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A p-adic measure attached to the zeta functions associated with two elliptic modular forms. I

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§ 0. Introduction

Let p be a prime number. The aim of this paper is to construct a p-adic bounded measure of several variables, which establishes the p-adic interpolation of the special values of the Rankin product of two elliptic modular forms of different weight. Let N be an arbitrary positive integer. Let f be a cusp form of weight $k \geqq 2$ for the congruence subgroup $\Gamma_0(N)$ with character ψ modulo N, which is, in addition, a primitive form (=normalized new form of level dividing N). Let g be a modular form of weight $l<k$ for $\Gamma_0(N)$, with character ω. Write $e(z)=\exp(2\pi i z)$. Suppose that the Fourier expansions of f and g are given by

$$f=\sum_{n=1}^{\infty} a(n)\, e(nz), \qquad g=\sum_{n=0}^{\infty} b(n)\, e(nz).$$

The Rankin product of f and g is defined by

$$(0.1) \qquad \mathscr{D}_N(s,f,g)=L_N(2s+2-k-l,\omega\psi)\sum_{n=1}^{\infty} a(n)\, b(n)\, n^{-s},$$

where $L_N(2s+2-k-l,\omega\psi)$ denotes the Dirichlet L-series of $\omega\psi$ with the Euler factors at the primes dividing N removed from its Euler product. It is well known that $\mathscr{D}_N(s,f,g)$ has a holomorphic continuation over the whole complex plane as a function of s. Moreover, when the Fourier coefficients $b(n)$ of g are algebraic numbers (note that the Fourier coefficients of f are automatically algebraic because f is primitive), Shimura [25, 26] has proven the basic result that

$$(0.2) \qquad \frac{\mathscr{D}_N(m,f,g)}{\pi^{2m+1-l}\langle f,f\rangle_N} \text{ is algebraic for all integers } m \text{ with } l \leqq m < k;$$

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$$\langle f, f\rangle_N = \int_{B(N)} |f(z)|^2\, y^{k-2}\, dx\, dy,$$

where $B(N)$ denotes a fundamental domain for $\Gamma_0(N)$. Let $\bar{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let Ω denote the completion of an algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$, and we normalize its valuation $|\ |_p$ by $|p|_p = p^{-1}$. We fix once and for all an embedding

$$(0.3) \qquad i\colon \bar{\mathbb{Q}} \to \Omega$$

(when there is no danger of confusion, we will omit i from our subsequent formulae). We assume for the rest of the paper that the form f is *ordinary* for p (or more correctly i) in the following sense

(0.4) *the image under i of the p-th Fourier coefficient of f is a unit in Ω.*

Let V be a vector space over $\mathbb{Q}$, and let $n\colon V \to \mathbb{Q}$ be a positive definit quadratic form on V. Define a symmetric bilinear form $S\colon V \times V \to \mathbb{Q}$ by

$$S(u, v) = n(u+v) - n(u) - n(v).$$

(We note that $n(v) = \frac{1}{2} S(v, v)$). Fix a lattice I in V so that $n(v) \in \mathbb{Z}$ for all $v \in I$. It is then clear that $S(u, v) \in \mathbb{Z}$ for all u and v in I, and hence, if we define

$$I^* = \{v \in V \mid S(v, I) \subset \mathbb{Z}\},$$

we have $I^* \supset I$. Write M for the smallest positive integer such that $M n(I^*) \subset \mathbb{Z}$. This integer M is called the level of I, and we note that I^*/I is annihilated by M. Throughout this paper except in § 1, we assume that *the dimension of V over $\mathbb{Q}$ is even.* Let

$$\eta\colon V \to \bar{\mathbb{Q}}$$

be a spherical function on V with algebraic values (see § 1), and let

$$\phi\colon V \to \bar{\mathbb{Q}}$$

be an arbitrary locally constant function for the p-adic topology on I such that the theta series

$$\theta(z) = \sum_{v \in I} \phi(v)\, \eta(v)\, e(n(v)\, z)$$

gives a modular form of weight l and of character ξ. We now take g to be the theta series $\theta(z)$ and assume that ϕ factors through $I/p^\beta I$ for a positive integer $\beta \geqq 1$. Take $N = M p^{2\beta}$. It is known that the level of θ divides N. Composing ϕ, η and n with the embedding i, we obtain continuous functions from $I_p = I \otimes_{\mathbb{Z}} \mathbb{Z}_p$ to Ω, which we denote by the same symbols. Let C be the divisor of N which is the exact level of f (i.e. the conductor of f), and define the root number $W(f)$ by

$$f|_k \begin{pmatrix} 0 & -1 \\ C & 0 \end{pmatrix} = W(f)\, f^\rho,$$

where $f^\rho = \sum_{n=1}^{\infty} \bar{a}(n) e(nz)$ is the complex conjugate form of f. Write

$$M = M' p^\lambda, \qquad C = C' p^\mu,$$

where $(M', p) = (C', p) = 1$ (note that C' divides M' because C divides N).

Theorem 0.1. *Assume that $\mu \geqq 1$. For each integer $b > 1$, with $(b, N) = 1$, there exists a unique bounded measure φ_b on $I_p = I \otimes_{\mathbf{Z}} \mathbf{Z}_p$ with values in Ω satisfying the following interpolation property: for every integer r with $0 \leqq 2r < k - l$, we let $j = l + 2r$ and we have that the value of the p-adic integral*

$$\int_{I_p} \phi\, \eta\, n^r\, d\varphi_b$$

is given by the image under i of

$$W(f)^{-1}\, t\,(1 - b^{k-j} \psi\, \bar{\xi}(b))\, a(p)^{\mu - \lambda - 2\beta} \frac{\mathscr{D}_N(j - r, f|_k \gamma, \theta|_l \tau)}{\pi^{j+1} \langle f, f \rangle_C},$$

where $\gamma = \begin{pmatrix} M'/C' & 0 \\ 0 & 1 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, *and*

$$t = (\sqrt{-1})^{k+j}\, 2^{1-k-j}\, p^{(\mu - \lambda)(1 - k/2) + \beta j}\, M^{(j-k)/2 + 1}\, \Gamma(j - r)\, \Gamma(r + 1).$$

A slightly stronger result, including the case when p does not divide C, is given in § 2. We also obtain results on the p-adic interpolation of the values (0.2) when g runs over the twists of a modular form (of weight strictly less than k) by all Dirichlet characters whose conductor is a power of p (see Theorem 2.2). Moreover, in a later paper, we shall show that one can naturally extend φ_b to a measure $\mathbf{Z}_p^\times \times I_p$ by allowing the p-ordinary form f to vary.

Our motivation for studying these p-adic measures has been our desire to investigate the Iwasawa theory of certain p-adic Lie extensions of number fields, which arise from abelian varieties and modular forms. Some work has been done in this direction in the complex multiplication case (see [5] and [29]), but the non-abelian theory remains shrouded in mystery.

Here is a summary of the contents of the paper. The detailed statements of our results are given in § 2. As far as the construction of the measure φ_b is concerned, we first construct a measure on I_p with values in the space of p-adic modular forms. This measure can be thought of as a p-adic convolution of the Katz's Eisenstein measure in [12] with the p-adic measure attached to a theta series. The measure φ_b is then obtained by combining this measure with a bounded linear form on the space of p-adic modular forms, which is studied in § 4 (our hypothesis that p is ordinary for f is essential for the construction of this linear form). We make use of Shimura's differential operators [25] to evaluate the p-adic integral as in the theorem.

Notation

Let $\mathfrak{H}$ be the upper half complex plane. Then the group $GL_2^+(\mathbf{R})$ of real 2×2 matrices with positive determinant acts on $\mathfrak{H}$ via linear fractional transformations. If $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ belongs to $GL_2^+(\mathbb{R})$ and $f(z)$ is any function on $\mathfrak{H}$, we define, for each $k\in\mathbb{Z}$,

$$(f|_k \gamma)(z)=(\det(\gamma))^{k/2} f(\gamma(z))(cz+d)^{-k}.$$

For each positive integer N, let $\Gamma_0(N)$ (resp. $\Gamma_1(N)$) be the subgroup of $SL_2(\mathbb{Z})$ consisting of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c\equiv 0 \bmod N$ (resp. $c\equiv 0 \bmod N$, $a\equiv d\equiv 1 \bmod N$). If Γ denotes either of these two subgroups of $SL_2(\mathbb{Z})$, we write $\mathcal{M}_k(\Gamma)$ for the space of holomorphic modular forms of weight k for Γ, and $\mathcal{S}_k(\Gamma)$ for the space of cusp forms of weight k for Γ. As usual, for each character ψ modulo N, we write

$$\mathcal{M}_k(\Gamma_0(N),\psi)=\left\{f\in\mathcal{M}_k(\Gamma_1(N))\,\middle|\, f|_k\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\psi(d)f \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N)\right\},$$

and we put $\mathcal{S}_k(\Gamma_0(N),\psi)=\mathcal{S}_k(\Gamma_1(N))\cap\mathcal{M}_k(\Gamma_0(N),\psi)$. Finally, we recall that the automorphism group of $\mathbb{C}$ has a natural action on $\mathcal{M}_k(\Gamma)$ given by

$$\left(\sum_{n=0}^{\infty} a(n)\,e(nz)\right)^{\sigma}=\sum_{n=0}^{\infty} a(n)^{\sigma}\,e(nz)$$

for each automorphism σ of $\mathbb{C}$.

§ 1. Theta series

Our aim in this section is to briefly recall those transformation formulae of θ-series defined by positive definite quadratic forms, which will be used later in the paper. See Shimura [21], § 2 for further details.

In this section, we use the notation defined in Introduction, and we allow the dimension of the quadratic space V to be odd. Let κ denote the half of the dimension of V; therefore, κ is a positive integer or half a positive integer. We also write S for the natural extension of S to a $\mathbb{C}$-bilinear form on $V\otimes_{\mathbb{Q}}\mathbb{C}$. Throughout this section, we write $\eta\colon V\to\mathbb{C}$ for an arbitrary complex-valued spherical function on V. We recall that this means that either η is homogenous of degree $\leqq 1$, or that η can be expressed as follows: there exist finitely many w in $V\otimes_{\mathbb{Q}}\mathbb{C}$ with $n(w)=0$ such that

$$\eta(v)=\sum_{w} c(w)\,S(w,v)^{\alpha},$$

where $c(w)\in\mathbb{C}$ and α is an integer $\geqq 2$. In general, we write α for the degree of η (or, as it is often called, the order of η). Write Φ for any complex-valued function on I^*/I. For any function $h\colon I^*\to\mathbb{C}$, we define formally

(1.1) $$\theta(h)(z)=\sum_{v\in I^*} h(v)\,e(n(v)\,z).$$

When $h=\Phi\eta$, this series converges, and defines a holomorphic function on $\mathfrak{H}$. We define an action of $\Gamma_0(M)$ on the set of all functions $\Phi\colon I^*/I\to\mathbb{C}$ via

(1.2) $$(\gamma\cdot\Phi)(v)=e(db\,n(v))\,\Phi(dv),$$

where $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. If m,n are non-zero integers, let the quadratic residue symbol $\left(\frac{m}{n}\right)$ be as defined on p. 442 of [21]. Moreover, we let $\varepsilon_d=1$ if $d\equiv 0,1,2 \bmod 4$, and $\varepsilon_d=\sqrt{-1}$ if $d\equiv 3 \bmod 4$. For each non-zero complex number x, we fix $x^{1/2}$ by taking its argument to be in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right]$. Finally, let $\Delta=[I^*:I]$.

Proposition 1.1. *The function $\theta(\Phi\eta)(z)$ satisfies the transformation formula:*

(1.3) $$\theta(\Phi\eta)(\gamma(z))=\left(\frac{\Delta}{d}\right)\left(\frac{2c}{d}\right)^{2\kappa}\varepsilon_d^{-2\kappa}(cz+d)^{\kappa+\alpha}\,\theta((\gamma\cdot\Phi)\,\eta)(z),$$

where $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in\Gamma_0(M)$.

The proof of this proposition is essentially contained in [21]. Note, however, that Shimura supposes that $4|M$ and then proves (1.3) for $\gamma=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with $c\equiv 0 \bmod \frac{M}{2}$ and $b\equiv 0 \bmod 2$. To derive (1.3) from Shimura's result, one needs only to verify (using the Poisson summation formula) the invariance, relative to weight $\kappa+\alpha$, of $\theta(\Phi\eta)(z)$ under the matrices

$$\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \quad\text{and}\quad \begin{pmatrix}1 & 0\\ M & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -M\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}.$$

We omit the details.

§ 2. Statement of main results

We begin by defining the space on which our p-adic measure exists. As in § 1, let V be a quadratic space over $\mathbb{Q}$. We shall now assume that V has *even* dimension 2κ over $\mathbb{Q}$ (i.e. that κ is an integer). As before, I will denote a lattice in V with $n(I)\subset\mathbb{Z}$, I^* the dual lattice, and M the least positive integer such that $M\,n(I^*)\subset\mathbb{Z}$. For each integer $\nu\geq 0$, $p^\nu I$ is a lattice with level $Mp^{2\nu}$. Define

$$X=\varprojlim_{\nu} I^*/p^\nu I.$$

In addition, let $\mathscr{W}=\{v\in I^*|\,n(v)\in\mathbb{Z}\}$, and put

$$W=\varprojlim_{\nu}\mathscr{W}/p^\nu I.$$

Plainly W is a subset of X, and the quadratic form n has a natural extension n: $W \to \mathbb{Z}_p$.

Let η: $V \to \bar{\mathbb{Q}}$ be an arbitrary spherical function on V of degree $\alpha \geqq 0$, taking algebraic values. Composing η with the fixed embedding (0.3), we obtain a unique extension of η by continuity to a function from W to Ω, which we again denote by η. Note that the group

$$Z = \varprojlim_{v} (\mathbb{Z}/M p^v \mathbb{Z})^\times$$

has a natural action on the space X, which leaves stable W. Let ϕ: $W \to \bar{\mathbb{Q}}$ be an arbitrary locally constant function satisfying the following property: there exists a character χ of finite order of Z such that

$$\phi(z w) = \chi(z) \phi(w) \qquad (z \in Z,\ w \in W). \tag{2.1}$$

We then define the θ-series

$$\theta(\phi \eta)(z) = \sum_{w \in \mathscr{W}} \phi(w) \eta(w) e(n(w) z).$$

Put

$$\xi(a) = \chi(a) \left(\frac{-1}{a}\right)^\kappa \left(\frac{\Delta}{a}\right),$$

where the symbols on the right are Legendre symbols, and $\Delta = [I^* : I]$. Proposition 1.1 shows that there exists $\beta \geqq 0$ such that the conductor of ξ divides $M p^\beta$ and $\theta(\phi \eta)$ belongs to $\mathscr{M}_{\kappa+\alpha}(\Gamma_0(M p^\beta), \xi)$. In the following, β will denote any fixed integer satisfying this property with $\beta \geqq 1$.

