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p-ADIC L-FUNCTIONS FOR UNITARY GROUPS

ELLEN EISCHEN, MICHAEL HARRIS, JIANSHU LI, AND CHRISTOPHER SKINNER

Abstract. This paper completes the construction of p-adic L-functions for unitary groups. More precisely, in 2006, the last three named authors proposed an approach to constructing such p-adic L-functions (Part I). Building on more recent results, including the first named author's construction of Eisenstein measures and p-adic differential operators, Part II of the present paper provides the calculations of local ζ-integrals occurring in the Euler product (including at p). Part III of the present paper develops the formalism needed to pair Eisenstein measures with Hida families in the setting of the doubling method.

Contents

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1. Introduction

This paper completes the construction of p-adic L-functions for unitary groups. More precisely, in 2006, the last three named authors proposed an approach to constructing such p-adic L-functions (Part I). Building on more recent results, including the first named author's construction of Eisenstein measures and p-adic differential operators, Part II of the present paper provides the calculations of local ζ-integrals occurring in the Euler product (including at p). Part III of the present paper develops the formalism needed to pair Eisenstein measures with Hida families in the setting of the doubling method.

The construction of p-adic L-functions consists of several significant steps, including studying certain ζ-integrals occurring in the Euler products of the corresponding $\mathbb{C}$-valued L-functions (one of the main parts of this paper, which involves certain careful choices of local data and which is the specific step about which we are most frequently asked by others in the field) and extending and adapting earlier constructions of p-adic L-functions (e.g. Hida's work in [Hid96a], which recovers Katz's construction from [Kat78] as a special case). We also note that the last three named authors had already computed local zeta integrals for sufficiently regular data as far back as 2003, but the computations were not included in [HLS06] for lack of space. Since then, a new approach to choosing local data and computing local zeta integrals at primes dividing p has allowed us to treat the general case. These are the computations presented here.

In Section 1.1, we put this paper in the context of the full project to construct p-adic L-functions (which comprises the present paper and [HLS06]), and we describe the key components and significance of the broader project. The exposition in the present paper, especially the description of the geometry, was written especially carefully to provide a solid foundation for future work both by the authors of this paper and by other researchers in the field.

1.1. **About the project.**

Very precise and orderly conjectures predict how certain integer values of L-functions of motives over number fields, suitably modified, fit together into p-adic analytic functions (e.g. [Coa89, CPR89, Pan94, Hid96a]). These functions directly generalize the p-adic zeta function of Kubota and Leopoldt that has played a central role in algebraic number theory, through its association with Galois cohomology, in the form of Iwasawa's Main Conjecture. Such p-adic L-functions have been defined in a number of settings. In nearly all cases they are attached to automorphic forms rather than to motives; no systematic way is known to obtain information about special values of motivic L-functions unless they can be identified with automorphic L-functions. However, the procedures for attaching L-functions to automorphic forms other than Hecke characters are by no means orderly; any given L-function can generally be obtained by a number of methods that have no relation to one another, and in general no obvious relation to the geometry of motives. And while these procedures are certainly precise, they also depend on arbitrary choices: the L-function is attached abstractly to an automorphic representation, but as an analytic function it can only be written down after choosing a specific automorphic form, and in general there is no optimal choice.

When Hida developed the theory of analytic families of ordinary modular forms he also expanded the concept of p-adic L-functions. Hida's constructions naturally gave rise to analytic functions in which the modular forms are variables, alongside the character of $GL(1)$ that plays the role of the s variable in the complex L-function. This theory has also been generalized, notably to overconvergent modular forms, for example in the work of Stevens (unpublished, but see [Ste00]) and Panchishkin [Pan03, Pan06]. There seems to be a consensus among experts on how this should go in general, but as far as we know no general conjectures have been made public. This is in part because constructions of p-adic families are no more orderly than the construction of automorphic L-functions, except in the cases Hida originally studied: families are realized in the coherent or topological cohomology[1] of a locally symmetric space; but the connection of the latter to motives is tenuous and in many cases purely metaphoric.

The present project develops one possible approach to the construction of p-adic L-functions. We study complex L-functions of automorphic representations of unitary groups of n-dimensional hermitian spaces, by applying the doubling method of Garrett and Piatetski-Shapiro-Rallis [Gar84, GPSR87] to the automorphic representations that contribute to the coherent cohomology of Shimura varieties in degree 0; in other words, to holomorphic modular forms. When $n = 1$, we recover Katz's theory of p-adic L-functions of Hecke characters [Kat78], and much of the analytic theory is an adaptation of Katz's constructions to higher dimensions. For general n, the theory of ordinary families of holomorphic modular forms on Shimura varieties of PEL type has been developed by Hida, under hypotheses on the geometry of compactifications that have subsequently been proved by Lan. It is thus no more difficult to construct p-adic L-functions of Hida families than to study the p-adic versions of complex L-functions of individual automorphic representations. Interpreting our results poses a special challenge, however. The conjectures on motivic p-adic L-functions are formulated in a framework in which the Betti realization plays a central role, in defining complex as well as p-adic periods used to normalize the special values. Betti cohomology exists in the automorphic setting as well, but it cannot be detected by automorphic methods. The doubling method provides

[1]In principle, completed cohomology in Emerton's sense could also be used for this purpose, and would give rise to more general families. As far as we know p-adic L-functions have not yet been constructed in this setting.

a substitute: the cup product in coherent cohomology. Here one needs to exercise some care. Shimura proved many years ago that the critical value at $s = 1$ of the adjoint L-function attached to a holomorphic modular form f equals the Petersson square norm $\langle f, f\rangle$, multiplied by an elementary factor. If one takes this quantity as the normalizing period, the resulting p-adic adjoint L-function is identically equal to 1. Hida observed that the correct normalizing period is not $\langle f, f\rangle$ but rather the product of (normalized) real and imaginary periods; using this normalization, one obtains a p-adic adjoint L-function whose special values measures congruences between f and other modular forms. This is one of the fundamental ideas in the theory of deformations of modular forms and Galois representations; but it seems to be impossible to apply in higher dimensions, because the real and imaginary periods are defined by means of Betti cohomology. One of the observations in the present project is that the integral information provided by these Betti periods can naturally be recovered in the setting of the doubling method, provided one works with Hida families that are free over their corresponding Hecke algebras, and one assumes that the Hecke algebras are Gorenstein. These hypotheses are not indispensable, but they make the statements much more natural, and we have chosen to adopt them as a standard; some of the authors plan to indicate in a subsequent paper what happens when they are dropped.

This approach to families is the first of the innovations of the present project, in comparison with the previous work [HLS06] of the last three named authors. We stress that the Gorenstein hypothesis, suitably interpreted, is particularly natural in the setting of the doubling method. Our second, most important innovation, is the use of the general Eisenstein measure constructed by the first named author in [Eis15, Eis14].

In order to explain the contents of this project more precisely, we remind the reader what is expected of a general theory of p-adic L-functions. We are given a p-adic analytic space Y and a subset Y^{class} of points such that, for each $y \in Y^{class}$ there is a motive M_y, and possibly an additional datum r_y (a *refinement*) such that 0 is a critical value of the L-function $L(s, M_y)$. The p-adic L-function is then a meromorphic function L_p on Y whose values at $y \in Y^{class}$ can be expressed in terms of $L(0, M_y)$. More precisely, there is a p-adic period $p(M_y, r_y)$ such that $\frac{L_p(y)}{p(M_y,r_y)}$ is an algebraic number, and then we have the relation

$$\frac{L_p(y)}{p(M_y, r_y)} = Z_\infty(M_y) Z_p(M_y, r_y) \cdot \frac{L(0, M_y)}{c^+(M_y)}. \tag{1.1.1}$$

Here $c^+(M_y)$ is the period that appears in Deligne's conjecture on special values of L-functions, so that $\frac{L(0,M_y)}{c^+(M_y)}$ is an algebraic number, while Z_∞ and Z_p are correction factors that are built out of Euler factors and ε-factors of the zeta function of M_y at archimedean primes and primes dividing p, respectively.

In our situation, we start with a CM field $\mathcal{K}$ over $\mathbb{Q}$, a quadratic extension of a totally real field $\mathcal{K}^+$, and an n-dimensional hermitian vector space $V/\mathcal{K}$. Then Y is the space of pairs (λ, χ), where λ runs through the set of ordinary p-adic modular forms on the Shimura variety $Sh(V)$ attached to $U(V)$ and χ runs through p-adic Hecke characters of $\mathcal{K}$; both λ and χ are assumed to be unramified outside a finite set S of primes of $\mathcal{K}$, including those dividing p, and of bounded level at primes not dividing p. Because we are working with nearly ordinary forms, the ring $\mathcal{O}(Y)$ of holomorphic functions on Y is finite over some Iwasawa algebra, and the additional refinement is superfluous. In the project, λ denotes a character of Hida's ordinary Hecke algebra. If $(\lambda,\chi)\in Y^{class}$ then

- $\lambda=\lambda_\pi$ for some automorphic representation π of $U(V)$; it is the character of the ordinary Hecke algebra acting on vectors that are spherical outside S and (nearly) ordinary at primes dividing p;
- χ is a Hecke character of type A_0;
- the standard L-function $L(s,\pi,\chi)$ has a critical value at $s=0$.

(By replacing χ by its multiples by powers of the norm character, this definition accomodates all critical values of $L(s,\pi,\chi)$.) Under hypotheses to be discussed below, the automorphic version of Equation (1.1.1) is particularly simple to understand:

$$L_p(\lambda_\pi,\chi)=c(\pi)\cdot Z_\infty(\pi,\chi)Z_p(\pi,\chi)Z_S\cdot\frac{L(0,\pi,\chi)}{Q_{\pi,\chi}} \tag{1.1.2}$$

The left-hand side is the specialization to the point (λ_π,χ) of an element $L_p\in\mathcal{O}(Y)$. The right hand side is purely automorphic. The L-function is the standard Langlands L-function of $U(V)\times GL(1)_{\mathcal{K}}$. Its analytic and arithmetic properties have been studied most thoroughly using the doubling method. If $U(V)$ is the symmetry group of the hermitian form $\langle\cdot,\cdot\rangle_V$ on V, let $-V$ be the space V with the hermitian form $-\langle\cdot,\cdot\rangle_V$, and let $U(-V)$ and $Sh(-V)$ be the corresponding unitary group and Shimura variety. The groups $U(V)$ and $U(-V)$ are canonically isomorphic, but the natural identification of $Sh(-V)$ with $Sh(V)$ is anti-holomorphic; thus holomorphic automorphic forms on $Sh(V)$ are identified with anti-holomorphic automorphic forms, or coherent cohomology classes of top degree, on $Sh(-V)$, and vice versa. The space $W=V\oplus(-V)$, endowed with the hermitian form $\langle\cdot,\cdot\rangle_V\oplus-\langle\cdot,\cdot\rangle_V$, is always maximally isotropic, so $U(W)$ has a maximal parabolic subgroup P with Levi factor isomorphic to $GL(n)_{\mathcal{K}}$. To any Hecke character χ of $\mathcal{K}$ one associates the family of degenerate principal series

$$I(\chi,s)=Ind_{P(\mathbb{A})}^{U(W)(\mathbb{A})}\chi\circ\det\cdot\delta_P^{-s/n}$$

and constructs the meromorphic family of Eisenstein series $s\mapsto E(\chi,s,f,g)$ with $f=f(s)$ a section of $I(\chi,s)$ and $g\in U(W)(\mathbb{A})$. On the other hand, $U(V)\times U(-V)$ naturally embeds in $U(W)$. Thus if ϕ and ϕ' are cuspidal automorphic forms on $U(V)(\mathbb{A})$ and $U(-V)(\mathbb{A})$, respectively, the integral

$$I(\phi,\phi',f,s)=\int_{[U(V)\times U(-V)]}E(\chi,s,f,(g_1,g_2))\phi(g_1)\phi'(g_2)\chi^{-1}(\det(g_2))dg_1dg_2,$$

defines a meromorphic function of s. Here $[U(V)\times U(-V)]=U(V)(F)\backslash U(V)(\mathbb{A})\times U(-V)(F)\backslash U(-V)(\mathbb{A})$, $g_1\in U(V)(\mathbb{A})$, $g_2\in U(-V)(\mathbb{A})$, and dg_1 and dg_2 are Tamagawa measures.

The doubling method asserts that, if π is a cuspidal automorphic representation of $U(V)$ and $\phi \in \pi$, then $I(\phi, \phi', f, s)$ vanishes identically unless $\phi' \in \pi^\vee$; and if $\langle \phi, \phi' \rangle \neq 0$, then the integrals $I(\phi, \phi', f, s)$ unwind and factor as an Euler product whose unramified terms give the standard L-function $L(s + \frac{1}{2}, \pi, \chi)$ and (as f, ϕ, ϕ' vary) provide the meromorphic continuation and functional equation of the standard L-function. Another way to look at this construction is to say that the *Garrett map*

$$\phi \mapsto G(f, \phi, s)(g_2) = \chi^{-1} \circ \det(g_2) \cdot \int_{U(V)(F)\backslash U(V)(\mathbb{A})]} E(\chi, s, f, (g_1, g_2))\phi(g_1) dg_1$$

is a linear transformation from the automorphic representation π of $U(V)$ to π viewed as an automorphic reprepresentation of $U(-V)$; and the matrix coefficients of this linear transformation give the adelic theory of the standard L-function. We develop a theory that allows us to interpret these matrix coefficients integrally in Hida families, under special hypotheses on the localized Hecke algebra described below. Note that when π is an anti-holomorphic representation of $U(V)$, its image under the Garrett map is π, but viewed as a *holomorphic* representation of $U(-V)$.

The factor $Q_{\pi,\chi}$ is a product of several terms, of which the most important is a normalized Petersson inner product of holomorphic forms on $U(V)$. Although it arises naturally as a feature of the doubling method, its definition involves some choices that are reflected in the other terms. The local term Z_S, in our normalization, is a local volume multiplied by a local inner product (depending on the choices). The correction factors Z_∞ and Z_p are explicit local zeta integrals given by the doubling method. The archimedean factor has not been evaluated explicitly, except when π is associated to a holomorphic modular form of scalar weight (by Shimura) or, more generally, of weight that is "half scalar" at every archimedean place (by Garrett) [Shi97, Gar08]. In the present paper we leave it unspecified; it depends only on the archimedean data (the weights) and not on the Hecke eigenvalues.

The explicit calculation of the local term Z_p is our third major innovation **and one of the key pieces of the current paper**, and it occupies the longest single section of this paper (Section 4). It has the expected form: a quotient of a product of Euler factors (evaluated at s) by another product of Euler factors (evaluated at $1 - s$) multiplied by a local ε factor and a volume factor. The key observation is that the denominator arises by applying the Godement-Jacquet local functional equation to the input data. This is the step in the construction that owes the most to (adelic) representation theory. The input data for the Eisenstein measure represent one possible generalization of Katz's construction in [Kat78]. The local integral has been designed to apply to overconvergent families as well as to ordinary families; one of us plans to explore this in future work. The precise form of the local factor at a prime w dividing p depends on the signatures of the hermitian form at the archimedean places associated to p as part of the ordinary data; this appears mysterious but in fact turns out to be a natural reflection of the PEL structure at primes dividing p, or alternatively of the embedding of the ordinary locus of the Shimura variety attached to (two copies of) $U(V)$ in that attached to the doubled group.

A different calculation of the local term had been carried out at the time of [HLS06]. It was not published at the time because of space limitations. It was more ad hoc than the present version and applied only when the adelic local components at primes dividing p of an ordinary form could be identified as an explicit function in a principal series. The present calculation is more uniform and yields a result in the expected form, multiplied by a volume factor that we have not evaluated explicitly in the present paper.

Before explaining the final factor $c(\pi)$ it is preferable to explain the special hypotheses underlying the formula 1.1.2, which represent the fourth innovation in this project. The point (λ_π, χ) belongs to a Hida family, which for the present purposes means a *connected* component, which we denote $Y_{\pi,\chi}$, of the space Y; in other contexts one works with an *irreducible* component. The ring of functions on $Y_{\pi,\chi}$ is of the form $\Lambda\hat{\otimes}\mathbb{T}_\pi$, where Λ is an Iwasawa algebra attached to χ and $\mathbb{T}_\pi$ is the localization of the big Hecke algebra at the maximal ideal attached to π. The principal hypotheses are that $\mathbb{T}_\pi$ is Gorenstein, and that the module of ordinary modular forms (or its $\mathbb{Z}_p$-dual, to be more precise) is free over $\mathbb{T}_\pi$. There are also local hypotheses that correspond to the hypothesis of minimal level in the Taylor-Wiles theory of deformations of modular Galois representations. These hypotheses make it possible to define L_p as an element of $\mathcal{O}_{Y_{\pi,\chi}}$. The presence of the factor $c(\pi)$ is a sign that L_p is not quite the p-adic L-function; $c(\pi)$ is a generator of the *congruence ideal* which measures congruences between λ_π and other characters $\lambda_{\pi'}$ of $\mathbb{T}_\pi$ (of the same weight and level). The specific generator $c(\pi)$ depends on the same choices used to define $Q_{\pi,\chi}$, so that the product on the right-hand side is independent of all choices.

In the absence of the special hypotheses, it is still possible to define L_p in the fraction field of $\Lambda\hat{\otimes}\mathbb{T}_\pi$, but the statement is not so clean. In any case, the p-adic valuations of $c(\pi)$ are in principle unbounded, and so the p-adic interpolation of the normalized critical values of standard L-functions is generally given by a meromorphic function on Y.

1.1.1. *Clarifications.* The above discussion has artificially simplified several points. The Shimura variety is attached not to $U(V)$ but rather to the subgroup, denoted $GU(V)$, of the similitude group of V with rational similitude factor. All of the statements above need to be modified to take this into account, and this is done in the paper. This detail plagues the paper from beginning to end, as it seems at least to some degree also to plague every paper on Shimura varieties attached to unitary groups. One can hope that a far-sighted colleague will find an efficient way to do away with this.

What we called the moduli space of PEL type associated to V is in general a union of several isomorphic Shimura varieties, indexed by the defect of the Hasse principle; p-adic modular forms are most naturally defined on a single Shimura variety rather than on the full moduli space. We need the moduli space in order to define p-adic modular forms, but in the computations we work with a single fixed Shimura variety.

Although the p-adic L-functions are attached to automorphic forms on unitary (similitude) groups, they are best understood as p-adic analogues of the standard L-functions of cuspidal automorphic representations of $GL(n)$. The passage from unitary groups to $GL(n)$ is carried out by means of stable base change. A version of this adequate for our applications was developed by Labesse in [Lab11]. Complete results, including precise multiplicity formulas, were proved by Mok for quasi-split unitary groups [Mok13]; however, we need to work with unitary groups over totally real fields with arbitrary signatures, and the quasi-split case does not suffice. The general case is presently being completed by Kaletha, Minguez, Shin, and White, and we have assumed implicitly that Arthur's multiplicity conjectures are known for unitary groups. The book [KMSW14] works out the multiplicities of tempered representations and is probably sufficient for the purposes of the present project.

From the standpoint of automorphic representations of $GL(n)$, the ordinary hypothesis looks somewhat special; in fact, the critical values of L-functions of $GL(n)$ can be interpreted geometrically on unitary groups of different signatures, and the ordinary hypotheses for these different unitary groups represent different branches of a p-adic L-function that can only be related to one another in a general overconvergent family. The advantage of restricting our attention to ordinary families is that the p-adic L-functions naturally belong to integral Hecke algebras. To add to the confusion, however, Hida's theory of (nearly) ordinary modular forms applies to holomorphic automorphic representations, but the doubling method requires us to work with *antiholomorphic* representations. The eigenvalues of the U_p-operators on representations do not coincide with those on their holomorphic duals; for lack of a better terminology, we call these representations *anti-ordinary.* Keeping track of the normalizations adds to the bookkeeping but involves no essential difficulty.

1.1.2. *What this project does not accomplish.* Although we have made an effort to prove rather general theorems, limitations of patience have induced us to impose restrictions on our results. Here are some of the topics we have not covered.

First of all, we have not bothered to verify that the local and global terms in Equation (1.1.2) correspond termwise with those predicted by the general conjectures on p-adic L-functions for motives. The correspondence between automorphic representations and (de Rham realizations of) motives is not straightforward; we expect to address this issue in a subsequent paper. However, until we find a simple way to compute the archimedean term $Z_\infty(\pi, \chi)$ explicitly, we will not be able to compare it with anything motivic.

We have also not attempted to analyze the local factors at ramified finite primes for π and χ. The geometry of the moduli space has no obvious connection to the local theory of the doubling method. Moreover, a complete treatment of ramified local factors requires a p-integral version of the doubling method. This may soon be available, thanks to work of Minguez, Helm, Emerton-Helm, and Moss, but for the moment we have preferred to simplify our presentation by choosing local data that give simple volume factors for the local integrals at bad primes.

One of us plans to adapt the methods of the present project to general overconvergent families, where Hida theory is no longer appropriate. On the other hand, the methods of

Hida theory do apply to more general families than those we consider. In [Hid98], Hida introduces the notion of P-*ordinary* modular forms on a reductive group G, where P denotes a parabolic subgroup of G. One obtains the usual (nearly) ordinary forms when $P = B$ is a Borel subgroup; in general, for P of p-adic rank r, the P-ordinary forms vary in an r-dimensional family, up to global adjustments (related to Leopoldt's conjecture in general). Most importantly, a form can be P-ordinary without being B-ordinary. Our theory applies to P-ordinary forms as well; we hope to return to this point in the future.

Our p-adic L-function, when specialized at a classical point corresponding to the automorphic representation π, gives the corresponding value of the classical complex L-function, divided by what appears to be the correctly normalized complex period invariant, and multiplied by a factor $c(\pi)$ measuring congruences between π and other automorphic representations. This is a formal consequence of the Gorenstein hypothesis and is consistent with earlier work of Hida and others on p-adic L-functions of families. It is expected that the factor $c(\pi)$ is the specialization at π of the "genuine" p-adic L-function that interpolates normalized values at $s = 1$ of the adjoint L-function (of π, or one of the Asai L-functions for its base change to $GL(n)$). As far as we know, no one has constructed this p-adic adjoint L-function in general. We do not know how to construct a p-adic analytic function on the ordinary family whose specialization at π equals $c(\pi)$, not least because $c(\pi)$ is only well-defined up to multiplication by a p-adic unit. Most likely the correct normalization will have to take account of p-adic as well as complex periods.

Finally, we have always assumed that our base field $\mathcal{K}$ is unramified at p. This hypothesis is unnecessary, thanks to Lan's work in [Lan14], but it simplifies a number of statements.

1.2. **History.** Work on this paper began in 2001 as a collaboration between two of the authors, around the time of a visit by one of us (M.H.) to the second one (J.-S. L.) in Hong Kong. The initial objective was to study congruences between endoscopic and stable holomorphic modular forms on unitary groups. The two authors were soon joined by a third (C. S.), and a report on the results was published in [HLS05]. The subsequent article [HLS06] carried out the first part of the construction of a p-adic analytic function for a single automorphic representation. Because p-adic differential operators had not yet been constructed for unitary group Shimura varities, this function only provided the p-adic interpolation for the right-most critical value of the L-function, and only applied to scalar-valued holomorphic modular forms. Moreover, although the local computation of the zeta integrals at primes dividing p, which was not included in [HLS06], was based on similar principles to the computation presented here, it had only been completed for ramified principal series and only when the conductors of the local inducing characters were aligned with the slopes of the Frobenius eigenvalues. After the fourth author (E.E.) had defined p-adic differential operators in [Eis12, Eis16] and constructed the corresponding Eisenstein measure in [Eis15, Eis14], it became possible to treat general families of holomorphic modular forms and general ramification.

The delay in completing the paper, for which the authors apologize, can be attributed in large part to the difficulty of reconciling the different notational conventions that had accumulated over the course of the project. In the meantime, Xin Wan had constructed certain p-adic L-functions in the same setting in [Wan15], by a method based on computation of Fourier-Jacobi coefficients, as in [SU14]. More recently, Zheng Liu has constructed p-adic L-functions for symplectic groups [Liu16]. Among other differences, Zheng makes consistent use of the theory of nearly overconvergent p-adic modular forms, thus directly interpreting nearly holomorphic Eisenstein series as p-adic modular forms; and her approach to the local zeta integrals is quite different from ours.

1.3. **Contents and structure of this paper.** After establishing notation and conventions in Section 1.4 below, we being in Section 2 by recalling the theory of modular forms on unitary groups, as well as Hida's theory of p-adic modular forms on unitary groups. This section has carefully set up the framework needed for our project and will likely also provide a solid foundation for others working in this area. In Section 3, we discuss the geometry of restrictions of automorphic forms, since the restriction of an Eisenstein series is a key part of the doubling method (Section 4.1) used to construct L-functions. In Section 4, we discuss the doubling method. This section also contains the local zeta calculations mentioned at the beginning of the introduction. The most important of these is the calculation at primes dividing p (Section 4.3), which is also the longest single step of this paper. In the Appendix A, we pay special attention to a special case, definite unitary groups. In this special case, the doubling method can be re-expressed as a finite sum over values of automorphic forms at CM points. Indeed, this is a generalization of the approach taken by Katz to construct p-adic L-functions for CM fields (which is closely related to the rank 1 case of our situation) in [Kat78]. Thus, it can be helpful to view the statements in this special case. Section 5 provides statements about measures, which depend on the local data chosen in Section 4. A formalism for relating duality pairings to complex conjugation and to the action of Hecke algebras is developed in Section 6; this is extended to Hida families in Section 7, which also begins the formalism for construction of p-adic L-functions in families. Section 8 establishes the relation between p-adic and C^∞-differential operators, and develops the local theory of ordinary and anti-ordinary vectors in representations at p-adic places. Finally, Section 9 states and proves the main theorems about the existence of the p-adic L-function.

1.4. **Notation and conventions.**

1.4.1. *General notation.* Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ and let the complex embeddings of a number field $F \subset \overline{\mathbb{Q}}$ be $\Sigma_F = \mathrm{Hom}(F, \mathbb{C})$; so $\Sigma_F = \mathrm{Hom}(F, \overline{\mathbb{Q}})$. Throughout, $\mathcal{K} \subset \overline{\mathbb{Q}}$ is a CM field with ring of integers $\mathcal{O}$, and $\mathcal{K}^+$ is the maximal totally real subfield of $\mathcal{K}$. The non-trivial automorphism in $\mathrm{Gal}(\mathcal{K}/\mathcal{K}^+)$ is denoted by c. Given a place v of $\mathcal{K}$, the conjugate place $c(v)$ is usually denoted $\bar{v}$.

Let p be a fixed prime that is unramified in $\mathcal{K}$ and such that every place above p in $\mathcal{K}^+$ splits in $\mathcal{K}$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ and fix an embedding $incl_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$.

Let $\overline{\mathbb{Z}}_{(p)} \subset \overline{\mathbb{Q}}$ be the valuation ring for the valuation determined by $incl_p$. Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ and let $\mathcal{O}_{\mathbb{C}_p}$ be the valuation ring of $\mathbb{C}_p$ (so the completion of $\overline{\mathbb{Z}}_{(p)}$). Let $\iota_p : \mathbb{C} \xrightarrow{\sim} \mathbb{C}_p$ be an isomorphism extending $incl_p$.

For any $\sigma \in \Sigma_{\mathcal{K}}$ let $\mathfrak{p}_\sigma$ be the prime of $\mathcal{O}$ determined by the embedding $incl_p \circ \sigma$. Note that $c(\mathfrak{p}_\sigma) = \mathfrak{p}_{\sigma c}$. For a place w of $\mathcal{K}$ over p we will write $\mathfrak{p}_w$ for the corresponding prime of $\mathcal{O}$. Let Σ_p be a set containing exactly one place of $\mathcal{K}$ over each place of $\mathcal{K}^+$ over p.

Let $\mathbb{Z}(1) \subset \mathbb{C}$ be the kernel of the exponential map $\exp : \mathbb{C} \to \mathbb{C}^\times$. This is a free $\mathbb{Z}$-module of rank one with non-canonical basis $2\pi\sqrt{-1}$. For any commutative ring R let $R(1) = R \otimes \mathbb{Z}(1)$.

In what follows, when (G, X) is a Shimura datum, an automorphic representation of G is defined to be a $(\mathfrak{g}, K) \times G(\mathbf{A}_f)$-module where K is *the stabilizer of a point* in X; in particular, K contains the center of $G(\mathbb{R})$ but *does not* generally contain a full maximal compact subgroup. In this way, holomorphic and antiholomorphic representations are kept separate. This is of fundamental importance for applications to coherent cohomology and thus to our construction of p-adic L-functions.

1.4.2. *Measures and pairings.* We will need to fix a Haar measure dg on the adèle group of a reductive group G over a number field F. For the sake of definiteness we take dg to be Tamagawa measure. In this paper we will not be so concerned with the precise choice of measure, because we will not be calculating local zeta integrals at archimedean primes explicitly, but we do want to be consistent. When we write $dg = \prod_v dg_v$, where v runs over places of F and dg_v is a Haar measure on the F_v-points $G(F_v)$, we will want to make the following additional hypotheses:

Hypotheses 1.4.3. (1) *At all finite places v at which the group G is unramified, dg_v is the measure that gives volume 1 to a hyperspecial maximal compact subgroup.*
(2) *At all finite places v at which the group G is isomorphic to $\prod_i GL(n_i, F_{i,w_i})$, where F_{i,w_i} is a finite extension of F_v with integer ring $\mathcal{O}_i$, (whether or not F_{i,w_i} is ramified over the corresponding completion of $\mathbb{Q}$), dg_v is the measure that gives volume 1 to the group $\prod_i GL(n_i, \mathcal{O}_i)$.*
(3) *At all finite places v, the values of dg_v on open compact subgroups are rational numbers.*
(4) *At archimedean places v, we choose measures such that $\prod_v dg_v$ is Tamagawa measure.*

Let $Z_G \subset G$ denote the center of G, and let $Z \subset Z_G(\mathbf{A})$ be any closed subgroup such that $Z_G(\mathbf{A})/Z$ is compact; for example, one can take Z to be the group of real points of the maximal F-split subgroup of Z_G. We choose a Haar measure on Z that satisfies the conditions of 1.4.3 if Z is the group of adèles of an F-subgroup of Z_G. The measure dg defines a bilinear pairing $\langle , \rangle$ on $L^2(Z \cdot G(F)\backslash G(\mathbf{A}))$; if $f_1(zg)f_2(zg) = f_1(g)f_2(g)$ for all $z \in Z$, we write

$$\langle f_1, f_2 \rangle_Z = \int_{Z \cdot G(F)\backslash G(\mathbf{A})} f_1(g) f_2(g) dg, \tag{1.4.1}$$

and if not, we set $\langle f_1, f_2\rangle_Z = 0$.

Suppose π and $\pi^\vee$ are irreducible cuspidal automorphic representations of G. Then $\langle,\rangle_Z : \pi\otimes\pi^\vee \to \mathbb{C}$ is a canonically defined pairing. Now suppose we have factorizations

$$fac_\pi : \pi \xrightarrow{\sim} \otimes'_v \pi_v,\ fac_{\pi^\vee} : \pi^\vee \xrightarrow{\sim} \otimes'_v \pi_v^\vee \tag{1.4.2}$$

where π_v is an irreducible representation of $G(F_v)$. Assume moreover that we are given non-degenerate pairings of $G(F_v)$-spaces

$$\langle,\rangle_{\pi_v} : \pi_v \otimes \pi_v^\vee \to \mathbb{C} \tag{1.4.3}$$

for all v. Then there is a constant $C = C(dg, fac_\pi, fac_{\pi^\vee}, \prod_v \langle,\rangle_{\pi_v})$ such that, for all vectors $\varphi \in \pi$, $\varphi^\vee \in \pi^\vee$ that are factorizable in the sense that

$$fac_\pi(\varphi) = \otimes_v \varphi_v;\ fac_{\pi^\vee}(\varphi^\vee) = \otimes_v \varphi_v^\vee$$

we have

$$\langle \varphi, \varphi^\vee\rangle_Z = C(dg, fac_\pi, fac_{\pi^\vee}, \prod_v \langle,\rangle_{\pi_v}) \prod_v \langle \varphi_v, \varphi_v^\vee\rangle_{\pi_v} \tag{1.4.4}$$

When G is quasi-split and unramified over F_v and π_v is a principal series representation, induced from a Borel subgroup $B \subset G(F_v)$, we choose a hyperspecial maximal compact subgroup $K_v \subset G(F_v)$ and define the *standard local pairing* to be:

$$\langle f, f^\vee\rangle_{\pi_v} = \int_{K_v} f(g_v) f^\vee(g_v) dg_v. \tag{1.4.5}$$

In situation (2) of Hypotheses 1.4.3, we take $K_v = \prod_i GL(n_i, \mathcal{O}_i)$.

Part II: zeta integral calculations

2. Modular forms and p-adic modular forms on unitary groups

This section introduces details about modular forms and p-adic modular forms on unitary groups that we will need for our applications. For alternate discussions of modular forms and p-adic modular forms on unitary groups, see [Hid04, CEF+16].

2.1. PEL moduli problems: generalities.

By a PEL datum we will mean a tuple $P = (B, *, \mathcal{O}_B, L, \langle\cdot,\cdot\rangle, h)$ where

- B is a semisimple $\mathbb{Q}$-algebra with positive involution $*$, the action of which we write as $b \mapsto b^*$;
- $\mathcal{O}_B$ is a $*$-stable $\mathbb{Z}$-order in B;
- L is a $\mathbb{Z}$-lattice with a left $\mathcal{O}_B$-action and a non-degenerate alternating pairing $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}(1)$ such that $\langle bx, y\rangle = \langle x, b^* y\rangle$ for $x, y \in L$ and $b \in \mathcal{O}_B$;
- $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}_B\otimes\mathbb{R}}(L\otimes\mathbb{R})$ is a homomorphism such that $\langle h(z)x, y\rangle = \langle x, h(\bar{z})y\rangle$ for $x, y \in L\otimes\mathbb{R}$ and $z \in \mathbb{C}$ and $-\sqrt{-1}\langle\cdot, h(\sqrt{-1})\cdot\rangle$ is positive definite and symmetric.

For the purposes of subsequently defining p-adic modular forms for unitary groups we assume that the PEL data considered also satisfy:

- B has no type D factor;
- $\langle\cdot,\cdot\rangle : L\otimes\mathbb{Z}_p \times L\otimes\mathbb{Z}_p \to \mathbb{Z}_p(1)$ is a perfect pairing;
- $p \nmid \mathrm{Disc}(\mathcal{O}_B)$, where $\mathrm{Disc}(\mathcal{O}_B)$ is the discriminant of $\mathcal{O}_B$ over $\mathbb{Z}$ defined in [Lan13, Def. 1.1.1.6]; this condition implies that $\mathcal{O}_B\otimes\mathbb{Z}_{(p)}$ is a maximal $\mathbb{Z}_{(p)}$-order in B and that $\mathcal{O}_B\otimes\mathbb{Z}_p$ is a product of matrix algebras.

We associate a group scheme $G = G_P$ over $\mathbb{Z}$ with such a PEL datum P: for any $\mathbb{Z}$-algebra R

$$G(R) = \{(g,\nu)\in \mathrm{GL}_{\mathcal{O}_B\otimes R}(L\otimes R)\times R^\times \;:\langle gx, gy\rangle = \nu\langle x,y\rangle\;\forall x,y\in L\otimes R\}.$$

Then $G_{/\mathbb{Q}}$ is a reductive group, and by our hypotheses with respect to p, $G_{/\mathbb{Z}_p}$ is smooth and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact of $G(\mathbb{Q}_p)$.

Let $F\subset\mathbb{C}$ be the reflex field of $(L,\langle\cdot,\cdot\rangle,h)$ (or of P) as defined in [Lan13, 1.2.5.4] and let $\mathcal{O}_F$ be its ring of integers. Let $\Box = \{p\}$ or $\varnothing$, and let $\mathbb{Z}_{(\Box)}$ be the localization of $\mathbb{Z}$ at the primes in $\Box$. Let $S_\Box = \mathcal{O}_F\otimes\mathbb{Z}_{(\Box)}$. Let $K^\Box\subset G(\mathbb{A}_f^\Box)$ be an open compact subgroup and let $K\subset G(\mathbb{A}_f)$ be $K^\Box$ if $\Box=\varnothing$ and $G(\mathbb{Z}_p)K^\Box$ otherwise. Suppose that K is neat, as defined in [Lan13, Def. 1.4.1.8]. Then, as explained in [Lan13, Cor. 7.2.3.10], there is a smooth, quasi-projective $S_\Box$-scheme $\mathrm{M}_K = \mathrm{M}_K(P)$ that represents the functor on local noetherian $S_\Box$-schemes that assigns to such a scheme T the set of equivalence classes of quadruples (A,λ,ι,α) where

- A is an abelian scheme over T;
- $\lambda : A\to A^\vee$ is a prime-to-$\Box$ polarization;
- $\iota:\mathcal{O}_B\otimes\mathbb{Z}_{(p)}\to \mathrm{End}_T A\otimes\mathbb{Z}_{(\Box)}$ such that $\iota(b)^\vee\circ\lambda = \lambda\circ\iota(b^*)$;
- α is a $K^\Box$-level structure: this assigns to a geometric point t on each connected component of T a $\pi_1(T,t)$-stable $K^\Box$-orbit of $\mathcal{O}_B\otimes\mathbb{A}_f^\Box$-isomorphisms

  $$\alpha_t : L\otimes\mathbb{A}_f^\Box \xrightarrow{\sim} H_1(A_t,\mathbb{A}_f^\Box)$$

  that identify $\langle\cdot,\cdot\rangle$ with a $\mathbb{A}_f^{\Box,\times}$-multiple of the symplectic pairing on $H_1(A_t,\mathbb{A}_f^\Box)$ defined by λ and the Weil-pairing;
- $\mathrm{Lie}_T A$ satisfies the Kottwitz determinant condition defined by $(L\otimes\mathbb{R},\langle\cdot,\cdot\rangle,h)$ (see [Lan13, Def. 1.3.4.1]);

and two quadruples (A,λ,ι,α) and $(A',\lambda',\iota',\alpha')$ are equivalent if there exists a prime-to-$\Box$ isogeny $f : A\to A'$ such that λ equals $f^\vee\circ\lambda'\circ f$ up to some positive element in $\mathbb{Z}_{(\Box)}^\times$, $\iota'(b)\circ f = f\circ\iota(b)$ for all $b\in\mathcal{O}_B$, and $\alpha' = f\circ\alpha$.

2.2. **PEL moduli problems related to unitary groups.** Suppose

$$P = (B, *, \mathcal{O}_B, L, \langle\cdot,\cdot\rangle, h)$$

is a PEL datum as in Section 2.1 with

- $B = \mathcal{K}^m$, the product of m copies of $\mathcal{K}$ (that is, $B_i = \mathcal{K}$);
- $*$ is the involution acting as c on each factor of $\mathcal{K}$;
- $\mathcal{O}_B \cap \mathcal{K} = \mathcal{O}$.

We say such a P is *of unitary type.* By maximality, $\mathcal{O}_B \otimes \mathbb{Z}_{(p)} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_m = \mathcal{O}_{(p)} \times \cdots \times \mathcal{O}_{(p)}$ (each $\mathcal{O}_i$ is a maximal $\mathbb{Z}_{(p)}$-order in $\mathcal{K}$), so $\mathcal{O}_B \otimes \mathbb{Z}_p \cong \prod_{w|p} \prod_{i=1}^m \mathcal{O}_w$. Let $e_i \in \mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ be the idempotent projecting B to the ith copy of $\mathcal{K}$. Let $n_i = \dim_{\mathcal{K}} e_i(L \otimes \mathbb{Q})$.

The homomorphism h determines a pure Hodge structure of weight -1 on $V = L \otimes \mathbb{C}$. Let $V^0 \subset V$ be the degree 0 piece of the Hodge filtration; this is an $\mathcal{O}_B \otimes \mathbb{C}$-submodule. For each $\sigma \in \Sigma_{\mathcal{K}}$, let $a_{\sigma,i} = \dim_{\mathbb{C}} e_i(V^0 \otimes_{\mathcal{O}\otimes\mathbb{C},\sigma} \mathbb{C})$. Let $b_{\sigma,i} = n_i - a_{\sigma,i}$. We call the collection of pairs $\{(a_{\sigma,i}, b_{\sigma,i})_{\sigma\in\Sigma_{\mathcal{K}}}\}$, the signature of h. Note that $(a_{\sigma c,i}, b_{\sigma c,i}) = (b_{\sigma,i}, a_{\sigma,i})$. The following fundamental hypothesis will be assumed throughout:

Hypothesis 2.2.1 (Ordinary hypothesis)**.**

$$\mathfrak{p}_\sigma = \mathfrak{p}_{\sigma'} \implies a_{\sigma,i} = a_{\sigma',i}.$$

For $w|p$ a place of $\mathcal{K}$, we can then define $(a_{w,i}, b_{w,i}) = (a_{\sigma,i}, b_{\sigma,i})$ for any $\sigma \in \Sigma_{\mathcal{K}}$ such that $\mathfrak{p}_w = \mathfrak{p}_\sigma$. Let $\mathcal{O}_{B,w} = \mathcal{O}_B \otimes_{\mathcal{O}} \mathcal{O}_w$ and $L_w = L \otimes_{\mathcal{O}} \mathcal{O}_w$. We fix an $\mathcal{O}_B \otimes \mathbb{Z}_p$-decomposition $L \otimes \mathbb{Z}_p = L^+ \oplus L^-$ such that

- $L^+ = \prod_{w|p} L_w^+$ is an $\mathcal{O}_B \otimes \mathbb{Z}_p = \prod_{w|p} \mathcal{O}_{B,w}$-module with $\mathrm{rank}_{\mathcal{O}_w} e_i L_w^+ = a_{w,i}$ (so $L_p^- = \prod_{w|p} L_w^-$ with $\mathrm{rank}_{\mathcal{O}_w} e_i L_w^- = b_{w,i}$ and $L_w = L_w^+ \oplus L_w^-$);
- $L_w^\pm$ is the annihilator of $L_{\bar{w}}^\pm$ for the perfect pairing $\langle\cdot,\cdot\rangle : L_w \times L_{\bar{w}} \to \mathbb{Z}_p(1)$.

Over $\mathbb{Z}_p$ there is a canonical isomorphism

$$\mathrm{GL}_{\mathcal{O}_B\otimes\mathbb{Z}_p}(L \otimes \mathbb{Z}_p) \xrightarrow{\sim} \prod_{w|p} \prod_{i=1}^m \mathrm{GL}_{\mathcal{O}_w}(e_i L_w), \quad g \mapsto (g_{w,i}), \tag{2.2.1}$$

induced by the $\mathcal{O}_B \otimes \mathbb{Z}_p = \prod_{w|p} \mathcal{O}_{B,w}$-decomposition $L \otimes \mathbb{Z}_p = \prod_{w|p} L_w$. This in turn induces

$$G_{/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w\in\Sigma_p} \prod_{i=1}^m \mathrm{GL}_{\mathcal{O}_w}(e_i L_w), \quad (g,\nu) \mapsto (\nu, (g_{w,i})). \tag{2.2.2}$$

We fix a decomposition of $e_i L_w^+$ as a direct sum of copies of $\mathcal{O}_w$. Taking $\mathbb{Z}_p$-duals via $\langle\cdot,\cdot\rangle$ yields a decomposition of $e_i L_{\bar{w}}^-$ as a direct sum of copies of $\mathcal{O}_{\bar{w}} \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}_w, \mathbb{Z}_p)$ (the $\mathcal{O}_B$-action on $\mathrm{Hom}_{\mathbb{Z}_p}(E_{i,w}, \mathbb{Z}_p)$ factors through $e_i \mathcal{O}_B \otimes \mathbb{Z}_{(p)}$ and is given by $b\phi(x) = \phi(b^* x)$). The choice of these decompositions determines isomorphisms

$$\begin{gathered}\mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w^+) \cong \mathrm{GL}_{a_{w,i}}(\mathcal{O}_w), \quad \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w^-) \cong \mathrm{GL}_{b_{w,i}}(\mathcal{O}_w), \\ \text{and} \quad \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w) \cong \mathrm{GL}_{n_i}(\mathcal{O}_w).\end{gathered} \tag{2.2.3}$$

With respect to these isomorphisms, the embedding

$$\mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w^+) \times \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w^-) \hookrightarrow \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w) = \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w^+ \oplus e_i L_w^-)$$

is just the block diagonal map $(A, B) \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

2.3. **Connections with unitary groups and their Shimura varieties.** We recall how PEL data of unitary type naturally arise from unitary groups. Let $\mathcal{V} = (V_i, \langle\cdot,\cdot\rangle_{V_i})_{1\le i\le m}$ be a collection of hermitian pairs over $\mathcal{K}$: V_i is a finite-dimensional $\mathcal{K}$-space and $\langle\cdot,\cdot\rangle_{V_i} : V_i \times V_i \to \mathcal{K}$ is a hermitian form relative to $\mathcal{K}/\mathcal{K}^+$. Let $\delta \in \mathcal{O}$ be totally imaginary and prime to p, and put $\langle\cdot,\cdot\rangle_i = \mathrm{trace}_{\mathcal{K}/\mathbb{Q}}\delta\langle\cdot,\cdot\rangle_{V_i}$. Let $L_i \subset V_i$ be an $\mathcal{O}$-lattice such that $\langle L_i, L_i\rangle_i \subset \mathbb{Z}$ and $\langle\cdot,\cdot\rangle_i$ is a perfect pairing on $L_i \otimes \mathbb{Z}_p$. Such an L_i exists because of our hypotheses on p and its prime divisors in $\mathcal{K}$ and on δ. For each $\sigma \in \Sigma_{\mathcal{K}}$, $V_{i,\sigma} = V_i \otimes_{\mathcal{K},\sigma} \mathbb{C}$ has a $\mathbb{C}$-basis with respect to which $\langle\cdot,\cdot\rangle_{i,\sigma} = \langle\cdot,\cdot\rangle_{V_{i,\sigma}}$ is given by a matrix of the form $\mathrm{diag}(1_{r_{i,\sigma}}, -1_{s_{i,\sigma}})$. Fixing such a basis, let $h_{i,\sigma} : \mathbb{C} \to \mathrm{End}_{\mathbb{R}}(V_{i,\sigma})$ be $h_{i,\sigma}(z) = \mathrm{diag}(z1_{r_{i,\sigma}}, \bar{z}1_{s_{i,\sigma}})$. Let $\Sigma = \{\sigma \in \Sigma_{\mathcal{K}} \ : \ \mathfrak{p}_\sigma \in \Sigma_p\}$. Then Σ is a CM type of $\mathcal{K}$, and we let $h_i = \prod_{\sigma\in\Sigma} h_{i,\sigma} : \mathbb{C} \to \mathrm{End}_{\mathcal{K}^+\otimes\mathbb{R}}(V_i \otimes \mathbb{R}) = \prod_{\sigma\in\Sigma}\mathrm{End}_{\mathbb{R}}(V_{i,\sigma})$. Let $B = \mathcal{K}^m$, $*$ the involution that acts by c on each $\mathcal{K}$-factor of B, $\mathcal{O}_B = \mathcal{O}^m$, $L = \prod_i L_i$ with the ith factor of $\mathcal{O}_B = \mathcal{O}^m$ acting by scalar multiplication on the ith factor of L, $\langle\cdot,\cdot\rangle = \sum_i \langle\cdot,\cdot\rangle_i$, and $h = \prod_i h_i$. Then $P = (B, *, \mathcal{O}_B, L, \langle\cdot,\cdot\rangle, h)$ is a PEL datum of unitary type as defined above. Note that $(a_{\sigma,i}, b_{\sigma,i})$ equals $(r_{i,\sigma}, s_{i,\sigma})$ if $\sigma \in \Sigma$ and otherwise equals $(s_{i,\sigma}, r_{i,\sigma})$. Over $\mathbb{Q}$, the group G associated with P is just the unitary similitude group denoted $GU(V_1 \times \cdots \times V_m)$ in [Har93]. The reflex field of this PEL datum P is just the field

$$F = \mathbb{Q}[\{\sum_{\sigma\in\Sigma_{\mathcal{K}}} a_{\sigma,i}\sigma(a) \ : \ a \in \mathcal{K}, i = 1,..,m\}] \subset \mathbb{C}.$$

This follows, for example, from [Lan13, Cor. 1.2.5.6]. Note that F is contained in the Galois closure $\mathcal{K}'$ of $\mathcal{K}$ in $\mathbb{C}$.

As explained in [Kot92, §8] (see also Equations (2.7.1) below), over the reflex field F, a moduli space $\mathrm{M}_{K/F}$ associated with P is the union of $|\ker^1(\mathbb{Q}, G)|$ copies of the canonical model of the Shimura variety $S_K(G, X_P)$ associated to (G, h_P, K); here (G, X_P) is the Shimura datum for which $h_P = h \in X_P$ and $\ker^1(\mathbb{Q}, G) \coloneqq \ker\left(H^1(\mathbb{Q}, G) \to \prod_v H^1(\mathbb{Q}_v, G)\right)$. More precisely, the elements of $\ker^1(\mathbb{Q}, G)$ classify isomorphism classes of hermitian tuples $\mathcal{V}' = (V_i', \langle\cdot,\cdot\rangle_{V_i'})_{1\le i\le m}$ that are locally isomorphic to $\mathcal{V}$ at every place of $\mathbb{Q}$. Let $\mathcal{V} = \mathcal{V}^{(1)}, ..., \mathcal{V}^{(k)}$ be representatives for these isomorphism classes. Then $\mathrm{M}_{K/F}$ is naturally a disjoint union of F-schemes indexed by the $\mathcal{V}^{(i)}$: $\mathrm{M}_{K/F} = \sqcup \mathrm{M}_{K,\mathcal{V}^{(i)}}$. The scheme $\mathrm{M}_{K,\mathcal{V}} = \mathrm{M}_{K,\mathcal{V}^{(1)}}$ is the canonical model of $S_K(G, X_P)$, and for each i there is an F-automorphism of $\mathrm{M}_{K/F}$ mapping $\mathrm{M}_{K,\mathcal{V}}$ isomorphically onto $\mathrm{M}_{K,\mathcal{V}^{(i)}}$. In [Kot92], Kottwitz only treats the case where $m = 1$, but the reasoning is the same in the general case.

If $m = 1$ and $\dim_{\mathcal{K}} V_1$ is even, then the group G satisfies the Hasse principle (that is, $\ker^1(\mathbb{Q}, G) = 0$). In this case M_K is an integral model of the Shimura variety $S_K(G, X_P)$. If $\dim_{\mathcal{K}} V_1$ is odd or $m \ge 1$, this is no longer the case. However, for applications to automorphic forms, we only need one copy of $S_K(G, X_P)$. We let $\mathrm{M}_{K,L}$ be the scheme theoretic closure of the F-scheme $\mathrm{M}_{K,\mathcal{V}}$ in M_K; this is a smooth, quasi-projective $S_\square$-scheme. We let

$$s_L : \mathrm{M}_{K,L} \hookrightarrow \mathrm{M}_K \tag{2.3.1}$$

be the inclusion. We will refer to M_K as the *moduli space* and $\mathrm{M}_{K,L}$ as the *Shimura variety*.

Remark 2.3.1. For any PEL datum P, Lan has explained how the canonical model of the Shimura variety $S_K(G, X_P)$ is realized as an open and closed subscheme of $\mathrm{M}_{K/F}$ [Lan12, §2] [LS13, §1.2], with a smooth, quasi-projective S_0-model provided by its scheme-theoretic closure in M_K. This is just the model described above.

2.3.2. *Base points.* Suppose $m = 1$. Let $(V, \langle\cdot,\cdot\rangle_V) = (V_1, \langle\cdot,\cdot\rangle_{V_1})$, and let $n = \dim_{\mathcal{K}} V$. Suppose $\mathcal{K}_1, ..., \mathcal{K}_r$ are finite CM extensions of $\mathcal{K}$ with $\sum_{i=1}^r [\mathcal{K}_i : \mathcal{K}] = n$. For $i = 1, ..., r$, let $J_{0,i}$ be the Serre subtorus (defined in, e.g., [CCO14, Definition A.4.3.1]) of $\mathrm{Res}_{\mathcal{K}_i/\mathbb{Q}}\mathbb{G}_m$ and let $\nu_i : J_{0,i} \to \mathbb{G}_m$ be its similitude map. Let $J_0' \subset \prod_{i=1}^r J_{0,i}$ be the subtorus defined by equality of all the ν_i. Let $V_i' = \mathcal{K}_i$, viewed as a $\mathcal{K}$-space of dimension $[\mathcal{K}_i : \mathcal{K}]$. Each V_i' can be given a $\mathcal{K}_i$-hermitian structure such that $\oplus_i V_i'$ is isomorphic to V as an hermitian space over $\mathcal{K}$. Such an isomorphism determines an embedding of J_0' in G. Moreover, with respect to such an embedding, there exists a point $h_0 \in X_P$ that factors through the image of $J_0'(\mathbb{R})$ in $G(\mathbb{R})$. The corresponding embedding of Shimura data $(J_0', h_0) \to (G, X)$ defines a CM Shimura subvariety of $\mathrm{M}_{K,L}$.

For the case $\mathcal{K}_i = \mathcal{K}$ for all i (so $r = n$), we write $J_0^{(n)}$ for J_0'; this corresponds to a PEL datum as in Section 2.1 with $B = \mathcal{K}^n$. The base point $h \in X_P$ is called *standard* if it factors through an inclusion of $J_0^{(n)}$. We henceforward assume that the base point h in the PEL datum P is standard. This will guarantee that later constructions involving Harish-Chandra modules are rational over the Galois closure of $\mathcal{K}$.

Concretely, the assumption that h is standard just means that V has a $\mathcal{K}$-basis with respect to which $\langle\cdot,\cdot\rangle_V$ is diagonalized and that each h_σ has image in the diagonal matrices with respect to the induced basis of $V \otimes_{\mathcal{K},\sigma} \mathbb{C}$.

2.4. **Toroidal compactifications.** One of the main results of [Lan13] is the existence of smooth toroidal compactifications of M_K over $S_\square$ associated to certain smooth projective polyhedral cone decompositions (which we do not make precise here); when $\square = \varnothing$ this was already known. We denote such a compactification by $\mathrm{M}_{K,\Sigma}^{\mathrm{tor}}$. There is a notion of one polyhedral cone decomposition refining another that partially orders the Σ's. If Σ' refines Σ, then there is a canonical proper surjective map $\pi_{\Sigma',\Sigma} : \mathrm{M}_{K,\Sigma'}^{\mathrm{tor}} \to \mathrm{M}_{K,\Sigma}^{\mathrm{tor}}$ that is the identity on M_K. We write $\mathrm{M}_K^{\mathrm{tor}}$ for the tower of compactications $\{\mathrm{M}_{K,\Sigma}^{\mathrm{tor}}\}_\Sigma$. In certain situations (e.g., changing the group K, defining Hecke operators) it is more natural work to work with this tower, avoiding making specific compatible choices of Σ or having to vary the 'fixed' choices.

If $K_1^\square \subset K_2^\square$ then the natural map $\mathrm{M}_{K_1} \to \mathrm{M}_{K_2}$ extends canonically to a map (of towers) $\mathrm{M}_{K_1}^{\mathrm{tor}} \to \mathrm{M}_{K_2}^{\mathrm{tor}}$. Similarly, if $g \in G(\mathbb{A}_f^\square)$, then the map $[g] : \mathrm{M}_{gKg^{-1}} \to \mathrm{M}_K$, $(A, \lambda, \iota, \alpha) \mapsto (A, \lambda, \iota, \alpha g)$, extends canonically to a map $\mathrm{M}_{gKg^{-1}}^{\mathrm{tor}} \to \mathrm{M}_K^{\mathrm{tor}}$. This defines a right action of $G(\mathbb{A}_f^\square)$ on the tower (of towers!) $\{\mathrm{M}_K^{\mathrm{tor}}\}_{K^\square \subset G(\mathbb{A}_f^\square)}$.

In the setting of Section 2.3, we let $\mathrm{M}^{\mathrm{tor}}_{K,L,\Sigma}$ be the scheme-theoretic closure of $\mathrm{M}_{K,\mathcal{V}}$ in $\mathrm{M}^{\mathrm{tor}}_{K,\Sigma}$. This is a smooth toroidal compactification of the Shimura variety $\mathrm{M}_{K,L}$; the base change to F is just the usual toroidal compactification of the canonical model. We continue to denote by s_L the induced inclusion $\mathrm{M}^{\mathrm{tor}}_{K,L,\Sigma} \subset \mathrm{M}^{\mathrm{tor}}_{K,\Sigma}$. Varying Σ and K as above induces maps between the $\mathrm{M}^{\mathrm{tor}}_{K,L,\Sigma}$. We let $\mathrm{M}^{\mathrm{tor}}_{K,L}$ be the tower $\{\mathrm{M}^{\mathrm{tor}}_{K,L,\Sigma}\}_\Sigma$. The action of $G(\mathbb{A}_f^\square)$ on $\{\mathrm{M}^{\mathrm{tor}}_K\}_{K^\square \subset G(\mathbb{A}_f^\square)}$ induces an action on $\{\mathrm{M}^{\mathrm{tor}}_{K,L}\}_{K^\square \subset G(\mathbb{A}_f^\square)}$.

Our convention will be to describe constructions over $\mathrm{M}^{\mathrm{tor}}_K$ as though $\mathrm{M}^{\mathrm{tor}}_K$ were a single scheme. The reader should bear in mind that this means a tower of such constructions over each $\mathrm{M}^{\mathrm{tor}}_{K,\Sigma}$. In particular, when we define a sheaf $\mathcal{F}$ over $\mathrm{M}^{\mathrm{tor}}_K$ (or some similar tower of schemes), this will be a sheaf $\mathcal{F}_\Sigma$ on each $\mathrm{M}^{\mathrm{tor}}_{K,\Sigma}$ such that there is a natural map $\pi^*_{\Sigma',\Sigma} : \mathcal{F}_\Sigma \to \mathcal{F}_{\Sigma'}$ for any Σ' that refines Σ. By $H^i(\mathrm{M}^{\mathrm{tor}}_K, \mathcal{F})$ we will mean the direct limit $\varinjlim_\Sigma H^i(\mathrm{M}^{\mathrm{tor}}_{K,\Sigma}, \mathcal{F}_\Sigma)$. In practice, the maps of cohomology groups appearing in such a limit will all be isomorphisms.

2.5. **Level structures at p.** Let $H = \mathrm{GL}_{\mathcal{O}_B \otimes \mathbb{Z}_p}(L^+)$. The identification (2.2.3) determines an isomorphism

$$H \xrightarrow{\sim} \prod_{w|p} \prod_{i=1}^{m} \mathrm{GL}_{a_{w,i}}(\mathcal{O}_w). \tag{2.5.1}$$

Let $B_H \subset H$ be the $\mathbb{Z}_p$-Borel that corresponds via this isomorphism with the product of the upper-triangular Borels and let B_H^u be its unipotent radical. Let $T_H = B_H/B_H^u$; this is identified by isomorphism (2.5.1) with the diagonal matrices.

Suppose $\square = \{p\}$. Let $\mathcal{A}$ be the semiabelian scheme over $\mathrm{M}^{\mathrm{tor}}_K$ and let $\mathcal{A}^\vee$ be its dual. We define $\overline{\mathrm{M}}_{K_r}$ to be the scheme over $\mathrm{M}^{\mathrm{tor}}_{i,K}$ whose S-points classify the $B_H^u(\mathbb{Z}_p)$-orbits of $\mathcal{O}_B \otimes \mathbb{Z}_p$-injections $\phi : L^+ \otimes \mu_{p^r} \xrightarrow{\sim} \mathcal{A}^\vee[p^r]_{/S}$ of group schemes with image an isotropic subgroup scheme. We write M_{K_r} for its restriction over M_K. The group $B_H(\mathbb{Z}_p)$ acts on $\overline{\mathrm{M}}_{K_r}$ on the right through its quotient $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. We let $\overline{\mathrm{M}}_{K_r,L}$ be the pullback of $\overline{\mathrm{M}}_{K_r}$ over $\mathrm{M}^{\mathrm{tor}}_{K,L}$ and let $\mathrm{M}_{K_r,L}$ be the pullback over $\mathrm{M}_{K,L}$. Generally, the scheme $\overline{\mathrm{M}}_{K_r}$ (resp. $\overline{\mathrm{M}}_{K_r,L}$) is étale and quasi-finite but not finite over $\mathrm{M}^{\mathrm{tor}}_K$ (resp. $\overline{\mathrm{M}}^{\mathrm{tor}}_{K_r,L}$). We continue to denote by s_L the inclusions $\mathrm{M}_{K_r,L} \hookrightarrow \mathrm{M}_{K_r}$ and $\overline{\mathrm{M}}_{K_r,L} \hookrightarrow \overline{\mathrm{M}}_{K_r}$ determined by these restrictions.

Let $B^+ \subset G_{/\mathbb{Z}_p}$ be the Borel that stabilizes L^+ and such that

$$B^+ \twoheadrightarrow \mathbb{G}_m \times B_H \subset \mathbb{G}_m \times H, \tag{2.5.2}$$

where the map to the first factor is the similitude character ν and the map to the second is projection to H. Let $B^u \subset B^+$ be the unipotent radical. Let $I_r^0 \subset G(\mathbb{Z}_p)$ consist of those g such that $g \bmod p^r \in B^+(\mathbb{Z}_p/p^r\mathbb{Z}_p)$, and let $I_r \subset I_r^0$ consist of those g projecting under the surjection (2.5.2) to an element in $(\mathbb{Z}_p/p^r\mathbb{Z}_p)^\times \times B_H^u(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Then $I_r^0/I_r \xrightarrow{\sim} T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. The choice of a basis of $\mathbb{Z}(1)$ naturally identifies $\mathrm{M}_{K_r/F}$ (resp. $\mathrm{M}_{K_r,L}$) with $\mathrm{M}_{I_rK^p/F}$ (resp. $\mathrm{M}_{I_rK^p,L/F} = S_{I_rK^p}(G, X_P)$), and $\overline{\mathrm{M}}_{K_r/F}$ (resp. $\overline{\mathrm{M}}_{K_r,L/F}$)

is the normalization of $\mathrm{M}^{\mathrm{tor}}_{K\,/F}$ (resp. $\mathrm{M}^{\mathrm{tor}}_{K,L/F}$) in $\mathrm{M}_{K_r/F}$ (resp. $\mathrm{M}_{K_r,L}/F$). Since it should therefore cause no ambiguity, we also put $K_r = I_r K^p$. We similarly put $K_r^0 = I_r^0 K_p$.

Note that under the isomorphisms (2.2.2) and (2.2.3), B^+ is identified with the group

$$B^+ \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w\in\Sigma_p}\prod_{i=1}^m \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \mathrm{GL}_{n_i}(\mathcal{O}_w) \;:\; \begin{matrix} A \in \mathrm{GL}_{a_{w,i}}(\mathcal{O}_w) \text{ is upper-triangular} \\ D \in \mathrm{GL}_{b_{w,i}}(\mathcal{O}_w) \text{ is lower-triangular} \end{matrix} \right\}. \tag{2.5.3}$$

2.6. **Modular forms.** We define spaces of modular forms for the groups G and various Hecke operators acting them.

2.6.1. *The groups G_0 and H_0.* Let $V = L\otimes\mathbb{C}$. The homomorphism h defines a pure Hodge structure $V = V^{-1,0}\oplus V^{0,-1}$ of weight -1. Let $W = V/V^{0,-1}$. This is defined over the reflex field F. Let $\Lambda_0 \subset W$ be an $\mathcal{O}_B$-stable $S_\square$-submodule such that $\Lambda_0\otimes_{S_\square}\mathbb{C} = W$. Let $\Lambda_0^\vee = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Lambda_0, \mathbb{Z}_{(p)}(1))$ with $\mathcal{O}_B\otimes S_\square$-action: $(b\otimes s)f(x) = f(b^*sx)$. Put $\Lambda = \Lambda_0\oplus\Lambda_0^\vee$, and let $\langle\cdot,\cdot\rangle_{can} : \Lambda\times\Lambda \to \mathbb{Z}_{(p)}(1)$ be the alternating pairing

$$\langle (x_1,f_1),(x_2,f_2)\rangle_{can} = f_2(x_1) - f_1(x_2).$$

Note that Λ_0 and $\Lambda_0^\vee$ are isotropic submodules of Λ. Note also that the $\mathcal{O}_B$-action on Λ is such that $\langle bx,y\rangle_{can} = \langle x, b^*y\rangle_{can}$. Let G_0 be the group scheme over $S_\square$ such that for any $S_\square$-algebra R

$$G_0(R) = \left\{ (g,\nu) \in \mathrm{GL}_{\mathcal{O}_B\otimes R}(\Lambda\otimes_{S_\square} R)\times R^\times \;:\; \begin{matrix} \langle gx, gy\rangle_{can} = \nu\langle x,y\rangle_{can}, \\ \forall x,y\in\Lambda\otimes_{S_\square} R \end{matrix} \right\}.$$

Let $H_0 \subset G_0$ be the stabilizer of the polarization $\Lambda = \Lambda_0\oplus\Lambda_0^\vee$. The projection $H_0 \to \mathbb{G}_m\times\mathrm{GL}_{\mathcal{O}_B\otimes S_\square}(\Lambda_0^\vee)$ is an isomorphism (the projection to $\mathbb{G}_m$ is the similitude factor ν). There is a canonical isomorphism $V\cong\Lambda\otimes_{S_\square}\mathbb{C}$ of $\mathcal{O}_B\otimes\mathbb{C}$-modules that identifies $V^{-1,0}$ with $\Lambda_0\otimes_{S_\square}\mathbb{C}$ and $V^{0,-1}$ with $\Lambda_0^\vee\otimes_{S_\square}\mathbb{C}$ and the pairing $\langle\cdot,\cdot\rangle$ with $\langle\cdot,\cdot\rangle_{can}$, and so identifies $G_{/\mathbb{C}}$ with $G_{0/\mathbb{C}}$. Let $C\subset G_{/\mathbb{R}}$ be the centralizer of the homomorphism h and set $U_\infty = U_h := C(\mathbb{R})$. The identification of $G_{/\mathbb{C}}$ with $G_{0/\mathbb{C}}$ identifies $C(\mathbb{C})$ with $H_0(\mathbb{C})$.

2.6.2. *The canonical bundles.* Let $\mathcal{A}$ be the semiabelian scheme over $\mathrm{M}_K^{\mathrm{tor}}$ and $\mathcal{A}^\vee$ its dual. Let ω be the $\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}$-dual of $\mathrm{Lie}_{\mathrm{M}_K^{\mathrm{tor}}}\mathcal{A}^\vee$. The Kottwitz determinant condition is equivalent to ω being locally isomorphic to $\Lambda_0^\vee\otimes_{S_\square}\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}$ as an $\mathcal{O}_B\otimes\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}$-module. Let

$$\mathcal{E} = \mathrm{Isom}_{\mathcal{O}_B\otimes\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}}\left((\omega,\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}(1)),(\Lambda_0^\vee\otimes_{S_\square}\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}},\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}(1))\right).$$

This is an H_0-torsor over $\mathrm{M}_K^{\mathrm{tor}}$. Let $\pi:\mathcal{E}\to\mathrm{M}_K^{\mathrm{tor}}$ be the structure map. Then $\pi_*\mathcal{O}_{\mathcal{E}}$ is an H_0-bundle on $\mathrm{M}_K^{\mathrm{tor}}$. Let R be an $S_\square$-algebra. A global section f of this bundle over $\mathrm{M}_{K/R}$ can be viewed as a functorial rule assigning to a pair $(\underline{A},\varepsilon)$ over an R-algebra S an element $f(\underline{A},\varepsilon)\in S$. Here $\underline{A}$ is a tuple classified by $\mathrm{M}_K(S)$ and ε is a corresponding element of $\mathcal{E}(S)$. We let $\mathcal{E}_r = \mathcal{E}\times_{\mathrm{M}_K^{\mathrm{tor}}}\overline{\mathrm{M}}_{K_r}$ and let $\pi_r:\mathcal{E}_r\to\overline{\mathrm{M}}_{K_r}$ be its structure map. Sections of the bundle $\pi_{r,*}\mathcal{O}_{\mathcal{E}_r}$ have interpretations as functorial rules of pairs $(\underline{X},\varepsilon)$, where $\underline{X} = (\underline{A},\phi)$ is a tuple classified by $\mathrm{M}_{K_r}(S)$ and ε is a corresponding element in $\mathcal{E}_r(S)$.

2.6.3. *Representations of H_0 over S_0.* Let $F' \subset \overline{\mathbb{Q}}$ be the compositum of F and the Galois closure $\mathcal{K}'$ of $\mathcal{K}$, and $\mathfrak{p} \subset \mathcal{O}_{F'}$ be the prime determined by $incl_p$. Let

$$S_0 = S_\square \otimes_{\mathcal{O}_{F,(p)}} \mathcal{O}_{F',(\mathfrak{p}')}.$$

(So $S_0 = F'$ if $\square = \varnothing$ and $S_0 = \mathcal{O}_{F',(\mathfrak{p}')}$ if $\square = \{p\}$.) The isomorphism $\mathcal{O} \otimes S_0 \xrightarrow{\sim} \prod_{\sigma \in \Sigma_{\mathcal{K}}} S_0$, $a \otimes s \mapsto (\sigma(a)s)_{\sigma \in \Sigma_{\mathcal{K}}}$, induces a decomposition

$$\mathcal{O}_B \otimes S_0' \xrightarrow{\sim} \mathcal{O}_B \otimes_{\mathcal{O}} (\mathcal{O} \otimes S_0) \xrightarrow{\sim} \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathcal{O}_B \otimes_{\mathcal{O},\sigma} S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathcal{O}_{B,\sigma}.$$

This in turn induces $\mathcal{O}_B \otimes S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \mathcal{O}_{B,\sigma}$-decompositions $\Lambda_0 \otimes_{S_0} S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \Lambda_{0,\sigma}$ and $\Lambda_0^\vee \otimes_{S_0} S_0 = \prod_{\sigma \in \Sigma_{\mathcal{K}}} \Lambda_{0,\sigma}^\vee$. The pairing $\langle \cdot, \cdot \rangle_{can}$ identifies $\Lambda_{0,\sigma c}^\vee = \mathrm{Hom}_{\mathbb{Z}_{(p)}}(\Lambda_{0,\sigma}, \mathbb{Z}_{(p)}(1))$.

