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NON-ARCHIMEDEAN RANKIN L-FUNCTIONS AND THEIR FUNCTIONAL EQUATIONS

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ABSTRACT. A functional equation is established for the S-adic L-functions obtained by non-Archimedean interpolation of the special values of the convolution of two cusp forms on the upper half-plane.

Bibliography: 33 titles.

§1. Introduction and statement of results

1.1. Let p be a prime, and let S be a finite set of primes including p. In this article we construct non-Archimedean Rankin L-functions which interpolate S-adically the special values of the convolution of two cusp forms on the complex upper half-plane, and we also establish a functional equation which these L-functions satisfy. Let N be an arbitrary natural number. We consider a cusp form f of weight $k \geq 2$ for the congruence subgroup $\Gamma_0(N)$ with Dirichlet character ψ mod N. We suppose that f is primitive, i.e., it is a normalized newform of some level C_f dividing N; C_f is called the *conductor* of f. Let g be another primitive cusp form of conductor C_g and weight $l < k$ for $\Gamma_0(N)$ with Dirichlet character ω mod N. We set $e(z) = e^{2\pi i z}$, and we let

$$f = \sum_{n=1}^{\infty} a(n)e(nz), \qquad g(z) = \sum_{n=1}^{\infty} b(n)e(nz) \tag{1.1}$$

be the Fourier expansions of f and g. The *Rankin convolution* of the modular forms f and g is introduced by means of the equality

$$\mathscr{D}(s, f, g) = L_N(2s + 2 - k - l, \omega\psi)L(s, f, g), \tag{1.2}$$

where

$$L(s, f, g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s},$$

and $L_N(s, \omega\psi)$ denotes the Dirichlet L-series with character $\omega\psi$, with the subscript N indicating that the factors corresponding to the prime divisors of N are omitted from the Euler product. A classical method of Rankin and Selberg [25], [30] enables one to construct a holomorphic continuation of the function $\mathscr{D}(s, f, g)$ to the whole

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complex plane and prove that it satisfies a functional equation, which in the simplest case $N = 1$ has the form

$$\Psi(s, f, g) = (-1)^k \Psi(k + l - 1 - s, f, g), \tag{1.3}$$

where

$$\Psi(s, f, g) = \gamma(s) \mathscr{D}(s, f, g), \tag{1.4}$$

and $\gamma(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s + 1 - l)$ denotes the Γ-factor.

It later became clear that the same method can be used to establish the following algebraicity property for special values of $\mathscr{D}(s, f, g)$ (see [26], [28], [6], and [33]): the numbers

$$\Psi(l + r, f, g)(\pi^{1-l}\langle f, f\rangle)^{-1} \in \overline{\mathbf{Q}} \tag{1.5}$$

are algebraic for all integers r for which $0 \le r \le k - l - 1$. Here $\langle f, f\rangle = \langle f, f\rangle_{C_f}$ is the Petersson inner product, i.e.,

$$\langle f, f\rangle_{C_f} = \int_{H/\Gamma_0(C_f)} |f(z)|^2 y^{k-2}\, dx\, dy, \qquad z = x + iy,$$

where $H/\Gamma_0(C_f)$ is a fundamental domain for the upper half-plane H modulo the action of $\Gamma_0(C_f)$. The integers $s = l + r$ in (1.5) are "critical" in the sense of Deligne [14]: they are precisely the values of s for which both of the functions $\gamma(s)$ and $\gamma(k + l - 1 - s)$ do not have poles.

We propose to use an S-adic version of the Rankin-Selberg method in order to obtain a non-Archimedean interpolation of the numbers (1.5). The paper is conceptually similar to Hida's article [18], in which p-adic modular forms are used to find a method for constructing the p-adic interpolation of half of the critical values ($s = l + r$ with $0 \le r \le (k - l)/2 - 1$). We extend this result to all of the remaining points of the critical strip $l \le \mathrm{Re}(s) \le k - 1$ and to a set of several primes in S. Our construction is more explicit in the sense that, instead of using p-adic modular forms, we give congruences between usual modular forms. Other variants on our construction are contained in [9]–[12] and [27].

1.2. The *domain of definition* of the S-adic zeta-functions is the $\mathbf{C}_p$-analytic Lie group

$$X_S = X(\mathbf{Z}_S^\times),$$

where $X(G) = \mathrm{Hom}_{\mathrm{contin}}(G, \mathbf{C}_p^\times)$ denotes the group of p-adic characters of the topological group G; $\mathbf{C}_p = \hat{\overline{\mathbf{Q}}}_p$ is the *Tate field*, the completion of the algebraic closure of the field $\mathbf{Q}_p$ of p-adic numbers, with the p-adic absolute value $|\cdot|_p$ normalized by the condition that $|p|_p = p^{-1}$; and $\mathbf{Z}_S^\times = \bigoplus_{q \in S} \mathbf{Z}_q^\times$ is the group of units of the S-adic completion $\mathbf{Z}_S$ of the ring of rational integers, see [1] and [3]. We set

$$U = \{x \in \mathbf{Z}_p^\times \mid x \equiv 1 \pmod{p^\nu}\},$$

where $\nu = 1$ or 2 depending on whether $p < 2$ or $p = 2$; then we have the decomposition

$$X_S = X\left((\mathbf{Z}/p^\nu\mathbf{Z})^\times \times \prod_{q \ne p} \mathbf{Z}_q^\times\right) \times X(U). \tag{1.6}$$

An *analytic structure* on the subgroup $X(U) \subset X_S$ is defined by means of the isomorphism

$$\varphi \colon X(U) \xrightarrow{\sim} T = \{t \in \mathbf{C}_p^\times \mid |t - 1|_p < 1\},$$

under which $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topological generator of the multiplicative group U. An analytic function $F: T \to \mathbf{C}_p$ is defined to be the sum of a series $\sum_{i=0}^{\infty} a_i(t-1)^i$, $a_i \in \mathbf{C}_p$, which converges for all $t \in T$, and then the notion of analyticity is extended to all of X_S using translation of the argument by elements of the group. If we fix an imbedding $i_p: \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, then the characters of finite order $\chi \in X_S^{\text{tors}} \subset X_S$, which form a discrete subgroup of X_S^{tors}, can be identified with Dirichlet characters (denoted by the same symbol χ) whose conductor C_χ is divisible only by primes in S. The symbol x_p denotes the composition of the natural projection $\mathbf{Z}_S^\times \to \mathbf{Z}_p^\times$ with the imbedding $\mathbf{Z}_p^\times \hookrightarrow \mathbf{C}_p^\times$ and the inclusion $x_p \in X_S$. For $\chi \in X_S^{\text{tors}}$ we let $S(\chi)$ be the support (the set of prime divisors) of the conductor C_χ. Recall that any bounded $\mathbf{C}_p$-analytic function on X_S is uniquely determined by its values $F(\chi_0\chi)$, where $\chi_0 \in X_S$ is fixed, and χ runs through all elements of X_S^{tors} with the possible exception of finitely many, in each analytic component of the decomposition (1.6). If μ is a bounded $\mathbf{C}_p$-valued measure on $\mathbf{Z}_S^\times$ (see §2), then the non-Archimedean Mellin transform, defined as

$$L_\mu(x) = \mu(x) = \int_{\mathbf{Z}_S^\times} x\, d\mu, \qquad x \in X_S, \tag{1.7}$$

gives a bounded $\mathbf{C}_p$-analytic function $L_\mu: X_S \to \mathbf{C}_p$.