As in the introduction, let $f = \sum_{n=1}^{\infty} a(n) e(n z)$ be a fixed primitive cusp form of weight $k \geqq 2$ with conductor C, and character ψ modulo C. We define the Petersson inner product of f with $g \in \mathscr{M}_k(\Gamma_0(C), \psi)$ by

$$\langle g, f \rangle_C = \int_{\mathfrak{H}/\Gamma_0(C)} \overline{g(z)} f(z) y^{k-2} \, dx \, dy.$$

We now fix the embedding (0.3) of $\bar{\mathbb{Q}}$ into Ω and assume that this embedding i satisfies the condition (0.4), i.e. that $i(a(p))$ is a unit in Ω. We then write γ for the unique root of the Euler factor

$$X^2 - i(a(p)) X + i(\psi(p)) p^{k-1}$$

which is not a unit in Ω (hence $\gamma = 0$ if p divides C). We now define the modular form $f_0(z)$ to be either $f(z)$ or $f(z) - i^{-1}(\gamma) f(p z)$, according as p does or does not divide the conductor C of $f(z)$. It is well known (see [28, p. 88] and Lemma 3.3 in the next section) that $f_0(z)$ is a common eigenform of all Hecke operators $T(n)$ $(n \geqq 1)$ of level $p C$, including those with n dividing $p C$. Moreover, $f_0(z)$ is a unique ordinary form of level $p C$ with the same n-th Fourier coefficient as $f(z)$ for every n prime to p (see Lemma 3.3). Let C_0 be the smallest possible level of f_0, i.e. $C_0 = C$ or $p C$ according as p does or does not divide the conductor C of f. We then define non-negative integers μ, λ, and C', M' prime to p, by

(2.3) $$C_0 = C' p^{\mu}, \quad M = M' p^{\lambda}, \quad (M', p) = (C', p) = 1.$$

Finally, we impose the hypothesis

(2.4) *C' divides M'.*

This assumption is not very restrictive, since it can always be achieved by replacing I by a suitable sub-lattice. Let us also write

$$\gamma = \begin{pmatrix} M'/C' & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau_{\beta} = \begin{pmatrix} 0 & -1 \\ M p^{\beta} & 0 \end{pmatrix}.$$

Theorem 2.1. *For each integer $b > 1$, with $(b, Mp) = 1$, there exists a unique bounded measure φ_b on W with values in Ω satisfying the following interpolation property: for each non-negative integer r with $0 \leqq 2r + \alpha < k - \kappa$, we let $j = \kappa + \alpha + 2r$, and we have that the value of the p-adic integral*

$$\int_W \phi\, \eta\, n^r \, d\varphi_b$$

is given by the image under i of

(2.5) $$t(1 - b^{k-j} \psi\, \bar{\xi}(b))\, a(p, f_0)^{\mu - \lambda - \beta} \frac{\mathscr{D}_{Mp^{\beta}}(j - r, f_0|_k \gamma, \theta(\phi\, \eta)|_{\kappa + \alpha} \tau_{\beta})}{\pi^{j+1} \langle h, f_0 \rangle_{C_0}},$$

where $a(p, f_0)$ is the p-th *Fourier coefficient of f_0,*

$$h = f_0^{\rho}|_k \begin{pmatrix} 0 & -1 \\ C_0 & 0 \end{pmatrix},$$

and

$$t = t(r, \alpha, \beta) = (\sqrt{-1})^{k+j}\, 2^{1-k-j}\, p^{(\mu - \lambda)(1 - k/2) + \beta j/2}\, M^{(j-k)/2 + 1}\, \Gamma(j - r)\, \Gamma(r + 1).$$

Here are several remarks about this theorem, whose proof is given in §7. Firstly, it is easy to see directly that (2.5) does not depend on the choice of β. Secondly, the uniqueness of φ_b follows from the fact that any locally constant function on W is a finite sum of those satisfying the condition (2.1). Finally, we note that we do not give the p-adic interpolation at all of the special values

$$\mathscr{D}_{Mp^{\beta}}(m, f_0|_k \gamma, \theta(\phi\, \eta)|_{\kappa + \alpha} \tau_{\beta}),$$

with m an integer satisfying $\kappa + \alpha \leqq m < k$, where algebraicity is known. This can be partly remedied by using the functional equation (see §9 for the discussion of a special case of this functional equation), but this does not cover all aspects of this interesting question.

We next discuss a result in which g is no longer assumed to be a theta series. Let

$$g = \sum_{n=0}^{\infty} b(n)\, e(nz)$$

be an arbitrary modular form of weight $l<k$ for $\Gamma_0(N)$ with character ω and assume that

$$b(n)\in\bar{\mathbb{Q}} \quad \text{for all } n\geqq 0.$$

Define a compact ring Y by

$$Y=\varprojlim_{\nu}\mathbb{Z}/Np^{\nu}\mathbb{Z}, \qquad Y^{\times}=\varprojlim_{\nu}(\mathbb{Z}/Np^{\nu}\mathbb{Z})^{\times}.$$

Let ϕ: $Y\to\bar{\mathbb{Q}}$ be an arbitrary locally constant function on Y with the property: there is a character χ of finite order of the group $Y^{\times}$ such that

$$(2.6) \qquad \phi(zy)=\chi(z)\phi(y) \quad (z\in Y^{\times},\ y\in Y).$$

We then define the twist of g by

$$g(\phi)=\sum_{n=0}^{\infty}\phi(n)\,b(n)\,e(nz).$$

Put $\xi=\chi^2\omega$, and write $N=N'p^{\lambda}$ with $(N',p)=1$. Then it is known that $g(\phi)$ belongs to $\mathcal{M}_l(\Gamma_0(NN'p^{\beta}),\xi)$ for a sufficiently large $\beta\geqq 1$. Now we fix such a $\beta\geqq 1$. Parallel to (2.4), we assume that

(2.7) *C' divides N'.*

and write

$$\gamma=\begin{pmatrix}N'^2/C' & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad \tau_{\beta}=\begin{pmatrix}0 & -1\\ NN'p^{\beta} & 0\end{pmatrix}.$$

Theorem 2.2. *For each integer $b>1$ prime to Np, there exists a unique bounded measure φ_b on Y with values in Ω satisfying the following property: for each non-negative integer r with $0\leqq r<(k-l)/2$, we let $j=l+2r$, and we have that the value of the p-adic integral*

$$\int_Y \phi(y)\,y_p^r\,d\varphi_b(y)$$

is given by the image under i of

$$t(1-b^{k-j}\psi\,\bar{\xi}(b))\,a(p,f_0)^{\mu-\lambda}\frac{\mathcal{D}_{Mp^{\beta}}(j-r,f_0|_k\gamma,g(\phi)|_l\tau_{\beta})}{\pi^{j+1}\langle h,f_0\rangle_{C_0}},$$

where y_p is the projection of $y\in Y=(\mathbb{Z}/N'\mathbb{Z})\times\mathbb{Z}_p$ to the factor $\mathbb{Z}_p$, and

$$t=t(r,\beta)=(\sqrt{-1})^{k+j}2^{1-k-j}p^{(\mu-\lambda)(1-k/2)+\beta j/2}(NN')^{(j-k)/2+1}\Gamma(j-r)\Gamma(r+1).$$

Since this theorem can be proven in a similar fashion as in the proof of Theorem 2.1, merely a sketch of the proof will be given in §8. By taking the Eisenstein series in [25, (4.3)] as g of Theorem 2.2, we see that $\mathcal{D}_N(s,f,g)$ is a product of Mellin transforms of f and its twist. This suggests to us a relation between our measure and those constructed by Mazur-Swinnerton-Dyer [16] and by Manin [14, 15]. It is an interesting problem to clarify these relations.

§ 3. Some results on Fourier coefficients

For every modular form f, we hereafter write $a(n, f)$ for the n-th Fourier coefficient of f, namely,

$$f(z) = \sum_{n=0}^{\infty} a(n, f)\, e(nz).$$

Terminology. We define a *normalized eigenform* of level N to be a non-zero common eigenform in $\mathcal{M}_k(\Gamma_1(N))$ of all Hecke operators $T(n)$ for $\Gamma_1(N)$ (including those with n dividing N) such that $f|T(n) = a(n, f)\, f$ for all n. We say that a form f in $\mathcal{S}_k(\Gamma_1(N))$ is *primitive* if there exists a divisor C of N such that (i) f is a new form (in the sense of Miyake [17]) of level C, and (ii) f is a normalized eigenform of level C. The number C is called the *conductor* of f. We say that a normalized eigenform f of level N is *ordinary* for p (or more precisely, for the embedding i fixed in (0.3) if p divides N and if $i(a(p, f))$ is a unit in Ω (i.e. $|i(a(p, f))|_p = 1$). (It is technically important for us to insist that p divides N in our definition of ordinary forms.)

If there is no danger of confusion, we hereafter drop the embedding i from our notation, when we consider algebraic numbers of $\mathbb{C}$ in the field Ω.

Let f be a primitive form of conductor C of weight k and with character ψ. Let $C(\psi)$ be the conductor of the character ψ, and define non-negative integers t and s by

$$C = C' p^t, \qquad C(\psi) = C'(\psi)\, p^s,$$

where $(C', p) = (C'(\psi), p) = 1$.

Proposition 3.1. *If* $a(p, f)$ *is a unit in* Ω (i.e. $|a(p, f)|_p = 1$), *then we have either* $t = s$ *or* $k = 2$, $t = 1$ *and* $s = 0$.

Before proving this fact, we recall the following result in Doi-Miyake [7, Th. 4.6.17], whose proof we recall because [7] is written in Japanese.

Lemma 3.2. *Let* ψ_0 *be the primitive character modulo* $C(\psi)$ *associated with* ψ. *Then we have*

(3.1 a) $$a(p, f)\, a(p, f)^{\rho} = p^{k-1} \qquad \text{if } t = s,$$

(3.1 b) $$a(p, f)^2 = \psi_0(p)\, p^{k-2} \qquad \text{if } t = 1 \text{ and } s = 0,$$

(3.1 c) $$a(p, f) = 0 \qquad \text{if } t \geqq 2 \text{ and } t > s,$$

where ρ *denotes complex conjugation.*

The facts (3.1 a, b) can be proven in exactly the same manner as in Asai [1, Lemma 3], where these are shown for every square-free conductor C. A proof of (3.1 c) is as follow: put

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,\middle|\, a \equiv d \equiv 1 \bmod (C/p),\ c \equiv 0 \bmod C \right\}.$$

Then it is well known that, if p^2 divides C,

$$\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1(C/p)=\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma \quad \text{as a subset of } M_2(\mathbb{Z}),$$

and thus the Hecke operator $T(p)$ takes $\mathcal{S}_k(\Gamma)$ into $\mathcal{S}_k(\Gamma_1(C/p))$. Then, the assumption of (3.1c) shows that $f\in\mathcal{S}_k(\Gamma)$ and we know that

$$a(p,f)\,f=f\,|\,T(p)=f\left|\left[\Gamma\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1(C/p)\right]\right.$$

is of level C/p. Since C is the smallest possible level of f, we know that $a(p,f)\,f=0$ and therefore $a(p,f)$ must vanish.

Now we can prove Proposition 3.1. When $k\geqq 2$, the proposition is a direct consequence of Lemma 3.2. We now assume that $k=1$. Since $a(p,f)$ must be an algebraic integer, the case (3.1b) is impossible, whence the proposition is true for $k=1$.

Lemma 3.3. *Suppose that the weight k of f is greater than or equal to 2 and that $|a(p,f)|_p=1$. Then, there is a unique ordinary form f_0 of weight k such that $a(n,f)=a(n,f_0)$ except for those n divisible by p. Moreover, f_0 is explicitly given by*

$$f_0(z)=\begin{cases} f(z) & \text{if } p \text{ divides } C,\\ f(z)-\gamma f(pz) & \text{if } C \text{ is prime to } p,\end{cases}$$

where γ is the unique root of $X^2-a(p,f)X+\psi(p)p^{k-1}$ with $|\gamma|_p<1$.

Proof. The cusp form given as above is known to be a normalized eigenform of level pC (cf. [28, Remark 3.59, p. 88]). Thus our task is to show that f_0 is ordinary and that f_0 is unique. If there is a normalized eigenform with the same Fourier coefficients as f except for those for which n is divisible by p, such a form must belong to $\mathcal{S}_k(\Gamma_0(Cp^\nu),\psi)$ for a suitable ν by the theory of primitive forms (cf. [17] and [4]). Put

$$(3.1)\qquad U(Cp^\nu,f)=\{g\in\mathcal{S}_k(\Gamma_1(Cp^\nu))\,|\,g\,|\,T(l)=a(l,f)\,g \text{ except for finitely many primes } l\},$$

and $f^{(n)}(z)=f(p^n z)$ for $0\leqq n\in\mathbb{Z}$. Then, it is known (e.g. [17]) that $\{f^{(0)},\dots,f^{(\nu)}\}$ gives a basis of $U(Cp^\nu,f)$. Let β and γ be the roots of $X^2-a(p,f)X+\psi(p)p^{k-1}$ with $|\beta|_p=1$ and $|\gamma|_p<1$. Then, we can choose another basis of $U(Cp^\nu,f)$ in the following manner:

(3.2a) If C is prime to p, then we put

$f_0(z)=f(z)-\gamma f(pz)$, $\quad f_1(z)=f(z)-\beta f(pz)$,

$f_2(z)=f_0(z)-\beta f_0(pz)$ and $f_n(z)=f_2(p^{n-2}z)$

for $2\leqq n\leqq\nu$;

(3.2b) If p divides C, then we put

$f_0=f$, $\quad f_1(z)=f(z)-a(p,f)\,f(pz)$ and $f_n(z)=f(p^{n-1}z)$

for $1\leqq n\leqq\nu$.

In the case (3.2a), f_0, f_1 and f_2 are normalized eigenforms (of level Cp^ν for every $\nu \geqq 2$) and their eigenvalues for $T(p)$ are β, γ and 0, respectively. Thus f_0 is ordinary, but neither f_1 nor f_2 can be ordinary. Similarly, f_0 and f_1 are normalized eigenforms in the case (3.2b) with eigenvalues $a(p,f)$ and 0, respectively. For the action of $T(p)$ on f_n for a general n, we know that

$$f_n | T(p) = f_{n-1}$$

if $n \geqq 3$ in the case (3.2a) and if $n \geqq 2$ in the case (3.2b) (cf. [28, p. 88]). This shows that for any $\nu \geqq 1$, the operator $T(p)$ is nilpotent on $\sum_{n=1}^{\nu} \mathbb{C} f_n$ or $\sum_{n=2}^{\nu} \mathbb{C} f_n$ according as p does or does not divide C. Thus the uniqueness of f_0 follows from this if we prove the $\mathbb{C}$-linear independence of $\{f_0, \dots, f_\nu\}$ in $U(Cp^\nu, f)$. We consider the matrix $(a(p^i, f_j))_{0 \leqq i, j \leqq \nu}$ of the Fourier coefficients of f_n at the powers of the prime p; namely, it is equal to

(3.3)
$$\left(\begin{array}{cc|c} 1 & 1 & \\ \beta & \gamma & \\ \beta^2 & \gamma^2 & 1_{\nu-1} \\ \vdots & \vdots & \\ \beta^{\nu-2} & \gamma^{\nu-2} & \\ \hline \beta^{\nu-1} & \gamma^{\nu-1} & \\ \beta^\nu & \gamma^\nu & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|c} 1 & \\ a(p,f) & \\ a(p,f)^2 & 1_\nu \\ \vdots & \\ a(p,f)^{\nu-1} & \\ \hline a(p,f)^\nu & 0 \end{array}\right)$$

according as p does not or does divide C. Here 1_ν is the $n \times n$ identity matrix. Since $\beta \neq \gamma$ and $a(p,f) \neq 0$, these matrices are non-singular and thus $\{f_0, \dots, f_\nu\}$ gives a basis. Q.E.D.

§ 4. *p*-adic Modular Forms and Hecke Operators

We begin by recalling the definition of the space of p-adic modular forms in a manner rather more similar to Serre [19] than Katz [11, 12]. Let Γ denote either of the two congruence subgroups $\Gamma_0(N)$ or $\Gamma_1(N)$ for a positive integer N. For any subring A of $\bar{\mathbb{Q}}$, define an A-module $\mathcal{M}_k(\Gamma; A)$ (resp. $\mathcal{M}_k(\Gamma_0(N), \psi; A)$ for each Dirichlet character ψ modulo N with values in A) to be the subspace of $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{M}_k(\Gamma_0(N), \psi)$) consisting of all modular forms with A-rational Fourier coefficients. For every modular form $f = \sum_{n=0}^{\infty} a(n,f)\, e(nz)$ with algebraic Fourier coefficients, define a p-adic norm $|f|_p$ of f by

$$|f|_p = \operatorname{Sup}_n |a(n,f)|_p.$$

It is well known (see [24, Th. 1] and [28, Th. 3.52]) that the norm $|f|_p$ is a well defined real number. Let K_0 be a finite extension of $\mathbb{Q}$, and K be the closure of K_0 in Ω (relative to the fixed embedding i: $\bar{\mathbb{Q}} \to \Omega$ of (0.3)). Let $\mathcal{M}_k(\Gamma; K)$ (resp. $\mathcal{M}_k(\Gamma_0(N), \psi; K)$) denote the completion of $\mathcal{M}_k(\Gamma; K_0)$ (resp. $\mathcal{M}_k(\Gamma_0(N), \psi; K_0)$) for the norm $|\ |_p$. Then these spaces become Banach spaces over K.