Since S_0 is a PID, $e_i\Lambda_{0,\sigma}$ and $e_i\Lambda_{0,\sigma}^\vee$ are free S_0-modules, of respective ranks $a_{\sigma,i}$ and $b_{\sigma,i}$. We fix an S_0-basis of $e_i\Lambda_{0,\sigma}$. By duality, this determines an S_0-basis of $e_i\Lambda_{0,\sigma c}^\vee$. This yields an isomorphism

$$H_{0/S_0} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \prod_{i=1}^m \mathrm{GL}_{\mathcal{O}_i \otimes_{\mathcal{O},\sigma} S_0'}(e_i \Lambda_{0,\sigma}^\vee) \cong \mathbb{G}_m \times \prod_{\sigma \in \Sigma_{\mathcal{K}}} \prod_{i=1}^m \mathrm{GL}_{b_{\sigma,i}}(S_0). \tag{2.6.1}$$

Let $B_{H_0} \subset H_{0/S_0}$ be the S_0-Borel that corresponds via the isomorphism (2.6.1) to the product of the lower-triangular Borels. Let $T_{H_0} \subset B_{H_0}$ be the diagonal torus and let $B_{H_0}^u \subset B_{H_0}$ be the unipotent radical. We say that a character κ of T_{H_0} that is defined over an S_0-algebra R is a *dominant character of* T_{H_0} if it is dominant with respect to the opposite (so upper-triangular) Borel $B_{H_0}^{\mathrm{op}}$. Via the isomorphism (2.6.1), the characters of T_{H_0} can be identified with the tuples $\kappa = (\kappa_0, (\kappa_{\sigma,i})_{\sigma \in \Sigma_{\mathcal{K}}, 1 \le i \le m})$, $\kappa_0 \in \mathbb{Z}$ and $\kappa_{\sigma,i} = (\kappa_{\sigma,i,j}) \in \mathbb{Z}^{b_{\sigma,i}}$, and the dominant characters are those that satisfy

$$\kappa_{\sigma,i,1} \ge \cdots \ge \kappa_{\sigma,i,b_{\sigma,i}}, \quad \forall \sigma \in \Sigma_{\mathcal{K}}, \quad i = 1, \dots, m. \tag{2.6.2}$$

The identification is just

$$\kappa(t) = t_0^{\kappa_0} \cdot \prod_{\sigma \in \mathcal{K}} \prod_{i=1}^m \prod_{j=1}^{b_{\sigma,i}} t_{\sigma,i,j}^{\kappa_{\sigma,i,j}},$$
$$t = (t_0, (\mathrm{diag}(t_{\sigma,i,1}, \dots, t_{\sigma,i,b_{\sigma,i}}))_{\sigma \in \Sigma_{\mathcal{K}}, 1 \le i \le m}) \in T_{H_0}.$$

Given a dominant character κ of T_{H_0} over an S_0-algebra R, let

$$W_\kappa(R) = \{\phi : H_{0/R} \to \mathbb{G}_a \ : \ \phi(bh) = \kappa(b)\phi(h), \ b \in B_{H_0}\},$$

where κ is extended trivially to $B_{H_0}^u$. If R is a flat S_0-algebra then this is an R-model of the irreducible algebraic representation of H_0 of highest weight κ with respect to $(T_{H_0}, B_{H_0}^{\mathrm{op}})$. Let $w \in W(T_{H_0}, H_{0/S_0})$ be the longest element in the Weyl group and let $\kappa^\vee$ be the dominant character of T_{H_0} defined by $\kappa^\vee(t) = \kappa(w^{-1}t^{-1}w)$. The dual

$$W_\kappa^\vee(R) = \mathrm{Hom}_R(W_\kappa(R), R)$$

is, for a flat S_0-algebra R, an R-model of the representation with highest weight $\kappa^\vee$.

The submodule $W_\kappa(R)^{B^u_{H_0}}$ is a free R-module of rank one spanned by ϕ_κ, the function with support containing the big cell $B_{H_0}wB_{H_0}$ (and equal to the big cell if κ is regular) and such that $\phi_\kappa(wB^u_{H_0}) = 1$; $w\phi_\kappa$ is a highest-weight vector. The module $W^\vee_\kappa$ is generated over R as an H_0-representation by the functional ℓ_κ = (evaluation at 1); $w\ell_\kappa$ is a highest-weight vector. Also,

$$\mathrm{Hom}_{H_0}(W^\vee_\kappa(R), W_{\kappa^\vee}(R)) = R$$

with basis the homomorphism that sends ℓ_κ to $\phi_{\kappa^\vee}$.

For future reference, we also note that via the isomorphism (2.6.1) the identification of $C(\mathbb{C})$ with $H_0(\mathbb{C})$ identifies

$$U_\infty = C(\mathbb{R}) \xrightarrow{\sim} \{(h_0, (h_\sigma)_{\sigma\in\Sigma_\mathcal{K}}) \in H_0(\mathbb{C}) \ : \ h_0 \in \mathbb{R}^\times, h_0{}^t\bar{h}_\sigma^{-1} = h_\sigma\}, \tag{2.6.3}$$

where the '‾' denotes complex conjugation on $\mathbb{C}$. That is, U_∞ is identified with the subgroup of the product $\prod_{\sigma\in\Sigma_\mathcal{K}} GU(b_\sigma)$ of unitary similitude groups in which all the similitude factors agree.

2.6.4. *The modular sheaves.* Let R be a S_0-algebra and κ a dominant R-character of T_{H_0}. Let

$$\omega_{\kappa,M} = \pi_*\mathcal{O}_\mathcal{E}[\kappa] \quad \text{and} \quad \omega_{r,\kappa,M} = \pi_{r,*}\mathcal{O}_{\mathcal{E}_r}[\kappa]$$

be the subsheaves on $\mathrm{M}^{\mathrm{tor}}_{K}{}_{/R}$ and $\overline{\mathrm{M}}_{K_r/R}$, respectively, on which B_{H_0} acts via κ. We let

$$\omega_\kappa = s_{L,!}s_L^*\omega_{\kappa,M} \quad \text{and} \quad \omega_{r,\kappa} = s_{L,!}s_L^*\omega_{r,\kappa,M}. \tag{2.6.4}$$

These are the respective restrictions to the Shimura varieties $\mathrm{M}_{K,L}$ of the sheaves $\omega_{\kappa,M}$ and $\omega_{r,\kappa,M}$, extended by zero to the full moduli space. We will use the same notation to denote the restriction of these sheaves over $\mathrm{M}_{K,L}$ and $\mathrm{M}_{K_r,L}$.

2.6.5. *Modular forms over S_0 of level K.* Let R be a S_0-algebra. The R-module of modular forms (on G) over R of weight κ and level K is

$$M_\kappa(K;R) = H^0(\mathrm{M}^{\mathrm{tor}}_{K}{}_{/R}, \omega_\kappa).$$

The Köcher principle [Lan16] and the definition (2.6.4) implies that

$$M_\kappa(K;R) = H^0\left(\mathrm{M}_{K,L/R}, \omega_\kappa\right) \tag{2.6.5}$$

except when $F_0 = \mathbb{Q}$ and $G^{\mathrm{der}}/\mathbb{Q} \cong SU(1,1)$. However, in this exceptional case the toroidal compactifications are the same as the minimal compactification and therefore canonical; we leave it to the reader to make the necessary adjustments to our arguments in this case (or to find them in the literature). By (2.6.5) a modular form $f \in M_\kappa(K;R)$ can be viewed as a functorial rule assigning to a pair $(\underline{A}, \varepsilon)$ over an R-algebra S (and which is an S-valued point of the Shimura variety[2]) $\mathrm{M}_{K,L}$) an element[3] $f(\underline{A},\varepsilon) \in S$ and satisfying $f(\underline{A}, b\varepsilon) = \kappa(b)f(\underline{A},\varepsilon)$ for $b \in B_{H_0}(S)$.

[2]The 'rule' is just zero when the point is not on the Shimura variety; see the next footnote.

[3]This element will be zero if $\underline{A}$ is not an S-valued point of the Shimura variety $\mathrm{M}_{K,L}$.

Let $D_\infty = \mathrm{M}_K^{\mathrm{tor}} - \mathrm{M}_K$. The R-module of cuspforms (on G) over R of weight κ and level K is the submodule

$$S_\kappa(K;R) = H^0\left(\mathrm{M}_K^{\mathrm{tor}}{}_{/R}, \omega_\kappa(-D_\infty)\right)$$

of $M_\kappa(K;R)$.

2.6.6. *Modular forms over $S_0[\psi]$ with Nebentypus ψ.* Let $\psi : T_H(\mathbb{Z}_p) \to \overline{\mathbb{Q}}^\times$ be a character factoring through $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Suppose R is an $S_0[\psi]$-algebra. We define the R-module of modular forms (on G) over R of weight κ, level K_r, and character ψ to be

$$M_\kappa(K_r, \psi; R) = \left\{ f \in H^0\left(\overline{\mathrm{M}}_{K_r/R}, \omega_{r,\kappa}\right) \ : \ t\cdot f = \psi(t) f \ \ \forall t \in T_H(\mathbb{Z}_p)\right\}.$$

If p is not a zero-divisor in R, then the Köcher principle implies[4]

$$M_\kappa\left(K_r, \psi; R\left[\frac{1}{p}\right]\right) = \left\{ f \in H^0\left(\mathrm{M}_{K_r/R\left[\frac{1}{p}\right]}, \omega_{r,\kappa}\right) \ : \ t\cdot f = \psi(t) f \ \ \forall t \in T_H(\mathbb{Z}_p)\right\}. \tag{2.6.6}$$

A section $f \in M_\kappa(K_r,\psi;R)$ can be interpreted as a functorial rule assigning to a pair $(\underline{X}, \varepsilon)$ (which is an S-valued point of the Shimura variety[5] $\mathrm{M}_{K_r,L}$) an element[6] $f(\underline{X},\varepsilon) \in S$, where $\underline{X} = (\underline{A}, \phi)$, satisfying $f(\underline{A}, \phi\circ t, b\varepsilon) = \psi(t)\kappa(b) f(\underline{X},\varepsilon)$ for all $t \in T_H(\mathbb{Z}_p)$ and $b \in B_{H_0}(S)$.

When p is not a zero-divisor in R we define the submodule of cuspforms of character ψ to be

$$S_\kappa(K_r,\psi;R) = M_\kappa(K_r,\psi;R) \cap S_\kappa\left(K_r; R\left[\frac{1}{p}\right]\right).$$

2.6.7. *The actions of $G(\mathbb{A}_f^\square)$ and $G(\mathbb{A}_f^p)$.* The action of $G(\mathbb{A}_f^\square)$ on $\{\mathrm{M}_K^{\mathrm{tor}}\}_{K^\square}$ gives an action of $G(\mathbb{A}_f^\square)$ on

$$\varinjlim_{K^\square} M_\kappa(K;R) \quad \text{and} \quad \varinjlim_{K^\square} S_\kappa(K:R).$$

Similarly, the action of $G(\mathbb{A}_f^p)$ extends to an action on $\{\overline{\mathrm{M}}_{K_r}\}_{K^p}$, giving an action of $G(\mathbb{A}_f^p)$ on

$$\varinjlim_{K^p} M_\kappa(K_r,\psi;R) \quad \text{and} \quad \varinjlim_{K^p} S_\kappa(K_r,\psi;R).$$

The submodules fixed by $K^\square$ (resp. K^p) are just the modular forms and cuspforms of weight κ and level K (resp. prime-to-p level K^p).

[4]Again, there is an exception when $F_0 = \mathbb{Q}$ and $G^{der}/\mathbb{Q} \cong SU(1,1)$.

[5]AGAIN: see next footnote

[6]This element is just 0 if $\underline{X}$ is not an S-point of $\mathrm{M}_{K_r,L}$.

2.6.8. *Hecke operators away from p.* Let $K_j = G(\mathbb{Z}_p)K_j^p \subset G(\mathbb{A}_f)$, $j = 1, 2$, be neat open compact supgroups. For $g \in G(\mathbb{A}_f^p)$ we define Hecke operators

$$[K_2 g K_1] : M_\kappa(K_1; R) \to M_\kappa(K_2; R),$$
$$[K_{2,r} g K_{1,r}] : M_\kappa(K_{1,r}, \psi; R) \to M_\kappa(K_{2,r}, \psi; R)$$

through the action of $G(\mathbb{A}_f^p)$ on the modules of modular forms:

$$[K_{2,r} g K_{1,r}] f = \sum_{g_j} [g_j]^* f, \quad K_2^p g K_1^p = \sqcup_{g_j} g_j K_1^p. \tag{2.6.7}$$

In particular,

$$([K_{2,r} g K_{1,r}] f)(A, \lambda, \iota, \alpha K_2^p, \phi, \varepsilon) = \sum_{g_j} f(A, \lambda, \iota, \alpha g_j K_1^p, \phi, \varepsilon). \tag{2.6.8}$$

These actions map cuspforms to cuspforms.

When $K_2 = K_1$ is understood we write $T(g)$ instead of $[K_1 g K_1]$ and $T_r(g) = [K_{1,r} g K_{1,r}]$; we drop the subscript r when that is also understood.

2.6.9. *Hecke operators at p.* If p is invertible in R (so R is a $\mathbb{Q}_p$-algebra) we define Hecke operators $T(g) = [KgK]$ and $T_r(g) = [K_r g K_r]$ on the spaces of modular forms and cuspforms over R just as we did in 2.6.8. We single out some particular operators: for $w \in \Sigma_p$, $1 \le i \le m$, $1 \le j \le n_i$, we let $t^+_{w,i,j} \in B^+(\mathbb{Q}_p)$ be the element identified via (2.5.3) with $(1, (t_{w',i',j}))$ where

$$t_{w',i',j} = \begin{cases} \mathrm{diag}(p1_j, 1_{n-j}) & w = w', i = i', j \le a_w \\ \mathrm{diag}(p1_{a_w}, 1_{n-j}, p1_{j-a_w}) & w = w', i = i', j > a_w \\ 1_n & \text{otherwise.} \end{cases}$$

Note that $t^+_{w,i,j}$ has the property that

$$t^+_{w,i,j} I^0_r t^{+,-1}_{w,i,j} \subset I^0_r.$$

Let $t^-_{w,i,j} = (t^+_{w,i,j})^{-1}$. We put

$$U_{w,i,j} = K_r t^+_{w,i,j} K_r, \quad U^-_{w,i,j} = K_r t^-_{w,i,j} K_r; \tag{2.6.9}$$

Remark 2.6.10. To define the actions of these Hecke operators on higher coherent cohomology of automorphic vector bundles it is necessary to use the class of smooth projective polyhedral cone decompositions used to define toroidal compactifications in [Lan13, Lan14]. For holomorphic forms this is generally superfluous because of the Koecher principle [Lan16].

2.6.11. *Comparing spaces of modular forms of different weight.* Given an integer a, let κ_a be the weight $\kappa_a = (a, (0))$. We define a modular form $f_a \in M_{\kappa_a}(K; R)$ by the rule $f_a(\underline{A}, \varepsilon) = \lambda^a$, where $(\underline{A}, \varepsilon)$ is a pair over an R-algebra S and ε acts as multiplication by $\lambda \in S^\times$ on $S(1)$.

Let $\kappa = (\kappa_0, (\kappa_{\sigma,i})$ be a weight, and put $\kappa' = (\kappa_0 + a, (\kappa_{\sigma,i}))$. The there are isomorphisms

$$M_\kappa(K;R) \overset{f\mapsto f_a\cdot f}{\to} M_{\kappa'}(K;R) \quad \text{and} M_\kappa(K_r,\psi;R) \overset{f\mapsto f_a\cdot f}{\to} M_{\kappa'}(K_r,\psi;R).$$

These maps induce isomorphisms on spaces of cuspforms, and the Hecke operators $T(g)$ satisfy

$$f_a \cdot T(g)f = ||\nu(g)||^a T(g)(f_a \cdot f).$$

2.7. **Complex uniformization.** We relate the objects defined so far to the usual complex analytic description of modular forms on Shimura varieties.

2.7.1. *The spaces.* Let X be the $G(\mathbb{R})$-orbit under conjugation of the homomorphism h. Recall that the stabilizer of h is the group $U_\infty = C(\mathbb{R})$, so there is a natural identification $G(\mathbb{R})/C(\mathbb{R}) \xrightarrow{\sim} X$, $g \mapsto ghg^{-1}$, which gives X the structure of a real manifold. Let $P_0 \subset G_0$ be the stabilizer of Λ_0. Via the identification of $G_{/\mathbb{C}}$ with $G_{0/\mathbb{C}}$, which identifies $C(\mathbb{C})$ with $H_0(\mathbb{C})$, X is identified with an open subspace of $G_0(\mathbb{C})/P_0(\mathbb{C})$, which gives X a complex structure. There are natural complex analytic identifications

$$\begin{aligned} \mathrm{M}_K(\mathbb{C}) =& G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K \\ \mathrm{M}_{K_r}(\mathbb{C}) =& G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_r, \end{aligned} \tag{2.7.1}$$

where the class of $(h', g) \in X \times G(\mathbb{A}_f)$ corresponds to the equivalence class of the tuple $\underline{A}_{h',g} = (A_{h'}, \lambda_{h'}, \iota, \eta_g)$ (or $\underline{X}_{h',g} = (\underline{A}_{h',g}, \phi_g)$) consisting of

- the abelian variety $A_{h'} = L \otimes \mathbb{R}/L$ with the complex structure on $L\otimes\mathbb{R}$ being that determined by h'; its dual abelian variety is $A^\vee_{h'} := L\otimes\mathbb{R}/L^\#$, where again $L\otimes\mathbb{R}$ has the complex structure defined by h' and where $L^\# = \{x \in L\otimes\mathbb{R} : \langle x, L\rangle \subseteq \mathbb{Z}(1)\}$;
- $\lambda_{h'} : A_{h'} \to A^\vee_{h'}$ is the isogeny induced by the identity map on $L\otimes\mathbb{R}$;
- ι is induced from the canonical action of $\mathcal{O}_B$ on L;
- η_g is the K (or K_r-orbit) of the translation by g map $g : L\otimes\mathbb{A}_f \xrightarrow{\sim} L\otimes\mathbb{A}_f = H_1(A_{h'}, \mathbb{A}_f)$.

This paramaterization is for $\square = \varnothing$. For $\square = \{p\}$ we require $g_p \in G(\mathbb{Z}_p)$ and

- η_g is the K^p-orbit of the translation by g map $g^p : L\otimes\mathbb{A}^p_f \xrightarrow{\sim} L\otimes\mathbb{A}^p_f = H_1(A_{h'}, \mathbb{A}^p_f)$;
- in the case of M_{K_r}, ϕ_g is the $B^u_H(\mathbb{Z}_p)$-orbit of the map $L^+\otimes\mu_{p^r} \hookrightarrow A^\vee_{h'}[p^r] = \frac{1}{p^r}L^\#/L^\# = L^\#\otimes\mathbb{Z}_p/\left(p^rL^\#\otimes\mathbb{Z}_p\right)$, $v\otimes e^{2\pi\sqrt{-1}/p^r} \mapsto g_pv \bmod \left(p^rL^\#\otimes\mathbb{Z}_p\right)$.

Here we are using that the simple factors of $G^{der}_{/\mathbb{R}}$ are all of type A (see [Kot92] for how this enters into the identifications (2.7.1)).

2.7.2. *Classical modular forms.* The dual of the Lie algebra of $A^\vee_{h'}$ is $\omega_{A^\vee_{h'}} = \mathrm{Hom}_\mathbb{C}(L\otimes\mathbb{R}, \mathbb{C})$ with the complex structure on $L\otimes\mathbb{R}$ being that determined by h'. Recalling that $L\otimes\mathbb{R} \xrightarrow{\sim} W = \Lambda_0 \otimes_{S_0} \mathbb{C}$ is a $\mathbb{C}$-linear isomorphism for the complex structure on $L\otimes\mathbb{R}$ determined by h, we find that there is a canonical $\mathcal{O}_B\otimes\mathbb{C}$-identification ε_0 :

$\omega_{A_h^\vee} \xrightarrow{\sim} \Lambda_0^\vee \otimes_{S_0} \mathbb{C}$. If $h' = ghg^{-1}$, then $\varepsilon_{h'}(\lambda) = \varepsilon_0(g^{-1}\lambda)$ is an $\mathcal{O}_B \otimes \mathbb{C}$-identification of $\omega_{A_{h'}^\vee}$ with $\Lambda_0^\vee \otimes_{S_0} \mathbb{C}$. The complex points of the H_0-torsors $\mathcal{E}/\mathrm{M}_K$ and $\mathcal{E}_r/\mathrm{M}_{K_r}$ are then given by

$$\begin{aligned}\mathcal{E}(\mathbb{C}) =& G(\mathbb{Q})\backslash G(\mathbb{R}) \times H_0(\mathbb{C}) \times G(\mathbb{A}_f)/U_\infty K \\ \mathcal{E}_r(\mathbb{C}) =& G(\mathbb{Q})\backslash G(\mathbb{R}) \times H_0(\mathbb{C}) \times G(\mathbb{A}_f)/U_\infty K_r,\end{aligned} \tag{2.7.2}$$

with the class of $(g, x, g_f) \in G(\mathbb{R})\times H_0(\mathbb{C})\times G(\mathbb{A}_f)$ corresponding to the class of $(\underline{A}_{ghg^{-1},g_f}, (x\varepsilon_0(g^{-1}\cdot), \nu(x)))$ and $(\underline{X}_{ghg^{-1},g_f}, (x\varepsilon_0(g^{-1}\cdot), \nu(x)))$, respectively.

As $\mathbb{C}$ is a $\mathbb{Z}_p$-algebra via ι_p, a weight κ modular form over $\mathbb{C}$ is therefore identified with a smooth function $\varphi : G(\mathbb{A}) \times H_0(\mathbb{C}) \to \mathbb{C}$ such that $\varphi(\gamma guk, bxu) = \kappa(b)\varphi(g,x)$ for $\gamma \in G(\mathbb{Q})$, $g \in G(\mathbb{A})$, $x \in H_0(\mathbb{C})$, $u \in U_\infty$, $b \in B_{H_0}(\mathbb{C})$, and $k \in K$ or K_r. The space

$$W_\kappa(\mathbb{C}) = \{\phi : H_0(\mathbb{C}) \to \mathbb{C} \ : \ \phi \text{ holomorphic}, \phi(bx) = \kappa(b)\phi(x) \ \forall b \in B_{H_0}(\mathbb{C})\}$$

is the irreducible $\mathbb{C}$-representation of H_0 of highest weight κ with respect to $(T_{H_0}, B_{H_0}^{\mathrm{op}})$, so a weight κ modular form is also identified with a smooth function $f : G(\mathbb{A}) \to W_\kappa(\mathbb{C})$ such that $f(\gamma guk) = u^{-1}f(g)$ for $\gamma \in G(\mathbb{Q})$, $u \in U_\infty$, and $k \in K$ or K_r. Here U_∞ acts on $W_\kappa(\mathbb{C})$ as $u\phi(x) = \phi(xu)$. The connection between f and φ is $f(g)(x) = \varphi(g,x)$. The condition that the modular form is holomorphic can be interepreted as follows. Let $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))_{\mathbb{C}}$, and let $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$ be the Cartan decomposition for the involution $h(\sqrt{-1})$: $ad\,h(\sqrt{-1})$ acts as $\pm\sqrt{-1}$ on $\mathfrak{p}^\pm$. The identification of $G(\mathbb{C})$ with $G_0(\mathbb{C})$ identifies $\mathrm{Lie}(P_0(\mathbb{C}))$ with $\mathfrak{k} \oplus \mathfrak{p}^+$, and so f corresponds to a holomorphic form if and only if $\mathfrak{p}^- * f = 0$.

Let $\psi : T_H(\mathbb{Z}_p) \to \overline{\mathbb{Q}}^\times$ be a finite character that factors through $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. The condition that a modular form have character ψ becomes $f(gt) = \psi(t)f(g)$ for all $t \in T_H(\mathbb{Z}_p)$, where the action of t comes via (2.5.2).

2.7.3. *Hecke operators.* The actions of the Hecke operators in 2.6.8 and 2.6.9 correspond to the following actions on the functions $f : G(\mathbb{A}) \to W_\kappa(\mathbb{C})$: the action of $[K_2gK_1]$ is just

$$f(g) \mapsto \sum_{g_j} f(gg_j), \quad K_2gK_1 = \sqcup g_jK_1, \tag{2.7.3}$$

and similarly with K_i replaced by $K_{i,r}$.

2.8. **Igusa towers.** Let $\square = \{p\}$. Let $\mathcal{A}$ be the semiabelian scheme over $\mathrm{M}_K^{\mathrm{tor}}/S_\square$ and let ω be the $\mathcal{O}_{\mathrm{M}_K^{\mathrm{tor}}}$-dual of the Lie algebra of $\mathcal{A}^\vee$. Recall that the hypothesis (2.2.1) implies that the completion of $incl_p(S_\square)$ is $\mathbb{Z}_p$; in this way we consider $\mathbb{Z}_p$ an $S_\square$-algebra. Let $k > 0$ be so large that the kth-power of the Hasse invariant has a lift to a section $E \in M_{\det^k}(K;\mathbb{Z}_p)$. Put

$$\mathrm{S}_m = \mathrm{M}_{K,L}^{\mathrm{tor}}[\frac{1}{E}]_{/\mathbb{Z}_p/p^m\mathbb{Z}_p}.$$

Let $\mathrm{S}_m^0 = \mathrm{M}_{K,L}[\frac{1}{E}]_{/\mathbb{Z}_p/p^m\mathbb{Z}_p}$; this is an open subscheme of S_m. For $n \geq m$ let $\mathrm{T}_{n,m}/\mathrm{S}_m$ be the finite étale scheme over S_m such that for any S_m-scheme S

$$\mathrm{T}_{n,m}(S) = \mathrm{Isom}_S(L^+ \otimes \mu_{p^n}, \mathcal{A}^\vee[p^n]^\circ),$$

where the isomorphisms are of finite flat group schemes over S with $\mathcal{O}_B \otimes \mathbb{Z}_p$-actions. The scheme $\mathrm{T}_{n,m}$ is Galois over S_m with Galois group canonically isomorphic to $H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. The collection $\{\mathrm{T}_{n,m}\}_n$ is called the Igusa tower over S_m.

2.9. **p-adic modular forms.** Let $\mathrm{D}_{n,m}$ be the preimage of $\mathrm{D}_m = \mathrm{S}_m - \mathrm{S}_m^0$ in $\mathrm{T}_{n,m}$. For a p-adic ring R (that is, $R = \varprojlim_m R/p^m R$), let

$$V_{n,m}(R) = H^0(\mathrm{T}_{n,m}/R, \mathcal{O}_{\mathrm{T}_{n,m}}) \quad \text{and} \quad V_{n,m}^{\mathrm{cusp}}(R) = H^0(\mathrm{T}_{n,m}/R, \mathcal{O}_{\mathrm{T}_{n,m}}(-\mathrm{D}_{n,m})).$$

The group $H(\mathbb{Z}_p)$ acts on each through its quotient $H(\mathbb{Z}_p/p^n\mathbb{Z}_p)$, the Galois group of $\mathrm{T}_{n,m}/\mathrm{S}_m$. The R-module of p-adic modular forms (for G) over R of level K^p is

$$V(K^p, R) = \varprojlim_m \varinjlim_n V_{n,m}(R)^{B_H^u(\mathbb{Z}_p)},$$

and the R-module of p-adic cuspforms (for G) over R of level K^p is

$$V(K^p, R)^{\mathrm{cusp}} = \varprojlim_m \varinjlim_n V_{n,m}^{\mathrm{cusp}}(R)^{B_H^u(\mathbb{Z}_p)}.$$

The group $T_H(\mathbb{Z}_p) = B_H(\mathbb{Z}_p)/B_H^u(\mathbb{Z}_p)$ acts on these modules.

A p-adic modular form over R can be viewed as a functorial rule that assigns an element of a p-adic R-algebra S to each tuple $(\underline{A}, \phi)$ over S, where $\underline{A} = (\underline{A}_m) \in \varprojlim \mathrm{S}_m(S)$ and $\phi = (\phi_{n,m}) \in \varprojlim_m \varprojlim_n \mathrm{T}_{n,m}(S)$ with each $\phi_{n,m}$ over $\underline{A}_m$.

2.9.1. *p-adic modular forms of weight κ and character ψ.* Let $\mathcal{K}' \subset \overline{\mathbb{Q}}_p$ be the extension of $\mathbb{Q}_p$ generated by the images of all the embeddings of $\mathcal{K}$ into $\overline{\mathbb{Q}}_p$, and let $\mathcal{O}'$ be its ring of integers. Let

$$\kappa = (\kappa_{\sigma,i})_{\sigma \in \Sigma_{\mathcal{K}}, 1 \leq i \leq m}, \ \kappa_{\sigma,i} \in \mathbb{Z}^{a_{\sigma,i}}.$$

We denote also by κ the $\mathcal{O}'$-valued character of $T_H(\mathbb{Z}_p)$ defined by

$$\kappa(t) = \prod_{w|p} \prod_{\substack{\sigma \in \Sigma_{\mathcal{K}} \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \prod_{i=1}^{m} \prod_{j=1}^{a_{\sigma,i}} \sigma(t_{w,i,j})^{\kappa_{\sigma,i,j}},$$

$$t = (\mathrm{diag}(t_{w,i,1}, \ldots, t_{w,i,a_{w,i}}))_{w|p, 1 \leq i \leq m} \in T_H(\mathbb{Z}_p).$$

If $\psi : T_H(\mathbb{Z}_p) \to \overline{\mathbb{Q}}_p^\times$ is a finite-order character, then we define an $\mathcal{O}'[\psi]$-valued character κ_ψ of $\mathrm{T}_H(\mathbb{Z}_p)$ by $\kappa_\psi(t) = \psi(t)\kappa(t)$. For R a p-adic ring that is also an $\mathcal{O}'[\psi]$-algebra, the spaces of p-adic modular forms and cuspforms of weight κ and character ψ are

$$V_\kappa(K^p, \psi, R) = \{f \in V(K^p, R) \ : \ t \cdot f = \kappa_\psi(t) f \ \ \forall t \in T_H(\mathbb{Z}_p)\}$$

and

$$V_\kappa^{\mathrm{cusp}}(K^p, \psi, R) = \{f \in V^{\mathrm{cusp}}(K^p, R) \ : \ t \cdot f = \kappa_\psi(t) f \ \ \forall t \in T_H(\mathbb{Z}_p)\}.$$

As a functorial rule, a p-adic modular form of weight κ and character ψ satisfies $f(\underline{A},\phi\circ t)=\kappa_\psi(t)f(\underline{A},\phi)$ for all $t\in T_H(\mathbb{Z}_p)$.

2.9.2. *The action of $G(\mathbb{A}_f^p)$.* The action of $G(\mathbb{A}_f^p)$ on $\{\mathrm{M}_K^{\mathrm{tor}}\}_{K^p}$ induces an action on $\{\mathrm{S}_m\}_{K^p}$ and on $\{\mathrm{T}_{n,m}\}_{n,K^p}$, and these actions give an action of $G(\mathbb{A}_f^p)$ on

$$\varprojlim_{K^p} V(K^p,R) \quad\text{and}\quad \varprojlim_{K^p} V_\kappa(K^p,\psi,R)$$

and on their submodules of cuspforms. The submodules fixed by K^p are just the p-adic modular forms and cuspforms of weight κ and prime-to-p level K^p.

2.9.3. *Hecke operators away from p.* Let $K_j^p\subset G(\mathbb{A}_f^p)$, $j=1,2$, be neat open compact supgroups. For $g\in G(\mathbb{A}_f^p)$ we define a Hecke operator $[K_2^pgK_1^p]$ on the spaces of p-adic modular forms and cuspforms just as in Section 2.6.8.

2.9.4. *Modular forms as p-adic modular forms.* Let $\square=\{p\}$. Under Hypothesis 2.2.1, the completion of $incl_p(S_{(p)})$ is $\mathbb{Z}_p$, so $incl_p$ identifies $\mathbb{Z}_p$ as an $S_{(p)}$-algebra and $\mathcal{O}'$ as an S_0-algebra. As $\mathcal{O}_B\otimes\mathbb{Z}_{(p)}=\mathcal{O}_{(p)}^m$, we have

$$\mathcal{O}_B\otimes\mathcal{O}'=(\mathcal{O}_{(p)}\otimes\mathcal{O}')^m=\prod_{w|p}\prod_{i=1}^m\mathcal{O}_w\otimes\mathcal{O}'\xrightarrow{\sim}\prod_{w|p}\prod_{\substack{\sigma\in\Sigma_\mathcal{K}\\ \mathfrak{p}_\sigma=\mathfrak{p}_w}}\prod_{i=1}^m\mathcal{O}=\prod_{\sigma\in\Sigma_\mathcal{K}}\prod_{i=1}^m\mathcal{O}'.$$

The choices in Sections 2.6.3 and 2.2 induce $\mathcal{O}_B\otimes\mathcal{O}'$-decompositions

$$\Lambda_0\otimes_{S_\square}\mathcal{O}'=\prod_{\sigma\in\Sigma_\mathcal{K}}\prod_{i=1}^m e_i\Lambda_{0,\sigma}\otimes_{S_0}\mathcal{O}'=\prod_{\sigma\in\Sigma_\mathcal{K}}\prod_{i=1}^m(\mathcal{O}')^{a_{\sigma,i}}$$

and

$$L^+\otimes_{\mathbb{Z}_p}\mathcal{O}'=\prod_{w|p}\prod_{i=1}^m e_iL_w\otimes_{\mathbb{Z}_p}\mathcal{O}'=\prod_{w|p}\prod_{i=1}^m(\mathcal{O}_w\otimes_{\mathbb{Z}_p}\mathcal{O}')^{a_{w,i}}=\prod_{w|p}\prod_{\substack{\sigma\in\Sigma_\mathcal{K}\\ \mathfrak{p}_\sigma=\mathfrak{p}_w}}\prod_{i=1}^m(\mathcal{O}')^{a_{\sigma,i}}.$$

Equating these identifications yields an $\mathcal{O}_B\otimes\mathcal{O}'$-identification $\Lambda_0\otimes_{S_\square}\mathcal{O}'=L^+\otimes_{\mathbb{Z}_p}\mathcal{O}'$. Recalling that $H_0\subset G_0$ is the stabilizer of the polarization $\Lambda=\Lambda_0\oplus\Lambda_0^\vee$ and hence that $H_{0/\mathcal{O}'}\xrightarrow{\sim}\mathbb{G}_m\times\mathrm{GL}_{\mathcal{O}_B\otimes\mathcal{O}'}(\Lambda_0\otimes_{S_\square}\mathcal{O}')$, this then determines an isomorphism

$$H_{0/\mathcal{O}'}\xrightarrow{\sim}\mathbb{G}_m\times H_{/\mathcal{O}'}$$

which is given explicitly in terms of (2.5.1) and (2.6.1) by

$$H_{0/\mathcal{O}'}\ni(\nu,(g_{\sigma,i})_{\sigma\in\Sigma_\mathcal{K}})\mapsto(\nu,(\prod_{\substack{\sigma\in\Sigma_\mathcal{K}\\ \mathfrak{p}_\sigma=\mathfrak{p}_w}}\nu\cdot{}^tg_{\sigma c,i}^{-1})_{w|p})\in\mathbb{G}_m\times H_{/\mathcal{O}'},\tag{2.9.1}$$

where we have used the identification $\mathrm{GL}_r(\mathcal{O}_w\otimes_{\mathbb{Z}_p}\mathcal{O}')\xrightarrow{\sim}\prod_{\sigma\in\Sigma_\mathcal{K},\mathfrak{p}_\sigma=\mathfrak{p}_w}\mathrm{GL}_r(\mathcal{O}')$. This identifies $B_{H_0/\mathcal{O}'}=\mathbb{G}_m\times B_{H/\mathcal{O}'}$, $B^u_{H_0/\mathcal{O}'}=B^u_{H/\mathcal{O}'}$, and $T_{H_0/\mathcal{O}'}=\mathbb{G}_m\times T_{H/\mathcal{O}'}$.

To each weight $\kappa=(\kappa_0,(\kappa_{\sigma,i}))$ as in (2.6.3), we associate a κ_p as in 2.9.1:

$$\kappa_p=(\kappa_{\sigma c,i}).$$

Note that $\kappa_{\sigma c,i} \in \mathbb{Z}^{b_{\sigma c,i}} = \mathbb{Z}^{a_{\sigma,i}}$. If $t \in T_H(\mathbb{Z}_p)$, $t = (\mathrm{diag}(t_{w,i,1},\dots,t_{w,i,a_{w,i}})$, then

$$\kappa_p(t) = \prod_{w|p} \prod_{\substack{\sigma\in\Sigma_{\mathcal{K}} \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \prod_{i=1}^{m} \prod_{j=1}^{a_{w,i}} \sigma(t_{w,i,j})^{\kappa_{\sigma c,i,j}}.$$

Note that if $x = (t_0,t) \in \mathbb{Z}_p^\times \times T_H(\mathbb{Z}_p) \subset T_{H_0}(\mathcal{O}')$, then

$$\kappa(x) = t_0^{c_0}\kappa_p(t^{-1}), \quad c_0 = \kappa_0 + \sum_{\sigma,i,j} \kappa_{\sigma,i,j}.$$

As we explain in the following, for $\psi : T_H(\mathbb{Z}_p) \to \mathbb{Q}_p$ a finite order character and R a p-adic $\mathcal{O}'[\psi]$-algebra, if κ satisfies the inequalities (2.6.2), then the modular forms over R of weight κ and character ψ are p-adic modular forms of weight κ and character ψ.

Fixing $\mathbf{G}_m = \mathrm{Spec}\mathbb{Z}[x,\frac{1}{x}]$ yields an identification $\mu_{p^n} = \mathrm{Spec}\mathbb{Z}[x,\frac{1}{x}]/(x^{p^n}-1)$ for each $n \geq 1$, and hence an identification $\mathrm{Lie}_{\mathbb{Z}}(\mu_{p^n}) = \mathbb{Z}x\frac{d}{dx}$. For any scheme S, this identifies $\mathrm{Lie}_S(\mu_{p^n})$ with $\mathcal{O}_S$, compatibly as n varies. If $n \geq m$, S is a $\mathbb{Z}_p/p^m\mathbb{Z}_p$-scheme, and $\phi \in \mathrm{T}_{n,m}(S)$, then this identification gives an isomorphism

$$\mathrm{Lie}(\phi) : L^+ \otimes \mathcal{O}_S = L^+ \otimes \mathrm{Lie}_S(\mu_{p^n}) \xrightarrow{\sim} \mathrm{Lie}_S(\mathcal{A}^\vee_{/S}[p^n]^\circ) = \mathrm{Lie}_S\mathcal{A}^\vee_{/S}.$$

The identification $\Lambda_0 \otimes \mathbb{Z}_p = L^+$ gives $(\mathrm{Lie}(\phi)^\vee, id) \in \mathcal{E}_n(S)$. If $f \in M_\kappa(K_r,\psi;R)$ for R a p-adic $\mathcal{O}'[\psi]$-algebra, then the value of the p-adic modular form $f_{p\text{-adic}}$ determined by f on a (p-adic) test object $(\underline{A},\phi)$ over a p-adic R-algebra S is

$$f_{p\text{-adic}}(\underline{A},\phi) = \varprojlim_m f(\underline{A}_m, \phi_{m,m,r}, (\mathrm{Lie}(\phi_{m,m,r})^\vee, id)) \in \varprojlim_m S/p^mS = S,$$

where for $n \geq \max\{r,m\}$, $\phi_{n,m,r}$ is the isomorphism $L^+\otimes\mu_{p^r} \xrightarrow{\sim} \mathcal{A}^\vee_{/S}[p^r]^\circ$ determined by $\phi_{n,m}$. If $t \in T_H(\mathbb{Z}_p)$ then $\mathrm{Lie}(\phi\circ t)^\vee = t^{-1}\cdot\mathrm{Lie}(\phi)^\vee$, so

$$(t\cdot f_{p\text{-adic}})(\underline{A},\phi) = \varprojlim f(\underline{A}_m, \phi_{m,m,t}\circ t, (\mathrm{Lie}(\phi_{m,m}\circ t)^\vee, id)) = \psi(t)\kappa_p(t)f_{p\text{-adic}}(\underline{A},\phi),$$

hence $f_{p\text{-adic}}$ is a p-adic modular form of weight κ_p and character ψ. Clearly, if f is a cuspform, then $f_{p\text{-adic}}$ is a p-adic cuspform[7]. Also, the corresponding R-module homomorphisms

(2.9.2) $$M_\kappa(K_r,\psi;R) \hookrightarrow V_{\kappa_p}(K^p,\psi,R) \quad\text{and}\quad S_\kappa(K_r,\psi;R) \hookrightarrow V^{\mathrm{cusp}}_{\kappa_p}(K^p,\psi,R)$$

are compatible with Hecke operators in the sense that

(2.9.3) $$(T(g)\cdot f)_{p\text{-adic}} = ||\nu(g)||^{-\kappa_0}T(g)\cdot f_{p\text{-adic}}$$

for $g \in G(\mathbb{A}_f^p)$.

Note that if $\kappa' = (\kappa_0 + a, (\kappa_{\sigma,i}))$, then $\kappa'_p = \kappa_p$. Furthermore, for $f \in M_\kappa(K;R)$ and $f' = f_af \in M_{\kappa'}(K;R)$ (see 2.6.11),

$$f_{p\text{-adic}} = f'_{p\text{-adic}}.$$

[7]A modular form can be a p-adic cuspform but not be cuspidal. A simple example is the the critical p-stabilization $E^*_{2k}(z) = E_{2k}(z) - E_{2k}(pz)$ of the level 1 weight $2k \geq 4$ Eisenstein series E_{2k}.

2.9.5. *Hecke operators at p.* Hida ([Hid04, 8.3.1]) has defined an action of the double cosets $u_{w,i,j} = B_H^u(\mathbb{Z}_p)t_{w,i,j}B_H^u(\mathbb{Z}_p)$ on the modules of p-adic modular forms and cusp-forms; this action is defined via correspondences on the Igusa tower (see also [SU02]). Moreover, as Hida shows, if R is a p-adic domain in which p is not zero, κ as in Section 2.9.1, and $f \in M_\kappa(K_r,\psi;R)$, then $u_{w,i,j}\cdot f \in M_\kappa(K_r,\psi;R)$ and

$$u_{w,i,j}\cdot f = |\kappa_{norm}(t_{w,i,j})|_p^{-1}U_{w,i,j}\cdot f, \quad \kappa_{norm} = (\kappa_{\sigma,i'} - b_{\sigma,i'}). \tag{2.9.4}$$

We put

$$u_p = \prod_{w\in\Sigma_p}\prod_{i=1}^{m}\prod_{j=1}^{n_i} u_{w,i,j}$$

and define a projector

$$e = \varinjlim_n u_p^{n!}. \tag{2.9.5}$$

2.9.6. *Ordinary forms.* Let R be a p-adic ring. The submodules of ordinary p-adic forms over R are

$$V^{\mathrm{ord}}(K^p,R) = eV(K^p,R), \quad \text{and} \quad V^{\mathrm{ord,cusp}}(K^p,R) = eV^{\mathrm{cusp}}(K^p,R),$$

and those of weight κ and character ψ are

$$M_\kappa^{\mathrm{ord}}(K_r,\psi;R) = eM_\kappa(K_r,\psi;R), \quad S_\kappa^{\mathrm{ord}}(K_r,\psi;R) = eS_\kappa(K_r,\psi;R),$$

$$V_\kappa^{\mathrm{ord}}(K^p,\psi,R) = eV_\kappa(K^p,\psi,R), \quad V_\kappa^{\mathrm{ord,cusp}}(K^p,\psi,R) = eV_\kappa^{\mathrm{cusp}}(K^p,\psi,R).$$

Hida's classicality theorem for ordinary forms establishes that if R is a finite $\mathcal{O}'[\psi]$-domain (resp. a finite $\mathcal{O}'$-domain) then

$$\begin{gathered} V_\kappa^{\mathrm{ord,cusp}}(K^p,\psi,R) = S_\kappa^{\mathrm{ord}}(K_r,\psi;R) \\ (\text{resp. } V_\kappa^{\mathrm{ord,cusp}}(K^p,R) = S_\kappa^{\mathrm{ord}}(K_r;R)) \\ \text{if } \kappa_{\sigma,i,a_{\sigma,i}} + \kappa_{\sigma c,i,b_{\sigma,i}} \gg 0 \ \forall\sigma\in\Sigma_{\mathcal{K}}, 1\le i\le m. \end{gathered} \tag{2.9.6}$$

This theorem is proved in [Hid04] assuming conditions denoted (G1)-(G3), which were subsquently proved by Lan in [Lan13]. Let R be as in Equation (2.9.6) and let O^+ denote the integral closure of $\mathbb{Z}_{(p)}$ in R. The fraction field $\mathrm{Frac}(O^+)$ of O^+ is a number field over which $S_\kappa(K_r,\psi;R)\otimes\mathbb{Q}$ has a rational model, given by the space of $\mathrm{Frac}(O^+)$-rational cusp forms of type κ and level K_r. The intersection of this space with $S_\kappa^{\mathrm{ord}}(K_r,\psi;R)$ is an O^+-lattice $S_\kappa^{\mathrm{ord}}(K_r,\psi;O^+)$. Given any embedding $\iota: O^+\hookrightarrow\mathbb{C}$, the image of $S_\kappa^{\mathrm{ord}}(K_r,\psi;O^+)$ in the space $S_\kappa(K_r,\psi;\mathbb{C})$ will be called the space of ordinary complex cusp forms (relative to ι) of type κ and level K_r.

2.10. **Measures and Λ-adic families.** We recall p-adic measures and their connections with Hida's theory of Λ-adic modular forms.

2.10.1. *p-adic measures.* Let R and R' be p-adic rings with R an R'-algebra. The space of R-valued measures on $T_H(\mathbb{Z}_p)$ is

$$\mathrm{Meas}(T_H(\mathbb{Z}_p);R) = \mathrm{Hom}_{R'}(C(T_H(\mathbb{Z}_p),R'),R),$$

where $C(T_H(\mathbb{Z}_p),R')$ is the R'-module of continuous R'-valued functions on $T_H(\mathbb{Z}_p)$. This is independent of the intermediate algebra R' since $C(T_H(\mathbb{Z}_p),R') = C(T_H(\mathbb{Z}_p),\mathbb{Z}_p)\hat{\otimes}_{\mathbb{Z}_p}R'$. The R-module of R-valued measures is naturally identified with $R[[T_H(\mathbb{Z}_p)]]$; the identification of a measure μ with an element f of the completed group ring is such that for any continuous homomorphism $\chi : T_H(\mathbb{Z}_p) \to R^\times$, $\mu(\chi) = \chi(f)$, where $\chi(f)$ is the image of f under the homomorphism $R[[T_H(\mathbb{Z}_p)]] \to R$ induced by χ.

2.10.2. *Λ-adic forms.* Let

$$\Lambda_H = \mathcal{O}'[[T_H(\mathbb{Z}_p)]].$$

Both $V(K_p,R)$ and $V^{cusp}(K_p,R)$, R a p-adic $\mathcal{O}'$-algebra, are Λ_H-modules via the actions of $T_H(\mathbb{Z}_p)$ on them. A Λ_H-adic modular form over R is a $\mu \in \mathrm{Meas}(T_H(\mathbb{Z}_p);V(K_p;R))$ such that $\mu(t\cdot f) = t\cdot\mu(f)$ for all $t \in \Lambda_H$. In particular, it follows that if R is an $\mathcal{O}'[\psi]$-algebra, then $\mu(\kappa_\psi) \in V_\kappa(K_p,\psi,R)$. A Λ_H-adic cuspform is defined in the same way, replacing the p-adic modular forms with cuspforms. Similarly, an ordinary Λ_H-adic modular forms or cuspform is also defined in the same way, replacing the modular forms and cuspforms with the ordinary forms. Clearly, if μ is a Λ-adic modular form, then $e\mu$ (the composition of μ with the R-linear projector $V(K_p,R) \to eV(K_p,R) = V^{\mathrm{ord}}(K_p,R)$) is an ordinary Λ_H-adic form. Let

$$\mathcal{S}^{\mathrm{ord}}(K_p,R) = \{\text{ordinary } \Lambda_H\text{-adic cuspforms } \mu \in \mathrm{Meas}(T_H(\mathbb{Z}_p);V^{\mathrm{ord},cusp}(K_p,R))\}.$$

The Hecke operators in 2.9.3 and 2.9.5 act on $\mathcal{S}^{\mathrm{ord}}(K_p,R)$ through their actions on $V^{\mathrm{ord},cusp}(K_p,R)$.

Let $\Delta \subset T_H(\mathbb{Z}_p)$ be the torsion subgroup. Since p is unramified in $\mathcal{K}$ by hypothesis, (2.5.1) induces an identification

$$\Delta \xrightarrow{\sim} \prod_{w|p}\prod_{i=1}^{m}(k_w^\times)^{a_{w,i}}$$

where k_w is the residue field of $\mathcal{O}_w$. In particular, Δ has order prime-to-p, so $\mathcal{S}^{\mathrm{ord}}(K_p,R)$ decomposes as a direct sum of isotypical pieces for the $\mathcal{O}'$-characters $\omega \in \hat{\Delta}$ of Δ:

$$\mathcal{S}^{\mathrm{ord}}(K_p,R) = \oplus_{\omega\in\hat{\Delta}}\mathcal{S}^{\mathrm{ord}}_\omega(K_p,R).$$

Let $W \subset T_H(\mathbb{Z}_p)$ be a free $\mathbb{Z}_p$-complement to Δ: $T_H(\mathbb{Z}_p) = \Delta\times W$. Then $\Lambda_H = \mathcal{O}'[[\Delta\times W]] = \Lambda[\Delta]$, where

$$\Lambda = \mathcal{O}'[[W]].$$

Each $\mathcal{S}^{\mathrm{ord}}_\omega(K_p,R)$ is a Λ-module.

Let $R \subset \overline{\mathbb{Q}}_p$ be a finite $\mathcal{O}'$-algebra and let

$$\Lambda_R = \Lambda\otimes_{\mathcal{O}'}R = R[[W]].$$

Hida [Hid04] has proven that

$$\mathcal{S}^{\mathrm{ord}}_{\omega}(K_p, R) \text{ is a free } \Lambda_R\text{-module of finite rank,} \tag{2.10.1}$$

and for any finite character $\psi : W \to \overline{\mathbb{Q}}_p^{\times}$ trivial on $W^{p^{r-1}}$ and κ as in 2.9.1 satisfying the restriction in (2.9.6),

$$\mathcal{S}^{\mathrm{ord}}_{\omega}(K_p, R) \otimes_R R[\psi]/\mathfrak{p}_{\kappa\psi}\mathcal{S}^{\mathrm{ord}}_{\omega}(K_p, R) \otimes_R R[\psi] \overset{\mu\mapsto\mu(\kappa\psi)}{\longrightarrow} S^{\mathrm{ord}}_{\kappa_*}(K_r, \omega\omega_\kappa\psi, R[\psi]) \tag{2.10.2}$$

is an isomorphism, where $\mathfrak{p}_{\kappa\psi}$ is the kernel of the homomorphism $\Lambda_R \otimes_R R[\psi] \to R[\psi]$ induced by the character $\kappa\psi$ and $\omega_\kappa \in \hat{\Delta}$ is $\kappa|_{\Delta}$.

Clearly, one can include types in the definition of Λ-adic cuspforms, and we write the module of Λ-adic cuspforms of type W_S as $\mathcal{S}^{\mathrm{ord}}(K_p, W_S, R)$. That the analogs of the maps (2.10.2) in this context are also isomorphisms follows from the fact that $\mathfrak{p}_{\kappa\psi}$ is generated by a regular sequence.

2.11. **Similitude components.** For the purposes of comparing geometric constructions with analytic computations later, we need to decompose some of the objects previously defined with respect to the similitude map

$$\nu : G \to \mathbb{G}_m, \quad g = (g', \nu') \mapsto \nu'.$$

2.11.1. *Connected components.* The set of connected components $\pi_0(\mathrm{M}_K/R) = \pi_0(\mathrm{M}_K^{\mathrm{tor}}/R)$ is represented by the R-points of a finite étale scheme $\pi_0(\mathrm{M}_K)$ over $S_\square$. Let $T_0 = G/G^{\mathrm{der}}$, and let $R_\square \subset \overline{\mathbb{Q}}$ be a finite integral normal extension of $S_\square$ over which $\pi_0(\mathrm{M}_K)$ is constant (in general, this will depend on K). Then

$$\pi_0(\mathrm{M}_K)(R_\square) = \pi_0(\mathrm{M}_K/\mathbb{C}) = T_0(\mathbb{Q})\backslash T_0(\mathbb{A}_f)/K_T,$$

where K_T is the image of $K \subset G(\mathbb{A}_f)$ in $T_0(\mathbb{A}_f)$ and the last identification, using the complex uniformization (2.7.1), sends the connected component containing $(h', g) \in X \times G(\mathbb{A}_f)$ to the class of the image of g in $T_0(\mathbb{A}_f)$.

2.11.2. *Similitude components.* The similitude map factors through T_0, so putting

$$C_K = \nu(G(\mathbb{Q}))\backslash\nu(G(\mathbb{A}_f))/\nu(K),$$

there is a surjection $\pi_0(\mathrm{M}_K)(R_\square) \twoheadrightarrow C_K$ that sends the component containing (h, g) to the class of $\nu(g)$. Given $\alpha \in C_K$, for any $R_\square$-scheme R we let M_K^{α}/R and $\mathrm{M}_K^{\mathrm{tor},\alpha}/R$ be the base change to R of the union of the connected components of M_K/R and $\mathrm{M}_K^{\mathrm{tor}}/R$, respectively, over α. For $R = \mathbb{C}$ this is just the set of points (h', g) with $\nu(g) = \alpha$. Similarly, we let $\mathrm{M}_{K_r}^{\alpha}$ and $\overline{\mathrm{M}}_{K_r}^{\alpha}$ be the pullbacks of M_{K_r} and $\overline{\mathrm{M}}_{K_r}$, respectively, over M_K^{α} and $\mathrm{M}_K^{\mathrm{tor},\alpha}$. Since $\nu(I_rK^p) = \nu(K)$, these definitions for $\mathrm{M}_{I_rK^p}^{\alpha}/R$ coincide when p is invertible in R. We also put $\S_m^{\alpha} = \mathrm{M}_K^{\mathrm{tor},\alpha}[1/E]$ and let $\mathrm{T}_{n,m}^{\alpha}/\S_m^{\alpha}$ be the corresonding component of the Igusa tower; the latter is a Galois cover with Galois group $H(\mathbb{Z}_p/p^n\mathbb{Z}_p)$.

2.11.3. *Similitude components of modular forms.* Let R be an $R_\square$-algebra. By restricting to each $\mathrm{M}_K^{\mathrm{tor},\alpha}/R$ or $\overline{\mathrm{M}}_{K_r}^{\alpha}/R$ (that is, pulling the canonical bundles back to these components) we obtain a decomposition of the R-modules of modular forms

$$M_\kappa(K;R) = \bigoplus_{\alpha\in C_K} M_\kappa(K;R)^\alpha \tag{2.11.1}$$

and

$$M_\kappa\left(K_r,\psi;R\right) = \bigoplus_{\alpha\in C_K} M_\kappa\left(K_r,\psi;R\right)^\alpha, \tag{2.11.2}$$

where f belongs to the α-summand if it is zero on any $(\underline{X},\varepsilon)$ with $\underline{X}$ not in $\mathrm{M}_{K_r}^{\alpha}(R')$ (R' an R-algebra). For $R = \mathbb{C}$, in terms of the complex uniformization, this means that as a function of $(h',g)\in X\times G(\mathbb{A}_f)$ or $g' = g_\infty g \in G(\mathbb{A}) = G(\mathbb{R})G(\mathbb{A}_f)$, f vanishes unless the image of $\nu(g)$ in C_K is α. The modules of cuspforms decompose similarly.

If R is also a p-adic ring, then there is a similar decomposition of the modules of p-adic modular forms obtained by restricting to $\mathrm{T}_{n,m}^{\alpha}$:

$$V\left(K^p,R\right) = \bigoplus_{\alpha\in C_K} V\left(K^p,R\right)^\alpha \tag{2.11.3}$$

and

$$V_\kappa\left(K^p,\psi,R\right) = \bigoplus_{\alpha\in C_K} V_\kappa\left(K^p,\psi,R\right)^\alpha. \tag{2.11.4}$$

There are similar decompositions of the spaces of p-adic cuspforms. The p-adic modular form defined by a modular form in belongs to $V_\kappa(K^p,\psi,R)^\alpha$ if and only if the form belongs to $M_{\kappa_*}(K_r,\psi;R)^\alpha$.

The α-components of the modular or p-adic modular forms are not in general stable under the Hecke operators $[K^pgK^p]$ but are if $\nu(g)$ belongs to the class of 1 in C_K (so if $\nu(g)=1$). In particular, they are stable under the operators $U_{w,i,j}$ and $u_{w,i,j}$ (when these operators are defined), and the isomorphism (2.9.6) can be refined as

$$\begin{gathered} V_\kappa^{\mathrm{ord,cusp}}(K_p,\psi,R)^\alpha = S_{\kappa_*}^{\mathrm{ord}}(K_r,\psi;R)^\alpha \\ \text{if}\ \ \kappa_{\sigma,i,a_{\sigma,i}} + \kappa_{\sigma c,i,b_{\sigma,i}} \geq n_i r_i\ \ \forall\sigma\in\Sigma_{\mathcal{K}}, 1\leq i\leq m. \end{gathered} \tag{2.11.5}$$

2.11.4. *The definite case.* Suppose P is in the definite case as in Section A below. In term of the spaces of functions in that section, the condition of being in the α-component always unwinds to meaning that the functions are zero on g such that $\nu(g)$ does not belong to α. All the modules of functions introduced in Section A can be decomposed in this way over C_K and we again use the superscript 'α' to denote the corresponding components.

3. The PEL problem and restriction of forms

In this section, we discuss restrictions of modular forms from a larger unitary group to a product of unitary groups, which is important for interpreting the doubling method (first introduced in Section 4.1) geometrically.

3.1. **The PEL problems.** Let $P = (\mathcal{K}, c, \mathcal{O}, L, \langle\cdot,\cdot\rangle, h)$ be a PEL problem of unitary type associated with a Hermitain pair $(V, \langle\cdot,\cdot\rangle_V)$ as in Sections 2.1 and 2.2 together with all the associated objects, choices, and conventions from Section 2. In particular, the index m equals 1. In what follows we will consider four unitary PEL data $P_i = (B_i, *_i, \mathcal{O}_{B_i}, L_i, \langle\cdot,\cdot\rangle_i, h_i)$ together with $\mathcal{O}_{B_i}\otimes\mathbb{Z}_p$ decompositions $L_i\otimes\mathbb{Z}_p = L_i^+\oplus L_i^-$:

- $P_1 = P = (\mathcal{K}, c, \mathcal{O}, L, \langle\cdot,\cdot\rangle, h)$, $L_1^\pm = L^\pm$;
- $P_2 = (\mathcal{K}, c, \mathcal{O}, L, -\langle\cdot,\cdot\rangle, h(\bar{\cdot}))$, $L_2^\pm = L^\mp$;
- $P_3 = (\mathcal{K}\times\mathcal{K}, c\times c, \mathcal{O}\times\mathcal{O}, L_1\oplus L_2, \langle\cdot,\cdot\rangle_1\oplus\langle\cdot,\cdot\rangle_2, h_1\oplus h_2)$, $L_3^\pm = L_1^\pm\oplus L_2^\pm$;
- $P_4 = (\mathcal{K}, c, \mathcal{O}, L_3, \langle\cdot,\cdot\rangle_3, h_3)$, $L_4^\pm = L_3^\pm$.

Given the hypotheses, there should be no confusion with the subscript 'i' being used in this section for the objects associated to the PEL problem P_i.

The reflex fields for P_1, P_2 and P_3 are all equal to the reflex field F of P. The reflex field of P_4 is $\mathbb{Q}$. We put $G_i = G_{P_i}$ for $i = 1, ..., 4$ and $H_i = \mathrm{GL}_{\mathcal{O}_{B_i}\otimes\mathbb{Z}_p}(L_i^+)$. Then $G_1 = G_2$ and there are obvious, canonical inclusions $G_3\hookrightarrow G_4$ and $G_3\hookrightarrow G_1\times G_2$ which induce the obvious, canonical inclusions $H_3\hookrightarrow H_4$ and $H_3\hookrightarrow H_1\times H_2$. For $K\subset G_i(\mathbb{A}_f)$ a neat open compact with $K = G_i(\mathbb{Z}_p)K^\Box$ if $\Box = \{p\}$, let $\mathrm{M}_{i,K} = \mathrm{M}_K(P_i)$ be the moduli scheme over $S_\Box$.

The choice of the $\mathcal{O}_w$-decomposition of $L_w^\pm$ determines $\mathcal{O}_{B_i,w}$-decompositions of the modules $L_{i,w}^\pm = L_i\otimes_{\mathcal{O}\otimes\mathbb{Z}_p}\mathcal{O}_w$ and so determines isomorphisms

$$G_{i/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m\times\prod_{w\in\Sigma_p}\begin{cases}\mathrm{GL}_n(\mathcal{O}_w) & i = 1,2\\ \mathrm{GL}_n(\mathcal{O}_w)\times\mathrm{GL}_{nr}(\mathcal{O}_w) & i = 3\\ \mathrm{GL}_{2n}(\mathcal{O}_w) & i = 4\end{cases} \tag{3.1.1}$$

and

$$H_{i/\mathbb{Z}_p} \xrightarrow{\sim} \prod_{w|p}\begin{cases}\mathrm{GL}_{a_w}(\mathcal{O}_w) & i = 1\\ \mathrm{GL}_{b_w}(\mathcal{O}_w) & i = 2\\ \mathrm{GL}_{a_w}(\mathcal{O}_w)\times\mathrm{GL}_{b_w}(\mathcal{O}_w) & i = 3\\ \mathrm{GL}_n(\mathcal{O}_w) & i = 4.\end{cases} \tag{3.1.2}$$

The canonical inclusions in the preceding paragraph just correspond to the identity map on the similitude factors and the obvious inclusions of the GL-parts (being the diagonal map in the case of the inclusions $G_3\hookrightarrow G_4$ and $H_3\hookrightarrow H_4$.)

Let $K_i^\square \subset G_i(\mathbb{A}_f^\square)$ be neat open compact subgroups. Let $K_i = K_i^\square$ if $\square = \varnothing$ and $K_i = G_i(\mathbb{Z}_p)K_i^\square$ otherwise. If $K_3^\square \subset K_4^\square \cap G_3(\mathbb{A}_f^\square)$, then there is a natural $S_\square$-morphism

$$\mathrm{M}_{3,K_3} \to \mathrm{M}_{4,K_4}, \quad \underline{A} = (A,\lambda,\iota,\alpha) \mapsto \underline{A}_4 = (A,\lambda,\iota\circ\mathrm{diag},\alpha K_4^\square), \tag{3.1.3}$$

where $\mathrm{diag} : K \hookrightarrow K \oplus K$ is the diagonal embedding. Let $e_i \in \mathcal{O}\oplus\mathcal{O}$, $i = 1,2$, be the idempotent corresponding to the projection to the ith factor. If $K_3^\square \subset (K_1^\square \times K_2^\square) \cap G_3(\mathbb{A}_f^\square)$, then there is a natural $S_\square$-morphism

$$\begin{aligned} &\mathrm{M}_{3,K_3} \to \mathrm{M}_{1,K_1} \times_{S_\square} \mathrm{M}_{2,K_2}, \\ &\underline{A} = (A,\lambda,\iota,\alpha) \mapsto (\underline{A}_1,\underline{A}_2) = (A_1,\lambda_1,\iota_1,\alpha_1)\times(A_2,\lambda_2,\iota_2,\alpha_2), \end{aligned} \tag{3.1.4}$$

where $A_i = \iota(e_i)A$ (so $A = A_1\times A_2$), $\lambda_i = \iota^\vee(e_i)\circ\lambda\circ\iota(e_i)$, ι_i is the restriction of ι to the ith factor, and $\alpha_{i,s} : L_i\otimes\mathbb{A}_f^\square \xrightarrow{\sim} H_1(A_{i,s},\mathbb{A}_f^\square)$ is the restriction of α_s to $L_i\otimes\mathbb{A}_f^\square \subset L_3\otimes\mathbb{A}_f^\square = (L_1\otimes\mathbb{A}_f^\square)\oplus(L_2\otimes\mathbb{A}_f^\square)$ composed with the projection $H_1(A_s,\mathbb{A}_f^\square)\to H_1(A_{i,s},\mathbb{A}_f^\square)$.

For suitably compatible choices of polyhedral cone decompositions, the morphisms (3.1.3) and (3.1.4) extend to maps of toroidal compactifications [Har89].

3.1.1. *Level structures at p.* The definitions of level structures at p in Section 2.5 for the PEL problems P_i are compatible, and the morphisms (3.1.3) and (3.1.4) extend to $S_\square$-morphisms with each M_{i,K_i} replaced by $\mathrm{M}_{i,K_{i,r}} = \mathrm{M}_{K_{i,r}}(P_i)$.

3.1.2. *The canonical bundles.* To define the groups $G_{0,i}$ and $H_{0,i}$ as in Section 2.6.1 in a compatible manner, we need to specify the choice of the $\Lambda_{0,i} \subset W_i = V_i/V_i^{0,-1}$, where $V_i = L_i\otimes\mathbb{C}$ with the Hodge structure defined by the complex structure on $L_i\otimes\mathbb{R}$ determined by h_i. As $V_1 = V$ with the same Hodge structure we take $\Lambda_{0,1} = \Lambda_0$, but since $V_2 = V_1$ with the Hodge indices reversed (so $V_2^{0,-1} = V_1^{-1,0}$) we take $\Lambda_{0,2}$ to be the image of $\Lambda_0^\vee$ in $W_2 = V_2/V_2^{0,-1} = V_1/V_1^{-1,0}$ using the canonical identification $V_1^{0,-1} = V^{0,-1} \cong \Lambda_0^\vee\otimes_{S_0}\mathbb{C}$. Then $\Lambda_1 = \Lambda$ with its canonical pairing, and $\Lambda_2 = \Lambda_0^\vee\oplus(\Lambda_0^\vee)^\vee = \Lambda$ with its canonical pairing. We then set $\Lambda_{0,3} = \Lambda_{0,4} = \Lambda_{0,1}\oplus\Lambda_{0,2}$ and $\Lambda_3 = \Lambda_4 = \Lambda_1\oplus\Lambda_2$.

The fixed decompositions of Λ_0 and $\Lambda_0^\vee$ as $\mathcal{O}_B\otimes\mathbb{Z}_p$-modules then determine compatible isomorphisms

$$H_{0,i/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{w|p} \begin{cases} \mathrm{GL}_{b_w}(\mathcal{O}_w) & i = 1 \\ \mathrm{GL}_{a_w}(\mathcal{O}_w) & i = 2 \\ \mathrm{GL}_{b_w}(\mathcal{O}_w)\times\mathrm{GL}_{a_w}(\mathcal{O}_w) & i = 3 \\ GL_n(\mathcal{O}_w) & i = 4. \end{cases} \tag{3.1.5}$$

There are canonical inclusions $H_{0,3}\hookrightarrow H_{0,4}$ and $H_{0,3}\hookrightarrow H_{0,1}\times H_{0,2}$ which correspond to the obvious inclusions under the isomorphisms (3.1.5): the indentity map on the $\mathbb{G}_m$-factor and the diagonal mapping and identity map, respectively, on the GL-factors. This gives similar inclusions among the (lower-triangular) Borels $B_{H_{0,i}}$ and the (diagonal) tori $T_{H_{0,i}}$. In particular, a dominant character κ of $T_{H_{0,4}}$ or a pair $\kappa = (\kappa_1,\kappa_2)$ consisting of dominant characters κ_1 of $T_{H_{0,1}}$ and κ_2 of $T_{H_{0,2}}$ restricts to a dominant character of $T_{H_{0,3}}$, which we also denote by κ.

Let $\pi_i : \mathcal{E}_i \to \mathrm{M}_{K_i}$ be the canonical bundle. The maps (3.1.3) and (3.1.4) extend to maps of bundles

$$\mathcal{E}_3 \to \mathcal{E}_4, \quad (A,\lambda,\iota,\alpha,\varepsilon) \mapsto (A,\lambda,\iota\circ \mathrm{diag}, \alpha K_4^{\square}, \varepsilon), \tag{3.1.6}$$

and

$$\begin{aligned} \mathcal{E}_3 \to & \mathcal{E}_1 \times_{S_\square} \mathcal{E}_2, \\ (A,\lambda,\iota,\alpha,\varepsilon) \mapsto & (A_1,\lambda_1,\iota_1,\alpha_1,\varepsilon_1)\times(A_2,\lambda_2,\iota_2,\alpha_2,\varepsilon_2), \end{aligned} \tag{3.1.7}$$

where $\varepsilon_i = e_i \circ \varepsilon \circ \iota(e_i)$. There are similar maps of the bundles $\mathcal{E}_{i,r} = \mathcal{E}_i \times_{\mathrm{M}^{\mathrm{tor}}_{i,K_i}} \overline{\mathrm{M}}_{i,K_{i,r}}$ with level structure at p.