1.3. For a precise statement of the results we introduce the notation

$$g(\chi) = \sum_{n=1}^{\infty} \chi(n) b(n) e(nz)$$

for the cusp form g twisted by a Dirichlet character $\chi \in X_S^{\text{tors}}$. Using the imbedding $i_p: \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$, we construct an S-adic interpolation of the numbers

$$i_p\left(\frac{\Psi(l+r, f, g(\chi))}{\pi^{1-l}\langle f, f\rangle}\right),$$

$r = 0, 1, \dots, k-l-1$ obtaining a bounded $\mathbf{C}_p$-analytic function on X_S. The most essential assumption is that f is a p-ordinary form, i.e., that $a(p)$ is a p-adic unit in $\mathbf{C}_p$:

$$|i_p(a(p))|_p = 1. \tag{1.8}$$

In addition, we suppose that

$$(C_f, M_0) = (C_g, M_0) = 1, \qquad M_0 = \prod_{q \in S} q, \tag{1.9}$$

$$(C_f, C_g) = 1, \tag{1.10}$$

and we set $C = C_f C_g$. We let $\alpha(q)$ denote the root of the Hecke polynomial $X^2 - a(q)X + \psi(q)q^{k-1}$, for which $|i_p(\alpha(q))|_p = 1$ for $q \in S$, and we let $\alpha'(q)$ be the other root. Then the numbers

$$\hat{\alpha}(q) = \overline{\psi(q)}\alpha(q), \quad \hat{\alpha}'(q) = \overline{\psi(q)}\alpha'(q), \qquad q \in S, \tag{1.11}$$

coincide with the roots of the complex conjugate polynomial

$$X^2 - \overline{a(q)}X + \overline{\psi(q)}q^{k-1} = (X - \hat{\alpha}(q))(X - \hat{\alpha}'(q))$$

because the Hecke operator $T(q)$ ($q \nmid C_f$) acting on the space $\mathscr{S}_k(C, \psi)$ of weight k cusp forms for $\Gamma_0(C)$ with Dirichlet character ψ mod C is ψ-Hermitian. Similarly, if

$$X^2 - b(q)X + \omega(q)q^{l-1} = (X - \beta(q))(X - \beta'(q)),$$

then

$$\hat{\beta}(q) = \overline{\omega(q)}\beta(q), \qquad \hat{\beta}'(q) = \overline{\omega(q)}\beta'(q) \tag{1.11a}$$

are the roots of the polynomial $X^2 - \overline{b(q)}X + \overline{\omega(q)}q^{l-1}$.

We extend the definition of $\alpha(n), \alpha'(n), \beta(n)$, and $\beta'(n)$ by multiplicativity to all natural numbers n. We set

$$f^\rho = \sum_{n=1}^{\infty} \overline{a(n)} e(nz) \in \mathscr{S}_k(C, \overline{\psi}), \qquad g^\rho = \sum_{n=1}^{\infty} \overline{b(n)} e(nz) \in \mathscr{S}_l(C, \overline{\omega}).$$

From the theory of newforms (see [2] and [22]) it follows that

$$f|_k W(C_f) = \Lambda(f) f^\rho, \qquad g|_l W(C_g) = \Lambda(g) g^\rho, \tag{1.12}$$

where $W(A) = \left(\begin{smallmatrix} 0 & -1 \\ A & 0 \end{smallmatrix}\right)$, and $\Lambda(f)$ and $\Lambda(g)$ are the constants in the functional equations of the series

$$L(s, f) = \sum_{n=1}^{\infty} a(n) n^{-s} = \prod_q [(1 - \alpha(q)q^{-s})(1 - \alpha'(q)q^{-s})]^{-1},$$

$$L(s, g) = \sum_{n=1}^{\infty} b(n) n^{-s} = \prod_q [(1 - \beta(q)q^{-s})(1 - \beta'(q)q^{-s})]^{-1}.$$

If the conductor C_χ of the primitive Dirichlet character χ is prime to C_g, then the cusp form $g(\chi) \in \mathscr{S}_l(C_g C_\chi^2, \omega\chi^2)$ is a primitive form of conductor $C_g C_\chi^2$, and

$$\Lambda(g(\chi)) = \omega(C_\chi)\chi(C_g)\frac{G(\chi)^2}{C_\chi}\Lambda(g),$$

where

$$G(\chi) = \sum_{u \bmod C_\chi} \chi(u) e\left(\frac{u}{C_\chi}\right)$$

is the Gauss sum.

1.4. FUNDAMENTAL THEOREM. *Under the assumptions* (1.8)–(1.10), *there exists a bounded* $\mathbf{C}_p$*-analytic function*

$$\Psi\colon X_S \to \mathbf{C}_p, \qquad \Psi(x) = \Psi_S(x; f, g), \tag{1.13}$$

which is uniquely determined by the following condition: for all characters $\chi \in X_S^{\mathrm{tors}}$ *and all integers* r *with* $0 \le r \le k - l - 1$, *the value* $\Psi(\chi x_p^r)$ *is given by the image under* i_p *of the following algebraic number:*

$$\omega(C_\chi)\frac{G(\chi)^2 C_\chi^{l+2r-1}}{\alpha(C_\chi)^2} \cdot \frac{\Psi(l + r, f, g^\rho(\overline{\chi}))}{\pi^{1-l}\langle f, f\rangle_{C_f}} A(r, \chi), \tag{1.14}$$

where

$$A(r, \chi) = \prod_{q \in S \setminus S(\chi)} [(1 - \chi(q)\alpha^{-1}(q)\beta(q)q^r)(1 - \chi(q)\alpha^{-1}(q)\beta'(q)q^r) \times (1 - \chi^{-1}(q)\alpha'(q)\hat{\beta}(q)q^{-l-r})(1 - \chi^{-1}(q)\alpha'(q)\hat{\beta}'(q)q^{-l-r})]. \tag{1.15}$$

1.5. THEOREM (non-Archimedean functional equation). *Suppose that the conditions of the fundamental theorem are fulfilled, and along with $\Psi(x)$ consider the function $\hat{\Psi}(x) = \Psi_S(x; f^\rho, g^\rho)$, obtained by replacing (f, g, α) by $(f^\rho, g^\rho, \hat{\alpha})$. Then the functional equation*

$$\Psi(x) = A_{f,g}\frac{C^{k-l-1}}{x(C)^2}\hat{\Psi}(x_p^{k-l-1}x^{-1}) \tag{1.16}$$

holds for all $x \in X_S$; here

$$A_{f,g} = i_p((-1)^{l+1}\overline{\omega(C_f)}\psi(C_g)\Lambda(f)^2\Lambda(g^\rho)^2). \tag{1.17}$$

1.6. The proof of Theorems 1.4 and 1.5 makes constant use of the classical Rankin-Selberg method, which has two essential components:

1) the Euler expansion of the convolution [25], [28]:

$$\begin{aligned}\mathscr{D}(s, f, g) = \prod_q [(1 - \alpha(q)\beta(q)q^{-s})(1 - \alpha(q)\beta'(q)q^{-s})\\ \times (1 - \alpha'(q)\beta(q)q^{-s})(1 - \alpha'(q)\beta'(q)q^{-s})]^{-1};\end{aligned} \tag{1.18}$$

2) the integral representation of the convolution [28], [33]: for $s \in \mathbf{C}$ with $\mathrm{Re}(s) \gg 0$ we have

$$2(4\pi)^{-s}\Gamma(s)\mathscr{D}(s, f, g) = \langle f^\rho, gE(z, s-k+1)\rangle_C, \tag{1.19}$$

where the Petersson inner product on the right contains the Eisenstein series

$$E(z, s) = y^s \sum_{(c,d)\in\mathbf{Z}^2\setminus\{(0,0)\}} \psi\omega(d)(Ccz + d)^{-(k-l)}|Ccz + d|^{-2s},$$

which is absolutely convergent for $\mathrm{Re}(2s) + k - l > 2$ and can be analytically continued to all $s \in \mathbf{C}$.

Property 1) is a consequence of an elementary lemma on series expansions of rational functions: if

$$\sum_{i=0}^{\infty} A_i X^i = [(1 - \alpha X)(1 - \alpha' X)]^{-1}, \qquad \sum_{i=0}^{\infty} B_i X^i = [(1 - \beta X)(1 - \beta' X)]^{-1},$$

then

$$\sum_{i=0}^{\infty} A_i B_i X^i = \frac{1 - \alpha\alpha'\beta\beta' X^2}{(1 - \alpha\beta X)(1 - \alpha\beta' X)(1 - \alpha'\beta X)(1 - \alpha'\beta' X)}; \tag{1.20}$$

it is obtained by applying (1.20) to all of the Euler factors in (1.18).

1.7. Plan of the paper. The functions $\Psi(x)$ in the fundamental theorem are constructed as non-Archimedean Mellin transforms of certain bounded $\mathbf{C}_p$-valued measures on the group $\mathbf{Z}_S^\times$. In §2 we recall the general properties of measures and distributions. The non-Archimedean measures in Theorem 1.4 are obtained from complex-valued distributions which we construct in §3 directly from the definition of the convolutions (1.2) in the form of a series. In §4 we obtain an integral representation for these distributions using the Rankin-Selberg method. In §5 we use a holomorphic projection operator to derive from this the algebraicity and integrality properties of the values of these distributions which enable us to complete the proof of the fundamental theorem and the derivation of Theorem 1.5 from the fundamental theorem. The S-adic functional equation is proved using the uniqueness property of the functions $\Psi(x)$ and the Archimedean functional equation [6], [22].

1.8. According to the general program described by Manin in [4] and [5], non-Archimedean L-functions must, alongside the more traditional Archimedean (complex) L-functions, correspond to objects of various sorts: automorphic forms, algebraic varieties over number fields, representations of Galois groups. Relations between different types of L-functions give connections between these objects and provide essentially new arithmetic identities, while the method of making the identifications is related to comparisons between the different cohomology theories for these objects [14].