Let $X_{/\mathbb{Q}}$ be the compactified canonical model of $\mathfrak{H}/\Gamma$ defined over $\mathbb{Q}$ [28, 6.7 and (7.3.5)]. Then the space $\mathcal{M}_k(\Gamma; K_0)$ (resp. $\mathcal{M}_k(\Gamma; K_0)\otimes_{K_0}K$) can be identified with the space of global sections over K_0 (resp. K) of a certain line bundle on $X_{/\mathbb{Q}}$ rational over $\mathbb{Q}$ (cf. [6, VII.3], [23, Th. 6] and [24, Th. 3]). Let A be either of the two fields K_0 or K. From this interpretation of these spaces, we know the following three facts: $\mathcal{M}_k(\Gamma; A)$ and $\mathcal{M}_k(\Gamma_0(N), \psi; A)$ are finite dimensional; $\mathcal{M}_k(\Gamma; K)=\mathcal{M}_k(\Gamma; K_0)\otimes_{K_0}K$, $\mathcal{M}_k(\Gamma_0(N), \psi; K)=\mathcal{M}_k(\Gamma_0(N), \psi; K_0)\otimes_{K_0}K$; $\mathcal{M}_k(\Gamma; K)$ and $\mathcal{M}_k(\Gamma_0(N), \psi; K)$ are determined independently of the choice of the dense subfield K_0. Furthermore, the abstract Hecke ring introduced in [28, (3.3.3) and Th. 3.34] acts naturally on $\mathcal{M}_k(\Gamma; A)$ and $\mathcal{M}_k(\Gamma_0(N), \psi; A)$. This action on $\mathcal{M}_k(\Gamma; K_0)$ and $\mathcal{M}_k(\Gamma_0(N), \psi; K_0)$ is induced from the usual action of the Hecke operators $T(n)$ and $T(n, n)$ on $\mathcal{M}_k(\Gamma)$ and $\mathcal{M}_k(\Gamma_0(N), \psi)$ as in [28, 3.4, 3.5]. See below for the precise definition of the action of these operators.

By writing q for $e(z)$, we can embed $\mathcal{M}_k(\Gamma; K_0)$ into $K_0[[q]]$. Then we may regard $\mathcal{M}_k(\Gamma; K)$ as the closure of $\mathcal{M}_k(\Gamma; K_0)$ in $K[[q]]$. Thus every element of $\mathcal{M}_k(\Gamma; K)$ has a unique q-expansion. For $f=\sum_{n=0}^{\infty} a(n, f)q^n \in \mathcal{M}_k(\Gamma; K)$, the norm of f is again given by $\mathrm{Sup}_n |a(n, f)|_p$. Let $\mathcal{O}_K$ denote the ring of p-adic integers in K, and define

$$\mathcal{M}_k(\Gamma; \mathcal{O}_K)=\{f\in\mathcal{M}_k(\Gamma; K) \mid |f|_p \leqq 1\}=\mathcal{M}_k(\Gamma; K)\cap\mathcal{O}_K[[q]],$$
$$\mathcal{M}_k(\Gamma_0(N), \psi; \mathcal{O}_K)=\{f\in\mathcal{M}_k(\Gamma_0(N), \psi; K) \mid |f|_p \leqq 1\}.$$

These spaces are complete normed $\mathcal{O}_K$-modules of finite rank. Let

$$\mathcal{M}_k(\Gamma; \Omega)=\mathcal{M}_k(\Gamma; K)\otimes_K \Omega, \qquad \mathcal{M}_k(\Gamma_0(N), \psi; \Omega)=\mathcal{M}_k(\Gamma_0(N), \psi; K)\otimes_K \Omega.$$

As already seen, these spaces do not depend on the choice of the subfield K of Ω. All the definitions as above for modular forms can be formulated naturally for cusp forms and the corresponding spaces of cusp forms will be written as $\mathcal{S}_k(\Gamma; K)$, $\mathcal{S}_k(\Gamma_0(N), \psi; K)$, etc.

Let N be an arbitrary positive integer and ψ be a character modulo N. Let A denote either of the field K or the ring $\mathcal{O}_K$. Put

$$\mathcal{M}_k(N; A)=\bigcup_{n=0}^{\infty}\mathcal{M}_k(\Gamma_1(Np^n); A),$$

$$\mathcal{M}_k(N, \psi; A)=\bigcup_{n=0}^{\infty}\mathcal{M}_k(\Gamma_0(Np^n), \psi; A).$$

Clearly, these spaces do not depend on the p-primary part of N. Let $\overline{\mathcal{M}}(N; A)$ (resp. $\overline{\mathcal{M}}_k(N, \psi; A)$) be the completion of $\mathcal{M}_k(N; A)$ (resp. $\mathcal{M}_k(N, \psi; A)$) for the norm $|\ |_p$. Any element of $\overline{\mathcal{M}}(N; K)$ will be called a p-adic modular form. The suffix "k" is dropped for the notation "$\overline{\mathcal{M}}(N; A)$", because, as a subspace of $A[[q]]$, the space $\overline{\mathcal{M}}(N; A)$ is determined independently of the weight k if $k\geqq 2$. This fact is implicit in the papers of Katz and Serre on p-adic modular forms, but we refrain from discussing it in detail, since we do not need this fact later.

However, the space $\overline{\mathcal{M}}_k(N, \psi; A)$ does depend on the weight k and the suffix "k" must be retained (cf. [12, Lemma 5.4.10]).

Let us give here an explicit description of the action of the Hecke operators $T(l)$ and $T(l, l)$ of level N for primes l. For any integer n prime to N, let $\sigma_n \in \Gamma_0(N)$ be the matrix with $\sigma_n \equiv \begin{pmatrix} * & * \\ 0 & n \end{pmatrix} \bmod N$. As shown in Deligne-Rapoport [6, VII, Cor. 3.11] and Katz [12, 5.3.2], the action: $f \mapsto f|_k \sigma_n$ of σ_n on $\mathcal{M}_k(\Gamma_1(N); K)$ leaves $\mathcal{M}_k(\Gamma_1(N); \mathcal{O}_K)$ stable. Then the action of the Hecke operators $T(l)$ and $T(l, l)$ for primes l on $\mathcal{M}_k(\Gamma_1(N); K)$ is given by

$$(4.1)\qquad \begin{aligned} a(n, f \mid T(l)) &= \begin{cases} a(ln, f) + l^{k-1} a\left(\frac{n}{l}, f|_k \sigma_l\right) & \text{if } l \text{ is prime to } N, \\ a(ln, f) & \text{of } l \text{ divides } N, \end{cases} \\ a(n, f \mid T(l, l)) &= \begin{cases} l^{k-2} a(n, f|_k \sigma_l) & \text{if } l \text{ is prime to } N, \\ 0 & \text{if } l \text{ divides } N. \end{cases} \end{aligned}$$

When N is divisible by p, (4.1) shows that $\mathcal{M}_k(\Gamma_1(N); \mathcal{O}_K)$ and $\mathcal{M}_k(\Gamma_0(N), \psi; \mathcal{O}_K)$ are stable under the operators $T(l)$ and $T(l, l)$. Let A denote either of the field K or the ring $\mathcal{O}_K$, and let $\mathcal{H}_k(\Gamma_0(N), \psi; A)$ (resp. $\mathcal{H}_k(\Gamma_1(N); A)$) be *the A-subalgebra of the ring of all A-linear endomorphisms of $\mathcal{M}_k(\Gamma_0(N), \psi; A)$ (resp. $\mathcal{M}_k(\Gamma_1(N); A)$) generated by $T(l)$ and $T(l, l)$ for all primes l.* Especially, we know that (if p divides N)

$$|f \mid T|_p \leqq |f|_p \qquad \text{for every } T \in \mathcal{H}_k(\Gamma_1(N); \mathcal{O}_K),$$

and therefore, any operator in $\mathcal{H}_k(\Gamma_1(N); \mathcal{O}_K)$ is uniformly continuous. These algebras are the Hecke algebras of the corresponding spaces of modular forms.

Next we consider the Hecke algebras of the space of p-adic modular forms. The restriction of operators in $\mathcal{H}_k(\Gamma_1(Np^n); \mathcal{O}_K)$ (resp. $\mathcal{H}_k(\Gamma_0(Np^n), \psi; \mathcal{O}_K)$) to the subspace $\mathcal{M}_k(\Gamma_1(Np^m); \mathcal{O}_K)$ (resp. $\mathcal{M}_k(\Gamma_0(Np^m), \psi; \mathcal{O}_K)$) for $n \geqq m \geqq 1$ gives a $\mathcal{O}_K$-algebra homomorphism of $\mathcal{H}_k(\Gamma_1(Np^n); \mathcal{O}_K)$ (resp. $\mathcal{H}_k(\Gamma_0(Np^n), \psi; \mathcal{O}_K)$) onto $\mathcal{H}_k(\Gamma_1(Np^m); \mathcal{O}_K)$ (resp. $\mathcal{H}_k(\Gamma_0(Np^m), \psi; \mathcal{O}_K)$). This fact follows from [28, Th. 3.34–5]. Taking the projective limit of these morphisms, we obtain compact topological algebras:

$$(4.2)\qquad \begin{aligned} \mathcal{H}(N; \mathcal{O}_K) &= \varprojlim_n \mathcal{H}_k(\Gamma_1(N p^n); \mathcal{O}_K), \\ \mathcal{H}_k(N, \psi; \mathcal{O}_K) &= \varprojlim_n \mathcal{H}_k(\Gamma_0(N p^n), \psi; \mathcal{O}_K) \end{aligned}$$

which naturally act on $\mathcal{M}_k(N; A)$ and $\mathcal{M}_k(N, \psi; A)$ for $A = K$ or $\mathcal{O}_K$. The action of $\mathcal{H}(N; \mathcal{O}_K)$ (resp. $\mathcal{H}_k(N, \psi; \mathcal{O}_K)$) can be naturally extended to an action on $\overline{\mathcal{M}}(N; A)$ (resp. $\overline{\mathcal{M}}_k(N, \psi; A)$) by the uniform continuity.

Let us now introduce the idempotent e attached to $T(p)$ in the Hecke algebra. Let R denote either of the two algebras $\mathcal{H}_k(\Gamma_0(Np^m), \psi; \mathcal{O}_K)$ or $\mathcal{H}_k(\Gamma_1(Np^m); \mathcal{O}_K)$ for $m \geqq 1$. Then the algebra R/pR over the field $\mathbb{F}_p$ with p-elements is commutative and finite dimensional [28, Th. 3.51]. The image $\tilde{T}(p)$

of $T(p)$ in R/pR can be decomposed into the unique sum $s+n$ of a semi-simple element s and a nilpotent element n of R/pR. Thus, for a sufficiently large integer r, the element $\tilde{T}(p)^{p^r}$ coincides with s^{p^r} and becomes semi-simple. Then, we can choose a positive integer u so that $\tilde{T}(p)^{p^r u}$ gives an idempotent of R/pR. This idempotent can be lifted to a unique idempotent e_m of R (cf. [3, III.4.6]). In fact, this idempotent can be given as a p-adic limit in R by

$$(4.3) \qquad e_m = \lim_{r\to\infty} T(p)^{p^r u}.$$

This idempotent is clearly independent of the choice of the integer u, and its construction is plainly compatible with the projective limit (4.2). Thus we can define an idempotent e of $\mathcal{H}(N;\mathcal{O}_K)$ and $\mathcal{H}_k(N,\psi;\mathcal{O}_K)$ by the projective limit $\varprojlim_m e_m$. For any module $\mathcal{M}$ over these Hecke algebras, we define the ordinary part $\mathcal{M}^\circ$ of $\mathcal{M}$ to be the corresponding component $e\mathcal{M}$ for the idempotent e. A remarkable fact is

Proposition 4.1. *The ordinary part $\bar{\mathcal{M}}_k^\circ(N,\psi;\mathcal{O}_K)$ of the space $\bar{\mathcal{M}}_k(N,\psi;\mathcal{O}_K)$ is free of finite rank over $\mathcal{O}_K$. Moreover, let $C(\psi)$ be the conductor of the Dirichlet character ψ, and define positive integers N' and s by*

$$N=N'p^r \quad \text{and} \quad s=\max(s',1) \quad \text{for} \quad C(\psi)=C'(\psi)\,p^{s'},$$

where $(N',p)=(C'(\psi),p)=1$. Then the ordinary part $\bar{\mathcal{M}}_k^\circ(N,\psi;\mathcal{O}_K)$ is contained in $\mathcal{M}_k(\Gamma_k(N'p^s),\psi;\mathcal{O}_K)$.

Proof. As shown in the proof of Lemma 3.2, the Hecke operator $T(p)^m$ for a sufficiently large integer m takes $\mathcal{M}_k(\Gamma_0(Np^n),\psi;\mathcal{O}_K)$ into $\mathcal{M}_k(\Gamma_0(N'p^s),\psi;\mathcal{O}_K)$ for each $n\geqq 1$. Then the assertion is clear from the definition (4.3) of the idempotent e.

In contrast with this result, the ordinary part of $\bar{\mathcal{M}}(N;\mathcal{O}_K)$ is usually of infinite rank. The relation between ordinary forms and the idempotent e is given as follows: Let f be an element of $\mathcal{M}_k(N,\psi;K)$ and let C be the smallest possible level of f. Assume that $f|T_C(p)=af$ with $a\in K$ for the Hecke operator $T_C(p)$ of level C.

Lemma 4.2. *If C is divisible by p, then the image $f|e$ of f under e is either f itself or 0 according as the eigenvalue a is or is not a unit in $\mathcal{O}_K$.*

Proof. We may assume that f is a modular form for $\Gamma_0(Np^m)$ for a suitable positive integer m. Note that the action of the operator $T(p)$ of level Np^m on f is the same as that of $T_C(p)$. Then, with the notation of (4.3), the eigenvalue of e at f is given by the p-adic limit

$$\lim_{r\to\infty} a^{p^r u}.$$

The lemma is obvious from this.

From this lemma, it is clear that, especially when f is a normalized form with level divisible by p, then

$$f \text{ is ordinary if and only if } f|e=f.$$

Lemma 4.3. *Put, for each non-negative integer m,*

$$f^{(m)}(z)=f(p^m z)=\sum_{n=0}^{\infty} a(n,f)\, q^{p^m n},$$

and for a positive integer n,

$$U=\sum_{m=0}^{n} K f^{(m)}.$$

Then the subspace U of $\mathcal{M}_k(N;K)$ is stable under the idempotent e. Moreover, assume that either $k\geqq 2$ or p divides C. Then, if a is not a unit in $\mathcal{O}_K$, the space U is annihilated by e.

Proof. Let $T(p)$ be the Hecke operator in $\mathcal{H}(N,\psi;\mathcal{O}_K)$. Note that $T(p)$ and $T_C(p)$ are different if C is prime to p. It is well known (e.g. [28, p. 88]) that

$$f^{(m)}|T(p)=f^{(m-1)} \quad \text{for } m\geqq 1,$$

$$f^{(0)}|T(p)=\begin{cases} a f^{(0)} & \text{if } p \text{ divides } C,\\ a f^{(0)}-\psi_0(p)\, p^{k-1} f^{(1)} & \text{if } C \text{ is prime to } p,\end{cases}$$

where ψ_0 is the primitive character associated with ψ. This shows that U and even its $\mathcal{O}_K$-lattice $U(\mathcal{O}_K)=\sum_{m=0}^{n} \mathcal{O}_K f^{(m)}$ are stable under $T(p)$ and hence, under e. Let $\mathfrak{P}$ be the prime ideal of $\mathcal{O}_K$. Assume that $a\in\mathfrak{P}$ (i.e. that a is not a unit). Then the above formulae show that if $k\geqq 2$ or p divides C,

$$U(\mathcal{O}_K)|T(p)^{n+1}\subset \mathfrak{P}\, U(\mathcal{O}_K).$$

Then the second assertion follows from the definition (4.3) of e.

