

# Automorphic Galois representations and the cohomology of Shimura varieties

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**Abstract.** The first part of this report describes the class of representations of Galois groups of number fields that have been attached to automorphic representations. The construction is based on the program for analyzing cohomology of Shimura varieties developed by Langlands and Kottwitz. Using  $p$ -adic methods, the class of Galois representations obtainable in this way can be expanded slightly; the link to cohomology remains indispensable at present. It is often possible to characterize the set of Galois representations that can be attached to automorphic forms, using the modularity lifting methods initiated by Wiles a bit over 20 years ago. The report mentions some applications of results of this kind. The second part of the report explains some recent results on critical values of automorphic  $L$ -functions, emphasizing their relation to the motives whose  $\ell$ -adic realizations were discussed in the first part.

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## 1. Introduction

Algebraic number theory has benefited immeasurably over the past four decades from the applications of the methods and results of the Langlands program to the study of Galois representations attached to automorphic forms. Yet Galois representations do not figure prominently in Langlands's original conjectures, apart from the complex Galois representations that are the object of the Artin conjecture. There seems to be no completely precise statement in the literature of a *Langlands reciprocity conjecture* – a bijection between representations of Galois groups with values in the  $\ell$ -adic points of reductive groups, subject to certain natural restrictions (including a version of irreducibility), and of automorphic representations of related reductive groups – although number theorists believe there should be such a conjecture and have a general idea of how it should go. The best general account of this question is still contained in the expanded version [69] of Taylor's 2002 ICM talk.

The first objective of the present survey is to describe the results in the direction of reciprocity obtained since the publication of [69]. Construction of the correspondence in one direction – from automorphic representations to Galois representations – has progressed considerably, even in directions that could not have been expected ten years ago. All of the Galois representations associated to automorphic representations have been constructed, either directly or by  $p$ -adic interpolation, using the cohomology of Shimura varieties. This source of Galois representations has been or soon will be exhausted, and new methods will

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need to be invented in order to find the Galois representations attached to automorphic representations that cannot be related in any way to cohomology of Shimura varieties, notably the representations of Galois groups of number fields that are not totally real nor CM.

Little was known at the time of [69] regarding the converse direction, the problem of proving that a given Galois representation  $\rho$  is attached to automorphic forms, when  $\dim \rho > 2$ . Now there is a mature theory of *automorphy lifting theorems*, in the spirit of the results developed by Wiles for his proof of Fermat's Last Theorem, applying in all dimensions. The attempt to complete this theory represents one of the most active branches of algebraic number theory, and is largely responsible for the rapid growth of interest in the  $p$ -adic local Langlands program.

Let  $K$  be a number field. The Galois group  $\Gamma_K := \text{Gal}(\overline{\mathbb{Q}}/K)$  acts on the  $p$ -adic étale cohomology of an algebraic variety or motive  $M$  defined over  $K$ , and this action determines the  $L$ -function  $L(s, M)$ . Theoretical considerations guarantee that the  $p$ -adic Galois representations on the cohomology of most algebraic varieties cannot be realized in the cohomology of Shimura varieties; for example, the cohomology of a generic hypersurface cannot be obtained in this way. Present methods, therefore, cannot prove the analytic continuation of  $L(s, M)$  for most motives arising from geometry. When the Galois representation is attached to an automorphic form, on the other hand, then so is  $L(s, M)$ , and this implies analytic (or at least meromorphic) continuation of the latter. Moreover, the conjectures concerned with the values at integer points of  $L(s, M)$  (of Deligne, Beilinson, or Bloch-Kato) can be studied with the help of automorphic forms. Everything one knows in the direction of the Birch-Swinnerton-Dyer Conjecture, for example, has been proved by means of this connection. There has been a great deal of activity in this direction as well, especially in connection with the growth of the "relative" theory of automorphic forms (the relative trace formula and conjectures of Gan-Gross-Prasad, Ichino-Ikeda, and Sakellaridis-Venkatesh). The second part of this paper reviews some of the recent results on special values of  $L$ -functions.

The conjectures on special values of complex  $L$ -functions are accompanied by conjectures on the existence of  $p$ -adic analytic functions interpolating their normalized special values. The article concludes with a few speculative remarks about automorphic  $p$ -adic  $L$ -functions.

## 2. Automorphic forms and Galois representations

**2.1. Construction of automorphic Galois representations.** Class field theory classifies abelian extensions of a number field  $K$  in terms of the structure of the idèle class group  $GL(1, K) \backslash GL(1, \mathbb{A}_K)$ . In doing so it also identifies 1-dimensional representations of  $\Gamma_K$  with continuous characters of the idèle class group. Non-abelian class field theory can be traced back to the 1950s, when Eichler and Shimura realized that 2-dimensional  $\ell$ -adic Galois representations could be attached to classical cusp forms that are eigenvalues of the Hecke algebra. A conjectural classification of  $n$ -dimensional  $\ell$ -adic Galois representations, in terms of the Langlands program, was formulated in Taylor's 2002 ICM talk (cf. [69]). We review this conjecture quickly. For any finite set  $S$  of places of  $K$ , let  $\Gamma_{K,S}$  be the Galois group of the maximal extension of  $K$  unramified outside  $S$ . Taylor adopts the framework of Fontaine and Mazur, who restrict their attention in [25] to continuous representations  $\rho : \Gamma_K \rightarrow GL(n, \mathbb{Q}_\ell)$  satisfying the following two axioms:

1.  $\rho$  factors through  $\Gamma_{K,S}$  for some finite set  $S$  of places of  $K$  (usually containing the primes dividing  $\ell$ );
2. For all primes  $v$  of  $K$  of residue characteristic  $\ell$ , the restriction of  $\rho$  to a decomposition group  $G_v \subset \Gamma_K$  at  $v$  is *de Rham* in the sense of Fontaine.

A  $\rho$  satisfying these two conditions is either called *geometric* or *algebraic*, depending on the context. Condition (1) guarantees that, at all but finitely many primes  $v$  of  $K$ , the restriction  $\rho_v$  of  $\rho$  to a decomposition group  $G_v$  is determined up to equivalence, and up to semisimplification, by the characteristic polynomial  $P_v(\rho, T)$  of the conjugacy class  $\rho(\text{Frob}_v) \in GL(n, \overline{\mathbb{Q}}_\ell)$ . One of the Fontaine-Mazur conjectures implies that there is a number field  $E$  such that all  $P_v(\rho, T)$  have coefficients in  $E$ ; by choosing an embedding  $\iota : E \hookrightarrow \mathbb{C}$  we may thus define  $P_v(\rho, T)$  as a polynomial of degree  $n$  in  $\mathbb{C}[T]$  with non-vanishing constant term. The set of such polynomials is in bijection with the set of (equivalence classes of) irreducible smooth representations  $\Pi_v$  of  $GL(n, K_v)$  that are *spherical*: the space of vectors in  $\Pi_v$  that are invariant under the maximal compact subgroup  $GL(n, \mathcal{O}_v) \subset GL(n, K_v)$ , where  $\mathcal{O}_v$  is the ring of integers in  $K_v$ , is non-trivial and necessarily one-dimensional. We let  $\Pi_v(\rho)$  be the spherical representation corresponding to  $P_v(\rho, T)$ .

An irreducible representation  $\Pi_v(\rho)$  of  $GL(n, K_v)$  can be attached to  $\rho$  for primes  $v \in S$  as well. If  $v$  is not of residue characteristic  $\ell$ , the restriction of  $\rho$  to  $G_v$  gives rise by a simple procedure to an  $n$ -dimensional representation  $WD(\rho, v)$  of the Weil-Deligne group  $WD_v$  at  $v$ . The local Langlands correspondence [41, 43] is a bijection between  $n$ -dimensional representations of  $WD_v$  and irreducible smooth representations of  $GL(n, K_v)$ , and we obtain  $\Pi_v(\rho)$  using this bijection. If  $v$  divides  $\ell$ , condition (2) allows us to define  $WD(\rho, v)$  by means of Fontaine’s  $D_{pst}$  functor. Fontaine’s construction also provides a set of Hodge-Tate numbers  $HT(\rho, v)$  for each archimedean prime  $v$ . This datum, together with the action of a complex conjugation  $c_v$  in a decomposition group  $G_v$  when  $v$  is a real prime, defines an  $n$ -dimensional representation  $\rho_v$  of the local Weil group  $W_v$ , and thus an irreducible  $(\mathfrak{g}_v, U_v)$ -module  $\Pi_v(\rho)$ , where  $\mathfrak{g}_v$  is the (complexified) Lie algebra of  $G(K_v)$  and  $U_v$  is a maximal compact subgroup of  $G(K_v)$ . We let  $\Pi(\rho)$  denote the restricted direct product (with respect to the  $GL(n, \mathcal{O}_v)$ -invariant vectors at finite primes outside  $S$ ) of the  $\Pi_v(\rho)$ , as  $v$  ranges over all places of  $K$ .