Our non-Archimedean L-functions correspond to automorphic representations of holomorphic type for the group $\mathrm{GL}(2) \times \mathrm{GL}(2)$, and the question of an arithmetic interpretation of these L-functions is very interesting. The non-Archimedean Kubota-Leopoldt zeta-function [21] is given by a certain power series in the ring $\Lambda = \mathbf{Z}_p[[T]]$, and, according to the main conjecture of Iwasawa [19], which was proved by Mazur and Wiles [24], this series generates the module of relations of a certain Λ-module of ideal class p-groups in a tower of cyclotomic fields. Another example is given by the non-Archimedean L-function which interpolates the special values of the symmetric square of the L-function of an elliptic curve over the field of rational numbers. It is conjectured that these non-Archimedean L-functions describe relations in the Λ-module of Selmer p-groups of this curve in a certain tower of extensions generated by the coordinates of points of p-primary order (see [13]). These functions have been constructed by the author's method (see [10], [12], and [27]), and they are closely connected with the functions considered in the present article: they correspond to certain automorphic representations of the adele group $\mathrm{GL}(3)$.

1.9. Additional preliminary facts and notation. Let $H = \{z = x + iy | y > 0\}$ denote the complex upper half-plane, on which the group $\mathrm{GL}_2^+(\mathbf{R})$ of real 2×2 matrices with positive determinant acts by fractional linear transformations. For any function $f: H \to \mathbf{C}$, any

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathrm{GL}_2^+(\mathbf{R})$$

and any natural number k we have an action of weight k:

$$(f|_k\gamma)(z) = \det \gamma^{k/2} f(\gamma(z))(c_\gamma z + d_\gamma)^{-k}.$$

For any natural number N we have the following congruence subgroups:

$$\Gamma_0(N) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) | c_\gamma \equiv 0 \mod N\},$$
$$\Gamma_1(N) = \{\gamma \in \Gamma_0(N) | a_\gamma \equiv d_\gamma \equiv 1 \mod N\},$$
$$\Gamma(N) = \{\gamma \in \mathrm{SL}_2(\mathbf{Z}) | \gamma \equiv 1_2 \mod N\}.$$

If Γ is any of these groups, then $\mathscr{M}_k(\Gamma)$ denotes the complex vector space of holomorphic modular forms of weight k for Γ, and $\mathscr{S}_k(\Gamma)$ denotes the subspace of cusp forms. If ψ mod N is a Dirichlet character, we set

$$\mathscr{M}_k(N, \psi) = \{f \in \mathscr{M}_k(\Gamma_1(N)) | f|_k\gamma = \psi(d_\gamma)f \text{ for all } \gamma \in \Gamma_0(N)\},$$
$$\mathscr{S}_k(N, \psi) = \mathscr{S}_k(\Gamma_1(N)) \cap \mathscr{M}_k(N, \psi).$$

For an arbitrary modular form $h \in \mathscr{M}_k(N, \psi)$ with $k \geq 1$ and $f \in \mathscr{S}_k(N, \psi)$ one has the Petersson inner product, defined by

$$\langle f, h \rangle_N = \int_{H/\Gamma_0(N)} \overline{f(z)} h(z) y^{k-2}\, dx\, dy,$$

where $H/\Gamma_0(N)$ is a fundamental domain for the action of $\Gamma_0(N)$.

Operators acting on a modular form

$$f = \sum_{n=1}^{\infty} a(n)e(nz) \in \mathcal{S}_k(N, \psi).$$

If d is a natural number, then

$$f|U(d) = \sum_{n=1}^{\infty} a(dn)e(nz) = d^{k/2-1} \sum_{u \bmod d} f|_k \begin{pmatrix} 1 & u \\ 0 & d \end{pmatrix},$$

$$f|V(d) = f(dz) = d^{-k/2} f|_k \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{S}_k(Nd, \psi),$$

$$f^{\rho}(z) = \overline{f(-\overline{z})} = \sum_{n=1}^{\infty} \overline{a(n)} e(nz), \qquad f|W(d) = f|_k \begin{pmatrix} 0 & -1 \\ d & 0 \end{pmatrix},$$

$$f^{\rho}, f|W(N) \in \mathcal{S}_k(N, \overline{\psi}).$$

The Hecke operators $T(d): \mathcal{M}_k(N, \psi) \to \mathcal{M}_k(N, \psi)$ for $(d, N) = 1$ can be defined by

$$f|T(d) = \sum_{d_1|d} \psi(d_1) d_1^{k-1} f|U(d_1)V(d/d_1).$$

By a primitive cusp form

$$f = \sum_{n=1}^{\infty} a(n)e(nz)$$

of conductor N we mean a normalized eigenfunction $f \in \mathcal{S}_k(N, \psi)$ for the Hecke operators which is a newform, i.e., it is orthogonal to all of the images of the maps $V(d): \mathcal{S}_k(N/d, \psi) \to \mathcal{S}_k(N, \psi)$ for $d|N$. In this case, f is uniquely determined by the eigenvalues of the Hecke operators $T(p)$ (with the possible exception of a finite number), and then we automatically have $f|T(p) = a(p)f$ and $f|U(q) = a(q)f$ respectively for all $p \nmid N$ and for $q|N$. We have the Euler product

$$L(s, f) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{q|N} (1 - a(q)q^{-s})^{-1} \prod_{p \nmid N} (1 - a(p)p^{-s} + \psi(p)p^{k-1-2s})^{-1}.$$

§2. Distributions, measures, and abstract Kummer congruences

2.1. Distributions. Let R be a commutative associative ring, and let $\mathcal{A}$ be an R-module. We consider a profinite (i.e., compact and totally disconnected) topological space Y. Then Y is a projective limit of finite sets:

$$Y = \varprojlim Y_i \qquad (\pi_{ij}: Y_i \to Y_j,\ i, j \in I,\ i \geq j), \tag{2.1}$$

where I is a directed set, the π_{ij} are surjective homomorphisms, and canonical projections $\pi_i: Y \to Y_i$ are defined for all $i \in I$. We consider the R-module Step(Y, R) consisting of all R-valued locally constant functions $\varphi: Y \to R$.

DEFINITION. A *distribution on* Y *with values in the* R*-modules* $\mathcal{A}$ is a homomorphism of R-modules

$$\mu: \text{Step}(Y, R) \to \mathcal{A}. \tag{2.2}$$

For $\varphi \in \text{Step}(Y, R)$ we use the notation

$$\mu(\varphi) = \int_Y \varphi \, d\mu = \int_Y \varphi(y) \, d\mu(y).$$

Any distribution μ can be given by a set of functions $\mu^{(i)}: Y_i \to \mathscr{A}$, satisfying the following finite additivity condition:

$$\mu^{(i)}(x) = \sum_{x \in \pi_{ij}^{-1}(y)} \mu^{(i)}(x), \qquad y \in Y_j,\ x \in Y_i. \tag{2.3}$$

To do this it suffices to set $\mu^{(j)}(y) = \mu(\delta_{i,x}) \in \mathscr{A}$, where $\delta_{i,x}$ is the characteristic function of the preimage $\pi_i^{-1}(x)$.

The following is a useful criterion for when an arbitrary set of functions $\mu^{(i)}: Y_i \to \mathscr{A}$ satisfies the finite additivity condition (2.3). For any function $\varphi_j: Y_j \to R$ and any $i \geq j$, we define the functions $\varphi_i = \varphi_j \circ \pi_{ij}$, $\varphi = \varphi_j \circ \pi_j \in \text{Step}(Y, R)$. Then the set of functions $\{\mu^{(i)}: Y_i \to \mathscr{A}\}$ satisfies (2.3) if and only if, for all $j \in I$ and all $\varphi_j: Y_j \to R$,

the sum $\mu(\varphi) = \mu^{(i)}(\varphi_i) = \sum_{y \in Y_i} \varphi_i(y)\mu^{(i)}(y)$ *does not depend on* i, $i \geq j$, *for* i *sufficiently large*; (2.4)

here it is sufficient to verify (2.4) for any "basis" family of functions. For example, if $Y = G = \varprojlim G_i$ is a profinite abelian group and R is an integral domain containing all roots of 1 of degree dividing the order of G (perhaps a transfinite cardinal), then one need only verify (2.4) for all characters of finite order $\chi: G \to R^\times$, since the orthogonality relation implies that their linear span over $R \otimes \mathbf{Q}$ coincides with $\text{Step}(G, R \otimes \mathbf{Q})$ (see [23]).

2.2. Measures. If R is a topological ring, then let $\mathscr{C}(Y, R)$ denote the topological module of continuous R-valued functions on Y.