3.1.3. *The Igusa towers.* Let $\mathrm{T}_{n,m,i}/\S_{m,i}$, $i = 1,..,4$, be the Igusa tower for M_{i,K_i} as in 2.8. The maps (3.1.3) and (3.1.4) extend to maps of Igusa towers in the obvious ways:

$$\mathrm{T}_{n,m,3} \to \mathrm{T}_{n,m,4}, \quad (\underline{A},\phi) \mapsto (\underline{A}_4,\phi) \tag{3.1.8}$$

and

$$\mathrm{T}_{n,m,3} \to \mathrm{T}_{n,m,1} \times_{\mathbb{Z}_p} \mathrm{T}_{n,m,2}, \quad (\underline{A},\phi) \mapsto ((\underline{A}_1,\phi_1),(\underline{A}_2,\phi_2)), \tag{3.1.9}$$

where ϕ_i is the restriction of ϕ to $L_i^+ \otimes \mu_{p^n}$ composed with the projection to $\mathcal{A}_i^\vee[p^n]^\circ$.

Remark 3.1.4. As explained in [HLS06, Section 2.1.11], the inclusion (3.1.8) *does not* restrict on complex points to the map i_3 of Shimura varieties determined by the inclusion of G_3 in G_4. For each prime w of F^+ dividing p, let

$$\gamma_{V_w} = \begin{pmatrix} 1_{a_w} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{b_w} \\ 0 & 0 & 1_{a_w} & 0 \\ 0 & 1_{b_w} & 0 & 0 \end{pmatrix} \in G_4(F_w^+);$$

$\gamma_{V_p} = (\gamma_{V_w})_{w|p} \in G_4(F_p^+)$. Then the inclusion (3.1.8) is given by i_3 composed with right translation by γ_{V_p}. (See map (4.3.10).)

When working with p-adic modular forms in subsequent sections, we will consider all the $T_{n,m,i}$ simultaneously, $i = 1,2,3,4$. The collection $\{T_{n,m,i}\}$, or equivalently $\varprojlim_m \varprojlim_n T_{n,m,i}$ will be denoted Ig_i, $1 \le i \le 4$. Thus, if $K_i^p, i = 1,2,3,4$, are prime-to-p level subgroups of $G_i(\mathbf{A}_f)$, with $K_3^p \subset K_4^p$, $K_3^p \subset K_1^p \times K_2^p$, we similarly define Igusa varieties ${}_{K_i^p}Ig_i$ and inclusions

$$\gamma_{V_p} \circ i_3 : {}_{K_3^p}Ig_3 \to {}_{K_4^p}Ig_4; \; i_4 : {}_{K_3^p}Ig_3 \to {}_{K_1^p}Ig_1 \times {}_{K_2^p}Ig_2 \tag{3.1.10}$$

3.1.5. *Similitude components.* Let $S_\square \subseteq R_\square \subset \overline{\mathbb{Q}}$ be a finite normal extension of $S_\square$ such that $\pi_0(\mathrm{M}_{K_3})/R$ is constant. (The same is then true of $\pi_0(\mathrm{M}_{K_i})/R$, $i = 1,2$). The maps (3.1.3) and (3.1.4) and the maps (3.1.8) and (3.1.9) can be refined as maps of similitude components over an $R_\square$-algebra R. In particular, (3.1.4) and (3.1.4) induce

$$\mathrm{M}^\alpha_{K_3} \to \mathrm{M}^\alpha_{K_1} \times_R \mathrm{M}^\alpha_{K_2} \tag{3.1.11}$$

and

$$\mathrm{T}^{\alpha}_{n,m,3} \to \mathrm{T}^{\alpha}_{n,m,1} \times_R \mathrm{T}^{\alpha}_{n,m,2}, \tag{3.1.12}$$

where $\alpha \in C_{K_3}$ defines elements in C_{K_1} and C_{K_2} by projection (since $\nu(K_3) \subset \nu(K_1) \cap \nu(K_2)$). If $\nu(K_1) = \nu(K_2)$ and $K_3 = (K_1 \times K_2) \cap G_3(\mathbb{A}_f)$, then both (3.1.11) and (3.1.12) are isomorphisms; in particular M_{K_3} and $\mathrm{T}_{n,m,3}$ are identified with unions of connected components of $\mathrm{M}_{K_1} \times_R \mathrm{M}_{K_2}$ and $\mathrm{T}_{n,m,1} \times_R \mathrm{T}_{n,m,2}$.

3.2. **Restrictions of forms.** The maps between the various moduli spaces and bundles induce maps between spaces of modular forms.

3.2.1. *Restricting modular forms.* Let R be a $\mathbb{Z}_p$-algebra and κ either a dominant character of $T_{H_{0,4}}$ or a pair $\kappa = (\kappa_1, \kappa_2)$ consisting of dominant characters κ_1 of $T_{H_{0,1}}$ and κ_2 of $T_{H_{0,2}}$. Then the maps (3.1.6) and (3.1.7) yield maps of modular forms

$$res_1 : M_\kappa(K_4; R) \to M_\kappa(K_3; R),$$

and

$$res_2 : M_{\kappa_1}(K_1; R) \otimes_R M_{\kappa_2}(K_2; R) \to M_\kappa(K_3; R).$$

Let ψ be either a $\overline{\mathbb{Q}}_p^\times$-valued character of $T_{H_4}(\mathbb{Z}_p/p^r\mathbb{Z}_p)$ or a pair $\psi = (\psi_1, \psi_2)$ consisting of a $\overline{\mathbb{Q}}_p^\times$-valued character ψ_1 of $T_{H_1}(\mathbb{Z}_p/p^r\mathbb{Z}_p)$ and ψ_2 of $T_{H_2}(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Then ψ defines a character of $T_{H_3}(\mathbb{Z}_p/p^r\mathbb{Z}_p)$ that we continue to denote ψ. Let R be a $\mathbb{Z}_p[\psi]$-algebra. The analogs of the maps (3.1.6) and (3.1.7) for level structures at p yield maps

$$res_3 : M_\kappa\left(K_{4,r}, \psi; R\right) \to M_\kappa\left(K_{3,r}, \psi; R\right),$$

and

$$res_4 : M_{\kappa_1}\left(K_{1,r}, \psi_1; R\right) \otimes_R M_{\kappa_2}\left(K_{2,r}, \psi_2; R\right) \to M_\kappa\left(K_{3,r}, \psi; R\right).$$

Let $R_\square$ be as in 3.1.5. If R is also an $R_\square$-algebra, then the maps res_i restricted to maps of similitude components. In particular, if $\nu(K_1) = \nu(K_2)$ and $K_3 = (K_1 \times K_2) \cap G_3(\mathbb{A}_f)$ and R is an $R_\square$-algebra, then res_2 and res_4 induce isomorphisms

$$M_{\kappa_1}(K_1; R)^\alpha \otimes_R M_{\kappa_2}(K_2; R)^\alpha \xrightarrow{\sim} M_\kappa(K_3; R)^\alpha$$

and

$$M_{\kappa_1}\left(K_{1,r}, \psi_1; R\right)^\alpha \otimes_R M_{\kappa_2}\left(K_{2,r}, \psi_2; R\right)^\alpha \xrightarrow{\sim} M_\kappa\left(K_{3,r}, \psi; R\right)^\alpha,$$

and hence isomorphisms

$$\bigoplus_{\alpha \in C_{K_3}} M_{\kappa_1}(K_1; R)^\alpha \otimes_R M_{\kappa_2}(K_2; R)^\alpha \xrightarrow{\sim} M_\kappa(K_3; R)$$

and

$$\bigoplus_{\alpha \in C_{K_3}} M_{\kappa_1}\left(K_{1,r}, \psi_1; R\right)^\alpha \otimes_R M_{\kappa_2}\left(K_{2,r}, \psi_2; R\right)^\alpha \xrightarrow{\sim} M_\kappa\left(K_{3,r}, \psi; R\right).$$

Remark 3.2.2. We write

$$M_{\kappa_1}(K_{1,r},\psi_1;R)\,[\otimes]_R M_{\kappa_2}(K_{2,r},\psi_2;R)$$

for the image of res_4 in $M_\kappa(K_{3,r},\psi;R)$, and use the notation $[\otimes]$ more generally for restrictions of this kind from (classical or p-adic) modular forms on $G_1\times G_2$ to forms on G_3.

3.2.3. *Restrictions of classical forms.* In terms of the complex uniformizations (2.7.2), the restrictions (3.1.6) and (3.1.7) correspond to the maps induced by the canonical inclusions of G_3 and $H_{0,3}$ into G_4 and $H_{0,4}$ and into $G_1\times G_2$ and $H_{0,1}\times H_{0,2}$, respectively. In particular, if $\varphi: G_4(\mathbb{A})\times H_{0,4}(\mathbb{C})\to\mathbb{C}$ corresponds to a weight κ modular form on G_4 of level K_4, then the image of φ under res_1 or res_3 corresponds to the restriction of φ to $G_3(\mathbb{A})\times H_{0,3}(\mathbb{C})$. Moreover, if φ corresponds to $f: G_4(\mathbb{A})\to W_{\kappa,4}(\mathbb{C})$ (we include the subscript 'i' to indicate that $W_{\kappa,i}$ is the irreducible representation of $H_{0,i}$ of highest weight κ), then its image under res_1 or res_3 is just the restriction of f to $G_3(\mathbb{A})$ composed with the projection $W_{\kappa,4}(\mathbb{C})\to W_{\kappa,3}(\mathbb{C})$, $\phi\mapsto\phi|_{H_{0,3}(\mathbb{C})}$. The same holds for the maps res_2 and res_4.

3.2.4. *Restrictions of p-adic forms.* The maps (3.1.8) and (3.1.9) induce the obvious restriction maps on modules of p-adic modular forms - which we also denote by res_i - compatible with weights κ and characters ψ in the obvious way, as well as with the inclusion of spaces of modular forms and with restriction to similitude components. In particular, the isomorphisms described above extend to isomorphism of spaces of p-adic modular forms (with the tensor product $\otimes_R$ replaced by the completed tensor product $\hat{\otimes}_R$).

3.2.5. *Base point restrictions.* Let $V=V_i$ for $i\in\{1,2,3,4\}$, $G=G_{P_i}$ the corresponding unitary similitude group, so that (G,X) is the Shimura datum associated to the moduli problem P_i. Let J_0' be a torus as in section 2.3.2, and let $(J_0',h_0)\to(G,X)$ be the morphism of Shimura data defined there. Say (J_0',h_0) is *ordinary* if the points in the image of the map $S(J_0',h_0)\to S(G,X)$ of Shimura varieties reduce to points corresponding to ordinary abelian varieties. If (J_0',h_0) is ordinary, then it has an associated Igusa tower, denoted $T_{n,m}(J_0',h_0)$ for all n,m. We have $T_{0,m}(J_0',h_0)=S_m(J_0',h_0)$, in the obvious notation, which is the reduction modulo p^m of an integral model of $S(J_0',h_0)$; each $T_{n,m}(J_0',h_0)$ is finite over the corresponding S_m.

Moreover, letting $T_{n,m}(G,X)=T_{n,m}(P_i)$ in the obvious notation, there is a morphism of Igusa towers

$$T_{n,m}(J_0',h_0)\to T_{n,m}(G,X). \tag{3.2.1}$$

Thus for any r there are restriction maps $res_{J_0',h_0}: M_\kappa(K_{i,r},R)\to M_\kappa((J_0',h_0),R)$, in the obvious notation; the image is contained in forms of level r on $S(J_0',h_0)$, in an appropriate sense, but we don't specify the level. The restriction maps behave compatibly with respect to classical, complex, and p-adic modular forms; the restriction map for p-adic modular forms is denoted res_{p,J_0',h_0}. In order to formulate a precise statement, we write $V_{\kappa_p}((G,X);K^p,R)$ for p-adic modular forms of weight κ_p and level K^p on the Igusa tower for $S(G,X)$, and $V_{\kappa_p}((J_0',h_0),R)$ for the corresponding object for $S(J_0',h_0)$ (the level away from p is not specified).

Proposition 3.2.6. *Let $(J_0',h_0) \to (G,X)$ be a morphism of Shimura data, with J_0' a torus, and suppose (J_0',h_0) is ordinary. Let $G = G_{P_i}$ for $i = 1,2,3,4$, and let κ be a dominant weight; let κ_p be the corresponding p-adic weight, as in 2.9.1. Let R be a p-adic ring.*

(i) The following diagram is commutative:

$$\begin{array}{ccc} M_\kappa(K_{i,r},R) & \xrightarrow{R_{\kappa,G,X}} & V_{\kappa_p}(K_i^p,R) \\ \Big\downarrow{\scriptstyle res_{J_0',h_0}} & & \Big\downarrow{\scriptstyle res_{p,J_0',h_0}} \\ M_\kappa((J_0',h_0),R) & \xrightarrow{R_{\kappa,J_0',h_0}} & V_{\kappa_p}((J_0',h_0),R) \end{array}$$

Here the horizontal maps are the ones defined in (2.9.2)

(ii) Let $f \in M_\kappa(K_{i,r},R)$. Suppose for every ordinary CM pair (J_0',h_0) mapping to (G,X), the restriction $res_{J_0',h_0}(f) = 0$. Then $f = 0$.

Proof. Point (i) is an immediate consequence of the definitions; point (ii) follows from the Zariski density of the ordinary locus in the integral model of $S(G,X)$ [Wed99]. □

4. Eisenstein series and zeta integrals

4.1. **Eisenstein series and the doubling method.** We begin this section by introducing certain Eisenstein series and (global) zeta functions. Then we choose specific local data and compute local zeta integrals (whose product gives the global zeta function).

We assume throughout this section that we are in the setting of Section 3. In particular, there is a hermitian pair $(V,\langle\cdot,\cdot\rangle_V)$ over $\mathcal{K}$ such that $V = L_1 \otimes \mathbb{Q}$ and $\langle\cdot,\cdot\rangle_1 = \mathrm{trace}_{\mathcal{K}/\mathbb{Q}}\delta\langle\cdot,\cdot\rangle_V$. Then $G_1/\mathbb{Q}$ is the unitary similitude group of the pair $(V,\langle\cdot,\cdot\rangle_V)$. Let $(W,\langle\cdot,\cdot\rangle_W)$ be the hermitian pair with $W = V \oplus V$ and $\langle\cdot,\cdot\rangle_W = \langle\cdot,\cdot\rangle_V \oplus -\langle\cdot,\cdot\rangle_V$. Then $G_4/\mathbb{Q}$ is the unitary similitude group of the pair $(W,\langle\cdot,\cdot\rangle_W)$. Most of the constructions to follow take place on the group $G_4/\mathbb{Q}$, which we denote throughout by G for ease of notation. We write Z_i to denote the center of G_i.

An important observation is that $G_2(\mathbb{A}) = G_1(\mathbb{A})$, so a function or representation of one of these groups can be viewed as a function or representation of the other; we use this repeatedly.

In part to aid with the comparison with calculations in the literature, we introduce the unitary groups $U_i = \ker(\nu : G_i \to \mathbb{G}_m)$.

Let $n = \dim_{\mathcal{K}} V$. Let S_0 be the set of primes dividing either the discriminant of the pairing $\langle \cdot, \cdot \rangle_1$ or the discriminant of $\mathcal{K}$.

Plan of this section. We begin by recalling the general setup for Siegel-Eisenstein series on G and the zeta integrals in the context of the doubling method, explaining how the global integral factors as a product over primes of $\mathcal{K}$. The local factors fall into three classes, which are treated in turn. The factors at non-archimedean places prime to p are the easiest to address: in Section 4.2 we recall the unramified factors, which have been known for more than 20 years, and explain how to choose data at ramified places to trivialize the local integrals.

Factors at primes dividing p are computed in Section 4.3. This is the most elaborate computation in the paper. The local data defining the Eisenstein series have to be chosen carefully to be compatible with the p-adic Eisenstein measure which is recalled in Section 5.3. The local data for the test forms on G_3 are chosen to be *anti-ordinary vectors*, a notion that will be defined explicitly in 8.2.5, and that provide the local expression of the fact, built in to Hida theory, that the test forms are naturally *dual* to ordinary forms. The result of the computation is given in Theorem 4.3.11: we obtain p-stabilized Euler factors, as predicted by conjectures.

Sections 4.4 and 4.5 are devoted to the local integrals at archimedean places. Much of the material here is a review of the theory of holomorphic differential operators developed elsewhere, and of classical invariant theory. We prove in particular (Proposition 4.5.5) that the archimedean zeta integrals do not vanish; as explained in the introduction, in most cases we do not know explicit formulas for these integrals.

4.1.1. *The Siegel parabolic.* Let $V^d = \{(x,x) \in W \ : x \in V\}$ and $V_d = \{(x,-x) \in W \ : \ x \in V\}$, so $W = V_d \oplus V^d$ is a polarization of $\langle \cdot, \cdot \rangle_W$. Projection to the first summand fixes identifications of V^d and V_d with V. Let $P \subset G$ be the stabilizer of V^d; this is a maximal $\mathbb{Q}$-parabolic, the Siegel parabolic. Let $M \subset P$ be the stabilizer of the polarization $W = V_d \oplus V^d$ and $N \subset P$ the group fixing both V^d and W/V^d, so M is a Levi subgroup and N the unipotent radical. Denote by Δ the canonical map $\Delta : P \to \mathrm{GL}_{\mathcal{K}}(V^d) = \mathrm{GL}_{\mathcal{K}}(V)$. Then $M \xrightarrow{\sim} \mathrm{GL}_{\mathcal{K}}(V) \times \mathbb{G}_m$, $m \mapsto (\Delta(m), \nu(m))$; the inverse map is $(A, \lambda) \mapsto m(A, \lambda) = \mathrm{diag}(\lambda A^*, A)$, where $A^* = {}^tA^c$ is the transpose of the conjugate under the action of c. Also, fixing a basis for V gives an identification $\Delta' : N \xrightarrow{\sim} \mathrm{Her}_n(\mathcal{K})$, where Her_n denotes the space of $n \times n$ hermitian matrices; with respect to this basis and the polarization above, we obtain an identification $N \xrightarrow{\sim} \begin{pmatrix} 1_n & \Delta'(N) \\ 0 & 1_n \end{pmatrix} \subseteq \mathrm{GL}_{2n}(\mathcal{K})$.

The modulus character of P is $\delta_P(\cdot) = |\det \circ \Delta(\cdot)|^n$.

4.1.2. *Induced representations.* Let $\chi = \otimes \chi_w$ be a character of $\mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times$. For $s \in \mathbb{C}$ let

$$I(\chi, s) = \mathrm{Ind}_{P(\mathbb{A})}^{G(\mathbb{A})} \left(\chi(\det \circ \Delta(\cdot)) \cdot \delta_P^{-s/n}(\cdot) \cdot |\nu(\cdot)|^{-sn/2} \right),$$

with the induction smooth and unitarily normalized. This factors as a restricted tensor product

$$I(\chi,s) = \otimes_v I_v(\chi_v,s),$$

with v running over the places of $\mathbb{Q}$, $I_v(\chi_v,s)$ the analogous local induction from $P(\mathbb{Q}_v)$ to $G(\mathbb{Q}_v)$, and $\chi_v = \otimes_{w|v}\chi_w$.

4.1.3. *Eisenstein series.* For $f \in I(\chi,s)$ we form the *standard (non-normalized)* Eisenstein series,

$$E(f,g) = \sum_{\gamma\in P(\mathbb{Q})\backslash G(\mathbb{Q})} f(\gamma g).$$

If $Re(s)$ is sufficiently large, this converges absolutely and uniformily on compact subsets and defines an automorphic form on $G(\mathbb{A})$. Given a unitary character χ and a Siegel-Weil section $f \in \mathrm{Ind}(\chi,s)$, we put

$$f_s := f_{\chi,s} := f$$
$$E_f(s,g) := E_{f_s}(g).$$

The Eisenstein series $E_f(s,g)$ have a meromorphic continuation in s.

4.1.4. *Zeta integrals.* Denote by $\mathcal{O}_{\mathcal{K}^+}$ the ring of integers of $\mathcal{K}^+$. For $i = 1,2,3,4$, we write $U_i(\mathbb{A}) = \prod'_v U_{i,v}$, with the (restricted) products over all the places of $\mathcal{K}^+$ and $U_{i,v}$ the points of groups defined over $\mathcal{O}_{\mathcal{K}^+_v}$. Similarly, we write $G(\mathbb{A}) = G_\infty \times \prod'_q G_q$ and $P(\mathbb{A}) = P_\infty \times \prod'_q P_q$, where the (restricted) products are over rational primes q. We can nevertheless write

$$G_p = \mathbb{Q}_p^\times \times \prod_{w\in\Sigma_p} G_w; P_p = \mathbb{Q}_p^\times \times \prod_{w\in\Sigma_p} P_w.$$

Let π be an irreducible cuspidal automorphic representation of $G_1(\mathbb{A})$, and let $\pi^\vee$ be its contragredient. Let S_π be the set of finite primes v in $\mathcal{O}_{\mathcal{K}^+}$ for which π_v is ramified. Before introducing the zeta integral for π, we would like to explain what it means for a function in π to be factorizable over places in $\mathcal{K}^+$. However, G_1 is a $\mathbb{Q}$-group that is not the restriction of scalars of a group over $\mathcal{K}^+$. We therefore choose an irreducible $U_1(\mathbb{A})$-constituent $\underline{\pi} \subset \pi$ that occurs in the space of automorphic forms on U_1, the dual $\underline{\pi}^\flat$; note that $\pi^\vee$ and $\pi^\flat$ coincide upon restriction to $U_1(\mathbb{A})$. We assume $\underline{\pi}$ contains the spherical vectors for K^{S_π}. It is well-known (and follows from the unfolding computation recalled below) that *the standard L-function does not depend on this choice.* We fix non-zero unramified vectors $\varphi_{w,0}$ and $\varphi'_{w,0}$ in π_w and $\pi_w^\vee$, respectively, for all finite places w outside S_π, and choose factorizations as in (1.4.2) compatible with the unramified choices:

$$\underline{\pi} \xrightarrow{\sim} \underline{\pi}_\infty \otimes \underline{\pi}_f;\ \underline{\pi}_f \xrightarrow{\sim} \underline{\pi}^{S_\pi,p} \otimes \underline{\pi}_p \otimes \underline{\pi}_{S_\pi}; \underline{\pi}_p \xrightarrow{\sim} \otimes_{w|p} \underline{\pi}_w;\ \underline{\pi}_{S_\pi} \xrightarrow{\sim} \otimes_{w\in S_\pi} \underline{\pi}_w; \tag{4.1.1}$$

and analogous factorizations for $\underline{\pi}^\flat$. We also think of $\pi^\flat$ as an anti-holomorphic automorphic representation of G_2. Let $\varphi \in \underline{\pi}^{K^{S_\pi}}$, $\varphi^\flat \in \underline{\pi}^{\flat,K^S}$; we think of φ and $\varphi^\flat$ as forms on G_1 and G_2, respectively. We suppose they decompose as tensor products with respect to the above factorizations:

$$\varphi = \otimes_v \varphi_v;\ \varphi^\flat = \otimes_v \varphi_v^\flat \tag{4.1.2}$$

with φ_v and $\varphi_v^\flat$ equal to the chosen $\varphi_{v,0}$ and $\varphi'_{v,0}$ when $v \notin S_\pi$. we write equalities but the formulas we write below depend on the factorizations in (4.1.1) and its counterpart for $\underline{\pi}^\flat$.

In Sections 4.3 (resp. 4.4-4.5), we will choose specific local components at primes dividing p (resp. at archimedean places). These will turn out to be anti-ordinary (resp. anti-holomorphic) vectors:

$$\varphi_p := \otimes_{w|p}\ \varphi_w = \phi^{\text{a-ord}}_{w,d,\pi_w}; \varphi_p^\flat := \otimes_{w|p}\ \phi^{\text{a-ord}}_{w,d,\pi_w^\flat} = \varphi^{\flat,ord}_{\pi_p} \tag{4.1.3}$$

and

$$\varphi_\infty := \otimes_{\sigma|\infty}\ \varphi_\sigma = \varphi_{\kappa_\sigma,-}; \varphi_\infty^\flat := \otimes_{\sigma|\infty}\ \varphi_\sigma^\flat = \varphi_{\kappa_\sigma^\flat,-}. \tag{4.1.4}$$

The meaning of the notation in (4.1.3) and (4.1.4) will be explained in sections 8.2.5 and 4.4.14, respectively.

Having made the choice of irreducible constituent $\underline{\pi}$, we will henceforth forget about the choice. In order not to make the notation too difficult to read, we will use π to denote an irreducible $U_1(\mathbb{A})$ representation, but we will mean an irreducible constitutent of the restriction of a representation of G_1.

We also fix local $U_1(\mathcal{K}_v^+)$-invariant pairings $\langle\cdot,\cdot\rangle_{\pi_v} : \pi_v \times \pi_v^\vee \to \mathbb{C}$ for all v such that $\langle \varphi_{v,0}, \varphi_{v,0}^\flat\rangle_{\pi_v} = 1$ for all $v \notin S_\pi$.

Let $f = f_s(\bullet) \in I(\chi, s)$. Let $\varphi \in \pi$ and $\varphi^\flat \in \pi^\flat$ be factorizable vectors as above. The zeta integral for f,φ, and $\varphi^\flat$ is

$$I(\varphi,\varphi^\flat,f,s) = \int_{Z_3(\mathbb{A})G_3(\mathbb{Q})\backslash G_3(\mathbb{A})} E_f(s,(g_1,g_2))\chi^{-1}(\det g_2)\varphi(g_1)\varphi^\flat(g_2)d(g_1,g_2).$$

By the cuspidality of φ and φ' this converges absolutely for those values of s at which $E_f(s,g)$ is defined and defines a meromorphic function in s (holomorphic wherever $E_f(s,g)$ is). Moreover, it follows from the unfolding in [GPSR87] that $(\varphi,\varphi^\flat) \mapsto I(\varphi,\varphi^\flat,f,s)$ defines a $G_1(\mathbb{A})$-invariant pairing between π and π'. By the multiplicity one hypothesis 7.1.5, this implies that

Fact 4.1.5. *If $\langle\varphi,\varphi^\flat\rangle = 0$ then $I(\varphi,\varphi^\flat,f,s) = 0$ for all s.*

So we suppose $\langle\varphi,\varphi^\flat\rangle \neq 0$. Then $\langle\varphi_v \otimes \varphi_v^\flat\rangle_{\pi_v} \neq 0$ for all v. For $Re(s)$ sufficiently large, 'unfolding' the Eisenstein series then yields

$$I(\varphi,\varphi^\flat,f,s) = \int_{U_1(\mathbb{A})} f_s(u,1)\langle\pi(u)\varphi,\varphi^\flat\rangle_\pi du.$$

Denote by f_U the restriction of f to $U_4(\mathbb{A})$. Henceforward we assume $f_U(g) = \otimes_v f_v(g_v)$ with

$$f_v = f_{v,s} \in I_v(\chi_v,s), \chi_v = \otimes_{w|v}\chi_w.$$

Then the last expression for $I(\varphi, \varphi^\flat, s)$ factors as

$$(4.1.5)\qquad \begin{aligned} I(\varphi, \varphi^\flat, f, s) &= \prod_v I_v(\varphi_v, \varphi_v^\flat, f_v, s) \cdot \langle \varphi, \varphi^\flat \rangle, \text{ where} \\ I_v(\varphi_v, \varphi_v^\flat, f_v, s) &= \frac{\int_{U_{1,v}} f_{v,s}(u, 1) \langle \pi_v(u)\varphi_v, \varphi_v^\flat \rangle_{\pi_v} du}{\langle \varphi_v, \varphi_v^\flat \rangle_{\pi_v}}. \end{aligned}$$

By hypothesis, the denominator of the above fraction equals 1 whenever $v \notin S$.

As in Section 2.3, let $\Sigma = \{\sigma \in \Sigma_{\mathcal{K}} \ : \ \mathfrak{p}_\sigma \in \Sigma_p\}$. This is a CM type for $\mathcal{K}$. Throughout the remainder of this section, we take $\chi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ to be a unitary character such that $\chi_\infty = \otimes_{\sigma \in \Sigma} \chi_\sigma$ is given by

$$(4.1.6)\qquad \chi_\infty\left((z_\sigma)\right) = \prod_{\sigma \in \Sigma} z_\sigma^{-(k_\sigma + 2\nu_\sigma)} \left(z_\sigma \overline{z_\sigma}\right)^{\frac{k_\sigma}{2} + \nu_\sigma}, \quad (z_\sigma) \in \prod_{\sigma \in \Sigma} \mathbb{C}^\times,$$

where $k = (k_\sigma) \in \mathbb{Z}_{\geq 0}^{\Sigma}$, and $(\nu_\sigma) \in \mathbb{Z}^{\Sigma}$.

For the remainder of this section, we choose specific local Siegel-Weil sections $f_v \in I_v(\chi_v, s)$ and compute the corresponding local zeta integrals (whose product is the Euler product of the global zeta function discussed at the beginning of this section).

4.2. **Local zeta integral calculations at nonarchimedean places $v \nmid p$.** Let $S_{\mathrm{ram}} = S_\pi \cup S_\chi \cup S_{\mathcal{K}}$, where S_χ denotes the set of finite primes v in $\mathcal{O}_{\mathcal{K}^+}$ for which $\chi_v = \otimes_{w|v} \chi_w$ is ramified and $S_{\mathcal{K}}$ denotes the set of finite primes in $\mathcal{O}_{\mathcal{K}^+}$ that ramify in $\mathcal{K}$. Let S be a finite set of finite primes in $\mathbb{Q}$ such that $p \notin S$ and such that for all rational primes ℓ, if a prime in $\mathcal{K}^+$ above ℓ is in S_{ram}, then $\ell \in S$. Let S' be the set of primes of $\mathcal{K}^+$ lying above the primes of S.

4.2.1. *Unramified case.* For the moment, assume that $\ell \neq p$ is a finite place of $\mathbb{Q}$ such that $\ell \notin S$. Then $K_\ell := G_4(\mathbb{Z}_\ell)$ is a hyperspecial maximal compact of $G(\mathbb{Q}_\ell) = G_4(\mathbb{Q}_\ell) = \prod_{v|\ell} G_{4,v}$, and we choose $f_\ell = \otimes_{v|\ell} f_v \in I_\ell(\chi_\ell, s)$ to be the unique K_ℓ-invariant function such that $f_\ell(K_\ell) = 1$. These sections are used to construct the Eisenstein measure in [Eis15]. For each prime $v \notin S'$, let φ_v and φ_v' be the normalized spherical vectors such that $\langle \varphi_v, \varphi_v' \rangle_{\pi_v} = 1$. The primes $v \notin S'$ fall into two categories: split and inert. For split places $v \notin S'$, $U_{1,v} \cong \mathrm{GL}_n(\mathcal{K}_v^+)$; the zeta integral computations in this case reduce to those in [Jac79] and [GPSR87, Section 6]. For inert places $v \notin S'$, the computations were completed in [Li92, Section 3]. In either case, we have

$$d_{n,v}(s, \chi_v) I_v\left(\varphi_v, \varphi_v', f_v, s\right) = L_v\left(s + \frac{1}{2}, \pi_v, \chi_v\right),$$

where[8]

$$d_{n,v}(s,\chi_v) = d_{n,v}(s) = \prod_{r=0}^{n-1} L_v\left(2s+n-r, \chi_v\,|_{\mathcal{K}^+}\,\eta_v^r\right),$$

η_v is the character on $\mathcal{K}_v^+$ attached by local class field theory to the extension $\mathcal{K}_w/\mathcal{K}_v^+$ (where w is a prime of $\mathcal{K}$ lying over v), and $L_v(s,\pi_v,\chi_v)$ denotes the value at s of the standard local Langlands Euler factor attached to the unramified representation π_v of $U_{1,v}$, the unramified character χ_v of $\mathcal{K}_v$, and the standard representation of the L-group of $U_{1,v}$. As noted on [HLS06, p. 45][9], for each $v \notin S'$,

$$L_v(s,\pi_v,\chi_v) = L_v(s, \mathrm{BC}(\pi_v)\otimes\chi_v\circ\det),$$

where BC denotes the local base change from $U_{1,v}$ to $\mathrm{GL}_n(\mathcal{K}_v)$ and the right hand side is the standard Godement-Jacquet Euler factor.

4.2.2. *Ramified case.* Now, assume that $\ell \in S$, and let $v \in S'$ be a prime lying over ℓ. By [HLS06, p. 45], $P_v\cdot(U_{1,v}\times 1_n) \subseteq P_v\cdot U_{3,1}$ is open in $U_{4,v}$. Since the big cell $P_v w P_v$ is also open in $U_{4,v}$, we see that $(P_v\cdot(U_{1,v}\times 1_n))\cap P_v w P_v$ is open in $U_{4,v}$. As noted in [HLS06, Equation (3.2.1.5)], $P_v w = P_v\cdot(-1_n,1_n) \subseteq P_v\cdot U_{3,v}$ and $P_v\cap(U_{1,v}\times 1_n) = (1_n,1_n)\in U_{3,v}$. Therefore $(P_v\cdot(U(V)\times 1_n))\cap P_v w P_v$ is an open neighborhood of w in $P_v w P_v$ and hence is of the form $P_v w\mathfrak{U}$ for some open subset $\mathfrak{U}$ of the unipotent radical N_v of P_v. Let $\varphi_v\in\pi_v$ and $\varphi'_w\in\pi'_v$ be such that $\langle\varphi_v,\varphi'_v\rangle_{\pi_v} = 1$. Let K_v be an open compact subgroup of $G_{1,v}$ that fixes φ_v.

For each place $v\in S'$, let L_v be a small enough lattice so that $\mathfrak{U}_v$ contains the open subgroup $N(L_v)$ of N_v defined by

$$N(L_v) = \left\{\begin{pmatrix}1_n & x\\ 0 & 1_n\end{pmatrix} \mid x\in L_v\right\}$$

(where we identify N with $\Delta'(N)$ as in Section 4.1.1) and so that

$$P_v w N(L_v) \subseteq P_v\cdot(-1_n\cdot K_v\times 1_n) \subseteq P_v\cdot U_{3,v}.$$

Then

$$P_v w N(L_v) = P_v\cdot(\mathcal{U}_v\times 1_n)$$

for some open neighborhood neighborhood $\mathcal{U}_v$ of -1_n contained in the open subset $-1_n\cdot K_v$ of $U_{1,v}$. Let δ_{L_v} denote the characteristic function of $N(L_v)$. As explained on [HLS06, p. 55], for each finite place v of $\mathcal{K}^+$, there is a Siegel section f_{L_v} supported on $P_v w P_v$ such that

$$f_{L_v}(wx) = \delta_{L_v}(x)$$

[8]From the formula for $d_{n,v}(s)$ given in [Li92, Section 6], it appears that there is a typographical error in the exponent in the formula for $d_{n,v}$ given in [HLS06, Equation (3.1.2.5)]. More precisely, according to the final formula in [Li92, Theorem 3.1], the $n-1$ should not appear in the exponent in [HLS06, Equation (3.1.2.5)].

[9]There is a typographical error on [HLS06, p. 45]. Although [HLS06, p. 45] gives a base change to GL_m, the base change should actually be to GL_n.

for all $x \in N_v$.

For each of the primes $v \in S'$, we define a local Siegel section $f_v \in I(\chi_v, s)$ by

$$f_v = f_{L_v}^-,$$

where

$$f_{L_v}^-(g) = f_{L_v}\left(g \cdot (-1,1)\right).$$

for all $g \in U_{4,v}$. (Note that $f_{L_v}^-$ is just a translation by $(-1,1) \in U_{3,v} = U_{1,v} \times U_{2,v} = U_{1,v} \times U_{1,v}$ of local Siegel sections discussed in [HLS06, Sections (3.3.1)-(3.3.2)] and that, where nonzero, the Fourier coefficients associated to $f_{L_v}^-$ are the same as the Fourier coefficients associated to similar Siegel sections discussed in [Eis15, Section 2.2.9] and [Shi97]. Therefore, this minor modification of the choice of Siegel sections in [HLS06, Eis15, Shi97] will not affect the p-adic interpolation of the q-expansion coefficients of the Eisenstein series that is necessary to construct an Eisenstein measure.)

Lemma 4.2.3. *Let $v \in S'$, and let $f_v = f_{L_v}^-$. Then*

$$I_v\left(\varphi_v, \varphi_v^\flat, f_v, \chi\right) = \text{volume}\left(\mathcal{U}_v\right).$$

Proof. The support of $f_{L_v}^-$ in $U_{1,v} \times 1_n$ is $-1_n \cdot \mathcal{U}_v \times 1_n$, and for $g \in U_{1,v} \times 1_n$,

$$f_{L_v}^-(g) = \delta_{-1_n \cdot \mathcal{U}_v \times 1_n}(g)$$

where $\delta_{-1_n \cdot \mathcal{U}_v \times 1_n}$ denotes the characteristic function of $-1_n \cdot \mathcal{U}_v \times 1_n$. Since $\pi_v(g)\varphi_v = \varphi_v$ for all $g \in K_v \supseteq -1_n \cdot \mathcal{U}_v$, we therefore see that

$$\begin{aligned} I_v\left(\varphi_v, \varphi_v^\flat, f_v, \chi\right) &= \frac{\int_{-1_n \cdot \mathcal{U}_v} \langle \varphi_v, \varphi_v^\flat \rangle_{\pi_v} dg}{\langle \varphi_v, \varphi_v^\flat \rangle_{\pi_v}} \\ &= \text{volume}\left(\mathcal{U}_v\right). \end{aligned}$$

□

4.3. **Local zeta integral calculations at places dividing p.**

Plan of this section. We begin by choosing local Siegel-Weil sections at the primes w dividing p that are compatible with the Eisenstein measure, and then turn to choosing test vectors (anti-ordinary vectors) in the local representations π_w and $\pi_w^\flat$. The last six pages or so contain explicit matrix calculations that reduce the zeta integral to a product of integrals of Godement-Jacquet type, which can then be computed explicitly.

The reader may observe that the representations π_w and $\pi_w^\flat$, like the automorphic representations of which they are local components, are logically prior to the local Siegel-Weil sections, inasmuch as our goal is to define p-adic L-functions of (ordinary) families and the Eisenstein measure is a means to this end. One of the subtleties of this construction is that a global automorphic representation π automatically picks out the function whose integral is the desired value of the Eisenstein measure. This is unfortunately concealed in the technical details of the construction, but the reader should be able to spot the principle at work in the section 7.3.

The calculations presented here are more general than those needed for our construction of the p-adic L-functions of ordinary families. The p-adic place w is assigned to an archimedean place σ and thus to a signature (a_w, b_w) of the unitary group at σ; but we also introduce partitions of a_w and b_w. These partitions can be used to study the variation of p-adic L-functions in P-*ordinary families*, where P is a parabolic subgroup of $G_1(\mathbb{Q}_p)$. However, this application has been postponed in order not to make the paper any longer than it already is, and we restrict our attention to the usual ordinary families, corresponding to $P = B$ a Borel subgroup.

4.3.1. *Definition of the Siegel-Weil sections.* With a few minor changes, the description of the Siegel-Weil section at p given below is the same as in [Eis15, Eis14]. For $w|p$ a place of $\mathcal{K}$ and U a $\mathcal{K}$-space we let $U_w = U \otimes_{\mathcal{K}} \mathcal{K}_w$.

To describe the section f_p we make use of the isomorphisms 2.2.2. The isomorphism for G_4 identifies $G(\mathbb{Q}_p)$ with $\mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} \mathrm{GL}_{\mathcal{K}_w}(W_w)$ and $P(\mathbb{Q}_p)$ with $\mathbb{Q}_p^\times \times \prod_{w\in\Sigma_p} P_n(\mathcal{K}_w)$ with $P_n \subset \mathrm{GL}_{\mathcal{K}}(W)$ the parabolic stabilizing V^d. So $M(\mathbb{Q}_p)$ is identified with $\mathbb{Q}_p^\times \times \prod_{w\in\Sigma_p} \mathrm{GL}_{\mathcal{K}_w}(V_{d,w}) \times \mathrm{GL}_{\mathcal{K}_w}(V_w^d)$ (the factors embedded diagonally in $\mathrm{GL}_{\mathcal{K}_w}(W_w)$), and $N(\mathbb{Q}_p)$ is identified with $\prod_{v\in\Sigma_p} N_n(\mathcal{K}_w)$ with $N_n \subset P_n$ the unipotent radical.

For $w \in \Sigma_p$ let $\chi_{w,1} = \chi_w$ and $\chi_{w,2} = \chi_{\bar{w}}^{-1}$, where we identify $\mathcal{K}_w = \mathcal{K}^+_{w^+} = \mathcal{K}_{\bar{w}}$ and where $w^+ = w|_{\mathcal{K}^+} = \bar{w}|_{\mathcal{K}^+}$. The pair $(\chi_{w,1}, \chi_{w,2})$ determines a character

$$\psi_w : P_n(\mathcal{K}_w) \to \mathbb{C}^\times, \quad \psi_w\left(\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right)\right) = \chi_{w,1}(\det D)\,\chi_{w,2}(\det A).$$

Here we have written an element of P_n with respect to the direct sum decomposition $W = V_d \oplus V^d$. We put

$$\psi_{w,s} = \left(\left(\begin{smallmatrix} A & B \\ 0 & D\end{smallmatrix}\right)\right) = \chi_{w,1}(\det D)\,\chi_{w,2}(\det A)\left|A^{-1}D\right|_w^{-s}$$

Given $\otimes_{w\in\Sigma_p} f_{w,s} \in \otimes_{w\in\Sigma_p} \mathrm{Ind}_{P_n(\mathcal{K}_w)}^{\mathrm{GL}_{\mathcal{K}_w}(W_w)}(\psi_{w,s})$, we set

$$f_{p,s}(g) = |\nu|_p^{-sn/2} \otimes_{w\in\Sigma_p} f_{w,s}(g_w), \quad g = (\nu, (g_w)) \in G(\mathbb{Q}_p). \tag{4.3.1}$$

Then, as explained in [Eis15], $f_p \in I_p(\chi_p, s)$.

The choice of a level structure at p for the PEL problem P_1 amounts to choosing an $\mathcal{O}_w$-basis of $L_{1,w}$, and hence a $\mathcal{K}_w$-basis of V_w, for each $w \in \Sigma_p$. This then determines a $\mathcal{K}_w$-basis of V_w^d and $V_{d,w}$, via their identifications with V_w, and hence a $\mathcal{K}_w$-basis[10] of $W_w = V_{d,w} \oplus V_w^d$. This basis identifies $\mathrm{Isom}_{\mathcal{K}_w}(V_w^d, V_w)$, $\mathrm{Isom}_{\mathcal{K}_w}(V_{d,w}, V_w)$, and an ordered choice of this basis identifies $\mathrm{GL}_{\mathcal{K}_w}(V_w)$ with $\mathrm{GL}_n(\mathcal{K}_w)$. This ordered basis also identifies $\mathrm{GL}_{\mathcal{K}_w}(W_w)$ with $\mathrm{GL}_{2n}(\mathcal{K}_w)$, $P_n(\mathcal{K}_w)$ with the subgroup of upper-triangular $n \times n$-block matrices and $M_n(\mathcal{K}_w)$ with the subgroup of diagonal $n\times n$-block matrices.

[10] This is not in general the basis corresponding to the the level structure for P_4 determined by that for P_1.

Let $w \in \Sigma_p$. To each Schwartz function $\Phi_w : \mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) \to \mathbb{C}$ (so Φ_w has compact support), we attach a Siegel-Weil section $f^{\Phi_w} \in \mathrm{Ind}_{P_n(\mathcal{K}_w)}^{\mathrm{GL}_{2n}(\mathcal{K}_w)} \psi_{w,s}$ as follows. Consider the decomposition

$$\mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) = \mathrm{Hom}_{\mathcal{K}_w}(V_w, V_{d,w}) \oplus \mathrm{Hom}_{\mathcal{K}_w}(V_w, V_w^d), \quad X = (X_1, X_2).$$

Let

$$\mathbf{X} = \left\{X \in \mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) | X(V_w) = V_w^d\right\} = \left\{(0, X) | X : V_w \xrightarrow{\sim} V_w^d\right\}.$$

For $X \in \mathbf{X}$, the composition $V_w \xrightarrow{X} V_w^d \xrightarrow{\sim} V_w$, where the last arrow comes from the fixed identification of V^d with V, is an isomorphism of V_w with itself. This identifies $\mathbf{X}$ with $\mathrm{GL}_{\mathcal{K}_w}(V_w)$.

We define the section $f^{\Phi_w} \in \mathrm{Ind}_{P_n(\mathcal{K}_w)}^{\mathrm{GL}_{2n}(\mathcal{K}_w)} \psi_{w,s}$ by[11]

$$f^{\Phi_w}(g) := \chi_{2,w}(\det g) |\det g|_w^{\frac{n}{2}+s} \int_{\mathbf{X}} \Phi(Xg) \chi_{1,w}^{-1} \chi_{2,w}(\det X) |\det X|_w^{n+2s} d^\times X. \tag{4.3.2}$$

Linear operations are viewed here as acting on the vector space W_w on the right. We recall that $\mathbf{X}$ is identified with $\mathrm{GL}_n(\mathcal{K}_w)$; $d^\times X$ is the measure identified with the right Haar measure on the latter. To define the Siegel sections $f_{w,s}$, we make specific choices of the Schwartz functions Φ_w.

Let (a_w, b_w) be the signature associated to $w|p$ and $L_1, \langle \cdot, \cdot \rangle_1$. For each $w \in \Sigma_p$, fix partitions

$$a_w = n_{1,w} + \cdots + n_{t(w),w} \quad \text{and} \quad b_w = n_{t(w)+1,w} + \cdots + n_{r(w),w}.$$

Let $\mu_{1,w}, \ldots, \mu_{r(w),w}$ be characters of $\mathcal{O}_w^\times$, and let $\mu_w = (\mu_{1,w}, \ldots, \mu_{r(w),w})$ and $\mu = \prod_{w \in \Sigma_p} \mu_w$. We view each character $\mu_{i,w}$ as a character of $\mathrm{GL}_{n_{i,w}}(\mathcal{O}_w)$ via composition with the determinant. Let

$$\nu_{i,w} = \chi_{1,w}^{-1} \chi_{2,w} \mu_{i,w}, \quad i = 1, ..., r(w),$$

and let $\nu_w = \left(\nu_{1,w}, \ldots, \nu_{r(w),w}\right)$.

Let $\mathfrak{X}_w \subset M_n(\mathcal{O}_w)$ comprise the matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, with $A \in M_{a_w}(\mathcal{O}_w)$ and $D \in M_{b_w}(\mathcal{O}_w)$, such that the determinant of the leading principal $n_{1,w} + \cdots + n_{i,w}$-th minor of A is in $\mathcal{O}_w^\times$ for $i = 1, \ldots, t(w)$ and the determinant of the leading principal $n_{t(w)+1,w} + \cdots + n_{i,w}$-th minor of D is in $\mathcal{O}_w^\times$ for $i = t(w) + 1, \ldots, r(w)$. Let A_i be the determinant of the leading principal i-th minor of A and D_i the determinant of the leading principal i-th minor of D. Define $\phi_{\nu_w} : M_n(\mathcal{K}_w) \to \mathbb{C}$ to be the function supported on $\mathfrak{X}$ and defined for

[11] The minor difference between the definitions of the Siegel section at p in Equation (4.3.2) in this paper and in [Eis15, Equation (21)] is due to the fact that we use normalized induction in the present paper, while we did not use normalized induction in [Eis15].

$X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{X}$ by

$$\phi_{\nu_v}(X) = \nu_{t(w),w}(A) \cdot \prod_{i=1}^{t(w)-1} (\nu_{i,w} \cdot \nu_{i+1,w}^{-1})(A_{n_{1,w}+\cdots+n_{i,w}})$$
$$\nu_{r(w),w}(D) \cdot \times \prod_{i=t(w)+1}^{r(w)-1} (\nu_{i,w} \cdot \nu_{i+1,w}^{-1})(D_{n_{t(w)+1,w}+\cdots+n_{i,w}}).$$

Let

$$t \geq \max_{w\in\Sigma_p, 1\leq i\leq r(w)} (1, \mathrm{ord}_w(\mathrm{cond}(\mu_{i,w})), \mathrm{ord}_w(\mathrm{cond}(\chi_w)))), \tag{4.3.3}$$

and let $\Gamma_w = \Gamma_w(t) \subset \mathrm{GL}_n(\mathcal{O}_w)$ be the subgroup of $\mathrm{GL}_n(O_w)$ consisting of matrices whose terms below the $n_{i,w} \times n_{i,w}$-blocks along the diagonal are in $\mathfrak{p}_w^t$ and such that the upper right $a_w \times b_w$ block is also in $\mathfrak{p}_w$. For each matrix $m \in \Gamma_w$ with $n_{i,w} \times n_{i,w}$-blocks m_i running down the diagonal, we define

$$\mu_w(m) = \prod_i \mu_{i,w}(\det(m_i)).$$

Let $\Phi_{1,w}$ be the function on $M_{n\times n}(\mathcal{K}_w)$ supported on $\Gamma_w(t)$ and such that $\Phi_{1,w}(x) = \mu_w(x)$ for all $x \in \Gamma_w(t)$. Let $\Phi_{2,w}$ be the function on $M_{n\times n}(\mathcal{K}_w)$ defined by

$$\Phi_{2,w}(x) = \hat{\phi}_{\nu_w}(x) = \int_{M_{n\times n}(\mathcal{K}_w)} \phi_{\nu_w}(y) e_w(-\mathrm{trace}\, y^t x) dy. \tag{4.3.4}$$

Note that $\hat{\phi}_{\nu_w}$ is the Fourier transform of ϕ_{ν_w}, as discussed in [Eis15, Lemma 10].

For $X = (X_1, X_2) \in \mathrm{Hom}_{\mathcal{K}_w}(V_w, W_w) = \mathrm{Hom}_{\mathcal{K}_w}(V_w, V_{d,w}) \oplus \mathrm{Hom}_{\mathcal{K}_w}\left(V_w, V_w^d\right)$, let

$$\Phi_w(X) = \Phi_{\chi,\mu,w}(X_1, X_2) = \mathrm{vol}(\Gamma_w)^{-1} \Phi_{1,w}(-X_1) \cdot \Phi_{2,w}(2X_2).$$

Recall that we have identified X_1 and X_2 with matrices through a choice of basis for V_w (coming from the level structure at p for P_1). Note that $\Phi_{\chi,\mu,w}$ is a partial Fourier transform in the second variable in the sense of [Eis15, Lemma 10]. We then define

$$f_{w,s} := f_w^{\chi,\mu} := f^{\Phi_w} = f^{\Phi_{\chi,\mu,w}}. \tag{4.3.5}$$

We then define $f_{p,s} \in I_p(\chi_p, s)$ by (4.3.1).

The following lemma describes the support of $\Phi_{1,w}$ and $\Phi_{2,w}$.

Lemma 4.3.2.

(i) *For* $\gamma_1, \gamma_2 \in \Gamma_w$,

$$\phi_{\nu_w}({}^t\gamma_1 X \gamma_2) = \mu_w(\gamma_1\gamma_2)\chi_{1,w}^{-1}\chi_{2,w}(\det \gamma_1\gamma_2)\phi_{\nu_w}(X).$$

(ii) *For* $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *with* $A \in M_{a_w\times a_w}(\mathcal{K}_w)$, $B \in M_{a_w\times b_W}(\mathcal{K}_w)$, $C \in M_{b_w\times a_w}(\mathcal{K}_w)$, *and* $D \in M_{b_w\times b_w}(\mathcal{K}_w)$,

$$\Phi_{2,w}(X) = \Phi_w^{(1)}(A)\Phi_w^{(2)}(B)\Phi_w^{(3)}(C)\Phi_w^{(4)}(D),$$

with

$$\Phi_w^{(2)} = \mathrm{char}_{M_{a_w\times b_w}(\mathcal{O}_w)}, \quad \Phi_w^{(3)} = \mathrm{char}_{M_{b_w\times a_w}(\mathcal{O}_w)}$$
$$\mathrm{supp}(\Phi_w^{(1)}) \subseteq \mathfrak{p}_w^{-t} M_{a_w\times a_w}(\mathcal{O}_w), \ \ \mathrm{supp}(\Phi_w^{(4)}) \subseteq \mathfrak{p}_w^{-t} M_{b_w\times b_w}(\mathcal{O}_w).$$

Here t is as in Inequality (4.3.3).

Proof. Part (i) follows immediately from the definition of ϕ_{ν_w}. It remains to prove part (ii). We have

$$\begin{aligned}\Phi_{2,w}(X) &= \int_{M_n(\mathcal{K}_w)} \phi_{\nu_w}(Y) e_w\left(-\mathrm{trace}\, Y \left(\begin{smallmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD\end{smallmatrix}\right)\right) dY \\ &= \int_{\mathfrak{X}} \phi_{\nu_w}(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)) e_w\left(-\mathrm{trace}\left(\alpha^t A + \beta^t B + \gamma^t C + \delta^t D\right)\right) d\alpha d\beta d\gamma d\delta \\ &= \Phi_w^{(1)}(A)\, \Phi_w^{(2)}(B)\, \Phi_w^{(3)}(C)\, \Phi_w^{(4)}(D),\end{aligned}$$

where

$$\Phi_w^{(2)}(B) = \int_{M_{a_w\times b_w}(\mathcal{O}_w)} e_w(-\mathrm{trace}\, \beta^t B) d\beta = \mathrm{char}_{M_{a_w\times b_w}(\mathcal{O}_w)}(B),$$

$$\Phi_w^{(3)}(C) = \int_{M_{b_w\times a_w}(\mathcal{O}_w)} e_w(-\mathrm{trace}\, \gamma^t C) d\gamma = \mathrm{char}_{M_{a_w\times b_w}(\mathcal{O}_w)}(C),$$

$$\Phi_w^{(1)}(A) = \sum_{x = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right) \in \mathfrak{X} \bmod \mathfrak{p}_w^t} \phi_{\nu_w}(x) e_w(-\mathrm{trace}\, \alpha^t A) \mathrm{char}_{\mathfrak{p}_w^{-t} M_{a_w\times a_w}(\mathcal{O}_w)}(A),$$

and

$$\Phi_w^{(4)}(D) = \sum_{x = \left(\begin{smallmatrix} 1 & 0 \\ 0 & \delta\end{smallmatrix}\right) \in \mathfrak{X} \bmod \mathfrak{p}_w^t} \phi_{\nu_w}(x) e_w(-\mathrm{trace}\, \delta^t D) \mathrm{char}_{\mathfrak{p}_w^{-t} M_{b_w\times b_w}(\mathcal{O}_w)}(D).$$

□

4.3.3. *Local induced representations.* Now that we have chosen a Siegel section at each prime $w \in \Sigma_p$, we proceed to the zeta integral calculations at w.

First, we introduce additional notation. Let $B_{a_w} \subseteq \mathrm{GL}_{a_w}$ be the standard parabolic subgroup associated to $a_w = n_{1,w} + \cdots + n_{t(w),w}$. Let $B_{b_w} \subseteq \mathrm{GL}_{b_w}$ be the standard parabolic subgroup associated to $b_w = n_{t(w)+1,w} + \cdots + n_{r(w),w}$. Let $R_{a_w,b_w} \subseteq \mathrm{GL}_n$ be the standard parabolic subgroup associated to $n = a_w + b_w$. For $a = (a_w)_w$ and $b = (b_w)_w$, let $B_a = \prod_{w\in\Sigma_p} B_{a_w}(\mathcal{K}_w)$, $B_b = \prod_{w\in\Sigma_p} B_{b_w}(\mathcal{K}_w)$, and $R_{a,b} = \prod_{w\in\Sigma_p} R_{a_w,b_w}(\mathcal{K}_w)$. Let L_{a_w,b_w} denote the Levi subgroup of R_{a_w,b_w}. Let $R = \prod_{w\in\Sigma_p} R_w(\mathcal{K}_w)$, where

$$R_w = \left\{ g = \begin{pmatrix} A & B \\ 0 & D\end{pmatrix} \in R_{a_w,b_w} \mid A \in B_{a_w}, D \in B_{b_w}^{\mathrm{op}} \right\}.$$

We define $N^{\mathrm{op}}_{a_w,b_w}(\mathfrak{p}_w\mathcal{O}_w) \subseteq R^{\mathrm{op}}_{a_w,b_w}$ to be the subgroup consisting of unipotent matrices in R_{a_w,b_w} such that the lower left $b_w \times a_w$-block lies in $M_{b_w\times a_w}(\mathfrak{p}_w\mathcal{O}_w)$.

In terms of the characters $\mu_{i,w}$ used to define the Siegel section at w, we define characters $\mu'_{i,w}$ by

$$\mu'_{i,w} = \begin{cases} \chi_{2,w}^{-1}\mu_{i,w}^{-1}, & \text{if } 1 \le i \le t(w) \\ \chi_{1,w}^{-1}\mu_{i,w}, & \text{if } t(w)+1 \le i \le r(w), \end{cases}$$

for all $w \in \Sigma_p$. Let

$$\begin{aligned} \mu'_w &= \otimes_{i=1}^{r(w)} \mu'_{i,w} \\ \mu' &= \otimes_{w\in\Sigma_p} \mu'_w. \end{aligned}$$

For all $w \in \Sigma_p$, let

$$\pi_w = \mathrm{Ind}_{R_{a_w,b_w}}^{\mathrm{GL}_n} \pi_{a_w} \otimes \pi_{b_w},$$

where π_{a_w} and π_{b_w} are representations of $\mathrm{GL}_{a_w}(\mathcal{K}_w)$ and $\mathrm{GL}_{b_w}(\mathcal{K}_w)$, respectively. Let $\mu'_{a_w} = \otimes_{i=1}^{t(w)} \mu'_{i,w}$ and $\mu'_{b_w} = \otimes_{i=t(w)+1}^{r(w)} \mu'_{i,w}$. We let

$$\begin{aligned} \pi_p &= \otimes_{w\in\Sigma_p} \pi_w \\ \pi_a &= \otimes_{w\in\Sigma_p} \pi_{a_w} \\ \pi_b &= \otimes_{w\in\Sigma_p} \pi_{b_w} \\ \mu'_a &= \otimes_{w\in\Sigma_p} \mu'_{a_w} \\ \mu'_b &= \otimes_{w\in\Sigma_p} \mu'_{b_w}. \end{aligned}$$

Let $\tilde{\mu}' = \otimes_{w\in\Sigma_p} \tilde{\mu}'_v$ denote the contragredient representation of μ', and define $\tilde{\pi}$, $\tilde{\pi}_a$, $\tilde{\pi}_b$, $\tilde{\mu}'_a$, and $\tilde{\mu}'_b$ by replacing μ' by $\tilde{\mu}'$. Note that in this paragraph, by Ind, we mean normalized induction.

4.3.4. *Local congruence subgroups and anti-ordinary test vectors.* Let t be as in Inequality (4.3.3), and let

$$d \ge 2t. \tag{4.3.6}$$

Consider the following groups:

$$\begin{aligned} \Gamma_{R,w} &= \left\{\gamma \in \mathrm{GL}_n(\mathcal{O}_w) \mid \gamma \bmod \mathfrak{p}_w^d \in R_w\left(\mathcal{O}/\mathfrak{p}_w^d\mathcal{O}\right)\right\} \\ \Gamma_R &= \prod_{w\in\Sigma_p} \Gamma_{R,w} \\ \Gamma_{a_w,w} &= \left\{\gamma \in \mathrm{GL}_{a_w}(\mathcal{O}_w) \mid \gamma \bmod \mathfrak{p}_w^d \in B_{a_w}\left(\mathcal{O}/\mathfrak{p}_w^d\mathcal{O}\right)\right\} \\ \Gamma_a &= \prod_{w\in\Sigma_p} \Gamma_{a_w,w} \\ \Gamma_{b_w,w} &= \left\{\gamma \in \mathrm{GL}_{b_w}(\mathcal{O}_w) \mid \gamma \bmod \mathfrak{p}_w^d \in B_{b_w}\left(\mathcal{O}/\mathfrak{p}_w^d\mathcal{O}\right)\right\} \\ \Gamma_b &= \prod_{w\in\Sigma_p} \Gamma_{b_w,w} \end{aligned}$$

We define $\varphi \in \pi$ as follows. Let $\varphi_a = \otimes_{w\in\Sigma_p}\varphi_{a_w} \in \pi_a$ be a section that satisfies

$$\pi_{a_w}(\gamma)\varphi_{a_w} = \mu'_{a_w}(\gamma)\varphi_{a_w}$$

for all $\gamma \in \Gamma_{a_w,w}$ and $w \in \Sigma_p$. Similarly, let $\varphi_b = \otimes_{w\in\Sigma_p}\varphi_{b_w} \in \pi_b$ be a section that satisfies

$$\pi_{b_w}(\gamma)\varphi_{b_w} = \mu'_{b_w}(\gamma)\varphi_{b_w} \tag{4.3.7}$$

for all $\gamma \in {}^t\Gamma_{b_w,w}$ and $w \in \Sigma_p$.

Definition 4.3.5. *(a) Viewed as an element of* $\otimes_{w\in\Sigma_p}\mathrm{Ind}_{R_{a_w,b_w}}^{\mathrm{GL}_n}\pi_{a_w}\otimes\pi_{b_w}$, *let* $\varphi = \otimes_{w\in\Sigma_p}\varphi_w \in \pi$ *be a section such that*

- φ_w *is supported on* $R_{a_w,b_w}\Gamma_{R,w}$*;*
- φ_w *is normalized by the property*
$$\varphi_w(1) = \varphi_{a_w}\otimes\varphi_{b_w},$$
for all $w \in \Sigma_p$ *and satisfies the invariance condition*
-
$$\varphi\left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right) = \varphi_a\otimes\varphi_b$$
for all $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix} \in \Gamma_R$.

(b) Similarly, we define $\varphi' \in \tilde{\pi}$ *as follows. Let* $\tilde{\varphi}_a = \otimes_{w\in\Sigma_p}\tilde{\varphi}_{a_w} \in \tilde{\pi}_a$ *be a section that satisfies*

$$\tilde{\pi}_{a_w}(\gamma)\tilde{\varphi}_{a_w} = \tilde{\mu}'_{a_w}(\gamma)\tilde{\varphi}_{a_w}$$

for all $\gamma \in {}^t\Gamma_{a_w,w}$ *and* $w \in \Sigma_p$*. Similarly, let* $\tilde{\varphi}_b = \otimes_{w\in\Sigma_p}\tilde{\varphi}_{b_w} \in \tilde{\pi}_b$ *be a section that satisfies*

$$\tilde{\pi}_{b_w}(\gamma)\tilde{\varphi}_{b_w} = \tilde{\mu}'_{b_w}(\gamma)\tilde{\varphi}_{b_w} \tag{4.3.8}$$

for all $\gamma \in \Gamma_{b_w,w}$ *and* $w \in \Sigma_p$*. Viewed as an element of* $\otimes_{w\in\Sigma_p}\mathrm{Ind}_{R_{a_w,b_w}}^{\mathrm{GL}_n}\tilde{\pi}_{a_w}\otimes\tilde{\pi}_{b_w}$, *let* $\varphi' = \otimes_{w\in\Sigma_p}\varphi'_w \in \pi$ *be a section such that*

- φ'_w *is supported on* $R_{a,b}{}^t\Gamma_R$
- φ'_w *is normalized by the property*
$$\varphi'_w(1) = \tilde{\varphi}_{a_w}\otimes\tilde{\varphi}_{b_w},$$
for all $w \in \Sigma_p$ *and satisfies the invariance condition*
-
$$\varphi'\left(\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}\right) = \tilde{\varphi}_a\otimes\tilde{\varphi}_b \tag{4.3.9}$$
for all $\begin{pmatrix}1 & 0\\ x & 1\end{pmatrix} \in {}^t\Gamma_R$.

We shall use the invariance conditions in Definition 4.3.5 in our computations of the local zeta integrals later in this section.

Remark 4.3.6. The functions denoted φ_w and φ'_w above depend on the integer d introduced in (4.3.6) (which determines the level of the subgroups Γ_R) as well as the representations π_w and $\tilde{\pi}_w$ (in subsequent sections the latter will be denoted $\pi_w^\flat$). They are uniquely determined by the support, normalizaion, and invariance conditions of Definition 4.3.5. For reasons that will be explained in detail in Section 8.2, and specifically in Lemma 8.2.7, in the global applications the vectors will be denoted $\phi^{\text{a-ord}}_{w,d,\pi_w}$ and $\phi^{\text{a-ord}}_{w,d,\pi_w^\flat}$, respectively.

4.3.7. *The main calculation.* The ordered $\mathcal{K}_w$-basis for V_w chosen above (that comes from the choice of a level structure for P_1) determines a $\mathcal{K}_w$-basis for $W_w = V_w \oplus V_w$. This ordered basis for $W_w = V_w \oplus V_w$ identifies $\mathrm{GL}_{\mathcal{K}_w}(W_w)$ with $\mathrm{GL}_{2n}(\mathcal{K}_w)$ and identifies $\mathrm{GL}_{\mathcal{K}_w}(V_w) \times \mathrm{GL}_{\mathcal{K}_w}(V_w) \subseteq \mathrm{GL}_{\mathcal{K}_w}(V_w \oplus V_w)$ with $\mathrm{GL}_n(V_w) \times \mathrm{GL}_n(\mathcal{K}_w) \subseteq \mathrm{GL}_{2n}(\mathcal{K}_w)$. Note that this is a different identification of $\mathrm{GL}_{\mathcal{K}_w}(W_w)$ with $\mathrm{GL}_{2n}(\mathcal{K}_w)$ from the identification coming from the decomposition $W_w = V_{d,w} \oplus V_w^d$. Recall the Siegel section $f_w^{\chi,\mu}$ defined in Equation (4.3.5). In the computation of the zeta integrals, we replace $f_w^{\chi,\mu}$ with the following translation of $f_w^{\chi,\mu}$:

$$g \mapsto f_w^{\chi,\mu}\left(g\begin{pmatrix}\frac{1}{2}\cdot 1_n & \frac{-1}{2}\cdot 1_n\\ 1_n & 1_n\end{pmatrix}\begin{pmatrix}1_{a_w} & 0 & 0 & 0\\ 0 & 0 & 0 & 1_{b_w}\\ 0 & 0 & 1_{a_w} & 0\\ 0 & 1_{b_w} & 0 & 0\end{pmatrix}\right). \tag{4.3.10}$$

The matrices in Equation (4.3.10) are given with respect to the identification of $\mathrm{GL}_{\mathcal{K}_w}(V_w \oplus V_w)$ with $\mathrm{GL}_{2n}(\mathcal{K}_w)$ introduced at the beginning of this paragraph.