If  $v$  is an archimedean place of  $K$ , the Harish-Chandra homomorphism identifies the center  $Z(\mathfrak{g}_v)$  with the symmetric algebra of a Cartan subalgebra  $\mathfrak{t}_v \subset \mathfrak{g}_v$ . The maximal ideals of  $Z(\mathfrak{g}_v)$  are in bijection with linear maps  $Hom(\mathfrak{t}_v, \mathbb{C})$ . The *infinitesimal character* of an irreducible  $(\mathfrak{g}_v, U_v)$ -module  $\Pi_v$  is the character defining the action of  $Z(\mathfrak{g}_v)$  on  $\Pi_v$ ; its kernel is a maximal ideal of  $Z(\mathfrak{g}_v)$ , and thus determines a linear map  $\lambda_{\Pi_v} \in Hom(\mathfrak{t}_v, \mathbb{C})$ . In [17], Clozel defines an irreducible  $(\mathfrak{g}_v, U_v)$ -module  $\Pi_v$  to be *algebraic* if  $\lambda_{\Pi_v}$  belongs to the lattice in  $Hom(\mathfrak{t}_v, \mathbb{C})$  spanned by the highest weights of finite-dimensional representations. Denote by  $|\bullet|_v$  the  $v$ -adic absolute value,  $|\bullet|_{\mathbb{A}}$  the adèle norm. The following corresponds to Conjectures 3.4 and 3.5 of [69].

**Conjecture 2.1.**

- (1) Let  $\rho : \Gamma_K \rightarrow GL(n, \overline{\mathbb{Q}}_\ell)$  be an irreducible geometric Galois representation. Then the local component

$$\Pi_v(\rho) \left( \frac{1-n}{2} \right) := \Pi_v(\rho) \otimes |\bullet|_v^{\frac{1-n}{2}} \circ \det$$

is algebraic at each archimedean prime  $v$  of  $K$ , and the representation  $\Pi_v(\rho)$  of  $GL(n, \mathbf{A}_K)$  occurs in the space of cusp forms on  $GL(n, K) \backslash GL(n, \mathbf{A}_K)$ .

- (2) Conversely, let  $\Pi$  be a cuspidal automorphic representation of  $GL(n, \mathbf{A}_K)$ . Suppose  $\Pi_v(\frac{1-n}{2})$  is algebraic for every archimedean place  $v$  of  $K$ . Then for each prime  $\ell$ , there exists an irreducible geometric  $n$ -dimensional representation

$$\rho_{\ell, \Pi} : \Gamma_K \rightarrow GL(n, \overline{\mathbb{Q}}_{\ell})$$

such that

$$\Pi \left( \frac{1-n}{2} \right) := \Pi \otimes | \bullet |_{\mathbf{A}}^{\frac{1-n}{2}} \circ \det \xrightarrow{\sim} \Pi(\rho_{\ell, \Pi}).$$

The Galois representations  $\rho_{\ell, \Pi}$  are called *automorphic*.<sup>1</sup> Quite a lot is known about this conjecture when  $K$  is either a CM field or a totally real field, almost exclusively in the *regular* case, when  $\lambda_{\Pi}$  is the infinitesimal character of an irreducible finite-dimensional representation of  $G(K_v)$  for all archimedean  $v$ . Let  $S$  be a finite set of primes of  $K$ , let  $\rho$  be an  $n$ -dimensional  $\ell$ -adic representation of  $\Gamma_K$ , and say that  $\Pi$  and  $\rho_{\ell}$  correspond away from  $S$  if  $\Pi_v = \Pi_v(\rho)$  for  $v \notin S$ . The following theorem represents the current state of knowledge regarding part (b) of Conjecture 2.1; part (a) will be treated in the next section. In its details it may already be obsolete by the time of publication.

**Theorem 2.2.** *Let  $K$  be a CM field or a totally real field. Let  $\Pi$  be a cuspidal automorphic representation of  $GL(n, \mathbf{A}_K)$ . Suppose  $\Pi_v$  is algebraic and regular for every archimedean place  $v$  of  $K$ .*

- (a) *Let  $S$  be the set of finite primes at which  $\Pi$  is ramified. If  $\ell$  is a rational prime, let  $S(\ell)$  denote the union of  $S$  with the set of primes of  $K$  dividing  $\ell$ . For each prime  $\ell$ , there exists a completely reducible geometric  $n$ -dimensional representation*

$$\rho_{\ell, \Pi} : \Gamma_K \rightarrow GL(n, \overline{\mathbb{Q}}_{\ell})$$

such that  $\Pi(\frac{1-n}{2})$  and  $\rho_{\ell, \Pi}$  correspond away from  $S(\ell)$ .

- (b) *Suppose  $\Pi$  is polarized, in the following sense:*

- (1) *If  $K$  is a CM field,*

$$\Pi^{\vee} \xrightarrow{\sim} \Pi^c,$$

where  $^c$  denotes the action of complex conjugation acting on  $K$

- (2) *If  $K$  is totally real,*

$$\Pi^{\vee} \xrightarrow{\sim} \Pi \otimes \omega$$

for some Hecke character  $\omega$  of  $GL(1, \mathbf{A}_K)$ .

Here  $^{\vee}$  denotes contragredient. Then there is a compatible family of  $n$ -dimensional representations  $\rho_{\ell, \Pi}$  satisfying (b) of 2.1. Moreover,  $\rho_{\ell, \Pi}$  is de Rham, in the sense of Fontaine, at all primes  $v$  dividing  $\ell$ .

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<sup>1</sup>When  $G$  is a reductive algebraic group, Buzzard and Gee have conjectured a correspondence between automorphic representations of  $G$  that satisfy an algebraicity condition at archimedean places and compatible systems of  $\ell$ -adic representations with values in the Langlands  $L$ -group of  $G$  [9]. The relation of this conjecture with Conjecture 2.1 is a bit subtle; two different algebraicity conditions are relevant to the conjecture.

**2.1.1.  $p$ -adic approximation.** To forestall certain kinds of cognitive dissonance, we switch from  $\ell$ -adic to  $p$ -adic representations in this section. Part (b) of Theorem 2.2 has been proved over the course of several decades by a number of people. For general  $n$ -dimensional representations, the most relevant references are [17, 41, 48] for work before the proof by Laumon and Ngô of the Fundamental Lemma; and [16, 19, 20, 51, 56, 62, 64] for results based on the Fundamental Lemma. I refer the reader to the discussion in [35], and take this opportunity to insist on the centrality of Labesse's results in [51] and earlier papers, which are inexplicably omitted from some accounts.<sup>2</sup>

Under the polarization hypothesis of case (b), most  $\rho_{p,\Pi}$  are realized in the cohomology of Shimura varieties  $S(G)$  attached to appropriate unitary groups  $G$ . Some important representations are nevertheless missing when  $n$  is even. To complete the proof of (b), the missing representations are constructed by  $p$ -adic approximation. One needs to show that  $\Pi$  is in some sense the limit of a sequence of  $\Pi_i$  that do satisfy the strong regularity hypothesis<sup>3</sup> For  $n = 2$  two approximation methods had been applied: Wiles used the ideas due to Hida, while Taylor obtained the most complete results by adapting ideas of Ribet. In the intervening years, the theory of *eigenvarieties*, which originated in the work of Coleman and Mazur, had been developed to define  $p$ -adic families of automorphic forms in a very general setting. Chenevier's thesis [14] generalized the approximation method of Wiles to attach  $p$ -adic Galois representations of dimension  $n > 2$  to non-ordinary  $\Pi$ , using eigenvarieties. Its extension in the book [6] with Bellaïche, and the subsequent article [15] were almost sufficient to construct the missing  $\rho_{p,\Pi}$  as the limit of  $\rho_{p,\Pi_i}$  as above. The final steps in the construction, and the proofs of most of the local properties of 2.1, were carried out in [16], using a descent argument introduced by Blasius and Ramakrishnan in [8] and extended by Sorensen in [65]. The remaining local properties – determination of local  $\ell$ -adic and  $p$ -adic monodromy of  $\rho_{p,\Pi}$  were not known when [35] was written; they were obtained in most cases in [4] and completed in [12, 13].

Part (a) of Theorem 2.2 is much more recent. The first result of this type was obtained for  $GL(2)$  over imaginary quadratic fields by Taylor in [67], following his joint work [40] with Soudry and the author; this was extended to general CM fields by Mok [54]. The proof of part (a) in [38] starts with an old idea of Clozel. Let  $K$  be a CM field and let  $K^+ \subset K$  be the fixed field under complex conjugation. Let  $G_n$  be the unitary group of a  $2n$ -dimensional hermitian space over  $K$ , and assume  $G_n$  is quasi split. Then  $G_n$ , viewed by restriction of scalars as an algebraic group over  $\mathbb{Q}$ , contains a maximal parabolic subgroup  $P_n$  with Levi factor isomorphic to  $R_{K/\mathbb{Q}}GL(n)_K$ . Let  $S(n, K)$  be the locally symmetric space attached to  $GL(n, \mathbf{A}_K)$ . Since  $K$  is a CM field,  $S(n, K)$  is not an algebraic variety, and therefore its  $\ell$ -adic cohomology does not carry a representation of any Galois group. If  $\Pi$  is a cuspidal automorphic representation of  $GL(n, \mathbf{A}_K)$  that is polarized, then the twisted trace formula attaches to  $\Pi$  a collection (an  $L$ -packet) of automorphic representations of the unitary group  $G$  mentioned above; thus  $\Pi$  transfers to the cohomology of the  $S(G)$ , and this is where the Galois representation is realized (in nearly all cases).