Hereafter, let f be a primitive form of conductor C in $\mathcal{M}_k(\Gamma_0(N),\psi;K_0)$. Thus N is a multiple of C. Put, for each integer $n\geqq 1$,

$$U(Np^n,f;K)=\{g\in\mathcal{M}_k(\Gamma_1(Np^n);K)\,|\,g|T(l)=a(l,f)g \text{ except for finitely many primes } l\}.$$

Define non-negative integers t, r, N' and C' by

$$N=N'p^r \quad\text{and}\quad C=C'p^t,$$

where $(N',p)=(C',p)=1$.

Proposition 4.4. *Assume that $k\geqq 2$ and $|a(p,f)|_p=1$. Let f_0 be the ordinary form associated with f defined in Lemma 3.3. Then we have, for every $n\geqq 1$,*

$$e\, U(Np^n,f;K)=\sum_{0<t|N'/C'} K f_0(tz).$$

Before proving this result, let us give some remarks. Firstly, the ordinary part $e\,U(Np^n,f;K)$ does not depend on the integer n, and we have

(4.4a) $\qquad \dim_K e\,U(Np^n,f;K)=1$ *if and only if* $C'=N'$.

Secondly, let P be the set of all primitive forms in $\mathcal{M}_k(N,\psi;\Omega)$ whose p-th Fourier coefficients are units in Ω. Then P is a finite set, and thus we may assume that $P\subset\mathcal{M}_k(N,\psi;K)$ by replacing K by its finite extension if necessary. By the theory of primitive forms, Propositions 4.1 and 4.4 show

$$\text{(4.4b)}\qquad \bar{\mathcal{M}}_k^{\circ}(N,\psi;K)=\sum_{g\in P} e\,U(Np,g;K).$$

Thirdly, let P_0 be the subset of P consisting of all elements with the property (4.4a), and put

$$U_0=\sum_{g\in P_0} K g_0,$$

where g_0 is the ordinary form associated with g. Write $\mathcal{H}_0$ for the subalgebra of the ring of all K-linear endomorphisms of U_0 generated over K by all Hecke operators $T(l)$ and $T(l,l)$. Then

(4.4c) *$\mathcal{H}_0$ is a semi-simple algebra over K.*

In view of these properties, one may regard the ordinary forms g_0 for $g\in P_0$ as an analogue of primitive forms of conductor N in the theory of ordinary forms.

Now we shall prove Proposition 4.4. Let us take a basis $\{f_m\}_{m=0,\dots,j}$ of $U(Cp^j,f)$ as in (3.2a, b). For $j=n+r-t$, we see $Np^n/Cp^j=N/Cp^{r-t}=N'/C'$. We see then from [17] that

$$U(Np^n,f;K)=\sum_{m=0}^{j}\sum_{0<t|N'/C'} K f_m(tz).$$

Since t is prime to p, the operation: $g(z)\mapsto g(tz)$ commutes with the Hecke operator $T(p)$, and hence, with the idempotent e. This shows that

$$e\,U(Np^n,f;K)=\sum_{0<t|N'/C'} K f_0(tz),$$

since f_0 is a unique ordinary form in $U(Cp^j,f)$ and f_m with $m\geqq 1$ is annihilated by e.

Let f be a primitive form of conductor C, of weight $k\geqq 2$ and with character ψ. Assume that $|a(p,f)|_p=1$. Now we are ready to define a continuous linear form ℓ_f (attached to f) on $\bar{\mathcal{M}}_k(C,\psi;K)$ into K. Let f_0 be the ordinary form associated with f and let C_0 be the exact level of f_0. By Proposition 4.4 (or more precisely by (4.4c)), the natural ring homomorphism of $\mathcal{H}_k(\Gamma_0(C_0),\psi;K)$ onto K, which assign $a(n,f_0)$ to $T(n)$, is split, and thus, there is a simple direct summand of the Hecke algebra $\mathcal{H}_k(\Gamma_0(C_0),\psi;K)$, isomorphic to K, through which this morphism factors. Let A be the subalgebra of this Hecke algebra which is the complementary direct summand. Namely, we have the algebra direct sum decomposition:

$$\text{(4.5)}\qquad \mathcal{H}_k(\Gamma_0(C_0),\psi;K)\cong A\oplus K.$$

Let 1_f be the idempotent corresponding to the direct summand K of (4.5). Note that the idempotent e sends $\bar{\mathcal{M}}_k(C_0,\psi;K)$ into $\mathcal{M}_k(\Gamma_0(C_0),\psi;K)$ by

Proposition 4.1. Then the linear form ℓ_f: $\overline{\mathcal{M}}_k(C_0, \psi; K) \to K$ is defined by

(4.6) $$\ell_f(g) = a(1, g|e\,1_f) \quad \textit{for } g \in \overline{\mathcal{M}}_k(C_0, \psi; K),$$

where $a(1, g|e\,1_f)$ is the first q-expansion coefficient of $g|e\,1_f$.

Proposition 4.5. *Assume that K_0 contains all the Fourier coefficients of the ordinary form f_0. Then, the linear form ℓ_f has values in the finite algebraic number field K_0 on $\mathcal{M}_k(\Gamma_0(C_0 p^n), \psi; K_0)$ for every $n \geqq 0$. Furthermore, we have*

$$\ell_f(g) = a(p, f_0)^{-n} p^{n(k-1)} \frac{\langle h_n, g\rangle_{C p^n}}{\langle h, f_0\rangle_{C_0}} \quad (g \in \mathcal{M}_k(\Gamma_0(C_0 p^n), \psi; K_0)),$$

where $h = f_0^\rho|_k \begin{pmatrix} 0 & -1 \\ C_0 & 0 \end{pmatrix}$ and $h_n(z) = h(p^n z)$ for the complex conjugation ρ.

Proof. First we shall deal with the case: $n=0$. We know that $\mathcal{H}_k(\Gamma_0(C_0), \psi; K) = \mathcal{H}_k(\Gamma_0(C_0), \psi; K_0) \otimes_{K_0} K$. Since K_0 contains the eigenvalues for f_0 of all Hecke operators, the decomposition (4.5) is induced from the similar decomposition:

$$\mathcal{H}_k(\Gamma_0(C_0), \psi; K_0) \cong A_0 \oplus K_0 \quad \text{(algebra direct sum)}.$$

Thus, by definition, the linear form ℓ_f has values in K_0 on $\mathcal{M}_k(\Gamma_0(C_0), \psi; K_0)$. Now we consider the general case: $n > 0$. As explained in the proof of Lemma 3.2, the operator $T(p)^n$ takes $\mathcal{M}_k(\Gamma_0(C_0 p^n), \psi; K_0)$ into $\mathcal{M}_k(\Gamma_0(C_0), \psi; K_0)$. By the definition of 1_f, it commutes with $T(p)$ and e. Thus, we have

$$g|T(p)^n e\,1_f = g|e\,1_f\,T(p)^n = a(p, f_0)^n g|e\,1_f \quad (g \in \mathcal{M}_k(\Gamma_0(C_0 p^n), \psi; K_0)).$$

This shows that

(4.7) $$\ell_f(g) = a(p, f_0)^{-n} \ell_f(g|T(p)^n) \in K_0 \quad (g \in \mathcal{M}_k(\Gamma_0(C_0 p^n), \psi; K_0)).$$

Next, we shall show that

$$\ell_f(g) = \langle h, g\rangle_{C_0} / \langle h, f_0\rangle_{C_0} \quad \text{for } g \in \mathcal{M}_k(\Gamma_0(C_0), \psi; K_0).$$

We can naturally extend ℓ_f to a linear form of $\mathcal{M}_k(\Gamma_0(C_0), \psi)$ with values in $\mathbb{C}$ so that it coincides with the original one on K_0-rational modular forms. We denote it by the same symbol. Since Eisenstein series are contained in the kernel of ℓ_f, we can find an element h' in $\mathcal{S}_k(\Gamma_0(C_0), \psi)$ so that

$$\langle h', g\rangle_{C_0} = \ell_f(g) \quad \text{for all } g \in \mathcal{M}_k(\Gamma_0(C_0), \psi).$$

For each primitive form $\mathscr{g}$ in $\mathcal{S}_k(\Gamma_0(C_0), \psi)$, let

$$U(\mathscr{g}) = \{g \in \mathcal{S}_k(\Gamma_0(C_0), \psi) \mid g|T(l) = a(l, \mathscr{g})g \text{ except for finitely many primes } l\}.$$

Then, we have the well known orthogonal decomposition under $\langle\ ,\ \rangle$:

$$\mathcal{S}_k(\Gamma_0(C_0), \psi) = \bigoplus_{\mathscr{g}} U(\mathscr{g}).$$

Therefore, h' must be in $U(f)$, since ℓ_f annihilates $U(g)$ for $g \neq f$. Put $\tau = \begin{pmatrix} 0 & -1 \\ C_0 & 0 \end{pmatrix}$ and let $T^*(m)$ be the adjoint operator of $T(m)$ under $\langle\ ,\ \rangle$. Then, it is known (cf. [28, Chap. 3] and [7, Th. 4.5.5]) that, as operators on $\mathcal{S}_k(\Gamma_0(C_0), \psi)$,

$$\tau^2 = (-1)^k, \qquad T^*(m) = \tau^{-1} \circ T(m) \circ \tau \qquad \text{for all } m > 0,$$

and if m is prime to C_0,

$$T^*(m) = \overline{\psi(m)}\, T(m).$$

Especially, f^ρ is an eigenform of $T^*(m)$ with eigenvalue $a(m, f)$ if m is prime to C_0. Thus, we see that, for m prime to C_0,

$$h \mid T(m) = (f_0^\rho \mid T^*(m))|_k \tau = a(m, f)\, h.$$

This shows that $0 \neq h \in U(f)$. When C is divisible by p, then $C_0 = C$, $f_0 = f$ and $U(f) = \mathbb{C} f$. Then, it is obvious that

$$h' = h / \overline{\langle h, f_0 \rangle}.$$

We now assume that C is prime to p. Take a basis $\{f_0, f_1\}$ of $U(f)$ as in (3.2a). Then, we know that $f_0 \mid T(p) = \beta f_0$ and $f_1 \mid T(p) = \gamma f_1$ for the elements β and γ of K_0 with $|\beta|_p = 1$ and $|\gamma|_p < 1$. Thus $\ell_f(f_0) = 1$ and $\ell_f(f_1) = 0$. Then, in order to see $h' = h / \overline{\langle h, f_0 \rangle}$, what we have to show is the vanishing:

$$\langle h, f_1 \rangle = 0.$$

Since $\beta \neq \gamma$, this is a consequence of the following equality:

$$\begin{aligned} \gamma \langle h, f_1 \rangle &= \langle h, f_1 \mid T(p) \rangle = \langle h \mid T^*(p), f_1 \rangle \\ &= \langle (f_0^\rho \mid T(p))|_k \tau, f_1 \rangle = \langle \beta^\rho h, f_1 \rangle = \beta \langle h, f_1 \rangle. \end{aligned}$$

This shows the last assertion for $n = 0$. For $g \in \mathcal{M}_k(\Gamma_0(C_0 p^n), \psi)$ with each $n > 0$, we know $g \mid T(p)^n \in \mathcal{M}_k(\Gamma_0(C_0), \psi)$, and we have

$$\begin{aligned} a(p^n, f_0) \langle h, f_0 \rangle_{C_0} \ell_f(g) &= \langle h, f_0 \rangle_{C_0} \ell_f(g \mid T(p)^n) \quad \text{by (4.7)} \\ &= \langle h, g \mid T(p)^n \rangle_{C_0} \\ &= \left\langle h, g \,\middle|\, \left[\Gamma_0(C_0 p^n) \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} \Gamma_0(C_0) \right] \right\rangle_{C_0} \\ &= \left\langle h \,\middle|\, \left[\Gamma_0(C_0) \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(C_0 p^n) \right], g \right\rangle_{C_0 p^n} \quad \text{by [28, (3.4.5)].} \end{aligned}$$

Since $\Gamma_0(C_0) \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(C_0 p^n) = \Gamma_0(C_0) \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix}$, we have

$$h \left| \left[\Gamma_0(C_0) \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(C_0 p^n) \right] \right. = p^{n(k-1)} h(p^n z).$$

This shows the last assertion for every $n \geqq 0$.

§5. Differential operators

In this section, we recall some of Shimura's results on differential operators on $\mathfrak{H}$, and prove several additional facts. Define the differential operators on $\mathfrak{H}$ by

$$\delta_s = \frac{1}{2\pi\sqrt{-1}}\left(\frac{s}{2\sqrt{-1}\,y} + \frac{\partial}{\partial z}\right),$$

$$d = \frac{1}{2\pi\sqrt{-1}}\frac{\partial}{\partial z} = q\frac{d}{dq} \qquad (q = e(z),\ z = x + \sqrt{-1}\,y),$$

$$\delta_s^r = \delta_{s+2r-2} \cdots \delta_{s+2}\,\delta_s \qquad \text{for } 0 \leqq r \in \mathbb{Z},$$

where we understand that $\delta_s^0 = 1$ is the identity operator. These operators satisfy

(5.1) $$\delta_{s+t}(fg) = g\,\delta_s(f) + f\,\delta_t(g) \quad \text{and} \quad \delta_k^r(f|_k\gamma) = (\delta_k^r f)|_{k+2r}\gamma$$

for $\gamma \in GL_2^+(\mathbb{R})$ and every positive integer k [23, (1.5), (1.8)]. The relation between δ and d is given in [27, (1.16a, b)] as

(5.2) $$\delta_s^r = \sum_{0 \leqq t \leqq r} \binom{r}{t} \frac{\Gamma(s+r)}{\Gamma(s+t)} (-4\pi y)^{t-r} d^t.$$

Let K_0 be a subfield of $\bar{\mathbb{Q}}$ and l and m be positive integers. Let $g \in \mathcal{M}_l(\Gamma_0(N), \xi; K_0)$ and $h \in \mathcal{M}_m(\Gamma_0(N), \chi; K_0)$. Then we have

(5.3) $$g\,\delta_m^r h = \sum_{n=0}^{r} \delta_{k-2n}^n g_n \quad \textit{with elements } g_n \textit{ of } \mathcal{M}_{k-2n}(\Gamma_0(N), \xi\chi; K_0)$$

for $k = l + m + 2r$.

These modular forms g_n are uniquely determined by g and h. This fact is shown in [25, Lemma 7]. We write $H(g\,\delta_m^r h)$ for g_0 in (5.3), and call it the holomorphic projection of $g\,\delta_m^r h$. This terminology is justified by the property given in [25, Lemma 6] (see also [27, Lemma 2.3]):

(5.4) $$\langle f, g\,\delta_m^r h\rangle_N = \langle f, H(g\,\delta_m^r h)\rangle_N \quad \textit{for every element } f \textit{ of } \mathcal{S}_k(\Gamma_0(N), \xi\chi).$$

Here the Petersson inner product $\langle f, g\,\delta_m^r h\rangle_N$ is defined as usual, since $g\,\delta_m^r h$ transforms under $\Gamma_0(N)$ as if it were an element of $\mathcal{M}_k(\Gamma_0(N), \xi\chi)$.

Lemma 5.1. *Let* $\mathcal{O}_{K_0} = \{x \in K_0 \,|\, |x|_p \leqq 1\}$, *and suppose that* $g \in \mathcal{M}_l(N; \mathcal{O}_{K_0})$ *and* $h \in \mathcal{M}_m(N; \mathcal{O}_{K_0})$. *Then, we can find a positive integer* C *independently of* g *and* h *such that*

$$CH(g\,\delta_m^r h) \in \mathcal{M}_k(N; \mathcal{O}_{K_0}) \qquad (k = l + m + 2r).$$

The integer C *depends only on* l, m *and* r.