To avoid cumbersome notation, we will denote $\Phi_{\chi,\mu,w}$ by Φ for the remainder of this section. The identification $\mathcal{K}_w = \mathcal{K}^+_{w^+}$ identifies the representation π_w with a representation π_{w^+}, sections φ_{w^+} and φ'_{w^+} with φ_w and φ'_w, respectively, and a pairing $\langle,\rangle_{\pi_w}$ with the pairing $\langle,\rangle_{w^+}$. Plugging in the translation of the section $f_w^{\chi,\mu}$ given in (4.3.10) yields

(4.3.11)

$$I_{w^+}\left(\varphi_{w^+},\varphi'_{w^+},f_{w^+},s\right) = I_w\left(\varphi,\varphi',\chi,\mu\right)$$

$$:= \int_{\mathrm{GL}_n(\mathcal{K}_w)}\chi_{2,w}(g)\left|\det g\right|_w^{s+\frac{n}{2}}\int_{\mathrm{GL}_n(\mathcal{K}_w)}\Phi\left((Xg,X)\begin{pmatrix}1_{a_w} & 0 & 0 & 0\\ 0 & 0 & 0 & 1_{b_w}\\ 0 & 0 & 1_{a_w} & 0\\ 0 & 1_{b_w} & 0 & 0\end{pmatrix}\right) \tag{4.3.12}$$
$$\times\, \chi_{1,w}^{-1}\chi_{2,w}(\det X)\left|\det X\right|_w^{2s+n}\langle\pi_w(g)\varphi_w,\varphi'_w\rangle_{\pi_w}d^\times X d^\times g.$$

We put

$$I_p\left(\varphi,\varphi',\chi,\mu\right) := \prod_{w\in\Sigma_p} I_w\left(\varphi,\varphi',\chi,\mu\right) = \prod_{w\in\Sigma_p} I_{w^+}\left(\varphi_{w^+},\varphi'_{w^+},f_{w^+},\chi\right).$$

Given $g, X \in \mathrm{GL}_n(\mathcal{K}_w)$, we denote by $Z_1 = (Z_1', Z_1'')$ and $Z_2 = (Z_2', Z_2'')$ the matrices in $M_{n\times n}(\mathcal{K}_w) = M_{n\times a_w}(\mathcal{K}_w) \times M_{n\times b_w}(\mathcal{K}_w)$ given by

$$Z_1 = Xg = \left[Z_1', Z_1''\right]$$
$$Z_2 = X = \left[Z_2', Z_2''\right],$$

with $Z_1', Z_2' \in M_{n\times a_w}(\mathcal{K}_w)$ and $Z_1'', Z_2'' \in M_{n\times b_w}(\mathcal{K}_w)$. So

$$\Phi((Xg, g)) = \mathrm{volume}(\Gamma_w)^{-1} \Phi_{1,w}\left(Z_1', Z_2''\right) \Phi_{2,w}\left(Z_2', Z_1''\right),$$

and

$$\begin{aligned}\left\langle \pi_w(g)\varphi_w, \varphi_w'\right\rangle_{\pi_w} &= \left\langle \pi_w(Xg)\varphi_w, \tilde{\pi}_v(X)\varphi_w'\right\rangle_{\pi_w} \\ &= \left\langle \pi_w(Z_1)\varphi_w, \tilde{\pi}_v(Z_2)\varphi_w'\right\rangle_{\pi_w}.\end{aligned}$$

Therefore,

(4.3.13)

$$I_w\left(\varphi, \varphi', \chi, \mu\right) = \mathrm{volume}(\Gamma_w)^{-1} \int_{\mathrm{GL}_n(\mathcal{K}_w)} \int_{\mathrm{GL}_n(\mathcal{K}_w)} \chi_{2,w}(\det Z_1)\, \chi_{1,w}^{-1}(\det Z_2)\, |\det(Z_1 Z_2)|_w^{s+\frac{n}{2}}$$
$$\times \Phi_{1,w}\left(Z_1', Z_2''\right) \Phi_{2,w}\left(Z_2', Z_1''\right) \left\langle \pi_w(Z_1)\varphi_w, \tilde{\pi}_w(Z_2)\varphi_w'\right\rangle_{\pi_w} d^\times Z_1 d^\times Z_2.$$

Now we take the integral over the following open subsets of full measure. We take the integral in Z_1 over

$$\left\{\begin{pmatrix}1 & 0\\ C_1 & 1\end{pmatrix}\begin{pmatrix}A_1 & 0\\ 0 & D_1\end{pmatrix}\begin{pmatrix}1 & B_1\\ 0 & 1\end{pmatrix} \mid C_1, {}^tB_1 \in M_{b_w\times a_w}(\mathcal{K}_w), A_1 \in \mathrm{GL}_{a_w}(\mathcal{K}_w), D_1 \in \mathrm{GL}_{b_w}(\mathcal{K}_w)\right\},$$

with the measure

$$\left|\det A_1^{b_w} \det D_1^{-a_w}\right|_w dC_1 d^\times A_1 d^\times D_1 dB_1.$$

We take the integral in Z_2 over

$$\left\{\begin{pmatrix}1 & B_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}A_2 & 0\\ 0 & D_2\end{pmatrix}\begin{pmatrix}1 & 0\\ C_2 & 1\end{pmatrix} \mid C_2, {}^tB_2 \in M_{b_w\times a_w}(\mathcal{K}_w), A_2 \in \mathrm{GL}_{a_w}(\mathcal{K}_w), D_2 \in \mathrm{GL}_{b_w}(\mathcal{K}_w)\right\},$$

with the measure

$$\left|\det A_2^{-b_w} \det D_2^{a_w}\right|_w dC_2 d^\times A_2 d^\times D_2 dB_2.$$

So

$$\Phi_{1,w}\left(Z_1', Z_2''\right) = \Phi_{1,w}\left(\begin{pmatrix}A_1 & B_2D_2\\ C_1 & D_2\end{pmatrix}\right) \tag{4.3.14}$$

$$\Phi_{2,w}\left(Z_2', Z_1''\right) = \Phi_{2,w}\left(\begin{pmatrix}A_2 + B_2D_2C_2 & A_1B_1\\ D_2C_2 & C_1A_1B_1 + D_1\end{pmatrix}\right). \tag{4.3.15}$$

Let $\Gamma_{a_w,w}(t)$ be defined similarly to the group $\Gamma_{a_w,w}$ but with d replaced by t. Let $\Gamma_{b_w,w}(t)$ be defined similarly to the group $\Gamma_{b_w,w}$ but with d replaced by t. So since

$d \geq 2t \geq t$,

$$\Gamma_{a_w,w}(t) \supseteq \Gamma_{a_w,w}$$
$$\Gamma_{b_w,w}(t) \supseteq \Gamma_{b_w,w}.$$

Note that since the nontrivial $G_w := G_{w^+}$-equivariant pairing $\langle , \rangle_{\pi_w}$ is unique up to a constant, there exists a constant $\aleph_w$ such that for all $\xi_w \in \pi_w$ and $\xi'_w \in \tilde{\pi}_w$,

$$\aleph_w \langle \xi_w, \xi'_w \rangle_{\pi_w} = \int_{G_w} \langle \xi_w(x), \xi'_w(x) \rangle_{\pi_{a_w} \otimes \pi_{b_w}} dx,$$

where

$$\langle , \rangle_{\pi_{a_w} \otimes \pi_{b_w}} : \pi_{a_w} \otimes \pi_{b_w} \times \tilde{\pi}_{a_w} \otimes \tilde{\pi}_{b_w} \to \mathbb{C}$$

is the unique nontrivial $L_{a_w,b_w}(\mathcal{K}_w)$-equivariant pairing on $(\pi_{a_w} \otimes \pi_{b_w}) \times (\tilde{\pi}_{a_w} \otimes \tilde{\pi}_{b_w})$ such that

$$\langle \varphi_w(1), \varphi'_w(1) \rangle_{\pi_{a_w} \otimes \pi_{b_w}} = 1.$$

Now, let

$$\langle , \rangle_{\pi_{a_w}} : \pi_{a_w} \times \tilde{\pi}_{a_w} \to \mathbb{C}^\times$$
$$\langle , \rangle_{\pi_{b_w}} : \pi_{b_w} \times \tilde{\pi}_{b_w} :\to \mathbb{C}^\times$$

denote the unique $\mathrm{GL}_{a_w}(\mathcal{K}_w)$- and $\mathrm{GL}_{b_w}(\mathcal{K}_w)$-equivariant pairings, respectively, such that

$$\langle \varphi_{a_w}, \tilde{\varphi}_{a_w} \rangle_{\pi_{a_w}} = 1 \quad (4.3.16)$$
$$\langle \varphi_{b_w}, \tilde{\varphi}_{b_w} \rangle_{\pi_{b_w}} = 1.$$

So by the uniqueness (up to a constant) of the equivariant pairing $\langle , \rangle_{\pi_{a_w} \otimes \pi_{b_w}}$, we have the factorization

$$\langle , \rangle_{\pi_{a_w} \otimes \pi_{b_w}} = \langle , \rangle_{\pi_{a_w}} \cdot \langle , \rangle_{\pi_{b_w}}.$$

Proposition 4.3.8. *The product* $\Phi_{1,w}\left(\begin{pmatrix} A_1 & B_2D_2 \\ C_1 & D_2 \end{pmatrix}\right) \Phi_{2,w}\left(\begin{pmatrix} A_2 + B_2D_2C_2 & A_1B_1 \\ D_2C_2 & C_1A_1B_1 + D_1 \end{pmatrix}\right)$ *is zero unless all of the following conditions are met:*

$$A_1 \in \Gamma_{a_w,w}(t)$$
$$C_1 \in \mathfrak{p}_w^t M_{b_w \times a_w}(\mathcal{O}_w)$$
$$D_2 \in \Gamma_{b_w,w}(t)$$
$$B_2 \in \mathfrak{p}_w^t M_{a_w \times b_w}(\mathcal{O}_w)$$
$$C_2 \in M_{b_w \times a_w}(\mathcal{O}_w)$$
$$A_2 \in \mathfrak{p}_w^{-t} M_{a_w \times a_w}(\mathcal{O}_w)$$
$$B_1 \in M_{a_w \times b_w}(\mathcal{O}_w)$$
$$D_1 \in \mathfrak{p}_w^{-t} M_{b_w \times b_w}(\mathcal{O}_w).$$

When all of the above conditions are met, we have the following factorization at each prime $w \in \Sigma_p$:

$$\Phi_{1,w}\left(Z_1', Z_2''\right)\Phi_{2,w}\left(Z_2', Z_1''\right)\langle \pi_w\left(Z_1\right)\varphi_w, \tilde{\pi}_v\left(Z_2\right)\varphi_w'\rangle_{\pi_w} = J_1 \cdot J_2, \tag{4.3.17}$$

where

$$J_1 = \chi_{2,w}\left(\det A_1\right)^{-1}\Phi_w^{(4)}\left(D_1\right)\left|\det D_1^{a_w}\right|_w^{1/2}\left\langle \varphi_{b_w}, \tilde{\pi}_{b_w}\left(D_1^{-1}\right)\tilde{\varphi}_{b_w}\right\rangle_{\pi_{b_w}} \tag{4.3.18}$$

$$J_2 = \chi_{1,w}\left(\det D_2\right)\Phi_w^{(1)}\left(A_2\right)\left|\det A_2^{b_w}\right|_w^{1/2}\langle \varphi_{a_w}, \tilde{\pi}_{a_w}\left(A_2\right)\tilde{\varphi}_{a_w}\rangle_{\pi_{a_w}}. \tag{4.3.19}$$

Proof. By Lemma 4.3.2 and the definition of $\Phi_{1,w}$, the product

$$\Phi_{1,w}\left(\begin{pmatrix} A_1 & B_2 D_2 \\ C_1 & D_2 \end{pmatrix}\right)\Phi_{2,w}\left(\begin{pmatrix} A_2 + B_2 D_2 C_2 & A_1 B_1 \\ D_2 C_2 & C_1 A_1 B_1 + D_1 \end{pmatrix}\right)$$

is zero unless all of the above conditions are met. For the remainder of the proof, we will work only with matrices meeting the above conditions. We now prove the second statement of the proposition. Note that when the above conditions are met,

$$\pi_w\left(Z_1\right)\varphi_w = \pi_w\left(\begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix}\right)\mu_w'\left(\begin{pmatrix} A_1 & 0 \\ 0 & 1 \end{pmatrix}\right)\varphi_w$$
$$\tilde{\pi}_w\left(Z_1\right)\varphi_w' = \tilde{\pi}_w\left(\begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix}\right)\left(\mu_w'\right)^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & D_2 \end{pmatrix}\right)\varphi_w'.$$

So

$$\Phi_{1,w}\left(Z_1', Z_2''\right)\Phi_{2,w}\left(Z_2', Z_1''\right)\langle \pi_w\left(Z_1\right)\varphi_w, \tilde{\pi}\left(Z_2\right)\varphi_w'\rangle_{\pi_w}$$
$$= \chi_{2,w}^{-1}\left(\det A_1\right)\chi_{1,w}\left(\det D_2\right)\left\langle \pi_w\left(\begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\left(\begin{pmatrix} A_2 & B_2 \\ 0 & 1 \end{pmatrix}\right)\varphi_w'\right\rangle_{\pi_w}.$$

Let $A \in M_{a_w}\left(\mathcal{K}_w\right)$, $D \in M_{b_w}\left(\mathcal{K}_w\right)$, $C \in M_{b_w \times a_w}\left(\mathcal{K}_w\right)$, and $D \in M_{a_w \times b_w}\left(\mathcal{K}_w\right)$ be matrices such that

$$\begin{pmatrix} 1 & -B_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C_1 & D_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}.$$

Then

$$A = 1 - B_2 C_1 \in 1 + \mathfrak{p}_w^{2t} M_{a_w}\left(\mathcal{O}_w\right)$$
$$CA = C_1 \in \mathfrak{p}_w^t M_{b_w \times a_w}\left(\mathcal{O}_w\right)$$
$$AB = -B_2 D_1 \in M_{a_w \times b_w}\left(\mathcal{O}_w\right).$$

So

$$C \in \mathfrak{p}_w^t M_{b_w \times a_w}\left(\mathcal{O}_w\right)$$
$$B \in M_{a_w \times b_w}\left(\mathcal{O}_w\right)$$
$$D = D_1 - CAB = \left(1 + CB_2\right)D_1 \in \left(1 + p^{2t} M_{b_w}\left(\mathcal{O}_w\right)\right)D_1.$$

Therefore, applying the invariance conditions of Definition 4.3.5 we obtain

$$\begin{aligned}\left\langle \pi_w\left(\begin{pmatrix}1 & 0\\ C_1 & D_1\end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & B_2\\ 0 & 1\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w} &= \left\langle \pi_w\left(\begin{pmatrix}1 & -B_2\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ C_1 & D_1\end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & 1\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w}\\ &= \left\langle \pi_w\left(\begin{pmatrix}1 & 0\\ C_1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & D_1\end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\begin{pmatrix}A_2 & 0\\ 0 & 1\end{pmatrix}\varphi'_w\right\rangle_{\pi_w}\\ &= \left\langle \pi_w\left(\begin{pmatrix}1 & 0\\ 0 & D_1\end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -C_1A_2 & 1\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w}\\ &= \left\langle \pi_w\left(\begin{pmatrix}1 & 0\\ 0 & D_1\end{pmatrix}\right)\varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & 1\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w}\\ &= \left\langle \varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & D_1^{-1}\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w}.\end{aligned}$$

Since $\langle\varphi_w, \varphi'_w\rangle_{\pi_w} = 1$, it follows that

$$\aleph_w = \int_{\mathrm{GL}_n(\mathcal{O}_w)} \langle\varphi_w(x), \varphi'_w(x)\rangle_{\pi_{a_w}\otimes\pi_{b_w}} dx.$$

Now, the support of φ_w in $\mathrm{GL}_n(\mathcal{O}_w)$ is

$$V(d) = \left\{\begin{pmatrix}1 & B\\ 0 & 1\end{pmatrix}\begin{pmatrix}A & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}1 & 0\\ C & 1\end{pmatrix} \mid C \in \mathfrak{p}_w^d M_{b_w\times a_w}, A \in \mathrm{GL}_{a_w}(\mathcal{O}_w), D \in \mathrm{GL}_{b_w}(\mathcal{O}_w), B \in M_{a_w\times b_w}(\mathcal{O}_w)\right\}.$$

Applying the definitions of φ_w and φ'_w, we obtain

$$\int_{\mathrm{GL}_n(\mathcal{O}_w)} \langle\varphi_w(x), \varphi'_w(x)\rangle_{\pi_{a_w}\otimes\pi_{b_w}} dx = \mathrm{volume}(V(d), dg)\,\langle\varphi_w(1), \varphi'_w(1)\rangle_{\pi_{a_w}\otimes\pi_{b_w}}.$$

Since we defined $\langle,\rangle_{\pi_{a_w}\otimes\pi_{b_w}}$ so that $\langle\varphi_w(1), \varphi'_w(1)\rangle_{\pi_{a_w}\otimes\pi_{b_w}} = 1$, we therefore obtain

$$\aleph_w = \mathrm{volume}(V(d), dg). \tag{4.3.20}$$

Note that since $d \geq 2t$, $A_2 \in \mathfrak{p}_w^{-\frac{d}{2}} M_{a_w}(\mathcal{O}_w)$ and $D_1 \in \mathfrak{p}_w^{-\frac{d}{2}} M_{b_w}(\mathcal{O}_w)$. Then once again using the definitions of φ_w and φ'_w and the fact that the support of φ_w in $\mathrm{GL}_n(\mathcal{O}_w)$ is $V(d)$, we see that

$$\begin{aligned}&\left\langle \varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & D_1^{-1}\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w}\\ &= \mathrm{volume}(V(d), dg)\,\aleph_w^{-1}\cdot\left\langle \varphi_w(1), \left|\det A_2^{b_w}\det D_1^{a_w}\right|_w^{1/2}\tilde{\pi}_{a_w}(A_2)\otimes\tilde{\pi}_{b_w}\left(D_1^{-1}\right)\varphi'_w(1)\right\rangle_{\pi_{a_w}\otimes\pi_{b_w}}.\end{aligned}$$

So by Equation (4.3.20),

$$\left\langle \varphi_w, \tilde{\pi}_w\left(\begin{pmatrix}A_2 & 0\\ 0 & D_1^{-1}\end{pmatrix}\right)\varphi'_w\right\rangle_{\pi_w} = \left|\det A_2^{b_w}\det D_1^{a_w}\right|_w^{1/2}\left\langle \varphi_w(1), \tilde{\pi}_{a_w}(A_2)\otimes\tilde{\pi}_{b_w}\left(D_1^{-1}\right)\varphi'_w(1)\right\rangle_{\pi_{a_w}\otimes\pi_{b_w}}.$$

Consequently,

$$\begin{aligned}&\left\langle\varphi_w(1),\tilde{\pi}_{a_w}\left(A_2\right)\otimes\tilde{\pi}_{b_w}\left(D_1^{-1}\right)\varphi_w'(1)\right\rangle_{\pi_{a_w}\otimes\pi_{b_w}}\\&=\left\langle\varphi_{a_w},\tilde{\pi}_{a_w}\left(A_2\right)\tilde{\varphi}_{a_w}\right\rangle_{\pi_{a_w}}\cdot\left\langle\varphi_{b_w},\tilde{\pi}_{b_w}\left(D_1^{-1}\right)\tilde{\varphi}_{b_w}\right\rangle_{\pi_{b_w}}.\end{aligned}$$

So

$$\Phi_{1,w}\left(Z_1',Z_2''\right)\Phi_{2,w}\left(Z_2',Z_1''\right)\left\langle\pi_w\left(Z_1\right)\varphi_w,\tilde{\pi}_w\left(Z_2\right)\tilde{\varphi}_w\right\rangle_{\pi_w}=J_1J_2,$$

where

$$J_1=\chi_{2,w}\left(\det A_1\right)^{-1}\Phi_w^{(4)}\left(D_1\right)\left|\det D_1^{a_w}\right|_w^{1/2}\left\langle\varphi_{b_w},\tilde{\pi}_{b_w}\left(D_1^{-1}\right)\tilde{\varphi}_{b_w}\right\rangle_{\pi_{b_w}}\tag{4.3.21}$$

$$J_2=\chi_{1,w}\left(\det D_2\right)\Phi_w^{(1)}\left(A_2\right)\left|\det A_2^{b_w}\right|_w^{1/2}\left\langle\varphi_{a_w},\tilde{\pi}_{a_w}\left(A_2\right)\tilde{\varphi}_{a_w}\right\rangle_{\pi_{a_w}}.\tag{4.3.22}$$

□

Corollary 4.3.9. *The integral* $I_w\coloneqq I_w\left(\varphi,\varphi',\chi,\mu\right)$ *in Equation* (4.3.11) *factors as*

$$\begin{aligned}I_w&=I_1\cdot I_2,\\I_1&=\int_{\mathrm{GL}_{b_w}(\mathcal{K}_w)}\chi_{2,w}\left(\det D\right)\Phi_w^{(4)}(D)\left|\det D\right|_w^{s+\frac{b_w}{2}}\left\langle\pi_{b_w}(D)\varphi_{b_w},\tilde{\varphi}_{b_w}\right\rangle_{\pi_{b_w}}d^\times D\\I_2&=\int_{\mathrm{GL}_{a_w}(\mathcal{K}_w)}\chi_{1,w}^{-1}\left(\det A\right)\Phi_w^{(1)}(A)\left|\det A\right|_w^{s+\frac{a_w}{2}}\left\langle\varphi_{a_w},\tilde{\pi}_{a_w}\left(A\right)\tilde{\varphi}_{a_w}\right\rangle_{\pi_{a_w}}d^\times A.\end{aligned}$$

Proof. By Equation (4.3.13) and Proposition 4.3.8,

(4.3.23)

$$\begin{aligned}I_w=\mathrm{volume}\left(\Gamma_w\right)^{-1}&\int_{A_1,A_2,B_1,B_2,C_1,C_2,D_1,D_2}\chi_{2,w}\left(\det\left(A_1D_1\right)\right)\chi_{1,w}^{-1}\left(\det\left(A_2D_2\right)\right)\\&\times\left|\det\left(A_1D_1A_2D_2\right)\right|_w^{s+\frac{n}{2}}J_1J_2\\&\times\left|\det A_1^{b_w}\det D_1^{-a_w}\right|_w\left|\det A_2^{-b_w}\det D_2^{a_w}\right|_w d^\times A_1d^\times A_2dB_1dB_2dC_1dC_2d^\times D_1d^\times D_2,\end{aligned}$$

where J_1 and J_2 are defined as in Equations (4.3.21) and (4.3.22), respectively, and

$$\begin{aligned}
&A_1 \in \Gamma_{a_w}(t)\\
&C_1 \in \mathfrak{p}_w^t \prod_{v|p} M_{b_w\times a_w}(\mathcal{O}_w)\\
&D_2 \in \Gamma_{b_w}(t)\\
&B_2 \in \mathfrak{p}_w^t \prod_{v|p} M_{a_w\times b_w}(\mathcal{O}_w)\\
&C_2 \in \prod_{v|p} M_{b_w\times a_w}(\mathcal{O}_w)\\
&A_2 \in \mathfrak{p}_w^{-c} \prod_{v|p} M_{a_w\times a_w}(\mathcal{O}_w)\\
&B_1 \in \prod_{v|p} M_{a_w\times b_w}(\mathcal{O}_w)\\
&D_1 \in \mathfrak{p}_w^{-c} \prod_{v|p} M_{b_w\times b_w}(\mathcal{O}_w).
\end{aligned}$$

Note that for such A_1 and D_2, $|\det A_1|_w = |\det D_2|_w = 1$. Applying Equations (4.3.21) and (4.3.22), we therefore see that the integrand in Equation (4.3.23) equals

$$\begin{aligned}
&\chi_{2,w}(\det D_1)\,\Phi_w^{(4)}(D_1)\langle \pi_{b_w}(D_1)\varphi_{b_w}, \tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} |D_1|_w^{s+\frac{b}{2}}\\
&\qquad\times \chi_{1,w}^{-1}(\det A_2)\,\Phi_w^{(1)}(A_2)\langle \varphi_{a_w}, \tilde{\pi}_{a_w}(A_2)\tilde{\varphi}_{a_w}\rangle_{\pi_{a_w}} |A_2|_w^{s+\frac{a}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_w =& \operatorname{volume}(\Gamma_w)^{-1}\operatorname{volume}(\Gamma_w)\left(\operatorname{volume}\left(\prod_{v|p} M_{a_w\times b_w}(\mathcal{O}_w)\right)\right)^2\\
&\times \int_{\mathrm{GL}_{b_w}(\mathcal{K}_w)} \chi_{2,w}(\det D_1)\,\Phi_w^{(4)}(D_1)\langle \pi_{b_w}(D_1)\varphi_{b_w}, \tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} |D_1|_w^{s+\frac{b}{2}}\, d^\times D_1\\
&\times \int_{\mathrm{GL}_{a_w}(\mathcal{K}_w)} \chi_{1,w}^{-1}(\det A_2)\,\Phi_w^{(1)}(A_2)\langle \varphi_{a_w}, \tilde{\pi}_{a_w}(A_2)\tilde{\varphi}_{a_w}\rangle_{\pi_{a_w}} |A_2|_w^{s+\frac{a}{2}}\, d^\times A_2\\
=& \int_{\mathrm{GL}_{b_w}(\mathcal{K}_w)} \chi_{2,w}(\det D)\,\Phi_w^{(4)}(D)\langle \pi_{b_w}(D)\varphi_{b_w}, \tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} |D|_w^{s+\frac{b}{2}}\, d^\times D\\
&\times \int_{\mathrm{GL}_{a_w}(\mathcal{K}_w)} \chi_{1,w}^{-1}(\det A)\,\Phi_w^{(1)}(A)\langle \varphi_{a_w}, \tilde{\pi}_{a_w}(A)\tilde{\varphi}_{a_w}\rangle_{\pi_{a_w}} |A|_w^{s+\frac{a}{2}}\, d^\times A.
\end{aligned}$$

□

4.3.10. *The main local theorem.* In Theorem 4.3.11, we calculate the integrals I_1 and I_2 from Corollary 4.3.9.

Theorem 4.3.11. *The integrals I_1 and I_2 are related to familiar L-functions as follows.*

$$I_1 = \frac{L\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w} \otimes \chi_{2,w}^{-1}\right)}$$
$$I_2 = \frac{\varepsilon\left(-s+\frac{1}{2}, \pi_{a_w} \otimes \chi_{1,w}\right) L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w} \otimes \chi_{1,w}^{-1}\right)}{L\left(-s+\frac{1}{2}, \pi_{a_w} \otimes \chi_{1,w}\right)}.$$

Consequently,
(4.3.24)
$$I_w = \frac{L\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w} \otimes \chi_{2,w}^{-1}\right)} \frac{\varepsilon\left(-s+\frac{1}{2}, \pi_{a_w} \otimes \chi_{1,w}\right) L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w} \otimes \chi_{1,w}^{-1}\right)}{L\left(-s+\frac{1}{2}, \pi_{a_w} \otimes \chi_{1,w}\right)}$$
$$:= L(s+\frac{1}{2}, ord, \pi_w, \chi_w)$$

Here as above we are writing $\pi_w = Ind_{R_{a_w,b_w}}^{GL_n} \pi_{a_w} \otimes \pi_{b_w}$.

Proof. The integrals I_1 and I_2 are of the same form as the "Godement-Jacquet" integral defined in [Jac79, Equation (1.1.3)]. Applying the "Godement-Jacquet functional equation" in [Jac79, Equation (1.3.7)], we obtain

$$I_1 = \frac{L\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w} \otimes \chi_{2,w}^{-1}\right)}$$
$$\times \int_{\mathrm{GL}_{b_w}(\mathcal{K}_w)} \left(\Phi_w^{(4)}\right)^{\wedge}(D) \left|\det D\right|_w^{-s+\frac{b_w}{2}} \chi_{2,w}^{-1}(D) \langle \varphi_{b_w}, \tilde{\pi}_{b_w}(D)\tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} d^{\times}D, \tag{4.3.25}$$

where $\left(\Phi_w^{(4)}\right)^{\wedge}$ denotes the Fourier transform of $\Phi_w^{(4)}$. The support of $\left(\Phi_w^{(4)}\right)^{\wedge}$ is $\mathrm{GL}_{b_w}(\mathcal{O}_w)$, and

$$\left(\Phi_w^{(4)}\right)^{\wedge}(D) = \phi_{\nu_v}\left(\begin{pmatrix} 1_{a_w} & 0 \\ 0 & D \end{pmatrix}\right) \tag{4.3.26}$$

for all $D \in \mathrm{GL}_{b_w}(\mathcal{O}_w)$.

Applying Equations (4.3.7) and (4.3.8), we obtain

$$\begin{aligned} \langle \varphi_{b_w}, \tilde{\pi}_{b_w}(D)\tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} &= \left(\mu'_{b_w}(D)\right)^{-1} \langle \varphi_{b_w}, \tilde{\varphi}_{b_w}\rangle_{\pi_{b_w}} \\ &= \left(\mu'_{b_w}(D)\right)^{-1} \end{aligned}$$

for all $D \in {}^t\Gamma_{b_w}\Gamma_{b_w} = \prod_{w\in\Sigma} \mathrm{GL}_{b_w}(\mathcal{O}_w)$. Plugging this into Equation (4.3.25) and applying Equation (4.3.26), we obtain

$$I_1 = \frac{L\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w} \otimes \chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w} \otimes \chi_{2,w}^{-1}\right)}.$$

The computation of I_2 is similar. Applying the Godement Jacquet functional equation [Jac79, Equation (1.3.7)] again and applying Equation (4.3.16), we obtain

$$I_2 = \frac{\varepsilon\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right) L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right)}{L\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right)}.$$

□

Remark 4.3.12. Let ω_{a_w} denote the central quasi-character of π_{a_w}. Then

$$\varepsilon\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right) = \frac{\omega_{a_w}(-1)}{\varepsilon\left(s+\frac{1}{2}, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right)}.$$

So we may rewrite I_2 and I_w as

$$I_2 = \frac{\omega_{a_w}(-1) L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right)}{\varepsilon\left(s+\frac{1}{2}, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right) L\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right)}$$

$$I_w = \omega_{a_w}(-1)\frac{L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right)}{\varepsilon\left(s+\frac{1}{2}, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right) L\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right)} \frac{L\left(s+\frac{1}{2}, \pi_{b_w}\otimes\chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w}\otimes\chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w}\otimes\chi_{2,w}^{-1}\right)}.$$

Therefore, the Euler factor at p, which we denoted in Equation (4.3.24)

$$\prod_{w|p} L(s+\frac{1}{2}, ord, \pi_w, \chi_w)$$

can also be written

(4.3.27)

$$\prod_{v|p} \omega_{a_w}(-1)\frac{L\left(\frac{1}{2}+s, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right)}{\varepsilon\left(s+\frac{1}{2}, \tilde{\pi}_{a_w}\otimes\chi_{1,w}^{-1}\right) L\left(-s+\frac{1}{2}, \pi_{a_w}\otimes\chi_{1,w}\right)} \frac{L\left(s+\frac{1}{2}, \pi_{b_w}\otimes\chi_{2,w}\right)}{\varepsilon\left(s+\frac{1}{2}, \pi_{b_w}\otimes\chi_{2,w}\right) L\left(-s+\frac{1}{2}, \tilde{\pi}_{b_w}\otimes\chi_{2,w}^{-1}\right)}.$$

Remark 4.3.13. Note the similarity of the form of the zeta integral at p in Equation (4.3.27) with the form of the modified Euler factor at p for the p-adic L-functions predicted by Coates in [Coa89, Section 2, Equation 18b].

4.4. **Holomorphic representations of enveloping algebras and anti-holomorphic vectors.**

4.4.1. *Holomorphic and anti-holomorphic modules.* Throughout this section, we identify Σ with $\Sigma_{\mathcal{K}^+}$, and we identify each element $\sigma \in \Sigma$ with the restriction $\sigma|_{\mathcal{K}^+}$. To simplify notation, we let $G^* = GU_1 = R_{\mathcal{K}/\mathbb{Q}} GU(V)$ where $GU(V)$ denotes the full unitary similitude group of V. Thus $G^*(\mathbb{R}) = \prod_{\sigma\in\Sigma_{\mathcal{K}^+}} G_\sigma$, with $G_\sigma = GU(V)_{\mathcal{K}^+_\sigma} \simeq GU(a_\sigma, b_\sigma)$. For any $h : R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}}) \to G^*_{\mathbb{R}}$ as in Section 2.1, the image of h is contained in the subgroup G of $(g_\sigma, \sigma \in \Sigma_{\mathcal{K}^+})$ for which the similitude factor $\nu(g_\sigma)$ is independent of σ, and it is to this latter subgroup that the Shimura variety is attached.

Let $U_\infty = C(\mathbb{R}) \subseteq G(\mathbb{R}) = G_4(\mathbb{R})$ and X be as in Section 2.7.1. We assume U_∞ is the centralizer of a *standard* h as in Section 2.3.2; let $U_\sigma \subset G_\sigma$ denote its intersection with

G_σ and let $K^o_\sigma \subset U_\sigma$ denote its maximal compact subgroup; K^o_σ is isomorphic to the product of compact unitary groups $U(a_\sigma) \times U(b_\sigma)$. We have

$$\mathfrak{k}_\sigma := Lie(U_\sigma) = \mathfrak{z}_\sigma \oplus Lie(K^o_\sigma)$$

where $\mathfrak{z}_\sigma$ is the $\mathbb{R}$-split center of $\mathfrak{g}_\sigma := Lie(GU(a_\sigma, b_\sigma))$. We let $U(\mathfrak{g}_\sigma)$ denote the enveloping algebra of $\mathfrak{g}_\sigma$.

For $\sigma \in \Sigma_{\mathcal{K}^+}$, we write the Harish-Chandra decomposition

$$\mathfrak{g}_\sigma = \mathfrak{k}_\sigma \oplus \mathfrak{p}^-_\sigma \oplus \mathfrak{p}^+_\sigma.$$

Because h was chosen to be standard, this decomposition is naturally defined over $\sigma(\mathcal{K}) \subset \mathbb{C}$. For any irreducible representation $(\tau_\sigma, W_{\tau_\sigma})$ of U_σ of $G_\sigma := G(\mathcal{K}^+_\sigma)$, we let

$$\mathbb{D}^\flat(\tau_\sigma) = U(\mathfrak{g}_\sigma) \otimes_{U(\mathfrak{k}_\sigma \oplus \mathfrak{p}^-_\sigma)} W_{\tau_\sigma} \tag{4.4.1}$$

We have assumed that our chosen h takes values in a rational torus $T (= J_{0,n}) \subset G$ (so that (T, h) is a CM Shimura datum), and let $T_\sigma \subset G_\sigma$ be the σ-component of $T(\mathbb{R})$, $\mathfrak{t}_\sigma$ its Lie algebra. We choose a positive root system R^+_σ for T_σ so that the roots on $\mathfrak{p}^+_\sigma$ are positive, and let $\mathfrak{b}^+_\sigma$ be the corresponding Borel subalgebra.

Let $R^{+,c}_\sigma \subset R^+_\sigma$ be the set of positive compact roots. The highest weight of τ_σ relative to $R^{+,c}_\sigma$ can be denoted $\kappa_\sigma = (c_\sigma; \kappa_{1,\sigma} \geq \cdots \geq \kappa_{a_\sigma,\sigma}; \kappa^c_{1,\sigma} \geq \cdots \geq \kappa^c_{b_\sigma,\sigma}) \in \mathbb{Z} \times \mathbb{Z}^{a_\sigma} \times \mathbb{Z}^{b_\sigma}$, where c_σ is the character of $\mathfrak{z}_\sigma$. We call $(\tau_\sigma, W_{\tau_\sigma})$ *strongly positive* if there exists an irreducible representation $\mathcal{W}^\vee_\sigma$ of G_σ, with highest weight $\mu = (-c_\sigma; a_1 \geq \cdots \geq a_n) \in \mathbb{Z} \times \mathbb{Z}^n$ relative to R^+_σ, such that, setting $a = a_\sigma$ and $b = b_\sigma$,

$$(a_1, \ldots, a_n) = (-\kappa^c_{b,\sigma} - a, \ldots, -\kappa^c_{1,\sigma} - a; -\kappa_{a,\sigma} + b, \ldots, -\kappa_{1,\sigma} + b); \tag{4.4.2}$$

in other words, if and only if $-\kappa^c_{1,\sigma} - a \geq -\kappa_{a,\sigma} + b$. The contragredient of $\mathbb{D}^\flat(\tau_\sigma)$ is denoted

$$\mathbb{D}_c(\tau_\sigma) = \mathbb{D}^\flat(\tau_\sigma)^\vee \cong U(\mathfrak{g}_\sigma) \otimes_{U(\mathfrak{k}_\sigma \oplus \mathfrak{p}^+_\sigma)} W^\vee_{\tau_\sigma}. \tag{4.4.3}$$

It is the complex conjugate representation of $\mathbb{D}^\flat(\tau_\sigma)$ with respect to the $\mathbb{R}$-structure on $\mathfrak{g}_\sigma$; we call this the *anti-holomorphic representation* of type τ_σ.

In what follows, we usually write $\mathbb{D}(\kappa_\sigma)$ instead of $\mathbb{D}(\tau_\sigma)$. It is well known that if τ_σ is strongly positive then $\mathbb{D}(\kappa_\sigma)$ (resp. $\mathbb{D}_c(\kappa_\sigma)$) is the $(U(\mathfrak{g}_\sigma), U_\sigma)$-module of a *holomorphic* (resp. *anti-holomorphic*) *discrete series* representation of G_σ, and moreover that

$$\dim H^{ab}(\mathfrak{g}_\sigma, U_\sigma; \mathbb{D}(\kappa_\sigma) \otimes \mathcal{W}^\vee_\sigma) = \dim H^{ab}(\mathfrak{g}_\sigma, U_\sigma; \mathbb{D}_c(\kappa_\sigma) \otimes \mathcal{W}_\sigma) = 1$$

with $\mathcal{W}^\vee_\sigma$ the representation with highest weight given by 4.4.2, and $\mathcal{W}_\sigma$ its dual, with highest weight

$$(c_\sigma; -a_n, \ldots, -a_1) = (c_\sigma; \kappa_{1,\sigma} - b; \ldots, \kappa_{a,\sigma} - b, \kappa^c_{1,\sigma} + a, \ldots, \kappa^c_{b,\sigma} + a). \tag{4.4.4}$$

The *minimal U_σ-type* of $\mathbb{D}^\flat(\kappa_\sigma)$ (resp. of $\mathbb{D}_c(\kappa_\sigma)$) is the subspace

$$1 \otimes W_{\tau_\sigma} \subset U(\mathfrak{g}_\sigma) \otimes_{U(\mathfrak{k}_\sigma \oplus \mathfrak{p}^-_\sigma)} W_{\tau_\sigma} \text{ (resp. } 1 \otimes W^\vee_{\tau_\sigma} \subset U(\mathfrak{g}_\sigma) \otimes_{U(\mathfrak{k}_\sigma \oplus \mathfrak{p}^+_\sigma)} W^\vee_{\tau_\sigma}).$$

The minimal U_σ-type of $\mathbb{D}^\flat(\kappa_\sigma)$ (resp. of $\mathbb{D}_c(\kappa_\sigma)$ is also called the space of *holomorphic vectors* (resp. *anti-holomorphic vectors*).

4.4.2. *Canonical automorphy factors and representations.* The $(U(\mathfrak{g}_\sigma), U_\sigma)$ module $\mathbb{D}^\flat(\kappa_\sigma)$ can be realized as a subrepresentation of the right regular representation on $C^\infty(G_\sigma)$ generated by a canonical automorphy factor. We recall this construction below when $G_\sigma = G_{4,\sigma} \simeq GU(n,n)$ and τ_σ is a scalar representation.

Let M_n be the affine group scheme of $n\times n$-matrices over $Spec(\mathbb{Z})$, $M_n = Spec(\mathcal{P}(n))$. For $\sigma \in \Sigma$, let $\mathcal{P}(n)_\sigma$ denote the base change of $\mathcal{P}(n)$ to $\mathcal{O}_\sigma = \sigma(\mathcal{O}_\mathcal{K})$. Corresponding to the factorization $G^*(\mathbb{R}) = \prod_\sigma G_\sigma$, we write $X = \prod_{\sigma\in\Sigma} X_\sigma$. The maximal parabolic P_n, together with U_σ, defines an unbounded realization of a connected component $X_\sigma^+ \subset X_\sigma$ as a tube domain in $\mathfrak{p}^+_{4,\sigma}$ ([Har86] (5.3.2)). A choice of basis for L_1, together with the identification of V with V_d and V^d introduced in Section 4.1.1, identifies $\mathfrak{p}^+_{4,\sigma}$ with $M_n(\mathbb{C})$ and therefore identifies X^+_σ with a tube domain in $M_n(\mathbb{C})$. Let $\daleth_\sigma \in X^+_\sigma$ be the fixed point of U_σ. Without loss of generality, we may assume $\daleth_\sigma$ to be a diagonal matrix with values in $\sigma(\mathcal{K})$ whose entries have trace zero down to $\mathcal{K}^+$. Then X^+_σ is identified with the standard tube domain

$$X_{n,n} := \left\{ z \in M_n(\mathbb{C}) \mid \daleth_\sigma \left({}^t\bar{z} - z\right) > 0 \right\}.$$

With respect to this identification, any $g_\sigma = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix} \in G_\sigma$ acts by $g_\sigma(z) = (a_\sigma z + b_\sigma)(c_\sigma z + d_\sigma)^{-1}$. (Here $a_\sigma, b_\sigma, c_\sigma$, and d_σ are $n\times n$ matrices.)

For $z = (z_\sigma)_{\sigma\in\Sigma} \in X = \prod_{\sigma\in\Sigma} X_\sigma$ and $g = (g_\sigma)_{\sigma\in\Sigma} \in \prod_{\sigma\in\Sigma} G(E_\sigma)$, let

$$J'(g_\sigma, z_\sigma) = \overline{c_\sigma}\cdot {}^t z_\sigma + \overline{d_\sigma}$$
$$J'(g,z) = \prod_{\sigma\in\Sigma} J'(g_\sigma, z_\sigma)$$
$$J(g_\sigma, z_\sigma) = c_\sigma z_\sigma + d_\sigma$$
$$J(g,z) = \prod_{\sigma\in\Sigma} J(g_\sigma, z_\sigma)$$

Let

$$j_{g_\sigma}(z_\sigma) = j(g_\sigma, z_\sigma) = \det J(g_\sigma, z_\sigma)$$
$$(= \nu(g_\sigma)^{-n}\det(g_\sigma)\det(J'(g_\sigma,z_\sigma)) = \nu(g_\sigma)^n \det(\overline{g_\sigma})^{-1}\det(J'(g_\sigma,z_\sigma))$$
$$j_g(z) = j(g,z) = \prod_{\sigma\in\Sigma} j_{g_\sigma}(z_\sigma).$$

Fix $\sigma\in\Sigma$. For $g \in G_\sigma$, let

$$J(g) = J(g, \daleth_\sigma);\, J'(g) = J'(g, \daleth_\sigma).$$

These are C^∞-functions on G_σ with values in $\mathrm{GL}(n,\mathbb{C})$, and any polynomial function of J and J' is annihilated by $\mathfrak{p}^-_\sigma$ and is contained in a finite-dimensional $\mathfrak{k}_\sigma$ subrepresentation of $C^\infty(G_\sigma)$. Similarly, let

$$j(g) = \det(J(g));\;\; j'(g) = \det(J'(g)),$$

viewed as C^∞-functions on G_σ with values in $\mathbb{C}^\times$.

Let $\chi = \|\bullet\|^m \cdot \chi_0$ be an algebraic Hecke character of $\mathcal{K}$, where $m \in \mathbb{Z}$ and

$$\chi_{0,\sigma}(z) = z^{-a(\chi_\sigma)}\bar{z}^{-b(\chi_\sigma)}.$$

for any archimedean place σ. Define $\mathbb{D}^2(\chi_\sigma) = \mathbb{D}^2(m, \chi_{0,\sigma})$ to be the holomorphic $(Lie(G_{4,\sigma}), U_\sigma)$-module with highest U_σ-type

$$\Lambda(\chi_\sigma) = \Lambda(m,\chi_{0,\sigma}) = (m-b(\chi_\sigma), m-b(\chi_\sigma), \ldots, m-b(\chi_\sigma); -m+a(\chi_\sigma), \ldots, -m+a(\chi_\sigma); \bullet)$$

in the notation of [Har97, (3.3.2)]. Here $\bullet$ is the character of the $\mathbb{R}$-split center of U_σ (denoted c in [Har97]), which we omit to specify because it has no bearing on the integral representation of the L-function. We define a map of $(U(\mathfrak{g}_\sigma), U_\sigma)$-modules

$$\iota(\chi_\sigma) : \mathbb{D}^2(\chi_\sigma) \to C^\infty(G_\sigma) \tag{4.4.5}$$

as follows. Let $v(\chi_\sigma)$ be the tautological generator of the $\Lambda(m,\chi_{0,\sigma})$-isotypic subspace (highest U_σ-type subspace) of $\mathbb{D}^2(m,\chi_\sigma)$. Let

$$\iota(\chi_\sigma)(v(\chi_\sigma)) = J_{\chi_\sigma}(g) := j(g)^{-m+a(\chi_\sigma)} \cdot j'(g)^{-m+b(\chi_\sigma)}\nu(g)^{n(m+a(\chi_\sigma)+b(\chi_\sigma))}$$

and extend this to a map of $U(\mathfrak{g}_\sigma)$-modules. Let $C(G_\sigma, \chi_\sigma)$ denote the image of $\iota(\chi_\sigma)$.

Remark 4.4.3. Note that J_{χ_σ} depends only on the archimedean character $\chi_\sigma = \|\bullet\|_\sigma^m \chi_{0,\sigma}$.

We will only take m in the closed right half-plane bounded by the center of symmetry of the functional equation of the Eisenstein series, as in [Har08]. For such m, the restriction of $\mathbb{D}^2(m,\chi_{0,\sigma})$ to $U_{3,\sigma} = U(a_\sigma, b_\sigma) \times U(b_\sigma, a_\sigma)$ decomposes as an infinite direct sum of irreducible holomorphic discrete series representations of the kind introduced in 4.4.1:

$$\mathbb{D}^2(m,\chi_{0,\sigma}) = \bigoplus_{\kappa_\sigma \in C_3(m,\chi_{0,\sigma})} \mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_{0,\sigma}) = \bigoplus_{\kappa_\sigma \in C_3(\chi_\sigma)} \mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_\sigma) \tag{4.4.6}$$

where $C_3(\chi_\sigma) = C_3(m,\chi_{0,\sigma})$ is a countable set of highest weights:

(4.4.7)

$$C_3(\chi_\sigma) = \{(-m+b(\chi_\sigma) - r_{b_\sigma}, \ldots, -m+b(\chi_\sigma) - r_1; m-a(\chi_\sigma) + s_1, \ldots, m-a(\chi_\sigma) + s_{a_\sigma})\}$$

where

$$r_1 \geq r_2 \geq \cdots \geq r_{b_\sigma} \geq 0; s_1 \geq s_2 \geq \cdots \geq s_{a_\sigma} \geq 0. \tag{4.4.8}$$

(Compare [Har97, Lemma 3.3.7] when $a(\chi_s) = 0$.)

For each $\sigma \in \Sigma$, we define

$$(\alpha(\chi_\sigma), \beta(\chi_\sigma)) = (-m+b(\chi_\sigma), \ldots, -m+b(\chi_\sigma); m-a(\chi_\sigma), \ldots, m_a(\chi_\sigma)) \in \mathbb{Z}^{a_\sigma+b_\sigma}$$

and let

$$(\alpha(\chi), \beta(\chi)) = (\alpha(\chi_\sigma), \beta(\chi_\sigma))_{\sigma\in\Sigma} \tag{4.4.9}$$

For $\kappa = (\kappa_\sigma)_{\sigma\in\Sigma}$, with $\kappa_\sigma \in C_3(\chi_\sigma)$, we define

$$\begin{aligned} \rho_\sigma &= \kappa_\sigma - (\alpha(\chi_\sigma), \beta(\chi_\sigma)) = (-r_{b_\sigma}, \ldots, -r_1; s_1, \ldots, s_{a_\sigma}); \\ \rho_\sigma^\upsilon &= (r_1, \ldots, r_{b_\sigma}; s_1, \ldots, s_{a_\sigma}); \rho = (\rho_\sigma)_{\sigma\in\Sigma}; \rho^\upsilon = (\rho_\sigma^\upsilon)_{\sigma\in\Sigma} \end{aligned} \tag{4.4.10}$$

The involution υ on the parameters (r_i, s_j) corresponds to an algebraic involution, also denoted υ, of the torus T.

The algebraic characters ρ, ρ^{υ}, and κ all determine one another and will be used in the characterization of the Eisenstein measure in subsequent sections.

Note that the twist by $\chi_{0,\sigma}$ coincides with the twist by χ_σ because the norm of the determinant is trivial on $U(b_\sigma, a_\sigma)$. We prefer to write the twist by χ_σ, which is more appropriate for parametrizing automorphic representations of unitary similitude groups.

Lemma 4.4.4. *For such χ, the map $\iota(\chi_\sigma)$ of 4.4.5 is injective for all σ. In particular, the image $C(G_\sigma, \chi_\sigma)$ of $\iota(\chi_\sigma)$ is a free $U(\mathfrak{p}_\sigma^+) \xrightarrow{\sim} S(\mathfrak{p}_\sigma^+)$-module of rank 1.*

Proof. Indeed, $\mathbb{D}^2(\chi_\sigma)$ is always a free rank one $U(\mathfrak{p}_\sigma^+)$-module, and for m in the indicated range is irreducible as $U(\mathfrak{g}_\sigma)$-module. Since $\iota(\chi_\sigma)$ is not the zero homomorphism, it is therefore injective. □

Definition 4.4.5. *Let $\kappa = (\kappa_\sigma, \sigma \in \Sigma)$, where for each σ, κ_σ is the highest weight of an irreducible representation τ_σ of U_σ. Let $(\chi_\sigma, \sigma \in \Sigma)$ be the archimedean parameter of an algebraic Hecke character χ of $\mathcal{K}$. The pair (κ, χ) (or the triple (κ, m, χ_0)) is* critical *if $\kappa_\sigma \in C_3(\chi_\sigma)$ for all $\sigma \in \Sigma$.*

If π is an anti-holomorphic automorphic representation of G_1 of type κ, we say (π, χ) is critical *if (κ, χ) is critical.*

Remark 4.4.6. When $\mathcal{K}$ is imaginary quadratic, the discussion in [Har97, Section 3] shows that, for fixed π and χ, the set of m such that (π, m, χ_0) is critical is exactly the set of critical values of $L(s + \frac{1}{2}, \pi, \chi)$ greater than or equal to the center of symmetry of the functional equation. The same considerations show that this is true for an arbitrary CM field. The verification is simple but superfluous unless one wants to compare the results of the present paper to conjectures on critical values of L-functions.

Let $v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^\flat \otimes \chi_\sigma}$ denote a highest weight vector in the minimal K_3-type of $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_\sigma)$, relative to a choice of compact maximal tori in $U_{3,\sigma}$ as in 4.4.1. The holomorphic module $\mathbb{D}^2(\chi_\sigma)$ is a free rank one module over $U(\mathfrak{p}_4^+)$, generated by $v(\chi_\sigma) \in \Lambda(\chi_\sigma)$. There is therefore a unique element $\delta_{\chi_\sigma,\kappa_\sigma} \in U(\mathfrak{p}_4^+)$ such that

$$\delta_{\chi_\sigma,\kappa_\sigma} \cdot v(\chi_\sigma) = v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^\flat \otimes \chi_\sigma}. \tag{4.4.11}$$

The differential operator $\delta_{\chi_\sigma,\kappa_\sigma}$ depends on the choice of basis vectors but is otherwise well-defined up to scalar multiples. The module $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_\sigma)$ has a natural rational structure over the field of definition $E(\tau_\sigma, \chi_\sigma)$ of $\tau_\sigma \boxtimes \tau_\sigma^\flat \otimes \chi_\sigma$. Let $span(v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^\flat \otimes \chi_\sigma})$ denote the $E(\tau_\sigma, \chi_\sigma)$-line in $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_\sigma)$ spanned by the indicated vector. We always choose $v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^\flat \otimes \chi_\sigma}$ to be rational over $E(\tau_\sigma, \chi_\sigma)$.

4.4.7. *Holomorphic projection.* We let $pr_{\kappa,\sigma} : \mathbb{D}^2(\chi_\sigma) \to \mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^\flat \otimes \chi_\sigma)$ denote the natural projection and

$$pr_{\kappa,\sigma}^{hol} = pr_{\kappa,\sigma}^{hol;a_\sigma,b_\sigma} : \mathbb{D}^2(\chi_\sigma) \to span(v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^\flat \otimes \chi_\sigma})$$

denote $pr_{\kappa,\sigma}$ followed by orthogonal projection on the highest weight component of the holomorphic subspace. Let

$$\mathbb{D}^2(\chi_\sigma)^{hol;a_\sigma,b_\sigma} = \bigoplus_{\kappa_\sigma\in C_3(m,\chi_\sigma)} im(pr^{hol}_{\kappa,\sigma})$$

and let

$$pr^{hol} = \bigoplus pr^{hol}_{\kappa,\sigma} : \mathbb{D}^2(\chi_\sigma) \to \mathbb{D}^2(\chi_\sigma)^{hol;a_\sigma,b_\sigma}.$$

Because we have chosen h standard, the enveloping algebra $U(\mathfrak{g}_\sigma)$ and its subalgebra $U(\mathfrak{p}^+_{4,\sigma}) \simeq S(\mathfrak{p}^+_{4,\sigma})$ have models over $\mathcal{O}_\sigma$. and we define an isomorphism of $\mathcal{O}_\sigma$ algebras

$$S(\mathfrak{p}^+_{4,\sigma}) \xrightarrow{\sim} \mathcal{P}(n)_\sigma \tag{4.4.12}$$

using the identification of section 4.4.2.

Let $n = a_\sigma + b_\sigma$ be a signature at σ. We write $X \in M_n$ in the form $X = \begin{pmatrix} A(X) & B(X) \\ C(X) & D(X) \end{pmatrix}$ with $A(X) \in M_{a_\sigma}$ (an $a_\sigma \times a_\sigma$) block, $D(X) \in M_{b_\sigma}$, and $B(X)$ and $C(X)$ rectangular matrices. With respect to this decomposition and the isomorphism (4.4.12) we obtain a natural map

$$j(a_\sigma, b_\sigma) : \mathcal{P}(a_\sigma)_\sigma \otimes \mathcal{P}(b_\sigma)_\sigma \hookrightarrow \mathcal{P}(n)_\sigma \xrightarrow{\sim} U(\mathfrak{p}^+_{4,\sigma}).$$

For $i = 1, \dots, a_\sigma$ (resp. $j = 1, \dots, b_\sigma$) let $\Delta_i(X)$ (resp. $\Delta'_j(X)$) be the element of $\mathcal{P}(a_\sigma)_\sigma$ (resp. $\mathcal{P}(b_\sigma)_\sigma$) given by the ith minor of A (resp. the jth minor of D) starting from the upper left corner. Let $r_{1,\sigma} \geq \dots \geq r_{a_\sigma,\sigma} \geq r_{a_\sigma+1,\sigma} = 0$, $s_{1,\sigma} \geq \dots \geq s_{b_\sigma,\sigma} \geq s_{b_\sigma+1,\sigma} = 0$ be descending sequences of integers as in Inequalities (4.4.8). Let

$$\tilde{r}_{i,\sigma} = r_{i,\sigma} - r_{i+1,\sigma}, i = 1, \dots, a_\sigma; \tilde{s}_{j,\sigma} = r_{j,\sigma} - r_{j+1,\sigma}, j = 1, \dots, b_\sigma.$$

and define $p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma) \in \mathcal{P}(n)_\sigma$ by

$$p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)(X) = j(a_\sigma, b_\sigma)(\prod_{i=1}^{a_\sigma} \Delta_i(X)^{\tilde{r}_{i,\sigma}} \cdot \prod_{j=1}^{b_\sigma} \Delta'_j(X)^{\tilde{s}_{j,\sigma}}) \tag{4.4.13}$$

Let $\delta(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma) \in U(\mathfrak{p}^+_\sigma)$ be the differential operator corresponding to $p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)$ under the isomorphism (4.4.12).

The group $GL(a_\sigma) \times GL(a_\sigma)$ (resp. $GL(b_\sigma) \times GL(b_\sigma)$) acts on $\mathcal{P}(a_\sigma)_\sigma$ (resp. $\mathcal{P}(b_\sigma)_\sigma$) by the map $(g_1, g_2)(X) = {}^t g_1^{-1} X g_2$, and the action preserves the grading by degree. With respect to the standard upper-triangular Borel subgroups, we can index representations of $GL(a_\sigma)$ (resp. $GL(b_\sigma)$) by their highest weights, which are a_σ-tuples of integers $r_1 \geq r_2 \geq \dots \geq r_{a_\sigma}$ (resp. b_σ-tuples $s_1 \geq s_2 \geq \dots \geq s_{b_\sigma}$). The following is a statement of classical Schur-Weyl duality:

Lemma 4.4.8. *Let $u = a_\sigma$ or b_σ. As a representation of $GL(u) \times GL(u)$, the degree d-subspace $\mathcal{P}(u)^d_\sigma \subset \mathcal{P}(u)_\sigma$ decomposes as the direct sum*

$$\mathcal{P}(u)_\sigma \xrightarrow{\sim} \bigoplus_\mu [F^{\mu,\vee} \otimes F^\mu]$$

where μ runs over r-tuples $c_1 \geq c_2 \geq \cdots \geq c_u \geq 0$ such that $\sum_i c_i = d$. Moreover, if $\mu = c_1 \geq c_2 \geq \cdots \geq c_u \geq c_{u+1} = 0$, the highest weight space $\mathcal{F}^{\mu,+} \subset [F^{\mu,\vee} \otimes F^{\mu}]$ is spanned by the polynomial $\Delta^{\mu} = \prod_{i=1}^{r} \Delta_i^{c_i - c_{i+1}}$.

Proof. This is the case $n = k = r$ of Theorem 5.6.7 of [GW09]. □

Define the (one-dimensional) highest weight space $\mathcal{F}^{\mu,+}$ as in the statement of the lemma, and write

$$\mathcal{P}(u)_{\sigma}^{+} = \bigoplus_{\mu} \mathcal{F}^{\mu,+}.$$

Recall the notation of (4.4.10).

Corollary 4.4.9. *Let (κ,χ) be critical. For each $\sigma \in \Sigma$, there is a unique $a_\sigma + b_\sigma$-tuple*

$$\rho_{\sigma}^{\upsilon} = (r_{1,\sigma} \geq \cdots \geq r_{a_\sigma,\sigma} \geq 0; s_{1,\sigma} \geq \cdots \geq s_{b_\sigma,\sigma} \geq 0)$$

as above such that

$$pr_{\kappa,\sigma}^{hol}(\delta(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{s}) \cdot v(\chi_\sigma)) = P_{\kappa_\sigma,\chi,\sigma} \cdot v_{\kappa_\sigma} \otimes v_{\kappa_\sigma^{\flat} \otimes \chi_\sigma}$$

with $P_{\kappa_\sigma,\chi,\sigma}$ a non-zero scalar in $E(\tau_\sigma,\chi_\sigma)^{\times}$.

We write

$$D(\rho_{\sigma}^{\upsilon}) = D(\kappa_\sigma,\chi_\sigma) = \delta(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{\sigma}), D(\rho^{\upsilon}) = D(\kappa,\chi) = \prod_{\sigma} D(\kappa_\sigma,\chi_\sigma)$$

and

$$D^{hol}(\rho_{\sigma}^{\upsilon}) = D^{hol}(\kappa_\sigma,\chi_\sigma) = pr_{\kappa,\sigma}^{hol}\delta(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{\sigma}), D^{hol}(\rho^{\upsilon}) = D^{hol}(\kappa,\chi) = \prod_{\sigma} D^{hol}(\kappa_\sigma,\chi_\sigma)$$

for these choices of $(r_{i,\sigma}; s_{j,\sigma})$. Then for all $\kappa^{\dagger} \leq \kappa$ there exist unique elements $\delta(\kappa,\kappa^{\dagger}) \in U(\mathfrak{p}_3^{+})$, defined over algebraic number fields, such that

$$D(\kappa,\chi) = \sum_{\kappa^{\dagger} \leq \kappa} \delta(\kappa,\kappa^{\dagger}) \circ D^{hol}(\kappa',\chi);$$

$\delta(\kappa,\kappa)$ is the scalar $\prod_{\sigma} P_{\kappa_\sigma,\chi,\sigma}$.

Proof. Consider $j(a_\sigma,b_\sigma)(\mathcal{P}(a_\sigma)_{\sigma}^{+} \otimes \mathcal{P}(b_\sigma)_{\sigma}^{+}) \subset \mathcal{P}(n)_{\sigma}$. This is the space spanned by the $p(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{\sigma})$ defined in (4.4.13). Let $\delta(a_\sigma,b_\sigma)^{+} \subset U(\mathfrak{p}_{\sigma}^{+})$ be the subspace identified with $j(a_\sigma,b_\sigma)(\mathcal{P}(a_\sigma)_{\sigma}^{+} \otimes \mathcal{P}(b_\sigma)_{\sigma}^{+})$ by the isomorphism (4.4.12). The decomposition (4.4.6) is based on the fact that the composition

$$\delta(a_\sigma,b_\sigma)^{+} \otimes v(\chi_\sigma) \hookrightarrow \mathbb{D}^2(\chi_\sigma) \overset{pr^{hol}}{\to} \mathbb{D}^2(\chi_\sigma)^{hol;a_\sigma,b_\sigma}$$

is an isomorphism. See the disscussion in section 7.11 of [Har86].

This *does not say* that $\delta(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{\sigma}) \otimes v(\chi_\sigma)$ lies in the highest weight space of the holomorphic subspace of the direct factor $\mathbb{D}(\kappa_\sigma) \otimes \mathbb{D}(\kappa_\sigma^{\flat} \otimes \chi_\sigma)$ corresponding to the $a_\sigma + b_\sigma$-tuple $(\underline{\tilde{r}}_{\sigma}, \underline{\tilde{s}}_{\sigma})$; but it does say that its projection on that highest weight space is non-trivial. This is equivalent to the first statement of the corollary. The remaining statements are formal consequences of the decomposition (4.4.6) and the fact that the decomposition is rational over an appropriate reflex field, cf. Lemma 7.3.2 of [Har86]. □

4.4.10. *Differential operators on C^∞-modular forms.* Let $\chi = \|\bullet\|^m\chi_0$ be an algebraic Hecke character of $\mathcal{K}$, as before. We view G_4 as the rational similitude group of a maximally isotropic hermitian space V_4; this allows us to write $Sh(V_4)$ for the corresponding Shimura variety. Let $\Lambda(\chi) = (\Lambda(\chi_\sigma), \sigma \in \Sigma)$ be the character of U_∞ whose restriction to U_σ is $\Lambda(\chi_\sigma)$. Let $L(\chi)$ be the 1-dimensional space on which U_∞ acts by $\Lambda(\chi)$; it can be realized over a number field $E(\chi_\infty)$ which depends only on χ_∞. The dual of the highest U_∞-type $\Lambda(\chi)$, restricted to the intersection of U_∞ with $G_4(\mathbb{R})$, defines an automorphic line bundle $\mathcal{L}(\chi)$ on $Sh(V_4)$ with fiber at the fixed point h of U_∞ isomorphic to $L(\chi)$. If $\pi = \pi_\infty \otimes \pi_f$ is an automorphic representation of G_4, with π_∞ a $(Lie(G_4), U_\infty)$ module isomorphic to $\mathbb{D}^2(\chi) = \otimes_{\sigma\in\Sigma}(\mathbb{D}^2(\chi_\sigma)$ and π_f an irreducible smooth representation of the finite adeles of G_4, then there a canonical embedding

$$\pi_f \xrightarrow{\sim} \pi_f \otimes H^0(\mathfrak{P}_h, U_\infty; \mathbb{D}^2(\chi) \otimes L(\chi)) \hookrightarrow H^0(Sh(V_4)^{tor}, \mathcal{L}(\chi)^{can}). \tag{4.4.14}$$

Write $\Omega = \Omega_{Sh(V_4)}$ for the cotangent bundle. For any integer $d \geq 0$, and for any ring $\mathcal{O}$, let $\mathcal{P}(n)^d(\mathcal{O})$ denote the $\mathcal{O}$-module of $\mathcal{O}$-valued polynomials on the matrix space M_n, and let $\mathcal{P}(n)^{d,*}(\mathcal{O}) = Hom_{\mathcal{O}}(\mathcal{P}(n)^d, \mathcal{O})$ denote the dual $\mathcal{O}$-module. There is a canonical action of U_∞ on $\mathcal{P}(n)^d$, for every d, defined over the field of definition $E(h)$ of the standard CM point h stabilized by U_σ, and even over its integer ring. The Maass operator of degree d, as defined in section 7.9 of [Har86] is a C^∞-differential operator

$$\delta_\chi^d : \mathcal{L}(\chi) \to \mathcal{L}(\chi) \otimes Sym^d\Omega \tag{4.4.15}$$

We can view the target of δ_χ^d as the automorphic vector bundle attached to the representation $L(\chi) \otimes \mathcal{P}(n)^{d,*}$ of U_∞, using the identification of section 4.4.2 as in (4.4.12). We use the same notation to denote the action on the space $\mathcal{A}(G_4)$ of (not necessarily cuspidal) automorphic forms on G_4:

$$\delta_\chi^d : \mathcal{A}(G_4, L(\chi)) \to \mathcal{A}(G_4, L(\chi) \otimes \mathcal{P}(n)^{d,*}) \tag{4.4.16}$$

where the notation denotes automorphic forms with values in the indicated vector space. For any polynomial $\phi \in \mathcal{P}(n)^d$ we thus obtain a differential operator

$$\delta_\chi^d(\phi) : \mathcal{A}(G_4, L(\chi)) \to \mathcal{A}(G_4, L(\chi) \otimes \mathcal{P}(n)^{d,*}); \delta_\chi^d(\phi)(f) = [\delta_\chi^d(f) \otimes \phi] \tag{4.4.17}$$

where the bracket denotes contraction $\mathcal{P}(n)^{d,*} \otimes \mathcal{P}(n)^d \to E(h)$.

Finally, for each σ define sequences $\underline{\tilde{r}}_\sigma$ and $\underline{\tilde{s}}_\sigma$ as in section 4.4.7; let $\underline{\tilde{r}} = (\underline{\tilde{r}}_\sigma)$, $\underline{\tilde{s}} = (\underline{\tilde{s}}_\sigma)$. Suppose $\sum_\sigma[\sum_i \tilde{r}_{i,\sigma} + \sum_j \tilde{s}_{j,\sigma}] = d$. Then we define $p(\underline{\tilde{r}}, \underline{\tilde{s}}) = \prod_\sigma p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)$ where the factors are as in (4.4.13), and let

(4.4.18)

$$\delta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}}) = \delta_\chi^d(p(\underline{\tilde{r}}, \underline{\tilde{s}})) : H^0(Sh(V_4)^{tor}, \mathcal{L}(\chi)^{can}) \to H^{0,\infty}(Sh(V_4), \mathcal{L}(\chi) \otimes Sym^d\Omega).$$

Here $H^{0,\infty}$ denotes the space of C^∞-sections. Under the isomorphisms (4.4.14), $\delta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}})$ is identified with the operator on the left hand side deduced from multiplying by the element

$p(\underline{\tilde{r}}, \underline{\tilde{s}})$, viewed as an element of $Sym^d\mathfrak{p}_4^+$, which maps $H^0(\mathfrak{P}_h, U_\infty; \mathbb{D}^2(\chi) \otimes L(\chi)) = \otimes_\sigma \mathbb{C}v_{\chi_\sigma} \otimes L(\chi)$ to $p(\underline{\tilde{r}}, \underline{\tilde{r}}) \otimes \otimes_\sigma \mathbb{C}v_{\chi_\sigma} \otimes L(\chi) \in \mathbb{D}^2(\chi) \otimes L(\chi))$.

The holomorphic differential operators of Corollary 4.4.9 define operators on automorphic forms, as follows. Let $S^\infty_{\kappa,V}(K_1, \mathbb{C})$ denote the space of C^∞ modular forms of type κ on $Sh(V_1)$, of level K_1, and define $S^\infty_{\kappa^\flat,-V}(K_2, \mathbb{C})$ analogously. The following Proposition restates Proposition 7.11.11 of [Har86]:

Proposition 4.4.11. *Let (κ, χ) be critical as in Corollary 4.4.9. Fix a level subgroup $K_4 \subset G_4(\mathbf{A}_f)$ and a subgroup $K_1 \times K_2 \subset G_3(\mathbf{A}_f) \cap K_4$. There are differential operators*

$$D(\kappa, \chi) : H^0({}_{K_4}Sh(V_4)^{tor}, \mathcal{L}(\chi)^{can}) \to S^\infty_{\kappa,V}(K_1, \mathbb{C}) \otimes S^\infty_{\kappa^\flat,-V}(K_2, \mathbb{C}) \otimes \chi \circ \det;$$

$$D^{hol}(\kappa, \chi) : H^0({}_{K_4}Sh(V_4)^{tor}, \mathcal{L}(\chi)^{can}) \to S_{\kappa,V}(K_1, \mathbb{C}) \otimes S_{\kappa^\flat,-V}(K_2, \mathbb{C}) \otimes \chi \circ \det$$

which give the operators $\delta^d_\chi(\underline{\tilde{r}}, \underline{\tilde{s}})$ and $pr^{hol} \circ \delta^d_\chi(\underline{\tilde{r}}, \underline{\tilde{s}})$ upon pullback to functions on $G_4(\mathbf{A})$ and restriction to $G_3(\mathbf{A})$.

4.4.12. *The Hodge polygon.* If π is a cuspidal automorphic representation of G whose component at σ is an anti-holomorphic discrete series representation of the form $\mathbb{D}_c(\tau_\sigma)$, then its base change Π to an automorphic representation of $\mathrm{GL}(n)_\mathcal{K}$ (ignoring the split center) is cuspidal, cohomological, and satisfies $\Pi^\vee \xrightarrow{\sim} \Pi^c$, and therefore the associated ℓ-adic Galois representations have associated motives (in most cases), realized in the cohomology of Shimura varieties attached to unitary groups, with specified Hodge structures. In what follows, we fix σ and attach a Hodge structure to the anti-holomorphic representation $\mathbb{D}_c(\tau)$, according to the rule used to assign a motive to Π. The Hodge structure is pure of weight $n-1$ and has the following Hodge types, each with multiplicity one:

(4.4.19)

$$(\kappa_1 - b + n - 1, b - \kappa_1), \ldots, (\kappa_a, n - 1 - \kappa_a), (n - 1 - \kappa^c_b, \kappa^c_b + a), \ldots, (-\kappa^c_1, \kappa^c_1 + n - 1),$$
$$(\kappa^c_1 + n - 1, -\kappa^c_1), \ldots, (\kappa^c_b + a, n - 1 - \kappa^c_b), (n - 1 - \kappa_a, \kappa_a), \ldots, (b - \kappa_1, \kappa_1 - b + n - 1).$$

Label the pairs in (4.4.19) (p_i, q_i), $i = 1, \ldots, 2n$, in order of appearance; thus (p_i, q_i) is in the top row if and only if $i \leq n$.

Hypothesis 4.4.13 (Critical interval hypothesis)**.** *We assume that the weights (κ, κ^c) are adapted to the signature (a, b) in the sense that, for every pair (p_i, q_i) in the collection (4.4.19), $p_i \neq q_i$ and $p_i > q_i$ if and only if $i \leq n$.*

One checks that Hypothesis 4.4.13 holds if and only if $2\kappa_a > n-1$ and $-2\kappa^c_1 > n-1$. We define the *Hodge polygon* $Hodge(\kappa, \kappa^c) = Hodge(\mathbb{D}_c(\tau))$, to be the polygon in the right half-plane connecting the vertices (i, p_i) with (p_i, q_i) the pairs in (4.4.19).

