges probably not for all values of $s$) is called the *Mellin transform*.



For example, if $\zeta(s) = \sum_{n\geq 1} n^{-s}$ is the Riemann zeta function, then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(y) = 1/(1-e^{-y})$:

$$\zeta(s)\Gamma(s) = \sum_0^\infty \frac{1}{1-e^{-y}} y^s \frac{dy}{y}, \qquad (4.25)$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. For an arbitrary function of type

$$f(z) = \sum_{n=1}^\infty a(n) e^{2i\pi nz}$$

with $z = x + iy \in \mathbb{H}$ in the upper half plane $\mathbb{H}$ and with the growth condition $a(n) = \mathrm{O}(n^c)$ $(c > 0)$ on its Fourier coefficients, we see that the zeta function

$$L(s, f) = \sum_{n=1}^\infty a(n) n^{-s},$$

essentially coincides with the Mellin transform of $f(z)$, that is

$$\frac{\Gamma(s)}{(2\pi)^s} L(s, f) = \int_0^\infty f(iy) y^s \frac{dy}{y}. \qquad (4.26)$$

Both sides of the equality (4.26) converge absolutely for $\mathrm{Re}(s) > 1 + c$. The identities (4.25) and (4.26) are immediately deduced from the well known integral representation for the gamma-function

$$\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}, \tag{4.27}$$

where $\frac{dy}{y}$ is a measure on the group $\mathbb{R}_+^\times$ which is invariant under the group translations (Haar measure). The integral (4.27) is absolutely convergent for $\mathrm{Re}(s) > 0$ and it can be interpreted as the integral of the product of an additive character $y \mapsto e^{-y}$ of the group $\mathbb{R}^{(+)}$ restricted to $\mathbb{R}_+^\times$, and of the multiplicative character $y \mapsto y^s$, integration is taken with respect to the Haar measure $\mathrm{d}y/y$ on the group $\mathbb{R}_+^\times$.



In the theory of the non-Archimedean integration one considers the group $\mathbb{Z}_S^\times$ (the group of units of the $S$-adic completion of the ring of integers $\mathbb{Z}$) instead of the group $\mathbb{R}_+^\times$, and the Tate field $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$) instead of the complex field $\mathbb{C}$. The domain of definition of the $p$-adic zeta functions is the $p$-adic analytic group

$$X_S = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_S^\times, \mathbb{C}_p^\times) = X(\mathbb{Z}_S^\times), \tag{4.28}$$

where:

$$\mathbb{Z}_S^\times \cong \oplus_{q \in S} \mathbb{Z}_q^\times,$$

and the symbol

$$X(G) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_p^\times) \tag{4.29}$$

denotes the functor of all $p$-adic characters of a topological group $G$ (see [Vi76]).

## The analytic structure of $X_S$

Let us consider in detail the structure of the topological group $X_S$. Define

$$U_p = \{x \in \mathbb{Z}_p^\times \quad | \quad x \equiv 1 \pmod{p^\nu}\},$$

where $\nu = 1$ or $\nu = 2$ according as $p > 2$ or $p = 2$. Then we have the natural decomposition

$$X_S = X\left((\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times\right) \times X(U_p). \tag{4.30}$$

The analytic dstructure on $X(U_p)$ is defined by the following isomorphism (which is equivalent to a special choice of a local parameter):

$$\varphi : X(U_p) \xrightarrow{\sim} T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\},$$

where $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topoplogical generator of the multiplicative group $U_p \cong \mathbb{Z}_p$. An arbitrary character $\chi \in X_S$ can be uniquely represented in the form $\chi = \chi_0\chi_1$ where $\chi_0$ is trivial on the component $U_p$, and $\chi_1$ is trivial on the other component

$$(\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times.$$



The character $\chi_0$ is called the *tame component*, and $\chi_1$ the *wild component* of the character $\chi$. We denote by the symbol $\chi_{(t)}$ the (wild) character which is uniquely determined by the condition

$$\chi_{(t)}(1+p^\nu) = t$$

with $t \in \mathbb{C}_p$, $|t|_p < 1$.
In some cases it is convenient to use another local coordinate $s$ which is analogous to the classical argument $s$ of the Dirichlet series:

$$\begin{array}{ccc} \mathrm{O}_p & \longrightarrow & X_S \\ s & \longmapsto & \chi^{(s)}, \end{array}$$

where $\chi^{(s)}$ is given by $\chi^{(s)}((1+p^\nu)^\alpha) = (1+p^\nu)^{\alpha s} = \exp(\alpha s \log(1+p^\nu))$. The character $\chi^{(s)}$ is defined only for such $s$ for which the series exp is $p$-adically convergent (i.e. for $|s|_p < p^{\nu - 1/(p-1)}$). In this domain of values of the argument we have that $t = (1+p^\nu)^s - 1$. But, for example, for $(1+t)^{p^n} = 1$ there is certainly no such value of $s$ (because $t \neq 1$), so that the $s$-coordonate parametrizes a smaller neighborhood of the trivial character than the $t$-coordinate (which parametrizes all wild characters) (see [Ma73], [Man76]).