DEFINITION. A *measure on* Y *with values in a topological* R*-module* $\mathscr{A}$ is a continuous homomorphism of R-modules $\mu: \mathscr{C}(Y, R) \to \mathscr{A}$.

The restriction of μ to the R-submodule $\text{Step}(Y, R) \subset \mathscr{C}(Y, R)$ is a distribution, which we denote by the same symbol. We take for R a closed subring of $\mathbf{C}_p$, and we let $\mathscr{A}$ be a complete normed R-module with norm $|\cdot|_{\mathscr{A}}$. Then the condition that a distribution (set of functions) $\mu^{(i)}: Y_i \to \mathscr{A}$ extend to an $\mathscr{A}$-valued measure on Y is equivalent to boundedness of $\mu^{(i)}$: for some constant $B > 0$ and all $i \in I$ and $x \in Y_i$, we have $|\mu^{(i)}(x)|_{\mathscr{A}} < B$. The proof of this is easy using the non-Archimedean property of the norm $|\cdot|_{\mathscr{A}}$ (see [3] and [20]). In particular, if $\mathscr{A} = R = \mathscr{O}_p = \{x \in \mathbf{C}_p \mid |x|_p \leq 1\}$ is the ring of integers in the Tate field, then $\mathscr{O}_p$-valued distributions are the same as $\mathscr{O}_p$-valued measures.

The most important tool in our non-Archimedean construction is the following criterion for the existence of a measure with prescribed properties.

2.3. Abstract Kummer congruences (compare with [20], p. 258). Let $\{f_i\}$ be a family of functions $f_i \in \mathscr{C}(Y, \mathscr{O}_p)$ such that the $\mathbf{C}_p$-linear span of $\{f_i\}$ is everywhere dense in $\mathscr{C}(Y, \mathbf{C}_p)$. Let $\{a_i\}$ be a family of elements $a_i \in \mathscr{O}_p$. Then the existence of an $\mathscr{O}_p$-valued measure μ with $\int_Y f_i\, d\mu = a_i$ is equivalent to the following congruence condition: for any set of elements $b_i \in \mathbf{C}_p$ of which only finitely many are nonzero,

$$\text{if } \sum b_i f_i(y) \in p^n \mathscr{O}_p \ (y \in Y) \text{ then } \sum b_i a_i \in p^n \mathscr{O}_p. \tag{2.5}$$

2.4. Non-Archimedean Mellin transform. Let $X_S = \text{Hom}_{\text{contin}}(\mathbf{Z}_S^\times, \mathbf{C}_p^\times)$ be the $\mathbf{C}_p$-analytic Lie group in §1.2, and let μ be a bounded $\mathbf{C}_p$-valued measure on $\mathbf{Z}_S^\times$. Then the non-Archimedean Mellin transform is defined by

$$L_\mu(X) = \mu(x) = \int_{\mathbf{Z}_S^\times} x\, d\mu, \tag{2.6}$$

which gives a bounded $\mathbf{C}_p$-analytic function $L_\mu: X_S \to \mathbf{C}_p$. In fact, boundedness of the function (2.6) follows from the definition, and analyticity reflects the general property that the integral (2.6) depends analytically on the parameter $x \in X_S$. The converse is also true: any bounded $\mathbf{C}_p$-analytic function on X_S is the Mellin transform of some measure μ; these measures with the convolution operation form an algebra, which essentially coincides with the Iwasawa algebra (see [1]–[3]).

§3. Complex-valued distributions corresponding to convolutions of cusp forms

3.1. We first define the complex-valued distributions associated with the convolutions $\Psi(s, f, g)$ by introducing the auxiliary cusp form

$$f_0 = \sum_{d|M_0} \mu(d)\alpha'(d)f(dz) = \sum_{n=1}^{\infty} a(n, f_0)e(nz), \tag{3.1}$$

where, as before, $f = \sum_{n=1}^{\infty} a(n)e(nz) \in \mathscr{S}_k(C_f, \psi)$ is a primitive cusp form. The definition (3.1) is equivalent to the following identity for the corresponding Dirichlet series:

$$L(s, f_0) = \prod_{q|M_0} (1 - \alpha'(q)q^{-s})L(s, f), \tag{3.2}$$

in which

$$L(s, f_0) = \sum_{n=1}^{\infty} a(n, f_0)n^{-s}.$$

From (3.1) it immediately follows that $f_0 \in \mathscr{S}_k(C_f M_0, \psi)$ and we have the general multiplicativity property

$$a(Mn, f_0) = \alpha(M)a(n, f_0) \tag{3.3}$$

for all natural numbers M with $S(M) \subset S$. In fact, the Dirichlet series (3.2) has an Euler product, in which the factors corresponding to primes $q \in S$ have degree 1.

3.2. PROPOSITION. a) *For every $s \in \mathbf{C}$ there exists a complex-valued distribution $\Psi_{S,s}$ on the group $\mathbf{Z}_S^\times$ which is uniquely given by the following condition: for an arbitrary Dirichlet character $\chi: \mathbf{Z}_S^\times \to \mathbf{C}^\times$ of conductor C_χ one has the equality*

$$\Psi_{S,s}^{(M)}(\chi) = \frac{(M'M_0)^{s-l/2}C_f^{s-l/2}\chi(C_g)\Lambda(g)^{-1}}{\alpha(M'M_0)} \cdot \frac{\Psi(s, f_0|V(C_f), g_{M_0}(\chi)|W(C_0M'))}{\pi^{1-l}(f, f)_{C_f}}, \tag{3.4}$$

where M and M' are arbitrary natural numbers for which

$$M_0C_\chi|M, \quad M_0^2C_\chi^2|M', \quad S(M) = S(M') = S,$$

$$g_{M_0}(\chi) = \sum_{\substack{n=1 \\ (n, M_0)=1}}^{\infty} \chi(n)b(n)e(nz) \in \mathscr{S}_l(C_gC_\chi^2M_0^2, \omega\chi^2),$$

$$g_{m_0}(\chi)|W(C_0M')(z) = (\sqrt{C_0M'z})^{-l}g_{M_0}(\chi)(-1/C_0M'z) \in \mathscr{S}_l(C_0M', \overline{\omega\chi}^2). \tag{3.5}$$

b) *Let $A(s-l, \chi)$ be the product defined in* (1.15). *Then, for all M and M' as in* a),

$$\Psi_{S,s}^{(M)}(\chi) = \frac{\omega(C_\chi)G(\chi)^2C_\chi^{2s-l-1}}{\alpha(C_\chi)^2} \cdot \frac{\Psi(s, f, g^\rho(\overline{\chi}))}{\pi^{1-l}(f, f)_{C_f}}A(s-l, \chi). \tag{3.6}$$

REMARK. By the criterion (2.4), part a) follows from b), since the right side of (3.6) does not depend on M or M'.

3.3. To prove part b), we simplify the right side of (3.4), where, by the definition (1.3),

$$\begin{aligned}\Psi(s, f_0|V(C_f), g_{M_0}(\chi)|W(C_0M')) \\ = (2\pi)^{2s}\Gamma(s)\Gamma(s+1-l)L_{M_0C}(2s+2-k-l, \psi\overline{\omega\chi}^2) \\ \times L(s, f_0|V(C_f), g_{M_0}(\chi)|W(C_0M')).\end{aligned} \tag{3.7}$$

We define the numbers $A(n)$ and $B(n)$ to be the coefficients in the Dirichlet series

$$\sum_{n=1}^{\infty} A(n)n^{-s} = L(s, f_0), \tag{3.8}$$

$$\sum_{n=1}^{\infty} B(n)n^{-s} = L(s, g_{M_0}(\chi)|W(C_gM_0^2C_\chi^2)). \tag{3.8a}$$

Then, by the multiplicativity property (3.3),

$$A(nM_1) = \alpha(M_1)A(n) \quad \text{for } S(M_1) \subset S. \tag{3.9}$$

We set $M' = M_0C_\chi^2M_1$. From the definition (3.5) it then follows that

$$\begin{aligned}g_{M_0}(\chi)|W(C_0M') &= (M_1C_f)^{l/2}g_{M_0}(\chi)|W(C_gM_0^2C_\chi^2)V(M_1C_f) \\ &= (M_1C_f)^{l/2}\sum_{n=1}^{\infty} B(n)e(M_1C_fnz).\end{aligned} \tag{3.10}$$