4.4.14. *Specific anti-holomorphic vectors.* When τ_σ is strongly positive with highest weight $\kappa = \kappa_\sigma$, we write $\mathbb{D}(\kappa) = \mathbb{D}(\tau_\sigma)$, $\mathbb{D}_c(\kappa) = \mathbb{D}_c(\tau_\sigma)$ when it's clear that κ is a weight and τ_σ is an irreducible representation. Let π be a cuspidal automorphic representation of G with $\pi_\sigma = \mathbb{D}_c(\kappa)$ as above. In the computation of the zeta integral, we use a factorizable automorphic form $\varphi = \otimes_v \varphi_v \in \pi$, with φ_v a vector in the minimal U_σ-type $1 \otimes W^\vee_{\tau_\sigma}$ of $\mathbb{D}_c(\kappa)$. In practice, we choose φ_v to be either the highest weight vector $\varphi_{\kappa,+}$ or the lowest weight vector $\varphi_{\kappa,-}$ in $1 \otimes W^\vee_{\tau_\sigma}$. If w_0 is the longest element of the Weyl group of T_σ relative to R^+_σ, then $\varphi_{\kappa,+}$ (resp. $\varphi_{\kappa,-}$) is an eigenvector for T_σ of weight $-w_0(\kappa)$ (resp. of weight $-\kappa$).

4.5. **Local zeta integrals at archimedean places.**

4.5.1. *Choices of local data.* This material has been covered at length in [Har97] and [Har08], so we can afford to be brief. Notation for induced representations is as in Section 4.1.2 above. The notation for holomorphic representations is as in Section 4.4.2. An easy computation, similar to that in [Har97], yields

Lemma 4.5.2. *As subspaces of* $C^\infty(G_\sigma)$, $\iota(m,\chi_\sigma)(\mathbb{D}^2(m,\chi_\sigma)) \subset I_\sigma(m - \frac{n}{2}, \chi)$.

Remark 4.5.3. Note that we have omitted similitude factors here. Strictly speaking, these should be included; but they do not change the theory in any significant way.

4.5.4. *Non-vanishing of* I_∞. Let σ be an archimedean place, $f_\sigma = f_\sigma(\chi_\sigma, c) \in I(\chi_{u,\sigma}, m)$ the local section at σ. We assume f_σ is of the form

$$f_\sigma(\chi_\sigma, c, g) = B(\chi_\sigma, \kappa_\sigma) D(\kappa_\sigma, m, \chi_{u,\sigma}) J_{m,\chi_{u,\sigma}}(g), g \in G_{4,\sigma} \tag{4.5.1}$$

where $J_{m,\chi_{0,\sigma}} \in C^\infty(G_4)$ is the canonical automorphy factor introduced in section 4.4.2 and $B(\chi_\sigma, \kappa_\sigma)$ is a non-zero scalar. Let $\varphi_\sigma \otimes \varphi^\flat_\sigma$ be an anti-holomorphic vector in the highest weight subspace of the minimal K_σ-type of $\pi_\sigma \otimes \pi^\flat_\sigma$.

Proposition 4.5.5. *The local factor* $I_\sigma(\varphi_\sigma, \varphi^\flat_\sigma, f_\sigma, m)$ *is not equal to* 0.

Proof. If $D(\kappa_\sigma, \chi_\sigma)$ is replaced by $D^{hol}(\kappa_\sigma, \chi_\sigma)$ in (4.5.1), this follows from Remark (4.4)(iv) of [Har08]. Since $\varphi_\sigma \otimes \varphi^\flat_\sigma$ is an anti-holomorphic vector, the pairing of (the Eisenstein section) $D(\kappa_\sigma, \chi_\sigma) J_{m,\chi_\sigma,\sigma}$ with (the highest weight vector) $\varphi_\sigma \otimes \varphi^\flat_\sigma$ factors through the projection of $D(\kappa_\sigma, \chi_\sigma) J_{m,\chi_\sigma,\sigma}$ onto $D^{hol}(\kappa_\sigma, \chi_\sigma) J_{m,\chi_\sigma,\sigma}$. The Proposition is thus a consequence of Corollary 4.4.9. □

When the extreme K-type $\tau_\sigma = \tau_{a_\sigma,\sigma} \otimes \tau_{b_\sigma,\sigma}$ in π_σ is one-dimensional, the archimedean zeta integrals have been computed in [Shi97, Shi00]. Garrett has shown in [Gar08] that the archimedean zeta integrals are algebraic up to a predictable power of the transcendental number π. When at least one of the two factors ($\tau_{a_\sigma,\sigma}$, $\tau_{b_\sigma,\sigma}$) of the extreme K-type is one-dimensional, the archimedean zeta integrals are given precisely on [Gar08, p. 12]; and furthermore, Garrett showed in [Gar08] that when both factors are scalars, the archimedean zeta integrals are non-zero algebraic numbers. The computations of the zeta integrals have not been carried out in the more general case (i.e. when neither $\tau_{a_\sigma,\sigma}$ nor $\tau_{b_\sigma,\sigma}$ is one-dimensional), but in any case, the zeta integrals at σ depend only upon the local data at σ.

The following result is due to Garrett [Gar08].

Proposition 4.5.6. *Let $I_\sigma(\chi_\sigma,\kappa_\sigma)$ be the local zeta integral*

$$I_\sigma(\chi_\sigma,\kappa_\sigma) = I_\sigma(\varphi_\sigma,\varphi_\sigma^\flat,f_\sigma,m),$$

where $\varphi_\sigma = \varphi_{\kappa_\sigma,-}$, $\varphi_\sigma^\flat = \varphi_{\kappa_\sigma^\flat,-}$ and f_σ is given by (4.5.1). Then $I_\sigma(\chi_\sigma,\kappa_\sigma)$ is a non-zero algebraic number.

Remark 4.5.7. When κ_σ is a scalar representation, Shimura obtains an explicit formula for the local zeta integral. In general, as explained at the end of [Har08, Section 5], Garrett's calculation actually determines the value of the integral up to an element of a specific complex embedding of the CM field F. In that paper F is imaginary quadratic, but the same reasoning applies in general. Undoubtedly the calculation actually gives a rational number, but the method is based on the choice of rational structures on U_σ and the aforementioned differential operators. We do not need to use this more precise information here.

4.6. **The global formula.** We have now computed all the local factors of the Euler product (4.1.5). The Proposition below summarizes the result of our computation. Bear in mind that, although we write $\varphi\in\pi$, we actually mean that $\varphi\in\underline{\pi}$, where the latter is the irreducible $U_1(\mathbb{A})$ constituent of π chosen as in (4.1.1).

First, write $\chi = \|\bullet\|^m\cdot\chi_u$ with χ_u a unitary Hecke character of $\mathcal{K}$. Denote by χ^+ the restriction of χ_u to the idèles of $\mathcal{K}^+$; it is a character of finite order. Let $\eta = \eta_{\mathcal{K}/\mathcal{K}^+}$ denote the quadratic idèle class character of $\mathcal{K}^+$ attached to the quadratic extension $\mathcal{K}/\mathcal{K}^+$. For any finite place v of $\mathcal{K}^+$, define the Euler factor

$$D_v(\chi) = \prod_{r=0}^{n-1} L_v(2m+n-r,\chi^+\cdot\eta^r).$$

For any finite set S of finite places, let

$$D^S(\chi) = \prod_{v\notin S} D_v(\chi);\ \ D(\chi) = D^{\varnothing}(\chi), \tag{4.6.1}$$

where the product is taken over finite places.

Proposition 4.6.1. *Let the test vectors $\varphi\in\pi$ and $\varphi^\flat\in\pi^\flat$ be chosen to be factorizable vectors as in (4.1.2), with the local components at p and ∞ given as in (4.1.3) and (4.1.4), respectively. Assume the local components at finite places outside $S = S_\pi$ are unramified vectors, and the local choices at ramified places are as in 4.2.2. Moreover, assume the Siegel-Weil section $f_s\in I(\chi,s)$ is chosen as in the preceding sections. Write $\chi = \|\bullet\|^m\chi_u$. Then we have the equality*

$$D(\chi)\cdot I(\varphi,\varphi^\flat,f,s) = \langle\varphi,\varphi^\flat\rangle\cdot I_p(\chi,\kappa)I_\infty(\chi,\rho^v)I_S L^S(s+\frac{1}{2},\pi,\chi_u)$$

where

$$I_S = \prod_{v\in S} D_v(\chi)\cdot \mathrm{volume}(\mathcal{U}_v),$$

$$I_\infty(\chi,\kappa) = \prod_\sigma I_\sigma(\chi_\sigma,\kappa_\sigma)$$

is the product of factors described in Proposition 4.5.6,

$$I_p = L_p(s, \mathrm{ord}, \pi, \chi) \coloneqq \prod_{w|p} L(s, \mathrm{ord}, \pi_w, \chi_w),$$

and $\langle \bullet, \bullet \rangle$ *is the* L^2 *inner product on cusp forms.*

A. Appendix: The definite case, revisited

We take another look at the definite case. This is the case where $a_{\sigma,i}b_{\sigma,i} = 0$ for all $\sigma \in \Sigma_{\mathcal{K}}$ and all $i = 1,\dots,m$. In this case the schemes M_K, the modular forms, and the p-adic modular forms have simple descriptions. For each $i = 1,\dots,m$, let $\Sigma_i = \{\sigma \in \Sigma_{\mathcal{K}} : a_{\sigma,i} > 0\}$; this is a CM type for $\mathcal{K}$. Let $\Sigma_{p,i} = \{w|p : w \text{ is determined by } incl_p \circ \sigma \text{ for some } \sigma \in \Sigma_i\}$. Note that $w \in \Sigma_{p,i}$ if and only if $\bar{w} \notin \Sigma_{p,i}$, as $w \in \Sigma_{p,i}$ if and only if $a_{w,i} > 0$ which holds if and only if $a_{\bar{w},i} = b_{w,i} = 0$. In particular, Σ_i is ordinary at $incl_p$ in the sense of [Kat78, Section (5.1)]. Furthermore, it follows from the isomorphism (2.2.2) that

$$G_{/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{i=1}^{m} \prod_{w \in \Sigma_{p,i}} \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w), \quad (g,\nu) \mapsto (\nu, (g_{w,i})). \tag{A.0.2}$$

Since $e_i L_w = e_i L_w^+$ if $w \in \Sigma_{p,i}$ and otherwise $e_i L_w = e_i L_w^-$, $L^+ = \prod_{i=1}^m \prod_{w \in \Sigma_{p,i}} e_i L_w$, it follows that

$$H = \mathrm{GL}_{\mathcal{O}_B \otimes \mathbb{Z}_p}(L^+) \xrightarrow{\sim} \prod_{i=1}^{m} \prod_{w \in \Sigma_{p,i}} \mathrm{GL}_{\mathcal{O}_{i,w}}(e_i L_w), \tag{A.0.3}$$

and hence

$$G_{/\mathbb{Z}_p} \xrightarrow{\sim} \mathbb{G}_m \times H. \tag{A.0.4}$$

Combining isomorphism (A.0.4) with isomorphism (2.9.1) yields an identification

$$G_{/\mathbb{Z}_p} \xrightarrow{\sim} H_{0/\mathbb{Z}_p}. \tag{A.0.5}$$

A.1. **Some CM abelian varieties.** The CM type Σ_i defines a complex structure on $V_i = \mathcal{O} \otimes \mathbb{R}$; the complex structure is that defined by transport of structure via the identification $V_i = \prod_{\sigma \in \Sigma_i} \mathbb{C}$, $x \otimes r \mapsto (\sigma(x) r)$. Then the canonical projection $V_i \xrightarrow{\sim} W_i = \mathcal{O} \otimes \mathbb{C} / (\mathcal{O} \otimes \mathbb{C})^0$ is a $\mathbb{C}$-linear isomorphism, where the superscript 0 denotes the degree 0 part of the Hodge filtration on $\mathcal{O} \otimes \mathbb{C} = V_i \otimes_{\mathbb{R}} \mathbb{C}$ for the given complex structure on V_i.

Let A_i be the abelian variety $A_i = V_i / \mathcal{O}$ with the complex structure on V_i being that defined by Σ_i. Then A_i has complex multiplication by $\mathcal{O}$, which acts through its canonical action on V_i; let $\iota_i : \mathcal{O} \hookrightarrow \mathrm{End}(A_i)$ be this action. A natural p^r-level structure on A_i is given by

$$\phi_{i,r} : \left(\prod_{w \in \Sigma_{p,i}} \mathcal{O}_w \right) \otimes \mu_{p^r} \xrightarrow{\sim} \prod_{w \in \Sigma_{p,i}} \frac{1}{p^r} \mathcal{O}_w / \mathcal{O}_w \hookrightarrow \prod_{w|p} \frac{1}{p^r} \mathcal{O}_w / \mathcal{O}_w = \frac{1}{p^r} \mathcal{O} / \mathcal{O} = A_i[p^r],$$

$$(v_w) \otimes e^{2\pi\sqrt{-1}/p^r} \mapsto \left(\frac{v_w}{p^r} \bmod \mathcal{O}_w \right)_{w \in \Sigma_{p,i}}.$$

By the theory of complex multiplication, each triple $(A_i, \iota_i, \phi_{i,r})$ has a model over $\overline{\mathbb{Q}}$, unique up to isomorphism. Since Σ_i is ordinary for $incl_p$ these triples have good reduction at the place determined by $incl_p$ (and at all places not dividing p) and so extend to triples over $\overline{\mathbb{Z}}_{(p)}$ that are compatible with varying r; we continue to denote the triples as $(A_i, \iota_i, \phi_{i,r})$. Note that over a p-adic $\overline{\mathbb{Z}}_{(p)}$-algebra (e.g., $\mathcal{O}_{\mathbb{C}_p}$ or $\overline{\mathbb{Z}}_{(p)}/p^m\overline{\mathbb{Z}}_{(p)}$) the image of $\phi_{i,r}$ is $A_i[p^r]^\circ$.

Let $\Lambda_i \subset W_i$ be the image of $\mathcal{O}\otimes S_0$; this is an $\mathcal{O}$-stable free S_0-submodule such that $\Lambda_i \otimes_{S_0} \mathbb{C} = W_i$. Let $\Omega_{A_i} = \mathrm{Lie}(A_i)^\vee$. Over $\mathbb{C}$, there is a canonical isomorphism of $\mathcal{O}\otimes\mathbb{C} = \prod_{\sigma\in\Sigma_i}\mathbb{C}$-modules

$$\Omega_{A_i/\mathbb{C}} = \mathrm{Hom}_{\mathbb{C}}(V_i, \mathbb{C}) = \prod_{\sigma\in\Sigma_i}\mathbb{C}e_{\sigma,i}, \quad e_{\sigma,i}(x\otimes r) = \sigma(x)r,$$

which determines an isomorphism of rank one $\mathcal{O}\otimes\mathbb{C}$-modules

$$\begin{aligned}\varepsilon_i : \Omega_{A_i/\mathbb{C}} &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(\Lambda_i, \mathbb{C}(1)) = \mathrm{Hom}_{S_0}\left(\Lambda_i, \overline{\mathbb{Z}}_{(p)}(1)\right)\otimes_{\overline{\mathbb{Z}}_{(p)}}\mathbb{C},\\ (a_\sigma e_\sigma)_{\sigma\in\Sigma_i} &\mapsto \left(x\otimes s \mapsto 2\pi\sqrt{-1}\sum_{\sigma\in\Sigma_i} a_\sigma\sigma(x)s\right).\end{aligned} \tag{A.1.1}$$

Similarly, over $\overline{\mathbb{Z}}_{(p)}$ there is an isomorphism of $\mathcal{O}\otimes\overline{\mathbb{Z}}_{(p)} = \prod_{\sigma\in\Sigma_{\mathcal{K}}}\overline{\mathbb{Z}}_{(p)}$-modules

$$\Omega_{A_i/\overline{\mathbb{Z}}_{(p)}} = \prod_{\sigma\in\Sigma_i}\overline{\mathbb{Z}}_{(p)}\omega_{\sigma,i}$$

with the $\omega_{\sigma,i}$'s determined up to elements of $\overline{\mathbb{Z}}_{(p)}^\times$, and this determines an isomorphism

$$\begin{aligned}\varepsilon_{A_i} : \Omega_{A_i/\overline{\mathbb{Z}}_{(p)}} &\xrightarrow{\sim} \mathrm{Hom}_{S_0}\left(\Lambda_i, \overline{\mathbb{Z}}_{(p)}(1)\right),\\ (a_\sigma\omega_{\sigma,i})_{\sigma\in\Sigma_i} &\mapsto \left(x\otimes s\mapsto 2\pi\sqrt{-1}\sum_{\sigma\in\Sigma_i}a_\sigma\sigma(x)s\right),\end{aligned} \tag{A.1.2}$$

Comparing (A.1.1) and (A.1.2) yields periods $\Omega_i = (\Omega_{\sigma,i}) \in (\mathcal{O}\otimes\mathbb{C})^\times$ such that

$$\varepsilon_i = \Omega_i\cdot\varepsilon_{A_i}. \tag{A.1.3}$$

Note that Ω_i is only determined up to an element of $(\mathcal{O}\otimes\overline{\mathbb{Z}}_{(p)})^\times$; it depends on the choice of the $\omega_{\sigma,i}$'s.

A.2. **The moduli spaces.** In the definite case, the space $X = \{h\}$ consists of a single element. Let $L_i = L\cap e_i(L\otimes\mathbb{Z}_{(p)})$. Then $L' = \prod_{i=1}^m L_i \subset L$ is an $\mathcal{O}_B$-stable lattice and the inclusion $L'\subset L$ has finite, prime-to-p index. Let $L_i^\# \subset e_i(L\otimes\mathbb{Z}_{(p)})$ be the lattice dual to L_i, that is $L_i^\# = \{x\in e_i(L\otimes\mathbb{Z}_{(p)}) \ : \ \langle x, L_i\rangle\subseteq\mathbb{Z}(1)\}$. Then $L_i\subset L_i^\#$ with finite, prime-to-p index, and $L'' = \prod_{i=1}^m L_i^\# \subset L\otimes\mathbb{Z}_{(p)}$ is an $\mathcal{O}_B$-stable lattice containing $L^\#$ with finite, prime-to-p index. For $g\in G(\mathbb{A}_f)$ with $g_p\in G(\mathbb{Z}_p)$, the tuple $\underline{A}_{h,g} = (A_h, \lambda_h, \iota, \eta_g)$, $g\in G(\mathbb{A}_f)$, is equivalent to the tuple $\underline{A}_g = (A, \lambda, \iota', \eta_g')$ with

- $A = L'\otimes\mathbb{R}/L' = \prod_{i=1}^m L_i\otimes_{\mathcal{O}} A_i$; the dual abelian variety is $A^\vee = L''\otimes\mathbb{R}/L'' = \prod_{i=1}^m L_i^\#\otimes_{\mathcal{O}} A_i$;
- λ the isogeny induced by the inclusion $L'\subseteq L''$ and the identity map on $L'\otimes\mathbb{R} = L''\otimes\mathbb{R}$; in particular, $\lambda = \prod_{i=1}^m incl_i\otimes id$, where $incl_i$ is the inclusion $L_i\subset L_i^\#$;
- ι' induced from the action of $\mathcal{O}_B$ on L; in particular, $\iota' = \prod_{i=1}^m can_i\otimes\iota_i$, where can_i is the canonical action of $\mathcal{O}_B$ on L_i;

- η'_g the K^p-orbit of the translation by g^p map $g^p : L \otimes \mathbb{A}_f^p \xrightarrow{\sim} L \otimes \mathbb{A}_f^p = L' \otimes \mathbb{A}_f^p = H_1(A, \mathbb{A}_f^p)$.

The equivalence of $\underline{A}_{h,g}$ and $\underline{A}_g$ is given by the prime-to-p isogeny $A \to A_h$ induced by the inclusion $L' \subset L$ and the identity map on $L \otimes \mathbb{R} = L' \otimes \mathbb{R}$.

Since each (A_i, ι_i) is defined over $\overline{\mathbb{Z}}_{(p)}$, so is $\underline{A}_g$. It follows that

$$\mathrm{M}_{K/\overline{\mathbb{Z}}_{(p)}} = \coprod_{G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K} \operatorname{Spec} \overline{\mathbb{Z}}_{(p)}, \tag{A.2.1}$$

with the $\overline{\mathbb{Z}}_{(p)}$-point corresponding to the class of $g \in G(\mathbb{A}_f)$ representing the equivalence class of the tuple $\underline{A}_g$. In fact, this disjoint union holds with $\overline{\mathbb{Z}}_{(p)}$ replaced by the integral closure of S_0 in some finite extension of the reflex field F (which will depend on the compact open subgroup K in general).

A similar description holds with K replaced by K_r. Note that $L^+ = \prod_{i=1}^m \prod_{w \in \Sigma_{p,i}} L_w = \prod_{i=1}^m \prod_{w \in \Sigma_{p,i}} L_i \otimes_{\mathcal{O}} \mathcal{O}_w$, from which it follows that for $g \in G(\mathbb{A}_f)$ with $g_p \in G(\mathbb{Z}_p)$ the tuple $\underline{X}_{h,g} = \left(\underline{A}_{h,g}, \phi_g\right)$ is equivalent to $\underline{X}_g = \left(\underline{A}_g, \phi'_{r,g}\right)$ with $\phi'_{r,g}$ the g_p-translate of

$$\phi'_r : L^+ \otimes \mu_{p^r} = \prod_{i=1}^m L_i \otimes_{\mathcal{O}} \left(\prod_{w \in \Sigma_{p,i}} \mathcal{O}_w \right) \otimes \mu_{p^r} \hookrightarrow A^\vee [p^r] = \prod_{i=1}^m L_i^\# \otimes_{\mathcal{O}} A_i [p^r],$$

$$\phi'_r = \prod_{i=1}^m id_i \otimes \phi_{i,r}.$$

The action of g_p on L^+ is via the projection to H in isomorphism (A.0.4). The equivalence again comes from the prime-to-p isogeny $A \to A_h$ determined by the inclusion $L' \subset L$ and the identity map on $L' \otimes \mathbb{R} = L \otimes \mathbb{R}$. So

$$\mathrm{M}_{K_r} = \coprod_{G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_r} \operatorname{Spec} \overline{\mathbb{Z}}_{(p)} \tag{A.2.2}$$

with the $\overline{\mathbb{Z}}_{(p)}$-point corresponding to the class of $g \in G(\mathbb{A}_f)$ being the class of $\underline{X}_g$.

A.3. **Modular forms.** We assume that

$$\Lambda_0 = \prod_{i=1}^m L_i \otimes_{\mathcal{O}} \Lambda_i = \prod_{i=1}^m L_i^\# \otimes_{\mathcal{O}} \Lambda_i,$$

so $W = \Lambda_0 \otimes_{S_0} \mathbb{C} = \prod_{i=1}^m L_i^\# \otimes_{\mathcal{O}} \Lambda_i \otimes_{S_0} \mathbb{C} = \prod_{i=1}^m L_i^\# \otimes_{\mathcal{O}} V_i$. The canonical isomorphism $\varepsilon_0 : \Omega_{A^\vee_{/\mathbb{C}}} = \mathrm{Hom}_{S_0}(\Lambda_0, \mathbb{C}(1)) = \Lambda_0^\vee \otimes \mathbb{C}$ from Section 2.7.2 is just

$$\begin{aligned}\Omega_{A^\vee/\mathbb{C}} &= \prod_{i=1}^m \mathrm{Hom}_{\mathbb{Z}_{(p)}}(L_i^\# \otimes \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \otimes_{S_0} \Omega_{A_i/\mathbb{C}} \\ &= \prod_{i=1}^m \mathrm{Hom}_{\mathbb{Z}_{(p)}}(L_i^\# \otimes \mathbb{Z}_{(p)}, \mathbb{Z}_{(p)}) \otimes_{S_0} \mathrm{Hom}_{S_0}(\Lambda_i, \mathbb{C}(1)) \\ &= \prod_{i=1}^m \mathrm{Hom}_{S_0}((L_i^\# \otimes \Lambda_i, \mathbb{C}(1)) = \mathrm{Hom}_{S_0}(\Lambda_0, \mathbb{C}(1)) = \Lambda_0^\vee \otimes_{S_0} \mathbb{C},\end{aligned} \tag{A.3.1}$$

where the second identification is induced by the isomorphisms (A.1.1). Similarly, the isomorphisms (A.1.2) determine an isomorphism

$$\varepsilon_A : \Omega_{A^\vee/\overline{\mathbb{Z}}_{(p)}} \cong \Lambda_0^\vee \otimes_{S_0} \overline{\mathbb{Z}}_{(p)}, \tag{A.3.2}$$

and putting

$$\Omega = (\Omega_i) \in \prod_{i=1}^m (\mathcal{O} \otimes \mathbb{C})^\times \in \prod_{i=1}^m \mathrm{GL}_{\mathcal{O}_i \otimes \mathbb{C}}(e_i \Lambda_0^\vee \otimes_{S_0} \mathbb{C}) = \mathrm{GL}_{\mathcal{O}_B \otimes \mathbb{C}}(\Lambda_0^\vee \otimes_{S_0} \mathbb{C}),$$

with Ω_i as in (A.1.3), we have

$$\varepsilon_0 = \Omega \cdot \varepsilon_A. \tag{A.3.3}$$

As ε_0 trivializes the $\mathrm{GL}_{\mathcal{O}_B \otimes_{\mathbb{C}}}(\Lambda_0^\vee \otimes_{S_0} \mathbb{C})$-torsor $\mathrm{Isom}_{\mathcal{O}_B \otimes \mathbb{C}}(\Omega_{A^\vee/\mathbb{C}}, \Lambda_0^\vee \otimes \mathbb{C})$, the modular forms over $\mathbb{C}$ of level K and weight κ are identified with

$$M_\kappa(K; \mathbb{C}) = \left\{ \begin{matrix} f_\infty : G(\mathbb{Q}) \backslash G(\mathbb{A}) \to W_\kappa(\mathbb{C}) \\ f_\infty(guk) = u^{-1} f_\infty(g) \ \ \forall u \times k \in G(\mathbb{R})K \end{matrix} \right\}, \tag{A.3.4}$$

and those of character ψ are identified with

$$M_\kappa(K_r, \psi; \mathbb{C}) = \left\{ \begin{matrix} f_\infty : G(\mathbb{Q}) \backslash G(\mathbb{A}) \to W_\kappa(\mathbb{C}) \\ f_\infty(guk) = u^{-1} \psi(k_p) f_\infty(g) \ \ \forall u \times k \in G(\mathbb{R})K_r^0 \end{matrix} \right\}, \tag{A.3.5}$$

where ψ defines a character of K_r^0 via the isomorphism $I_r^0/I_r \xrightarrow{\sim} T_H(\mathbb{Z}_p/p\mathbb{Z}_p)$. Explicitly, a modular form f in the left-hand side of (A.3.4) or (A.3.5) is identified with a function f_∞ on the right-hand side such that the value of f on $(\underline{A}_{g_f}, (x\varepsilon_0(g_\infty^{-1} \cdot), \nu(x)))$ or $(\underline{X}_{g_f}, (x\varepsilon_0(g_\infty^{-1} \cdot), \nu(x)))$, respectively, equals $g_\infty^{-1} \cdot f_\infty(g_f)(x)$. This is just the complex uniformization given in 2.7.2 specialized to the definite case.

Similarly, ε_A trivializes $\mathrm{Isom}_{\mathcal{O}_B \otimes \overline{\mathbb{Z}}_{(p)}}(\Omega_{A^\vee/\overline{\mathbb{Z}}_{(p)}}, \Lambda_0^\vee \otimes_{S_0} \overline{\mathbb{Z}}_{(p)})$, and the modular forms over R, $\overline{\mathbb{Z}}_{(p)} \subseteq R \subseteq \mathbb{C}$, of weight κ are identified with those that take values in R on $\left(\underline{A}_{g_f}, (x\varepsilon_A, \nu(x))\right)$ or $\left(\underline{X}_{g_f}, (x\varepsilon_A, \nu(x))\right)$ for $x \in H_0(R)$ and $g_f \in G\left(\mathbb{A}_f\right)$ such that $g_p \in G(\mathbb{Z}_p)$. The relation (A.3.3) identifies the modular forms over R with those f_∞ in the right-hand sides of (A.3.4) and (A.3.5) such that

$$\kappa(\Omega)^{-1} f_\infty(g_f) \in W_\kappa(R)$$

for $g_f \in G(\mathbb{A}_f)$ with $g_p \in G(\mathbb{Z}_p)$. Restricting to $G(\mathbb{A}_f)$ we find that the spaces of modular forms of weight κ over $\overline{\mathbb{Q}}$ can be identified with the spaces

$$\mathcal{A}^{\mathrm{alg}}_\kappa(K;\overline{\mathbb{Q}}) = \left\{ \begin{matrix} f_a : G(\mathbb{A}_f) \to W_\kappa(\overline{\mathbb{Q}}) \\ f_a(\gamma g k) = \gamma^{-1}\cdot f_a(g) \;\; \forall \gamma \in G(\mathbb{Q}),\;\; k \in K \end{matrix} \right\} \tag{A.3.6}$$

and

$$\mathcal{A}^{\mathrm{alg}}_\kappa(K_r,\psi;\overline{\mathbb{Q}}) = \left\{ \begin{matrix} f_a : G(\mathbb{A}_f) \to W_\kappa(\overline{\mathbb{Q}}) \\ f_a(\gamma g k) = \gamma^{-1}\cdot \psi(k_p) f_a(g) \;\; \forall \gamma \in G(\mathbb{Q}),\;\; k \in K_r^0 \end{matrix} \right\}. \tag{A.3.7}$$

An f in $M_\kappa(K;\overline{\mathbb{Q}})$ or $M_\kappa(K_r,\psi;\overline{\mathbb{Q}})$ is then identified with functions f_a and f_∞ such that

$$f_a(g) = \kappa(\Omega)^{-1} f_\infty(g) \tag{A.3.8}$$

for $g \in G(\mathbb{A}_f)$. Clearly, the right-hand sides of (A.3.6) (resp. (A.3.7)) makes sense with $\overline{\mathbb{Q}}$ replaced with any $\mathbb{Q}$-algebra R (resp. $\mathbb{Q}[\psi]$-algebra R); the corresponding module is denoted $\mathcal{A}^{\mathrm{alg}}_\kappa(K;R)$ (resp. $\mathcal{A}^{\mathrm{alg}}_\kappa(K_r,\psi;R)$).

If R is also a $\mathbb{Q}_p$-algebra, then $\mathcal{A}^{\mathrm{alg}}_\kappa(K;R)$ and $\mathcal{A}^{\mathrm{alg}}_\kappa(K_r,\psi;R)$ can be identified, respectively, with

$$\mathcal{A}^{p-\mathrm{alg}}_\kappa(K;R) = \left\{ \begin{matrix} f_p : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to W_\kappa(R) \\ f_p(\gamma g k) = k_p^{-1} f_p(g) \;\; \forall k \in K \end{matrix} \right\} \tag{A.3.9}$$

and

$$\mathcal{A}^{p-\mathrm{alg}}_\kappa(K_r,\psi;R) = \left\{ \begin{matrix} f_p : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to W_\kappa(R) \\ f_p(\gamma g k) = k_p^{-1}\psi(k_p) f_p(g) \;\; \forall k \in K_r^0 \end{matrix} \right\}. \tag{A.3.10}$$

An f_a in $\mathcal{A}^{\mathrm{alg}}_\kappa(K;R)$ or $\mathcal{A}^{\mathrm{alg}}_\kappa(K_r,\psi;R)$ is identified with $f_p(g) = g_p^{-1} f_a(g)$, where the action of g_p on W_κ is through the identification (A.0.5). The right-hand side of (A.3.9) (resp. (A.3.10)) makes sense with R any $\mathbb{Z}_p$-algebra (resp. $\mathbb{Z}_p[\psi]$-algebra).

In the right-hand sides of (A.3.9) and (A.3.10), $W_\kappa(R)$ can be replaced with any $R[K]$-module. We write $\mathcal{M}^{p-\mathrm{alg}}_{\kappa^\vee}(K;R)$ and $\mathcal{M}^{p-\mathrm{alg}}_{\kappa^\vee}(K_r,\psi;R)$ for the modules of forms with $W_\kappa(R)$ replaced by $W_\kappa^\vee(R)$. For a $\mathbb{Q}_p$-algebra R, the isomorphism $W_\kappa^\vee(R) \cong W_{\kappa^\vee}(R)$, $\ell_\kappa \mapsto \phi_{\kappa^\vee}$, identifies these spaces with $\mathcal{A}^{p-\mathrm{alg}}_{\kappa^\vee}(K;R)$ and $\mathcal{A}^{p-\mathrm{alg}}_{\kappa^\vee}(K,\psi;R)$, respectively.

Let $[\cdot,\cdot]_\kappa : W_\kappa(R)\times W_\kappa^\vee(R) \to R$ be the canonical perfect R-pairing (so $[\phi,\ell] = \ell(\phi)$). Using this we define a perfect R-pairing

$$\begin{gathered} \langle\cdot,\cdot\rangle_\kappa : \mathcal{A}^{p-\mathrm{alg}}_\kappa(K_r,\psi;R)\times \mathcal{M}^{p-\mathrm{alg}}_{\kappa^\vee}(K_r,\psi^{-1};R) \to R, \\ \langle f_p, f_p'\rangle_\kappa = \sum_{x\in G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K_r^0} [f(x), f'(x)]_\kappa. \end{gathered} \tag{A.3.11}$$

Suppose $R \subset \mathbb{C}$ (which is a $\mathbb{Z}_p$-algebra via $incl_p$). Let $f \in M_\kappa(K_r,\psi;R)$ and $f' \in M_{\kappa^\vee}(K_r,\psi^{-1};R)$. Chasing through the correspondences and the identification of $W_\kappa^\vee(\mathbb{C})$ with $W_{\kappa^\vee}(\mathbb{C})$ yields

$$\langle f_p, f_p'\rangle_\kappa = \mathrm{vol}(K_r^0)^{-1}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})/G(\mathbb{R})} [f_\infty(x), f'_\infty(x)]_\kappa dx. \tag{A.3.12}$$

A.4. **Hecke operators.** The Hecke operators away from p and S are given by the usual double coset on all the spaces of forms in Section A.3 and satisfy

$$\langle [K_r g K_r] f_p, f'_p \rangle_{\kappa,S} = \langle f_p, [K_r g^{-1} K_r] f'_p \rangle_{\kappa,S}. \tag{A.4.1}$$

The same is true of the action of $U_{w,i,j}$ on the spaces of forms defined in Equations (A.3.4), (A.3.5), (A.3.6), and (A.3.7).

A.5. **p-adic modular forms.** The abelian varieties A_i and A are ordinary, so $\S_m$ is just $\mathrm{M}_K/\mathcal{O}_{\mathbb{C}_p}/p^m\mathcal{O}_{\mathbb{C}_p}$. As ϕ'_n is a trivialization of $A^\vee[p^n]^0$, the quotient $\mathrm{T}_{n,m}/B_H^u(\mathbb{Z}_p)$ is $\mathrm{M}_{K_n}/\mathcal{O}_{\mathbb{C}_p}/p^m\mathcal{O}_{\mathbb{C}_p}$. If R is a p-adic $\mathcal{O}_{\mathbb{C}_p}$-algebra, then the R-module of p-adic modular forms for G is identified with the limit

$$\begin{aligned} V(K^p, R) &= \varprojlim_m \varinjlim_n \left\{ \begin{matrix} f : G(\mathbb{Q})\backslash G(\mathbb{A}_f) \to R/p^m R \\ f(gk) = f(g)\ \forall k \in K_n \end{matrix} \right\} \\ &= \varprojlim_m \varinjlim_n \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n; R/p^m R), \end{aligned} \tag{A.5.1}$$

where $\mathbf{1}$ denotes the trivial character. Explicitly, the value of $f \in V(K^p, R)$ on the test object $\big(\underline{A}_g, (\phi'_{n,g})\big)$, $g \in G(\mathbb{A}_f)$, is $f(g)$.

As the isomorphism $\mathrm{Lie}((\phi_{i,n}))^\vee : \Omega_{A_i/\mathcal{O}_{\mathbb{C}_p}} \xrightarrow{\sim} \mathrm{Hom}_{S_0}(\Lambda_i, \mathcal{O}_{\mathbb{C}_p}(1))$ is an $\mathcal{O} \otimes \mathcal{O}_{\mathbb{C}_p}$-linear isomorphism of rank one $\mathcal{O} \otimes \mathcal{O}_{\mathbb{C}_p}$-modules, there is $\Omega_{p,i} \in (\mathcal{O} \otimes \mathcal{O}_{\mathbb{C}_p})^\times$ such that $\mathrm{Lie}((\phi_{i,n}))^\vee = \Omega_{p,i}\varepsilon_{A_i}$. It follows that $\Omega_p = (\Omega_{p,i}) \in \prod_{i=1}^m (\mathcal{O} \otimes \mathcal{O}_{\mathbb{C}_p})^\times$, which belongs to the center of $\mathrm{GL}_{\mathcal{O}_B \otimes \mathcal{O}_{\mathbb{C}_p}}(\Lambda_0^\vee \otimes_{S_0} \mathcal{O}_{\mathbb{C}_p}) \subset H_0(\mathcal{O}_{\mathbb{C}_p})$, satisfies

$$\varepsilon_p := \mathrm{Lie}((\phi'_n))^\vee = \Omega_p \cdot \varepsilon_A. \tag{A.5.2}$$

The realization of $f \in M_{\kappa_*}(K_r, \psi; R)$ as a p-adic modular form $f_{p-\mathrm{adic}}$ of weight κ is such that for $g \in G(\mathbb{A}_f)$

$$\begin{aligned} f_{p-\mathrm{adic}}(g) &= f(\underline{A}_g, \phi'_r \circ g_p, (g_p^{-1}\varepsilon_p, id)) \\ &= \kappa_*(\Omega_p) f(\underline{A}_g, \phi'_r \circ g_p, (g_p^{-1}\varepsilon_A, id)) \\ &= \kappa_*(\Omega_p) f(\underline{A}_g, \phi'_r \circ g_p, g_p^{-1}(\varepsilon_A, id)) \\ &= \kappa_*(\Omega_p) f_a(g)(g_p^{-1}) \\ &= \kappa_*(\Omega_p) f_p(g)(1). \end{aligned} \tag{A.5.3}$$

The right-hand side of (A.5.1) is defined for any p-adic algebra R and we denote the limit by $\mathcal{A}(K^p, R)$ and the submodule of weight κ and character ψ forms (for κ and ψ both R-valued) by $\mathcal{A}_\kappa(K^p, \psi, R)$. There is a natural map

$$ev : \mathcal{A}_{\kappa_*}^{p-\mathrm{alg}}(K_r, \psi; R) \to \mathcal{A}_\kappa(K^p, \psi, R), \quad f_p \mapsto (g \mapsto f_p(g)(1)).$$

If p is not a zero-divisor in R then this is an injection, but in general it is not.

Let

$$\mathcal{M}(K^p, R) = \varprojlim_m \varprojlim_n \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n; R/p^m R), \tag{A.5.4}$$

where the transition maps in n are the trace maps

$$\mathrm{Tr}_{n',n} : \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_{n'};-) \to \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n;-),$$
$$\mathrm{Tr}_{n',n} f(g) = \sum_{k \in K_n/K_{n'}} f(gk).$$

The perfect pairings $\langle\cdot,\cdot\rangle_{n,m} := \langle\cdot,\cdot\rangle_{\mathbf{1}}$ are compatible with $\mathrm{Tr}_{n',n}$ in the sense that for $f_p \in \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n; R/p^m R)$ and $f_p' \in \mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_{n'}; R/p^m R)$, $n' \geq n$,

$$\langle f_p, f_p' \rangle_{n',m} = \langle f_p, \mathrm{Tr}_{n',n} f_p' \rangle_{n,m},$$

and so, upon taking limits, they yield a perfect pairing

$$\langle\cdot,\cdot\rangle : \mathcal{A}(K^p, R) \times \mathcal{M}(K^p, R) \to R. \tag{A.5.5}$$

Note that $\mathcal{A}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n;-) = \mathcal{M}_{\mathbf{1}}^{p-\mathrm{alg}}(K_n;-)$.

The values $\langle ev(f_p),\cdot\rangle$ can be expressed in terms of $\langle ev(f),\cdot\rangle_{\kappa_*}$ as follows. Given an R-valued character ψ of $T_H(\mathbb{Z}_p)$ that factors throught $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$ and a dominant weight κ_* we put

$$ev_{\kappa,\psi}^\vee : \mathcal{M}(K^p, R) \to \mathcal{M}_{\kappa_*^\vee}^{p-\mathrm{alg}}\left(K_r, \psi^{-1}; R\right),$$
$$ev_{\kappa,\psi}^\vee(f) = \varprojlim_m ev_{\kappa,\psi}^\vee(f)_m, \quad ev_{\kappa,\psi}^\vee(f)_m(g) = \sum_{x \in K_r^0/K_t} f_{t,m}(gx)\psi(x)x \cdot \ell_\kappa,$$

where $f = \varprojlim_m \varprojlim_n f_{n,m}$ and $t \geq r$; $ev_{\kappa,\psi}^\vee(f)_m$ is easily seen to be independent of t. Then for $f_p \in \mathcal{A}_{\kappa_*}^{p-\mathrm{alg}}(K_r, \psi; R)$ and $f' \in \mathcal{M}(K^p, R)$,

$$\langle ev(f_p), f' \rangle = \langle f_p, ev_{\kappa,\psi}^\vee(f') \rangle_{\kappa_*}. \tag{A.5.6}$$

A.6. **Hecke operators again.** The Hecke operators away from p act via the usual double coset actions on all the spaces of forms defined in A.3 and A.5. Similarly, the action of $u_{w,i,j}$ on $V(K^p, R)$ is just the usual action of the double coset $U_{w,i,j}$. For the Hecke operators away from p these actions are compatible with the various identifications and the maps ev and $ev_{\kappa,\psi}^\vee$, while the operators at p satisfy the relation (2.9.4): for $f_a \in \mathcal{A}_{\kappa*}^{\mathrm{alg}}(K_r, \psi; R)$

$$|\kappa(t_{w,i,j})|_p^{-1} ev((U_{w,i,j} \cdot f_a)_p) = u_{w,i,j} \cdot ev(f_p). \tag{A.6.1}$$

Furthemore, the action of $u_{w,i,j}$ on any $f_p \in \mathcal{A}_{\kappa_*}^{p-\mathrm{alg}}(K_r, \psi; R)$ when p is not a zero-divisor in R is given as follows. Write $U_{w,i,j} = \sqcup b K_r$ with $b \in B^+(\mathbb{Q}_p)$. Given $x \in G(\mathbb{Z}_p)$, let $xb = b'x' \in B(\mathbb{Q}_p)G(\mathbb{Z}_p) = G(\mathbb{Q}_p)$. Then

$$u_{w,i,j} \cdot f_p(g)(x) = \sum_b \kappa_*(b'/t_{w,i,j}^+) f_p(gb)(x'). \tag{A.6.2}$$

For any p-adic R, the formula (A.6.2) defines an action of $u_{w,i,j}$ on f_p that is compatible with ev. So the ordinary projector e acts compatibly on these spaces.

Under the perfect pairings $\langle\cdot,\cdot\rangle_\kappa$ and $\langle\cdot,\cdot\rangle$, the ordinary idempotent e aquires an adjoint e' acting compatibly on $\mathcal{M}_{\kappa^\vee}^{p-\mathrm{alg}}\left(K_r, \psi^{-1}; R\right)$ and $\mathcal{M}(K^p, W_S^\vee, R)$ (that is, $ev_{\kappa,\psi}^\vee(e' \cdot f) = e' \cdot ev_{\kappa,\psi}^\vee(f)$). The idempotent e' can also be described in terms of Hecke operators.

Let $U'_{w,i,j} = K_r(t^+_{w,i,j})^{-1}K_r$ and $U_p = \prod_{w\in\Sigma_p}\prod_{i=1}^{m}\prod_{j=1}^{n_ir_i} U'_{w,i,j}$. Then the action of e' on $\mathcal{M}(K^p, W_S^\vee, R)$ is just $e' = \varinjlim_n (U_p)^{n!}$; the action of U_p is readily seen to be compatible with the trace maps $\mathrm{Tr}_{r',r}$.

A.7. **Ordinary forms.** Suppose R is a p-adic ring in which p is not a zero-divisor and ψ is an R-valued character factoring through $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$. Then the map ev induces an identification

$$ev : e\mathcal{A}^{p-\mathrm{alg}}_{\kappa_*}(K_r, \psi; R) \xrightarrow{\sim} e\mathcal{A}_\kappa(K^p, \psi, R). \tag{A.7.1}$$

This is deduced from the contraction property of u_p and the formula (A.6.2). For R a p-adic $\mathcal{O}_{\mathbb{C}_p}$-algebra this is just a restatement of (2.9.6). From this and the perfect pairings (A.3.11) and (A.5.5) it then follows that

$$ev^\vee_{\kappa,\psi} : e'\mathcal{M}(K^p, R) \xrightarrow{\sim} e'\mathcal{M}_{\kappa_*^\vee}\left(K_r, \psi^{-1}; R\right). \tag{A.7.2}$$

Part III: Ordinary families and p-adic L-functions

5. Measures and restrictions

This section focuses on measures and restrictions. In particular, Section 5.3 gives a measure whose values at certain specified characters are the Eisenstein series associated to the local data chosen when we calculated the zeta-integrals above.

5.1. **Measures: generalities.** Let X be a compact and totally disconnected topological space. For a p-adic ring R we let $C(X,R)$ be the R-module of continuous maps from X to R (continuous with respect of the p-adic topology on R). Note that $C(X,\mathbb{Z}_p)\hat{\otimes}_{\mathbb{Z}_p}R \xrightarrow{\sim} C(X,R)$. Let M be a p-adically complete R-module. Then by an M-valued measure on X we mean an element of the R-module

$$\mathrm{Meas}(X,M) = \mathrm{Hom}_{\mathbb{Z}_p}(C(X,\mathbb{Z}_p),M) = \mathrm{Hom}_R(C(X,R),M).$$

Suppose X is a profinite abelian group. Then $\mathrm{Meas}(X,R)$ is identified with the completed group ring $R[[X]]$ so that if a measure μ is identified with $f \in R[[X]]$, then for any continuous character $\chi : X \to R_1^\times$ with R_1 a p-adic R-algebra, $\mu(\chi) = \chi(f)$.

In particular $\mathrm{Meas}(X,R)$ is itself a ring. The following lemma is immediate:

Lemma 5.1.1. *Suppose $X = X_1 \times X_2$ is a product of profinite abelian groups. Then there is a natural isomorphism*

$$\mathrm{Meas}(X_1 \times X_2, R) \xrightarrow{\sim} \mathrm{Meas}(X_1, Meas(X_2,R))$$

If we write $X = \varprojlim_i X/X_i$, where $X = X_0 \supset X_1 \supset X_2 \supset \dots$ is a neighborhood basis of the identity consisting of open subgroups of X of finite index, then

$$\Lambda_{X,R} = \varprojlim_i R[X/X_i].$$

This is a compact topological ring. The following dictionary is well-known and due to Mazur:

Fact 5.1.2. *The identification of a measure μ on X with an element f of the $\Lambda_{X,R}$ has the property that, for any continuous homomorphism $\chi : X \to R^\times$,*

$$\int_X \chi d\mu := \mu(\chi) = \chi(f)$$

where $\chi(f)$ is the image of f under the homomorphism $\Lambda_{X,R} \to R$ induced by χ.

We let χ denote the homomorphism $\Lambda_{X,R} \to R$ of Fact 5.1.2; in this way χ defines an R-valued point of $\Lambda_{X,R}$.

In what follows, characters of X_1 will be Hecke characters, X_2 will be the group of integral points of a p-adic torus, whose characters parametrize weights of p-adic modular forms, and M will be the ring of p-adic modular forms. When X_2 is a point, the measure on $X = X_1$ will be an Eisenstein measure that pairs with modular forms of fixed weight, and in particular can be used to construct what we will call, loosely and somewhat abusively, a *p-adic L-function of one variable*, the variable Hecke character, attached to a fixed holomorphic automorphic representation. When X_2 is the group of points of a non-trivial torus, we will be constructing *p-adic L-function of two variables*, the second variable running through the points of a Hida family.

The following lemma is well-known.

Lemma 5.1.3. *Suppose X is a finite-dimensional compact p-adic Lie group. Let $R \subset \mathcal{O}_{\mathbb{C}_p}$ be a $\mathbb{Z}_p$-subalgebra. Let $\alpha : \hat{X} := \mathrm{Hom}(X, \mathcal{O}_{\mathbb{C}_p}^{\times}) \to \mathcal{O}_{\mathbb{C}_p}$ be a map from $\mathcal{O}_{\mathbb{C}_p}$-valued characters of X to $\mathcal{O}_{\mathbb{C}_p}$ with the property that, whenever $\chi_1, \chi_2 \in \hat{X}$*

$$\chi_1 \equiv \chi_2 \pmod{p^a} \Rightarrow \alpha(\chi_1) \equiv \alpha(\chi_2) \pmod{p^a}.$$

Then α extends to an $\mathcal{O}_{\mathbb{C}_p}$-valued measure μ_α on X. Moreover, if $\mu_\alpha(\phi) \in R$ for all $\phi \in C(X, R)$ then μ_α is obtained by extension of scalars from an R-valued measure on X.

5.2. **The space X_p.** For each integer $r > 0$, let

$$U_r = (\mathcal{O} \otimes \widehat{\mathbb{Z}}^{\{p\}})^{\times} \times (1 + p^r \mathcal{O} \otimes \mathbb{Z}_p) \subset (\mathcal{K} \otimes \widehat{\mathbb{Z}})^{\times}$$

and

$$X_p = \varprojlim_r \mathcal{K}^{\times} \backslash (\mathcal{K} \otimes \widehat{\mathbb{Z}})^{\times} / U_r.$$

This is the projective limit of the ray class groups of $\mathcal{K}$ of conductor (p^r). In particular, it is a profinite abelian group.

5.2.1. *Admissible measures on X_p.* We suppose now that we are in the situation of Section 3, and we freely use the notation and conventions introduced therein. Using the isomorphism (3.1.2) we identify $H_1(\mathbb{Z}_p)$ with $H_2(\mathbb{Z}_p)$ via $h_1 = (h_{1,w})_{w|p} \mapsto h_2 = (h_{2,w})_{w|p}$ with $h_{2,w} = h_{1,\bar{w}}$. This then identifies $T_{H_2}(\mathbb{Z}_p)$ with $T_{H_1}(\mathbb{Z}_p)$ and $T_{H_4}(\mathbb{Z}_p) = T_{H_3}(\mathbb{Z}_p) = T_{H_1}(\mathbb{Z}_p) \times T_{H_2}(\mathbb{Z}_p)$ with $T_{H_1}(\mathbb{Z}_p) \times T_{H_1}(\mathbb{Z}_p)$. In particular, the characters ψ of $T_{H_3}(\mathbb{Z}_p)$ are identified with pairs of characters (ψ_1, ψ_2) of $T = T_{H_1}(\mathbb{Z}_p)$.

Let

- $\kappa = (\kappa_\sigma)$ be an $\mathcal{O}'$-character of T as in Section 2.9.1 and let κ' be the $\mathcal{O}'$-character of $T_{H_3}(\mathbb{Z}_p)$ identified with the pair $(\kappa, \kappa^\vee)$;
- ψ be a finite-order $\overline{\mathbb{Q}}_p^{\times}$-valued character of $T(\mathbb{Z}_p)$;
- $K_i^p \subset G_i\left(\mathbb{A}_f^{\{p\}}\right)$, $i = 1, 2$, be open compact subgroups such that $\nu(K_1) = \nu(K_2)$;
- R be a p-adic $\mathcal{O}'[\psi]$-algebra.

For any finite-order $\overline{\mathbb{Q}}_p^{\times}$-valued character χ of X_p, let $\psi_\chi^{-1} = \psi^{-1} \cdot \chi \circ \det$, where by det we mean the map $\det : H_1(\mathbb{Z}_p) \to (\mathcal{O} \otimes \mathbb{Z}_p)^{\times} = \prod_{w|p} \mathcal{O}_w^{\times}$ that is the composition of the isomorphism (3.1.2) with the products of the determinants of each of the GL-factors, and let ψ'_χ be the character of $T_{H_3}(\mathbb{Z}_p)$ identified with the pair (ψ, ψ_χ^{-1}), . By an admissible R-measure on X_p of weight κ, character ψ, and level $K^p = K_3^p = (K_1^p \times K_2^p) \cap G_3(\mathbb{A}_f^p)$, we mean a measure $\mu(\cdot) = \mu(\kappa, \psi, \cdot) \in \mathrm{Meas}(X_p; V_{\kappa'}^{\mathrm{ord}}(K^p, R))$ such that for any finite-order $\overline{\mathbb{Q}}_p^\times$-valued character χ of X_p,

$$\mu(\chi) = \mu(\kappa, \psi, \chi) \in V_{\kappa'}^{\mathrm{ord}}\left(K^p, \psi'_\chi, R[\chi]\right).$$

Let R' be any p-adic R-algebra and ℓ an R-linear functional $\ell : V_{\kappa'}^{\mathrm{ord,cusp}}(K^p, R) \to R'$. Then

$$\mu_\ell(\cdot) = \mu_\ell(\kappa, \psi, \cdot) = \ell \circ \mu(\kappa, \psi, \cdot)$$

is an R'-valued measure on X_p. One useful way of defining such ℓ is as follows. As explained in Section 3.2.4, the space $V_{\kappa'}^{\mathrm{ord,cusp}}(K^p, R)$ is identified with the direct summand

$$\oplus_{\alpha \in C_K} V_\kappa^{\mathrm{ord,cusp}}\left(K_1^p, R\right)^\alpha \otimes_R V_{\kappa^\vee}^{\mathrm{ord,cusp}}\left(K_2^p, R\right)^\alpha.$$

Given $\ell = (\ell^\alpha)_{\alpha \in C_K}$ with each $\ell^\alpha = (\ell_1^\alpha, \ell_2^\alpha)$ a pair of R-linear functionals ℓ_1^α and ℓ_2^α of $V_\kappa^{\mathrm{ord,cusp}}(K_1^p, R)^\alpha$ and $V_{\kappa^\vee}^{\mathrm{ord,cusp}}(K_2^p, R)^\alpha$, respectively, ℓ determines an R-linear functional of $V_{\kappa'}^{\mathrm{ord,cusp}}(K^p, R)$ by

$$\ell(f_1^\alpha \otimes f_2^\alpha) = \ell_1^\alpha(f_1^\alpha)\ell_2^\alpha(f_2^\alpha).$$

From the definition of an admissible measure, it is clear that $\mu(\cdot)$ takes values in

$$\oplus_{\alpha \in C_K} V_\kappa^{\mathrm{ord,cusp}}(K_{1,r}, \psi, R)^\alpha \otimes_R V_{\kappa^\vee}^{\mathrm{ord,cusp}}(K_2^p, R)^\alpha$$

(for r sufficiently large), so it suffices to take ℓ_1^α to be a functional of $V_\kappa^{\mathrm{ord,cusp}}(K_{1,r}, \psi, R)$.

In the definite case (i.e., G_1 - and hence G_2 and G_3 - definite), we take $\mu(\kappa, \psi, \cdot)$ to be a $e\mathcal{A}_{\kappa'}(K^p, R)$-valued measure, making the corresponding modifications to the above definitions.

We will need a slight generalization of the above definition. Let

- $\rho = (\rho_\sigma)$ be an $\mathcal{O}'$-character of T as in Section 2.9.1 and let $\rho^\triangle$ be the $\mathcal{O}'$-character of $T_{H_3}(\mathbb{Z}_p)$ identified with the pair $(\rho, \rho^\vee)$;
- ψ be a finite-order $\overline{\mathbb{Q}}_p^\times$-valued character of $T(\mathbb{Z}_p)$;
- $K_i^p \subset G_i\left(\mathbb{A}_f^{\{p\}}\right)$, $i = 1, 2$, be open compact subgroups such that $\nu(K_1) = \nu(K_2)$;
- R be a p-adic $\mathcal{O}'[\psi]$-algebra.

Note that $\rho^\triangle$ and $(\rho, \rho^\flat)$ coincide as characters of $T_{H_3}(\mathbb{Z}_p)$.

For any finite-order $\overline{\mathbb{Q}}_p^\times$-valued character χ of X_p, let $\psi_\chi^{-1} = \psi^{-1} \cdot \chi \circ \det$, where by det we mean the map $\det : H_1(\mathbb{Z}_p) \to (\mathcal{O} \otimes \mathbb{Z}_p)^\times = \prod_{w|p} \mathcal{O}_w^\times$ that is the composition of the isomorphism (3.1.2) with the products of the determinants of each of the GL-factors, and let $\psi_\chi^\triangle$ be the character of $T_{H_3}(\mathbb{Z}_p)$ identified with the pair (ψ, ψ_χ^{-1}). Let (α, β) be a character of $T_{H_3}(\mathbb{Z}_p)$, written as a pair of characters of $T_{H_1}(\mathbb{Z}_p) \equiv T_{H_2}(\mathbb{Z}_p)$. By an admissible R-measure on X_p of weight ρ, character ψ, *shift* (α, β), and level $K^p = K_3^p =$

$(K_1^p \times K_2^p) \cap G_3(\mathbb{A}_f^p)$, we mean a measure $\mu(\cdot) = \mu(\rho, \psi, \cdot) \in \mathrm{Meas}(X_p; V^{\mathrm{ord}}_{\rho^\triangle \cdot (\alpha,\beta)}(K^p, R))$ such that for any finite-order $\overline{\mathbb{Q}}_p^\times$-valued character χ of X_p,

$$\int_{X_p} \chi d\mu := \mu(\chi) = \mu(\rho, \psi, \chi) \in V^{\mathrm{ord}}_{\rho^\triangle \cdot (\alpha,\beta)}\left(K^p, \psi'_\chi, R[\chi]\right). \tag{5.2.1}$$

5.2.2. *Pairings and measures on X_p: the definite case.* Suppose now that G_1 is definite. The same is then true of G_2 and G_3. Suppose $R \subset \mathcal{O}_{\mathbb{C}_p}$. Let $\mu(\cdot)$ be an admissible R-measure of weight κ, character ψ, and level K^p. Let $\langle \cdot, \cdot \rangle_i$, $i = 1, 2, 3$, be the perfect pairing (A.5.5) for G_i and K_i^p.

Let $\phi \in \mathcal{M}(K_1^p, R)$ and $x \in G_2(\mathbb{A}_f)$. Let $\delta_x \in \mathcal{M}(K_2^p, R)$ be such that $\langle f, \delta_x \rangle_2 = f(x)$ (so δ_x is the inverse limit of the characteristic function of the image of x in $G_2(\mathbb{Q}) \backslash G_2(\mathbb{A}_f)/K_{2,n}$). We define an R-measure $\mu(\phi, x, \cdot) = \mu(\kappa, \psi, \phi, x, \cdot)$ on X_p by setting $\mu(\phi, x, \cdot) = \mu_\ell(\cdot)$ with ℓ defined by

$$\ell_1^\alpha(f^\alpha) = \langle f^\alpha, \phi \rangle_1 \quad \text{and} \quad \ell_2^\alpha(f^\alpha) = \langle f^\alpha, \delta_x \rangle_2 = f^\alpha(x).$$

Then

$$\mu(\phi, x, -) = \langle \mu(\kappa, \psi, -), \phi \otimes \delta_x \rangle_3.$$

For a given χ, $x \mapsto \mu(\phi, x, \chi)$ belongs to $e\mathcal{A}_{\kappa^\vee}\left(K_2^p, \psi_\chi^{-1}, R[\chi]\right)$ and so $\mu(\phi, \cdot, \chi) = ev(f_{\chi,2})$ for some $f_{\chi,2} \in e\mathcal{A}^{p-\mathrm{alg}}_{\kappa^\vee}\left(K_{2,r}, \psi_\chi^{-1}; R[\chi]\right)$ (with r sufficiently large). For $\phi' \in \mathcal{M}(K_2^p, R)$,

$$\langle \mu(\phi, \cdot, \chi), \phi' \rangle_2 = \langle f_{\chi,2}, \phi_{\kappa,\psi} \otimes \phi'_{\kappa^\vee, \psi_\chi^{-1}} \rangle_{\kappa'}, \tag{5.2.2}$$

where $\phi_{\kappa,\psi} = ev^\vee_{\kappa,\psi}(\phi)$ and $\phi'_{\kappa^\vee . \psi_\chi^{-1}} = ev^\vee_{\kappa^\vee, \psi_\chi^{-1}}(\phi')$. Let $f_\chi \in e\mathcal{A}^{p-\mathrm{alg,cusp}}_{\kappa'}\left(K_r, \psi'_\chi; R[\chi]\right)$ be such that $ev_{\kappa', \psi'_\chi}(f_\chi) = \mu(\chi)$. The right-hand side can be rewritten as an integral:

$$\mathrm{vol}\left(K_r^0\right)^{-1} \int_{G_3(\mathbb{Q}) \backslash G_3(\mathbb{A})/G_3(\mathbb{R})} \left[f_{\chi,\infty}(g_1, g_2), \phi_{\kappa,\psi,\infty}(g_1) \otimes \phi'_{\kappa^\vee, \psi_\chi^{-1}, \infty}(g_2)\right]_{\kappa'} d(g_1, g_2)$$

where $f_{\chi,\infty}(g) = g_\infty^{-1} g_p \cdot f_\chi\left(g_f\right)$ is the corresponding automorphic form on $G_3(\mathbb{A})$ (we have used the identification of $\mathbb{C}_p$ with $\mathbb{C}$).

Remark. In practice the function $f_{\chi,\infty}$ will be known, and the preceding integral will be used to relate the value of the measure to an integral representation of an L-function.

5.2.3. *Pairings and measures on X_p: the indefinite case.* Suppose now that G_1 is not definite. The same is then true of G_2 and G_3. In this case there is no obvious pairing on the spaces of forms that preserves R-structures, so we define only $\mathbb{C}_p$-valued measures (using the fixed identification of $\mathbb{C}_p$ with $\mathbb{C}$). Later we discuss how this can in some situations be refined to an R-valued measure.

Let $R \subset \mathcal{O}_{\mathbb{C}_p}$ be a p-adic ring. Let $\mu(\cdot) = \mu(\kappa, \psi, \cdot)$ be an admissible R-measure of weight κ, character ψ, and level K^p. Recall that $V_\kappa^{\mathrm{ord,cusp}}\left(K_1^p, \psi, R\right) = eS^{\mathrm{ord}}_{\kappa'_*}\left(K_r, \psi'_\chi; R[\chi]\right)$. Let $\phi : G_1(\mathbb{A}) \to W_\kappa(\mathbb{C})^\vee = W_{\kappa^\vee}(\mathbb{C})$ be any smooth function such that $\phi(\gamma g u k) =$

$u^{-1}\cdot\psi^{-1}(k_p)\phi(g)$ for all $\gamma\in G_1(\mathbb{Q})$, $g\in G_1(\mathbb{A})$, $u\in U_{1,\infty}$, and $k\in K^0_{1,r}$, and let $x=(\underline{A},\phi)\in\varprojlim_m\varprojlim_n T_{n,m}(\mathbb{C}_p)$. We define a $\mathbb{C}_p$-valued measure $\mu(\phi,x,\cdot)=\mu(\kappa,\psi,\phi,x,\cdot)$ on X_p by setting $\mu(\phi,x,\cdot)=\mu_\ell(\cdot)$ with ℓ defined by

$$\ell_1^\alpha(f^\alpha)=\int_{G_1(\mathbb{Q})\backslash G_1(\mathbb{A})}[f^\alpha(g),\phi(g)]_\kappa\,dg\quad\text{and}\quad \ell_2^\alpha(f^\alpha)=f^\alpha(x).$$

In the definition of ℓ_1 we have identifed f^α with its corresponding automorphic form on $G_1(\mathbb{A})$ (see Section 2.7.2).

For a given χ, $x\mapsto\mu(\phi,x,\chi)$ belongs to $e\mathcal{A}_{\kappa^\vee}(K_2^p,\psi_\chi^{-1},\mathbb{C}_p)=eS_{\kappa^\vee}(K_{2,r},\psi^{-1},\mathbb{C}_p)$. Let $f_{\chi,2}$ be the corresponding automorphic form on $G_2(\mathbb{A})$. Similarly, let f_χ be the automorphic form on $G_3(\mathbb{A})$ corresponding to $\mu(\chi)\in V^{\mathrm{ord,cusp}}_{\kappa'}(K^p,\psi'_\chi,R[\chi])=eS_{\kappa'}(K_r,\psi'_\chi,R[\chi])$. Given any smooth function $\phi':G_2(\mathbb{A})\to W_{\kappa^\vee}(\mathbb{C})^\vee=W_\kappa(\mathbb{C})$ such that $\phi'(\gamma guk)=u^{-1}\cdot\psi_\chi(k_p)\phi'(g)$ for all $\gamma\in G_2(\mathbb{Q})$, $g\in G_2(\mathbb{A})$, $u\in U_{2,\infty}$, and $k\in K^0_{2,r}$,

$$\begin{aligned}\int_{G_2(\mathbb{Q})\backslash G_2(\mathbb{A})/G_2(\mathbb{R})}&[f_{\chi,2}(g),\phi'(g)]_{\kappa^\vee}dg\\&=\int_{G_3(\mathbb{Q})\backslash G_3(\mathbb{A})/G_3(\mathbb{R})}\left[f_\chi((g_1,g_2)),\psi(g_1)\otimes\psi'(g_2)\right]_{\kappa'}d(g_1,g_2).\end{aligned}$$

Remark. As in the definite case, in practice the function f_χ will be known, and the preceding integral will be used to relate the value of the measure to an integral representation of an L-function.

5.2.4. *Admissible measures on $X_p\times T_H$: two variables.* In this section we consider admissible measures of weight ρ and shift (α,β) where ρ and (α,β) are allowed to vary. This requires a slight adjustment to the notation of the previous section. More precisely, suppose we are given a homomorphism $sh:T_H(\mathbb{Z}_p)\to X_p$. By duality this gives a map $sh^*:C(X_p,R)\to C(T_H(\mathbb{Z}_p),R)$ for any ring R; sh^* takes characters to characters.

We also suppose we are given an algebraic automorphism $\upsilon:T_H\to T_H$. If ρ is a function on T_H, we let $\rho^\upsilon(t)=\rho(\upsilon(t))$.

We fix a tame level N_0 as in Section 5.2 and define $X_p=X_{p,N_0}$ as before. By an admissible R-measure on $X_p\times T_H$ of character ψ, shift sh, twist υ, and level $K^p=K_3^p=(K_1^p\times K_2^p)\cap G_3(\mathbb{A}_f^p)$, we mean a measure

$$\mu(\cdot)=\mu(\psi,sh,\cdot)\in\mathrm{Meas}(X_p,\mathrm{Meas}(T_H,V^{\mathrm{ord}}(K^p,R)))$$

such that for any finite-order $\overline{\mathbb{Q}}_p^\times$-valued character χ of X_p and any character ρ of T_H,

$$\mu(\chi)(\rho^\upsilon)=\mu(\psi,sh,\chi)(\rho^\upsilon)\in V^{\mathrm{ord}}_{\rho^\triangle\cdot sh^*(\chi)}\left(K^p,\psi_\chi^\triangle,R[\chi]\right).$$

5.3. **Eisenstein measures on $X_p\times T$.** Above, we discussed measures on X_p. Now, we recall certain measures on $X_p\times T$, namely the Eisenstein measures of [Eis15, Eis14, EFMV16]. Note that if we fix a character on T, then we recover the measures above on X_p. We briefly summarize the basic properties of the Eisenstein measures in [Eis15, Eis14, EFMV16], which - in fact - p-adically interpolate values of the Eisenstein series associated to the local data chosen above for the zeta integral calculations. As in Section 2.3, let $\Sigma = \{\sigma \in \Sigma_{\mathcal{K}} \ : \ \mathfrak{p}_\sigma \in \Sigma_p\}$. This is a CM type for $\mathcal{K}$. Throughout this section, we take $\chi : \mathcal{K}^\times \backslash \mathbb{A}_{\mathcal{K}}^\times \to \mathbb{C}^\times$ to be a unitary Hecke character.

Let $S_{\mathrm{ram}} = S_\pi \cup S_\chi \cup S_{\mathcal{K}}$, where S_χ denotes the set of finite primes v in $\mathcal{O}_{\mathcal{K}^+}$ for which $\chi_v = \otimes_{w|v}\chi_w$ is ramified and $S_{\mathcal{K}}$ denotes the set of finite primes in $\mathcal{O}_{\mathcal{K}^+}$ that ramify in $\mathcal{K}$. Let S be a finite set of finite primes in $\mathbb{Q}$ such that $p \notin S$ and such that for all rational primes ℓ, if a prime in $\mathcal{K}^+$ above ℓ is in S_{ram}, then $\ell \in S$. Let S' be the set of primes of $\mathcal{K}^+$ lying above the primes of S.

5.3.1. *Axiomatics of the Eisenstein measure.* The Eisenstein measures of [Eis15, Eis14, Eis16], as well as the local components of ordinary vectors in Hida families, have been reverse-engineered in order to meet the requirements of the construction of the p-adic L-functions. In this section we first present the axioms the Eisenstein measure is required to satisfy, and then explain how they are satisfied by the ones constructed in the references just cited.

The Eisenstein measure is, in the first place, a p-adic measure on the space $X_p \times T_H(\mathbb{Z}_p)$ with values in the space of p-adic modular forms on G_4. It is characterized by its specializations at classical points. Let Y_H be the formal scheme over $\mathbb{Z}_p$ whose points with values in a complete $\mathbb{Z}_p$-algebra R are given by $\mathrm{Hom}(X_p \times T_H(\mathbb{Z}_p), R^\times)$. Let $Y_H^{\mathrm{alg}} \subset Y_H(\mathbb{C}_p)$ be the set of pairs (χ, c), where $\chi : X_p \to R^\times$, for some $R \subset \mathbb{C}_p$, is the p-adic character associated to an algebraic Hecke character, denoted χ^{class}, and $c = \psi\rho^v$ is a locally algebraic character of $T_H(\mathbb{Z}_p)$: ρ is an algebraic character, v is an involution of T_H, as in (4.4.10) and ψ is a character of finite order. In other words, $c \in C_r(T_H(\mathbb{Z}_p), R)$ for some $r \geq 0$, in the notation of Lemma 7.2.2.