When  $\Pi$  is not polarized, one uses the theory of Eisenstein series for the parabolic group

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<sup>2</sup>Although complete base change from unitary groups remains to be established (the quasi-split case has recently been treated in [55]), Labesse proved the basic properties in the case of cohomological representations, without which the proof of Theorem 2.2 would have been impossible.

<sup>3</sup>Strictly speaking, the limits discussed here are taken relative to the Zariski topology on appropriate eigenvarieties, so the term “ $p$ -adic limit” would not be quite appropriate. In many cases the missing representations can indeed be obtained as actual limits in the  $p$ -adic topology, but as far as I know these cases have not been given an intrinsic characterization.

$P_n$  to attach a family  $E(s, \Pi)$  of automorphic representations of  $G_n$ , with  $s \in \mathbb{C}$ . Up to twisting  $\Pi$  by a positive integral power of the norm, we may assume  $E(s, \Pi)$  is regular at 0 and write  $E(\Pi) = E(0, \Pi)$ . Then  $E(\Pi)$  is also cohomological and (for nearly all positive integral twists) defines a non-trivial class in the cohomology of the Shimura variety  $S(G_n)$  attached to (the unitary similitude group of)  $G_n$ . The realization in  $p$ -adic étale cohomology of this Eisenstein class then defines a  $p$ -adic Galois representation. However, it is easy to see that the semisimplification of this representation is a sum of abelian characters, and therefore it cannot be used to construct the desired  $\rho_{p, \Pi}$ .

Some years later, Skinner (and independently Urban) revived Clozel's idea by suggesting that  $E(\Pi)$  might be realized as the limit in a  $p$ -adic family of a sequence of *cuspidal* cohomological automorphic representations  $\tau_i$  of  $G_n$ . One then considers the collection of  $2n$ -dimensional representations  $\rho_{p, \tau_i}$ . The symbol  $\chi_{E(\Pi)} = \lim_i \text{tr} \rho_{p, \tau_i}$  then makes sense as a  $\overline{\mathbb{Q}}_p$ -valued function on  $\Gamma_{K, S}$  for appropriate  $S$ , and because it is the limit of traces of genuine representations it defines a  $2n$ -dimensional *pseudorepresentation*. The latter notion is an abstraction of the invariance properties of the character of a representation, first constructed in the 2-dimensional case by Wiles, then defined by Taylor in general using results (especially results of Procesi) from invariant theory. Taylor's theory implies that  $\chi_{E(\Pi)}$  is the character of a unique  $2n$ -dimensional representation, and by varying  $\Pi$  among its abelian twists it can be shown by elementary methods that  $\chi_{E(\Pi)}$  breaks up as the sum of two  $n$ -dimensional pieces, one of which is the  $\rho_{p, \Pi}$  of Theorem 2.2.

The hard part is to obtain  $E(\Pi)$  as the limit of cuspidal  $\tau_i$ . What this means is that the eigenvalues of Hecke operators at primes at which  $\Pi$  is unramified are  $p$ -adic limits of the corresponding Hecke eigenvalues on  $\tau_i$ . In [38] this is achieved by realizing  $E(\Pi)$  in a  $p$ -adic cohomology theory that satisfies a short list of desirable properties. The most important properties are (i) the global cohomology is computed as the hypercohomology in the (rigid) Zariski topology of the de Rham complex and (ii) the cohomology has a weight filtration, characterized by the eigenvalues of an appropriate Frobenius operator. The cohomology theory chosen in [38] is a version of Berthelot's rigid cohomology (generalizing Monsky-Washnitzer cohomology). This is calculated on the complement, in the *minimal* (Baily-Borel) compactification  $S(G_n)^*$  of  $S(G_n)$ , of the vanishing locus of lifts (modulo increasing powers of  $p$ ) of the Hasse invariant. This complement is affinoid and therefore by (i) the cohomology can be computed by a complex whose terms are spaces of  $p$ -adic modular forms, in the sense of Katz. By analyzing the finiteness properties of this complex, and using the density of genuine holomorphic modular forms in the space of  $p$ -adic modular forms, [38] writes  $E(\Pi)$  as the limit of cuspidal  $\tau_i$ , as required.

About a year after the results of [38] were announced, Scholze discovered a more flexible construction based on a very different cohomology theory, the  $p$ -adic étale cohomology of *perfectoid spaces*. The topological constructions in [38] can in principle also lift torsion classes in the cohomology of the locally symmetric space attached to  $GL(n, \mathbf{A}_K)$  to torsion classes in the cohomology of  $S(G_n)$ , but rigid cohomology cannot detect torsion classes. The  $p$ -adic étale cohomology of perfectoid spaces does not have this defect, and Scholze's article [61] not only gives a new and more conceptual proof of the results of [38] but applies to torsion classes as well. Thus Scholze proved a long-standing conjecture, first formulated by Ash in [2], that has greatly influenced subsequent speculation on  $p$ -adic representations of general Galois groups. The reader is referred to Scholze's article in the current proceedings for more information about his results.

**Restrictions on Galois representations on the cohomology of Shimura varieties.** In part (b) of 2.2 the proof of the deepest local properties of the (polarized)  $\rho_{p,\Pi}$  at primes dividing  $p$  were proved by finding representations closely related to  $\rho_{p,\Pi}$  (the images under tensor operations) directly in the cohomology of Shimura varieties. When  $\Pi$  is not polarized, the  $\rho_{p,\Pi}$  are still constructed in [38] and [61] by a limiting process, starting from a family of  $\rho_{p,\Pi_i}$  of geometric origin, but there is every reason to believe (see below) that the  $\rho_{p,\Pi}$  and its images under tensor operations will almost never be obtained in the cohomology of Shimura varieties, and although they are expected to be geometric no one has the slightest idea where they might arise in the cohomology of algebraic varieties.

**Room for improvement.** The infinitesimal character  $\lambda_\Pi \in Hom(\mathfrak{t}_v, \mathbb{C})$  is regular provided it is orthogonal to no roots of  $\mathfrak{t}_v$  in  $\mathfrak{g}_v$ ; in other words, if it is contained in the interior of a Weyl chamber. The regularity hypothesis in Theorem 2.2 can sometimes be relaxed to allow non-degenerate limits of discrete series, whose infinitesimal characters lie on one or more walls of a Weyl chamber. The first result of this type is the Deligne-Serre theorem which attaches (Artin) representations of  $\Gamma_{\mathbb{Q}}$  to holomorphic modular forms of weight 1. This has recently been generalized by Goldring [28] to representations of  $GL(n)$  obtained by base change from holomorphic limits of discrete series of unitary groups.

**2.2. Reciprocity.** Number theorists can't complain of a shortage of Galois representations. The étale cohomology of algebraic varieties over a number field  $K$  provides an abundance of  $\ell$ -adic representations of  $\Gamma_K$  satisfying the two Fontaine-Mazur axioms. One of the Fontaine-Mazur conjectures predicts that any irreducible representation of  $\Gamma_K$  satisfying these axioms is equivalent to a constituent of  $\ell$ -adic cohomology of some (smooth projective) variety  $V$  over  $K$ . The reciprocity Conjecture 2.1 (a) has been tested almost exclusively for  $\rho$  arising from geometry in this way. The paradigmatic case in which  $K = \mathbb{Q}$  and  $V$  is an elliptic curve was discussed in the ICM talks of Wiles (in 1994) and Taylor (in 2002).<sup>4</sup> The Fontaine-Mazur conjecture itself has been solved in almost all 2-dimensional cases when  $K = \mathbb{Q}$  for  $\rho$  that take complex conjugation to a matrix with determinant  $-1$ . Two different proofs have been given by Kisin and Emerton; both of them take as their starting point the solution by Khare and Wintenberger of Serre's conjecture on 2-dimensional modular representations of  $\Gamma_{\mathbb{Q}}$ . All of these results are discussed in a number of places, for example in [24, 46, 47]. I will therefore concentrate on results valid in any dimension  $n$ .

Let  $\rho : \Gamma_K \rightarrow GL(n, \mathcal{O})$  be a continuous representation with coefficients in an  $\ell$ -adic integer ring  $\mathcal{O}$  with maximal ideal  $\mathfrak{m}$  and residue field  $k$ ; let  $\bar{\rho} : \Gamma_K \rightarrow GL(n, k)$  denote the reduction of  $\rho$  modulo  $\mathfrak{m}$ . We say  $\rho$  is *residually automorphic* if  $\bar{\rho} \xrightarrow{\sim} \bar{\rho}_{\ell,\Pi}$  for some cuspidal automorphic representation  $\Pi$  of  $GL(n, \mathbf{A}_K)$ . The method for proving reciprocity initiated by Wiles consists in proving theorems of the following kind:

**Theorem 2.3** (Modularity Lifting Theorem, prototypical statement). *Suppose  $\bar{\rho}$  is residually automorphic. Then every lift of  $\bar{\rho}$  to characteristic zero that satisfies axioms (1) and (2) of Fontaine-Mazur, as well as*

- (1) *a polarization condition;*
- (2) *conditions on the size of the image of  $\bar{\rho}$  (typically including the hypothesis that  $\bar{\rho}$  is*

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<sup>4</sup>The nomenclature associated with the conjecture in this particular case, which predates the Fontaine-Mazur conjecture, is a matter of considerable sociological and philosophical interest.