Taking (3.9) and (3.10) into account, we transform the convolution in (3.7) to the form

$$\begin{aligned}L(s, f_0|V(C_f), g_{M_0}(\chi)|W(C_0M')) \\ &= (M_1C_f)^{l/2}\sum_{n=1}^{\infty} A(nC_f^{-1})B(nM_1^{-1}C_f^{-1})n^{-s} \\ &= (M_1C_f)^{l/2}\sum_{n=1}^{\infty} A(nM_1)B(n)(C_fM_1n)^{-s} \\ &= (M_1C_f)^{l/2-s}\alpha(M_1)L(s, f_0, g_{M_0}(\chi)|W(C_gM_0^2C_\chi^2)) \\ &= \frac{\alpha(M')}{\alpha(M_0C_\chi^2)} \cdot \frac{M'^{l/2-s}C_f^{l/2-s}}{(M_0C_\chi^2)^{l/2-s}} L(s, f_0, g_{M_0}(\chi)|W(C_gM_0^2C_\chi^2)).\end{aligned} \tag{3.11}$$

This property suffices for the proof of a): if we substitute (3.11) in (3.4), we see that (3.4) does not depend on M or M'. In order to obtain the more precise formula (3.6), it is enough to establish the following equality:

$$\begin{aligned}\Psi(s, f_0, g_{M_0}(\chi)|W(C_gM_0^2C_\chi^2)) \\ = \alpha(M_0)^2M_0^{l-2s}A(s-l, \chi)\Lambda(g(\chi))\Psi(s, f, g^\rho(\overline{\chi})),\end{aligned} \tag{3.12}$$

where

$$\begin{aligned}g(\chi)|W(C_gC_\chi^2) = \Lambda(g(\chi))g^\rho(\overline{\chi}), \\ \Lambda(g(\chi)) = \omega(C_\chi)\chi(C_g)G(\chi)^2C_\chi^{-1}\Lambda(g)\end{aligned} \tag{3.13}$$

is the constant in the functional equation for $g(\chi) \in \mathscr{S}_l(C_gC_\chi^2\omega\chi^2)$.

3.4. To derive (3.12) we use the properties of the Möbius function $\mu(n)$:

$$\sum_{d|n} \mu(d) = \begin{cases} 0, & \text{if } n = 1, \\ 1, & \text{if } n > 1. \end{cases}$$

Consequently,

$$\begin{aligned} g_{M_0} &= \sum_{\substack{n=1 \\ (n,M_0)=1}}^{\infty} \chi(n)b(n)e(nz) \\ &= \sum_{d|M_0} \mu(d) \sum_{n=1}^{\infty} \chi(dn)b(dn)e(dn) \\ &= \sum_{d|M_0} \mu(d)g(\chi)|U(d)V(d). \end{aligned} \tag{3.14}$$

We now express the operators $U(d)$ in terms of the Hecke operators: for prime $q, q \nmid C_g$, we have

$$g(\chi)|T(q) = g(\chi)|U(q) + \omega\chi^2(q)q^{l-1}g(\chi)|V(q) = \chi(q)b(q)g(\chi), \tag{3.15}$$

$$\begin{aligned} g(\chi)|U(d) &= g(\chi)\left|\left[\prod_{q|d} \chi(q)b(q) - \chi^2(q)\omega(q)q^{l-1}V(q)\right]\right. \\ &= g(\chi)\left|\left[\sum_{d_1|d} \mu(d_1)\omega(d_1)d_1^{l-1}b(dd_1^{-1})\chi(dd_1)V(d_1)\right]\right., \end{aligned}$$

since d is square-free and $(d, C_g) = 1$. From (3.14) and (3.15) it now follows that

$$g_{M_0}(\chi) = \sum_{d_1|d|M_0} \mu(d)\mu(d_1)\omega(d_1)d_1^{l-1}b(dd_1^{-1})\chi(dd_1)g(\chi)|V(dd_1). \tag{3.16}$$

We apply the involution $W(C_g C_\chi^2)$ to the function (3.16), where we use the following obvious commutation property of matrix operators: for natural numbers A, A' and B with $A'|B$ we have

$$g|V(A')W(AB) = A^{l/2}A'^{-l}g|W(B)V(A/A'). \tag{3.17}$$

If in (3.17) we substitute $A = M_0^2$, $A' = dd_1$, and $B = C_g C_\chi^2$, we obtain

$$\begin{aligned} g(\chi)|V(dd_1)W(C_g M_0^2 C_\chi^2) &= M_0^l(dd_1)^{-l}g(\chi)|W(C_g C_\chi^2)V(M_0^2(dd_1)^{-1}) \\ &= M_0^l(dd_1)^{-l}\Lambda(g(\chi))g^\rho(\overline{\chi})|V(M_0^2(dd_1)^{-1}) \end{aligned} \tag{3.18}$$

by (3.13). Substituting (3.18) in (3.16), we have

$$\begin{aligned} &g_{M_0}(\chi)|W(C_g M_0^2 C_\chi^2) = M_0^l\Lambda(g(\chi)) \\ &\times \sum_{d_1|d|M_0} \mu(d)\mu(d_1)\omega(d_1)d_1^{l-1}b(dd_1^{-1})\chi(dd_1)(dd_1)^{-l}g^\rho(\overline{\chi})|V(M_0^2(dd_1)^{-1}). \end{aligned} \tag{3.19}$$

We apply (3.19) to prove (3.12):

$$\begin{aligned} &L(s, f_0, g_{M_0}(\chi)|W(C_g M_0^2 C_\chi^2)) \\ &\quad = M_0^l\Lambda(g(\chi)) \sum_{d_1|d|M_0} \mu(d)\mu(d_1)\omega(d_1)d_1^{l-1}b(dd_1^{-1})\chi(dd_1)(dd_1)^{-l} \\ &\qquad \times L(s, f_0, g^\rho(\overline{\chi})|V(M_0^2(dd_1)^{-1})). \end{aligned} \tag{3.20}$$

xamination of the Dirichlet series coefficients in (3.20) shows that

$$\begin{aligned} &L(s, f_0, g^\rho(\overline{\chi})|V(M_0^2(dd_1)^{-1})) \\ &\quad = (M_0^2(dd_1)^{-1})^{-s} L(s, f_0|U(M_0^2(dd_1)^{-1}), g^\rho(\overline{\chi})) \\ &\quad = \alpha(M_0^2(dd_1)^{-1})(M_0^2(dd_1)^{-1})^{-s} L(s, f_0, g^\rho(\overline{\chi})), \end{aligned} \tag{3.21}$$

so the convolution (3.12) is reduced to the form

$$\begin{aligned} &\Psi(s, f_0, g_{M_0}(\chi)|W(C_g M_0^2 C_\chi^2)) \\ &\quad = M_0^{l-2s}\alpha(M_0)^2\Lambda(g(\chi))\Psi(s, f_0, g^\rho(\overline{\chi})) \\ &\qquad \times \sum_{d_1|d|M_0} \mu(d)\mu(d_1)\omega(d_1)d_1^{l-1}\chi(dd_1)\alpha(dd_1)^{-1}(dd_1)^{s-l}b(dd_1^{-1}). \end{aligned} \tag{3.22}$$

it is not hard to find the Euler factors of this Dirichlet series, using the properties
) and (1.20) of the convolution, from which it follows that

$$, f_0, g^\rho(\overline{\chi})) = \Psi(s, f, g^\rho(\overline{\chi})) \prod_{q\in S}[(1 - (\overline{\chi}\alpha'\hat{\beta})(q)q^{-s})(1 - (\overline{\chi}\alpha'\hat{\beta}')(q)q^{-s})]. \tag{3.23}$$

ldition, from the definition of the Hecke polynomial

$$1 - b(q)X + \omega(q)q^{l-1}X^2 = (1 - \beta(q)X)(1 - \beta'(q)X)$$

ave

$$\begin{aligned} &\sum_{d_1|d|M_0} \mu(d)\mu(d_1)\omega(d_1)d_1^{l-1}b(dd_1^{-1})\alpha(dd_1)^{-1}\chi(dd_1)(dd_1)^{s-l} \\ &\quad = \prod_{q|M_0}[(1 - (\chi\alpha^{-1}\beta)(q)q^{s-l})(1 - (\chi\alpha^{-1}\beta')(q)q^{s-l})]. \end{aligned} \tag{3.24}$$

bining (3.22)–(3.24), we obtain (3.12). This completes the proof of Proposition

§4. Archimedean integral representation for the distributions

1. We shall prove an S-adic analog of the following classical integral formula of
kin (see (1.19)): for $F \in \mathscr{S}_k(N, \psi)$ and $G \in \mathscr{M}_l(N, \omega)$ we have

$$\Psi(s, F, G) = 2^{-1}\Gamma(s + 1 - l)\pi^{-s}\langle F^\rho, GE(s - k + 1)\rangle_N, \tag{4.1}$$

e

$$\begin{aligned} F^\rho(z) &= \overline{F(-\overline{z})} \in \mathscr{S}_k(N, \overline{\psi}), \\ E(s) &= E(z, s) = E(z, s; k - l, \psi\omega, N) \\ &= y^s {\sum}' \psi\omega(n)(mNz + n)^{-(k-l)-|2s|} \end{aligned} \tag{4.2}$$