Note that we are not requiring χ^{class} to be unitary here; rather, the variable "s" is included in the infinity type of χ; we fix an integer μ such that, for each $\sigma \in \Sigma$ we have $\chi_\sigma = \|\bullet\|_\sigma^\mu \chi_{0,\sigma}$, where $\chi_{0,\sigma} = (z_\sigma^{-a(\chi_\sigma)} \bar{z}_\sigma^{-b(\chi_\sigma)})$. This factorization is not unique; however, recall the set $C_3(\mu, \chi_\sigma)$ of (4.4.6). We assume we are given a subset $Y_H^{class} \subset Y_H^{\mathrm{alg}}$, determined by the following positivity condition:

$$(\chi, c) \in Y_H^{class} \Leftrightarrow \kappa_\sigma \in C_3(\mu, z_\sigma^{-a(\chi_\sigma)} \bar{z}_\sigma^{-b(\chi_\sigma)}) \forall \sigma \in \Sigma \tag{5.3.1}$$

This condition is independent of the choice of m as above, in other words is independent of the choice of factorization.

Now write $\chi = \|\bullet\|^m \cdot \chi_u$, and define the finite order idèle character χ^+ of $\mathcal{K}^+$, as in Section 4.6. We omit the expression of μ and χ_0 in terms of m and χ_u, and vice versa. Define the normalizing factor $D^S(\chi)$ and $D(\chi)$ as in 4.6.1

Definition 5.3.2. *Let K_i^p be an open compact subgroup of $G_i(\mathbb{A}_f^p)$, $i = 3, 4$, with $K_3^p \subset K_4^p$. Let S be the set of bad primes defined above. An axiomatic Eisenstein measure on $X_p \times T_H(\mathbb{Z}_p)$ of level S, relative to the set Y_H^{class}, of level K_4^p and with coefficients in R, is a measure $dEis$ with values in $V_3(K_3^p, R)$ such that, for every pair $(\chi = \|\bullet\|^m \cdot \chi_u, c =$*

$\psi\rho^v) \in Y_H^{class}$, there is a factorizable Siegel section

$$f(\chi,c) = \otimes'_v f_v(\chi_v,c) \ \in \ \bigotimes_v{}' \ I_v(\chi_{u,v},m)$$

and such that

- *If v is a finite place outside S, then χ_v is unramified for all $\chi \in Y_H^{class}$ and $f_v(\chi_v,c)$ is the unramified vector in $I_v(\chi_{u,v},m)$ with $f_v(\chi_v,c)(1) = 1$.*
- *If $v \in S$ then $f_v(\chi_v,c)$ is independent of the pair (χ,c).*
- *For any prime w dividing p and for any real prime $\sigma \in \Sigma_w$, the local section $f_\sigma(\chi_\sigma,c)$ depends only on χ_σ^{class} and κ_w (and on the choice of signature), and is of the form*

  $$f_\sigma(\chi_\sigma,c,g) = B(\chi_\sigma,\kappa_\sigma)D(\kappa_\sigma,m,\chi_{u,\sigma})J_{m,\chi_{u,\sigma}}(g), g \in G_{4,\sigma}$$

  where $J_{m,\chi_{0,\sigma}} \in C^\infty(G_4)$ is the canonical automorphy factor introduced in section 4.4.2 and $B(\chi_\sigma,\kappa_\sigma)$ is a non-zero scalar. In particular, $f_\sigma(\chi_\sigma,c,g)$ does not depend on the factorization of χ_σ. (This follows from Remark 4.4.3.
- *For any prime w dividing p, the local section $f_w(\chi_w,c)$ depends only on χ_w and ψ_w (and on the choice of signature).*
- *$\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,c)dEis = D^S(\chi)\cdot res_3 E_{f(\chi,c)}$ for all $(\chi,c) \in Y_H^{class}$, where $D^S(\chi)$ is the normalizing factor defined in* (4.6.1).

The measure $dEis$ is said to be normalized at S if instead of the last relation one has $\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,c)dEis = D(\chi)\cdot res_3 E_{f(\chi,c)}$ for all $(\chi,c) \in Y_H^{class}$.

One obtains a measure normalized at S from an unnormalized measure by multiplying by the appropriate product of local Euler factors at S. We write $D^?(\chi)$ for $? = S$ or empty if we haven't specified whether or not $dEis$ is taken to be normalized.

Definition 5.3.2 makes no mention of whether or not the measure $dEis$ contains a shift. The Eisenstein measure whose construction is recalled in section 5.4 comes with a shift that will be specified in Corollary 5.4.3.

We choose $f(\chi,c)$ meeting the conditions of Definition 5.3.2 in Sections 4.2.1 (local choices for $v \notin S$), 4.2.2 (local choices for $v \in S$), 4.5 (local choices for archimedean places) and 4.3 (local choices for $v \mid p$). Note that the choices at p and ∞ depend on the signature of the unitary group G_1. The existence of the Eisenstein measure itself is proved in [Eis15, Eis12]; see Theorem 5.4.1 below.

In the applications, the integrals of elements of Y_H^{class} against $dEis$ suffice to determine $dEis$ completely. We write

$$f^{holo}(\chi,c) = \otimes_{\sigma\in\Sigma_F} J_{m,\chi_{u,\sigma}} \otimes \otimes_{v\nmid\infty} f_v(\chi_v,c);$$

$$E^{holo}_{\chi_u,c}(m) = E_{f^{holo}(\chi_u,c)}(m). \tag{5.3.2}$$

Then the last condition of Definition 5.3.2 can be rewritten

(5.3.3)
$$\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,c)dEis = D^?(\chi)\cdot res_3 D(\kappa,m,\chi_u)E^{holo}_{\chi_u,c}(m), \forall(\chi=\|\bullet\|^m\cdot\chi_u,c)\in Y_H^{class},$$
where $D(\kappa,m,\chi_u)$ is as defined in Corollary 4.4.9.

5.4. **Existence of the axiomatic Eisenstein measure.** Let χ_{unitary} be a unitary Hecke character. Let $\chi=\chi_{\text{unitary}}|\cdot|_{\mathcal{K}/\mathbb{Q}}^{-k/2}$. So χ is a Hecke character of type A_0. Write $\chi=\prod_w\chi_w$. We obtain a p-adically continuous $\mathcal{O}_{\mathbb{C}_p}$-valued character $\tilde{\chi}$ on X_p as follows. Since χ is of type A_0, there are integers $k,\nu_\sigma\in\mathbb{Z}$ such that for each element $a\in\mathcal{K}^\times$,
$$\chi_\infty(a)=\prod_{\sigma\in\Sigma}\chi_\sigma(a)=\prod_{\sigma\in\Sigma}\left(\frac{1}{\sigma(a)}\right)^k\left(\frac{\overline{\sigma}(a)}{\sigma(a)}\right)^{\nu_\sigma}$$
with $\overline{\sigma}:=\sigma c$. Let $\tilde{\chi}_\infty:(\mathcal{K}\otimes\mathbb{Z}_p)^\times\to\overline{\mathbb{Q}}_p^\times$ be the p-adically continuous character such that
$$\tilde{\chi}_\infty(a)=incl_p\circ\chi_\infty(a)$$
for all $a\in\mathcal{K}$. So the restriction of $\tilde{\chi}_\infty$ to $(\mathcal{O}\otimes\mathbb{Z}_p)^\times$ is a $\mathcal{O}_{\mathbb{C}_p}^\times$-valued character. We define
$$\tilde{\chi}:X_p\to\mathcal{O}_{\mathbb{C}_p}^\times$$
by $\tilde{\chi}((a_w))=\tilde{\chi}_\infty\left((a_w)_{w|p}\right)\prod_{w\nmid\infty}\chi_w(a_w)$. Define $\nu=(\nu_\sigma)_{\sigma\in\Sigma}$.

For each $\sigma\in\Sigma$, let $n=a_\sigma+b_\sigma$ with $a_\sigma,b_\sigma\geq 0$ be a partition of n, and let $a_\sigma=n_{1,\sigma}+\cdots+n_{t(\sigma)}$ and $b_\sigma=n_{t(\sigma)+1}+\cdots+n_{r(\sigma)}$ be partitions of a_σ and b_σ, respectively. Let $a=(a_\sigma)_{\sigma\in\Sigma}$ and $b=(b_\sigma)_{\sigma\in\Sigma}$. Let ψ be a finite order character on $T_H(\mathbb{Z}_p)$. Let κ be a dominant character as in Section 2.6.3, and define ρ and ρ^υ as in (4.4.8), (4.4.10).

Let $c=\psi\cdot\rho^\upsilon$. We choose $f(\chi,c)$ to be a factorizable Siegel section meeting the conditions of Definition 5.3.2; the specific local sections will be as in Sections 4.2.1 (local choices for $v\notin S$), 4.2.2 (local choices for $v\in S$), and 4.3 (local choices for $v\mid p$), and 4.5 (local choices for archimedean places). Note that the choices at p and ∞ depend on the signature of the unitary group G_1. When ρ is trivial, the Eisenstein series associated to $f(\chi,c)=f(\chi,\psi)$ is holomorphic; in the notation of [Eis15], it is (a normalization of) the algebraic automorphic form denoted $G_{k,\nu,\chi_{\text{unitary}},\psi}$ (which arises over $\mathcal{O}$ but can be viewed over $\mathbb{C}$ by extending scalars) in [Eis14, Equation (32)].[12]

We denote by D here the C^∞ differential operator obtained by applying the Gauss-Manin connection composed with the Kodaira-Spencer isomorphism and then using the Hodge-de Rham splitting to project onto the submodule of holomorphic differentials; these differential operators were studied in [Eis12]. For each Σ-tuple of nonnegative

[12]The section $f(\chi,\psi)$ is the Siegel section associated to χ_{unitary}, k, ν, and ψ in [Eis14], and the associated Eisenstein series $E(f(\chi,\psi),\bullet)$ is the one denoted $E_{k,\nu}\left(\bullet,\chi,\psi,\frac{k}{2}\right)$ in [Eis15]. The Eisenstein series $E(f(\chi,\psi),\bullet)$ is normalized by a factor $D(n,K,\mathfrak{b},p,k)$ defined in [Eis15, Proposition 13] in order to cancel transcendental factors. Note that although we do not include (a,b) in the (already long) subscript for the Eisenstein series, the choice of $f(\chi,c)$ (and hence, the associated Eisenstein series) depends on the choice of (a,b).

integers $d = (d(\sigma))_{\sigma\in\Sigma}$, we write $D^d = \otimes_{\sigma\in\sigma} D^{d(\sigma)}$, where $D^{d(\sigma)}$ denotes D applied iteratively $d(\sigma)$ times to the σ-component of the module of automorphic forms. Applying D^d to $G_{k,\nu,\chi_{\text{unitary}},\psi}$, we obtain C^∞ (not necessarily holomorphic) Eisenstein series on $U(n,n)$. Similarly, by applying the p−adic differential operators θ in [Eis12] (where the differential operator is defined by applying the Gauss-Manin connection composed with the Kodaira-Spencer isomorphism and then using the unit root splitting to project modulo the unit root module) to $G_{k,\nu,\chi_{\text{unitary}},\psi}$, we obtain p-adic (not necessarily algebraic) automorphic forms on $U(n,n)$. We define θ^d analogously to how we defined D^d.

Like in Section 4.4.7, let $r_{1,\sigma} \geq \cdots \geq r_{a_\sigma,\sigma} \geq r_{a_\sigma+1,\sigma} = 0$, $s_{1,\sigma} \geq \cdots \geq s_{b_\sigma,\sigma} \geq s_{b_\sigma+1,\sigma} = 0$ be descending sequences of integers. Let ρ_σ^υ be the corresponding character on the torus T_H, and let

$$\tilde{r}_{i,\sigma} = r_{i,\sigma} - r_{i+1,\sigma}, i = 1, \dots a_\sigma; \tilde{s}_{j,\sigma} = r_{j,\sigma} - r_{j+1,\sigma}, j = 1, \dots b_\sigma.$$

Define $\rho^\upsilon := \prod_{\sigma\in\Sigma} \rho_\sigma^\upsilon$ and $\phi_\kappa := \otimes_{\sigma\in\Sigma} p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)$, with $p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)$ defined as in Equation (4.4.13) (and identified with a polynomial function on the tangent space of the moduli space). So $p(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)$ is a homogeneous polynomial of degree $d(\sigma)$ for some nonnegative integer $d(\sigma)$. We define $\theta^{(\kappa,a,b)}$ and $D^{(\kappa,a,b)}$ by

$$\theta^{(\kappa,a,b)}(f) := \theta^d(f)(\phi_\kappa)$$
$$D^{(\kappa,a,b)}(f) := D^d(f)(\phi_\kappa).$$

It follows from [Eis12, Section 10] (which extends [Kat78, Lemma 5.1.27] to unitary groups) that $\theta^{(\kappa,a,b)}(f)$ and $D^{(\kappa,a,b)}(f)$ agree at CM points, up to periods.

From the p-adic q-expansion principle and the description of the q-expansion coefficients given in [Eis15, Section 2], we obtain the following theorem.

Theorem 5.4.1 (The Eisenstein Measure)**.** *Recall the notation of Equation* (4.4.10)*. There is a measure* $\mathrm{Eis}_{a,b}$ *(dependent on a and b) on* $X_p \times T_H(\mathbb{Z}_p)$ *that takes values in the space of p-adic modular forms on* G_4 *and that satisfies*

$$\int_{X_p\times T_H(\mathbb{Z}_p)} \tilde{\chi}\psi \cdot \rho^\upsilon \mathrm{Eis}_{a,b} = \theta^{(\kappa,a,b)}\left(G_{k,\nu,\chi_{\text{unitary}},\psi}\right).$$

Remark 5.4.2. When $a_\sigma b_\sigma = 0$ for all $\sigma \in \Sigma$ (i.e. in the definite case), the measure in Theorem 5.4.1 is the Eisenstein measure from [Eis15, Theorem 20] and [Eis14, Section 5].

Corollary 5.4.3. *The measure* $dEis^{a,b}$*, defined by*

$$\int_{X_p\times T_H(\mathbb{Z}_p)} \tilde{\chi}\psi \cdot \rho^\upsilon d\mathrm{Eis}_{a,b} = res_3\theta^{(\kappa,a,b)}\left(G_{k,\nu,\chi_{\text{unitary}},\psi}\right).$$

is an axiomatic Eisenstein measure on values in $V_3(K_3^p, R)$*, with shift* $(1,\chi)$*.*

6. Serre duality, complex conjugation, and anti-holomorphic forms

6.1. **The Shimura variety** $Sh(V)$**.** Let $P = (\mathcal{K}, c, \mathcal{O}, L, \langle\cdot,\cdot\rangle, h)$ be a PEL problem of unitary type associated with a Hermitain pair $(V, \langle\cdot,\cdot\rangle_V)$ as in 2.1 and 2.2 together with all the associated objects, choices, and conventions from Section 2. However, since the number of factors m equals 1, the indexing subscript 'i' will disappear from our notation. Let $G = G_P$ be the group scheme over $\mathbb{Z}$ associated with P and let $X = X_P$ be the $G(\mathbb{R})$ conjugacy class of h. Let Z_G be the center of G. In this section we take $\square = \varnothing$, so the moduli problems are all considered over the reflex field F.

Given $K \subset G(\mathbb{A}_f) = GU(V)(\mathbb{A}_f)$, we now write ${}_KSh(V)$ for the Shimura variety associated with the Shimura datum[13] (G, X). So ${}_KSh(V)$ is just the F-scheme $\mathrm{M}_{K,L}$. We set

$$Sh(V) = \varprojlim_K {}_KSh(V) = \varprojlim_K \mathrm{M}_{K,L}.$$

The dimension of each ${}_KSh(V)$ is just the $\mathbb{C}$-dimension of X, which is

$$d = \frac{1}{2}\sum_{\sigma\in\Sigma_{\mathcal{K}}} a_\sigma b_\sigma.$$

At times we will be comparing constructions for both $Sh(V)$ and the Shimura variety $Sh(-V)$ for the pair $(V, -\langle\cdot,\cdot\rangle_V)$ (and the PEL problem $P^c = (\mathcal{K}, c, \mathcal{O}, L, -\langle\cdot,\cdot\rangle, h^c)$, where $h^c(z) = h(\bar{z})$). When it is important to distinguish which hermitian space an object is associated with, we will generally add a subscript 'V' (for the pair $(V, \langle\cdot,\cdot\rangle_V)$) or '$-V$' (for the pair $(V, -\langle\cdot,\cdot\rangle_V)$, if the notation does not already distinguish the space (such as is done by $Sh(V)$ and $Sh(-V)$). We will also be using the notation $G_1 = GU(V)$, $G_2 = GU(-V)$ as in 3.1.

6.1.1. *Automorphic vector bundles.* Recall that automorphic vector bundles on $Sh(V) = Sh(G, X)$ are defined by a $\otimes$-functor

$$G-\mathrm{Bun}(\hat{X}) \longrightarrow \mathrm{Bun}(Sh(V)),$$

where $\hat{X}$ is the compact dual of X, so a flag variety for G, and $G-\mathrm{Bun}$ is the $\otimes$-category of G-equivariant vector bundles. The base point $h \in X$ determines a point $P_h \in \hat{X}$; this is just the stabilizer of the Hodge filtration on $L \otimes \mathbb{R}$ determined by h. There is then a fibre functor $G-\mathrm{Bun}(\hat{X}) \to \mathrm{Rep}_{\mathbb{C}}(P_h) \cong \mathrm{Rep}_{\mathbb{C}}(P_0)$, where the last equivalence comes from the fixed identifications in 2.6.1. Given an irreducible representation W of P_0 that factors through the Levi quotient H_0 of P_0, we let ω_W be the corresponding automorphic vector bundle. Each such bundle has a canonical model over a number field $F(W)/F$ contained in F'. For $W = W_\kappa$ as in 2.6.3 (here and in the following we write W_κ for $W_\kappa(\mathbb{C})$), the vector bundle ω_κ defined in 2.6.4 is the base change to $\mathcal{K}'$ of the canonical model of ω_{W_κ}. In fact, the ω_κ, which are defined over the toroidal compactifications, are the canonical extensions of the automorphic vector bundles, and their twists by the ideal sheaves of the boundaries are the subcanonical bundles.

[13]If $a_\sigma b_\sigma = 0$ for all $\sigma \in \Sigma_{\mathcal{K}}$, then, properly speaking, the datum (G, X) does not satisfy the axioms of a Shimura variety as set out in [Del79]. Nevertheless, in this case, as the datum arises from a PEL problem P, the notion of the associated 'Shimura variety' still makes sense, following the conventions in [Lan12].

6.1.2. *Coherent cohomology and* $(\mathfrak{P}_h, K_h)$*-cohomology.* We will write $H^i(Sh(V), \omega_\kappa)$ instead of $H^0(Sh(V)^{tor}, \omega_\kappa)$, which is imperfect shorthand for

$$\varinjlim_{K,\Sigma} H^i({}_K Sh(V)_\Sigma, \omega_\kappa)$$

where the limit is taken over toroidal compactifications (indexed by Σ) at finite level (indexed by K). For $i = 0$, this is superfluous, by Koecher's principle, except possibly when $n = 2$ and $F = \mathbb{Q}$, and the reader can be trusted to supply the missing indices in this case. Likewise we write $H^i(Sh(V), \omega_\kappa^{\mathrm{sub}})$ for

$$\varinjlim_{K,\Sigma} H^i({}_K Sh(V)_\Sigma, \omega_\kappa(-D_\Sigma))$$

where $D_\Sigma = {}_K Sh(V)_\Sigma - {}_K Sh(V)$. We let

$$H^i_!(Sh(V), \omega_\kappa) = im\{H^i(Sh(V), \omega_\kappa^{\mathrm{sub}}) \to H^i(Sh(V), \omega_\kappa)\}.$$

Note that the ground field here can be taken to be any extension of $\mathcal{K}'$. Moreover, these definitions make sense over the ring $\mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$, provided we restrict to those K of the form $K = G(\mathbb{Z}_p)K^p$ or $K = I_r K^p$.

Over $\mathbb{C}$ the coherent cohomology can be computed in terms of Lie algebra cohomology. Let $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R}))_{\mathbb{C}}$, and let $\mathfrak{g} = \mathfrak{p}_h^- \oplus \mathfrak{k}_h \oplus \mathfrak{p}_h^+$ be the Harish-Chandra decomposition associated with h (the eigenvalue decomposition for the involution $ad\,h(\sqrt{-1})$). Let $\mathfrak{P}_h = \mathfrak{p}_h^- \oplus \mathfrak{k}_h$; this is just $\mathrm{Lie}(P_h(\mathbb{R}))_{\mathbb{C}}$ (so the Lie algebra of $P_h(\mathbb{C})$). We put

$$K_h = U_\infty = C(\mathbb{R}).$$

Let $\mathcal{A}_0(G)$ be the space of cuspforms on $G(\mathbb{A})$. Then over $\mathbb{C}$ there is a natural identification of $G(\mathbb{A}_f)$-modules:

$$H^i_!(Sh(V), \omega_\kappa) = H^i(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa). \tag{6.1.1}$$

Here we use the identifications of $P_h(\mathbb{C})$ with $P_0(\mathbb{C})$ and $C(\mathbb{C})$ with $H_0(\mathbb{C})$ to realize W_κ as a $(\mathfrak{P}_h, K_h)$-module. For $i = 0$ this just restates the identification, recalled in 2.7.2, of $S_\kappa(K, \mathbb{C})$ with the space of $U_\infty \times K$-invariant smooth functions $f : G(\mathbb{A}) \to W_\kappa$ that are annihilated by $\mathfrak{p}_h^-$.

6.1.3. *The* $\star$ *involution.* There is an anti-holomorphic involution $\star$ of $G - \mathrm{Bun}(\hat{X})$ that takes a G-equivariant bundle to the complex conjugate bundle; on representations of P_0 factoring through the Levi quotient H_0 (which has been identified over $\mathbb{C}$ with the stabilizer C in $G_{/\mathbb{R}}$ of h) it takes the irreducible representation W_κ to a representation $W_{\kappa^\star}$ whose restriction to the maximal compact subgroup of $U_\infty = C(\mathbb{R}) \subset H_0(\mathbb{C})$ is dual to the restriction of W_κ but whose restriction to $\mathbb{R}^\times \subset G(\mathbb{R})$ coincides with that of κ. Concretely, if κ is identified with the tuple $\kappa = (\kappa_0, (\kappa_\sigma))$, $\kappa_\sigma = (\kappa_{\sigma,1}, \ldots, \kappa_{\sigma,b_\sigma})$, then $\kappa^\star$ is the weight

$$\kappa^\star = (\kappa_0^\star, (\kappa_\sigma^\star)), \quad \kappa_0^\star = -\kappa_0 + a(\kappa), \ \kappa_\sigma^\star = (-\kappa_{\sigma,b_\sigma}, \ldots, -\kappa_{\sigma,1}) \tag{6.1.2}$$

and

$$W_{\kappa^\star} \cong W_\kappa^\vee \otimes \nu^{a(\kappa)},$$

where

$$a(\kappa) = 2\kappa_0 + \sum_{\sigma\in\Sigma_{\mathcal{K}}} \sum_{j=1}^{b_\sigma} \kappa_{\sigma,j}. \tag{6.1.3}$$

There is a unique, up to scalar multiple, c-semilinear, K_h-equivariant isomorphism $W_\kappa \xrightarrow{\sim} W_{\kappa^\star}$. Such an isomorphism is given explicitly by the map that sends $\phi \in W_\kappa$ to $\phi^\star \in W_{\kappa^\star}$, where if $h \in H_0(\mathbb{C})$ is identified with $(h_0, (h_\sigma)) \in \mathbb{C}^\times \times \prod_{\sigma\in\Sigma_{\mathcal{K}}} \mathrm{GL}_{b_\sigma}(\mathbb{C})$ via (2.6.1), then

$$\phi^\star(h) = h_0^{a(\kappa)} \cdot \overline{\phi((\bar{h}_0^{-1}, (w_\sigma {}^t\bar{h}_\sigma^{-1})))}.$$

Here $w_\sigma \in \mathrm{GL}_{b_\sigma}(\mathbb{C})$ is the longest element of the Weyl group of the standard pair and the overline $^-$ denotes complex conjugation. The K_h-invariance follows easily from (2.6.3).

The identification of $G(\mathbb{C})$ with $G_0(\mathbb{C})$ in 2.6.1 identifies $\mathrm{Lie}(P_0(\mathbb{C}))$ with $\mathfrak{P}_h$ and $\mathrm{Lie}(H^0(\mathbb{C}))$ with $\mathfrak{k}_h$. It then follows that the map $\phi \mapsto \phi^\star$ is $\mathfrak{P}_h$-equivariant, up to c-semilinearity.

The action of $h = (h_0, (h_\sigma)) \in H_0(\mathbb{C})$ on $\mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^\pm, \mathbb{C})$ is just multiplication by $h_0^{\mp d} \prod_{\sigma\in\Sigma_{\mathcal{K}}} \det(h_\sigma)^{\pm 2a_\sigma}$; this is just the character

$$\kappa_h^\pm = (\mp d, (\kappa_{h,\sigma}^\pm)), \quad \kappa_{h,\sigma}^\pm = (\pm 2a_\sigma, \ldots, \pm 2a_\sigma).$$

Then the $H_0(\mathbb{C})$-representation

$$\mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, W_{\kappa^\star}) = \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C}) \otimes_{\mathbb{C}} W_{\kappa^\star}$$

is naturally identified with W_{κ^D} (the identification depends on a choice of basis of the one-dimensional space $\wedge^d \mathfrak{p}_h^-$), where

$$\kappa^D = \kappa^\star + \kappa_h^+.$$

The Killing form on $\mathfrak{g}$ defines an $H_0(\mathbb{C})$-equivariant contraction map

$$\wedge^d \mathfrak{p}_h^- \otimes_{\mathbb{C}} \wedge^d \mathfrak{p}_h^+ \to \mathbb{C},$$

and so defines an $H_0(\mathbb{C})$-equivariant inclusion

$$i_{\kappa^\star} : W_{\kappa^\star} \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^- \otimes_{\mathbb{C}} \wedge^d \mathfrak{p}_h^+, W_{\kappa^\star}) = \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, W_{\kappa^D}).$$

6.2. **Complex conjugation and automorphic forms.** In this section we describe three actions of complex conjugation on spaces of modular forms. Each has an interpretation in Deligne's formalism for motives of absolute Hodge cycles, though we do not emphasize this here. We describe these actions in terms of $(\mathfrak{P}_h, K_h)$-cohomology as well in terms of coherent cohomology.

6.2.1. *Complex conjugation on automorphic forms.* Let π be a $(\mathfrak{g}, K_h)\times G(\mathbb{A}_f)$-representation occurring in the space $\mathcal{A}_0(G)$ of cuspforms on $G(\mathbb{A})$. We define $\bar{\pi}$ to be the complex conjugate representation; that is, $\bar{\pi}$ consists of the functions $\bar{\varphi}(g) = \overline{\varphi(g)}$ for $\varphi \in \pi$. The map $\pi \to \bar{\pi}$, $\varphi \mapsto \bar{\varphi}$, is c-semilinear and $K_h \times G(\mathbb{A}_f)$-equivariant, and even $\mathfrak{g}$-equivariant up to c-semilinearity. We then obtain a c-semilinear $G(\mathbb{A}_f)$-equivariant map

$$(\pi \otimes_{\mathbb{C}} W_\kappa)^{K_h} \xrightarrow{\varphi\otimes\phi\mapsto\bar{\varphi}\otimes\phi^\star} (\bar{\pi} \otimes_{\mathbb{C}} W_{\kappa^\star})^{K_h} \xrightarrow{id\otimes i_{\kappa^\star}} \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \otimes\bar{\pi} \otimes_{\mathbb{C}} W_{\kappa^D})^{K_h} \tag{6.2.1}$$

that is also $\mathfrak{P}_h$-equivariant, up to c-semilinearity. This induces a c-semilinear $G(\mathbb{A}_f)$-equivariant isomorphism

$$c_B : H^0(\mathfrak{P}_h, K_h; \pi \otimes_{\mathbb{C}} W_\kappa) \to H^d(\mathfrak{P}_h, K_h; \bar{\pi} \otimes_{\mathbb{C}} W_{\kappa^D}). \tag{6.2.2}$$

Taking π to be the space of cuspforms $\mathcal{A}_0(G)$ of $G(\mathbb{A})$ (so, in particular, $\bar{\pi} = \pi$), we obtain a c-semilinear $G(\mathbb{A}_f)$-equivariant isomorphism

$$c_B : H^0_!(Sh(V), \omega_\kappa) \xrightarrow{\sim} H^d_!(Sh(V), \omega_{\kappa^D}). \tag{6.2.3}$$

6.2.2. *Complex conjugation on $Sh(V)$.* Recall that

$$P^c = (\mathcal{K}, c, \mathcal{O}, L, -\langle\cdot,\cdot\rangle, h^c), \quad h^c(z) = h(\bar{z}),$$

is just the PEL datum of unitary type associated with the Hermitian pair $(V, -\langle\cdot,\cdot\rangle_V)$. The corresponding reflex field is $F_{-V} = cF_V = cF$, the complex conjugate of F. There is a canonical identification $G_{P^c} = G_P = G$. The respective stabilizers in $G(\mathbb{R})$ of h and h^c (action by conjugation) are the same: they both equal U_∞ (that is, $K_h = U_\infty = K_{h^c}$). Let $X = G(\mathbb{R})/U_\infty$. We then have identifications $X \xrightarrow{\sim} X_h = X_P$, $g \mapsto ghg^{-1}$, and $X \xrightarrow{\sim} X_{h^c} = X_{P^c}$, $g \mapsto gh^cg^{-1}$. Each of X_h and X_{h^c} have a complex structure, and the pullbacks of these complex structures to X are complex conjugates. In particular, the composition $X_h \xrightarrow{\sim} X \xrightarrow{\sim} X_{h^c}$ is an antiholomorphic map. So a holomorphic function on X_{h^c} defines an antiholomorphic function on X_h, and *vice versa.* This explains the map F_∞ in (6.2.7) below.

The automorphic sheaves on $Sh(-V)$ are associated to representations of the group $H_{0,-V}$, which is canonically identified with $H_{0,V} = H_0$ by switching the roles of Λ_0 and $\Lambda_0^\vee$). The analog of (2.6.1) for $H_{0,-V}$ is the isomorphism

$$H_{0,-V}{}_{/S_0} \xrightarrow{\sim} \mathbb{G}_m \times \prod_{\sigma\in\Sigma_{\mathcal{K}}} \mathrm{GL}_{\mathcal{O}\otimes_{\mathcal{O},\sigma}}(\Lambda_{0,\sigma}) \cong \mathbb{G}_m \times \prod_{\sigma\in\Sigma_{\mathcal{K}}} \mathrm{GL}_{a_\sigma}(S_0). \tag{6.2.4}$$

The identification $H_{0,V} = H_{0,-V}$ is given in terms of (2.6.1) and (6.2.4) by $(h_0, (h_\sigma)) \mapsto (h_0, (h_0{}^t h_{\sigma c}^{-1}))$. We have associated to each dominant character κ of the diagonal torus $T_{H_{0,-V}}$ of $H_{0,-V}$ a representation $W_{\kappa,-V}$ of $H_{0,-V}$ and hence a vector bundle $\omega_{\kappa,-V}$ on $Sh(-V)$. Given a dominant character $\kappa = (\kappa_0, (\kappa_\sigma))$ of $T_{H_{0,V}}$, we define a dominant character $\kappa^\flat = (\kappa_0, (\kappa_{\sigma c}))$ of $T_{H_{0,-V}}$. With respect to the canonical identification $H_{0,-V} = H_{0,V}$ described above, there is an explicit identification of H_0-representations

$$W_{\kappa^\flat,-V} \xrightarrow{\sim} W_{\kappa^\star,V}, \quad \phi \mapsto ((h,(h_\sigma)) \mapsto \phi(h_0, (w_\sigma h_0{}^t h_{\sigma c}^{-1}))).$$

The Harish-Chandra decompositions $\mathfrak{g} = \mathfrak{p}_h^- \oplus \mathfrak{k}_h \oplus \mathfrak{p}_h^+ = \mathfrak{p}_{h^c}^- \oplus \mathfrak{k}_{h^c} \oplus \mathfrak{p}_{h^c}^+$ satisfy $\mathfrak{p}_h^\pm = \mathfrak{p}_{h^c}^\mp$ and $\mathfrak{k}_h = \mathfrak{k}_{h^c}$. Let π be a $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$-representation occurring in the automorphic forms on $G(\mathbb{A})$. Then the natural map

$$(6.2.5) \qquad (\pi^{\mathfrak{p}_{h^c}^-} \otimes_{\mathbb{C}} W_{\kappa^\flat,-V})^{K_{h^c}} = (\pi^{\mathfrak{p}_h^+} \otimes_{\mathbb{C}} W_{\kappa^\star,V})^{K_h} \overset{id\otimes i_{\kappa^\star}}{\to} \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, \pi \otimes W_{\kappa^D,V})^{K_h}$$

induces a $\mathbb{C}$-linear $G(\mathbb{A}_f)$-equivariant isomorphism

$$(6.2.6) \qquad F_\infty : H_!^0(\mathfrak{P}_{h^c}, K_{h^c}; \pi \otimes_{\mathbb{C}} W_{\kappa^\flat,-V}) \to H_!^d(\mathfrak{P}_h, K_h; \pi \otimes_{\mathbb{C}} W_{\kappa^D,V}).$$

Taking π to be $\mathcal{A}_0(G)$ we then obtain a $\mathbb{C}$-linear $G(\mathbb{A}_f)$-equivariant isomorphism

$$(6.2.7) \qquad F_\infty : H_!^0(Sh(-V), \omega_{\kappa^\flat,-V}) \overset{\sim}{\longrightarrow} H_!^d(Sh(V), \omega_{\kappa^D,V}).$$

Note that **no complex conjugation is involved in this isomorphism**: F_∞ identifies a cohomology class on G_2 represented by a holomorphic modular form with a cohomology class represented by an anti-holomorphic modular form, simply because the groups G_1 and G_2 are canonically equal but the hermitian symmetric domains have opposite complex structure. In 6.3 it is explained how this isomorphism identifies Serre duality with the canonical pairing on $\mathcal{A}_0(G)$.

6.2.3. *The involution '$\dagger$' and the isomorphisms ${}_KSh(V) \cong {}_{K^\dagger}Sh(-V)$.* Recall that we have assumed that h is standard (see 2.3.2). This means that there is a $\mathcal{K}$-basis of V with respect to which the Hermitian pairing $\langle\cdot,\cdot\rangle_V$ is given by a diagonal matrix $D = \mathrm{diag}(d_1, ..., d_n)$, $d_1, ..., d_n \in \mathcal{K}^+$, and such that the image of h is diagonal with respect to the induced basis on each of the spaces $V_\sigma = V \otimes_{K,\sigma} \mathbb{C}$. Under the hypothesis that each prime above p in $\mathcal{K}^+$ splits in $\mathcal{K}$, it is always possible to choose such a $\mathcal{K}$-basis and the lattice L so that D is a diagonalization of the perfect Hermitian pairing on $L \otimes \mathbb{Z}_{(p)}$ induced by $\langle\cdot,\cdot\rangle_V$; we fix such a choice of $\mathcal{K}$-basis and a lattice L. Let $I : V \to V$ be the $\mathcal{K}^+$-involution of V that is just the action of c on the coordinates with respect to this fixed $\mathcal{K}$-basis. Note that $L \otimes \mathbb{Z}_{(p)}$ is I-stable, and the map induced by I on $L \otimes \mathbb{Z}_p$ interchanges L^+ and L^-.

With respect to the fixed $\mathcal{K}$-basis, $G_{/\mathbb{Q}}$ is identified with a subgroup of $\mathrm{Res}_{\mathcal{K}/\mathbb{Q}}\mathrm{GL}_n(\mathcal{K})$, and the action of c on $\mathcal{K}$ induces an automorphism $g \mapsto \bar{g}$ of $G_{/\mathbb{Q}}$ (note that $g^c = IgI$). This automorphism takes h to h^c and so maps U_∞ to itself. In particular, it induces an automorphism of X. The composition $X_h \overset{\sim}{\longrightarrow} X \overset{g\mapsto\bar{g}}{\longrightarrow} X \overset{\sim}{\longrightarrow} X_{h^c}$ (which is just $ghg^{-1} \mapsto \bar{g}h^c\bar{g}^{-1}$) is holomorphic. In particular, the induced map $Sh(V)(\mathbb{C}) \to Sh(-V)(\mathbb{C})$ is holomorphic and so a morphism of Shimura varieties over $\mathbb{C}$.

We modify this map at p, to more easily compare level structures. Recall that for each prime $w|p$ we fixed decompositions $L_w = L_w^+ \oplus L_w^-$ (see 2.2). We also fixed an $\mathcal{O}_w$-basis of each $L_w^\pm$, which gives an $\mathcal{O}_w$-basis of each L_w. We define level structures at p for P^c by taking $L_w^{c,\pm} = L_w^\pm$. Then $I_{w,-V}^0 = {}^tI_{w,V}^0 = {}^t(I_{w,V}^0)^{-1}$ with respect to this $\mathcal{O}_w$-basis of L_w. This chosen $\mathcal{O}_w$-basis of L_w may not be the $\mathcal{K}_w$-basis of $V \otimes_{\mathcal{K}} \mathcal{K}_w$ induced by the fixed $\mathcal{K}$-basis of V; let $\beta_w \in \mathrm{GL}_{\mathcal{K}_w}(V \otimes_{\mathcal{K}} \mathcal{K}_w) \cong \mathrm{GL}_n(\mathcal{K}_w)$ be an element taking the latter to former. Let $\delta_p = (1, D^t\beta_w^{-1}\beta_w^{-1})_{w\in\Sigma_p}) \in \mathbb{Q}_p^\times \times \prod_{w\in\Sigma_p} \mathrm{GL}_n(\mathcal{K}_w) \cong G(\mathbb{Q}_p)$, where the isomorphism is determined by the fixed $\mathcal{K}$-basis of V. Then

$$\bar{\delta}_p = \delta_p^{-1}, \quad \delta_p^{-1}\overline{G(\mathbb{Z}_p)}\delta_p = G(\mathbb{Z}_p), \quad \text{and} \quad \delta_p^{-1}\bar{I}^0_{r,V}\delta_p = I^0_{r,-V}. \tag{6.2.8}$$

We then define an automorphism $g \mapsto g^\dagger$ of $G(\mathbb{A}) \to G(\mathbb{A})$ by $g^\dagger = \nu(g)^{-1}\delta_p^{-1}\bar{g}\delta_p$. Given $K \subset G(\mathbb{A}_f)$ we let $K^\dagger$ be the image of K under $\dagger$. As a consequence of (6.2.8), if $K = G(\mathbb{Z}_p)K^p$, then $K^\dagger = G(\mathbb{Z}_p)\bar{K}^p$ and

$$(K_{r,V})^\dagger = K^\dagger_{r,-V}. \tag{6.2.9}$$

Consequently, the map $Sh(V)(\mathbb{C}) \to Sh(-V)(\mathbb{C})$ induced by $g \mapsto \bar{g}\delta_p$ identifies ${}_{K_{r,V}}Sh(V)$ with ${}_{K^\dagger_{r,-V}}Sh(-V)$. The following Proposition is then obvious.

Proposition 6.2.4. *The isomorphism* ${}_{K_{r,V}}Sh(V) \xrightarrow{\sim} {}_{K^\dagger_{r,-V}}Sh(-V)$ *is defined over* $\mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$. *On moduli problems it is given by the map that sends a tuple* $(A,\lambda,\iota,\alpha,\phi)$ *classified by* $M_{P,K_r,L}(R)$ *to the tuple* $(A,\lambda,\iota\circ c,\alpha\circ I,\phi\circ I)$ *classified by* $M_{P^c,K_r^\dagger,L}(R)$ *for any* $\mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$*-algebra* R.

The automorphism $g \mapsto g^\dagger$ takes $\mathfrak{p}_h^\pm$ to $\mathfrak{p}_{h^c}^\pm$ and P_h to P_{h^c}. The action of $g \mapsto g^\dagger$ on K_h is identified via (2.6.3) as $(h_0,(h_\sigma)) \mapsto (h_0^{-1},({}^th_\sigma^{-1}))$. Let

$$\kappa^\dagger = \kappa^\flat \cdot \|\nu\|^{-a(\kappa)},$$

so

$$W_{\kappa^\dagger,-V} \cong W_{\kappa^\vee,V} \cong W^\vee_{\kappa,V}.$$

The map

$$W_{\kappa,V} \overset{\phi\mapsto\phi^\dagger}{\longrightarrow} W_{\kappa^\dagger,-V} = W_{\kappa^\flat,-V} \otimes \|\nu\|^{-a(\kappa)}, \quad \phi^\dagger((h_0,(h_\sigma)) = \phi((h_0,(h_{\sigma c}))h_0^{-a(\kappa)},$$

satisfies $(k^\dagger\cdot\phi)^\dagger = k\cdot\phi^\dagger$ for all $k \in K_{h^c} = K_h$. It follows that under the isomorphism $Sh(V) \xrightarrow{\sim} Sh(-V)$ defined by $g \mapsto g^\dagger$, $\omega_{\kappa^\dagger,-V}$ pulls back to $\omega_{\kappa,V}$, and so there are $\mathbb{C}$-linear isomorphisms

$$F^\dagger : H^i_!(Sh(V),\omega_{\kappa,V}) \xrightarrow{\sim} H^i_!(Sh(-V),\omega_{\kappa^\dagger,-V}) \tag{6.2.10}$$

that are $G(\mathbb{A}_f)$-equivariant up to the action of the automorphism '†.' In particular, these induces isomorphisms

$$F^\dagger : H^i_!({}_{K_r}Sh(V),\omega_{\kappa,V}) \xrightarrow{\sim} H^i_!({}_{K_r^\dagger}Sh(-V),\omega_{\kappa^\dagger,-V}), \tag{6.2.11}$$

even over $\mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$-algebras $R \subset \mathbb{C}$. In particular, $F^\dagger$ restricts to an isomorphism

$$F^\dagger : S_{\kappa,V}(K_{r,V},\psi;R) \xrightarrow{\sim} S_{\kappa^\dagger,-V}(K^\dagger_{r,-V},\psi^\dagger;R) \tag{6.2.12}$$

for $R \subset \mathbb{C}$ any $\mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}[\psi]$-algebra, where $\psi^\dagger = \psi^{-1}$ if both are viewed as characters of the diagonal torus of the right side of (2.2.2) via the isomorphisms (2.2.3).

The action of $F^\dagger$ is described in terms of automorphic forms as follows. Let π be a $(\mathfrak{g},K_h)\times G(\mathbb{A}_f)$ representation occurring in the space of automorphic forms on $G(\mathbb{A})$. We define $\pi^\dagger$ to be the space of functions $\varphi^\dagger(g) = \varphi(g^\dagger)$ for $\varphi \in \pi$. The map $\pi \to \pi^\dagger$, $\varphi \mapsto \varphi^\dagger$, is $\mathbb{C}$-linear and is both $\mathfrak{g}$- and K_h-equivariant up to the action of the automorphism '†'. The map

$$(\pi \otimes_{\mathbb{C}} W_{\kappa,V})^{K_h} \overset{\varphi\otimes\phi\mapsto\varphi^\dagger\otimes\phi^\dagger}{\longrightarrow} (\pi^\dagger \otimes_{\mathbb{C}} W_{\kappa^\dagger,-V})^{K_{h^c}}$$

is then a $\mathbb{C}$-linear isomorphism that intertwines the actions of g and $g^\dagger$ for all $g \in G(\mathbb{A}_f)$. This induces a corresponding isomorphism

$$F^\dagger : H^i(\mathfrak{P}_h, K_h; \pi \otimes_{\mathbb{C}} W_{\kappa,V}) \overset{\sim}{\longrightarrow} H^i(\mathfrak{P}_{h^c}, K_{h^c}; \pi^\dagger \otimes_{\mathbb{C}} W_{\kappa^\dagger,-V}). \tag{6.2.13}$$

Taking $\pi = \mathcal{A}_0(G)$ (and so $\pi^\dagger = \pi$), we get $F^\dagger$ from before.

6.3. **Serre duality and pairing of automorphic forms.** Since

$$W_{\kappa^D} = \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, W_{\kappa^\star}) \cong \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C}) \otimes_{\mathbb{C}} W_\kappa^\vee \otimes \nu^{a(\kappa)},$$

the natural contraction $W_\kappa \otimes_{\mathbb{C}} W_\kappa^\vee \to \mathbb{C}$ gives a homomorphism of $H_0(\mathbb{C})$-representations

$$W_\kappa \otimes_{\mathbb{C}} W_{\kappa^D} \to \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, \mathbb{C}) \otimes \nu^{a(\kappa)}.$$

This induces a natural map

$$\omega_\kappa \otimes \omega_{\kappa^D} \ \to\ \Omega^d_{Sh(V)} \otimes L(\kappa),$$

where $L(\kappa)$ is the automorphic line bundle attached to the character $\nu^{a(\kappa)}$. Since the character is trivial on G^{der}, $L(\kappa)$ is topologically the $\mathcal{O}_{Sh(V)}$-bundle attached to the constant (trivial) sheaf, but the action of $G(\mathbb{A}_f)$ on $L(\kappa)$ is non-trivial. Fixing a level subgroup K and a toroidal compactification ${}_KSh(V) \hookrightarrow_K Sh(V)_\Sigma$, we can extend this to a natural pairing

$$\omega_\kappa \otimes \omega_{\kappa^D}^{\mathrm{sub}} \ \to\ \Omega^d_{{}_KSh(V)_\Sigma} \otimes L(\kappa)$$

and the analogous pairing on $\omega_\kappa^{\mathrm{can}} \otimes \omega_{\kappa^D}$. As in [Har90, Cor. 2.3], Serre duality therefore defines a perfect pairing

$$H^0_!(Sh(V), \omega_\kappa) \otimes H^d_!(Sh(V), \omega_{\kappa^D}) \ \to\ \varinjlim_{K,\Sigma} H^d({}_KSh(V)_\Sigma, \Omega^d_{Sh(V)_\Sigma} \otimes L(\kappa)) \tag{6.3.1}$$

The function $g \mapsto \|\nu(g)\|^{-a(\kappa)}$ defines a global section of $L(\kappa)^\vee$ and therefore an isomorphism

$$\varinjlim_{K,\Sigma} H^d({}_KSh(V)_\Sigma, \Omega^d_{Sh(V)_\Sigma} \otimes L(\kappa)) \overset{\sim}{\to} \varinjlim_{K,\Sigma} H^d({}_KSh(V)_\Sigma, \Omega^d_{Sh(V)_\Sigma}).$$

The right-hand side is isomorphic under the trace map to the space of functions $C(\pi_0(V))$ on the compact space $\pi_0(V)$ of similitude components of $Sh(V)$. Composing with the projection of $C(\pi_0(V))$ onto the invariant line $C(\pi_0(V))^{G(\mathbb{A})}$ – in other words, integration over $\pi_0(V)$ with respect to an invariant measure with rational total mass – we thus obtain a canonical perfect pairing:

$$\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}} : H^0_!(Sh(V), \omega_\kappa) \otimes H^d_!(Sh(V), \omega_{\kappa^D}) \ \to\ \mathbb{C} \tag{6.3.2}$$

Remark 6.3.1. In what follows, we will be using the Tamagawa number to normalize the Serre duality pairings. This is likely to introduce a factor of a power of 2 in a comparison of our results with those predicted by motivic conjectures.

The pairing $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$ can be described in terms of automorphic forms as follows. Let $\mathfrak{p} = \mathfrak{p}_h^+ \oplus \mathfrak{p}_h^-$. Then $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$ is just the pairing

$$H^0(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_\kappa) \otimes H^d(\mathfrak{P}_h, K_h; \mathcal{A}_0(G) \otimes W_{\kappa^D}) \to \mathbb{C}$$

defined by multiplication of cuspforms, contraction of the coefficients, and integration. More precisely, if we denote the contraction

$$W_\kappa \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^+, W_\kappa^\vee \otimes \nu^{a(\kappa)})) \to \mathrm{Hom}_{\mathbb{C}}(\wedge^{2d}\mathfrak{p}, \mathbb{C}(\nu^{a(\kappa)})),$$

by

$$\phi \otimes \phi' \mapsto [\phi, \phi'],$$

then for

$$\varphi \in (\mathcal{A}_0(G)^{\mathfrak{p}_h^-} \otimes_{\mathbb{C}} W_\kappa)^{K_h} \quad \text{and} \quad \varphi' \in \mathrm{Hom}_{\mathbb{C}}(\wedge^d \mathfrak{p}_h^-, \mathcal{A}_0(G) \otimes_{\mathbb{C}} W_{\kappa^D})^{K_h},$$

we normalize $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$ so that

$$\langle \varphi, \varphi' \rangle_\kappa^{\mathrm{Ser}} = \int_{G(\mathbb{Q})Z_G(\mathbb{R})\backslash G(\mathbb{A})} [\varphi(g), \varphi'(g)] \cdot \|\nu(g)\|^{-a(\kappa)} dg, \tag{6.3.3}$$

where dg is the Tamagawa measure. From $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$ we obtain the hermitian Petersson pairing:

$$\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Pet}} : H_!^0(Sh(V), \omega_\kappa) \times H_!^0(Sh(V), \omega_\kappa) \to \mathbb{C}, \quad \langle\cdot,\cdot\rangle_\kappa^{\mathrm{Pet}} = \langle \cdot, c_B(\cdot)\rangle_\kappa^{\mathrm{Ser}}. \tag{6.3.4}$$

6.3.2. *Integral structures on top cohomology.* Let $\mathcal{O} = \mathcal{O}_{\mathcal{K}',(\mathfrak{p}')}$ as in the previous section. Fix V and write $\omega_\kappa = \omega_{\kappa,V}$. The spaces $H_!^i({}_{K_r}Sh(V), \omega_\kappa)$ have natural integral structures over $\mathcal{O}$ with respect to $\mathcal{O}$-integral structures on the underlying schemes, for any i. However, because the special fibers become progressively more singular as r increases, we *do not choose* integral structures on the schemes. For cohomology in degree $i = 0$, we define the $\mathcal{O}$-structure on $H_!^0({}_{K_r}Sh(V), \omega_{\kappa,V})$ by $S_\kappa(K_r, \mathcal{O})$ as in §2, specifically in sections 2.5, 2.6.5, and especially 2.9. We then define the $\mathcal{O}$-structure on $H_!^d({}_{K_r}Sh(V), \omega_{\kappa^D})$ to be *dual* to the integral structure on $H_!^0({}_{K_r}Sh(V), \omega_\kappa)$ with respect to the pairing (6.3.2). In other words, for any $\mathcal{O}$-algebra R, we let

$$H_!^d({}_{K_r}Sh(V), \omega_{\kappa^D}, R) = Hom(S_\kappa(K_r, \mathcal{O}), R) \tag{6.3.5}$$

We will see in Lemma 7.2.12 that these integral structures are compatible with respect to the trace maps from level K_{r+1} to K_r.

6.4. **(anti-)holomorphic automorphic representations.** By an automorphic representation of G we will always mean an *irreducible* $(\mathfrak{g}, K_h) \times G(\mathbb{A}_f)$-representation ocurring in the space of automorphic forms on $G(\mathbb{A})$. This convention allows us to distinguish holomorphic representations from anti-holomorphic representations. (Note that K_h, which is the stabilizer of h in $G(\mathbb{R})$, need not project to the maximal compact in $G(\mathbb{R})/Z_G(\mathbb{R})$.)

6.4.1. *Holomorphic and anti-holomorphic cuspidal representations of type* (κ, K). Let π be a cuspidal automorphic representation of G (always assumed irreducible). Write $\pi = \pi_\infty \otimes \pi_f$, where π_f is an irreducible admissible representation of $G(\mathbb{A}_f)$ and π_∞ is an irreducible $(\mathfrak{g}, K_h)$-module. Let $K \subset G(\mathbb{A}_f)$ be an open compact. We say π is *holomorphic* (resp. *anti-holomorphic*) of type (κ, K) if $H^0(\mathfrak{P}_h, K_h; \pi_\infty \otimes_{\mathbb{C}} W_\kappa) \neq 0$ (resp. $H^d(\mathfrak{P}_h, K_h; \pi_\infty \otimes_{\mathbb{C}} W_{\kappa^D}) \neq 0$) and if $\pi_f^K \neq 0$. In this paper, we will only be concerned with π that are either holomorphic or anti-holomorphic. If π is holomorphic (resp. anti-holomorphic) of type (κ, K), then by our conventions $\bar{\pi}$ is anti-holomorphic (resp. holomorphic) of type (κ, K).

Note that, with G fixed, π can be either holomorphic or anti-holomorphic, but not both; however, the isomorphism F_∞ of (6.2.6) identifies anti-holomorphic representations of G_2 with holomorphic representations of G_1, and vice versa. Although Hida theory is generally understood to be a theory of p-adic variation of (ordinary) holomorphic modular forms, the nature of the doubling method makes it more natural for us to take our basic object π to be an *anti-holomorphic* (and anti-ordinary, see 6.5.6 below) cuspidal automorphic representation of G_1. Thus π is a *holomorphic* automorphic representation of G_2 but the natural object there is $\pi^\flat$, or $\bar{\pi}$, which is again anti-holomorphic. Because this is inevitably a source of confusion, reminders of these conventions have been inserted at strategic locations in the text.

Remark 6.4.2. If π is holomorphic or anti-holomorphic, then, by the considerations in [BHR94], π_f is always defined over a number field, say $E(\pi)$. We will always take $E(\pi)$ to contain $\mathcal{K}'$.

6.4.3. *The $\flat$ involution and the MWV involution $\dagger$.* Let π be a cuspidal automorphic representation of G. Let ξ_π be the central character π. If $(\pi_\infty \otimes_{\mathbb{C}} W_\kappa)^{K_h} \neq 0$ (for example, if π is holomorphic of type (κ, K)), then $\xi_{\pi,\infty}(t) = t^{a(\kappa)}$ for $t \in \mathbb{R}^\times$. Let

$$\pi^\flat = \pi^\vee \otimes |\xi_\pi \circ \nu| = \pi^\vee \otimes ||\nu||^{a(\kappa)}. \tag{6.4.1}$$

Because $\pi \otimes |\xi_\pi \circ \nu|^{-\frac{1}{2}}$ is unitary,

$$\pi^\flat \cong \bar{\pi}, \tag{6.4.2}$$

and when π occurs with multiplicity one, as we will generally assume, $\pi^\flat$ and $\bar{\pi}$ are the same spaces of automorphic forms. In particular, the operation $\pi \mapsto \pi^\flat$ is an involution of the set of cuspidal automorphic representations of G. If π is holomorphic, then $\pi^\flat$ is anti-holomorphic, and vice versa.

The involution $g \mapsto g^\dagger$ of G that was fixed in 6.2.2 is an involution of the type considered by Moeglin, Vigneras, and Waldspurger in [MVW87, Chapitre 4]. In particular, there is an element $h_0 \in \mathrm{GL}_{\mathcal{K}^+}(V)$ such that h_0 is c-semilinear for the $\mathcal{K}$-action on V and $\langle h_0 v, h_0 w\rangle_V = \langle w, v\rangle_V$ and such that $\bar{g} = h_0 g h_0^{-1}$; with respect to the fixed $\mathcal{K}$-basis of V, h_0 is just 'act-by-c on the coordinates'. Let $\pi = \otimes_{\ell \le \infty} \pi_\ell$ be an automorphic representation of G. If the hermitian pair $(V, \langle\cdot,\cdot\rangle_V)$ is unramified at ℓ, then it is a deep result proved in [MVW87, Chapitre 4] (cf. [HKS96]) that

$$\pi_\ell^\dagger \cong (\pi_\ell \circ \mathrm{Ad}(h_0)) \otimes (\xi_\pi^{-1} \circ \nu) \cong \pi^\vee. \tag{6.4.3}$$

In particular, if π satisfies strong multiplicity one – which we expect if all the places at which $(V, \langle\cdot,\cdot\rangle_V)$ is ramified all split in $\mathcal{K}/\mathcal{K}^+$ and its base change to $\mathrm{GL}_{n/\mathcal{K}}$ is cuspidal – then $\pi^\dagger \cong \pi^\vee$ and so $\pi^\dagger \otimes ||\nu||^{a(\kappa)} = \pi^\flat = \bar{\pi}$. In any event, (6.4.3) permits the Hecke actions on $\pi^\flat$ to be expressed in terms of the Hecke actions on $\pi^\dagger$, at least at the unramified primes ℓ. As will be explained later, the doubling method will pair π and (a twist) of $\pi^\flat$, but we will use the involution '†' to compare level structures and Hecke algebras. This partly motivates our putting

$$K^\flat = K^\dagger, \quad \psi^\flat = \psi^\dagger, \quad \text{and} \quad \kappa^\flat = \kappa^\dagger \cdot \nu^{a(\kappa)}. \tag{6.4.4}$$

6.4.4. *Relating $\langle\cdot,\cdot\rangle_\pi$ to $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$.* Let π be a holomorphic cuspidal automorphic representation of G of type (κ, K). Recall that the canonical pairing $\langle\cdot,\cdot\rangle_\pi : \pi \otimes \pi^\vee \to \mathbb{C}$ can be expressed as

$$\langle \varphi, \varphi'\rangle_\pi = \int_{G(\mathbb{Q})Z_G(\mathbb{R})\backslash G(\mathbb{A})} \varphi(g)\varphi'(g)dg, \quad \varphi \in \pi, \ \varphi' \in \pi^\vee. \tag{6.4.5}$$

The pairing $\langle\cdot,\cdot\rangle_\kappa^{\mathrm{Ser}}$ can be expressed in terms of $\langle\cdot,\cdot\rangle_\pi$ as follows.

Let $w_1, ..., w_m$ be a basis of W_κ and let $w_i^\vee, ..., w_m^\vee$ be the dual basis of $W_\kappa^\vee$. As W_{κ^D} is the twist of $W_\kappa^\vee$ by a character, the $\omega_i^\vee$ also defined a basis of W_{κ^D}. Let $\varphi \in (\pi^{\mathfrak{p}_h^-} \otimes_{\mathbb{C}} W_\kappa)^{K_h}$ and $\varphi' \in \mathrm{Hom}(\wedge^d \mathfrak{p}_h^-, \pi^\flat \otimes_{\mathbb{C}} W_{\kappa^D})^{K_h}$. Write $\varphi = \sum_i \varphi_i \otimes w_i$ and $\varphi' = \sum_j \varphi_j' \otimes w_j^\vee$. Then it follows from (6.3.3) that

$$\langle \varphi, \varphi'\rangle_\kappa^{\mathrm{Ser}} = \sum_i \langle \varphi_i, \varphi_i' \cdot ||\nu||^{-a(\kappa)}\rangle_\pi. \tag{6.4.6}$$

6.5. **Hecke algebras.** We continue to let $G = G_1 = GU(V)$ and we return to the notation of Section 2.9.4; thus classical modular forms are of weight κ. Fix a positive integer r as in 2.5 and a level subgroup $K = K_r^p = K^p \cdot K_r \subset G(\mathbb{A}_f)$. Henceforth we will write $T(g) = T_r(g)$ for the Hecke operators $[K_r^p g K_r^p]$ for $g \in G(\mathbb{A}_f^p)$; we have also introduced U-operators $U_{w,j}$ (the index i of 2.6.9 is superfluous because G is the unitary similitude group of a single hermitian space).

For any S_0-algebra $R \subset \mathbb{C}$, we let $\mathbf{T}_{K_r,\kappa,R}$ be the R-subalgebra of $\mathrm{End}_{\mathbb{C}}(S_\kappa(K_r;\mathbb{C})) = \mathrm{End}_{\mathbb{C}}(H^0({}_{K_r}Sh(V), \omega_\kappa))$ generated by the $U_{w,j,\kappa} = |\kappa'(t_{w,j})|_p^{-1} U_{w,j}$, where κ' is related to κ as in (2.9.4), and by the $T(g) = T_r(g)$ for $g \in G(\mathbb{A}_f^S)$, where $S = S(K^p)$ is the set of places at which K^p does not contain a hyperspecial maximal subgroup. We similarly define $\mathbf{T}^-_{K_r,\kappa,R}$ and $\mathbf{T}^d_{K_r,\kappa,R}$ by replacing $U_{w,j,\kappa}$ with $U^-_{w,j,\kappa} = |\kappa'(t_{w,j})|_p U^-_{w,j}$ and, in the second case, also replacing $S_\kappa(K_r;\mathbb{C})$ with $H^d({}_{K_r}Sh(V),\omega_\kappa) = H^d(Sh(V),\omega_\kappa)^{K_r}$. We will follow the convention of adding a subscript 'V' (reps. '$-V$') to notation if it is needed to indicate that it relates to the hermitian pair $(V,\langle\cdot,\cdot\rangle_V)$ (resp. $(V,-\langle\cdot,\cdot\rangle_V)$).

Lemma 6.5.1. *Let $R \subset \mathbb{C}$ be a subring.*

(i) *There exists a unique R-algebra isomorphism $\mathbf{T}_{K_r,\kappa,R} \xrightarrow{\sim} \mathbf{T}^d_{K_r,\kappa^D,R}$, $T \mapsto T^d$, such that $U^d_{w,j,\kappa} = U^-_{w,j,\kappa^D}$ and $T(g)^d = \|\nu(g)\|^{a(\kappa)} \cdot T(g^{-1})$.*

(ii) *There exists a unique R-algebra isomorphism $\mathbf{T}_{K_r,\kappa,V,R} \xrightarrow{\sim} \mathbf{T}_{K_r^\flat,\kappa^\flat,-V,R}$, $T \mapsto T^\flat$, such that $U^\flat_{w,j,\kappa} = U^{-1}_{w,n,\kappa^\flat} U_{w,n-j,\kappa^\flat}$ and $T(g)^\flat = T(g^\dagger) = T(\bar{g})$.*

(iii) *There exists a unique isomorphism $\mathbf{T}^-_{K_r,\kappa,R} \xrightarrow{\sim} \mathbf{T}^d_{K_r,\kappa^D,cR}$ that maps $r \in R$ to $c(r)$, U^-_{w,j,κ^D} to U^-_{w,j,κ^D}, and $T(g)$ to $T(g)$.*

Proof. Part (i) follows from Serre duality, part (ii) from the isomorphism $F^\dagger_\infty$, and part (iii) from the isomorphism c_B. □

For a nebentypus ψ of level r (a character of $T_H(\mathbb{Z}_p)$ that factors through $T_H(\mathbb{Z}_p/p^r\mathbb{Z}_p)$), we let $\mathbf{T}_{K_r,\kappa,\psi,R}$ and $\mathbf{T}^d_{K_r,\kappa,\psi,R}$ be the quotients of $\mathbf{T}_{K_r,\kappa,R}$ and $\mathbf{T}^d_{K_r,\kappa,R}$ upon restriction to the (invariant) subspaces $S_\kappa(K_r,\psi,\mathbb{C}) \subset S_\kappa(K_r,\mathbb{C})$ and $H^d({}_{K_r}Sh(V),\omega_\kappa)^\psi$, the subspace of $H^d({}_{K_r}Sh(V),\omega_\kappa)$ on which K_r^0 acts via ψ.

Lemma 6.5.2. *The isomorphisms in Lemma 6.5.1(i)-(ii) induce R-algebra isomorphisms*

$$\mathbf{T}_{K_r,\kappa,\psi,R} \xrightarrow{\sim} \mathbf{T}^d_{K_r,\psi^{-1},\omega_{\kappa^D},R} \quad \text{and} \quad \mathbf{T}_{K_r,\kappa,\psi,V,R} \xrightarrow{\sim} \mathbf{T}_{K_r^\flat,\kappa^\flat,\psi^\flat,-V,R}.$$

This is clear from the definitions.

The R-modules $S_\kappa(K_r;R)$ and $S_\kappa(K_r,\psi;R)$ are stable under the action of the Hecke operators $U_{w,j,\kappa}$ and $T(g)$, $g \in G(A_f^S)$. In particular, the cuspforms over $\mathbb{C}$ can be replaced by those over R in the definition of $\mathbf{T}_{K,\kappa,R}$ and $\mathbf{T}_{K,\kappa,\psi,R}$.

For any of these Hecke algebras $\mathbf{T}^?_\bullet$, we write $\mathbf{T}^{?,p}_\bullet$ for the subalgebra generated over the ring R by the $T(g)$, $g \in G(\mathbb{A}_f^S)$ (so omitting the $U_{w,j,\kappa}$ and $U^-_{w,j,\kappa}$). The isomorphisms of Lemmas 6.5.1 and 6.5.2 restrict to corresponding isomorphisms of these (p-depleted) Hecke rings.

If $R = S_0$, then we omit the subscript 'R' from our notation.

6.5.3. *The homomorphism λ^p_π, isotypical subspaces, and the multiplicity one hypothesis.* Let π be a holomorphic cuspidal representation of G of type (κ, K_r). Then the natural action of $\mathbf{T}^p_{K_r,\kappa}$ on π^{K_r} is given by a character that we denote λ^p_π; these homomorphisms are compatible under the natural projections $\mathbf{T}^p_{K_r,\kappa} \twoheadrightarrow \mathbf{T}^p_{K_{r'},\kappa}$, $r \geq r'$, so we do not include the r in our notation. Via the isomorphism $\mathbf{T}_{K_r,\kappa,V} \xrightarrow{\sim} \mathbf{T}_{K_r^\flat,\kappa^\flat,-V}$ of Lemma 6.5.1(ii), $\lambda^p_{\pi,V} = \lambda^p_\pi$ determines a homomorphism $\lambda^{p,\flat}_{\pi,V}$ of $\mathbf{T}^p_{K_r^\flat,\kappa^\flat,-V,R}$, which, by (6.4.3), satisfies

$$\lambda^{p,\flat}_{\pi,V} = \lambda^p_{\pi^\flat,-V}. \tag{6.5.1}$$

For an S_0-algebra $R \subset \mathbb{C}$, the homomorphism λ^p_π extends R-linearly to a homomorphism of the Hecke algebras over R; we use the same notation for this homomorphism.

We say that π satisfies the *multiplicity one hypothesis for* π if:

Hypothesis 6.5.4 (Multiplicity one hypothesis)**.** *For any holomorphic cuspidal $\pi' \neq \pi$ of type (κ, K_r), $\lambda^p_{\pi'} \neq \lambda^p_\pi$.*

This multiplicity one hypothesis for π is expected to hold if $S = S(K^p)$ consists only of places that are split in $\mathcal{K}/\mathcal{K}^+$ (so no local L-packets) and if the base change of π to $\mathrm{GL}_{n/\mathcal{K}}$ is cuspidal (so π is not obtained by endoscopic transfer from a non-trivial elliptic endoscopic group of G). When G is quasi-split this has been established by Mok [Mok13], and the general case has been proved under certain restrictive hypotheses and is being treated by Kaletha, Mínguez, Shin, and White. We will generally assume that π satisfies this multiplicity one hypothesis; this is not indispensable, but it simplifies some of the statements.

Let $E(\lambda^p_\pi)$ be the extension of the number field $E(\pi)$ generated by the values of λ^p_π; this is a finite extension of $E(\pi)$.

We fix a basis of the one-dimensional space $H^0(\mathfrak{P}_h, K_h; \pi_\infty \otimes_{\mathbb{C}} W_\kappa)$. Let $S_\kappa(K_r, \mathbb{C})(\pi)$ be the λ^p_π-isotypic subspace of $S_\kappa(K_r, \mathbb{C})$ for the action of $\mathbf{T}^p_{K_r,\kappa}$. There is then an embedding

$$j_\pi : H^0(\mathfrak{P}_h, K_h; \pi^{K_r} \otimes_{\mathbb{C}} W_\kappa) \cong \pi_f^{K_r} \hookrightarrow S_\kappa(K_r, \mathbb{C})(\pi)$$

of $\mathbf{T}^p_{K_r,\kappa}$-modules.

Lemma 6.5.5. *Let π be a holomorphic cuspidal automorphic representation of type (κ, K_r), and suppose π satisfies Hypothesis 6.5.4.*

(i) *The injection j_π defines an isomorphism*

$$j_\pi : \pi_S^{K_S} \otimes \pi_p^{I_r} \xrightarrow{\sim} S_\kappa(K_r, \mathbb{C})(\pi).$$

(ii) *Let λ be any extension of λ^p_π to a character of $\mathbf{T}_{K_r,\kappa,R}$. Let $R \subset \mathbb{C}$ be a finite extension of $E(\lambda^p_\pi)$ containing the values of λ, and let $S_\kappa(K_r, R)[\lambda]$ be the localization of the $\mathbf{T}_{K_r,\kappa,R}$-module $S_\kappa(K_r, R)$ at the prime ideal $\mathfrak{p}_\lambda \subset \mathbf{T}_{K_r,\kappa,R}$ that is the kernel of the character λ; in other words, $S_\kappa(K_r, R)[\lambda]$ is the λ-isotypic component of $S_\kappa(K_r, R)$. Then j_π defines an isomorphism*

$$j_\pi : \pi_S^{K_S} \otimes \pi_p^{I_r}[\lambda] \xrightarrow{\sim} S_\kappa(K_r, R)[\lambda] \otimes_R \mathbb{C} = S_\kappa(K_r, \mathbb{C}).$$

Here $\pi_p^{I_r}[\lambda]$ is the subspace of $\pi_p^{I_r}$ on which each $U_{w,j,\kappa}$ acts as $\lambda(U_{w,j,\kappa})$.

6.5.6. *The (anti-)ordinary projector and (anti-)ordinary Hecke algebra.* Suppose $R \subset \mathbb{C}$ is the localization of a finite S_0-algebra at the maximal prime determined by $incl_p$ or a p-adic algebra in the sense that $\iota_p(R)$ is p-adically complete.