## $p$-adic analytic functions on $X_S$

Recall that an analytic function $F : T \longrightarrow \mathbb{C}_p$ ($T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\}$), is defined as the sum of a series of the type $\sum_{i\geq 0} a_i(t-1)^i$ ($a_i \in \mathbb{C}_p$), which is assumed to be absolutely convergent for all $t \in T$. The notion of an analytic function is then obviously extended to the whole group $X_S$ by shifts. The function

$$F(t) = \sum_{i=0}^{\infty} a_i(t-1)^i$$

is bounded on $T$ iff all its coefficients $a_i$ are universally bounded. This last fact can be easily deduced for example from the basic properties of the Newton polygon of the series $F(t)$ (see [Kob80], [Vi76]). If we apply to these series the Weierstrass preparation theorem (see [Kob80], [Man71]), we see that in this case the function $F$ has only a finite number of zeroes on $T$ (if it is not identically zero).



In particular, consider the torsion subgroup $X_S^{\mathrm{tors}} \subset X_S$. This subgroup is discrete in $X_S$ and its elements $\chi \in X_S^{\mathrm{tors}}$ can be obviously identified with primitive Dirichlet characters $\chi \mod M$ such that the support $S(\chi) = S(M)$ of the conductor of $\chi$ is contained in $S$. This identification is provided by a fixed embedding denoted

$$i_p : \overline{\mathbb{Q}}^\times \hookrightarrow \mathbb{C}_p^\times$$

if we note that each character $\chi \in X_S^{\mathrm{tors}}$ can be factored through some finite factor group $(\mathbb{Z}/M\mathbb{Z})^\times$:

$$\chi : \mathbb{Z}_S^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \overset{i_p}{\hookrightarrow} \mathbb{C}_p^\times,$$

and the smallest number $M$ with the above condition is the conductor of $\chi \in X_S^{\mathrm{tors}}$.

The symbol $x_p$ will denote the composition of the natural projection $\mathbb{Z}_S^\times \to \mathbb{Z}_p^\times$ and of the natural embedding $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, so that $x_p \in X_S$ and all integers $k$ can be considered as the characters $x_p^k : y \longmapsto y^k$.

Let us consider a bounded $\mathbb{C}_p$-analytic function $F$ on $X_S$. The above statement about zeroes of bounded $\mathbb{C}_p$-analytic functions implies now that the function $F$ is uniquely determined by its values $F(\chi_0\chi)$, where $\chi_0$ is a fixed character and $\chi$ runs through all elements $\chi \in X_S^{\mathrm{tors}}$ with possible exclusion of a finite number of characters in each analyticity component of the decomposition (4.30). This condition is satisfied, for example, by the set of characters $\chi \in X_S^{\mathrm{tors}}$ with the $S$-complete conductor (i.e. such that $S(\chi) = S$), and even for a smaller set of characters, for example for the set obtained by imposing the additional assumption that the character $\chi^2$ is not trivial (see [Ma73], [Man76], [Vi76]).



## $p$-adic Mellin transform

Let $\mu$ be a (bounded) $\mathbb{C}_p$-valued measure on $\mathbb{Z}_S^\times$. Then the *non-Archimedean Mellin transform* of the measure $\mu$ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbb{Z}_S^\times} x \mathrm{d}\mu, \quad (x \in X_S), \tag{4.31}$$

which represents a bounded $\mathbb{C}_p$-analytic function

$$L_\mu : X_S \longrightarrow \mathbb{C}_p. \tag{4.32}$$

Indeed, the boundedness of the function $L_\mu$ is obvious since all characters $x \in X_S$ take values in $\mathcal{O}_p$ and $\mu$ also is bounded. The analyticity of this function expresses a general property of the integral (4.31), namely that it depends analytically on the parameter $x \in X_S$. However, we give below a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description will also imply that every bounded $\mathbb{C}_p$-analytic function on $X_S$ is the Mellin transform of a certain measure $\mu$.

## The Iwasawa algebra

Let $\mathrm{O}$ be a closed subring in $\mathrm{O}_p = \{z \in \mathbb{C}_p \quad | \quad |z|_p \le 1\}$,

$$G = \varprojlim_i G_i, \qquad (i \in I),$$

a profinite group. Then the canonical homomorphism $G_i \xleftarrow{\pi_{ij}} G_j$ induces a homomorphism of the corresponding group rings

$$\mathrm{O}[G_i] \longleftarrow \mathrm{O}[G_j].$$

Then the *completed group ring* $\mathrm{O}[[G]]$ is defined as the projective limit

$$\mathrm{O}[[G]] = \varprojlim_i \mathrm{O}[[G_i]], \qquad (i \in I).$$

Let us consider also the set $\mathrm{Dist}(G, \mathrm{O})$ of all $\mathrm{O}$-valued distributions on $G$ which itself is an $\mathrm{O}$-module and a ring with respect to multiplication given by the *convolution of distributions*, which is defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \longrightarrow \mathrm{O},$$