Eisenstein series of weight $k - l > 0$, and in (4.2) we have adopted the notation
$^{|s|} = z^{-k}|z|^{-s}$ for $k \in \mathbf{Z}$ and $s \in \mathbf{C}$ (see [14]); here the prime after the summation
ol indicates summation over all $(0, 0) \neq (m, n) \in \mathbf{Z}^2$.
pplying (4.1) in the case when

$$\begin{aligned} F &= f_0|V(C_f) \in \mathscr{S}_k(M_0C_f^2, \psi) \subset \mathscr{S}_k(C_0C_f, \psi), \\ G &= g_{M_0}(\chi)|W(C_0M') \in \mathscr{S}_l(C_0M', \overline{\omega}\overline{\chi}^2) \subset \mathscr{S}_l(C_0C_fM', \overline{\omega}\overline{\chi}^2), \end{aligned} \tag{4.3}$$

ansform the definition of the distributions (3.4) by means of the equality

$$\begin{aligned} &\Psi(s, f_0|V(C_f), g_{M_0}(\chi)|W(C_0M')) \\ &\quad = 2^{-1}\Gamma(s + 1 - l)\pi^{-s}\langle F^\rho, GE(s - k + 1)\rangle_{C_0C_fM'}, \end{aligned} \tag{4.4}$$

where

$$E(s) = E(z, s; k-l, \psi\overline{\omega\chi^2}, C_0C_fM')$$

is in the vector space $\tilde{\mathscr{M}}_{k-l}(C_0C_fM', \overline{\psi}\omega\chi^2)$, which consists of C^∞-modular forms for $\Gamma_0(C_0C_fM')$ with character $\overline{\psi}\omega\chi^2$. If we set

$$K(s) = G\cdot E(s) \in \tilde{\mathscr{M}}_k(C_0C_fM', \overline{\psi}), \tag{4.5}$$

then the formula for the values of the distribution (3.4) takes the form

$$\begin{aligned}\Psi_{S,s}^{(M)}(\chi) &= (M'M_0)^{s-l/2}\alpha(M'M_0)^{-1}C_f^{s-l/2}\overline{\chi}(C_g)\\ &\times \Lambda(g)^{-1}2^{-1}\Gamma(s+1-l)\pi^{-s}\frac{\langle F^\rho, K(s+1-k)\rangle_{C_0C_fM'}}{\pi^{1-l}\langle f,f\rangle_{C_f}}.\end{aligned} \tag{4.6}$$

4.2. We define the *trace operator* $\mathrm{Tr}_N^{NM'}: \tilde{\mathscr{M}}_k(NM', \overline{\psi}) \to \tilde{\mathscr{M}}_k(N, \overline{\psi})$ by the equality

$$K|\mathrm{Tr}_N^{NM'} = \sum_{\gamma\in\Gamma_0(NM')\backslash\Gamma_0(N)} K|_k\gamma, \tag{4.7}$$

where the set of representatives of the right cosets $\Gamma_0(NM')\backslash\Gamma_0(N)$ is chosen, for example, in the form $\{(\begin{smallmatrix}1 & 0\\ uN & 1\end{smallmatrix}) | u \bmod M'\}$. We apply the operator (4.7) to (4.6) with $N = C_0C_f$; as a result,

$$\langle F^\rho, K(s)\rangle_{C_0C_fM'} = \langle F^\rho, K(s)|\mathrm{Tr}_{C_0C_f}^{C_0C_fM'}\rangle_{C_0C_f}, \tag{4.8}$$

since

$$F \in \mathscr{S}_k(C_0C_f, \overline{\psi}).$$

We use the following identity from [6] for explicit computations:

$$K|\mathrm{Tr}_N^{NM'} = M'^{1-k/2}K|W(NM')U(M')W(N), \tag{4.9}$$

in which

$$K|U(M') = M'^{k/2-1}\sum_{u \bmod M'} K\Big|_k \begin{pmatrix}1 & u\\ 0 & M'\end{pmatrix}. \tag{4.10}$$

The identity (4.9) follows immediately from the matrix equality

$$\begin{pmatrix}1 & 0\\ uN & 1\end{pmatrix} = -(NM')^{-1}\begin{pmatrix}0 & -1\\ NM' & 0\end{pmatrix}\begin{pmatrix}1 & -u\\ 0 & M'\end{pmatrix}\begin{pmatrix}0 & -1\\ N & 0\end{pmatrix}.$$

We apply (4.9) in (4.8) with $N = C_0C_f$, obtaining

$$\langle F^\rho, K(s)\rangle_{C_0C_fM'} = M'^{1-k/2}\langle F^\rho, K'(s)|U(M')W(C_0C_f)\rangle_{C_0C_f}, \tag{4.11}$$

where

$$K'(s) = K(s)|W(C_0C_fM') \in \tilde{\mathscr{M}}_k(C_0C_fM', \psi)$$

is a function whose Fourier coefficients can be computed completely explicitly for special values of s (more precisely, for $l-k \le s \le 0$, $s \in \mathbf{Z}$). In fact, we first note that, by definition,

$$g_{M_0}(\chi)|W(C_0M')W(C_0C_fM') = (-1)^l C_f^{l/2} g_{M_0}(\chi)|V(C_f), \tag{4.12}$$

and we set $E'(z,s) = E(z,s)|W(C_0C_jM')$. The Fourier expansion of the Eisenstein series will be computed in §4.5, after which we obtain the Fourier expansion of the function

$$K'(s) = (-1)^l C_f^{l/2} g_{M_0}(\chi)|V(C_f)E'(z,s). \tag{4.13}$$

4.3. We now give a general result about Fourier expansions of Eisenstein series in terms of the Whittaker functions $W(y, \alpha, \beta)$, which for $y > 0$ and $\alpha, \beta \in \mathbf{C}$, $\mathrm{Re}(\beta) > 0$, are defined by the integral

$$W(y,\alpha,\beta) = \Gamma(\beta)^{-1}\int_0^\infty (u+1)^{\alpha-1}u^{\beta-1}e^{-yu}\,du \tag{4.14}$$

and for arbitrary $\alpha, \beta \in \mathbf{C}$ are defined by analytic continuation and the functional equation

$$W(y,\alpha,\beta) = y^{1-\alpha-\beta}W(y, 1-\beta, 1-\alpha).$$

If r is a nonnegative integer, then

$$W(y,\alpha,-r) = \sum_{i=0}^{r}(-1)^r\binom{r}{i}\frac{\Gamma(\alpha)}{\Gamma(\alpha-i)}y^{r-i}. \tag{4.15}$$

Suppose that N and m are integers, $s \in \mathbf{C}$, $N \geq 1$, and $\varphi: \mathbf{Z}/N\mathbf{Z} \times \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ is an arbitrary function. Then for $m + 2\,\mathrm{Re}(s) > 2$ one can define the Eisenstein series

$$E(z,s;m,\varphi) = \sum{}'\varphi(n_1,n_2)(n_1+n_2z)^{-m}|n_1+n_2z|^{-2s}, \tag{4.16}$$

which can be analytically continued in $s \in \mathbf{C}$ and satisfy a certain functional equation; here it is possible for m to be negative. In order to write out the Fourier expansion of the functions (4.16), we define a partial Fourier transform $P_\varphi(n_1, n_2)$ of $\varphi(c, d)$ by the formula

$$(P\varphi)(n_1,n_2) = \sum_{a \bmod N}\varphi(a,n_2)e(an_1/N), \tag{4.17}$$

which has inverse given by

$$(P^{-1}\varphi)(n_1,n_2) = N^{-1}\sum_{a \bmod N}\varphi(a,n_2)e(-an_1/N). \tag{4.17a}$$

For an arbitrary function $h: \mathbf{Z}/N\mathbf{Z} \to \mathbf{C}$ we set

$$h_{(m)}(x) = h(x) + (-1)^m h(-x), \qquad L(s,h) = \sum_{n=1}^{\infty}h(n)n^{-s}.$$

4.4. Proposition. *If s is an integer for which $m + s > 0$, $s \leq 0$, then one has the following Fourier expansion*:

$$\begin{aligned}
&\frac{N^{m+2s}\Gamma(m+s)}{(-2\pi i)^{m+2s}(-4\pi y)^{-s}}E(Nz,s;m,\varphi)\\
&\quad= \frac{(-4\pi y)^s\Gamma(m+s)}{\Gamma(m+2s)}L(1-m-2s, P\varphi(\cdot,0))\\
&\qquad+\frac{\Gamma(m+2s-1)}{(4\pi y)^{m+s-1}\Gamma(s)}L(m+2s-1, P\varphi(\cdot,0)_{(m)})\\
&\qquad+(4\pi y)^s\sum_{dd'>0}\mathrm{sgn}(d)d^{m+2s-1}P\varphi(d,d')W(4\pi dd'y, m+s, s)e(dd'z).
\end{aligned} \tag{4.18}$$