Proof. By applying (5.2) to the equality (5.3), we have

$$\sum_{0\leqq t\leqq r}\binom{r}{t}\frac{\Gamma(m+r)}{\Gamma(m+t)}(g\,d^t h)(-4\pi y)^{t-r}$$
$$=\sum_{n=0}^{r}\sum_{0\leqq t\leqq n}\binom{n}{t}\frac{\Gamma(k-n)}{\Gamma(k-2n+t)}(d^t g_n)(-4\pi y)^{t-n}.$$

We consider this to be an equality of polynomials in the variable $(-4\pi y)^{-1}$. By comparing the coefficients of $(-4\pi y)^{-t}$ for each $0\leqq t\leqq r$, we have

$$\binom{r}{t}\frac{\Gamma(m+r)}{\Gamma(m+t)}g\,d^t h=\sum_{n=r-t}^{r}\binom{n}{t-r+n}\frac{\Gamma(k-n)}{\Gamma(k-n+t-r)}d^{t-r+n}g_n. \tag{5.5}$$

When $t=0$, we see from (5.5) that $gh=\dfrac{\Gamma(m)\Gamma(k-r)}{\Gamma(m+r)\Gamma(k-2r)}g_r$. Define C_r to be the numerator of $\dfrac{\Gamma(m)\Gamma(k-r)}{\Gamma(m+r)\Gamma(k-2r)}$. Then $C_r g_r$ has Fourier coefficients in $\mathcal{O}_{K_0}$ whenever g and h have their Fourier coefficients in $\mathcal{O}_{K_0}$. Now, let j be an integer with $0\leqq j\leqq r$, and assume that there are positive integers C_n for $j<n\leqq r$ such that $C_n g_n$ has Fourier coefficients in $\mathcal{O}_{K_0}$ whenever g and h do. Then we see from (5.5) for $t=r-j$ that

$$\frac{\Gamma(k-j)}{\Gamma(k-2j)}g_j=\binom{r}{r-j}\frac{\Gamma(m+r)}{\Gamma(m+r-j)}g\,d^{r-j}h-\sum_{n=j+1}^{r}\binom{n}{n-j}\frac{\Gamma(k-n)}{\Gamma(k-n-j)}d^{n-j}g_n.$$

Since $C_n d^{n-j} g_n$ for every $n>j$ has coefficients in $\mathcal{O}_{K_0}$, we can find a positive integer C_j so that $C_j g_j$ has $\mathcal{O}_{K_0}$-integral Fourier coefficients whenever g and h are $\mathcal{O}_{K_0}$-integral. Thus, by induction on j, we obtain the lemma.

Lemma 5.2. *Suppose that $g\in\mathcal{M}_l(\Gamma_1(N);K_0)$ and $h\in\mathcal{M}_m(\Gamma_1(N);K_0)$, and define $g_n\in\mathcal{M}_{k-2n}(\Gamma_1(N);K_0)$ $(0\leqq n\leqq r,\ k=l+m+2r)$ for a positive integer r by (5.3). Put $g'=-\sum_{n=0}^{r-1}d^n g_{n+1}$. Then, the p-adic norm $|a(n,g')|_p$ of the Fourier coefficients of g' for all n is bounded, and we have that*

$$H(g\,\delta_m^r h)=g\,d^r h+dg'.$$

Moreover, $H(g\,\delta_m^r h)$ is a cusp form if $r>0$.

Proof. We see from (5.5) for $t=r$ that

$$g\,d^r h=H(g\,\delta_m^r h)+\sum_{n=1}^{r}d^n g_n=H(g\,\delta_m^r h)-dg'. \tag{5.6}$$

Since g_n is a K_0-rational modular form by (5.3), the norm $|g'|_p$ is a well defined real number; namely, $|a(n,g')|_p$ is bounded. For an arbitrary $\gamma\in SL_2(\mathbb{Q})$, by substituting $g|_l\gamma$ and $h|_m\gamma$ for g and h in (5.3), we see easily from (5.1) and (5.6) that

$$H(g\,\delta_m^r h)|_k\gamma=H[(g|_l\gamma)\,\delta_m^r(h|_m\gamma)]=(g|_l\gamma)d^r(h|_m\gamma)-\sum_{n-1}^{r}d^n(g_n|_{k-2n}\gamma).$$

This vanishes at $i\infty$, and therefore $H(g\,\delta_m^r h)$ is a cusp form when $r>0$.

Lemma 5.3. *For arbitrary elements g of $\mathcal{M}_l(\Gamma_1(N))$ and h of $\mathcal{M}_m(\Gamma_1(N))$, we have*

$$H(g\,\delta_m^r h) = (-1)^r H(h\,\delta_l^r g).$$

Proof. The assertion is trivially true for $r=0$; thus, we assume that $r>0$. Then $H(g\,\delta_m^r h)$ and $H(h\,\delta_l^r g)$ are cusp forms. For any C^∞-functions f and f' with the same automorphic property as elements of $\mathcal{M}_k(\Gamma_1(N))$, put

$$\langle f, f'\rangle = \int_{\mathfrak{H}/\Gamma_1(N)} \overline{f(z)}\, f'(z)\, y^{k-2}\, dx\, dy,$$

if it is well defined. Let ϕ and ψ be arbitrary C^∞-functions on $\mathfrak{H}$ satisfying $\phi|_i\gamma = \phi$ and $\psi|_j\gamma=\psi$ for all $\gamma\in\Gamma_1(N)$. If ϕ and ψ are slowly increasing in the sense of [27, (2.17)], then $\langle f, \phi\,\delta_j\psi\rangle$, $\langle f, \psi\,\delta_i\phi\rangle$ and $\langle f, \delta_{i+j}(\phi\psi)\rangle$ for every f of $\mathcal{S}_{i+j+2}(\Gamma_1(N))$ are finite. Especially, $\langle f, \delta_{i+j}(\phi\psi)\rangle$ vanishes by [27, Lemma 2.3]. In addition to this, we see from (5.1) that $\delta_{i+j}(\phi\psi) = \phi\,\delta_j\psi + \psi\,\delta_i\phi$. Then, we have

$$\text{(5.7)} \qquad \langle f, \phi\,\delta_j\psi\rangle = -\langle f, \psi\,\delta_i\phi\rangle \qquad \text{for every } f\in\mathcal{S}_{i+j+2}(\Gamma_1(N)).$$

Substituting $\delta_l^{r-n} g$ and $\delta_m^{n-1} h$ for ϕ and ψ in (5.7), we have

$$\langle f, (\delta_l^{r-n} g)(\delta_m^n h)\rangle = -\langle f, (\delta_l^{r-n+1} g)(\delta_m^{n-1} h)\rangle.$$

Then, by induction on n, we know

$$\langle f, g\,\delta_m^r h\rangle = (-1)^r \langle f, h\,\delta_l^r g\rangle.$$

Then (5.4) shows that, for all $f\in\mathcal{S}_k(\Gamma_1(N))$ $(k=l+m+2r)$,

$$\langle f, H(g\,\delta_m^r h)\rangle = (-1)^r \langle f, H(h\,\delta_l^r g)\rangle.$$

Since $H(g\,\delta_m^r h)$ and $H(h\,\delta_l^r g)$ are cusp forms, the non-degeneracy of the Petersson inner product on $\mathcal{S}_k(\Gamma_1(N))$ shows the lemma.

§ 6. Bounded measures with values in p-adic modular forms

Firstly, we recall the theory of bounded measures according to Mazur and Swinnerton-Dyer [16]. Let A be a closed subring of Ω. Let $\mathcal{M}$ be an A-module complete under a norm $|\ |_{\mathcal{M}}$ with the following properties: $|x|_{\mathcal{M}}=0$ if and only if $x=0$ $(x\in\mathcal{M})$; $|ax|_{\mathcal{M}} = |a|_p |x|_{\mathcal{M}}$ for $a\in A$ and $x\in\mathcal{M}$; $|x+y|_{\mathcal{M}} \leqq \max(|x|_{\mathcal{M}}, |y|_{\mathcal{M}})$. For our later use, A will be a finite extension of $\mathbb{Q}_p$ and $\mathcal{M}$ will be the space $\overline{\mathcal{M}}(N; A)$ of p-adic modular forms. Let T be a projective limit of finite discrete sets T_n. Let $\mathcal{C}(T; A)$ be the space of all continuous functions on T with values in A. We can define a norm $\|\phi\|$ of $\phi\in\mathcal{C}(T; A)$ by

$$\|\phi\| = \operatorname*{Sup}_{t\in T} |\phi(t)|_p.$$

Then $\mathcal{C}(T; A)$ becomes a complete normed A-module [2, X.1.6, X.3.3]. A linear functional ψ on $\mathcal{C}(T; A)$ into $\mathcal{M}$ is called a bounded measure on T with values in $\mathcal{M}$ if there is a positive constant B such that $|\psi(\phi)|_{\mathcal{M}} \leqq B\|\phi\|$ for all $\phi \in \mathscr{C}(T; A)$. Usually, the value $\psi(\phi)$ is written as $\int_T \phi\, d\psi$. For a point $t \in T_n$, let $\chi_{n,t}$ be the pull back to T of the characteristic function of the one point subset $\{t\}$ of T_n. Put $\psi_n(t) = \int_T \chi_{n,t}\, d\psi$. Since locally constant functions are dense in $\mathscr{C}(T; A)$, the measure ψ is uniquely determined by the system $\{\psi_n(t)\}_{n \geqq u, t \in T_n}$ for any given integer u. This system satisfies

(6.1) $$\sum_{\pi_{ij}(t)=s} \psi_i(t) = \psi_j(s) \quad \textit{for any } i \geqq j \geqq u \textit{ and every } s \in T_j,$$

where π_{ij} is the projection of T_i onto T_j. Conversely, if a system $\{\psi_n(t)\}_{n \geqq u, t \in T_n}$ satisfying (6.1) is given and if the norm $|\psi_n(t)|_{\mathcal{M}}$ is bounded independently of n and $t \in T_n$, then this system comes from a unique bounded measure on T. For a given measure ψ on T and any continuous function Φ in $\mathscr{C}(T; A)$, the product measure $\Phi \cdot \psi$ (or occasionally written as $\Phi\, d\psi$) of Φ and ψ is defined by

(6.2) $$(\Phi \cdot \psi)(\phi) = \psi(\Phi \phi) = \int_T \phi\, \Phi\, d\psi \quad \textit{for any } \phi \in \mathscr{C}(T; A).$$

Let M be an arbitrary positive integer and put

$$Z_\nu = (\mathbb{Z}/M p^\nu \mathbb{Z})^\times \quad \text{and} \quad Z = \varprojlim_\nu Z_\nu.$$

We shall introduce the Eisenstein measure on the space Z. Let us define an Eisenstein series for each $a \in Z_\nu$ by giving its Fourier expansion: for each positive integer m,

(6.3) $$E_{m,\nu}(a) = \zeta(1-m; a, M p^\nu) + \sum_{n=1}^{\infty} \Big(\sum_{\substack{d \mid n \\ d \equiv a \bmod M p^\nu}} \operatorname{sgn}(d)\, d^{m-1} \Big) e(nz),$$

where $\zeta(s; a, M p^\nu) = \sum_{0 < n \equiv a \bmod M p^\nu} n^{-s}$ is the partial zeta function modulo $M p^\nu$. It is known by Hecke [8] that the series of (6.3) belongs to $\mathcal{M}_m(\Gamma_1(M p^\nu); \mathbb{Q})$ if $M p^\nu > 2$. The system $\{E_{m,\nu}(a)\}_{\nu \geqq 2, a \in Z_\nu}$ for each m satisfies the condition (6.1), but their norms in $\mathcal{M}(M; \mathbb{Q}_p)$ are unbounded. For each integer $b > 1$ prime to $M p$, put

(6.4a) $$E^b_{m,\nu}(a) = E_{m,\nu}(a) - b^m E_{m,\nu}(b^{-1} a),$$

where we take the inverse b^{-1} in Z_ν considering b to be an element of Z_ν naturally. Define another system $\{\varepsilon^b_{m,\nu}(a)\}_{\nu \geqq 1, a \in Z_\nu}$ by

(6.4b) $$\varepsilon^b_{m,\nu}(a) = \zeta(1-m; a, M p^\nu) - b^m \zeta(1-m; b^{-1} a, M p^\nu) \in \mathbb{Q}.$$

Then, it is well known that $|\varepsilon^b_{m,\nu}(a)|_p$ is bounded independently of $a \in Z_\nu$ and ν (e.g. [13, Chap. 2]). Thus the system (6.4a, b) for each positive integer m gives bounded measures on Z with values in $\overline{\mathcal{M}}(N; \mathbb{Q}_p)$ and $\mathbb{Q}_p$, respectively. We will denote these measures by E^b_m and ε^b_m. The measures ε^b_m are related to the

Kubota-Leopoldt p-adic L-functions (cf. [9] and [13]), and the measure E_1^b is a one-dimensional part of the Eisenstein measure introduced in [11, 12].

Hereafter in this section, we return to the situation of Theorem 2.1. Especially, M denotes the level of the fixed lattice I of the quadratic space V. Recall $\mathcal{W}=\{v\in I^* \mid n(v)\in\mathbb{Z}\}$, $W_\nu=\mathcal{W}/p^\nu I$ for each positive integer ν and $W=\varprojlim_\nu W_\nu$. We shall now define a measure associated with the quadratic form n on W. Take a spherical function η: $V\to\bar{\mathbb{Q}}$ of degree α with algebraic values on V. By composing η with the embedding i of $\bar{\mathbb{Q}}$ into Ω fixed in (0.3), we extend it by continuity to a function on W into Ω. We denote the extension again by η. Fix a finite extension K of $\mathbb{Q}_p$ so that η has values in K. For each $w\in W_\nu$, put

$$\theta_\nu(w,\eta)=\sum_{\substack{v\in\mathcal{W}\\ v\equiv w \bmod p^\nu I}}\eta(v)\,e(n(v)\,z)\in\mathcal{M}_{\kappa+\alpha}(\Gamma_1(Mp^{2\nu});K). \tag{6.5}$$

Then, the system $\{\theta_\nu(w,\eta)\}_{\nu\geqq 0,\, w\in W_\nu}$ defines a measure on W with values in the K-Banach space $\bar{\mathcal{M}}(M;K)$. When η is the constant function with value 1 on V, this measure will be called the theta measure attached to the quadratic space V, and will be denoted by θ or $d\theta$. For any continuous function $\phi\in\mathscr{C}(W;K)$, the value $\theta(\phi)=\int_W \phi\,d\theta$ has the following q-expansion:

$$\theta(\phi)=\int_W \phi\,d\theta=\sum_{w\in\mathcal{W}}\phi(w)\,q^{n(w)}\in\bar{\mathcal{M}}(M;K).$$

Then, it is obvious that the product measure $\eta\cdot d\theta$ for the general spherical function η gives the measure attached to the system (6.5).

Next, we shall construct another bounded measure on W, which may be regarded as a convolution product of the theta measure and the Eisenstein measure. Write the level M of I as $M'p^\lambda$ with a positive integer M' prime to p, and let ω be a Dirichlet character modulo Mp^u for some integer $u\geqq -\lambda$. By definition, $Z_\nu=(\mathbb{Z}/Mp^\nu\mathbb{Z})^\times$ naturally acts on $I^*/p^\nu I$, and the subset W_ν of $I^*/p^\nu I$ is stable under the action of Z_ν. Thus, we can consider $\theta_\nu(aw,\eta)$ for $w\in W_\nu$ and $a\in Z_\nu$. For each non-negative integer r and each positive integer m, we define a system $\{\Phi_\nu(w)\}_{\nu,\,w\in W_\nu}$ by

$$\begin{aligned}\Phi_\nu(w)&=\Phi_\nu(w;r,m,\omega,\eta)\\ &=\sum_{a\in Z_\nu}\omega(a)\,H[\theta_\nu(aw,\eta)\,\delta_m^r\,E_{m,\nu}^b(a)]\in\mathcal{M}_k(\Gamma_1(Mp^{2\nu});K),\end{aligned} \tag{6.6}$$

where $k=\kappa+\alpha+m+2r$, δ_m^r is Shimura's differential operator defined in § 5 and H denotes the holomorphic projection map. We have to assume that $\nu\geqq 2$ and $\nu\geqq u$ in (6.6). By Lemma 5.1, we know

(6.7) $|\Phi_\nu(w)|_p$ *is bounded independently of* ν *and* $w\in W_\nu$.