*absolutely irreducible*); and

- (3) *ramification conditions at primes dividing  $\ell$  (typically including a regularity hypothesis)*

is automorphic. In particular, if  $\rho$  itself satisfies conditions (1), (2), and (3), then  $\rho$  is automorphic.

The method for proving such theorems is called the *Taylor-Wiles method* or the *Taylor-Wiles-Kisin method*, depending on context, and is named after its inventors in the setting when  $n = 2$ . The first theorems of this kind for arbitrary  $n$  were proved in [21, 70]. Together with the results of [39] they imply the Sato-Tate theorem for elliptic curves over  $\mathbb{Q}$  with non-integral  $j$ -invariant (see below). Subsequent improvements have allowed for less restrictive conditions in (2) and (3). The following theorem of Barnet-Lamb, Gee, Geraghty, and Taylor [3] represents the current state of the art.

**Theorem 2.4** (Modularity Lifting Theorem). *Let  $K$  be a CM field with totally real subfield  $K^+$ , and let  $c \in \text{Gal}(K/K^+)$  denote complex conjugation. Let  $\rho$  be as in 2.3. Suppose  $\ell \geq 2(n + 1)$  and  $K$  does not contain a primitive  $\ell$ -th root of 1. Suppose  $\rho$  satisfies axioms (1) and (2) of Fontaine-Mazur, as well as*

- (1)  $\rho^c \xrightarrow{\sim} \rho^\vee \otimes \mu$ , where  $\mu$  is an  $\ell$ -adic character of  $\Gamma_{K^+}$  such that  $\mu(c_v) = -1$  for every complex conjugation  $c_v$ ;
- (2) *The restriction of  $\bar{\rho}$  to  $\Gamma_{K(\zeta_\ell)}$  is absolutely irreducible; and*
- (3) *For any prime  $v$  of  $K$  dividing  $\ell$  the restriction  $\rho_v$  of  $\rho$  to the decomposition group  $\Gamma_v$  is potentially diagonalizable and is HT-regular:  $\rho_v$  has  $n$  distinct Hodge-Tate weights.*

*Suppose  $\rho$  is residually automorphic. Then  $\rho$  is automorphic.*

**Remark 2.5.** This is not the most general statement – there is a version of this theorem when  $K$  is totally real, and condition (2) can be replaced by adequacy.

**Remark 2.6.** The first novelty is the simplification of condition (2) on the image of  $\bar{\rho}$ : Thorne showed in [72] that the Taylor-Wiles-Kisin method works when the image of  $\bar{\rho}$  is what he called *adequate*, and this condition is implied by the irreducibility condition (2) as long as  $\ell \geq 2(n + 1)$ . The second novelty in 2.4 is the notion of potential diagonalizability. This is roughly the requirement that, after a finite base change,  $\rho_v$ , for  $v$  dividing  $\ell$ , is crystalline and can be deformed in a moduli space of crystalline representations to a sum of characters. It is known that  $\rho_v$  in the Fontaine-Laffaille range (the setting of [21, 70]) and ordinary  $\rho_v$  (the setting of [5, 27]) are potentially diagonalizable, but the condition is more general. In particular, it is preserved under finite ramified base change, which allows for considerable flexibility.

**2.3. Potential automorphy.** The need to assume residual automorphy places important restrictions on the application of theorems on the model of 2.3 to reciprocity. For some applications, however, it is enough to know that a given  $\rho$  is *potentially* residually automorphic: that  $\rho$  becomes residually automorphic after base change to an unspecified totally real or CM Galois extension  $K'/K$ . One can then often use a modularity lifting theorem to prove that  $\rho|_{\Gamma_{K'}}$  is automorphic, in other words that  $\rho$  is *potentially automorphic*. If  $\rho$  is attached to



a motive  $M$ , then  $L(s, \rho) = L(s, M)$  is given by an Euler product that converges absolutely in some right half-plane. An application of Brauer’s theorem on induced characters then implies that  $L(s, \rho)$  has a *meromorphic* continuation to the entire plane, and moreover (by a theorem due to Shahidi and to Jacquet-Piatetski-Shapiro-Shalika) that  $L(s, \rho)$  has no zeroes down to the right-hand edge of the critical strip.

Potential automorphy was introduced by Taylor in [68] in order to prove a potential version of the Fontaine-Mazur conjecture for 2-dimensional Galois representations. The method was generalized to higher dimensions in [39] and in subsequent work of Barnet-Lamb. The idea is the following. A theorem of the form 2.3 can be applied to an  $\ell$ -adic  $\rho$  that is residually automorphic. But it can also be applied if  $\rho = \rho_\ell$  is a member of a compatible family  $\{\rho_{\ell'}\}$  of  $\ell'$ -adic representations, where  $\ell'$  varies over all primes, provided at least one  $\rho_{\ell_1}$  in the family is known to be residually automorphic. It thus suffices to find a motive  $M$  of rank  $n$  such that

**Hypothesis 2.7.**  $\bar{\rho}_{\ell, M} \simeq \bar{\rho}$  and  $\bar{\rho}_{\ell_1, M}$  is known a priori to be residually automorphic for some  $\ell_1 \neq \ell$ .

Typically one assumes  $\bar{\rho}_{\ell', M}$  is induced from an algebraic Hecke character. The motives used in [39] are the invariants  $M_t$ , under a natural group action, in the middle-dimensional cohomology of the  $n - 1$ -dimensional hypersurfaces  $X_t$  with equation (depending on  $t$ , with  $t^{n+1} \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$ )

$$f_t(X_0, \dots, X_n) = (X_0^{n+1} + \dots + X_n^{n+1}) - (n + 1)tX_0 \dots X_n = 0 \tag{2.1}$$

This *Dwork family* of hypersurfaces was known to physicists for their role in the calculations that led to the formulation of the mirror symmetry conjectures [11]; and they were known to number theorists because Dwork had studied their cohomology in connection with  $p$ -adic periods.

The isomorphism class of  $X_t$  depends on  $t^{n+1}$  and one sees that their cohomology defines a hypergeometric local system over  $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ . Properties of this local system proved by a number of people, are used, together with a “local-global principle” due to Moret-Bailly, to find a  $t$  over a totally real (or CM) Galois extension  $K'/K$  such that  $M_t$  satisfies Hypothesis 2.7.

Applying the method of potential automorphy is not always automatic. One has to satisfy the conditions of Moret-Bailly’s theorem as well as conditions (1), (2), and (3) of 2.3. More details can be found in [35] (which was written, however, before the simplifications of [72] and [3]). Here are a few applications:

**Theorem 2.8.** *Let  $K = \mathbb{Q}$  and let  $\Pi$  be a cuspidal holomorphic automorphic representation of  $GL(2)_{\mathbb{Q}}$  (attached to an elliptic modular form of weight  $k \geq 2$ , say) to which one can associate a compatible family of 2-dimensional  $\ell$ -adic representations  $\rho_{\ell, \Pi}$ . Suppose  $\Pi$  is not obtained by automorphic induction from a Hecke character of an imaginary quadratic field. Then  $Sym^n \rho_{\ell, \Pi}$  is potentially automorphic for all  $n \geq 1$ .*

This theorem was proved first when  $k = 2$  in [21, 39, 70], assuming  $\Pi_v$  is a Steinberg representation for some  $v$ . This hypothesis was dropped, and was generalized to all  $k$  in [5]. It follows from the arguments of Serre in [63] and from the non-vanishing of  $L(s, Sym^n \rho)$  mentioned above, that this implies the *Sato-Tate conjecture* for elliptic modular forms [5, 21, 39, 70]:

**Theorem 2.9.** *Let  $f$  be an elliptic modular newform of weight  $k$  for  $\Gamma_0(N)$  (for some  $N$ ), and assume the  $\ell$ -adic Galois representations  $\rho_{\ell, f}$  attached to  $f$  are not dihedral. For any prime  $p$  not dividing  $N$ , let  $a_p(f)$  denote the eigenvalue of the normalized Hecke operator at  $p$  on  $f$ . Let  $\tilde{a}_p(f) = a_p(f)/2p^{\frac{k-1}{2}}$ , which is known to be a real number in the interval  $[-1, 1]$ . As  $p$  varies, the  $\tilde{a}_p(f)$  are equidistributed in  $[-1, 1]$  for the measure  $\sqrt{1-t^2}dt$ .*

*In particular, if  $E$  is an elliptic curve over  $\mathbb{Q}$  without complex multiplication, and  $1+p-a_p(E)$  is the number of points of  $E$  over  $\mathbb{F}_p$ , then the numbers  $a_p(E)/2p^{\frac{1}{2}}$  are equidistributed in  $[-1, 1]$  for the measure  $\sqrt{1-t^2}dt$ .*

The hypothesis that  $f$  has trivial nebentypus (is a form for  $\Gamma_0(N)$ ) is unnecessary and was only included to allow for a simple statement. A version of 2.8 for Hilbert modular forms was proved by Barnet-Lamb, Gee, and Geraghty, and they derived the corresponding version of Theorem 2.9. All of these results were subsumed in the following theorem of Patrikis and Taylor [59], a strengthening of one of the main theorems of [3]:

**Theorem 2.10.** *Let  $K$  be totally real (resp. CM) and let  $\{r_\lambda\}$  be a weakly compatible family of  $\lambda$ -adic representations of  $\Gamma_K$  (where  $\lambda$  runs over finite places of a number field  $M$ ). Assume the  $r_\lambda$  are pure of fixed weight  $w$  (the Frobenius eigenvalues at an unramified place of norm  $q$  are Weil  $q^{\frac{w}{2}}$ -numbers); that they are HT-regular; and that they satisfy an appropriate polarization condition. Then there is a finite totally real (resp. CM) Galois extension  $K'/K$  over which the family becomes automorphic.*

The Hodge-Tate multiplicities of  $n$ -dimensional  $\ell$ -adic representations realized on the cohomology of the Dwork family are at most 1; moreover,  $n$  has to be even, and each Hodge-Tate weight between 0 and  $n-1$  occurs. Griffiths transversality implies that such a condition is inevitable when Hodge structures vary in non-trivial families. This appears to restrict the applicability of the Dwork family to proving potential automorphy. However, it was observed in [34], and more generally in [5], that it suffices to prove that a given  $\rho_{\ell, \Pi}$  becomes automorphic after tensoring with the Galois representation obtained by induction from an automorphic Galois character attached to a Hecke character of an appropriate cyclic CM extension  $K'/K$ . This observation was applied in the proof of 2.9 and more systematically in [3], in both cases in order to replace the given Hodge-Tate weights of  $\rho$  by the set of weights adapted to the cohomology of the Dwork family.

**Remark 2.11.** Let  $f$  be as in Theorem 2.9 and  $\Pi$  the associated automorphic representation. Theorem 2.9 is equivalent to the assertion that, as  $p$  varies over primes unramified for  $\rho_{\ell, \Pi}$ , the conjugacy classes of  $\rho_{\ell, f}(Frob_p)$ , normalized so that all eigenvalues have complex absolute value 1, are equidistributed in the space of conjugacy classes of  $SU(2)$ . A version of the Sato-Tate conjecture can be formulated for a general motive  $M$ ;  $SU(2)$  is replaced by the derived subgroup of the compact real form of the Mumford-Tate group  $MT(M)$  of  $M$ . In order to prove this conjecture for more complicated  $MT(M)$  one would have to be able to prove the corresponding generalization of Theorem 2.8, with the symmetric powers replaced by the full set of equivalence classes of irreducible representations  $\sigma$  of  $MT(M)^{der}$ . But even if the  $\ell$ -adic representation  $\rho_{\ell, M}$  attached to  $M$  is HT-regular,  $\sigma \circ \rho_{\ell, M}$  is generally not HT-regular, and thus cannot be obtained by Theorem 2.2. Thus one has no way to start proving potential automorphy of  $\sigma \circ \rho_{\ell, M}$  once  $MT(M)^{0, der}$  is of rank greater than 1.

**2.3.1.  $p$ -adic realization of very general Galois representations.** It was mentioned above that the proof of 2.2 is completed by a  $p$ -adic approximation argument. One says more gen-

erally that a  $p$ -adic representation  $\rho : \Gamma_{K,S} \rightarrow GL(n, \overline{\mathbb{Q}}_p)$  for some  $S$  is  $p$ -adically automorphic if  $\rho = \lim_i \rho_i$  (for example, in the sense of pseudo-representations, where the limit can be in the Zariski or in the  $p$ -adic topology), where each  $\rho_i$  is an automorphic Galois representation of  $\Gamma_{K,S}$ . The theory of eigenvarieties shows that  $p$ -adically automorphic Galois representations vary in  $p$ -adic analytic families. The representations  $\rho_{p,\Pi}$  of 2.2 are  $HT$ -regular because  $\Pi$  is cohomological, but analytic families of  $p$ -adically automorphic Galois representations can specialize to representations that are Hodge-Tate but not regular, and to representations that are not Hodge-Tate at all.

One can ask whether a given  $\rho$  is  $p$ -adically automorphic. There are discrete obstructions; for example the set of ramified primes is finite in any  $p$ -adic family. There are also sign obstructions. The 2-dimensional Galois representations  $\rho_{\ell,f}$  attached to an elliptic modular form  $f$  are *odd*:  $\det \rho_{\ell,f}(c) = -1$  when  $c$  is complex conjugation. In other words, no representation  $\rho$  for which  $\det \rho_{\ell,f}(c) = 1$  can be obtained in the cohomology of a Shimura variety. The signature of complex conjugation is constant on  $p$ -adic analytic families of Galois representations, and therefore represents an obstruction to realizing such an *even* representation as a  $p$ -adically automorphic representation.

However, the direct sum of two even representations does not necessarily have such a sign obstruction. Similar discrete invariants characterize  $p$ -adically automorphic Galois representations in higher dimension, but they can be made to vanish upon taking appropriate direct sums. Say  $\rho$  is  $p$ -adically stably automorphic if  $\rho \oplus \rho'$  is  $p$ -adically automorphic for some  $\rho'$ . One knows what this means if  $K$  is a totally real or CM field. If not, let  $K_0 \subset K$  be the maximal totally real or CM subfield, and say a  $p$ -adic representation  $\rho$  is  $p$ -adically stably automorphic if  $\rho \oplus \rho'$  is the restriction to  $\Gamma_K$  of a  $p$ -adically automorphic representation of  $\Gamma_{K_0}$ .

**Question 2.12.** *Is every  $p$ -adic representation of  $\Gamma_K$  that satisfies the Fontaine-Mazur axioms stably  $p$ -adically automorphic?*

The main theorem of [30] states, roughly, that every  $p$ -adic representation of  $\Gamma_K$  is “stably potentially residually automorphic,” where the reader is invited to guess what that means.

One can often define analytic or geometric invariants of  $p$ -adic families by interpolation of their specializations to automorphic points. Thus one defines  $p$ -adic  $L$ -functions or Galois cohomology (Selmer groups) of  $p$ -adic families. Specializations to points not known to be automorphic (e.g., because they are not  $HT$ -regular) define invariants of the corresponding Galois representations.

### 2.3.2. Prospects for improvement.

- (a) Condition (1) in Theorem 2.4 corresponds to the polarization condition in (b) of Theorem 2.2. At present no one knows how to remove this condition and thus to prove the reciprocity conjecture for all representations constructed in Theorem 2.2 (see, however, the articles [10] of Calegari and Geraghty and [31] of Hansen). Removing condition (1) is sufficient, and probably necessary, to show that the  $\rho_{\ell,\Pi}$  of Theorem 2.2 are irreducible for (almost) all  $\ell$ .
- (b) Although we have seen that substitutes can be found for residual irreducibility in applications to compatible families, it remains a major obstacle for many applications. In addition to the argument applied in Skinner-Wiles for 2-dimensional representations of  $\Gamma_{\mathbb{Q}}$ , Thorne has recently found a new method based on level raising [73].

- (c) The article [70] replaces the very deep questions regarding congruences between automorphic forms of different levels (“level-raising”, which an earlier version of [21] proposed to solve by generalizing Ihara’s Lemma on congruences between elliptic modular forms) by a careful study of the singularities of certain varieties of tame representations of local Galois groups. But this comes at the cost of losing control of nilpotents in the deformation rings. In particular, current methods cannot classify liftings of  $\bar{\rho}$  to rings in which  $\ell$  is nilpotent. This may be important if one wants to extend the results of this section to the torsion representations constructed by Scholze.
- (d) Dieulefait has expanded on the ideas by Khare and Wintenberger to prove the Serre conjecture and has proved some astonishing results. For example, he has proved base change of elliptic modular forms to any totally real extension [23]. The methods of [46] and of [23] do not assume residual automorphy but actually prove it in the cases they consider. It is not yet known whether or not these methods can be applied in higher dimension.
- (e) The authors of [3] ask whether every potentially crystalline representation is potentially diagonalizable. An affirmative answer would expand the range of their methods. The regularity hypothesis of Condition (3) seems insuperable for the moment. At most one can hope to prove reciprocity for representations like those constructed in [28], with Hodge-Tate multiplicities at most 2. The recent proof by Pilloni and Stroth of the Artin conjecture for (totally odd) 2-dimensional complex representations of  $\Gamma_K$ , when  $K$  is totally real, is the strongest result known in this direction. As long as one has no method for constructing automorphic Galois representations with Hodge-Tate multiplicities 3 or greater, the reciprocity question for such representations will remain inaccessible.