The proof of the proposition is based on a classical computation of Hecke [17], a version of which is contained in Katz's paper [20]. It follows rather easily from the Fourier expansion of the function

$$F(z) = \sum_{n\in\mathbf{Z}}(z+n)^{-\alpha}(\overline{z}+n)^{-\beta} \qquad (z \in H). \tag{4.19}$$

Namely,

$$F(z) = \sum_{n\in\mathbf{Z}} t_n(y,\alpha,\beta)e(nx),$$

where the coefficients $t_n(y,\alpha,\beta)$ are given by the relations (see [31])

$$i^{\alpha-\beta}(2\pi)^{-\alpha-\beta}t_n(y,\alpha,\beta) = \begin{cases} n^{\alpha+\beta-1}e^{-2\pi ny}\Gamma(\alpha)^{-1}W(4\pi ny,\alpha,\beta), & n>0,\\ |n|^{\alpha+\beta-1}e^{-2\pi|n|y}\Gamma(\beta)^{-1}W(4\pi|n|y,\beta,\alpha), & n<0,\\ \Gamma(\alpha)^{-1}\Gamma(\beta)^{-1}\Gamma(\alpha+\beta-1)(4\pi y)^{1-\alpha-\beta}, & n=0,\end{cases}$$

which at the same time give an analytic continuation of the function $F(z) = F(z;\alpha,\beta)$ to all $(\alpha,\beta)\in\mathbf{C}^2$, whereas (4.19) converges only for $\mathrm{Re}(\alpha+\beta)>1$. We omit the details of the proof of (4.18).

4.5. We apply the result in Proposition 4.4 to the Eisenstein series in §4.2:

$$E(z,s) = E(z,s;m,\chi,N), \qquad E'(z,s) = (\sqrt{N}z)^{-m}E(-(Nz)^{-1},s).$$

For convenience we introduce the normalized Eisenstein series

$$\begin{aligned} G^*(z,s) &= \frac{N^{(m+2s)/2}\Gamma(m+s)}{(-2\pi i)^{m+2s}(-4\pi)^{-s}}E'(z,s) \\ &= \frac{\Gamma(m+s)}{(-2\pi i)^{m+2s}(-4\pi y)^{-s}}E(z,s;m,\varphi_2), \end{aligned} \tag{4.20}$$

where $\varphi_2(n_1,n_2) = \chi(n_2)$ and the series

$$E(z,s;m,\varphi_2) = \sum{}' \chi(c)(cz+d)^{-m-|2s|}$$

depends on the character χ but not on N. Then in (4.18) we have that $P\varphi_2(d,d') = N_\chi(d')$ or 0 depending on whether or not d is divisible by N. As a result, we obtain the expansion

$$\begin{aligned} G^*(z,s) &= (4\pi y)^s\varepsilon(s,m,\chi) \\ &\quad + (4\pi y)^s \sum_{dd'>0} \mathrm{sgn}(d)d^{m+2s-1}\chi(d')W(4\pi dd'y,m+s,s)e(dd'z). \end{aligned} \tag{4.21}$$

4.6. We apply (4.20) to (4.13) with s equal to $s-k+1$ and m equal to $k-l$. Then in (4.20) we have that $s+m$ is equal to $s-l+1$, $m+2s$ is equal to $2s+2-k-l$, and

$$\begin{aligned} E'(z,s-k+1) &= (C_0C_fM')^{-(2s+2-k-l)/2}(-1)^{s-l+1}i^{k-l} \\ &\quad \times 2^{k-l}\pi^{s-l+1}\Gamma(s-l+1)^{-1}G^*(z,s-k+1). \end{aligned} \tag{4.22}$$

Combining this with (4.6), (4.11), and (4.13), we obtain the following Archimedean integral representation for the distributions:

$$\Psi_{S,s}^{(M)}(\chi) = \gamma(M')\langle f,f\rangle_{C_f}^{-1}\langle F^\rho, K^*(s-k+1)|U(M')W(C_0C_f)\rangle_{C_0C_f}, \tag{4.23}$$

in which we have set

$$\begin{aligned} K^*(s) &= (-1)^s C_f^{-s}\overline{\chi}(C_g)C_g^{-s}g_{M_0}(\chi)|V(C_f)G^*(z,s), \\ G^*(z,s) &= G^*(z,s;k-l,\psi\overline{\omega}\overline{\chi}^2,C_0C_fM'), \end{aligned} \tag{4.24}$$

and

$$\gamma(M') = i^{-k+l}2^{k-l-1}C_f^{-1}C_g^{-(k-l)/2}\alpha(M'M_0)M_0^{k/2-1}\Lambda(g)^{-1} \tag{4.25}$$

a p-integral algebraic number. It follows from the Fourier expansion (4.21) that r integers s with $l-k<s\le 0$ we have

$$K^*(s)=\sum_{n=1}^{\infty}\sum_{\substack{Cn_1+n_2=n\\(n_1,M_0)=1}} d(n_1,n_2;y,s)e(nz), \tag{4.26}$$

iere for $(n,M_0)>1$ the Fourier coefficients are given by

$$d(n_1,n_2;y,s)=(-1)^s C_f^{-s}\overline{\chi}(C_g)C_g^{-s}\chi(n_1)b(n_1) \\ \times(4\pi y)^s\sum_{n_2=dd'}\psi\overline{\omega}\overline{\chi}^2(d')\mathrm{sgn}(d)d^{2s+k-l-1}W(4\pi n_2 y,s-l+k,s). \tag{4.27}$$

We now state the basic result of the section.

4.7. PROPOSITION. *In the above notation, for $s\in\mathbf{Z}$ with $l\le s\le k-1$ one has the [e]quality*

$$\Psi_{S,s}^{(M)}(\chi)=\gamma(M')\langle f,f\rangle_{C_f}^{-1}\langle f_0^\rho|V(C_f),\tilde K_{M'}(s-k+1)|W(C_0C_f)\rangle_{C_0C_f}, \tag{4.28}$$

[i]n which, for $s\in\mathbf{Z}$ with $1+l-k\le s\le 0$

$$\tilde K_{M'}(s)=\sum_{n=1}^{\infty}\sum_{Cn_1+n_2=M'n} d(n_1,n_2;s,\chi)e(nz)\in\mathcal{S}_k(C_0C_f,\overline{\psi})$$

[i]s a cusp form with algebraic Fourier coefficients given by

$$d(n_1,n_2;s,\chi)=(-1)^s C_f^{-s}\overline{\chi}(C_g)C_g^{-s}\chi(n_1)b(n_1) \\ \times\sum_{n_2=dd'}\psi\overline{\omega}\overline{\chi}^2(d')\mathrm{sgn}(d)d^{2s+k-l-1}P_s(n_2,n), \tag{4.29}$$

[a]nd

$$P_s(x,y)=\sum_{i=0}^{-s}(-1)^i\binom{-\ s}{i}\frac{\Gamma(k-l+s)\Gamma(k-i-1)}{\Gamma(k-l+s-i)\Gamma(k-1)}x^{-s-i}y^i \\ =x^{-s}+yQ_s(x,y)\in\mathbf{Z}[x,y]\qquad(-s\ge 0) \tag{4.30}$$

[i]s a polynomial with integer coefficients.

The proposition is proved using the holomorphic projection operator

$$\mathcal{HOL}:\tilde{\mathcal{M}}_k(C_0C_f,\psi)\to\mathcal{S}_k(C_0C_f,\psi), \tag{4.31}$$

[w]hich is defined by the condition

$$\langle h,\mathcal{HOL}(K)\rangle_{C_0C_f}=\langle h,K\rangle_{C_0C_f}$$

[f]or all $h\in\mathcal{S}_k(C_0C_f,\psi)$. We apply the general integral formula for the action of the [o]perator $\mathcal{HOL}$ (see [16] and [32]) to the function

$$\tilde K_{M'}(s)=\mathcal{HOL}(K^*(s)|U(M')).$$

[H]ere this function takes the form: for $n\in\mathbf{N}$, $Cn_1+n_2=M'n$

$$d(n_1,n_2;s,\chi)=\frac{(4\pi n)^{k-1}}{\Gamma(k-1)}\int_0^\infty d(n_1,n_2;y,s)e^{-2\pi ny}y^{k-2}\,dy. \tag{4.32}$$

The proof of the proposition is completed by integrating in (4.32), taking (4.27) and (4.15) into account, and also using the standard integral representation for the Γ-function.