In order to show that the system $\{\Phi_\nu(w)\}$ comes from a bounded measure, we have to check the condition (6.1). The calculation may be done as follows: for any $i\geqq j\geqq \max(u,2)$,

$$\sum_{\substack{w\in W_i\\ w\equiv x \bmod p^j I}} \Phi_i(w) = \sum_{a\in Z_i} \omega(a)\, H[(\sum_{\substack{w\in W_i\\ w\equiv x \bmod p^j I}} \theta_i(a\,w,\eta))\,\delta_m^r E_{m,i}^b(a)]$$

$$= \sum_{a\in Z_i} \omega(a)\, H[\theta_j(a\,x,\eta)\,\delta_m^r E_{m,i}^b(a)]$$

$$= \sum_{a\in Z_j} \omega(a)\, H[\theta_j(a\,x,\eta)\,\delta_m^r (\sum_{\substack{c\in Z_i\\ c\equiv a \bmod M p^j}} E_{m,j}^b(c))]$$

$$= \sum_{a\in Z_j} \omega(a)\, H[\theta_j(a\,x,\eta)\,\delta_m^r E_{m,j}^b(a)] = \Phi_j(x).$$

Let us denote by $\Phi = \Phi(r, m, \omega, \eta)$ for the measure defined by (6.6).

Lemma 6.1. *Let γ be an element of $\Gamma_0(M p^\nu)$ with $\gamma \equiv \begin{pmatrix} t & * \\ 0 & t^{-1}\end{pmatrix} \bmod M p^\nu$. Then, we have*

$$E_{m,\nu}(a)|_m\, \gamma = E_{m,\nu}(a\,t) \qquad \text{if } M p^\nu > 2.$$

Proof. Let χ be a Dirichlet character modulo $M p^\nu$ with $\chi(-1) = (-1)^m$, and put

$$E(\chi) = \tfrac{1}{2} L(1-m, \chi) + \sum_{n=1}^{\infty} (\sum_{0<d|n} \chi(d)\, d^{m-1})\, e(n\,z).$$

It is well known that $E(\chi)$ belongs to $\mathcal{M}_m(\Gamma_0(M p^\nu), \chi)$ if $M p^\nu > 2$ (cf. [8] and [25, (3.4)]). The explicit q-expansion (6.3) of $E_{m,\nu}(a)$ shows that

$$E_{m,\nu}(a) = 2\,|Z_\nu|^{-1} \sum_{\chi} \bar{\chi}(a)\, E(\chi),$$

where χ runs over all Dirichlet character modulo $M p^\nu$ with $\chi(-1) = (-1)^m$ and $|Z_\nu|$ denotes the number of elements in Z_ν. Then, we know

$$E_{m,\nu}(a)|_m\, \gamma = 2\,|Z_\nu|^{-1} \sum_{\chi} \bar{\chi}(a\,t)\, E(\chi) = E_{m,\nu}(a\,t).$$

This shows the lemma.

We know from Proposition 1.1 that, for $\gamma \in \Gamma_0(M p^{2\nu})$ with $\gamma \equiv \begin{pmatrix} t & * \\ 0 & t^{-1}\end{pmatrix} \bmod M p^{2\nu}$,

$$\theta_\nu(w, \eta)|_{\kappa+\alpha}\, \gamma = \chi_0(t)\, \theta_\nu(t\,w, \eta),$$

where $\chi_0(t) = \left(\dfrac{(-1)^\kappa \Delta}{t}\right)$ for $\Delta = [I^* : I]$ is the Legendre symbol. Then, Lemma 6.1 shows that $\Phi_\nu(w)$ belongs to $\mathcal{M}_k(\Gamma_0(M p^{2\nu}), \omega\chi_0; K)$ for $k = \kappa + \alpha + m + 2r$. Thus

(6.8) *$\Phi(r, m, \omega, \eta)$ has values in $\bar{\mathcal{M}}_k(M, \omega\chi_0; K)$ for $k = \kappa + \alpha + m + 2r$.*

We shall clarify possible relations between the measures Φ for various r, m, ω and η.

Proposition 6.2. *Let k be a positive integer greater than κ, and assume that the degree α of the spherical function η is less than $k - \kappa$. Then, we have*

$$\Phi(0, k-\kappa-\alpha, \omega, \eta) = \eta \cdot \Phi(0, k-\kappa, \omega, 1),$$

where we denote by the symbol "1" the constant function with value 1 on W.

Proof. Note that the two measures $\Phi(0, k-\kappa-\alpha, \omega, \eta)$ and $\Phi(0, k-\kappa, \omega, 1)$ have values in the same space $\bar{\mathcal{M}}_k(M, \omega\chi_0; K)$. Thus it is sufficient to show that

(6.9) $|\Phi_\nu(w; 0, k-\kappa-\alpha, \omega, \eta)-\eta(w)\Phi_\nu(w; 0, k-\kappa, \omega, 1)|_p$ *is convergent to* 0 *uniformly in* $w\in W$ *as* ν *approaches to the infinity.*

By definition, we have

$$E^b_{m,\nu}(a)=\varepsilon^b_{m,\nu}(a)+\sum_{n=1}^{\infty}\Big(\sum_{\substack{d|n\\ d\equiv a \bmod Mp^\nu}} \operatorname{sgn}(d)\, d^{m-1}-b\sum_{\substack{d|n\\ bd\equiv a \bmod Mp^\nu}} \operatorname{sgn}(bd)(bd)^{m-1}\Big)\, e(nz).$$

As for the constant term $\varepsilon^b_{m,\nu}(a)$, we have that, for every $a\in Z$,

$$|\varepsilon^b_{m,\nu}(a)-a_p^{m-1}\,\varepsilon^b_{1,\nu}(a)|_p\leqq p^{-\nu},$$

where $a_p\in\mathbb{Z}_p^\times$ is the projection of $a\in Z=(\mathbb{Z}/M'\mathbb{Z})^\times\times\mathbb{Z}_p^\times$ into the second factor. A similar inequality can be verified more easily for the non-constant terms of $E^b_{m,\nu}(a)$, and then, we have

$$|E^b_{m,\nu}(a)-a_p^{m-1}\,E^b_{1,\nu}(a)|_p\leqq p^{-\nu} \quad \text{for every } a\in Z.$$

Here, we use the norm $|\ |_p$ defined by $\left|\sum_{n=0}^{\infty} c(n)\, q^n\right|_p=\operatorname{Sup}_n |c(n)|_p$ for any element of $\mathcal{O}_K[[q]]$. Replacing η by its constant multiple if necessary, we may assume that η has values in the p-adic integer ring $\mathcal{O}_K$ of K. Note that $\eta(aw)=a^\alpha\eta(w)$ for any $a\in\mathbb{Z}$. Then, we have that, for any $w\in W$,

$$\text{(6.10)}\quad \left|\eta(w)\,\Phi_\nu(w; 0, k-\kappa, \omega, 1)-\sum_{\substack{a=1\\ (a,mp)=1}}^{Mp^\nu} a^{k-\kappa-\alpha-1}\,\omega(a)\,E^b_{1,\nu}(a)\,\eta(aw)\,\theta_\nu(aw, 1)\right|_p$$

$$=|\eta(w)\,\Phi_\nu(w; 0, k-\kappa, \omega, 1)-\eta(w)\sum_a a^{k-\kappa-1}\,\omega(a)\,E^b_{1,\nu}(a)\,\theta_\nu(aw, 1)|_p$$

$$=|\eta(w)\sum_a \omega(a)\,[E^b_{k-\kappa,\nu}(a)-a^{k-\kappa-1}\,E^b_{1,\nu}(a)]\,\theta_\nu(aw, 1)|_p$$

$$\leqq \operatorname{Sup}_a(|\eta(w)|_p\,|E^b_{k-\kappa,\nu}(a)-a^{k-\kappa-1}\,E^b_{1,\nu}(a)|_p\,|\theta_\nu(aw, 1)|_p)$$

$$\leqq p^{-\nu}.$$

Since η is a polynomial function, we can find a constant $C\geq 1$ so that if $v\equiv w \bmod p^\nu I_p$ for any $v, w\in W$, then

$$|\eta(v)-\eta(w)|_p\leqq C p^{-\nu}.$$

Then, the definition of $\theta_\nu(w, \eta)$ and $\theta_\nu(w, 1)$ in (6.5) shows that

$$|\theta_\nu(w, \eta)-\eta(w)\,\theta_\nu(w, 1)|_p\leqq C p^{-\nu} \quad \text{for every } w\in W.$$

Thus, we have that, for every $w\in W$,

$$(6.11)\quad \left|\Phi_\nu(w;0,k-\kappa-\alpha,\omega,\eta)-\sum_{\substack{a=1\\(a,Mp)=1}}^{Mp^\nu} a^{k-\kappa-\alpha-1}\omega(a)E^b_{1,\nu}(a)\eta(aw)\theta_\nu(aw,1)\right|_p$$
$$=|\Phi_\nu(w;0,k-\kappa-\alpha,\omega,\eta)-\sum_a \omega(a)E^b_{k-\kappa-\alpha,\nu}(a)\eta(aw)\theta_\nu(aw,1)$$
$$+\sum_a \omega(a)E^b_{k-\kappa-\alpha,\nu}(a)\eta(aw)\theta_\nu(aw,1)$$
$$-\sum_a a^{k-\kappa-\alpha-1}\omega(a)E^b_{1,\nu}(a)\eta(aw)\theta_\nu(aw,1)|_p$$
$$\leqq \operatorname{Sup}_a[|E^b_{k-\kappa-\alpha,\nu}(a)|_p\,|\theta_\nu(aw,\eta)-\eta(aw)\theta_\nu(aw,1)|_p,$$
$$|\eta(aw)\theta_\nu(aw,1)|_p\,|E^b_{k-\kappa-\alpha,\nu}(a)-a^{k-\kappa-\alpha-1}E^b_{1,\nu}(a)|_p]$$
$$\leqq Cp^{-\nu}.$$

Then, (6.9) follows from (6.10) and (6.11). Q.E.d.

For any $\phi\in\mathscr{C}(W;K)$, the value $\theta(\phi)=\sum_{w\in\mathscr{W}}\phi(w)q^{n(w)}$ is an element of $\bar{\mathscr{M}}(M;K)$. Then, it is plain that, for $0\leqq r\in\mathbb{Z}$,

$$d^r\theta(\phi)=\theta(n^r\phi),$$

where d is the differential operator $q\dfrac{d}{dq}$. It is known (e.g. [12, 5.8]) that the differential operator d takes $\bar{\mathscr{M}}(M;K)$ into itself. Extend the Hecke operator $T(p)$ to an operator on $K[[q]]$ by

$$\left(\sum_{n=0}^{\infty}a(n)q^n\right)\Bigg|T(p)=\sum_{n=0}^{\infty}a(np)q^n,$$

and put

$$\left|\sum_{n=0}^{\infty}a(n)q^n\right|_p=\operatorname{Sup}_n|a(n)|_p.$$

Then, we can define a valuation ring by

$$\mathscr{U}=\{F\in K[[q]]\,|\,|F|_p \text{ is finite}\}.$$

Then, $\mathscr{U}$ is stable under the differential operator d and the Hecke operator $T(p)$. We see easily that

$$(6.12)\qquad \lim_{m\to\infty}(dF)|T(p)^m=0 \quad \text{if } F\in\mathscr{U}.$$

Note that the space $\bar{\mathscr{M}}(M;K)$ may be regarded as a subspace of $\mathscr{U}$ through q-expansion. Then, the definition of the idempotent e of $\mathscr{H}(M;\mathcal{O}_K)$ in (4.3) shows that

(6.13) *The idempotent e can be naturally extended to an operator on $d\mathscr{U}+\bar{\mathscr{M}}(M;K)$ so that e annihilates $d\mathscr{U}$ $(\supset d\bar{\mathscr{M}}(M;K))$.*

We shall define the ordinary part $\Phi^0=\Phi^0(r,m,\omega,\eta)$ of the measure $\Phi(r,m,\omega,\eta)$ by

(6.14) $$\int_W \phi\, d\Phi^0(r, m, \omega, \eta) = e\Big[\int_W \phi\, d\Phi(r, m, \omega, \eta)\Big] \quad \text{for } \phi \in \mathscr{C}(W; K).$$

Then, the measure Φ^0 has values in the finite dimensional K-vector space $\bar{\mathscr{M}}_k^0(M, \omega\chi_0; K)$ (Proposition 4.1).

Proposition 6.3. *Let k and r be integers with $k > \kappa$ and $0 \leqq r < \frac{1}{2}(k-\kappa)$, and assume that the degree α of η is less than $k-\alpha-2r$. Then, we have*

$$\Phi^0(r, k-\kappa-\alpha-2r, \omega, \eta) = (-1)^r \eta\, n^r \cdot \Phi^0(0, k-\kappa, \omega, 1).$$

Proof. Put, for each $\phi \in \mathscr{C}(W; K)$,

$$\theta_\nu(w, \phi) = \sum_{\substack{v \equiv w \bmod p^\nu I \\ v \in \mathscr{W}}} \phi(w)\, q^{n(w)} \in \bar{\mathscr{M}}(M; K) \qquad (w \in W_\nu).$$

Then, we can define a bounded measure $\Phi(0, m, \omega, \phi)$ on W by the system

$$\Phi_\nu(w; 0, m, \omega, \phi) = \sum_{a \in Z_\nu} \omega(a)\, E^b_{m,\nu}(a)\, \theta_\nu(a w, \phi) \in \bar{\mathscr{M}}(M; K).$$

We can apply the argument which proves (6.9) to any homogenous polynomial ϕ on W in place of η there. Let us take $\eta\, n^r$ as ϕ. Then, in exactly the same manner as in the proof of (6.9), we obtain

(6.15) $$|\Phi_\nu(w; 0, k-\kappa-\alpha-2r, \omega, \eta\, n^r) - \eta(w)\, n(w)^r\, \Phi_\nu(w; 0, k-\kappa, \omega, 1)|_p$$

converges to 0 *uniformly in $w \in W$ as ν approaches to the infinity.*

For simplicity, we write Φ^0 for $\Phi^0(0, k-\kappa, \omega, 1)$. In order to prove the assertion, what we have to show is

(6.16) $|\Phi^0_\nu(w; r, k-\kappa-\alpha-2r, \omega, \eta) - (-1)^r (\eta\, n^r)(w)\, \Phi^0_\nu(w)|_p$ *converges to* 0 *uniformly in $w \in W$ as ν approaches to ∞.*

On the other hand, by Lemma 5.2, there is an element $g \in \bar{\mathscr{M}}(M; K)$ such that

$$H[E^b_{m,\nu}(a)\, \delta^r_{\kappa+\alpha}\, \theta_\nu(a w, \eta)] = E^b_{m,\nu}(a)\, d^r\, \theta_\nu(a w, \eta) + d g.$$

Then, (6.13) shows that $e(dg) = 0$, and we thus have

$$\begin{aligned} e[H(E^b_{m,\nu}(a)\, \delta^r_{\kappa+\alpha}\, \theta_\nu(a w, \eta))] &= e[E^b_{m,\nu}(a)\, d^r\, \theta_\nu(a w, \eta)] \\ &= e[E^b_{m,\nu}(a)\, \theta_\nu(a w, \eta\, n^r)]. \end{aligned}$$

This shows that

$$\begin{aligned} \Phi^0_\nu(w; r, m, \omega, \eta) &= \sum_{a \in Z_\nu} \omega(a)\, e[H(\theta_\nu(a w, \eta)\, \delta^r_m\, E^b_{m,\nu}(a))] \\ &= (-1)^r \sum_a \omega(a)\, e[H(E^b_{m,\nu}(a)\, \delta^r_{\kappa+\alpha}\, \theta_\nu(a w, \eta))] \quad \text{by Lemma 5.3} \\ &= (-1)^r \sum_a \omega(a)\, e[E^b_{m,\nu}(a)\, \theta_\nu(a w, \eta\, n^r)] \\ &= (-1)^r\, e[\Phi_\nu(w; 0, m, \omega, \eta\, n^r)]. \end{aligned}$$

Then, (6.16) follows from (6.15). Q.E.D.