### 3. Critical values of automorphic $L$ -functions

**3.1. Critical values and automorphic motives.** Let  $M$  be a (pure) motive of rank  $n$  over a number field  $K$ , with coefficients in a number field  $E$ . By restriction of scalars we can and will regard  $M$  as a motive of rank  $n[K : \mathbb{Q}]$  over  $\mathbb{Q}$ . The values at integer points of the  $L$ -function  $L(s, M)$  are conjectured to contain deep arithmetic information about  $M$ . If, for example,  $M = M(A)$  is the motive attached to the cohomology in degree 1 of an abelian variety  $A$ , then the value, or more generally the first non-vanishing derivative, of  $L(s, M(A))$  at  $s = 1$  is predicted by the Birch–Swinnerton-Dyer conjecture. This is the only *critical value* of  $L(s, M(A))$ , in the sense of Deligne (the importance of critical values had previously been noted by Shimura). Deligne formulated his conjecture on critical values in one of his contributions to the 1977 Corvallis conference. We follow Deligne in working with motives for absolute Hodge cycles; thus  $M$  is by definition a collection of compatible realizations in the cohomology of smooth projective algebraic varieties. The realization in  $\ell$ -adic cohomology gives the Galois representation  $\rho_{\ell, M}$  on an  $\ell$ -adic vector space  $M_{\ell}$ , and therefore determines  $L(s, M)$ . Extension of scalars from  $\mathbb{Q}$  to  $\mathbb{C}$  makes  $M$  a motive over  $\mathbb{C}$ , whose cohomology is thus a direct factor of the cohomology of a complex manifold, whose topological cohomology is a  $\mathbb{Q}$ -vector space called  $M_B$  (Betti realization). Complex conjugation on the points of  $M(\mathbb{C})$  acts on  $M_B$  as an involution  $F_{\infty}$ . As a motive over  $\mathbb{Q}$ ,  $M$  also has the algebraic de Rham cohomology, a  $\mathbb{Q}$ -vector space  $M_{dR}$  with a decreasing

Hodge filtration  $\dots F^q M_{dR} \subset F^{q-1} M_{dR} \dots$  by  $\mathbb{Q}$ -subspaces. For any integer  $m$  let  $M(m)$  denote the Tate twist  $M \otimes \mathbb{Q}(m)$ . Hodge theory defines comparison isomorphisms

$$I(m)_{M,\infty} : M(m)_B \otimes \mathbb{C} \xrightarrow{\sim} M(m)_{dR} \otimes \mathbb{C}.$$

This isomorphism does not respect the rational structures on the two sides. By restricting  $I(m)_{M,\infty}$  to the  $+1$ -eigenspace of  $F_\infty$  in  $M(m)_B$  and then projecting on a certain quotient  $M(m)_{dR}/F^q M(m)_{dR} \otimes \mathbb{C}$ , one defines an isomorphism between two complex vector spaces of dimension roughly half that of  $M$ , provided  $M(m)$  is *critical* in Deligne’s sense. The determinant of this isomorphism, calculated in rational bases of the two sides, is the *Deligne period*  $c_{\mathbb{Q}}^+(M(m))$ . It is a determinant of a matrix of integrals of rational differentials in  $M_{dR}$  over rational homology cycles, and is well defined up to  $\mathbb{Q}^\times$ -multiples. More generally, if  $M$  is a *motive with coefficients in* a number field  $E$  – in other words, if there are actions of  $E$  on each of the vector spaces  $M_B, M_{dR}, M_\ell$ , compatible with the comparison isomorphisms – then there is a Deligne period  $c_E^+(M(m))$  well-defined up to  $E^\times$ -multiples; moreover,  $L(s, M)$  then defines an element of  $E \otimes \mathbb{C}$ , as in [22]. In the following discussion we will drop the subscript and just write  $c^+(M(m))$  for the Deligne period with coefficients.

We call  $s = m$  a *critical value* of  $L(s, M)$  if  $M(m)$  is critical. The set of critical  $m$  can be read off from the Gamma factors in the (conjectural) functional equation of  $L(s, M)$  ([22], Definition 1.3). When  $M = M(A)$ ,  $s = 1$  is the only critical value. Deligne’s conjecture is the assertion that

**Conjecture 3.1** ([22]). *If  $m$  is a critical value of the motive  $M$  with coefficients in  $E$ , then*

$$L(m, M)/c^+(M(m)) \in E^\times.$$

Beilinson’s conjectures express the non-critical integer values of  $L(s, M)$  at non-critical integers in terms of the motivic cohomology (higher algebraic  $K$ -theory) of  $M$ . Automorphic methods give very little information about non-critical values of the  $L$ -functions of motives that can be related to automorphic forms, and this survey has nothing to say about them. On the other hand, the de Rham realizations of the motives that arise in the cohomology of Shimura varieties are given explicitly in terms of automorphic forms. One can therefore state versions of Deligne’s conjecture for certain of these motives entirely in the language of automorphic forms.<sup>5</sup> The literature on special values of  $L$ -functions is vast and a book-length survey is long overdue. Automorphic versions of Deligne’s conjecture represent a relatively small segment of the literature that is still too extensive for treatment in the space of this article. The proofs are generally quite indirect, not least because one can rarely write down  $M_B$  in terms of automorphic forms. When  $M$  is realized in the cohomology (with coefficients) of a Shimura variety  $S(G)$ , one can occasionally define non-trivial classes in  $M_B$  by projecting onto  $M$  the cycles defined by Shimura subvarieties  $S(G') \subset S(G)$ . Integrating differential forms on  $S(G) \times S(G)$  over the diagonal cycle  $S(G)$  amounts to computing a

<sup>5</sup>Strictly speaking, Deligne’s conjecture only makes sense in the setting of a theory of motives that is the subject of very difficult conjectures. For example, one expects that if  $M$  and  $M'$  are motives such that the triples  $(M_B, M_{dR}, I(m)_{M,\infty})$  and  $(M'_B, M'_{dR}, I(m)_{M',\infty})$  are isomorphic, then  $M$  and  $M'$  are isomorphic as motives. This would follow from the Hodge conjecture. Similarly, one assumes that  $L(s, M) = L(s, M')$  implies that  $M \simeq M'$ ; this would follow from the Tate conjecture.

Blasius’s proof of Deligne’s conjecture for  $L$ -functions of Hecke characters of CM fields is carried out within the framework of motives for absolute Hodge cycles. It is practically the only authentically motivic result known in this direction.

cohomological cup product. In this brief account we limit our attention to a class of motives whose Deligne periods can be factored as products of cup products of this kind.

Suppose  $K$  is a CM field. As explained in 2.1.1, most of the representations  $\rho_{\ell, \Pi}$  of  $\Gamma_K$  are realized in the cohomology of a Shimura varieties  $S(G)$  attached to unitary groups  $G$ . Along with the  $n$ -dimensional Galois representation this construction yields a candidate for the rank  $n$  motive  $M(\Pi)$ . Originally  $M(\Pi)$  is defined over  $K$ ; one obtains a motive  $RM(\Pi) = R_{K/\mathbb{Q}}M(\Pi)$  by restriction of scalars to  $\mathbb{Q}$ , taking into account the theorem of Borovoi and Milne on conjugation of Shimura varieties (the Langlands conjecture). The spaces  $RM(\Pi)_{dR}$  and  $RM(\Pi)_B$  satisfy analogues of conditions (1) and (3) of Theorem 2.4. The regularity condition (3) implies there is a set of integers  $q_1 < q_2 < \dots < q_n$  such that  $\dim_E F^q RM(\Pi)_{dR} / F^{q+1} RM(\Pi)_{dR} = 1$  if and only if  $q = q_i$  for some  $i$ , and the dimension is 0 otherwise. Here  $E = E(\Pi)$  is the field of coefficients of  $RM(\Pi)$  (more precisely,  $E$  is a finite product of number fields). We choose a non-zero  $\mathbb{Q}$ -rational  $E$ -basis  $\omega_i$  of  $F^{q_i} RM(\Pi)_{dR} / F^{q_i+1} RM(\Pi)_{dR}$ , view  $\omega_i$  as a (vector-valued) automorphic form on  $G(\mathbb{Q}) \backslash G(\mathbb{A})$ , and let  $Q_i(\Pi) = \langle \omega_i, \omega_i \rangle$  denote its appropriately normalized  $L_2$  inner product with itself.

**Conjecture 3.2.** *Up to multiplication by  $E^\times$ , each  $Q_i(\Pi)$  depends only on the automorphic representation  $\Pi$  of  $GL(n)$  and not on the realization in the cohomology of a Shimura variety.*

This conjecture is implied by the Tate conjecture. It has been verified in many cases for the (holomorphic) period  $Q_1(\Pi)$ . The author has partial results for general  $Q_i(\Pi)$ .