Let $U_{p,\kappa} = \prod_{w\in\Sigma_p}\prod_{j=1}^n U_{w,j,\kappa}$ and let $e_\kappa = \varinjlim_N U_{p,\kappa}^{N!}$ (as an operator). We call this the ordinary projector, and put $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,R} = e_\kappa \mathbf{T}_{K_r,\kappa,R}$ and $\mathbf{T}^{\mathrm{ord}}_{K,\kappa,\psi,R} = e_\kappa \mathbf{T}_{K_r,\kappa,\psi,R}$. Then $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,R}$ and $\mathbf{T}^{\mathrm{ord}}_{K,\kappa,\psi,R}$ are just the rings obtained by restricting the Hecke operators to the (stable) subspaces $S^{\mathrm{ord}}_\kappa(K_r;R)$ and $S^{\mathrm{ord}}_\kappa(K_r,\psi;R)$. For R not p-adic we define the latter modules to be the respective intersections of $S_\kappa(K_r;R)$ and $S_\kappa(K_r,\psi;)$ with the ordinary spaces over the p-adic completion of R (that is, the completion of $incl_p(R)$).

Similarly, let $U^-_{p,\kappa} = \prod_{w\in\Sigma_p}\prod_{j=1}^n U^-_{w,j,\kappa}$ and let $e^-_\kappa = \varinjlim_N (U^-_{p,\kappa})^{N!}$ (as an operator, when it exists). We call this the *anti-ordinary projector*, and put $\mathbf{T}^{\text{a-ord}}_{K_r,\kappa,R} = e^-_\kappa \mathbf{T}^d_{K_r,\kappa,R}$ and $\mathbf{T}^{\text{a-ord}}_{K,\kappa,\psi,R} = e^-_\kappa \mathbf{T}^d_{K,\kappa,\psi,R}$.

Lemma 6.5.7. *Suppose R is as above. The isomorphisms of Lemmas 6.5.1(i)-(ii) and 6.5.2 restrict to R-algebra isomorphisms:*

(i) $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,R} \xrightarrow{\sim} \mathbf{T}^{\text{a-ord}}_{K_r,\kappa^D,R}$ *and* $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,\psi,R} \xrightarrow{\sim} \mathbf{T}^{\text{a-ord}}_{K_r,\kappa,\psi^{-1},R}$,
(ii) $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,V,R} \xrightarrow{\sim} \mathbf{T}^{\mathrm{ord}}_{K_r^\flat,\kappa^\flat-V,R}$ *and* $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,\psi,V,R} \xrightarrow{\sim} \mathbf{T}^{\mathrm{ord}}_{K_r^\flat,\kappa^\flat,-V,R}$.

This is immediate from the definitions.

6.5.8. *Spaces of ordinary forms and the character λ_π.* Let π be a holomorphic cuspidal automorphic representation of G of type (κ, K_r). Let

$$\pi_p^{\mathrm{ord}} = e_\kappa \pi_p^{I_r}.$$

This space has dimension at most one and it does not depend on r, in the sense that $e_\kappa\pi_p^{I_r} = e_\kappa\pi_p^{I_{r'}}$ for all $r' \geq r$. This is a consequence of the following:

Theorem 6.5.9 (Hida)**.** *For any representation π_p of $G(\mathbb{Q}_p)$, the ordinary eigenspace $e_\kappa\pi_p^{I_r} \subset \pi_p^{I_r}$ is of dimension ≤ 1, for any r.*

This theorem is a variant of [Hid98, Corollary 8.3] (we thank Hida for this reference). The proof, an adaptation of Hida's, is given in Section 8.2 below.

Will will say that π is *ordinary* if $\pi_p^{\mathrm{ord}} \neq 0$. Note that π_p^{ord} is stable under the action of I_r^0, and so I_r^0 will act on π_p^{ord} (when it is non-zero) through a well-defined character ψ; we call its identification with a character of $T_H(\mathbb{Z}_p)$ the *ordinary nebentypus* of π.

The space

$$\pi_{p,r}^{\flat,\text{a-ord}} = e^-_{\kappa^D}\pi_p^{\flat,I_r} \subset \pi_p^\flat$$

is at most one-dimensional, and is non-zero (and so has dimension one) if and only if π_p^{ord} is non-zero. This follows from Lemma 8.2.6 below. While it is not generally true that $\pi_{p,r}^{\flat,\text{a-ord}}$ is independent of r, if $r' \geq r$ then Lemma 8.2.7 asserts that

$$\mathrm{trace}_{K_r/K_{r'}}\pi_{p,r'}^{\flat,\text{a-ord}} = \pi_{p,r}^{\flat,\text{a-ord}}.$$

Suppose that π is ordinary. We let λ_π be the (unique) extension of λ_π^p to the Hecke character giving the action of $\mathbf{T}_{K_r,\kappa}$ on $\pi_p^{\mathrm{ord}} \otimes \pi^{p,K^p}$. For R as in 6.5.6, this character factors through $\mathbf{T}_{K_r,\kappa,\psi,R}^{\mathrm{ord}}$ for ψ the ordinary nebentypus of π. Let $E(\lambda_\pi)$ be the finite extension of $E(\pi)$ generated by the values of λ_π, and let $R(\lambda_\pi)$ be the localization of the ring of integers of $E(\lambda_\pi)$ at the maximal ideal determined by $incl_p$; then λ_π is $R(\lambda_\pi)$-valued. Let $\bar{\lambda}_\pi$ be the reduction of λ_π modulo the maximal ideal of $R(\lambda_\pi)$; this can be viewed as taking values in the residue field of $\overline{\mathbb{Z}}_{(p)}$. We let

$$\mathcal{S}(K_r,\kappa,\pi) = \{\text{ordinary holomorphic } \pi' \text{ of type } (\kappa,K_r) \text{ such that } \bar{\lambda}_{\pi'} = \bar{\lambda}_\pi\}.$$

Lemma 6.5.10. *Let π be a holomorphic cuspidal automorphic representation of type (κ,K_r). Suppose π is ordinary. Suppose also that π satisfies Hypothesis 6.5.4. Let $R \subset \mathbb{C}$ be the localization of a finite extension of $R(\lambda_\pi)$ at the prime determined by $incl_p$ or the p-adic completion of such a ring. Let $E = R[\frac{1}{p}]$.*

(i) $S_\kappa^{\mathrm{ord}}(K_r;E)[\lambda_\pi] = e_\kappa S_\kappa(K_r;E)[\lambda]$ *and j_π restricts to an isomorphism*

$$j_\pi : \pi_p^{\mathrm{ord}} \otimes \pi_S^{K_S} \cong \pi_S^{K_S} \xrightarrow{\sim} S_\kappa^{\mathrm{ord}}(K_r;E) \otimes_E \mathbb{C}.$$

(ii) *Let $\mathfrak{m}_\pi$ be the maximal ideal of $\mathbf{T}_{K_r,\kappa,R}$ that is the kernel of the reduction of λ_π modulo the maximal ideal of R. Let $S_\kappa^{\mathrm{ord}}(K_r;R)_\pi$ be the localization of $S_\kappa^{\mathrm{ord}}(K_r;R)$ at $\mathfrak{m}_\pi$. Then*

$$S_\kappa^{\mathrm{ord}}(K_r;R)[\pi] = S_\kappa^{\mathrm{ord}}(K_r;R)_\pi \cap S_\kappa^{\mathrm{ord}}(K_r;E)[\lambda_\pi]$$

is identified by j_π with an R-lattice in $\pi_p^{\mathrm{ord}} \otimes \pi_S^{K_S} \cong \pi_S^{K_S}$, and $S_\kappa^{\mathrm{ord}}(K_r,R)_\pi$ is identified with an R-lattice in

$$\bigoplus_{\pi' \in \mathcal{S}(K_s,\kappa,\pi)} \pi_p^{',\mathrm{ord}} \otimes (\pi'_S)^{K_S}.$$

This last identification is via $\oplus_{\pi'}\lambda_{\pi'}$.

We also need a dual picture. Let

$$\hat{S}_\kappa(K_r;R) = \mathrm{Hom}_R(S_\kappa(K_r;R),R) \quad \text{and} \quad \hat{S}_\kappa^{\mathrm{ord}}(K_r;R) = \mathrm{Hom}_R(S_\kappa^{\mathrm{ord}}(K_r;R),R).$$

These are $\mathbf{T}_{K_r,\kappa,R}$-modules through the Hecke action on $S_\kappa(K_r;R)$, so $\hat{S}_\kappa^{\mathrm{ord}}(K_r,R)$ is a $\mathbf{T}_{K_r,\kappa,R}^{\mathrm{ord}}$-module. Serre duality identifies $\hat{S}_\kappa(K_r:R)$ with

$$H_{\kappa^D}^d(K_r,R) = \{\varphi \in H^d({}_{K_r}Sh(V),\omega_{\kappa^D}) \ : \ \langle S_\kappa(K_r;R),\varphi\rangle_\kappa^{\mathrm{Ser}} \subseteq R\}.$$

Let $S_\kappa^{\mathrm{ord},\perp}(K_r;R) \subset H_{\kappa^D}^d(K_r,R)$ denote the annihilator of $S_\kappa^{\mathrm{ord}}(K_r;R)$ with respect to this pairing. Then Serre duality identifies $\hat{S}_\kappa^{\mathrm{ord}}(K_r;R)$ with

$$H_{\kappa^D}^{d,\mathrm{ord}}(K_r,R) = \{\varphi \in H^d({}_{K_r}Sh(V),\omega_{\kappa^D})/S_\kappa^{\mathrm{ord},\perp}(K_r;R) \ : \ \langle S_\kappa^{\mathrm{ord}}(K_r;R),\varphi\rangle_\kappa^{\mathrm{Ser}} \subseteq R\}.$$

Each of these is a $\mathbf{T}_{K_r,\kappa,R}$-module through its action on $S_\kappa(K_r;R)$ or, equivalently, the isomorphism of Lemma 6.5.1(i), so $H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R)$ is a $\mathbf{T}^{\mathrm{ord}}_{K^r,\kappa,R}$-module.

Lemma 6.5.11. *The natural map $H^d_{\kappa^D}(K_r;R) \to H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R)$, which is just restriction to $S^{\mathrm{ord}}_\kappa(K_r;R)$, induces an isomorphism*

$$e^-_{\kappa^D} H^d_{\kappa^D}(K_r;R) \xrightarrow{\sim} H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R). \tag{6.5.2}$$

Proof. This is an immediate consequence of Lemma 8.2.4, (iii). □

Let π be a holomorphic cuspidal automorphic representation of G of type (κ, K_r). Then $\pi^\flat$ is antiholomorphic of type (κ, K_r). The choice of a basis of the one-dimensional space $H^d(\mathfrak{P}_h, K_h; \pi^\flat_\infty \otimes_{\mathbb{C}} W_{\kappa^D})$ determines an injection

$$j^\vee_{\pi^\flat} : H^d(\mathfrak{P}_h, K_h; \pi^{\flat,K_r} \otimes_{\mathbb{C}} W_{\kappa^D}) \cong \pi^{\flat,K_r} \hookrightarrow H^d_{\kappa^D}(K_r;\mathbb{C}) = H^d({}_{K_r}Sh(V), \omega_{\kappa^D}).$$

Lemma 6.5.12. *Let π, R, and E be as in Lemma 6.5.10. Let $H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)_\pi$ be the localization of $H^{d,\mathrm{ord}}_{\kappa^D,V}(K_r,R)$ at $\mathfrak{m}_{\pi^\flat}$, and let*

$$H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi] = H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R)_\pi \cap H^{d,\mathrm{ord}}_{\kappa^D}(K_r;E)[\lambda_\pi]$$

where the notation '$[\lambda_\pi]$' again denotes the $\lambda^\flat_\pi$-isotypic component.

(i) *The inclusion $j^\vee_{\pi^\flat}$ restricts to an isomorphism*

$$j^\vee_{\pi^\flat} : \pi^{\flat,\text{a-ord}}_{p,r} \otimes \pi^{\flat,K_S}_S \cong \pi^{K_S}_S \xrightarrow{\sim} H^{d,\mathrm{ord}}_{\kappa^D}(K_r,E)[\pi] \otimes_E \mathbb{C}.$$

(ii) *The map $j^\vee_{\pi^\flat}$ identifies $H^{d,\mathrm{ord}}(K_r;R)[\pi]$ with an R-lattice in $\pi^{\flat,\text{a-ord}}_{p,r} \otimes \pi^{\flat,K_S}_S$, and $H^{d,\mathrm{ord}}(K_r;R)_\pi$ is identified with an R-lattice in*

$$\oplus_{\pi' \in \mathcal{S}(K_r,\kappa,\pi)} \pi'^{,\flat,\mathrm{ord}}_{p,r} \otimes \pi'^{,\flat,K_S}_S.$$

This last identification is by $\oplus j^\vee_{\pi',\flat}$.

(iii) *Serre duality induces perfect $\mathbf{T}^{\mathrm{ord}}_{K_r,\kappa,R}$-equivariant pairings (with respect to the isomorphisms of Lemma 6.5.7)*

$$S^{\mathrm{ord}}_\kappa(K_r;R)[\pi] \otimes_R H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R)[\pi] \to R \quad \text{and} \quad S^{\mathrm{ord}}_\kappa(K_r;R)_\pi \otimes_R H^{d,\mathrm{ord}}_{\kappa^D}(K_r;R)_\pi \to R$$

We say π is *ordinary of type* (κ, K) if π is anti-holomorphic of type (κ, K) and if the image of j_π has non-trivial intersection with in $S^{\mathrm{ord}}_\kappa(K,R)$. In that case, λ_π, defined as above, takes values in a p-adic integer ring, say $\mathcal{O}_\pi$, with residue field $k(\pi)$, and we let $\bar\lambda_\pi : \mathbf{T}_{K,\kappa} \to k(\pi)$ denote the reduction of λ_π modulo the maximal ideal of $\mathcal{O}_\pi$.

6.5.13. *Change of level.* For future reference, we let $K = K^pI_r$, $K' = K^pK_{r'}$ with $r' \geq r$. For fixed κ we consider the inclusion

$$S^{\mathrm{ord}}_{\kappa,V}(K,R) \to S^{\mathrm{ord}}_{\kappa,V}(K',R) \tag{6.5.3}$$

and the dual map

$$\hat{S}^{\mathrm{ord}}_{\kappa,V}(K',R) \to \hat{S}^{\mathrm{ord}}_{\kappa,V}(K,R) \tag{6.5.4}$$

Lemma 6.5.14. *Let R be either a local $\mathbb{Z}_{(p)}[\lambda_\pi]$-algebra or a finite flat $\mathbb{Z}_p[\lambda_\pi]$-algebra. Then the image of the map* (6.5.3) *is an R-direct factor of $S^{\mathrm{ord}}_{\kappa,V}(K',R)$, identified with the submodule of $I_r/I_{r'}$-invariants of the latter. Moreover, the morphism* (6.5.4) *is surjective.*

Proof. The first assertion is obvious; the second is an immediate consequence of the first. □

6.6. **Normalized periods.** Fix the group G as above; we will be taking $G = G_1$ or $G = G_2$ later in this section. We are still assuming π to be an anti-holomorphic representation of G of type (κ, K). We let R be a local $\mathbb{Z}_{(p)}[\lambda_\pi]$-algebra and define $S^{\mathrm{ord}}_\kappa(K,R)[\pi]$ and $S^{\mathrm{ord}}_\kappa(K,R)_\pi$ just as in Lemma 6.5.10. The Petersson pairing is positive definite and hence defines perfect hermitian pairings $\langle\phi,\phi'\rangle_{P,\pi}$ and $\langle\phi,\phi'\rangle_P[\pi]$ on $S^{\mathrm{ord}}_\kappa(K,R)_\pi$ and $S^{\mathrm{ord}}_\kappa(K,R)[\pi]$, respectively.

Lemma 6.6.1. *The images*

$$L[\pi] = \langle S^{\mathrm{ord}}_\kappa(K,R)[\pi], S^{\mathrm{ord}}_\kappa(K,R)[\pi]\rangle_P[\pi]$$
$$L_\pi = \langle S^{\mathrm{ord}}_\kappa(K,R)[\pi], S^{\mathrm{ord}}_\kappa(K,R)_\pi\rangle_{P,\pi}$$

are rank one R-submodules of $\mathbb{C}$, generated by positive real numbers $Q[\pi]$ and Q_π, respectively.

Proof. This is a version of Schur's Lemma. The analogous statement is proved in [Har13a] when R is a finite extension of $\mathbb{Q}$. This implies that $L[\pi]\otimes\mathbb{Q}$ and $L_\pi\otimes\mathbb{Q}$ are finite rank one $\mathrm{Frac}(R)$-subspaces of $\mathbb{C}$. Since R is a discrete valuation ring, the result follows immediately from this. □

The numbers $Q[\pi]$ and Q_π are well-defined up to multiples by $R^\times$; they are respectively unnormalized and normalized periods for π. We can also write $Q[\pi, G]$ and $Q_{\pi,G}$ to emphasize the dependence on G (either G_1 or G_2). Let

$$S^{\mathrm{ord}}_\kappa(K,R)[\pi]^\perp \subset S^{\mathrm{ord}}_\kappa(K,R)_\pi$$

be the orthogonal complement to $S^{\mathrm{ord}}_\kappa(K,R)[\pi]$ with respect to $\langle\phi,\phi'\rangle_{P,\pi}$. This is the intersection of $S^{\mathrm{ord}}_\kappa(K,R)_\pi$ with $\oplus_{\pi'\neq\pi}S^{\mathrm{ord}}_\kappa(K,R[\frac{1}{p}])[\pi']$.

Definition 6.6.2. *Define the congruence ideal $C(\pi)\subset R$ to be the annihilator of*

$$S^{\mathrm{ord}}_\kappa(K,R)_\pi/S^{\mathrm{ord}}_\kappa(K,R)[\pi] + S^{\mathrm{ord}}_\kappa(K,R)[\pi]^\perp.$$

Lemma 6.6.3. *Let $c(\pi) \in R$ be a generator of $C(\pi)$. Then $c(\pi)Q_\pi = Q[\pi]$. More precisely, let f be any primitive element of $S_\kappa^{\mathrm{ord}}(K,R)[\pi]$; in other words, if $u \in R$ and and $ug = f$ for some $g \in S_\kappa^{\mathrm{ord}}(K,R)[\pi]$ then $u \in R^\times$. Let*

$$Q(f) = \langle f, f\rangle_P[\pi].$$

Then we can take $Q(f) = Q[\pi]$, $Q(f)$ is divisible in the R-module $L[\pi]$ by $c(\pi)$, and $c(\pi)^{-1}Q(f)$ generates L_π.

Proof. This is an elementary consequence of the definitions. □

More generally, the congruence ideal $C(\pi, M)$ can be defined for any $\mathbb{T}_{K,\kappa,R}$-module M as the annihilator of $M_\pi/(M[\pi] + M[\pi]^\perp)$, where the notation has the same meaning as above. In particular, we can define $C(\pi,\mathbb{T})$ to be the congruence ideal for $\mathbb{T}_{K,\kappa,R}$ considered as a free module over itself. We can also define $C(\pi, H^{d,\mathrm{ord}}_{\kappa^D,V}(K_r,R))$ or (equivalently) $C(\pi, \hat{S}_\kappa^{\mathrm{ord}}(K_r;R))$ by the same formula.

Remark 6.6.4. The congruence ideal $C(\pi)$ has a local component, due to possible congruences between the representation $\pi_p^{\flat,ord} \otimes \pi_S^{\flat,K_S}$ and the $\pi_p^{\flat,ord}\otimes(\pi'_S)^{\flat,K_S}$ for π' such that $\bar{\lambda}_\pi = \bar{\lambda}_{\pi'}$. Here if S has the property that, for every rational prime q, either all the primes of $\mathcal{K}^+$ dividing q split in $\mathcal{K}$ or none of them does, we can view the latter as representations of the (integral) Hecke algebra of K_S-biinvariant functions on $GU(V)(\mathbb{A}_{f,S})$. The separation of global and local components of $C(\pi)$ will need to be understood for applications, but it is not addressed here.

All of the above statements have variants in which $S_\kappa^{\mathrm{ord}}(K,R)$ is replaced by $S_\kappa^{\mathrm{ord}}(K,\psi,R)$, for some nebentypus character ψ. We leave the statements to the reader.

In what follows, R is a sufficiently large finite flat p-adic integer ring.

Definition 6.6.5. *Write $\mathbb{T} = \mathbb{T}_{K,\kappa,R,\pi}$. The $\mathbb{T}$-module $S_\kappa^{\mathrm{ord}}(K,R)_\pi$ is said to satisfy the Gorenstein hypothesis if the following conditions hold.*

- $\mathbb{T} \tilde{\to} \mathrm{Hom}_R(\mathbb{T},R)$ *as R-algebras.*
- $S_\kappa^{\mathrm{ord}}(K,R)_\pi$ *is free over $\mathbb{T}$.*

The $\mathbb{T}_{K,\kappa,R}$-module $S_\kappa^{\mathrm{ord}}(K,R)$ is said to satisfy the Gorenstein hypothesis if all its localizations at maximal ideals of $\mathbb{T}_{K,\kappa,R}$ satisfy the two conditions above.

The following is then obvious.

Lemma 6.6.6. *Assume $S_\kappa^{\mathrm{ord}}(K,R)_\pi$ satisfies the Gorenstein hypothesis. Then we have*

$$C(\pi,\mathbb{T}) = C(\pi) = C(\pi, H^{d,\mathrm{ord}}_{\kappa^D,V}(K_r,R)).$$

The congruence ideal for π can be calculated as follows. We assume the multiplicity one hypothesis, so that the localization of $\mathbb{T}$ at the kernel of λ_π is of rank 1 over R. Let

$e_1,\dots,e_n$ be an R-basis for $\mathbb{T}$, and let $e_1^*,\dots,e_n^*$ be the dual basis of $\mathrm{Hom}_R(\mathbb{T},R)$. Write $E=\mathrm{Frac}(R)$, and write

$$\mathbb{T}_E=\mathbb{T}\otimes_R E=\oplus E_i,$$

indexed by the maximal ideals λ_{π_i} of $\mathbb{T}$, with $\pi=\pi_1$. We assume R is sufficiently large that $E_1=E$. Choose a basis $d_1,\dots,d_n\in\mathbb{T}$ be a basis of $\mathbb{T}_E$, with d_1 an R-generator of $\mathbb{T}\cap E_1$ and $d_2,\dots,d_n$ an R-basis of $\mathbb{T}\cap\oplus_{i>1}E_i$. Write $e_i=\sum c_{ij}d_j$, with $c_{ij}\in E$. Then

$$C(\pi,\mathbb{T})=\sup_{c_{i1}\neq 0}\ -v(c_{i1}) \tag{6.6.1}$$

where v is the valuation on R.

The following lemma is then clear:

Lemma 6.6.7. *The second isomorphism of Lemma 6.5.2 takes $C(\pi)$ isomorphically to $C(\pi^\flat)$.*

It then follows from Lemmas 6.6.6 and 6.6.7, and (iii) of Lemma 6.5.12 that

Proposition 6.6.8. *Assume $S_\kappa^{\mathrm{ord}}(K,R)_\pi$ satisfies the Gorenstein hypothesis. Then under the isomorphisms of 6.5.7 all the ideals $C(\pi,G_1)$, $C(\pi^\flat,G_1)$, $C(\pi,G_2)$, $C(\pi^\flat,G_2)$, $C(\pi,\mathbb{T})$, $C(\pi,H^{d,\mathrm{ord}}_{\kappa^D,V}(K_r,R))$, etc. are identified. In particular, the congruence ideals attached to π and to $\pi^\flat$ are canonically identified whether π and $\pi^\flat$ are considered holomorphic and ordinary or anti-holomorphic and anti-ordinary.*

In particular, we can reformulate Lemmas 6.6.1 and 6.6.3 in terms of periods of anti-holomorphic anti-ordinary forms. Define the Petersson pairings $\langle\bullet,\bullet\rangle_P$ on $H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi]$ and $H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)_\pi$ by the usual L^2 integrals of anti-holomorphic forms. Then

Corollary 6.6.9. *Assume $S_\kappa^{\mathrm{ord}}(K,R)_\pi$ satisfies the Gorenstein hypothesis.*

(a) The images

$$\hat{L}[\pi]=\langle H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi],H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi]\rangle_P[\pi]$$
$$\hat{L}_\pi=\langle H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi],H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)_\pi\rangle_{P,\pi}$$

are rank one R-submodules of $\mathbb{C}$, generated by positive real numbers $\hat{Q}[\pi]$ and $\hat{Q}_\pi$, respectively. We can write

$$\hat{Q}[\pi]=\langle f,f\rangle_P[\pi]$$

for appropriate integral generators $f\in H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)[\pi]$.

(b) Moreover,

- $c(\pi)\hat{Q}_\pi=\hat{Q}[\pi]$, *and*
- $\hat{Q}_\pi=Q_\pi^{-1}$,

where the equalities are understood as in the statement of Lemma 6.6.3.

The second claim in (b) is an elementary consequence of the duality between $H^{d,\mathrm{ord}}_{\kappa^D}(K_r,R)_\pi$ and $S^{\mathrm{ord}}_\kappa(K,R)_\pi$.

Remark 6.6.10. The normalized period Q_π and the generator $c(\pi)$ of the congruence ideal are well defined up to units in R. However, this ambiguity is unsatisfactory; one expects there is a natural choice of global function c in $\mathbb{T}$ which is not a zero divisor and whose value at the classical point π generates $C(\pi)$. This would allow a uniform choice of periods Q_π.

Let $\mathcal{G}_n$ be the algebraic group introduced in [CHT08] as the target of the compatible family of ℓ-adic representations attached to π; it is the semidirect product of $GL(n)\times GL(1)$ with the Galois group of $\mathcal{K}/\mathcal{K}^+$. It is natural to expect that c can be taken to be a p-adic L-function attached to the adjoint representation on the Lie algebra of $\mathcal{G}_n$. The corresponding complex L-function has a single pair of critical values, interchanged by the functional equation, so the hypothetical p-adic L-function would be an element of $\mathbb{T}$, without any additional variation for twists by characters.

7. Families of ordinary p-adic modular forms and duality

7.1. **Big Hecke algebras and the control theorem.** In this section, we write $T = T_H = T_{H_1}(\mathbb{Z}_p)\equiv T_{H_2}(\mathbb{Z}_p)$, i.e. the torus introduced in Section 2.5. For the moment we look at families of p-adic modular forms on a group G, which in the applications will be G_1 or G_2. Fix an algebraic character κ of T_H and a tame level subgroup $K^p\subset G(\mathbb{A}_f^p)$ as above, and let $K = K_r^p$, $K' = K_{r'}^p$ with $r'>r$. Let R be a p-adic ring. The inclusion $S^{\mathrm{ord}}_\kappa(K,R)\subset S^{\mathrm{ord}}_\kappa(K',R)$ defines by restriction a map of ordinary Hecke algebras $\mathbb{T}_{K',\kappa,R}\to\mathbb{T}_{K,\kappa,R}$. Let $\mathbb{T}_{K^p,\kappa,R} = \varprojlim_r \mathbb{T}_{K_r^p,\kappa,R}$ with respect to the restriction maps. The following theorem is due to Hida:

Theorem 7.1.1. *For any pair of characters κ, κ^1, there is a canonical isomorphism*

$$\mathbb{T}_{K^p,\kappa,R}\xrightarrow{\sim}\mathbb{T}_{K^p,\kappa^1,R}.$$

Thus we can write $\mathbb{T}_{K^p,R}$ to designate any $\mathbb{T}_{K^p,\kappa,R}$ without fear of ambiguity. We will even write $\mathbb{T}=\mathbb{T}_{K^p,R}$ when there is no danger of ambiguity.

Remark 7.1.2. In the application to unitary groups this theorem and the next one are special cases of [Hid02, Theorem 7.1] and the results of [Hid04, Chapter 8]. Hida's Theorems 7.1.1 and 7.1.3 are proved assuming the conditions (G1)-(G3) mentioned in connection with (2.9.6).

Fix a cuspidal antiholomorphic automorphic representation π of G which is ordinary of type (κ,K) as in Section 6.5. We let $R=\mathcal{O}_\pi$. Let $\Lambda=\Lambda_H$, with coefficients in R. The homomorphisms $\lambda_\pi:\mathbb{T}_{K,\kappa,\mathcal{O}_\pi}\to\mathcal{O}_\pi$ and $\bar{\lambda}_\pi:\mathbb{T}_{K,\kappa,\mathcal{O}_\pi}\to k(\pi)$ lift to homomorphisms $L_\pi:\mathbb{T}_{K^p,\mathcal{O}_\pi}\to\mathcal{O}_\pi$ and $\bar{L}_\pi:\mathbb{T}_{K^p,\mathcal{O}_\pi}\to k(\pi)$. Let $\mathfrak{m}_\pi=\ker\bar{L}_\pi$, and let

$$\mathbb{T}_\pi=\mathbb{T}_{K^p,\mathcal{O}_\pi,\mathfrak{m}_\pi}$$

denote the localization. The intersection $\mathfrak{m}_\pi \cap \Lambda_{\mathcal{O}_\pi}$ is the maximal ideal defined by some tame character of T. We let Λ_π denote Λ_H with coefficients in $\mathcal{O}_\pi$.

Theorem 7.1.3. *(i) The Hecke algebra $\mathbb{T}_\pi$ is a free Λ_π-algebra of finite type.*

(ii) (Control theorem) Let $I_\kappa \subset \Lambda_\pi$ be the kernel of the map $\Lambda_\pi \to \mathbb{C}_p$ defined by the character κ. Suppose κ is sufficiently regular. Then the natural map

$$\mathbb{T}_\pi \otimes_\Lambda \Lambda_\pi/I_\kappa \to \mathbb{T}_{K_r^p,\kappa,\mathcal{O}_\pi,\pi}$$

is an isomorphism of algebras.

The *main involution* of Λ is the involution induced by the map $t \mapsto t^{-1}$ of T_H. The involution $\flat$ of $\mathbb{T}_{K^p,R}$ restricts to the main involution on Λ and induces a ($\flat$-linear) isomorphism $\flat : \mathbb{T}_\pi \xrightarrow{\sim} \mathbb{T}_{\pi^\flat}$ of Λ_π-algebras. Here and in what follows, for any r and K^p we will let $\mathbb{T}_\pi$ act on $\mathrm{Hom}_{\mathcal{O}_\pi}(S_\kappa^{\mathrm{ord}}(K_r^p,\mathcal{O}_\pi),\mathcal{O}_\pi)_{\mathfrak{m}_\pi}$ by the natural action twisted by $\flat$. We consider the following hypotheses:

Hypothesis 7.1.4. *(Gorenstein Hypothesis) Let $\hat{\mathbb{T}}_\pi = \mathrm{Hom}_{\Lambda_\pi}(\mathbb{T}_{\pi^\flat},\Lambda_\pi)$. Then*

- *$\hat{\mathbb{T}}_\pi$ is a free rank-one $\mathbb{T}_\pi$-module via the isomorphism $\flat : \mathbb{T}_\pi \xrightarrow{\sim} \mathbb{T}_{\pi^\flat}$.*
- *Let $\mathbb{T}_\pi$ act on $\mathrm{Hom}_{\mathcal{O}_\pi}(S_\kappa^{\mathrm{ord}}(K_r^p,\mathcal{O}_\pi),\mathcal{O}_\pi)_{\mathfrak{m}_\pi}$ by the natural action twisted by $\flat$. Then $\mathrm{Hom}_{\mathcal{O}_\pi}(\varinjlim_r S_\kappa^{\mathrm{ord}}(K_r^p,\mathcal{O}_\pi),\mathcal{O}_\pi)_{\mathfrak{m}_\pi}$ is a free $\mathbb{T}_\pi$-module.*

This is of course a variant of the hypothesis 6.6.5 of the previous section.

Hypothesis 7.1.5. *(Global Multiplicity One) Let $\pi' \in \mathcal{S}(K_r,\kappa,\pi)$. Then the representation π' occurs with multiplicity one in the cuspidal spectrum of G.*

7.1.6. *Local representation theory.* Henceforth, we abuse notation and write $\mathcal{O}$ for $\mathcal{O}_\pi$. (The ring of integers of $\mathcal{K}$ does not appear in the context in which we do this; so we will only be using $\mathcal{O}$ for $\mathcal{O}_\pi$ here.) We let I_π denote the image of the specialization map

$$S_\kappa^{\mathrm{ord}}(K_r^p,\mathcal{O}) \otimes_{\mathbb{T}_{K_r^p,\kappa,\mathcal{O}}} \mathbb{T}_{K_r^p,\kappa,\mathcal{O},\pi}/\ker(\lambda_\pi) \hookrightarrow \pi_{S^p}^{\flat,K^p}.$$

This is a free $\mathcal{O}$-lattice in $\pi_{S^p}^{\flat,K^p}$ (and *not* in π_{S^p}!). Fix a non-zero element $f_{p,\pi^\flat}^{\mathrm{ord}}$ of the 1-dimensional $\mathrm{Frac}(\mathcal{O})$-space $\pi_p^{\flat,ord}$. Then tensoring $\pi_{S^p}^{\flat,K^p}$ with $f_{p,\pi^\flat}^{\mathrm{ord}}$ identifies I_π with a $\mathcal{O}$-lattice in $\pi_p^{\flat,ord} \otimes \pi_{S^p}^{\flat,K^p}$.

The anti-ordinary subspace $\pi_p^{a-ord} \subset \pi_p^{I_r}$ is the tensor product over $w \mid p$ of the local anti-ordinary subspaces π_w^{a-ord}, which will be defined in Lemma 8.2.6. Let $\hat{I}_\pi = \mathrm{Hom}(I_\pi,\mathcal{O})$. The natural duality between $\pi_S^\flat$ and π_S identifies $\hat{I}_\pi$ with an $\mathcal{O}$-lattice in $\pi_p^{\text{a-ord}} \otimes \pi_{S^p}^{K^p}$, and thus defines a natural isomorphism

$$\hat{I}_\pi \xrightarrow{\sim} I_{\pi^\flat}. \tag{7.1.1}$$

The following hypothesis comes down to the assumption that π is *minimal* (in the sense of the Taylor-Wiles method) of level K^p with respect to deformations of its Galois representation.

Hypothesis 7.1.7. *(Minimality Hypothesis) For every pair (κ^1, r^1), there is an isomorphism of $\mathbb{T}_{K^p_{r^1},\kappa^1,\mathcal{O},\pi}$-modules*

$$\mathbb{T}_{K^p_{r^1},\kappa^1,\mathcal{O},\pi} \otimes \hat{I}_\pi \xrightarrow{\sim} \mathrm{Hom}(S^{\mathrm{ord}}_{\kappa^1}(K^p_{r^1}, \mathcal{O})_{\mathfrak{m}_\pi}, \mathcal{O}).$$

such that the following diagrams commute when $r^1 > r$:

$$\begin{array}{ccc} \mathbb{T}_{K^p_{r^1},\kappa^1,\mathcal{O},\pi} \otimes \hat{I}_\pi & \xrightarrow{\sim} & \mathrm{Hom}(S^{\mathrm{ord}}_{\kappa^1}(K^p_{r^1}, \mathcal{O})_{\mathfrak{m}_\pi}, \mathcal{O}) \\ \downarrow & & \downarrow \\ \mathbb{T}_{K^p_{r},\kappa^1,\mathcal{O},\pi} \otimes \hat{I}_\pi & \xrightarrow{\sim} & \mathrm{Hom}(S^{\mathrm{ord}}_{\kappa^1}(K^p_{r}, \mathcal{O})_{\mathfrak{m}_\pi}, \mathcal{O}) \end{array}$$

and such that the specialization at the $\mathcal{O}$-valued point λ_π:

$$\mathbb{T}_{K^p_r,\kappa,\mathcal{O},\pi}/\ker(\lambda_\pi) \otimes \hat{I}_\pi \xrightarrow{\sim} \mathrm{Hom}(S^{\mathrm{ord}}_{\kappa^1}(K^p_{r^1}, \mathcal{O})_{\mathfrak{m}_\pi}, \mathcal{O}) \otimes_{\mathbb{T}_{K^p_r,\kappa,\mathcal{O},\pi}} \mathbb{T}_{K^p_r,\kappa,\mathcal{O},\pi}/\ker(\lambda_\pi)$$

is just the tautological isomorphism $\hat{I}_\pi \xrightarrow{\sim} \mathrm{Hom}(I_\pi, \mathcal{O})$.

These hypotheses will be assumed in the statements of most of the results about p-adic L-functions in families; they will be verified when possible.

7.2. **Equivariant measures.** In this section we consider measures with values in p-adic modular forms on G_3. We fix a prime-to-p level subgroup $K^p \subset G_3(\mathbb{A}_f)$ and let $K_r = I_r \cdot K^p$ as before, where $I_r = I_{r,1} \times I_{r,2} \subset G_3(\mathbb{Q}_p)$ with $I_{r,i} \subset G_i(\mathbb{Q}_p)$, $i = 1, 2$. For $\mathcal{O}$ as above, we write $\mathcal{V} = V_3^{\mathrm{ord,cusp}}(K^p, \mathcal{O})$ for the corresponding space of ordinary p-adic cusp forms on G_3 with values in $\mathcal{O}$.

Remark 7.2.1. Although the Eisenstein measure does not generally take values in the space of cusp forms, even after ordinary projection, we will be localizing at a non-Eisenstein maximal ideal of the Hecke algebra. Much of the discussion below applies without change to measures with values in the space of ordinary p-adic forms.

We choose a sequence of congruence subgroups $T \supset \cdots \supset T_r \supset T_{r+1} \cdots$ such that $\cap_r T_r = \{1\}$. Let $\mathcal{I}_r \subset \Lambda$ be the augmentation ideal of T_r, and let $\Lambda_r = \Lambda/\mathcal{I}_r$. For $\mathcal{O}$ as above, let $C_r(T, \mathcal{O}) = C(T/T_r, \mathcal{O})$ be the (free) $\mathcal{O}$-module of T_r-invariant functions on T. Then there is a natural identification $\Lambda_r = \mathrm{Hom}_{\mathcal{O}}(C_r(T, \mathcal{O}), \mathcal{O})$; alternatively, viewing Λ_π as the algebra of distributions on T with coefficients in $\mathcal{O}$, and $C(T, \mathcal{O})$ the module of continuous $\mathcal{O}$-valued functions on T, the canonical pairing $\Lambda_\pi \otimes C(T, \mathcal{O}) \to \mathcal{O}$ restricts to a pairing $\Lambda_\pi \otimes C_r(T, \mathcal{O}) \to \mathcal{O}$ which factors through a perfect pairing $\Lambda_r \otimes C_r(T, \mathcal{O}) \to \mathcal{O}$.

Let $r_r : C_r(T, \mathcal{O}) \hookrightarrow C_{r+1}(T, \mathcal{O})$ be the canonical inclusion. The next lemma follows from the definitions. Note that $\mathcal{V} = V_3^{\mathrm{ord,cusp}}(K^p, \mathcal{O})$ is a $\Lambda_{\mathcal{O}}$-module by the action on the first factor. We fix an involution $\upsilon : T \to T$ and define $\rho^\upsilon = \rho \circ \upsilon$ for any function $\rho \in C(T, \mathcal{O})$.

Lemma 7.2.2. *Fix a character $\rho : T \to \mathcal{O}^\times$ and let $C_r(T, \mathcal{O}) \cdot \rho^\upsilon \subset C(T, \mathcal{O})$ denote multiples of ρ^υ by elements of $C_r(T, \mathcal{O})$. There is an equivalence between* *(i) $\mathcal{V}$-valued measures ϕ on T satisfying*

$$\phi(t\cdot f) = \upsilon(t)\cdot\phi(f), f \in C(T,\mathcal{O});$$

(ii) Collections $\phi_\rho = (\phi_{r,\rho})$ with

$$\phi_{r,\rho} \in \mathrm{Hom}_{\Lambda_\pi}(C_r(T,\mathcal{O})\cdot\rho^\upsilon,\mathcal{V}),$$

satisfying $r_r^*(\phi_{r+1,\rho}) = \phi_{r,\rho}$, where r_r^* is dual to r_r.

We let $\mathcal{I}_{r,\rho} \subset \Lambda_\pi$ be the annihilator of $C_r(T,\mathcal{O})\cdot\rho^\upsilon$, and let $\Lambda_{r,\rho} = \Lambda_\pi/\mathcal{I}_{r,\rho}$. Thus Lemma 7.2.2 identifies equivariant measures on T with twist υ with collections of linear forms on $\Lambda_{r,\rho}$ that are compatible with the natural projection maps as r varies.

Let $\hat{\mathcal{V}} = \mathrm{Hom}_{\mathcal{O}}(\mathcal{V},\mathcal{O})$ and let ϕ be a measure as above, which we assume to be the specialization at a character χ of X_p of an admissible measure in two variables with shift $sh^*(\chi) = (\alpha(\chi),\beta(\chi))$ and twist υ as in Section 5.2.4.

We write $\phi_{\chi,r,\rho}$ to indicate dependence on χ. For $\kappa = \rho\cdot\alpha(\chi)$ sufficiently regular,

$$Im(\phi_{\chi,r,\rho}) \subset S^{\mathrm{ord}}_{(\rho\cdot\alpha(\chi)),V}(K_r,\mathcal{O})[\otimes]S^{\mathrm{ord}}_{(\rho^\flat\cdot\beta(\chi)),-V}(K_r,\mathcal{O}) \tag{7.2.1}$$

where we continue to identify open compact subgroups of $GU(V)(\mathbb{A}_f) = GU(-V)(\mathbb{A}_f)$, and where the notation $[\otimes]$ is as in Remark 3.2.2.

More generally, we let X_p be a compact p-adic Lie group, and let $\Phi_X = (\Phi_a,\Phi_b) \in C(X_p,\mathcal{O})^2$. Say $\phi(\bullet)$ (the $\bullet$ is a place-keeper) is a *measure of type* Φ_X if

$$Im(\phi(\bullet)_{r,\rho}) \subset \left(S^{\mathrm{ord}}_{\rho,V}(K_r,\mathcal{O})\otimes\Phi_a\circ\det\right)[\otimes]\left(S^{\mathrm{ord}}_{\rho^\flat,-V}(K_r,\mathcal{O})\otimes\Phi_b\circ\det\right). \tag{7.2.2}$$

We also have

$$\hat{\mathcal{V}} = \varprojlim_r \mathrm{Hom}_{\mathcal{O}}(S^{\mathrm{ord}}_{\rho,V}(K_r,\mathcal{O})[\otimes]S^{\mathrm{ord}}_{\rho^\flat,-V}(K_r,\mathcal{O}),\mathcal{O}).$$

In the situation of (7.2.1), assuming $\kappa = \rho\cdot\alpha(\chi)$ is sufficiently regular, we thus have

$$Im(\phi_{\chi,r,\rho}) \subset \mathrm{Hom}_{\mathcal{O}}(\hat{S}^{\mathrm{ord}}_{(\rho\cdot\alpha(\chi)),V}(K_r,\mathcal{O}),S^{\mathrm{ord}}_{(\rho^\flat\cdot\beta(\chi)),-V}(K_r,\mathcal{O})).$$

The following hypothesis expresses a basic property of the Garrett map that is the basis of the doubling method for studying standard L-functions of classical groups.

Hypothesis 7.2.3.

$$Im(\phi_{\chi,r,\rho}) \subset \mathrm{Hom}_{\mathbb{T}_{r,\rho\cdot\alpha(\chi)}}(\hat{S}^{\mathrm{ord}}_{(\rho\cdot\alpha(\chi)),V}(K_r,\mathcal{O}),S^{\mathrm{ord}}_{(\rho\cdot\alpha(\chi))^\flat,-V}(K_r,\mathcal{O})\otimes\chi\circ\det).$$

Bear in mind that $\phi_{\chi,r,\rho}$ designates integration of functions locally equal to ρ^υ – not ρ – against the specialization at χ of a two-variable measure.

Remark 7.2.4. We sometimes write $\kappa = \rho \cdot \alpha(\chi)$ when we want to emphasize the weight of the specialized Hecke algebra rather than the weight of the character of T. Here and below the algebra $\mathbb{T}_\kappa = \mathbb{T}_{r,\rho\cdot\alpha(\chi)}$ ignores the twist by $\chi \circ \det$ at the end. One checks that incorporating the $\chi \circ \det$ into the subscript of the second S^{ord} replaces $\alpha(\chi)^\flat$ by the $\beta(\chi)$ of (7.2.1).

Now let π be an anti-holomorphic representation of G_1 of type $(\kappa = \rho\cdot\alpha(\chi), K_r)$. Let $\phi_{\chi,r,\rho,\pi}$ denote the composition of $\phi_{\chi,r,\rho}$ with projection on the localization at the ideal $\mathfrak{m}_\pi$ in the first variable. Bearing in mind our conventions for the subscripts $_\pi$ and $_{\pi^\flat}$, it then follows from Hypothesis 7.2.3 that

$$Im(\phi_{\chi,r,\rho,\pi}) \subset \mathrm{Hom}_{\mathcal{O}}(\hat{S}^{\mathrm{ord}}_{\kappa,V,\pi}(K_r,\mathcal{O}), S^{\mathrm{ord}}_{\kappa^\flat,-V,\pi^\flat}(K_r,\mathcal{O}) \otimes \chi \circ \det). \tag{7.2.3}$$

Now both $\hat{S}^{\mathrm{ord}}_{\kappa,V,\pi}(K_r,\mathcal{O})$ and $S^{\mathrm{ord}}_{\kappa^\flat,-V,\pi^\flat}(K_r,\mathcal{O})$ are $\mathbb{T}_\pi$-modules, and indeed the Gorenstein hypothesis guarantees that they are free $\mathbb{T}_{r,\kappa,\pi}$-modules (in the obvious notation) of the same rank. In the next few paragraphs they are denoted $S^{\mathrm{ord}}_{r,V,\pi}$ and $S^{\mathrm{ord}}_{r,-V,\pi^\flat}$ to save space, the character $(\kappa = \rho\cdot\alpha(\chi))$ being understood.

We rewrite Hypothesis 7.1.7 in this notation.

Hypothesis 7.2.5. *Let R be a $\mathcal{O}$-algebra, and let $\hat{\mathbb{T}}_{r,\kappa} = \mathrm{Hom}(\mathbb{T}_{r,\kappa}, R)$. In the above notation, there are isomorphisms of $\mathbb{T}_{r,\kappa,\pi}$-modules*

$$\mathbb{T}_{r,\kappa,\pi} \otimes \hat{I}_\pi \xrightarrow{\sim} \hat{S}^{\mathrm{ord}}_{r,V,\pi};$$

$$\hat{\mathbb{T}}_{r,\kappa,\pi} \otimes I_\pi \xrightarrow{\sim} S^{\mathrm{ord}}_{r,V,\pi}.$$

The first is $\mathbb{T}_\pi$ linear, the second $\mathbb{T}_\pi$ $\flat$-linear.

These isomorphisms are not unique, but they can be coordinated as follows. The second isomorphism of Hypothesis 7.2.5 for $-V$ provides a $\mathbb{T}_\pi$-linear isomorphism:

$$j : S^{\mathrm{ord}}_{r,-V,\pi^\flat} \xrightarrow{\sim} \hat{\mathbb{T}}_{r,\kappa^\flat,\pi^\flat} \otimes I_{\pi^\flat}.$$

Composing with an isomorphism

$$G : \hat{\mathbb{T}}_{r,(\kappa)^\flat,\pi^\flat} \xrightarrow{\sim} \mathbb{T}_{r,\kappa,\pi} \tag{7.2.4}$$

of ($\mathbb{T}_\pi$-linear) $\mathbb{T}_\pi$-modules given by the Gorenstein hypothesis (Hypothesis 7.1.4), this becomes a $\mathbb{T}_\pi$-linear isomorphism

$$j' : S^{\mathrm{ord}}_{r,-V,\pi^\flat} \xrightarrow{\sim} \mathbb{T}_{r,\kappa,\pi} \otimes I_{\pi^\flat} \xrightarrow{\sim} \mathbb{T}_{r,\kappa,\pi} \otimes \hat{I}_\pi$$

where the second arrow is isomorphism (7.1.1). Now compose j' with the first isomorphism of Hypothesis 7.2.5 to obtain

$$j_G : S^{\mathrm{ord}}_{r,-V,\pi^\flat} \xrightarrow{\sim} \hat{S}^{\mathrm{ord}}_{r,V,\pi}.$$

Thus

Lemma 7.2.6. *Given a choice of isomorphism G as in* (7.2.4)*, there is a unique isomorphism*

$$j_G : S^{\mathrm{ord}}_{r,-V,\pi^\flat} \xrightarrow{\sim} \hat{S}^{\mathrm{ord}}_{r,V,\pi}$$

of free $\mathbb{T}_\pi$-modules compatible with G as above.

Remark 7.2.7. These maps can be compared to the $\flat$-isomorphism $F^\flat : S^{\mathrm{ord}}_{r,V,\pi} \xrightarrow{\sim} S^{\mathrm{ord}}_{r,-V,\pi^\flat}$.

Thus Hypothesis 7.2.3 yields (with $\kappa = \rho \cdot \alpha(\chi)$ as above)

Hypothesis 7.2.8.

$$\begin{aligned} Im(\phi_{\chi,r,\rho}) \subset \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(\hat{S}^{\mathrm{ord}}_{r,V}, S^{\mathrm{ord}}_{r,-V} \otimes \chi \circ \det) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{r,\kappa\cdot\alpha(\chi)}}(\mathbb{T}_{r,\kappa} \otimes \hat{I}_\pi, \hat{\mathbb{T}}_{r,\kappa} \otimes I_{\pi^\flat}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(\mathbb{T}_{r,\kappa}, \hat{\mathbb{T}}_{r,\kappa}) \otimes End_R(I_{\pi^\flat}). \end{aligned}$$

Here we have tensored with $\chi^{-1} \circ \det$ in the first line.

In the remainder of this subsection we no longer need to localize at $\mathfrak{m}_\pi$. We write $C_r = C_r(T, \mathcal{O})$ and drop the $\mathcal{O}$'s from the notation for modules of ordinary cusp forms, and ignore the twists by $\chi \circ \det$ where relevant. The natural inclusion $C_r \hookrightarrow C_{r+1}$, together with the map $\iota_r^* : \hat{S}^{\mathrm{ord}}_{r+1,V} \to \hat{S}^{\mathrm{ord}}_{r,V}$ (dual to the tautological inclusion $\iota_r : S^{\mathrm{ord}}_{r,V} \hookrightarrow S^{\mathrm{ord}}_{r+1,V}$) defines a diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{T}_{r+1,\kappa}}(C_{r+1} \otimes \hat{S}^{\mathrm{ord}}_{r+1,V}, S^{\mathrm{ord}}_{r+1,-V}) & \xrightarrow{r_r^* \otimes id^*_{r+1}} & \mathrm{Hom}_{\mathbb{T}_{r+1,\kappa}}(C_r \otimes \hat{S}^{\mathrm{ord}}_{r+1,V}, S^{\mathrm{ord}}_{r+1,-V}) \\ & & \uparrow \iota_r^* \\ & & \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(C_r \otimes \hat{S}^{\mathrm{ord}}_{r,V}, S^{\mathrm{ord}}_{r,-V}) \end{array}$$

Here $id^*_{r+1} : \hat{S}^{\mathrm{ord}}_{r+1,V} \to \hat{S}^{\mathrm{ord}}_{r+1,V}$ is the identity map and ι_r^* is the dual to ι_r (applied in the contravariant variable). It follows from the equivariance hypothesis that the tensor products ($C_{r+1} \otimes \hat{S}^{\mathrm{ord}}_{r+1,V}$ and the other two) can be taken over Λ_π, and then $\mathrm{Hom}_{\mathbb{T}_{r+1,\kappa}}$ is relative to the action of the Hecke algebra on $\hat{S}^{\mathrm{ord}}_{r+1,V}$ and $S^{\mathrm{ord}}_{r+1,-V}$. Hypothesis 7.2.3 now implies that

Fact 7.2.9. *For all r, the image of $\phi_{r+1,\kappa}$ under $r_r^* \otimes id^*$ lies in $Im(\iota_r^*)$.*

We make a more precise hypothesis:

Hypothesis 7.2.10. *More precisely,*

$$(r_r^* \otimes id^*_{r+1})(\phi_{r+1,\kappa}) = \iota_r \circ \phi_{r,\kappa} \circ (id_{C_r} \otimes \iota_r^*)$$

as maps from $C_r \otimes \hat{S}^{\mathrm{ord}}_{r+1,V}$ to $S^{\mathrm{ord}}_{r+1,-V}$, where id_{C_r} is the identity map on C_r.

7.2.11. *Serre duality and change of level.* We can interpret the map ι_r^* with respect to the Serre duality pairing 6.3.2 as follows. In this section we let R be a finite $\mathbb{Z}$-algebra with p-adic completion $R \hookrightarrow \mathcal{O}$. Identify $S_{\kappa,V}(K_r, R)$ with an R-lattice in $H_!^0({}_{K_r}Sh(V), \omega_\kappa)$. Let $H_!^{0,ord}({}_{K_r}Sh(V), \omega_\kappa)$ be the $\mathbb{C}$-linear span of $S_{\kappa,V}^{\mathrm{ord}}(K_r, R)$ and let $H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D)$ be the corresponding quotient of $H_!^d({}_{K_r}Sh(V), \omega_\kappa^D)$; then the action of the Hecke algebra identifies $H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D)$ as a direct summand of $H_!^d({}_{K_r}Sh(V), \omega_\kappa^D)$ that is in a perfect pairing with $H_!^{0,ord}({}_{K_r}Sh(V), \omega_\kappa)$. We can thus identify $\hat{S}_{\kappa,V}^{\mathrm{ord}}(K_r, R)$ with an R-lattice in $H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D)$ in such a way that

$$\begin{aligned} \hat{S}_{\kappa,V}^{\mathrm{ord}}(K_r, R) &= H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D)(R) \\ &:= \{h \in H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D) \mid \langle f, h\rangle_\kappa^{\mathrm{Ser}} \in R \ \forall f \in S_{\kappa,V}^{\mathrm{ord}}(K_r, R)\} \end{aligned} \tag{7.2.5}$$

The following statements (Lemma 6.2.12, Proposition 6.2.13, and Definition 6.2.14) are written in terms of κ rather than $\rho \cdot \alpha(\chi)$, for simplicity.

Lemma 7.2.12. *With respect to the identification* (7.2.5)*, the map*

$$\iota_r^* : \hat{S}_{\kappa,V}^{\mathrm{ord}}(K_{r+1}, R) \to \hat{S}_{\kappa,V}^{\mathrm{ord}}(K_r, R)$$

is given by the trace map:

$$t_r(h) = \sum_{\gamma \in K_r/K_{r+1}} \gamma(h).$$

In particular, the trace map t_r defines a surjective homomorphism

$$H_!^{d,ord}({}_{K_{r+1}}Sh(V), \omega_\kappa^D)(R) \to H_!^{d,ord}({}_{K_r}Sh(V), \omega_\kappa^D)(R)$$

Proof. It suffices to prove that the map

$$H_!^d({}_{K_{r+1}}Sh(V), \omega_\kappa^D) \to H_!^d({}_{K_r}Sh(V), \omega_\kappa^D)$$

dual under pairing (6.3.2) to the inclusion of forms of level K_r in forms of level K_{r+1} is given by the trace. But since the duality pairing is just integration, this comes down to the following observation: the adelic integral of a K_{r+1} invariant function f against a K_r invariant function g is the same as the adelic integral of g against the sum of K_r/K_{r+1}-translates of f (and there is no need to correct the normalization of the measure).

The final assertion then follows from Lemma 6.5.14. □

Now we localize again at $\mathfrak{m}_\pi$. As in Hypothesis 7.2.8 we can identify

$$\begin{aligned} \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(C_r \otimes \hat{S}_{r,V}^{\mathrm{ord}}, S_{r,-V}^{\mathrm{ord}}) &\simeq \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(C_r \otimes \mathbb{T}_{r,\kappa} \otimes \hat{I}_\pi, \hat{\mathbb{T}}_{r,\kappa} \otimes I_{\pi^\flat}) \\ &= \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(C_r \otimes \mathbb{T}_{r,\kappa}, \hat{\mathbb{T}}_{r,\kappa}) \otimes End_R(I_{\pi^\flat}) \\ &\simeq \hat{\mathbb{T}}_{r,\kappa} \otimes End_R(I_{\pi^\flat}) \end{aligned}$$

with appropriate modifications to accomodate a function Φ_X as above. We are using Hypotheses 7.1.7 and 7.2.5 systematically.

Proposition 7.2.13. *With respect to the identifications*

$$\mathrm{Hom}_{\mathbb{T}_{r,\kappa,\flat}}(C_r \otimes \hat{S}^{\mathrm{ord}}_{r,V}, S^{\mathrm{ord}}_{r,-V}) \simeq \hat{\mathbb{T}}_{r,\kappa} \otimes End_R(I_{\pi^\flat}),$$

Hypothesis 7.2.10, and the isomorphism $G_r : \hat{\mathbb{T}}_{r,\kappa} \xrightarrow{\sim} \mathbb{T}_{r,\kappa}$ *of the Gorenstein hypothesis, the measure* $\{\phi_{r,\kappa}\}$ *defines an element*

$$L(\phi_\kappa) \in \varprojlim_r \mathbb{T}_{r,\kappa} \otimes I_{\pi^\flat} \otimes I_\pi \xrightarrow{\sim} \mathbb{T} \otimes End_R(I_{\pi^\flat})$$

Moreover, if κ^1 *is a second sufficiently regular character, then* $L(\phi_\kappa)$ *and* $L(\phi_{\kappa^1})$ *are identified with respect to the identifications* $\mathbb{T} \xrightarrow{\sim} \mathbb{T}_{K^p,\kappa,\mathcal{O}_\pi} \xrightarrow{\sim} \mathbb{T}_{K^p,\kappa^1,\mathcal{O}_\pi}$ *of Theorem 7.1.1. Thus the measures* $\{\phi_{r,\kappa}\}$ *and* $\{\phi_{r,\kappa^1}\}$ *define the same element* $L(\phi) \in \mathbb{T} \otimes End_R(I_{\pi^\flat})$*. Conversely, any such* $L(\phi)$ *defines a measure* $\{\phi_{r,\kappa}\}$ *for any sufficiently regular* κ*.*

Moreover, the element $L(\phi_\kappa)$ *does not depend on the choice of identifications in Hypotheses 7.1.7 and 7.2.5, provided they are compatible with the choice of* G_r *in the sense of Lemma 7.2.6.*

Proof. This is a consequence of Lemma 7.2.12 and follows by unwinding the definitions. □

Note that G_r is independent of choices, provided that they are compatible in the sense of Lemma 7.2.6.

The above construction adapts easily to accommodate the compact p-adic Lie group X_p. We have seen that a $\mathcal{V}$-valued measure on $X_p \times T$ is the same thing as a measure on X_p with values in $\mathcal{V}$-valued measures on T. In particular, one obtains a $\mathcal{V}$-valued measure on $X_p \times T$ from a collection, for all characters α of X_p, of $\mathcal{V}$-valued measures ϕ_α of type α on T satisfying the congruence properties of Lemma 5.1.3.

Definition 7.2.14. *Fix a level* r*, a character* κ*, and an* $\mathcal{O}$*-algebra* R*. Let* $\lambda : \mathbb{T} \to R$ *be a continuous homomorphism. Say* λ *is classical of level* p^r *and weight* κ *if it factors through a homomorphism (still denoted)* $\lambda : \mathbb{T}_{r,\kappa} \to R$*, which is of the form* λ_π *for some antiholomorphic automorphic representation* π *of type* (κ, K_r) *with* $K_r = K^p I_r$ *for some open compact* $K^p \subset G(\mathbf{A}_f^p)$*, as before*

Let $X(\kappa, r, R)$ *denote the set of classical homomorphisms of level* p^r *and weight* κ *with values in* R*; let* $X^{class}(R) = \cup_{\kappa,r} X(\kappa, r, R)$*. Any* $\lambda \in X^{class}(R)$ *is called classical (with values in* R*).*

When $R = \mathbb{T}_{r,\kappa}$*, we let* $\lambda_{taut} : \mathbb{T}_{r,\kappa} \to \mathbb{T}_{r,\kappa}$ *be the identity homomorphism. When* π *is a cuspidal anti-holomorphic representation of weight* κ *as above, let* $\lambda_{taut,\pi} : \mathbb{T}_{r,\kappa} \to \mathbb{T}_{r,\kappa,\pi}$ *be* λ_{taut} *followed by localization at* $\mathfrak{m}_\pi$*.*

When κ is sufficiently regular, the character λ_{taut} deserves to be called classical because its composition with any map from $\mathbb{T}_{r,\kappa}$ to a p-adic field is attached to a classical modular form of weight κ. The relationship between $L(\phi)$ and the elements $\phi_{\chi,r,\kappa}$ is given by the following proposition.

Proposition 7.2.15. *Let χ be a character of X_p. Let $\phi = d\phi(x,t)$ be a measure on $X_p \times T$ as in Section 5.2.4, with shift sh: $sh^*(\chi) = (\alpha(\chi), \beta(\chi))$ Let ρ be an algebraic character of T, $\kappa = \rho \cdot \alpha(\chi)$. Fix a cuspidal anti-holomorphic representation π of weight κ satisfying the hypotheses above. We consider $L(\phi_\chi) = \int_{X_p} \chi(x) d\phi(x,t)$, localized at $\mathfrak{m}_\pi$, as an element of $\mathbb{T} \otimes End_R(I_{\pi^\flat})$. Let $L(\phi_\chi, \kappa, r)$ denote the image of $L(\phi_\chi)$ in $\mathbb{T}_{r,\kappa} \otimes End_R(I_{\pi^\flat})$. Equivalently,*

$$L(\phi_\chi, \kappa, r) = \int_{X_p \times T} \chi \times \lambda_{taut,\pi} d\phi(x,t),$$

where integration against $\lambda_{taut,\pi}$ amounts to the projection

$$\mathbb{T} \twoheadrightarrow \mathbb{T} \otimes_\Lambda \Lambda_{r,\kappa} = \mathbb{T}_{r,\kappa}$$

followed by localization at $\mathfrak{m}_\pi$.

Then $L(\phi_\chi, \kappa, r)$ corresponds to the element

$$\phi_{\chi,r,\kappa} \in \mathrm{Hom}_{\mathbb{T}_{r,\kappa}}(C_r \otimes \hat{S}^{\mathrm{ord}}_{r,V}, S^{\mathrm{ord}}_{r,-V} \otimes \chi \circ \det)$$

under the identifications in Hypotheses 7.1.7 and 7.2.5, compatible with the Gorenstein isomorphism G_r (from Proposition 7.2.13).

Proof. This is just a restatement of the definition of the element $L(\phi) \in \mathbb{T} \otimes End_R(I_{\pi^\flat})$ introduced in Proposition 7.2.13. □

The following is now an elementary consequence of Proposition 7.2.13.

Proposition 7.2.16. (Abstract p-adic L-functions of families) *Let $\phi = d\phi(x,t)$ be a measure on $X_p \times T$ such that, for each character χ of X_p, $\int_{X_p} \chi(x) d\phi(x,t)$ is a $\mathcal{V}$-valued measure ϕ_χ of type χ satisfying Hypothesis 7.2.3. Fix a cuspidal anti-holomorphic representation π satisfying the hypotheses of the previous sections. Then there is an element $L(\phi) \in \Lambda_{X_p} \hat{\otimes} \mathbb{T} \otimes End_R(I_{\pi^\flat})$ such that, for every R-valued character χ of X_p, the image of $L(\phi)$ under the map*

$$\chi \otimes Id : \Lambda_{X_p} \hat{\otimes} \mathbb{T} \otimes End_R(I_{\pi^\flat}) \to \mathbb{T} \otimes End_R(I_{\pi^\flat})$$

given by contraction in the first factor, or equivalently integration against χ with respect to the first variable, is the element $L(\phi_\chi)$ of Proposition 7.2.15.

The following standard fact (see, for example, [Hid88, Lemma 3.3]) shows that the specializations of Proposition 7.2.15 determine the abstract L-function $L(\phi)$:

Lemma 7.2.17. *The $\mathcal{V}$-valued measure $\phi = \phi_\chi$ of type χ and the abstract L-function $L(\phi)$ are completely determined by their integrals against elements of the sets $X(\kappa, r, \mathcal{O}_{\mathbb{C}_p})$ for any fixed sufficiently regular κ.*

We write

$$End_R(I_{\pi^\flat}) = \hat{I}_{\pi^\flat} \otimes I_{\pi^\flat} \simeq \mathrm{Hom}(\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}, R). \tag{7.2.6}$$

Then for any $\varphi \otimes \varphi^\flat \in \hat{I}_\pi \otimes \hat{I}_{\pi^\flat}$ we have a tautological pairing

$$L(\chi, \phi, r, \kappa, \varphi \otimes \varphi^\flat) = [L(\phi_\chi, \kappa, r), \varphi \otimes \varphi^\flat]_{loc} \in \mathbb{T}_{r,\kappa}. \tag{7.2.7}$$

where $[\bullet, \bullet]_{loc}$ is the tautological pairing

$$\mathrm{Hom}(\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}, R) \otimes \hat{I}_\pi \otimes \hat{I}_{\pi^\flat} \to R.$$

We reformulate Proposition 7.2.15 in terms of Equation (7.2.7).

Proposition 7.2.18. *Let R be a p-adic ring. Let ϕ be an admissible R-measure on $X_p \times T$ as in Section 5.2.4. Assume Hypotheses 7.1.4, 7.1.5, and 7.1.7. Let $\varphi \otimes \varphi^\flat \in \hat{I}_\pi \otimes \hat{I}_{\pi^\flat}$ as above. Then there is a unique element $L(\phi, \varphi \otimes \varphi^\flat) \in \Lambda_{X_p,R} \hat{\otimes} \mathbb{T}$ such that, for any classical $\chi : X_p \to R^\times$ and any $\lambda \in X(\kappa, r, R)$ (with κ sufficiently regular), the image of $L(\phi, \varphi \otimes \varphi^\flat)$ under the map $\Lambda_{X_p,R} \hat{\otimes} \mathbb{T} \to R$ induced by the character $\chi \otimes \lambda$ equals $L(\chi, \phi, r, \kappa, \varphi \otimes \varphi^\flat)$.*

7.3. **Classical pairings in families.** The following is essentially obvious. The notation $\langle \cdot, \cdot \rangle_\kappa^{\mathrm{Ser}}$ is as in (6.3.2).

Lemma 7.3.1. *Let $h \in S_{\kappa,V}^{\mathrm{ord}}(K_r, \mathcal{O})$, $\varphi \in H_{\kappa^D}^{d,ord}(K_r, \mathcal{O})[\pi]$, in the notation of Section 6.5. Then the map*

$$\mathbb{T} \to \mathcal{O};\ A \mapsto \langle A(h), \varphi \rangle_\kappa^{\mathrm{Ser}}$$

takes A to $\lambda_\pi(A) \langle h, \varphi \rangle_\kappa^{\mathrm{Ser}}$.

Proof. We have

$$\langle A(h), \varphi \rangle_\kappa^{\mathrm{Ser}} = \langle h, A^\flat(\varphi) \rangle_\kappa^{\mathrm{Ser}} = \lambda_{\pi^\flat}(A^\flat) \langle h, \varphi \rangle_\kappa^{\mathrm{Ser}} = \lambda_\pi(A) \langle h, \varphi \rangle_\kappa^{\mathrm{Ser}}.$$

□

Note that h is not assumed to be an eigenform in Lemma 7.3.1. However, the pairing with an eigenform for $\lambda_{\pi^\flat}$ factors through the projection of h on the (dual) λ_π-eigenspace. In general, this projection can only be defined over $\mathcal{O}[\frac{1}{p}]$. We write $h = \sum_{\pi' \in \mathcal{S}(K_r, \kappa, \pi^\flat)} a_{\pi'} h_{\pi'}$ where $a_{\pi'} \in \mathcal{O}[\frac{1}{p}]$ and $h_{\pi'}$ is in the $\lambda_{\pi'}$-eigenspace for $\mathbb{T}$. Then under the hypotheses of the lemma,

$$\langle h, \varphi \rangle_\kappa^{\mathrm{Ser}} = a_\pi \langle h_\pi, \varphi \rangle_\kappa^{\mathrm{Ser}}. \tag{7.3.1}$$

where of course $h_\pi \in \pi^\flat$.

The denominator of a_π is bounded by the congruence ideal $C(\pi) = C(\pi^\flat)$. In what follows we are making use of Proposition 6.6.8 and Corollary 6.6.9.

Lemma 7.3.2. *Let $\varphi \in H^{d,ord}_{\kappa^D}(K_r, \mathcal{O})[\pi]$. Then the linear functional*

$$h \mapsto L_\varphi(h) := \langle h, \varphi \rangle^{\mathrm{Ser}}_\kappa$$

belongs to $\hat{S}^{\mathrm{ord}}_{\kappa,V}(K_r, \mathcal{O})[\pi]$. Moreover, the restriction of $L_\varphi(h)$ to $S^{\mathrm{ord}}_{\kappa,V}(K_r, \mathcal{O})[\pi]$ takes values in the congruence ideal $C(\pi) = C(\pi^\flat) \subset \mathcal{O}$.