(see the previous section) as follows:



We noticed above that the theorem 4 would imply a description of $\mathbb{C}_p$-analytic bounded functions on $X_S$ in terms of measures. Indeed, these functions are defined on analyticity components of the decomposition (4.30) as certain power series with $p$-adically bounded coefficients, that is, power series, whose coefficients belong to $\mathrm{O}_p$ after multiplication by some constant from $\mathbb{C}_p^\times$. Formulas for coefficients of these series can be also deduced from the proof of the theorem. However, we give a more direct computation of these coefficients in terms of the corresponding measures. Let us consider the component $aU_p$ of the set $\mathbb{Z}_S^\times$ where

$$a \in (\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q\neq} \mathbb{Z}_q^\times,$$

and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on $U_p$ defined by restriction of $\mu$ to the subset $aU_p \subset \mathbb{Z}_S^\times$.

Consider the isomorphism $U_p \cong \mathbb{Z}_p$ given by:

$$y = \gamma^x \quad (x \in \mathbb{Z}_p, y \in U_p),$$

with some choice of the generator $\gamma$ of $U_p$ (for example, we can take $\gamma = 1 + p^\nu$). Let $\mu'_a$ be the corresponding measure on $\mathbb{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbb{Z}_p} \binom{x}{i} d\mu'_a(x) = a_i, \tag{4.36}$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbb{C}_p$-span of the family

$$\left\{ \binom{x}{i} \right\} \quad (i \in \mathbb{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbb{Z}_p, \mathcal{O}_p)$ according to Mahler's interpolation theorem for continuous functions on $\mathbb{Z}_p$). Indeed, from the basic properties of the interpolation polynomials it follows that

$$\sum_i b_i \binom{x}{i} \equiv 0 \pmod{p^n} \quad (\text{for all } x \in \mathbb{Z}_p) \Longrightarrow b_i \equiv 0 \pmod{p^n}.$$

We can now apply the abstract Kummer congruences (see proposition 4.1), which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exists a measure with the property (4.36).



## Coefficients of power series and the Iwasawa isomorphism

We state that the Mellin transform $L_{\mu_a}$ of the measure $\mu_a$ is given by the power series $F_a(t)$ with coefficients as in (4.36), that is

$$\int_{U_p} \chi_{(t)}(y) \mathrm{d}\mu(ay) = \sum_{i=0}^{\infty} \left( \int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu'_a(x) \right) (t-1)^i \tag{4.37}$$

for all wild characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that (4.37) is valid for all characters of the type $y \longmapsto y^m$, where $m$ is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

which implies that

$$\int_{U_p} y^m \mathrm{d}\mu(ay) = \int_{\mathbb{Z}_p} \gamma^{mx} \mathrm{d}\mu'_a(x) = \sum_{i=0}^{\infty} \left( \int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu'_a(x) \right) (\gamma^m - 1)^i,$$

establishing (4.37).

## Example: Mazur's measure and the non-Archimedean Kubota-Leopoldt zeta function

Let us first consider a positive integer $c \in \mathbb{Z}_S^\times \cap \mathbb{Z}$, $c > 1$ coprime to all primes in $S$. Then for each complex number $s \in \mathbb{C}$ there exists a complex distribution $\mu_s^c$ on $G_S = \mathbb{Z}_S^\times$ which is uniquely determined by the following condition

$$\mu_s^c(\chi) = (1 - \chi^{-1}(c)c^{-1-s})L_{M_0}(-s, \chi), \tag{4.38}$$

where $M_0 = \prod_{q \in S} q$. Moreover, the right hand side of (4.38) is holomorphic for all $s \in \mathbb{C}$ including $s = -1$. If $s$ is an integer and $s \geq 0$ then according to criterion of proposition 4.1 the right hand side of (4.38) belongs to the field

$$\mathbb{Q}(\chi) \subset \mathbb{Q}^{\mathrm{ab}} \subset \overline{\mathbb{Q}}$$

generated by values of the character $\chi$.



Thus we get a distribution with values in $\mathbb{Q}^{\mathrm{ab}}$. If we now apply to (4.38) the fixed embedding $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ we get a $\mathbb{C}_p$-valued distribution $\mu^{(c)} = i_p(\mu_0^c)$ which turns out to be an $\mathrm{O}_p$-measure in view of proposition 4.1, and the following equality holds

$$\mu^{(c)}(\chi x_p^r) = i_p(\mu_r^c(\chi)).$$

This identity relates the special values of the Dirichlet $L$-functions at different non-positive points. The function

$$L(x) = \left(1 - c^{-1}x(c)^{-1}\right)^{-1} L_{\mu^{(c)}}(x) \quad (x \in X_S) \tag{4.39}$$

is well defined and it is holomorphic on $X_S$ with the exception of a simple pole at the point $x = x_p \in X_S$. This function is called the *non-Archimedean zeta-function of Kubota-Leopoldt*. The corresponding measure $\mu^{(c)}$ will be called the *S-adic Mazur measure*.