Given any motive  $M$  of rank  $n$  satisfying conditions (1) and (3) of 2.4 we can define invariants  $Q_i(M)$  in the same way, and a determinant factor  $q(M)$  (for this and what follows, see [32, 36], and section 4 of [29]). For any integer  $0 \leq r \leq n$  we write

$$P_{\leq r}(M) = q(M)^{-1} \cdot \prod_{i \leq r} Q_i(M).$$

Let  $M'$  be a second motive of rank  $n'$ , satisfying conditions (1) and (3) of 2.4. Then for any integer  $m$  critical for  $R_{K/\mathbb{Q}}(M \otimes M')$  there is a factorization (cf. [29] (4.11)):

$$c^+(R(M \otimes M')(m)) \sim (2\pi i)^{c(m, n, n')} \prod_{r=1}^n P_{\leq r}(M)^{a_r} \prod_{r'=1}^{n'} P_{\leq r'}(M')^{b_{r'}} \tag{3.1}$$

where  $\sim$  means that the ratio of the two sides lies in the multiplicative group of the coefficient field,  $c(m, n, n')$  is an explicit polynomial in  $m$  and the dimensions,  $0 \leq a_r := a(r, M, M')$ ,  $b_{r'} := b(r', M, M')$  and

$$\sum_r a_r \leq n'; \quad \sum_{r'} b_{r'} \leq n.$$

Defining  $\Pi$  as above, there is an (ad hoc) determinant factor  $q(\Pi)$ , and we let

$$P_{\leq r}(\Pi) = q(\Pi)^{-1} \cdot \prod_{i \leq r} Q_i(\Pi).$$

An automorphic version of Deligne’s conjecture is

**Conjecture 3.3.** *Let  $\Pi$  and  $\Pi'$  be cuspidal automorphic representations of  $GL(n)_K$  and  $GL(n')_K$ , satisfying the hypotheses of Theorem 2.2 (b). Let  $m$  be a critical value of*

$$L(s, R_{K/\mathbb{Q}}(M(\Pi) \otimes M(\Pi'))) = L(s - \frac{n + n' - 2}{2}, \Pi \times \Pi').$$

Then

$$L(m, R_{K/\mathbb{Q}}(M(\Pi) \otimes M(\Pi'))) \sim (2\pi i)^{c(m, n, n')} \prod_{r=1}^{n-1} P_{\leq r}(\Pi)^{a_r} \prod_{r'=1}^{n-2} P_{\leq r'}(\Pi')^{b_{r'}},$$

with  $a_r, b_{r'}$  as in (3.1).

The integers  $a_r$  and  $b_{r'}$  of (3.1) are determined purely by the relative position of the Hodge decompositions of  $M_{dR} \otimes \mathbb{C}$  and  $M'_{dR} \otimes \mathbb{C}$  (and don't depend on  $m$ ). Suppose  $M = RM(\Pi)$ ,  $M' = RM(\Pi')$ , with  $\Pi$  and  $\Pi'$  as in (3.3). The regularity hypotheses imply that there are finite-dimensional representations  $W(\Pi_\infty)$  and  $W'(\Pi'_\infty)$  of  $GL(n)_K$  and  $GL(n')_K$ , respectively, such that  $\Pi_\infty$  and  $W(\Pi_\infty)$  (resp.  $\Pi'_\infty$  and  $W'(\Pi'_\infty)$ ) have the same infinitesimal characters. The  $a_i$  and  $b_{i'}$  can be computed explicitly in terms of the highest weights of  $W(\Pi_\infty)$  and  $W'(\Pi'_\infty)$ . For example, suppose  $n' = n - 1$  and

$$Hom_{GL(n-1, K \otimes \mathbb{C})}(W(\Pi_\infty) \otimes W(\Pi'_\infty), \mathbb{C}) \neq 0. \tag{3.2}$$

Then  $a_i = b_{i'} = 1, 1 \leq i \leq n - 1; 1 \leq i' \leq n - 2; a_n = b_{n-1} = 0$ .

**Theorem 3.4.** *Suppose  $K$  is an imaginary quadratic field. Let  $\Pi$  and  $\Pi'$  be as in 3.3. Suppose moreover that the infinitesimal characters of  $\Pi_\infty$  and  $\Pi'_\infty$  satisfy 3.2 and are sufficiently regular. Then there are constants  $c'(m, \Pi_\infty, \Pi'_\infty)$  such that*

$$L(m, R_{K/\mathbb{Q}}(M(\Pi) \otimes M(\Pi')))/[c'(m, \Pi_\infty, \Pi'_\infty) \prod_{r=1}^{n-1} P_{\leq r}(\Pi) \prod_{r'=1}^{n-2} P_{\leq r'}(\Pi')] \in \overline{\mathbb{Q}} \tag{3.3}$$

for every critical value  $m$ .

This is a reinterpretation of Theorem 1.2 of [29]. There the invariants  $P_{\leq r}(\Pi)$  are replaced by complex numbers  $P^{(r)}(\Pi)$ , which are Petersson square norms of holomorphic automorphic forms on unitary Shimura varieties of different signatures (and it is shown that the quotient in (3.3) lies in a specific number field). Naturally one expects the constants  $c'(m, \Pi_\infty, \Pi'_\infty)$  to be powers of  $2\pi i$ . The Tate conjecture implies an identity between the two kinds of invariants, and this has been proved (up to unspecified archimedean factors, and up to  $\overline{\mathbb{Q}}$ -multiples) in [33] (and subsequent unpublished work).

The methods of [29] are based on interpreting the Rankin-Selberg integral for  $GL(n) \times GL(n - 1)$  as a cohomological cup product. Such arguments have been used previously by Mahnkopf and Raghuram; see [60] for the most general results in this direction. Earlier results on this problem were conditional on the conjecture that certain archimedean zeta integrals did not vanish identically. Sun's recent proof of this conjecture [66] has revived interest in the problem and one can expect rapid progress in the next few years. For general number fields one does not have the analogues of the invariants  $P_{\leq r}(\Pi)$  and the results of [60] are expressed in terms of period invariants obtained by comparing the cohomological rational structure of  $\Pi$  with one defined by Whittaker models. The (mild) regularity hypothesis of 3.4

is required in the comparison of these Whittaker period invariants with the motivic invariants  $P_{\leq r}(\Pi)$ . Similar arguments should suffice to treat the cases of Conjecture 3.3 for  $n' \leq n - 1$  that satisfy an analogue of (3.2), for general CM fields. (The case where  $n' = 1$  was treated by the author in a series of papers, starting with [32], and is used crucially in the proof of Theorem 3.4.) The full scope of the methods of [29] is not yet clear, but it is certain that it is not limited to the situation of (3.2). The identification of  $c'(m, \Pi_\infty, \Pi'_\infty)$  with the invariant  $(2\pi i)^{c(m, n, n-1)}$  is likely to follow from these methods as well.<sup>6</sup>

**3.2. How general are these results?** Only a restricted class of Galois representations can be obtained using the cohomology of Shimura varieties, and only those that can be realized directly in the cohomology are associated to motives that admit an automorphic interpretation. The Rankin-Selberg  $L$ -functions described in the previous section, along with a few related constructions (symmetric and exterior squares and adjoint  $L$ -functions), seem to be the only ones whose critical values can be analyzed by automorphic methods. Raghuram's results in [60] apply only under the hypothesis (3.2). It should be straightforward to generalize his methods to pairs  $\Pi, \Pi'$  where  $\Pi$  is cuspidal and  $\Pi'$  is an essentially tempered cohomological Eisenstein series, as in [29] (or earlier work of Mahnkopf). If Raghuram's results could be extended to cases where neither  $\Pi$  nor  $\Pi'$  is cuspidal, then the hypothesis (3.2) would be superfluous (in Theorem 3.4 as well).

A motivic analysis of critical values of Rankin-Selberg  $L$ -functions, as in Theorem 3.4, has thus far only been carried out for CM fields. Bhagwat has proved an analogue of the relation (3.1) when  $K = \mathbb{Q}$ , following earlier work of Yoshida (see the appendix to [60]) and similar factorizations must hold for totally real fields. As far as I know, no one has proposed automorphic interpretations of the terms that occur in Bhagwat's factorization. For  $\Pi$  satisfying the polarization condition as in (b) of Theorem 2.2 it should be possible to interpret some of them as periods of motives realized in the cohomology of Shimura varieties attached to special orthogonal groups of signature  $(2, n)$ . In the absence of a polarization condition, Shimura varieties seem to be of no help.

**3.3. Exact formulas for the central critical value.** The conjectures of Bloch-Kato and Fontaine-Perrin-Riou give exact formulas for special values of motivic  $L$ -functions. The algebraic quotients  $L(m, M)/c^+(M(m))$  and their generalizations to non-critical values are expressed explicitly as products of local and global algebraic factors defined in terms of Galois cohomology. For the central critical value these expressions generalize the Birch-Swinnerton-Dyer conjecture for the value at  $s = 1$  of  $L(s, M(A))$ , in the notation of the previous section.

Beginning with the thesis of Waldspurger, exact formulas have also been found for certain central values of automorphic  $L$ -functions. The conjecture of Ichino-Ikeda, and its version for unitary groups formulated by N. Harris, [42, 45] give exact formulas for central values in the framework of the Gan-Gross-Prasad conjectures [26]. In what follows  $K$  is a CM field. We change notation and let  $\Pi$  denote a cuspidal automorphic representation of  $GL(n)_K$  that descends to a (cuspidal)  $L$ -packet  $P_{\Pi, V}$  of a given  $G = U(V)$ , viewed as group over  $K^+$ , with  $\dim V = n$ . Similarly,  $\Pi'$  is an automorphic representation of  $GL(n-1)_K$  obtained by base change from a (cuspidal)  $L$ -packet  $P_{\Pi', V'}$  of  $G' = U(V')$ . It

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<sup>6</sup>Note added in proof. This has now been carried out, at least when the coefficients are sufficiently regular, by Lin Jie.



is assumed that  $V'$  embeds in  $V$  as a non-degenerate hermitian subspace of codimension 1. For any  $\pi \in P_{\Pi, V}$  and  $\pi' \in P_{\Pi', V'}$ , the pairing

$$I = \pi \otimes \pi' \rightarrow \mathbb{C} : f \otimes f' \mapsto \int_{G'(K^+) \backslash G'(\mathbf{A})} f(g')f'(g')dg', \quad f \in \pi, f' \in \pi' \quad (3.4)$$

is invariant under the diagonal action of  $G'(\mathbf{A})$ . One of the Gan-Gross-Prasad conjectures asserts that the space of such invariant pairings is of dimension 1 for exactly one pair  $(V, V')$  and one pair  $(\pi, \pi') \in P_{\Pi, V} \times P_{\Pi', V'}$ , and that the lucky pair is identified by a complicated formula involving root numbers. The non-archimedean part of this conjecture has been proved by R. Beuzart-Plessis, following the method used by Waldspurger to solve the analogous conjecture for special orthogonal groups [7, 74]. Thus if one fixes a non-trivial pairing  $B : \pi \otimes \pi' \rightarrow \mathbb{C}$ , the pairing  $I$  defined in 3.4 is a multiple of  $B$ . The Ichino-Ikeda Conjecture can be seen as a determination of this multiple. In the statement of the conjecture, the superscript  $\vee$  denotes contragredient; all integrals are taken with respect to Tamagawa measure.