Proof. The claims follow from Lemmas 7.3.1 and 6.6.7, respectively. □

The functional in the last lemma can be rewritten as an integral. Recall that $\hat{I}_\pi$ (resp. $\hat{I}_{\pi^\flat}$) was identified with an $\mathcal{O}$-lattice in $\pi_p^{\text{a-ord}} \otimes \pi_{S^p}^{K^p}$, (resp. $\pi_p^{\flat,\text{a-ord}} \otimes \pi_{S^p}^{\flat,K^p}$. Recall also that we have dropped the subscript π for the moment, and so we are writing $\mathcal{O}$ in place of $\mathcal{O}_\pi$. In order to facilitate comparison of the p-adic and complex pairings, we let R be a finite local $\mathbb{Z}_{(p)}[\lambda_\pi]$-subalgebra of $\mathbb{C}$ that admits an embedding as a dense subring of $\mathcal{O}$, and let $\hat{I}_{\pi^\flat,R}$ and $\hat{I}_{\pi,R}$ be free R-modules given with isomorphisms

$$\hat{I}_{\pi^\flat,R} \otimes_R \mathcal{O} \xrightarrow{\sim} \hat{I}_{\pi^\flat};\ \hat{I}_{\pi,R} \otimes_R \mathcal{O} \xrightarrow{\sim} \hat{I}_\pi.$$

The following lemma is then just a restatement of (6.3.3).

Lemma 7.3.3. *In the notation of the previous lemma, let $\varphi \in \hat{I}_\pi$. If we identify h as above with an element of $H^0(\mathfrak{P}_h(V), K_h; \mathcal{A}_0(G) \otimes W_\kappa)$ and φ with an element of $H^d(\mathfrak{P}_h(-V), K_h; \mathcal{A}_0(G) \otimes W_{\star(\kappa^F)^D})$, as in Equation* (6.1.1)*, we can rewrite*

$$L_\varphi(h) = \int_{G(\mathbb{Q})Z_G(\mathbb{R})\backslash G(\mathbb{A})} [h(g), \varphi(g)] ||\nu(g)^{-a(\kappa)}|| dg.$$

Proof. Abbreviate $[G_3] = G_3(\mathbb{Q})Z(\mathbb{R})\backslash G_3(\mathbb{A})$, $dg_2^\chi = \chi(\det(g_2))^{-1} dg_2$. By doubling the formula in Lemma 7.3.3 – in other words, by applying it to the group G_3 – we obtain

$$\begin{aligned} &L_{\varphi\otimes\varphi^\flat}\left(res_3 D(\kappa, m, \chi_0) E^{holo}_{f(\chi,\psi\rho^\upsilon)}(m)\right) \\ &= \int_{[G_3]} D(\kappa, m, \chi_0) E^{holo}_{f(\chi_0,\psi\rho^\upsilon)}((g_1, g_2), m)\, \varphi(g_1)\varphi^\flat(g_2) ||\nu(g_1)^{a(\kappa)}|| dg_1 dg_2^\chi. \end{aligned}$$

Comparing this with Equation (9.1.3) and the definition of the zeta integral, we obtain the equality. □

8. Local theory of ordinary forms

8.1. **p-adic and C^∞-differential operators.** The notation (κ, χ) and $(\underline{\tilde{r}}, \underline{\tilde{s}})$ is as in Corollary 4.4.9 and Proposition 4.4.11. Parts (a) and (b) of the following proposition are in [Eis16], to which we refer for explanation of undefined terms.

Proposition 8.1.1. *(a) For $(\underline{\tilde{r}}, \underline{\tilde{s}})$ and χ as in 4.4.18, and for any prime-to-p level subgroup K^p, there is a differential operator*

$$\theta^d_\chi(\underline{\tilde{r}}, \underline{\tilde{s}}) = \theta^d_\chi(p(\underline{\tilde{r}}, \underline{\tilde{s}})) : V_\chi(G_4, K^p, \mathcal{O}) \to V(G_4, K^p, \mathcal{O})$$

compatible with change of level subgroup, and with the following property: For any level K^p*, for any form* $f \in M_\chi(G_4, K^p, \mathcal{O})$*, and any ordinary CM pair* (J_0', h_0) *as in Section 3.2.5, we have the identity*

$$R_{\kappa,J_0',h_0} \circ res_{J_0',h_0} \circ \delta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}})(f) = res_{p,J_0',h_0} \circ \theta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}}) \circ R_{\kappa,G,X}(f)$$

in the notation of Proposition 3.2.6.

(b) Let (κ, χ) *be critical as in Corollary 4.4.9. Fix a level subgroup* $K_4 \subset G_4(\mathbf{A}_f)$ *and a subgroup* $K_1 \times K_2 \subset G_3(\mathbf{A}_f) \cap K_4$*. The composition of* $\theta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}})$ *with the pullback* $res_3 := (\gamma_{V_p} \circ \iota_3)^*$ *defines an operator*

$$\theta(\kappa, \chi) : V_\chi(G_4, K^p, \mathcal{O}) \to S_{\kappa,V}(K_1) \otimes S_{\kappa^\flat,-V}(K_2) \otimes \chi \circ \det;$$

which coincides with the operator $\delta_\chi^d(\underline{\tilde{r}}, \underline{\tilde{s}})$ *upon pullback to functions on* $G_4(\mathbf{A})$ *and restriction to* $G_3(\mathbf{A})$ *(with respect to the maps* (2.9.2) *for* G_3 *and* G_4*).*

(c) Under the hypotheses of (a) and (b), there is a differential operator

$$\theta^{hol}(\kappa, \chi) : V_\chi(G_4, K^p, \mathcal{O}) \to V(G_4, K^p, \mathcal{O})$$

whose composition with the pullback res_3 *coincides with the operator* $D^{hol}(\kappa, \chi)$ *upon pullback to functions on* $G_4(\mathbf{A})$ *and restriction to* $G_3(\mathbf{A})$*.*

Proof. As mentioned above, parts (a) and (b) are in [Eis16]. The third part follows from Eischen's construction as well: it follows (by induction on the size of κ) from the last part of Corollary 4.4.9 that the operator $D^{hol}(\kappa, \chi)$ is obtained by pullback of the differential operator attached to a polynomial $P^{hol}(\kappa, \chi) \in \otimes_\sigma \mathcal{P}(n)_\sigma$. One lets $\theta^{hol}(\kappa, \chi)$ be the differential operator on p-adic modular forms attached to the same polynomial. □

The following Corollary is the p-adic version of the last part of Corollary 4.4.9.

Corollary 8.1.2. *Under the hypotheses of the previous proposition, for all* $\kappa^\dagger \leq \kappa$ *there are differential operators* $\theta(\kappa, \lambda) : V_\chi(G_4, K^p, \mathcal{O}) \to V(G_4, K^p, \mathcal{O})$ *such that*

$$\theta(\kappa, \chi) = \sum_{\kappa^\dagger \leq \kappa} res_3 \theta(\kappa, \kappa^\dagger) \circ \theta^{hol}(\kappa^\dagger, \chi).$$

Proposition 8.1.3. *Let* $F \in H^0(Sh(G_4), \mathcal{L}(\chi))$*.*

Assume $\kappa, (\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma), m, \chi_\sigma$ *are all associated, and let* e_κ *be the ordinary projector of 2.9.5 attached to the weight* κ*, as in 6.5.6. Then*

$$(e_\kappa \circ \theta(\kappa, \chi))(F) = e_\kappa \circ pr_\kappa^{hol} \circ \delta(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)(F). \tag{8.1.1}$$

Proof. By Corollary 8.1.2, the left hand side equals

$$\sum_{\kappa^\dagger \leq \kappa} e_\kappa \circ res_3 \theta(\kappa, \kappa^\dagger) \circ \theta^{hol}(\kappa^\dagger, \chi).$$

It suffices to show that, for every ordinary CM point $a \in Ig_3$

(1) For $\kappa^\dagger < \kappa$, $e_\kappa \circ res_3\theta(\kappa,\kappa') \circ \theta^{hol}(\kappa^\dagger,\chi)(F) = 0$ upon restriction to a;
(2) $e_\kappa \circ res_3\theta(\kappa,\kappa) \circ \theta^{hol}(\kappa,\chi)(F) - e_\kappa \circ pr^{hol}_\kappa \circ \delta(\underline{\tilde{r}}_\sigma, \underline{\tilde{s}}_\sigma)(F) = 0$ upon restriction to a.

Part (2) is a consequence of (b) of 8.1.1. We show that the expression in (1) is arbitrarily divisible by p. More precisely,

Lemma 8.1.4. *For any $\kappa^\dagger < \kappa$, the ordinary projector $e_\kappa = \varinjlim_N U_{p,\kappa}^{N!}$ converges absolutely to 0 on $S_{\kappa^\dagger}(K_r;R)$.*

Proof. The point is that, for each w,j, $U_{w,j,\kappa} = |\kappa'(t_{w,j})|_p^{-1} U_{w,j}$, with κ' defined as in 2.6.11. Thus

$$U_{p,\kappa} = \prod_{w,j} |\kappa'^{,-1} \cdot \kappa^{\dagger,\prime}(t_{w,j})|_p \cdot U_{w,j,\kappa^\dagger}.$$

The condition $\kappa^\dagger < \kappa$ is equivalent to the condition that the p-adic valuation of $\prod_{w,j} \kappa'^{,-1} \cdot \kappa^{\dagger,\prime}(t_{w,j})$ is positive. Thus $U_{p,\kappa}$ has p-adic norm strictly less than 1 on $S_{\kappa^\dagger}(K_r;R)$, and it follows that $e_\kappa = \varinjlim_N U_{p,\kappa}^{N!}$ acts as 0 on $S_{\kappa^\dagger}(K_r;R)$. □

Part (1) above now follows from the fact that the ordinary projector commutes with the differential operators. □

8.2. **Ordinary representations and ordinary vectors.** For this section, let $G = G_1$. For each prime $w \mid p$, let $G_w = \mathrm{GL}_n(\mathcal{K}_w)$. Recall that by (2.2.2) and (2.2.3) there is an identification

$$G(\mathbb{Q}_p) \xrightarrow{\sim} \mathbb{Q}_p^\times \times \prod_{w \in \Sigma_p} G_w. \tag{8.2.1}$$

Let $B_w \subset \mathrm{GL}_n(K_w)$ be the (non-standard) Borel consisting of elements $g = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with $A \in \mathrm{GL}_{a_w}(\mathcal{K}_w)$ upper-triangular and $D \in \mathrm{GL}_{b_w}(\mathcal{K}_w)$ lower-triangular. Let $T_w \subset B_w$ be its diagonal subgroup and $B_w^u \subset B_w$ its unipotent radical. Let $I_{w,r}^0 \subset \mathrm{GL}_n(\mathcal{O}_w)$ be the subgroup of elements g such that $g \bmod p^r = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ with $A \in \mathrm{GL}_{a_w}(\mathcal{O}_w/p^r\mathcal{O}_w)$ upper-triangular and $D \in \mathrm{GL}_{b_w}(\mathcal{O}_w/p^r\mathcal{O}_w)$ lower-triangular (this is the mod p^r Iwahori subgroup relative to the Borel B_w). Let $I_{w,r} \subset I_{w,r}^0$ be the subgroup consisting of those g such that A and D are unipotent. Under the identification (8.2.1) the subgroups $I_r \subset I_r^0$ of $G(\mathbb{Z}_p)$ defined in Section 2.5 are identified as

$$I_r^0 \xrightarrow{\sim} \mathbb{Z}_p^\times \prod_{w \in \Sigma_p} I_{w,r}^0 \quad \text{and} \quad I_r \xrightarrow{\sim} \mathbb{Z}_p^\times \prod_{w \in \Sigma_p} I_{w,r}. \tag{8.2.2}$$

Let $\delta_w : B_w \to \mathbb{C}$ be the modulus character: if $t = \mathrm{diag}(t_1, ..., t_n) \in T_w$, then $\delta_w(t) = |t_1^{n-1} \cdots t_{a_w}^{b_w - a_w} t_{a_w+1}^{1-n} \cdots t_n^{b_w - 1 - a_w}|_p$.

8.2.1. *Ordinary holomorphic representations: local theory.* Let π be a cuspidal holomorphic representation of $G(\mathbb{A})$ of weight type (κ, K) as in Section 6.4.1 with $\kappa = (\kappa_\sigma)_{\sigma \in \Sigma_\mathcal{K}}$, $\kappa_\sigma \in \mathbb{Z}^{a_\sigma}$, assumed to satisfy:

$$\kappa_\sigma + \kappa_{\sigma c} \geq n, \quad \forall \sigma \in \Sigma_\mathcal{K}. \tag{8.2.3}$$

Let $\kappa_{norm} = (\kappa_{norm,\sigma})$ with $\kappa_{norm,\sigma} = \kappa_\sigma - b_\sigma$.

Via the identification (8.2.1), the p-constituent π_p of π is identified with a tensor product $\pi_p \cong \mu_p \otimes_{w\in\Sigma_p} \pi_w$ with μ_p a character of $\mathbb{Q}_p^\times$ and each π_w an irreducible admissible representation of G_w.

Recall that the Hecke operators $u_{w,j} = |\kappa_{norm}(t_{w,j})|_p^{-1} U_{w,j}$, $w \in \Sigma_p$ and $1 \le j \le n$, act on the spaces $\pi_f^{K_r} = \pi_p^{I_r} \otimes (\otimes_{\ell\neq p}\pi_\ell)^{K^p}$ through an action on the spaces $\pi_p^{I_r}$: $U_{w,j}$ acts on the latter spaces as the usual double coset operator $I_r t_{w,j}^+ I_r$, and, furthermore, the generalized eigenvalues of the $u_{w,j}$ are p-adically integral (cf. Section 2.6.9; since $m = 1$ the subscript i has been dropped from our notation, following our conventions). In particular, the ordinary projector $e = \lim_{m\to\infty}(\prod_{w\in\Sigma_p}\prod_{i=1}^n u_{w,j})^{m!}$ acts on each $\pi_p^{I_r}$. From the identification $\pi_p = \mu_p \otimes_{w\in\Sigma_p} \pi_w$ and (8.2.2) we find that $u_{w,j}$ acts on $\pi_p^{I_r} = \otimes_{w\in\Sigma}\pi_w^{I_{w,r}}$ via the action of the Hecke operator $u_{w,j}^{\mathrm{GL}} = |\kappa_{norm}(t_{w,j})|_p^{-1} U_{w,j}^{\mathrm{GL}}$ on $\pi_w^{I_{w,r}}$, where $U_{w,j}^{\mathrm{GL}}$ acts as the double coset operator $I_{w,r} t_{w,j} I_{w,r}$; here, $t_{w,j} \in T_w$ is the element defined in Section 2.6.9. It follows that the generalized eigenvalues of the action of the Hecke operators $u_{w,j}^{\mathrm{GL}}$ are p-adically integral, and $e_w = \lim_{m\to\infty}(\prod_{j=1}^n u_{w,j}^{\mathrm{GL}})^{m!}$ defines a projector on each $\pi_w^{I_{w,r}}$.

Suppose that π is ordinary at p. Recall that this means $\pi_p^{I_r} \neq 0$ if $r \gg 0$ and that, for any such r, there is at least one vector $0 \neq \phi \in \pi_p^{I_r}$ such that $e \cdot \phi = \phi$. We call such a ϕ an *ordinary vector*[14] *for* π_p. The existence of an ordinary vector is equivalent to the existence of a $\phi \in \pi_p^{I_r}$, $r \gg 0$, that is a simultaneous eigenvector for the Hecke operators $u_{w,j}$ and having the property that $u_{w,j} \cdot \phi = c_{w,j}\phi$ with $|c_{w,j}|_p = 1$. It follows from the identification $\pi_p = \mu_p \otimes_{w\in\Sigma_p} \pi_w$ that π_p being ordinary at p is equivalent to μ_p being unramified and each π_w being ordinary, in the sense that there exists $\phi_w \in \pi_w^{I_{w,r}}$, $r \gg 0$, such that $e_w \cdot \phi_w = \phi_w$; we call such a ϕ_w an *ordinary vector for* π_w. The existence of an ordinary vector for π_w is equivalent to

(a) $\pi_w^{I_{w,r}} \neq 0$ for all $r \gg 0$;
(b) for each r as in (a) there exists $0 \neq \phi_w \in \pi_w^{I_{w,r}}$ such that ϕ_w is a simultaneous eigenvector for the $u_{w,j}^{\mathrm{GL}}$, $1 \le j \le n$, and having the property that $u_{w,j}^{\mathrm{GL}} \cdot \phi_w = c_{w,j}\phi_w$ with $|c_{w,j}|_p = 1$.

Note that if $\phi_w \in \pi_w$, $w \in \Sigma_p$, are ordinary vectors and μ_p is unramified, then $\phi = \otimes_{w\in\Sigma_p}\phi_w \in \pi_p$ is an ordinary vector for π_p.

Lemma 8.2.2. *Let* $w \in \Sigma_p$. *Suppose* π_w *is an irreducible admissible representation of* G_w *such that* (a) *and* (b) *above hold for a weight* κ *satisfying inequality* (8.2.3).

[14]But note that this notion depends *a priori* on the character κ_{norm}, which in turn depends on κ and the signatures $(a_\sigma, b_\sigma)_{\sigma\in\Sigma_\mathcal{K}}$. It turns out that there is at most one κ_{norm} with respect to which a given π_p can be ordinary, but in general the signatures are not uniquely determined.

(i) *Up to multiplication by a scalar, there is a unique ordinary vector* $\phi_w^{\mathrm{ord}} \in \pi_w^{I_{w,r}}$; ϕ_w^{ord} *is necessarily independent of* $r \gg 0$.

(ii) *There exists a unique character* $\alpha_w : T_w \to \mathbb{C}^\times$ *such that* $\pi_w \hookrightarrow \mathrm{Ind}_{B_w}^{G_w}\alpha_w$ *is the unique irreducible subrepresentation and* ϕ_w^{ord} *is identified with the unique simultaneous* $U_{w,j}^{\mathrm{GL}}$*-eigenvector,* $1 \le j \le n$, *with support containing* $B_w I_{w,r}$, *for* $r \gg 0$. (*In particular,* $c_{w,j} = |\kappa_{norm}(t_{w,j})|_p^{-1}\delta_w^{-1/2}\alpha_w(t_{w,j})$.)

Proof. Our proof is inspired in part by the arguments in [Hid98, §5]. Let V be the space underlying the irreducible admissible representation π_w of $G_w = \mathrm{GL}_n(\mathcal{K}_w)$, and let V_{B_w} be the Jacquet module of V with respect to the unipotent radical B_w^u of the Borel B_w. Let $N = \cap_r I_{w,r}$; this is just $B_w^u \cap \mathrm{GL}_n(\mathcal{O}_w)$. For each $j = 1, ..., n$, let

$$t_j = \begin{cases} \mathrm{diag}(p1_j, 1_{n-j}) & j \le a_w \\ \mathrm{diag}(p1_{a_w}, 1_{n-j}, p1_{j-a_w}) & j > a_w. \end{cases}$$

We let the double coset $U_j = N t_j N$ act on $V^N = \cup_r V^{I_{w,r}}$ in the usual way: if $Nt_jN = \sqcup_i x_{i,j} N$ then $U_j \cdot v = \sum_i x_{i,j} \cdot v$. Then U_j acts on the subspace V^{I_r} as $U_{w,j}^{\mathrm{GL}}$. By the same arguments explaining [Hid98, (5.3)], V^N decomposes as $V^N = V_{nil}^N \oplus V_{inv}^N$, where the U_j act nilpotently on V_{nil}^N and are invertible on V_{inv}^N. Then, just as in [Hid98], the natural B_w-invariant projection $V \overset{v \mapsto \bar{v}}{\twoheadrightarrow} V_{B_w}$ induces an isomorphism

$$V_{inv}^N \xrightarrow{\sim} V_{B_v}, \quad v \mapsto \bar{v}, \tag{8.2.4}$$

that is equivariant for the action of the U_j.

Let $\phi \in V^{I_r}$ be an ordinary vector for some r: ϕ is an eigenvector for each $u_j = |\kappa_{norm}(t_j)|_p^{-1} U_j$ with eigenvalue c_j such that $|c_j|_p = 1$. In particular, $\phi \in V_{inv}^N$. As U_j acts on V_{B_w} via $\delta_w(t_j)^{-1} t_j$, it then follows from (8.2.4) that there must be a B_w-quotient

$$\iota : V_{B_w} \twoheadrightarrow \mathbb{C}(\lambda)$$

with $\lambda : T_w \xrightarrow{\sim} B_w/B_w^u \to \mathbb{C}$ is a character such that $\lambda(t_j) = |\kappa_{norm}(t_j)|_p \delta(t_j) c_j$ for all $j = 1, .., n$. Let $\alpha = \lambda\delta^{-1/2}$ and let $I(\alpha) = \mathrm{Ind}_{B_w}^{G_w}(\alpha)$ be the unitary induction of α to a representation of G_w. By [Cas95, Thm. 3.2.4],

$$\mathrm{Hom}_{G_w}(V, I(\alpha)) \xrightarrow{\sim} \mathrm{Hom}_B(V_B, \mathbb{C}(\lambda)), \quad \varphi \mapsto (\bar{v} \mapsto \varphi(v)(1)),$$

is an isomorphism, from which we conclude that there exists a non-zero G_w-homomorphism $V \hookrightarrow I(\alpha)$, $v \mapsto f_v$ (which is necessarily an injection since π_w is irreducible) such that

$$\iota(\bar{v}) = f_v(1). \tag{8.2.5}$$

By the characterization of λ, $\beta = |\kappa_{norm}|_p^{-1}\delta_w^{-1}\lambda = |\kappa_{norm}|_p^{-1}\delta_w^{-1/2}\alpha$ is a continuous character $T_w \to \mathbb{C}^\times$ such that each $\beta(t_j)$ is a p-adic unit. From the definition of the t_j it then follows easily that $\beta(t)$ is a p-adic unit for all $t \in T_w$. Let W be the Weil group of T_w in G_w. For $x \in W$, let $\beta_x = |\kappa_{norm}|_p^{-1}\delta_w^{-1/2}\alpha^x$, where $\alpha^x(t) = \alpha(xtx^{-1})$. We claim that the values of β_x are all p-adic units if and only if $x = 1$. If the values of β_x are all p-adic units, then

$$\beta_x/\beta^x(t) = |\kappa_{norm}(xtx^{-1}t^{-1})|_p^{-1}\delta_w(xtx^{-1}t^{-1})^{-1/2}$$

is a p-adic unit for all $t \in T_w$. As δ_w is the composition of $|\cdot|_p$ with an algebraic character of T_w, it follows that the above values must all be 1. That is, the character $\theta = |\kappa_{norm}|_p\delta_w^{-1/2}$ satisfies $\theta^x = \theta$. Recall that if κ is identified with a dominant tuple $(\kappa_0, (\kappa_\sigma)_{\sigma\in\Sigma_\mathcal{K}})$ as in (2.6.2) then

$$\kappa_{norm}(\mathrm{diag}(t_1, ..., t_n)) = \prod_{\substack{\sigma \\ \mathfrak{p}_\sigma = \mathfrak{p}_w}} \prod_{i=1}^{a_w} \sigma(t_i)^{\kappa_{\sigma,i}-b_w} \prod_{j=1}^{b_w} \sigma(t_{a_w+j})^{-\kappa_{\sigma c,j}+a_w}.$$

In particular, letting

$$m_i = \begin{cases} \sum_{\sigma,\mathfrak{p}_\sigma=\mathfrak{p}_w}(\kappa_{\sigma,i} - b_w) & i \le a_w \\ -\sum_{\sigma,\mathfrak{p}_\sigma=\mathfrak{p}_w}(\kappa_{\sigma c,i} - a_w) & i > a_w, \end{cases}$$

we have

$$|\kappa_{norm}(\mathrm{diag}(t_1, ..., t_n))|_p = \prod_{i=1}^{n} |t_i|_p^{m_i}.$$

It follows that

$$\theta(\mathrm{diag}(t_1, ..., t_n)) = |t_1^{m_1+\frac{n-1}{2}} \cdots t_{a_w}^{m_{a_w}+\frac{n-a_w}{2}} t_{a_w+1}^{m_{a_w+1}-\frac{b_w}{2}} \cdots t_n^{m_n+\frac{a_w-1-b_w}{2}}|_p^{-1}.$$

From the dominance of κ and the inequality (8.2.3) it follows that

$$m_1 \ge m_2 \ge \cdots \ge m_{a_w} \ge m_n \ge m_{n-1} \ge \cdots \ge m_{a_w+1},$$

and so

$$m_1 + \frac{n-1}{2} > \cdots > m_{a_w} + \frac{b_w - a_w}{2} > m_n + \frac{b_w - 1 - a_w}{2} > \cdots > m_{a_w+1} + \frac{1-n}{2}.$$

That is, θ is a regular character of T_w, and therefore $\theta^x = \theta$ if and only if $x = 1$. This completes that proof that the values of β_x are all p-adic units if and only $x = 1$.

As $\beta_x \ne \beta$ for all $x \ne 1$, the characters α^x, $x \in W$, are all distinct, and hence the Jacquet module $I(\alpha)_{B_w}$ of $I(\alpha)$ is a semi simple B_w-module and isomorphic to the direct sum $\oplus_{x\in W}\mathbb{C}(\alpha^x\delta^{1/2})$ (cf. [Hid98, Prop. 5.4]). The inclusion $V \hookrightarrow I(\alpha)$, $v \mapsto f_v$, induces a B_w-inclusion

$$V_{B_w} \hookrightarrow I(\alpha)_{B_w} \cong \oplus_{x\in W}\mathbb{C}(\alpha^x\delta_w^{1/2}). \tag{8.2.6}$$

It then follows from (8.2.4) that V_{inv}^N is a sum of one-dimensional simultaneous eigenspaces for the U_j that are in one-to-one correspondence with those characters $\alpha^x\delta^{-1/2}$, $x \in W$, that appear in V_B via (8.2.6); the eigenvalue of $u_j = |\kappa_{norm}(t_j)|_p^{-1}U_j$ on the eigenspace corresponding to $\alpha^x\delta^{1/2}$ is $\beta_x(t_j)$. As the values of β_x are not all p-adic units if $x \ne 1$, it follows that the space of ordinary vectors in V is one-dimensional; this proves part (i). It further follows that the ordinary eigenspace must project non-trivially to $\mathbb{C}(\lambda) = \mathbb{C}(\alpha\delta^{1/2})$ via the composition of (8.2.4) with ι, and that all other eigenspaces map to 0 under this composition. As this composition is just $v \mapsto f_v(1)$ by (8.2.5), part (ii) follows easily. □

Corollary 8.2.3. *Suppose κ satisfies (8.2.3) and π_p is ordinary. Up to multiplication by a scalar, there is a unique ordinary vector $\phi^{\mathrm{ord}} \in \pi_p^{I_r}$ for $r \gg 0$; ϕ^{ord} is necessarily independent of r. Furthermore, under the identification $\pi_p = \mu_p \otimes_{w\in\Sigma_p} \pi_w$, $\phi^{\mathrm{ord}} = \otimes_{w\in\Sigma_p}\phi_w^{\mathrm{ord}}$, with ϕ_w^{ord} as in Lemma 8.2.2.*

The following lemma will aid in the computation of certain local zeta integrals involving ordinary vectors.

Lemma 8.2.4. *Let w, π_w, and κ be as in Lemma 8.2.2. Let $\pi_w^\vee$ be the contragredient of π_w and $\langle\cdot,\cdot\rangle_w : \pi_w \times \pi_w^\vee \to \mathbb{C}$ the non-degenerate G_w-invariant pairing (unique up to scalar multiple).*

(i) *Let α_w be as in Lemma 8.2.2(ii). Then $\pi_w^\vee$ is isomorphic to the unique irreducible quotient of $\mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1}$: $\mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1} \twoheadrightarrow \pi_w^\vee$.*

(ii) *For $r \gg 0$, let $\phi_{w,r}^\vee \in \pi_w^\vee$ be the image of the vector in $\mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1}$ that is supported on B_wI_r. Then $c(\pi_w, r) := \langle\phi_w^{\mathrm{ord}}, \phi_{w,r}^\vee\rangle_w$ is non-zero and depends only on r.*

(iii) *Let $0 \neq \phi \in \pi_w^{I_r}$ with $e\cdot\phi = c(\phi)\phi_w^{\mathrm{ord}}$. Then*
$$\langle\phi, \phi_{w,r}^\vee\rangle_w = c(\phi)\langle\phi_w^{\mathrm{ord}}, \phi_{w,r}^\vee\rangle_w.$$

Proof. Part (i) follows from the identification of $\mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1}$ as the contragredient of $\mathrm{Ind}_{B_w}^{G_w}\alpha_w$ (cf. [Cas95, Prop. 3.1.2]). The pairing $\langle\cdot,\cdot\rangle : \mathrm{Ind}_{B_w}^{G_w}\alpha_w \times \mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1} \to \mathbb{C}$ corresponding to this identification is just integration over $\mathrm{GL}_n(\mathcal{O}_w) \subset G_w$:
$$\langle\varphi, \varphi'\rangle = \int_{\mathrm{GL}_n(\mathcal{O}_w)} \varphi(k)\varphi'(k)dk, \quad \varphi \in \mathrm{Ind}_{B_w}^{G_w}\alpha_w, \varphi' \in \mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1},$$
(cf. [Cas95, Prop. 3.1.3]). For part (ii), let $\varphi^{\mathrm{ord}} \in \mathrm{Ind}_{B_w}^{G_w}\alpha_w$ correspond to ϕ_w^{ord} as in Lemma 8.2.2(ii) and let $\varphi_r^\vee \in \mathrm{Ind}_{B_w}^{G_w}\alpha_w^{-1}$ be the function supported on B_wI_r. Then
$$\langle\phi_w^{\mathrm{ord}}, \phi_{w,r}^\vee\rangle_w = \int_{\mathrm{GL}_n(\mathcal{O}_w)} \varphi^{\mathrm{ord}}(k)\varphi_r^\vee(k)dk.$$
As $B_wI_r \cap \mathrm{GL}_n(\mathcal{O}_w) = I_r^0$, and since for $k = tk' \in I_r^0 = T_w(\mathcal{O}_w)I_r$ we have $\varphi^{\mathrm{ord}}(k)\varphi^\vee(k) = \alpha_w(t)\alpha_w^{-1}(t) = 1$, it then follows that
$$c(\pi_w, r) := \langle\phi_w^{\mathrm{ord}}, \phi_{w,r}^\vee\rangle_w = \int_{I_w^0} dk = \mathrm{vol}(I_r^0) \neq 0.$$
This proves part (ii).

For part (iii), write ϕ as a sum of simultaneous generalized $U_{w,j}^{\mathrm{GL}}$-eigenvectors:
$$\phi = c(\phi)\phi_w^{\mathrm{ord}} + \sum_{i=1}^m \phi_i, \quad e\cdot\phi_i = 0.$$

Let φ (resp. φ^{ord}, φ_i) be the function in $\mathrm{Ind}_{B_w}^{G_w}\alpha_w$ that corresponds to ϕ (resp. ϕ_w^{ord}, ϕ_i) as in Lemma 8.2.2(ii). Then, for $r \gg 0$, $\varphi_i|_{I_r^0} = 0$, and so

$$\begin{aligned}\langle \phi, \phi_{w,r}^\vee\rangle_w &= \int_{\mathrm{GL}_n(\mathcal{O}_w)} \varphi(k)\varphi_r^\vee(k)dk = \int_{I_r^0}\varphi(k)\varphi_r^\vee(k)dk \\ &= c(\phi)\int_{I_r^0}\varphi^{\mathrm{ord}}(k)\varphi_r^\vee(k) = c(\phi)\langle\phi_w^{\mathrm{ord}}, \phi_{w,r}^\vee\rangle_w.\end{aligned}$$

□

8.2.5. *Anti-ordinary anti-holomorphic representations: local theory.* Let π be an anti-holomorphic representation of $G(\mathbb{A})$ of type (κ, K) as in 6.4.1 with κ satisfying the inequality (8.2.3). This the case if and only if $\pi^\flat$ is a cuspidal holomorphic representation of type (κ, K) as considered in the preceding section.

For each $r > 0$ the Hecke operators $u_{w,j}^- = |\kappa_{norm}(t_{w,j})|_p U_{w,j}^-$, $w \in \Sigma_p$ and $1 \le j \le n$, act on the space $\pi_f^{K_r} = \pi_p^{I_r} \otimes (\otimes_{\ell\neq p}\pi_\ell)^{K^p}$ through an action on the space $\pi_p^{I_r}$: $U_{w,j}^-$ acts on $\pi_p^{I_r}$ as the usual double coset operator $I_r t_{w,j}^- I_r$. Furthermore, the generalized eigenvalues of the $u_{w,j}^-$ are p-adically integral. In particular, the *anti-ordinary* projector $e^- = \lim_{m\to\infty}(\prod_{w\in\Sigma_p}\prod_{i=1}^n u_{w,j}^-)^{m!}$ acts on $\pi_p^{I_r}$. From the identification $\pi_p = \mu_p \otimes_{w\in\Sigma_p}\pi_w$ (via the isomorphism (8.2.1)) we find that $u_{w,j}^-$ acts on $\pi_p^{I_r} = \otimes_{w\in\Sigma_p}\pi_w^{I_{w,r}}$ via the action of the Hecke operator $u_{w,j}^{\mathrm{GL},-} = |\kappa_{norm}(t_{w,j})|_p^{-1} U_{w,j}^{\mathrm{GL},-}$, where $U_{w,j}^{\mathrm{GL},-}$ acts as the double coset operator $I_{w,r}t_{w,j}^{-1}I_{w,r}$; here $t_{w,j} \in T_w$ is the element defined in Section 2.6.9. It follows that the generalized eigenvalues of the action of the Hecke operators $u_{w,j}^{\mathrm{GL},-}$ are p-adically integral and $e_w^- = \lim_{m\to\infty}(\prod_{j=1}^n u_{w,j}^{\mathrm{GL},-})^{m!}$ defines a projector on $\pi_w^{I_r}$.

We say that π is *anti-ordinary at p of level r* if $\pi_p^{I_r} \neq 0$ and there exists $0 \neq \phi \in \pi_p^{I_r}$ such that $e^- \cdot \phi = \phi$. We say that such a ϕ is an *anti-ordinary vector for π_p of level r*. Under the identification $\pi_p = \mu_p \otimes_{w\in\Sigma_p}\pi_w$, the existence of an anti-ordinary vector of level r in π is equivalent to μ_p being unramified and, for each $w \in \Sigma_p$, there existing $0 \neq \phi_w \in \pi_w^{I_{w,r}} \neq 0$ such that $e_w \cdot \phi_w = \phi_w$; we call such a ϕ_w an *anti-ordinary vector for π_w of level r*.

Lemma 8.2.6. *Let $w \in \Sigma_p$ and π_w be a constituent of π_p as above.*

- (i) *The representation π_w is anti-ordinary of some level r if and only if $\pi_w^\vee$ is ordinary, in which case π_w is anti-ordinary of all levels $r \gg 0$.*
- (ii) *If π_w is anti-ordinary of level r, then there exists a unique (up to nonzero scalar multiple) anti-ordinary vector $\phi_{w,r}^{\text{a-ord}} \in \pi_w^{I_{w,r}}$ of level r; it is characterized by $\langle\phi_{w,r}^{\text{a-ord}}, \phi_w^{\vee,\mathrm{ord}}\rangle_w \neq 0$ and $\langle\phi_{w,r}^{\text{a-ord}}, \phi\rangle_w = 0$ for all $\phi \in \pi_w^{I_{w,r}}$ belonging to a generalized eigenspace of some $u_{w,j}^{\mathrm{GL},-}$ with non-unit eigenvalue.*

Proof. Suppose π_w is anti-ordinary of some level r. Then $\pi_w^{I_{w,r}} \neq 0$ and there exists a simultaneous eigenvector $\phi_{w,r}^{\text{a-ord}} \in \pi_w^{I_{w,r}}$ for the $u_{w,j}^{\mathrm{GL},-}$ with p-adic unit eigenvalues $a(j,r)$.

Let $\langle\cdot,\cdot\rangle_w : \pi_w \times \pi_w^\vee \to \mathbb{C}$ be the G_w-equivariant pairing. Then we have

$$a(j,r)\langle\phi_{w,r}^{\text{a-ord}},\phi\rangle_w = \langle u_{w,j}^{\mathrm{GL},-}\cdot\phi_{w,r}^{\text{a-ord}},\phi\rangle_w = \langle\phi_{w,r}^{\text{a-ord}},u_{w,j}^{\mathrm{GL}}\cdot\phi\rangle_w \tag{8.2.7}$$

for all $\phi\in\pi_w^{\vee,I_{w,r}}$. It follows that the action of each $u_{w,j}^{\mathrm{GL}}$ on $\pi_w^{\vee,I_{w,r}}$ has an eigenspace with eigenvalue $a(j,r)$ (which is a p-adic unit). To see that there exists a simultaneous such eigenspace we use the commutativity of the $u_{w,j}^{\mathrm{GL}}$s: Let $V_{j-1}\subset\pi_w^{\vee,I_{w,r}}$ be a maximal subspace that is a simultaneous eigenspace for $u_{w,1}^{\mathrm{GL}},\dots,u_{w,j-1}^{\mathrm{GL}}$ with respective eigenvalues $a(1,r),\dots,a(j-1,r)$. Then by the commutativity of the $u_{w,j}^{\mathrm{GL}}$s, the identity (8.2.7) holds for all $\phi\in V_{j-1}$. In particular, there is a non-zero (maximal) subspace of $V_j\subset V_{j-1}$ which is an eigenspace for $u_{w,j}^{\mathrm{GL}}$ with eigenvalue $a(j,r)$. It follows from induction on j that there exists a non-zero simultaneous $u_{w,j}^{\mathrm{GL}}$-eigenvector $\phi\in\pi_w^{\vee,I_{w,r}}$, $j=1,...,n$, with p-adic unit eigenvalues $a(j,r)$. That is, $\pi_w^\vee$ is ordinary.

Conversely, suppose that $\pi_w^\vee$ is ordinary, and let $\phi_w^{\vee,\mathrm{ord}}\in\pi_w^\vee$ be an ordinary vector with $u_{w,j}^{\mathrm{GL}}$- eigenvalue $c(j)$ (which is a p-adic unit). Then for $r\gg0$ we have

$$c(j)\langle\phi,\phi_w^{\vee,\mathrm{ord}}\rangle_w = \langle\phi,u_{w,j}^{\mathrm{GL}}\cdot\phi_w^{\vee,\mathrm{ord}}\rangle_w = \langle u_{w,j}^{\mathrm{GL},-}\phi,\phi_w^{\vee,\mathrm{ord}}\rangle_w \tag{8.2.8}$$

for all $\phi\in\pi_w^{I_{w,r}}$. It follows from the non-degeneracy of $\langle\cdot,\cdot\rangle_w$ that there exists a $u_{w,j}^{\mathrm{GL},-}$-eigenvector $\phi_{j,r}\in\phi_w^{I_{w,r}}$ with eigenvalue $c(j)$. Using (8.2.8) and the commutativity of the $u_{w,j}^{\mathrm{GL},-}$ we find, as in the preceding proof of the ordinarily of $\pi_w^\vee$, that there exists a non-zero simultaneous $u_{w,j}^{\mathrm{GL},-}$-eigenvector $\phi\in\pi_w^{I_{w,r}}$, $j=1,...,n$, with p-adic unit eigenvalues $c(j)$. That is, π_w is anti-ordinary of level r for all $r\gg0$.

Suppose now that π_w is anti-ordinary of level r, and let $\phi_{w,r}^{\text{a-ord}}\in\pi_w^{I_{w,r}}$ be an anti-ordinary vector of level r. As shown above, $\pi_w^\vee$ is ordinary and $\phi_w^{\vee,\mathrm{ord}}\in\pi_w^{\vee,I_{w,r}}$. We note that

$$\pi_w^{\vee,I_{w,r}} = \mathbb{C}\phi_w^{\vee,\mathrm{ord}}\oplus V_1\oplus\cdots\oplus V_t$$

with each V_i a simultaneous generalized $u_{w,j}^{\mathrm{GL}}$-eigenspace with at least one of the (generalized) eigenvalues not a p-adic unit; this follows from the uniqueness of the ordinary vector (see Lemma 8.2.2(i)). Since (8.2.7) holds for all $\phi\in V_i$ it follows that $\langle\phi_{w,r}^{\text{a-ord}},V_i\rangle_w=0$. This proves that $\phi_{w,r}^{\text{a-ord}}\in\pi_w^{I_{w,r}}$ is characterized (up to non-zero scalar multiple) as stated in part (ii). The uniqueness also follows.

□

Using this we can deduce an analog of Lemma 8.2.2(ii):

Lemma 8.2.7. *Let $w\in\Sigma_p$ and π_w be a constituent of π_p as above. Suppose π_w is anti-ordinary. Then there exists a unique character $\beta_w : T_w\to\mathbb{C}^\times$ such that $\mathrm{Ind}_{B_w}^{G_w}\beta_w\twoheadrightarrow\pi_w$ is the unique irreducible quotient and the anti-ordinary vector $\phi_{w,r}^{\text{a-ord}}\in\pi_w^{I_{w,r}}$ of level r is (up to non-zero scalar multiple) the image of the vector in $\mathrm{Ind}_{B_w}^{G_w}$ with support $B_wI_{w,r}$.*

In particular, the $\phi^{\text{a-ord}}_{w,r}$, $r \gg 0$, *can be chosen to satisfy*

$$\mathrm{Tr}_{I_{w,r}/I_{w,r'}} \phi^{\text{a-ord}}_{w,r'} = \phi^{\text{a-ord}}_{w,r}, \quad r' \geq r.$$

Proof. Since π_w is anti-ordinary, it follows from Lemma 8.2.6(i) that $\pi_w^\vee$ is ordinary. Let α_w be the unique character of B_w associated with $\pi_w^\vee$ as in Lemma 8.2.2(ii). Let $\beta_w = \alpha_w^{-1}$. As $\pi_w^\vee$ is the unique irreducible subrepresentation of $\mathrm{Ind}_{B_w}^{G_w}\alpha_w$, π_w is the unique irreducible quotient of $\mathrm{Ind}_{B_w}^{G_w}\beta_w$. Furthermore, it follows from Lemmas 8.2.2(ii) and 8.2.4(ii-iii) that the image of the vector in $\mathrm{Ind}_{B_w}^{G_w}\beta_w$ that is supported on $B_w I_{w,r}$ satisfies the conditions that characterize $\phi^{\text{a-ord}}_{w,r}$ in Lemma 8.2.6(ii). The uniqueness of β_w easily follows from the uniqueness of α_w and Lemma 8.2.4. □

Corollary 8.2.8. *Suppose* κ *satisfies Inequality* (8.2.3). *Then* π_p *is anti-ordinary if and only if* $\pi_p^\flat$ *is ordinary, and up to multiplication by a scalar, there is a unique anti-ordinary vector* $\phi_r^{\text{a-ord}} \in \pi_p^{I_{w,r}}$ *of level* r *for each* $r \gg 0$. *Furthermore, under the identification* $\pi_p = \mu_p \otimes_{w\in\Sigma_p}$, $\phi^{\text{a-ord}} = \otimes_{w\in\Sigma_p}\phi_w^{\text{a-ord}}$, *with* $\phi_w^{\text{a-ord}}$ *as in Lemma 8.2.6.*

8.2.9. *The Newton polygon.* Let π be a holomorphic or anti-holomorphic cuspidal automorphic representation of $G(\mathbb{A})$, and let $\pi_p = \mu_p \otimes_{w\in\Sigma_p} \pi_w$ be the identification corresponding to (8.2.1). We assume that

$$\text{each } \pi_w \text{ is an irreducible subquotient of } \mathrm{Ind}_{B_w}^{G_w}\beta_w \tag{8.2.9}$$

for some character $\beta_w : T \to \mathbb{C}^\times$. We view β_w as n-tuple $\beta_w = (\beta_{w,1}, ..., \beta_{w,n})$ of characters of $\mathcal{K}_w^\times$, defined by $\beta_w(\mathrm{diag}(t_1, ..., t_n)) = \prod_{i=1}^n \beta_{w,i}(t_i)$; the characters $\beta_{w,i}$ are uniquely determined up to order. We define the *total Hecke polynomial* of π at w to be

$$H_w(T) = \prod_{i=1}^n (1 - \alpha_{w,i}(\varpi_w)T)(1 - \alpha_{w,i}^{-1}(\varpi_w)T) \tag{8.2.10}$$

The *Newton polygon* $\mathrm{Newt}(\pi, w)$ of π at w is the Newton polygon of $H_w(T)$. Note that

$$\mathrm{Newt}(\pi, w) = \mathrm{Newt}(\pi^\flat, w).$$

Let $\Sigma_w = \{\sigma \in \Sigma_{\mathcal{K}} \mid \mathfrak{p}_\sigma = \mathfrak{p}_w\}$. Let

$$\pi_{\Sigma_w} = \otimes_{\sigma\in\Sigma_w}\pi_\sigma = \otimes_{\sigma\in\Sigma_w}\mathbb{D}_c(\tau_\sigma)$$

in the notation of (4.4.3). Define the Hodge polygon $\mathrm{Hodge}(\pi, w)$ to be the polygon in the right half-plane with vertices $(i, \sum_{\sigma\in\Sigma_w} p_{i,\sigma})$, where $(p_{i,\sigma}, q_{i,\sigma})$ are the pairs introduced in section 4.4.12 for $\mathbb{D}_c(\tau_\sigma)$.

Proposition 8.2.10. *Suppose* π *is (anti-)holomorphic and (anti-)ordinary. Then* $\mathrm{Newt}(\pi_w)$ *and* $\mathrm{Hodge}(\pi_w)$ *meet at the midpoint* $(n, \sum_{\sigma\in\Sigma_w} p_{i,\sigma})$.

In motivic terms, this says that the motive obtained by restriction of scalars to $\mathbb{Q}$ of the motive attached to Π satisfies the *Panchishkin condition*, see [Pan94]. The proof is an elementary calculation and is omitted; it will not be used in what follows. Details will be provided in a future article, when the results obtained here are related to standard conjectures on p-adic L-functions.

8.3. **Global consequences of the local theory.** We assume the ordinary cuspidal anti-holomorphic representation π of G_1 is of type (κ, K_r), $K_r = K^p I_r$, and satisfies the Gorenstein, Minimality, and Global Multiplicity One Hypotheses of section 7.1. Let S be the set of finite primes, not dividing p, at which K^p is not hyperspecial maximal. We summarize the implications of the local theory for the identification of automorphic forms in π. We let $\pi^\flat$ denote the dual representation of π, viewed as a holomorphic automorphic representation of G_1. Let I_π and $\hat{I}_\pi$ be as in Section 7.1.6. We say that the anti-holomorphic cuspidal representation of G_1 is *in the family determined by* π if there is a non-trivial character $\lambda_{\pi'}$ of the Hecke algebra $\mathbb{T} = \mathbb{T}_\pi$ defining the action of the unramified Hecke operators on π'. Any such π' is assumed to be given with a factorization (1.4.2). The factors π'_w, for $w \mid p$, are all (tempered) subquotients of principal series representations.

In what follows, the Borel subalgebras $\mathfrak{b}^+_\sigma$ are chosen at archimedean places σ as in Section 4.4.1. The Minimality Hypothesis allows us to choose v_S uniformly for π' in the following proposition.

Proposition 8.3.1. *Fix an element $v_S \in \hat{I}_\pi$. Let π' be any anti-holomorphic representation, of type $(\kappa', K_{r'})$, in the family determined by π. Let $\varphi_{\kappa',-}$ denote a lowest weight vector in the anti-holomorphic subspace of π'_∞, as in* (4.1.4)*. For a finite prime $v \notin S \cup \Sigma_p$, let φ'_v be a fixed generator of the spherical subspace of π'_v and let $\varphi'^{,\flat}_v$ be the dual generator of the spherical subspace of $\pi'^{,\flat}_v$. Assume κ satisfies* (8.2.3)*. Then*

(1) *For $r'' >> 0$, there is, up to scalar multiples, a unique anti-ordinary anti-holomorphic vector $\varphi^{r''}(v_S, \pi') \in (\pi')^{K_{r''}}$ with factorization* (1.4.2) *given by*

$$fac_{\pi',\flat}(\varphi^{r''}(v_S, \pi')) = \varphi_{\kappa',-} \otimes \otimes_{v \notin S \cup \Sigma_p} \varphi'_v \otimes \otimes_{w|p} \phi^{a-ord}_{w,r''} \otimes v_S.$$

(2) *As r' varies, the $\varphi^{r'}(v_S, \pi') \in \pi'$ can be chosen so that, if $r'' >> 0$, then*

$$t_{r''} \varphi^{r''+1}(v_S, \pi') = \varphi^{r''}(v_S, \pi'),$$

where the trace map $t_{r''}$ is defined as in Lemma 7.2.12.

Proof. This follows directly from the results in the previous sections, in particular Lemmas 8.2.7 and 8.2.2. □

9. Construction of p-adic L-functions

Review of notation. We recall the notation from the previous sections, because some of it is admittedly counterintuitive. Our basic Shimura varieties are denoted $Sh(V)$ (attached to G_1) and $Sh(-V)$ (attached to G_2, which is isomorphic to G_1). Classical points of our Hida families correspond to cuspidal automorphic representations denoted π and

$\pi^\flat$, for $Sh(V)$ and $Sh(-V)$, respectively. With our conventions, π is an *antiholomorphic* automorphic representation of G_1, and therefore with respect to the isomorphism $G_2 \xrightarrow{\sim} G_1$ is a *holomorphic* automorphic representation of G_2. Correspondingly, $\pi^\flat$, which can be identified with the complex conjugate of π, is a holomorphic representation of G_1, and therefore gives rise to a holomorphic modular form – of weight κ, in practice – on $Sh(V)$; but $\pi^\flat$ is antiholomorphic on G_2. The input of the doubling integral is an *antiholomorphic* vector on G_3 which comes from a vector $w \in \pi \otimes \pi^\flat$, that will be identified shortly; this is paired with the Eisenstein measure, which takes values in the ring of p-adic modular forms on G_4 and which specializes to classical forms of weight $\kappa \otimes \kappa^\flat$ on G_3. We always assume that π and $\pi^\flat$ are *anti-ordinary* at all primes dividing p; in particular, the vector w has local components at p that are chosen to be anti-ordinary.

Since one is in the habit of thinking of Hida theory as a theory of families of holomorphic and ordinary forms, the following lemma may be welcome; in any case, it is implicit in the assumption that both π and $\pi^\flat$ are anti-ordinary.

Lemma 9.0.2. *Suppose π is an anti-ordinary and anti-holomorphic representation of G_1. Then the p-adic component π_p of π is also ordinary.*

Proof. The property of being ordinary is preserved under complex conjugation, and by twist by a power of the similitude character. On the other hand, duality exchanges ordinary with anti-ordinary representations, by Lemma 8.2.6. Since π is essentially unitary, it follows that it is both ordinary and anti-ordinary. □

More precisely still, the anti-ordinary subspace (or submodule) of $\pi \otimes \pi^\flat$ is denoted $\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}$. However, it is best to view $\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}$ as a trace compatible system

$$w_r \in \hat{S}^{ord}_{\kappa,V}(K_r, R)[\pi] \otimes \hat{S}^{ord}_{\kappa^\flat,-V}(K_r, R)[\pi^\flat];\ t_r^*(w_{r+1}) = w_r, \tag{9.0.1}$$

with notation as in Lemma 7.2.12. Thus, in what follows, $\varphi \otimes \varphi^\flat \in \pi \otimes \pi^\flat$, viewed as an anti-holomorphic form of level K_r on $Sh(V) \times Sh(-V)$, is taken to belong in $\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}$, which we now take (with respect to the factorization 4.1.1) to be the space

$$\bigotimes_{w|p}[\phi^{\text{a-ord}}_{w,r,\pi_w} \otimes \phi^{\text{a-ord}}_{w,r,\pi^\flat_w}] \otimes \bigotimes_{\sigma|\infty}[\varphi_{\kappa_\sigma,-} \otimes \varphi_{\kappa^\flat_\sigma,-}] \otimes \pi^{K^p}_{S^p} \otimes \pi^{\flat,K^p}_{S^p} \subset \pi \otimes \pi^\flat. \tag{9.0.2}$$

In other words, these test vectors have local components as in (4.1.2), (4.1.3), and (4.1.4). Moreover, we take our vector $\varphi \otimes \varphi^\flat$ to be integral over $\mathcal{O}$. By our choice in (9.0.2), this is then the antiordinary vector $w = w_r \in \pi \otimes \pi^\flat$ to which we referred above.

Note that the choice of $\varphi \otimes \varphi^\flat$, and therefore of w_r, depends on the level K_r of the vector w however, the corresponding system $\{w_r\}$ satisfies the trace compatibility relation (9.0.1) by Lemma 8.2.7 and Proposition 8.3.1. In particular, the value of the pairing with the Eisenstein measure is independent of this choice, and we can specifically take $r = d \geq 2t$ as in 4.3.6, and as required for the local calculation at primes dividing p.

9.1. **Pairings of axiomatic Eisenstein measures with Hida families.** We now apply the considerations of Section 7.3 to the integral over G_3. Given a fixed Hecke character χ, we let the parameters κ, ρ, ρ^{υ} determine one another as in (4.4.8), (4.4.10). Let $\phi_{r,\rho}$ be as in (ii) of Lemma 7.2.2, a measure on $T_H(\mathbb{Z}_p)$ of type χ for some p-adic Hecke character χ of X_p. Choose $\Xi \in C_r(T_H(\mathbb{Z}_p),R)\rho^{\upsilon} \subset C_r(T_H(\mathbb{Z}_p),\mathcal{O})\rho^{\upsilon}$ so that (cf. (7.2.1))

$$\phi_{r,\rho}(\Xi) \in S^{\mathrm{ord}}_{\kappa,V}(K_r,R) \otimes S^{\mathrm{ord}}_{\kappa^\flat,-V}(K_r,R) \otimes \chi \circ \det, \tag{9.1.1}$$

For $\varphi \otimes \varphi^\flat \in [\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}] \subset \pi \otimes \pi^\flat$ we define (in the obvious notation)

$$L_{\varphi\otimes\varphi^\flat}(\phi_{r,\rho})(\Xi)$$

by the natural pairing of $S^{\mathrm{ord}}_{\kappa,V}(K_r,R) \otimes S^{\mathrm{ord}}_{\kappa^\flat,-V}(K_r,R) \otimes \chi \circ \det$ with

$$H^{d,ord}_{\kappa^D}(K_r,R)[\pi] \otimes H^{d,ord}_{\kappa^\flat,D}(K_r,R)[\pi^\flat] \otimes \chi^{-1} \circ \det \simeq [\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}] \otimes \chi^{-1} \circ \det$$

(the characters χ and χ^{-1} cancel in the obvious way).

We apply this to the measure $Eis_{r,\rho,\chi}$ attached to

$$\Xi \mapsto \int_{X_p \times T_H(\mathbb{Z}_p)} (\chi,\Xi) dEis$$

by Lemma 7.2.2, with $dEis$ an axiomatic Eisenstein measure as above. First, we need to show that the discussion in Section 7.3 applies to this situation.

9.1.1. *Equivariance of the Garrett map.* If $\lambda : \mathbf{T}_{K,\kappa,R} \to \mathbb{C}$ is a character, let $\lambda^\flat(T) = \lambda(T^\flat)$, where $^\flat$ is the involution defined in 6.5.1. It follows from (6.5.1) that

Lemma 9.1.2. *Let π be a cuspidal automorphic representation of G of type (κ,K). Then*

$$\lambda_{\bar{\pi}} = \lambda_\pi^\flat.$$

Let π be cuspidal of type (κ,K), and let $\varphi \in \pi^K$ be an antiholomorphic vector. We pick a Hecke character χ as in Section 4.1.2. In Section 4.1.4 we defined the zeta integral

$$I(\varphi,\varphi',f,s) = \int_{Z_3(\mathbb{A})G_3(\mathbb{Q})\backslash G_3(\mathbb{A})} E_f(s,(g_1,g_2))\chi^{-1}(\det g_2)\varphi(g_1)\varphi'(g_2)d(g_1,g_2).$$

where $\varphi' \in \bar{\pi}$ and $E_f(s,g_1,g_2)$ is an Eisenstein series depending on a section $f \in I(\chi,s)$. We specialize s to a point m where $E_f(s,\bullet)$ is nearly holomorphic, in other words where the archimedean component f_∞ of f satisfies the hypotheses of Definition 5.3.2. We consider the *Garrett map*

(9.1.2)

$$G(f,\varphi)(g_2) = I(\varphi,f,m)(g_2) := \chi^{-1}(\det g_2) \int_{Z_1(\mathbb{A})G_1(\mathbb{Q})\backslash G_1(\mathbb{A})} E_f(m,(g_1,g_2))\varphi(g_1)dg_1.$$

When f is clear from context, we set $G(\varphi) := G(f,\varphi)$. One of the main observations of [Gar84, GPSR87] is that if $\varphi \in \pi$ then $I(\varphi,\varphi',f,s) \equiv 0$ unless $\varphi' \in \pi^\vee$, in other words that $G(\varphi) \in \mathrm{Hom}(\pi^\vee,\mathbb{C}) \simeq \pi$:

Theorem 9.1.3. *If $\varphi \in \pi$ then $G(\varphi) \in \pi$.*

The forms φ and $G(\varphi)$ are on the same group $GU(V) = GU(-V)$ but on different Shimura varieties. The restriction of $E_f(m,\bullet)$ is a holomorphic form on $Sh(V,-V)$, which means it pairs with an anti-holomorphic form on $Sh(V)$ to yield a holomorphic form on $Sh(-V)$. In terms of parameters, this becomes

Corollary 9.1.4. *The Garrett map defines a homomorphism*

$$I(\chi_f,m) \to \mathrm{Hom}_{\mathbf{T}_{K,\kappa}}(H^0_!({}_KSh(V),\omega_\kappa)^\vee, H^0_!({}_KSh(-V),\omega_{\star(\kappa^F)})),$$
$$\to \mathrm{Hom}_{\mathbf{T}_{K,\kappa}}(H^d_!(Sh_K(V),\omega^D_\kappa\otimes L(-\kappa)), H^0_!({}_KSh(-V),\omega_{\star(\kappa^F)})),$$

Equivalently, letting $\mathrm{Hom}_{\mathbf{T}_{K,\kappa,\flat}}$ *denote the space of* $\flat$*-antilinear homorphisms,* $c_{dR}^{-1}\circ G(\bullet,\bullet)$ *defines a homomorphism*

$$I(\chi_f,m) \to Hom_{\mathbf{T}_{K,\kappa,\flat}}(H^0_!({}_KSh(V),\omega_\kappa)^\vee, H^0_!({}_KSh(V),\omega_\kappa))$$

The factor $L(-\kappa)$ was reinserted in the second line in order to respect the Hecke algebra action. The action of $\mathbf{T}_{K,\kappa}$ on $L(-\kappa)$ factors through the similitude map.

Lemma 9.1.5. *Let dEis be an axiomatic Eisenstein measure as in Definition 5.3.2. Then dEis satisfies the equivariance property of Hypothesis 7.2.3.*

Proof. This corresponds to the equivariance property of the Garrett map stated in Corollary 9.1.4. □

9.1.6. *Pairings, continued.* Thanks to Lemma 9.1.5, we now proceed as in Section 7.3. In order to guarantee that our global pairings are compatible with the local calculations in Section 4, especially the local calculations at primes dividing p, we choose test vectors $c\varphi\in\pi$ and $\varphi^\flat\in\pi^\flat$ that are anti-holomorphic, anti-ordinary, and integral over $\mathcal{O}$, as above. Substituting $\psi\rho^\upsilon$ for Ξ in the above discussion, with $\psi\in C_r(T_H(\mathbb{Z}_p),R)$ for some $R\subset\mathcal{O}$ and ρ as above, we find

$$L_{\varphi\otimes\varphi^\flat}\left(\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,\psi\rho^\upsilon)dEis\right) = D(\chi)\cdot L_{\varphi\otimes\varphi^\flat}(res_3 D(\kappa,m,\chi_0)E^{holo}_{\chi_0,\psi\rho^\upsilon}(m)). \tag{9.1.3}$$

Proposition 9.1.7. *Assume* π *satisfies Hypotheses 7.1.3, 7.1.4, 7.1.5, and 7.1.7. Let* φ *and* $\varphi^\flat$ *be respectively elements of R-bases of* $\hat{I}_\pi$ *and* $\hat{I}_{\pi^\flat}$*. Suppose* $(\chi,\psi\rho^\upsilon)\in Y_H^{class}$*, with* $\psi\in C_r(T_H(\mathbb{Z}_p),R)$ *with* $\chi = ||\bullet||^m\chi_u$*,* $m\geq n$*. Then we have the equality*

$$L_{\varphi\otimes\varphi^\flat}\left(\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,\psi\rho^\upsilon)dEis\right) = D(\chi)\cdot I(\varphi,\varphi^\flat, D(\kappa,m,\chi_0)f^{holo}(\chi_u,\psi\rho^\upsilon),m)$$

Proof. Abbreviate $[G_3] = G_3(\mathbb{Q})Z(\mathbb{R})\backslash G_3(\mathbb{A})$, $dg_2^\chi = \chi(\det(g_2))^{-1}dg_2$. By doubling the formula in Lemma 7.3.3 – in other words, by applying it to the group G_3 – we obtain

$$L_{\varphi\otimes\varphi^\flat}\left(res_3 D(\kappa,m,\chi_0)E^{holo}_{f(\chi,\psi\rho^\upsilon)}(m)\right)$$
$$= \int_{[G_3]} D(\kappa,m,\chi_0)E^{holo}_{f(\chi_0,\psi\rho^\upsilon)}((g_1,g_2),m)\,\varphi(g_1)\varphi^\flat(g_2)||\nu(g_1)^{a(\kappa)}||dg_1 dg_2^\chi.$$

Comparing this with Equation (9.1.3) and the definition of the zeta integral, we obtain the equality. □

In view of our choices of local vectors in (9.0.2), Corollary 9.1.8 below is then a consequence of the local computations summarized in Proposition 4.6.1, and of the axiomatic properties of the Eisenstein measure summarized in Definition 5.3.2 and Corollary 5.4.3.

Corollary 9.1.8. *Under the hypotheses of Proposition 9.1.7, suppose φ and $\varphi^\flat$ admit the factorizations* (9.0.2),*. Let the parameters κ, ρ, ρ^υ determine one another as in Inequalities* (4.4.8) *and Equations* (4.4.10)*. Then we have the equality*

$$L_{\varphi\otimes\varphi^\flat}\left(\int_{X_p\times T_H(\mathbb{Z}_p)}(\chi,\psi\rho^\upsilon)dEis\right)=D(\chi)\prod_v I_v(\varphi_v,\varphi_v^\flat,f_v,m)$$
$$=\langle\varphi,\varphi^\flat\rangle\cdot I_p(\chi,\kappa)I_\infty(\chi,\rho^\upsilon)I_S L^S(m+\frac{1}{2},\pi,\chi_u)$$

where the factors are defined as in Proposition 4.6.1.

9.2. **Statement of the main theorem.** We reinterpret the identity in Corollary 9.1.8 in the language of Proposition 7.2.18.

Corollary 9.2.1. *Under the hypotheses of Corollary 9.1.8, there is a unique element $L(Eis,\varphi\otimes\varphi^\flat)\in\Lambda_{X_p,R}\hat{\otimes}\mathbb{T}$ such that, for any classical $\chi: X_p\to R^\times$, the image of $L(Eis,\varphi\otimes\varphi^\flat)$ under the map $\Lambda_{X_p,R}\hat{\otimes}\mathbb{T}\to R$ induced by the character $\chi\otimes\lambda_\pi$ equals*

$$\langle\varphi,\varphi^\flat\rangle\cdot I_p(\chi,\kappa)I_\infty(\chi,\rho^\upsilon)I_S L^S(m+\frac{1}{2},\pi,\chi_u).$$

Here λ_π is the character of $\mathbb{T}$ defined in section 6.5.8, and the local factors are defined as in Proposition 4.6.1.

In the language of Corollary 9.2.1 this admits the following reformulation. The statement is in terms of the highest weight κ of the (holomorphic) representation dual to π and a Hecke character χ. Let the algebraic characters κ, ρ, ρ^υ determine one another, relative to a given χ, as in Inequalities (4.4.8) and Equation (4.4.10).

Main Theorem 9.2.2. *Let π be a cuspidal antiholomorphic automorphic representation of G_1 which is ordinary of type (κ,K), and let $\mathbb{T}=\mathbb{T}_\pi$ be the corresponding connected component of the ordinary Hecke algebra. Let φ and $\varphi^\flat$ be respectively elements of R-bases of $\hat{I}_\pi$ and $\hat{I}_{\pi^\flat}$. Assume π satisfies the following Hypotheses:*

1. *Hypothesis 7.1.4 (the Gorenstein Hypothesis)*
2. *Hypothesis 7.1.5 (the Global Multiplicity One Hypothesis)*
3. *Hypothesis 7.1.7 (the Minimality Hypothesis)*

There is a unique element

$$L(Eis,\varphi\otimes\varphi^\flat)\in\Lambda_{X_p,R}\hat{\otimes}\mathbb{T}$$

with the following property. For any classical $\chi = \|\bullet\|^m\chi_u : X_p \to R^\times$, *the image of* $L(Eis, \varphi \otimes \varphi^\flat)$ *under the map* $\Lambda_{X_p,R}\hat{\otimes}\mathbb{T} \to R$ *induced by the character* $\chi \otimes \lambda_\pi$ *equals*

$$c(\pi)\cdot\frac{\langle\varphi,\varphi^\flat\rangle}{\hat{Q}[\pi]}I_\infty(\chi,\kappa)I_S L_p(m,\mathrm{ord},\pi,\chi_u)\frac{L^S(m+\frac{1}{2},\pi,\chi_u)}{Q_\pi}.$$

Here λ_π *is the character of* $\mathbb{T}$ *defined in Section 6.5.8.*

Proof. This follows from Corollary 9.2.1, after we write $\hat{Q}[\pi] = c(\pi)\hat{Q}_\pi = c(\pi)Q_\pi^{-1}$ as in (b) of Corollary 6.6.9. It follows from Corollary 6.6.9 that the factor $\frac{\langle\varphi,\varphi^\flat\rangle}{\hat{Q}[\pi]}$ is necessarily p-integral. □

9.3. **Comments on the main theorem.** Even in the setting of ordinary families of p-adic modular forms on unitary Shimura varieties, this should not be considered the definitive construction of p-adic L-functions. We list some aspects that call for refinement.

Remark 9.3.1 (The Gorenstein Hypothesis). It is often possible to verify the Gorenstein hypothesis when the residual Galois representation attached to π has sufficiently general image, using the Taylor-Wiles method. See [Pil11] and [Har13b] for examples. On the other hand, it is certainly not valid in complete generality. Since the Gorenstein condition is an open one, one can obtain a more general statement by replacing $\Lambda_{X_p,R}\hat{\otimes}\mathbb{T}$ by the fraction fields of its irreducible components. The method of this paper then provides p-adic meromorphic functions on each such components, which specialize at classical points as indicated in the Main Theorem.

Remark 9.3.2 (The Global Multiplicity One Hypothesis). This is already known for an automorphic representation of a unitary group such as G_1 whose base change to $GL(n)$ is cuspidal, thanks to [KMSW14].

Remark 9.3.3 (The Minimality Hypothesis). This was included for convenience, in order to work with a module $[\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}]$ that is locally constant on the Hida family. One can easily eliminate this hypothesis, but the statement is no longer so clean.

Remark 9.3.4 (Unspecified local factors). The volume factor I_S is a placekeeper. It might be more illuminating to replace I_S by

$$\tilde{I}_S = \prod_{v\in S} L_v(m+\frac{1}{2},\pi_v,\chi_{u,v})^{-1}I_S$$

and write the specialized value of the L-function

$$c(\pi)\cdot\frac{\langle\varphi,\varphi^\flat\rangle}{\hat{Q}[\pi]}\cdot I_\infty(\chi,\kappa)\tilde{I}_S L_p(m,\mathrm{ord},\pi,\chi_u)L(m+\frac{1}{2},\pi,\chi_u).$$

Here $L(s,\pi,\chi_u)$ denotes the standard L-function without the archimedean factors. Written this way, one sees that the inverted local Euler factors $L_v(m+\frac{1}{2},\pi_v,\chi_{u,v})^{-1}$ can give rise to exceptional zeroes.

Ideally one would like to choose an optimal vector in $[\hat{I}_\pi \otimes \hat{I}_{\pi^\flat}]$ and to adapt the local Eisenstein sections at primes in S to this choice. This would settle the issues of minimality and local factors simultaneously. At present we do not see how to carry this out.

Remark 9.3.5 (The congruence factors). It is expected at least under the Gorenstein hypothesis that a congruence factor $c(\pi)$ can be chosen to be the specialization at π of a canonical p-adic analytic function $\mathbf{c}$ that interpolates the normalized and p-stabilized value at $s = 1$ of the adjoint L-function $L(s, \pi, Ad)$. The factor $c(\pi)$ that appears in Main Theorem 9.2.2 depends on the choice of period Q_π, which in turn depends on the choice of f in Lemma 6.6.3. As π varies, the vector f can be chosen uniformly in the Hida family, but there is no obvious preferred choice. For this reason, one can only define the hypothetical analytic function $\mathbf{c}$ up to a unit in the Hecke algebra. This is a persistent problem in the theory, and it has been noted by Hida in [Hid96b].

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Ellen Eischen, Department of Mathematics, University of Oregon, Eugene, OR 97403, USA

E-mail address: `eeischen@uoregon.edu`

Michael Harris, Department of Mathematics, Columbia University, New York, NY 10027, USA

E-mail address: `harris@math.columbia.edu`

Jian-Shu Li, School of Mathematical Sciences, Shanghai Jiaotong University, Shanghai, China, and, Department of Mathematics, HKUST, Clear Water Bay, Kowloon, Hong Kong

E-mail address: `matom@ust.hk`

E-mail address: `jianshu@sjtu.edu.cn`

Christopher Skinner, Department of Mathematics, Princeton University, Fine Hall, Washington Road, Princeton, NJ 08544-1000, USA

E-mail address: cmcls@math.princeton.edu