§5. Algebraicity, integrality, and congruences for the distributions

5.1. We now specialize the general definition of the distribution in (3.4) to the case $s = l + r$, where $0 \le r \le k - l - 1$; it then follows from Proposition 4.7 that

$$\begin{aligned}\Psi^{(M)}_{S,l+r}(\chi) &= \gamma(M')\langle f, f\rangle_{C_f}^{-1}\langle F^\rho, \tilde{K}_{M'}(r-k+l+1)|W(C_0C_f)\rangle_{C_0C_f} \\ &= \gamma(M')\langle f, f\rangle_{C_f}^{-1}\langle F^\rho|W(C_0C_f), \tilde{K}_{M'}(r-k+l+1)\rangle_{C_0C_f}\end{aligned} \tag{5.1}$$

and all of the numbers (5.1) are algebraic. In fact, the cusp form $\tilde{K}_{M'}(r-k+l+1)$ in §4.7 with algebraic Fourier coefficients can be decomposed with respect to an orthogonal basis of the vector space $\mathscr{S}_k(C_0C_f, \psi)$ such that one of the basis vectors is the cusp form $f_0^\rho|W(C_0) = C_f^{k/2}F^\rho|W(C_0C_f)$. Here one must take into account that the inner products $\langle f, f\rangle_{C_f}$ and $\langle F^\rho|W(C_0C_f), F^\rho|W(C_0C_f)\rangle_{C_0C_f}$ differ only by an algebraic factor [29]. This fact can be established using the Euler products

$$L(s, f), \quad L(s, F^\rho|W(C_0C_f)), \tag{5.2}$$

which differ from one another only in a finite number of Euler factors; at the same time the corresponding inner products can be interpreted in terms of the special values of (5.2) at the point $s = k - 1$. This also implies that the linear functional

$$\mathscr{L}: K \mapsto \langle f, f\rangle_{C_f}^{-1}\langle F^\rho, K|W(C_0C_f)\rangle_{C_0C_f} \tag{5.3}$$

on the complex vector space $\mathscr{S}_k(C_0C_f, \psi)$ is defined over $\overline{\mathbf{Q}}$, i.e., for a finite set of natural numbers $n_i \in \mathbf{N}$ and fixed algebraic coefficients $\xi(n_i) \in \overline{\mathbf{Q}}$ we have

$$\mathscr{L}(K) = \sum_i \xi(n_i)a(n_i, K), \tag{5.4}$$

in which, by definition, $K(z) = \sum_{n=1}^{\infty} a(n, K)e(nz)$.

5.2. We now apply our fixed embedding $i_p: \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}_p$ to the algebraic numbers (5.1), as a result obtaining the $\mathbf{C}_p$-valued distributions $i_p(\Psi_{S,l+r})$. We set $\Psi_S = i_p(\Psi_{S,l})$. We note that, if we use the definition of the functional (5.3), the formulas (5.1) for the distributions take the form

$$\Psi^{(M)}_{S,l+r}(\chi) = \gamma(M')\mathscr{L}(\tilde{K}_{M'}(r-k+l+1)), \tag{5.5}$$

where M' is a sufficiently large natural number which is chosen depending on χ, $\gamma(M')$ was defined in (4.25), and we have the property $|i_p(\gamma(M'))|_p = 1$ by our assumption that the cusp form f is p-ordinary (see (1.8)):

$$|i_p(\alpha(q))|_p = 1 \quad (q \in S), \qquad |i_p(\alpha(M'))|_p = 1.$$

5.3. Proposition. a) *The $\mathbf{C}_p$-valued distributions $i_p(\Psi_{S,l+r})$ on $\mathbf{Z}_S^\times$ are bounded for all integers $r = 0, 1, \dots, k-l-1$.*

b) *The following S-adic equality holds:*

$$\int_{\mathbf{Z}_S^\times} \chi x_p^r \, d\Psi_S = \int_{\mathbf{Z}_S^\times} \chi \, di_p(\Psi_{S,l+r}). \tag{5.6}$$

Proposition 5.3 is proved using the abstract Kummer congruences in §2.3. Here we make use of the set of functions on $\mathbf{Z}_S^\times$ of the form χx_p^r, where $\chi \in X_S^{\mathrm{tors}}$,

$1, \ldots, k-l-1$. For any finite set of characters $\chi \in X_S^{\text{tors}}$ we choose a common of M and a sufficiently large M' so that the integral formula (5.5), coming ›roposition 4.7, holds for each of these characters. We now use the description of the linear functions $\mathscr{L}$. As a result we find that proving the abstract Kum- ›ngruences for the numbers $i_p(\Psi_{S,l+r}^{(M)}(\chi))$ is equivalent to proving the following ıences for the Fourier coefficients (4.29):

$$d(n_1, n_2; r-k+l+1, \chi) \qquad C_f n_1 + n_2 = M'n.$$

(4.29) and (4.30), along with the equality $C_f n_1 + n_2 = M'n$, it follows that

$$P_s(n_2, n) \equiv n_2^{k-l-1-r} \equiv (dd')^{k-l-1-r} \pmod{M'},$$
$$\chi(n_1) = \overline{\chi}(-C_f)\chi(n_2) = \overline{\chi}(-C_f)\chi(dd').$$

ɘquently, we have the congruences

$$_1, n_2; r-k+l+1, \chi)$$
$$\equiv \overline{\chi}(-C)(-C)^{-r+k-l-1} b(n_1) \sum_{n_2 = dd'} \overline{\chi}(d') d'^{k-l-1-r} \operatorname{sgn}(d) \chi(d) d^r \pmod{M'}. \tag{5.7}$$

ʼw remains only to note that the abstract Kummer congruences are obviously led for the expressions on the right in (5.7). Since we have considered the tions χx_p^r with $r = 0, 1, \ldots, k-l-1$ simultaneously, both parts of Proposition mmediately follow.

4. The non-Archimedean Rankin L-functions in the fundamental theorem can be constructed as the non-Archimedean Mellin transforms of the measure $\Psi_S =$ $_{s,l})$. These functions are bounded $\mathbf{C}_p$-analytic functions on X_S which are uniquely rmined by the special values $\Psi_S(\chi x_p^r)$. These values are given in Propositions) and 5.3b):

$$i_p^{-1}(\Psi(\chi x_p^r)) = \omega(C_\chi) G(\chi)^2 C_\chi^{l+2r-1} \alpha(C_\chi)^{-2} A(r, \chi) \frac{\Psi(l+r, f, g^\rho(\overline{\chi}))}{\pi^{1-l} \langle f, f \rangle_{C_f}},$$

ɛe $A(r, \chi)$ is the product in (1.15). Theorem 1.4 is proved.

5. To prove the S-adic functional equation (1.16), we use the Archimedean func- ıl equation for the convolutions: if $f \in \mathscr{S}_k(C_f, \psi)$ and $g \in \mathscr{S}_l(C_g, \omega)$ are prim- cusp forms and $(C_f, C_g) = 1$, then for any primitive Dirichlet character χ C_χ we have (see [22])

$$\Psi(s, f, g^\rho(\overline{\chi})) = B_\chi(s) \Psi(k+l-1-s, f^\rho, g(\chi)), \tag{5.9}$$

ɛe $(C_f, C_\chi) = (C_g, C_\chi) = 1$, and

$$B_\chi(s) = (C_f C_g C_\chi^2)^{k+l-2s} (-1)^k \overline{\omega}(C_f) \psi(C_g)$$
$$\times \overline{\chi}^2(C_f C_g) \psi\overline{\omega}(C_\chi) \frac{G(\overline{\chi})^4}{C_\chi^2} \Lambda(f)^2 \overline{\Lambda(g)^2},$$

ɛe $\Lambda(f)$ and $\Lambda(g)$ are the constants in the functional equations (1.12). In (5.9) we set $s = l+r$, and we compare the values of the functions $\Psi(x)$ and $\hat{\Psi}(x_p^{k-l-1} x^{-1})$ ıe points $x = \chi x_p^r$. Here we take into account that the product $A(r, \chi) =$ $\chi; f, g, \alpha)$ does not change if we replace $(f, g, \alpha, \beta, \chi, r)$ by $(f^\rho, g^\rho, \hat{\alpha}, \hat{\beta}, \overline{\chi}, k-l-$). As a result we find that the functional equation (1.16) holds for these special ɛs, and it remains once again to use the fact that bounded $\mathbf{C}_p$-analytic functions are uniquely determined by their values at the points $x = \chi x_p^r$. With this observation the proof of Theorem 1.5 is complete.

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