§ 7. Proof of Theorem 2.1

Before proving Theorem 2.1, we list some formulae among several Eisenstein series, which are found in [8, 21] and [26]. Let ω be a Dirichlet character modulo N for a positive integer N and m be a positive integer with $\omega(-1)=(-1)^m$. Define Eisenstein series by

$$J_{m,N}(z,s,\omega)=\sum_{0\neq(c,d)\in\mathbb{Z}^2}\omega(c)(cz+d)^{-m}|cz+d|^{-2s},$$
$$E_{m,N}(z,s;a,b)=\sum_{0\neq(c,d)\equiv(a,b)\bmod N}(cz+d)^{-m}|cz+d|^{-2s}$$

for $a, b\in\mathbb{Z}/N\mathbb{Z}$, and

$$E_{m,N}(\omega)=E_{m,N}(z,\omega)=\tfrac{1}{2}L_N(1-m,\omega)+\sum_{n=1}\Big(\sum_{0<d|n}\omega(d)d^{m-1}\Big)e(nz).$$

The series $J_{m,N}(z,s,\omega)$ and $E_{m,N}(z,s;a,b)$ have analytic continuations as functions of s. We write simply $E_{m,N}(z;a,b)$ and $J_{m,N}(z,\omega)$ for their values at $s=0$. As shown in [25, (2.4)] (see also [26, p. 217]), we know that, for every positive integer r,

$$J_{m+2r,N}(z,-r,\omega)=\frac{\Gamma(m)}{\Gamma(m+r)}(-4\pi y)^r\,\delta_m^r[J_{m,N}(z,\omega)], \tag{7.1}$$

where $y=\mathrm{Im}(z)$. The function $E_{m,N}(\omega)$ belongs to $\mathcal{M}_m(\Gamma_0(N),\omega)$ except in the case where $m=2$ and $N=1$. Let ω_0 be the primitive character modulo N_0 associated with ω, and define another positive integer N_1 by $N=N_0N_1$. Then, obviously, we have

$$E_{m,N}(z,\omega)=\sum_{0<t|N_1}\mu(t)\,\omega_0(t)\,t^{m-1}E_{m,N_0}(tz,\omega_0), \tag{7.2}$$

where μ denotes the Moebius function.

Lemma 7.1. *For* $\tau=\begin{pmatrix}0&-1\\N&0\end{pmatrix}$, *we have*

$$E_{m,N}(\omega)|_m\tau=\frac{\Gamma(m)\,N^{m/2}\,G(\omega_0)}{2(2\pi i)^m N_0}\sum_{0<t|N_1}\mu(t)\,\omega_0(t)\,t^{-1}J_{m,N_0}(t^{-1}N_1z,\bar{\omega}_0), \tag{7.3}$$

where $G(\omega_0)=\sum_{u=1}^{N_0}\omega_0(u)\,e\left(\frac{u}{N_0}\right)$ *is the Gauss sum for* ω_0.

Proof. It is known by Hecke [8] that

$$E_{m,N}(z;a,b)=\text{constant}+\frac{(-2\pi i)^m}{N^m\Gamma(m)}\cdot\sum_{\substack{jk>0\\ k\equiv a\bmod N}}j^{m-1}\,\mathrm{sgn}(j)\,e\left(\frac{j(b+kz)}{N}\right).$$

Write simply A for the constant

$$\frac{N_0^m\Gamma(m)}{2(-2\pi i)^m G(\bar{\omega}_0)}.$$

An easy calculation shows that

$$E_{m,N_0}(z,\omega_0)-A\sum_{a\in\mathbb{Z}/N_0\mathbb{Z}}\bar{\omega}_0(a)\,E_{m,N_0}(z;0,a)$$

is a constant; hence, we know

$$\begin{aligned}E_{m,N_0}(\omega_0)&=A\sum_{a\in(\mathbb{Z}/N_0\mathbb{Z})^\times}\bar{\omega}_0(a)\,E_{m,N_0}(z;0.\,a)\\&=A\cdot\sum_{0\neq(c,d)\in\mathbb{Z}^2}\bar{\omega}_0(d)(cN_0z+d)^{-m}|cN_0z+d|^{-2s}|_{s=0}.\end{aligned}$$

Then, we know from this formula that

$$E_{m,N_0}(\omega_0)|_m\begin{pmatrix}0&-1\\N_0&0\end{pmatrix}=N_0^{-m/2}AJ_{m,N_0}(z,\bar{\omega}_0). \tag{7.4}$$

This is a special case of (7.3) by the well known equality:

$$G(\omega_0)\,G(\bar{\omega}_0)=\omega_0(-1)N_0=(-1)^mN_0.$$

The formula (7.3) in general follows from (7.2) and (7.4) because of the identity:

$$\begin{pmatrix}t&0\\0&1\end{pmatrix}\tau=t\begin{pmatrix}0&-1\\N_0&0\end{pmatrix}\begin{pmatrix}N_1t^{-1}&0\\0&1\end{pmatrix}.$$

Now, we are ready to give a proof of Theorem 2.1. We use the same notation as in the theorem. Especially, M denotes the level of the lattice I of V, f is the fixed primitive form of conductor C, with character ψ and of weight $k>\kappa$. Assume that the p-th Fourier coefficient $a(p,f)$ of f is a unit in Ω. Let f_0 be the ordinary form associated with f defined in Lemma 3.3 and write C_0 for the smallest possible level of f_0. Define integers $\mu\geqq 1$ and $\lambda\geqq 0$ by

$$C_0=C'p^\mu,\qquad M=M'p^\lambda,$$

where $(C',p)=(M',p)=1$. Then, we assume that C' divides M'. Then, the space $\mathcal{M}_k(\Gamma_0(C_0),\psi)$ is a subspace of $\mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}),\psi)$, and we know that $Mp^{\mu-\lambda}/C_0=M'/C'$.

We first construct the measure φ_b in the theorem for each $b>1$ prime to Mp. Let K be a sufficiently large finite extension of the p-adic field $\mathbb{Q}_p$ which contains all the Fourier coefficients of f_0. Let χ_0 be the Dirichlet character modulo M defined by

$$\chi_0(a)=\left(\frac{(-1)^\kappa\Delta}{a}\right)\qquad\text{for }\Delta=[I^*:I].$$

Let $\Phi^0=\Phi^0(0,k-\kappa,\psi\chi_0,1)$ be the bounded measure on $\mathscr{C}(W;K)$ defined in (6.14). Then, the measure Φ^0 has values in the space $\overline{\mathcal{M}}_k^0(M,\psi;K)$, which is a subspace of $\mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}),\psi;K)$ (see Proposition 4.1). Let Tr denote the trace operator of $\mathcal{M}_k(\Gamma_0(Mp^{\mu-\lambda}),\psi;\Omega)$ onto $\mathcal{M}_k(\Gamma_0(C_0),\psi;\Omega)$ defined by

$$\mathrm{Tr}(g)=\sum_\gamma\bar{\psi}(\gamma)g|_k\gamma, \tag{7.5}$$

where γ runs over a representative set for $\Gamma_0(Mp^{\mu-\lambda})\backslash\Gamma_0(C_0)$ and $\overline{\psi\begin{pmatrix} a & b \\ c & d\end{pmatrix}} = \overline{\psi(d)}$. Then, Tr is a bounded linear operator. The finite extension K can be chosen so that the trace operator sends $\mathscr{M}_k(\Gamma_0(Mp^{\mu-\lambda}), \psi; K)$ onto $\mathscr{M}_k(\Gamma_0(C_0), \psi; K)$. Then, define the measure φ_b by

$$\int_W \phi\, d\varphi_b = \ell_f[\mathrm{Tr}(\int_W \phi\, d\Phi^0)] \quad \text{for } \phi \in \mathscr{C}(W; K), \tag{7.6}$$

where ℓ_f: $\bar{\mathscr{M}}_k(C_0, \psi; K) \to K$ is the bounded linear form associated with f given in (4.6).

Let η be an arbitrary algebraic valued spherical function on V with degree α less than $k-\kappa$, and ϕ be an algebraic valued locally constant function on W such that $\phi(aw) = \chi(a)\phi(w)$ for every $a \in Z$ and $w \in W$ with a character χ of finite order of Z. Define a Dirichlet character ξ by

$$\xi(a) = \chi(a)\chi_0(a) \qquad \text{for } a \in \mathbb{Z} \text{ prime to } Mp.$$

Then, for a sufficiently large $\beta \geqq 1$, the theta series $\theta(\phi\eta)$ belongs to $\mathscr{M}_{\kappa+\alpha}(\Gamma_0(Mp^\beta), \xi)$ by Proposition 1.1. We know fix such a $\beta \geqq 1$. Let r be an arbitrary integer with $0 \leqq 2r + \alpha < k - \kappa$. Now, we shall evaluate the integral $\int_W \phi\eta n^r d\varphi_b$ as in (2.5). We may assume that the functions η and ϕ on W have values in K. Take a positive integer ν so that ϕ factors through $W_\nu = \mathscr{W}/p^\nu I$. We may assume that $\nu \geqq \beta$ and $\nu \geqq \mu - \lambda$. Then, Proposition 6.3 shows

$$(-1)^r \int_W \phi\eta n^r d\Phi^0 = e[\sum_{w \in W_\nu} \phi(w)\Phi_\nu(w; r, m, \psi\chi_0, \eta)],$$

where $m = k - \kappa - \alpha - 2r$ and $\Phi_\nu(w)$ is as in (6.6). We see from (6.6) that

$$\begin{aligned} &\sum_{w \in W_\nu} \phi(w)\Phi_\nu(w; r, m, \psi\chi_0, \eta) \\ &= \sum_{w \in W_\nu} \phi(w) \sum_{a \in Z_\nu} \psi\chi_0(a) H[\theta_\nu(aw, \eta)\, \delta_m^r E_{m,\nu}^b(a)] \\ &= \sum_a \psi\chi_0(a) \sum_w \phi(a^{-1}w) H[\theta_\nu(w, \eta)\, \delta_m^r E_{m,\nu}^b(a)] \\ &= \sum_a \psi\chi_0 \bar{\chi}(a) H[\sum_w \phi(w)\theta_\nu(w, \eta)\, \delta_m^r E_{m,\nu}^b(a)] \\ &= H[\theta(\phi\eta)\, \delta_m^r(\sum_a \psi\bar{\xi}(a) E_{m,\nu}^b(a))]. \end{aligned} \tag{7.7}$$

Note that $E_{m,Mp^\beta}(\psi\bar{\xi}) = E_{m,Mp^\nu}(\psi\bar{\xi}) = \frac{1}{2}\sum_{a \in Z_\nu} \psi\bar{\xi}(a) E_{m,\nu}(a)$. Then, (7.7) is equal to

$$2(1 - b^m \psi\bar{\xi}(b)) H[\theta(\phi\eta)\, \delta_m^r E_{m,Mp^\beta}(\psi\bar{\xi})].$$

We have by the definition of φ_b that

$$(-1)^r \int_W \phi\eta n^r d\varphi_b = 2(1 - b^m\psi\bar{\xi}(b))\ell_f[\mathrm{Tr}\{e(H(\theta(\phi\eta)\, \delta_m^r E_{m,Mp^\beta}(\psi\bar{\xi})))\}].$$

Let us now choose a complete representative set R for

$$\Gamma_0(Mp^\beta)\backslash\Gamma_0(C_0 p^{\beta+\lambda-\mu}).$$

Note that $Mp^\beta = M'p^{\beta+\lambda}$, $C_0 p^{\beta+\lambda-\mu} = C'p^{\beta+\lambda}$, and that C' and M' are prime to p. Therefore, the set R may be regarded as a complete representative set for $\Gamma_0(Mp^{\mu-\lambda})\backslash\Gamma_0(C_0)$. Thus, one can extend the operator Tr defined in (7.5) to the trace operator of $\mathcal{M}_k(\Gamma_0(Mp^\beta),\psi;\Omega)$ onto $\mathcal{M}_k(\Gamma_0(C_0p^{\beta+\lambda-\mu}),\psi;\Omega)$; namely, we put

$$\mathrm{Tr}(g) = \sum_{\gamma\in R} \bar{\psi}(\gamma) g|_k\gamma \quad \text{for } g\in\mathcal{M}_k(\Gamma_0(Mp^\beta),\psi;\Omega).$$

Then, we see easily that

$$\mathrm{Tr}\circ T(p) = T(p)\circ \mathrm{Tr} \quad \text{and} \quad \mathrm{Tr}\circ e = e\circ \mathrm{Tr}.$$

Since $\mathrm{Tr}(g)|T(p)^{\beta+\lambda-\mu}$ for $g\in\mathcal{M}_k(\Gamma_0(Mp^\beta),\psi;\bar{\mathbf{Q}})$ belongs to $\mathcal{M}_k(\Gamma_0(C_0),\psi;\bar{\mathbf{Q}})$, Proposition 4.5 and (4.7) show that

$$\begin{aligned}\ell_f[\mathrm{Tr}(e(g))] &= \ell_f[e(\mathrm{Tr}(g))]\\ &= a(p,f_0)^{\mu-\beta-\lambda}\ell_f[\mathrm{Tr}(g)|T(p)^{\beta+\lambda-\mu}]\\ &= a(p,f_0)^{\mu-\beta-\lambda}p^{(\beta+\lambda-\mu)(k-1)}\frac{\langle h_{\beta+\lambda-\mu}, \mathrm{Tr}(g)\rangle_{C_0p^{\beta+\lambda-\mu}}}{\langle h, f_0\rangle_{C_0}},\end{aligned}$$

where $h = f_0^\rho|_k\begin{pmatrix}0 & -1\\ C_0 & 0\end{pmatrix}$ and $h_{\beta+\lambda-\mu}(z) = h(p^{\beta+\lambda-\mu}z)$. Note that, for $\tau = \begin{pmatrix}0 & -1\\ Mp^\beta & 0\end{pmatrix}$,

$$\langle h_{\beta+\lambda-\mu}, \mathrm{Tr}(g)\rangle_{C_0p^{\beta+\lambda-\mu}} = \langle h_{\beta+\lambda-\mu}, g\rangle_{Mp^\beta} = \langle h_{\beta+\lambda-\mu}|_k\tau, g|_k\tau\rangle_{Mp^\beta}.$$

Applying these formulae to $g = H[\theta(\phi\eta)\,\delta_m^r E_{m,Mp^\beta}^b(\psi\bar{\xi})]$, we have by (5.4)

$$(7.8)\qquad (-1)^r\int_W \phi\eta n^r d\varphi_b = 2(1-b^m\psi\bar{\xi}(b))p^{(\beta+\lambda-\mu)(k-1)}a(p,f_0)^{\mu-\beta-\lambda} \times\frac{\langle h_{\beta+\lambda-\mu}|_k\tau, (\theta(\phi\eta)\delta_m^r E_{m,Mp^\beta}(\psi\bar{\xi}))|_k\tau\rangle_{Mp^\beta}}{\langle h, f_0\rangle_{C_0}},$$

where $m = k-\kappa-\alpha-2r$.