**Conjecture 3.5** ([45]). *Let  $f \in \pi, f' \in \pi', f^\vee \in \pi^\vee, f'^\vee \in \pi'^\vee$ , and suppose all four vectors are factorizable. Then*

$$\frac{I(f, f') \cdot I(f^\vee, f'^\vee)}{\langle f, f^\vee \rangle_2 \langle f', f'^\vee \rangle_2} = 2^{-r} \prod_{v \in S} Z_v(f, f', f^\vee, f'^\vee) \cdot \Delta \cdot \frac{L(\frac{1}{2}, \Pi \times \Pi')}{L(1, \pi, Ad)L(1, \pi', Ad)}.$$

Here  $\langle \bullet, \bullet \rangle_2$  are the  $L_2$  pairings, the factor  $2^{-r}$  is trivial when  $\Pi$  and  $\Pi'$  are cuspidal but not in general,  $S$  is the set of ramified primes for  $\pi, \pi'$ , and the chosen vectors, including archimedean primes, the  $Z_v$  for  $v \in S$  are normalized integrals of matrix coefficients attached to the data,  $\Delta$  is a special value of a finite product of abelian  $L$ -functions (the  $L$ -function of the Gross motive), the numerator on the right-hand side is the Rankin-Selberg product for  $GL(n) \times GL(n - 1)$ , and the factors in the denominator are the Langlands  $L$ -functions for  $G$  and  $G'$  attached to the adjoint representations of their  $L$ -groups.

Here and elsewhere,  $L(s, \bullet)$  denotes the non-archimedean Euler product. The  $L$ -functions in the right-hand side are given the unitary normalization. Thus the completed  $L$ -function  $\Lambda(s) = L_\infty(s, \Pi \times \Pi') \cdot L(s, \Pi \otimes \Pi')$  in the numerator of the right-hand side always satisfies  $\Lambda(s) = \pm \Lambda(1 - s)$ . When  $\Pi$  and  $\Pi'$  satisfy (b) of 2.2, however, there is a second (motivic) normalization as well, in which the value  $s = \frac{1}{2}$  is replaced by an integer value, and all the values of  $L$ -functions that occur in the right-hand side are critical.

Conjecture 3.5 is of no interest when the sign is  $-1$ , because the numerator vanishes trivially. When the  $L$ -function is motivic, there have been proposals for an arithmetic substitute for the conjecture in this case, with  $L(\frac{1}{2}, \bullet)$  replaced by its derivative at  $s = \frac{1}{2}$ , along the lines of the Gross-Zagier conjecture and subsequent work. When the sign is  $+1$ , the conjecture refines the global Gan-Gross-Prasad conjecture, which asserts that  $L(\frac{1}{2}, \Pi \times \Pi') = 0$  if and only if the pairing  $I$  of 3.4 is trivial.

When  $L(\frac{1}{2}, \Pi \times \Pi') \neq 0$ , Conjecture 3.5 gives an exact expression for its value, provided one can make good choices of the test vectors  $f, f', f^\vee, f'^\vee$  and can control the local zeta integrals. It is natural to speculate that these zeta integrals can be interpreted in terms of local Galois cohomological information, and that when  $\Pi$  and  $\Pi'$  are attached to motives, the expressions on the two sides of Conjecture 3.5 can be matched termwise with corresponding expressions in the Bloch-Beilinson and Bloch-Kato conjectures. The local factor

$Z_v(f, f', f^\vee, f'^\vee)$  is the integral of the matrix coefficient of  $\pi_v$  attached to the pair  $(f_v, f_v^\vee)$  against the matrix coefficient of  $\pi'_v$  attached to  $(f'_v, f'^\vee_v)$ . The following question is deliberately vague.

**Question 3.6.** *For any given pair of local (ramified) representations  $\pi_v, \pi'_v$ , is there a quadruple  $f_v, f'_v, f_v^\vee, f'^\vee_v$  such that the local zeta integral  $Z_v(f, f', f^\vee, f'^\vee)$  exactly equals the local Galois-cohomological factor in the Bloch-Kato conjecture?*

As explained in [37], the expressions on the left-hand side are algebraic multiples of invariants called *Gross-Prasad periods* that depend only on  $\Pi$  and  $\Pi'$ , provided the test vectors are chosen to be rationally normalized (with respect to coherent cohomology). The denominators are closely related to the  $P_{\leq r}$  defined above. Combining Conjecture 3.5 with Conjecture 3.3, one gets conjectural expressions for the Gross-Prasad periods as well in terms of  $P_{\leq r}(\pi)$  and  $P_{\leq r'}(\pi')$ ; see [37], Conjecture 5.16.

In order to compare the local terms of Conjecture 3.5 with the Galois-cohomological data of the Bloch-Kato conjecture, *integral* normalizations of the test vectors are needed. It is well known, however, that even the module of elliptic modular forms with integral modular Fourier coefficients is not spanned by Hecke eigenfunctions. This is the phenomenon of *congruences* between Hecke eigenvalues for different automorphic representations, which is the subject of theorems of the form 2.3, and it is no less relevant to automorphic representations of groups other than  $GL(2)$ .

**3.3.1. Adjoint  $L$ -functions.** The denominator of the Ichino-Ikeda formula is relevant to the problem of integral normalization of test vectors. The point  $s = 1$  is the only critical value of the adjoint  $L$ -functions that occur there. Suppose  $\pi$  has an associated motive  $M(\Pi) = M(\pi)$ . Then for any prime  $\ell$ , the Bloch-Kato conjecture identifies the  $\ell$ -adic valuation of the quotient of  $L(1, \pi, Ad)$  by an (integrally normalized) Deligne period with the order of a Galois cohomology group that is supposed to count the number of  $\ell$ -adic deformations of the residual Galois representation  $\bar{\rho}_{\ell, \pi}$ . When  $n = 2$  and  $K$  is totally real, a version of this conjecture has been proved by Diamond-Flach-Guo and Dimitrov, combining the methods of Theorem 2.4 with the results of [44].

Hida's paper [44] was the starting point for his theory of families of modular forms, and was the first to establish a relation between the critical value of the adjoint  $L$ -function and congruences between modular forms. In dimension  $n > 2$ , the special cases of the Ichino-Ikeda conjecture proved by Wei Zhang in [75] are used in [29] to relate the Whittaker period of a  $\Pi$  satisfying (b) of Theorem 2.2 to  $L(1, \pi, Ad)$ , up to rational multiples. One hopes this provides a starting point for determining  $L(1, \pi, Ad)$  up to units in number fields, as required by the Bloch-Kato conjecture.

### 3.4. Two speculative remarks on automorphic $p$ -adic $L$ -functions.

**Remark 3.7.** Deligne's conjecture is the starting point of the construction of  $p$ -adic  $L$ -functions. The algebraic values on the left-hand side of the identity in 3.1, suitably normalized, are predicted to extend analytically whenever  $M$  and  $m$  vary in  $p$ -adic families. The literature is vast but fragmentary, and the author's ongoing project with Eischen, Li, and Skinner will only add one (rather bulky) fragment to the collection when it is finished. Current plans are limited to *ordinary* (Hida) families, but ultimately one expects the method to extend to completely general families. In particular, such  $p$ -adic  $L$ -functions could be

specialized to the “very general”  $p$ -adic representations of 2.3.1. Moreover, using Brauer induction, one could even attach a  $p$ -adic  $L$ -functions to a motivic Galois representation  $\rho_{p,M}$  that is potentially  $p$ -adically automorphic. Although such a function would have no obvious connection to the complex  $L$ -function of  $M$ , it could conceivably be related to the Galois cohomology of  $\rho_{p,M}$ .

**Remark 3.8.** One can study the behavior of the right-hand side of Conjecture 3.5 when  $\Pi$  and  $\Pi'$  vary in  $p$ -adic families. Given the right choice of data in the local zeta integrals at primes dividing  $p$ , the result should be a  $p$ -adic meromorphic function of  $\Pi$  and  $\Pi'$ . Can this function be constructed directly on the left-hand side of the identity?

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