On the other hand, Lemma 7.1 combined with (7.1) shows that

$$\begin{aligned}&(\delta_m^r E_{m,Mp^\beta}(\omega))|_{k-\kappa-\alpha}\tau\\ &\quad = Ty^{-r}\sum_{0<t|N_1}\mu(t)\,\omega_0(t)t^{-1}J_{k-\kappa-\alpha,N_0}(t^{-1}N_1z, -r, \bar{\omega}_0),\end{aligned}$$

where $\omega = \psi\bar{\xi}$, N_0 is the conductor of ω, ω_0 is the primitive character associated with ω, $N = Mp^\beta = N_0N_1$ and

$$T = N_0^{-1}G(\omega_0)\pi^{-m-r}2^{2r-m-1}(\sqrt{-1})^{2r-m}(Mp^\beta)^{m/2}\,\Gamma(m+r)$$

for $m=k-\kappa-\alpha-2r$. Note that

$$p^{(\beta+\lambda-\mu)(k-1)}h_{\beta+\lambda-\mu}|_k\tau=(-1)^k p^{(\beta+\lambda-\mu)(k/2-1)}f\delta^{\rho}|_k\gamma$$

for $\gamma=\begin{pmatrix} M'/C' & 0 \\ 0 & 1 \end{pmatrix}$. Applying these formulae to (7.8), we know that $\int_W \phi\eta n^r\, d\varphi_b$ is equal to

$$S(1-b^m\psi\bar{\xi}(b))\,a(p,f_0)^{\mu-\beta-\lambda}G(\omega_0)\langle h,f_0\rangle^{-1}$$
$$\cdot\sum_{0<t|N_1}\mu(t)\,\omega_0(t)t^{-1}\langle f_0|_k\tau,(\theta(\phi\eta))|_{\kappa+\alpha}\tau)J_{k-\kappa-\alpha,N_0}(t^{-1}N_1 z,-r,\bar{\omega}_0)y^{-r}\rangle_{Mp^{\beta}},$$

where

$$S=\pi^{-m-r}2^{-m-2r}(\sqrt{-1})^{2k-m}M^{m/2}p^{m\beta/2+(\beta+\lambda-\mu)(k/2-1)}N_0^{-1}\Gamma(m+r).$$

Then, the evaluation (2.5) follows from the formula given in [26, p. 217]:

$$\mathscr{D}_{Mp^{\beta}}(\kappa+\alpha+\theta,f_0|_k\gamma,\theta(\phi\eta)|_{\kappa+\alpha}\tau)$$
$$=U\sum_{0<t|N_1}\mu(t)\,\omega_0(t)t^{-1}\cdot\langle f\delta^{\rho}|_k\gamma,(\theta(\phi\eta)|_{\kappa+\alpha}\tau)J_{k-\kappa-\alpha,N_0}(t^{-1}N_1 z,-r,\bar{\omega}_0)y^{-r}\rangle_{Mp^{\beta}},$$

where

$$U=\pi^{k-r-2m+1}2^{2k-2r-2m-1}M^{m-1}p^{\beta(m-1)}N_0^{-1}G(\omega_0)\frac{\Gamma(m+r)}{\Gamma(\kappa+\alpha+r)\Gamma(r+1)}.$$

§ 8. A sketch of the Proof of Theorem 2.2

In this section, we use the same notation as in Theorem 2.2. Especially, $g=\sum_{n=0}^{\infty}b(n)e(nz)$ is the fixed modular form in $\mathcal{M}_l(\Gamma_0(N),\omega)$ with $b(n)\in\bar{\mathbb{Q}}$. Let v be a psoitive integer and ϕ be an arbitrary function on $Y_v=\mathbb{Z}/Np^v\mathbb{Z}$ with values in $\mathbb{C}$. Put

$$g(\phi)=\sum_{n=0}^{\infty}\phi(n)\,b(n)\,e(nz),$$

as a function on $\mathfrak{H}$.

Proposition 8.1. *For any* $\gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma_0(N^2p^{2v})$, *we have the following transformation formula:*

$$g(\phi)|_l\gamma=\omega(d)\,g(\phi_a),$$

where ϕ_a *is a function on* $\mathbb{Z}/Np^v\mathbb{Z}$ *defined by*

$$\phi_a(y)=\phi(a^{-2}y).$$

Proof (cf. [20, Lemma 2]). We simply write n for Np^v for a fixed $v\geqq 1$, and define a $n\times n$ matrix by

$$A=(e(xy/n))_{x,y\in Y_v}.$$

The matrix A is invertible, and thus we can find $x(u, y) \in \mathbb{C}$ for any pair $u, y \in Y_\nu$ so that

$$\sum_{u \in Y_\nu} x(u, y)\, e(uv/n) = \begin{cases} 1 & \text{if } v = y, \\ 0 & \text{otherwise.} \end{cases}$$

For any $t \in Y_\nu^\times$ and $v \in Y_\nu$, we have that

$$\sum_{u \in Y_\nu} x(tu, y)\, e(uv/n) = \sum_{u \in Y_\nu} x(u, y)\, e(t^{-1}uv/n) = \begin{cases} 1 & \text{if } t^{-1}v = y, \\ 0 & \text{otherwise.} \end{cases}$$

This shows

(8.1) $$x(tu, y) = x(u, ty) \quad \text{for every } t \in Y_\nu^\times.$$

Put

(8.2) $$g(y) = g_\nu(y) = \sum_{m \equiv y \bmod Np^\nu} b(m)\, e(mz) \quad \text{for } y \in Y_\nu.$$

Since we can express $g(\phi) = \sum_{y \in Y_\nu} \phi(y)\, g(y)$, our task is to show

(8.3) $$g(y)|_l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \omega(d)\, g(a^2 y) \quad \text{for every } y \in Y_\nu \left(\text{if } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n^2)\right).$$

For each $u \in Y_\nu$, take $u_0 \in \mathbb{Z}$ with $u_0 \equiv u \bmod n$, and put $\alpha_u = \begin{pmatrix} 1 & u_0/n \\ 0 & 1 \end{pmatrix}$. Then, by the definition of $x(u, y)$, we have

$$g(y) = \sum_{u \in Y_\nu} x(u, y)\, g|_l \alpha_u.$$

As in the proof of [28, Prop. 3.64], for each $u \in Y_\nu$, we can find $\gamma_u = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \Gamma_0(N)$ so that

$$\alpha_u \gamma = \gamma_u \alpha_{a^{-2}u} \quad \text{and} \quad d \equiv d' \bmod N \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right).$$

Then (8.3) can be shown as follows:

$$\begin{aligned} g(y)|_l \gamma &= \sum_{u \in Y_\nu} x(u, y)\, g|\gamma_u \alpha_{a^{-2}u} = \omega(d) \sum_u x(a^2 u, y)\, g|_l \alpha_u \\ &= \omega(d) \sum_u x(u, a^2 y)\, g|_l \alpha_u = \omega(d)\, g(a^2 y). \end{aligned}$$

Now we shall give a sketch of a proof of Theorem 2.2. Fix an integer $b > 1$ prime to Np, and let r and m be integers with $r \geqq 0$ and $m > 0$. Define the Eisenstein series $E^b_{m,\nu}(a)$ for each $a \in Y_\nu^\times$ by (6.3) and (6.4a) for N in place of M there. Write $N = N'p^\lambda$ with an integer N' prime to p and let ψ' be a Dirichlet character modulo Np^u for some $u \geqq 1$. Define, for each $y \in Y_\nu$ $(\nu \geqq u)$,

(8.4) $$\Phi_\nu(y) = \Phi_\nu(y; r, m, \psi') = \sum_{a \in Y_\nu^\times} \psi'(a)\, H[g_\nu(a^2 y)\, \delta^r_m E^b_{m,\nu}(a)].$$

Then, the system $\{\Phi_\nu(y)\}$ defines a bounded measure $\Phi(r,m,\psi')$ with values in $\bar{\mathcal{M}}_{l+m+2r}(NN',\psi'\omega;K)$ for a suitable finite extension K of $\mathbb{Q}_p$. We now define, parallel to (6.14), the ordinary part $\Phi^0(r,m,\psi')$ of the measure $\Phi(r,m,\psi')$ by

(8.5) $$\int_Y \phi\, d\Phi^0(r,m,\psi') = e\Big[\int_Y \phi\, d\Phi(r,m,\psi')\Big].$$

Then, the measure $\Phi^0(r,m,\psi')$ has values in the finite dimensional vector space $\mathcal{M}_{l+m+2r}(\Gamma_0(NN'p^u),\psi'\omega;K)$ by Proposition 4.1. Let k be an integer with $k>l$. If r is an integer with $0\leqq 2r<k-l$, we have, for any $\phi\in\mathscr{C}(Y;K)$,

(8.6) $$\int_Y \phi\, d\Phi^0(r,k-l-2r,\psi') = (-1)^r \int_Y \phi(y)\, y_p^r\, d\Phi^0(y;0,k-l,\psi')$$

where y_p is the projection of $y\in Y=\mathbb{Z}/N'\mathbb{Z}\times\mathbb{Z}_p$ to the factor $\mathbb{Z}_p$. This can be proved in exactly the same manner as in the proof of Propositions 6.2 and 6.3. Let f be a primitive form of weight $k>l$, of conductor C and with character ψ. Assume that $|a(p,f)|_p=1$ and that K contains all the Fourier coefficients of f. Let f_0 be the ordinary form associated with f and let C_0 be the smallest level of f_0. Write $C_0=C'p^u$ with an integer C' prime to p and assume the divisibility of N' by C'. Then the measure $\Phi^0=\Phi^0(0,k-l,\psi\bar{\omega})$ has values in the space $\mathcal{M}_k(\Gamma_0(N'^2p^u),\psi;K)$. Let Tr denotes the trace operator of $\mathcal{M}_k(\Gamma_0(N'^2p^u),\psi;K)$ onto $\mathcal{M}_k(\Gamma_0(C_0),\psi;K)$. Then, the bounded measure φ_b on Y in Theorem 2.2 can be defined by

$$\int_Y \phi\, d\varphi_b = \ell_f\Big[\mathrm{Tr}\Big(\int_Y \phi\, d\Phi^0\Big)\Big],$$

where ℓ_f is the linear form on $\bar{\mathcal{M}}_k(C_0,\psi;K)$ attached to f. The evaluation of the integral $\int_Y \phi(y)\, y_p^r\, d\varphi_b(y)$ for any locally constant function ϕ with (2.6) can be carried out in exactly the same fashion as in § 7.

§ 9. Functional equations of $\mathscr{D}_N(s,f,g)$

The functional equations and the meromorphy of the zeta functions $\mathscr{D}_N(s,f,g)$ was proved by Jacquet [10, Th. 19.14] for any primitive forms f and g through a representation theoretic generalization of Rankin's method [18]. However, the familiarity with the representation theory is necessary to understand his results; so, for the reader's convenience, we give here a brief exposition of this in a special case where the original method of [18] can be applied. The details of our arguments may be found in [21, 25, 26]. Let

$$f=\sum_{n=1}^{\infty} a(n)\,e(nz) \quad\text{and}\quad g=\sum_{n=1}^{\infty} b(n)\,e(nz)$$

be primitive forms of conductor $C(f)$ and $C(g)$, respectively. Let k and ψ (resp. l and ξ) be the weight and the character of f (resp. g). For any Dirichlet character ω, we write $C(\omega)$ for the conductor of ω. We now assume that:

(9.1a) *N is the least common multiple of $C(f)$ and $C(g)$;*

(9.1b) $k>l$;

(9.1c) $N=C(\psi\xi)$.

Now we define the root numbers $W(f)$ and $W(g)$ by

$$f|_k\begin{pmatrix}0 & -1\\ C(f) & 0\end{pmatrix}=W(f)f^\rho, \quad g|_l\begin{pmatrix}0 & -1\\ C(g) & 0\end{pmatrix}=W(g)g^\rho, \tag{9.2}$$

where ρ is the complex conjugation. Let $G(\omega)$ denote the Gauss sum for a primitive character ω and put $W(\omega)=G(\omega)/|G(\omega)|$. Write $M(g)=N/C(g)$ and $M(f)=N/C(f)$, and put

$$W(f,g)=a(M(g))^\rho\, b(M(f))^\rho\, W(f)\, W(g)\, W(\psi\xi).$$

Theorem 9.1. *Put*

$$R(s,f,g)=(2\pi)^{-2s}\Gamma(s)\Gamma(s+1-l)\mathscr{D}_N(s,f,g).$$

Then, $R(s,f,g)$ can be continued as an entire function on the whole complex plane and satisfies the functional equation:

$$\begin{aligned} &R(k+l-1-s,f,g)\\ &=(-1)^l\, W(f,g)N^{s-k-l+\frac{3}{2}}\, C(f)^{s-\frac{k}{2}}\, C(g)^{s-\frac{l}{2}}\, R(s,f^\rho,g^\rho). \end{aligned} \tag{9.3}$$

Even when $k=l$, a similar functional equation holds, but the holomorphy is not necessarily valid (see [22]).

Proof. Let us define an Eisenstein series of weight m and of character ω modulo N by

$$F_{m,N}(z,s,\omega)=\pi^{-s}y^s\Gamma(s+m)\sum_{a \bmod N}\omega(a)\,E_{m,N}(z,s;0,a).$$

Then, $F_{m,N}(z,s,\omega)$ is an entire function in s if $m>0$. If ω is primitive, it satisfies the functional equation:

$$F_{m,N}(z,1-m-s,\omega)=W(\omega)N^{3s+m-\frac{3}{2}}z^{-m}F_{m,N}(-1/Nz,s,\bar{\omega}) \tag{9.4}$$

(cf. [26, (19)]). On the other hand, we know from [26, (22)]

$$R(s,f,g)=2^{-1}\pi^{1-k}\int_{\mathfrak{H}/\Gamma_0(N)}\overline{f^\rho(z)}\,g(z)\,F_{m,N}(z,s+1-k,\psi\xi)y^{k-2}\,dx\,dy \tag{9.5}$$

for $m=k-l$. This shows the holomorphy of $R(s,f,g)$ on the whole complex plane. Since $\psi\xi$ is primitive by (9.1c), we know from (9.4) that

$$R(k+l-1-s,f,g)=A_1(s)\int_{\mathfrak{H}/\Gamma_0(N)}\overline{f^\rho}\,g\,F_{m,N}\left(\frac{-1}{Nz},s+1-k,\overline{\psi\xi}\right)z^{-m}y^{k-2}\,dx\,dy,$$

where

$$A_1(s)=2^{-1}\pi^{1-k}N^{3s-2k-l+\frac{3}{2}}W(\psi\xi).$$

Note that

$$f^\rho(-1/Nz)=(-1)^k\overline{W(f)}\,N^{k/2}M(f)^{k/2}f(M(f)z)z^k,$$
$$g(-1/Nz)=W(g)N^{l/2}M(g)^{l/2}g^\rho(M(g)z)z^l.$$

Substituting z for $-1/Nz$ in the formula of $R(k+l-1-s,f,g)$, we have by (9.5) that

$$R(k+l-1-s,f,g)=A_2(s)L(2s+2-k-l,\overline{\psi\xi})\sum_{n=1}^{\infty}a(n/M(f))^\rho\cdot b(n/M(g))^\rho n^{-s}$$

where

$$A_2(s)=(-1)^l\,W(f)\,W(g)\,W(\psi\xi)\,M(f)^{\frac{k}{2}}\,M(g)^{\frac{l}{2}}$$
$$\cdot N^{3s-\frac{3}{2}(k+l-1)}(2\pi)^{-2s}\Gamma(s)\Gamma(s+1-l).$$

Define complex numbers α_p, α'_p, β_p, β'_p for every prime p by the Euler products:

$$\sum_{n=1}^{\infty}a(n)^\rho n^{-s}=\prod_p[(1-\alpha_p p^{-s})(1-\alpha'_p p^{-s})]^{-1},$$

$$\sum_{n=1}^{\infty}b(n)^\rho n^{-s}=\prod_p[(1-\beta_p p^{-s})(1-\beta'_p p^{-s})]^{-1}.$$

Since N is the least common multiple of $C(f)$ and $C(g)$, $M(f)$ is prime to $M(g)$. Then, we know from [25, Lemma 1] that

$$\sum_{n=1}^{\infty}a(n/M(f))^\rho b(n/M(g))^\rho n^{-s}$$
$$=(M(f)M(g))^{-s}\sum_{n=1}^{\infty}a(M(g)n)^\rho b(M(f)n)^\rho n^{-s}$$
$$=(M(f)M(g))^{-s}\prod_p X_p^*(s)/Y_p(s),$$

where $X_p^*(s)$ and $Y_p(s)$ are given by

$$X_p^*(s)=\begin{cases}1-\alpha_p\alpha'_p\beta_p\beta'_p p^{-2s} & \text{if } p\nmid M(f)M(g),\\ a(p)^{\rho\,\mathrm{ord}_p(M(g))} & \text{if } p\mid M(g),\\ b(p)^{\rho\,\mathrm{ord}_p(M(f))} & \text{if } p\mid M(f),\end{cases}$$

$$Y_p(s)=(1-\alpha_p\beta_p p^{-s})(1-\alpha_p\beta'_p p^{-s})(1-\alpha'_p\beta_p p^{-s})(1-\alpha'_p\beta'_p p^{-s}).$$

The above expression of $X_p^*(s)$ for the prime factor p of $M(f)M(g)$ follows from [25, (3.1)], since $M(f)$ (resp. $M(g)$) is a divisor of $C(g)$ (resp. $C(f)$) by (9.1a). Then, by [25, Lemma 1], we have

$$\sum_{n=1}^{\infty}a(n/M(f))^\rho b(n/M(g))^\rho n^{-s}$$
$$=a(M(g))^\rho b(M(f))^\rho(M(f)M(g))^{-s}\sum_{n=1}^{\infty}a(n)^\rho b(n)^\rho n^{-s}.$$

This proves (9.3).

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