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Colloquium Paper

# Zeta functions and Eisenstein series on classical groups

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## ABSTRACT

Go to:

We construct an Euler product from the Hecke eigenvalues of an automorphic form on a classical group and prove its analytic continuation to the whole complex plane when the group is a unitary group over a CM field and the eigenform is holomorphic. We also prove analytic continuation of an Eisenstein series on another unitary group, containing the group just mentioned defined with such an eigenform. As an application of our methods, we prove an explicit class number formula for a totally definite hermitian form over a CM field.

## SECTION 1.

Go to:

Given a reductive algebraic group $G$ over an algebraic number field, we denote by $G_{\mathrm{A}}$, $G_{\mathrm{a}}$, and $G_{\mathrm{h}}$ its adelization, the archimedean factor of $G_{\mathrm{A}}$, and the nonarchimedean factor of $G_{\mathrm{A}}$. We take an open subgroup $D$ of $G_{\mathrm{A}}$ of the form $D = D_0G_{\mathrm{a}}$ with a compact subgroup $D_0$ such that $D_0 \cap G_{\mathrm{a}}$ is maximal compact in $G_{\mathrm{a}}$. Choosing a specific type of representation of $D_0 \cap G_{\mathrm{a}}$, we can define automorphic forms on $G_{\mathrm{A}}$ as usual. For simplicity we consider here the forms invariant under $D_0 \cap G_{\mathrm{h}}$. Each Hecke operator is given by $D\tau D$, with $\tau$ in a subset $\mathfrak{X}$ of $G_{\mathrm{A}}$, which is a semigroup containing $D$ and the localizations of $G$ for almost all nonarchimedean primes. Taking an automorphic form $\mathbf{f}$ such that $\mathbf{f}|D\tau D = \lambda(\tau)\mathbf{f}$ with a complex number $\lambda(\tau)$ for every $\tau \in \mathfrak{X}$ and a Hecke ideal character $\chi$ of $F$, we put

$$\mathfrak{T}(s,\ \boldsymbol{f},\ \chi) = \sum_{\tau\in D/X/D} \lambda(\tau)\chi(\nu_0(\tau))N(\nu_0(\tau))^{-s}, \qquad 1.1$$

where $\nu_0(\tau)$ is the denominator ideal of $\tau$ and $N(\nu_0(\tau))$ is its norm. Now our first main result is that if $G$ is symplectic, orthogonal, or unitary, then

$$\Lambda(s,\ \chi)\mathfrak{T}(s,\ \boldsymbol{f},\ \chi) = \prod_{p} W_p[\chi(p)N(p)^{-s}]^{-1}, \qquad 1.2$$

where $\Lambda(s, \chi)$ is an explicitly determined product of $L$-functions depending on $\chi$, $W_{\mathrm{p}}$ is a polynomial determined for each $v \in \mathbf{h}$ whose constant term is 1, and p runs over all the prime ideals of the basic number field. This is a purely algebraic result concerning only nonarchimedean primes.

Let $Z(s, \mathbf{f}, \chi)$ denote the right-hand side of Eq. 1.2. As our second main result, we obtain a product $\mathfrak{G}(s)$ of gamma factors such that $\mathfrak{G}Z$ can be continued to the whole $s$-plane as a meromorphic function with finitely many poles, when $G$ is a unitary group of an arbitrary signature distribution over a CM field, and $\mathbf{f}$ corresponds to holomorphic forms.

Now these problems are closely connected with the theory of Eisenstein series $E$ on a group $G'$ in which $G$ is embedded. To describe the series, let $\mathfrak{Z}'$ denote the symmetric space on which $G'$ acts. Then the series as a function of $(z, s) \in \mathfrak{Z}' \times \mathbf{C}$ can be given (in the classical style) in the form

$$E(z, s; \boldsymbol{f}, \chi) = \sum_{\alpha \in A} \delta(z, s, \boldsymbol{f}, \chi) \| \alpha, \quad A = (P \cap \Gamma) / \Gamma, \tag{1.3}$$

where $\Gamma$ is a congruence subgroup of $G'$, and $P$ is a parabolic subgroup of $G'$ which is a semidirect product of a unipotent group and $G \times GL_m$ with some $m$. The adelized version of $\delta$ will be explicitly described in Section 5. Now our third main result is that there exists an explicit product $\mathfrak{G}'$ of gamma factors and an explicit product $\Lambda'$ of $L$-functions such that $\mathfrak{G}'(s)\Lambda'(s)Z(s, \mathbf{f}, \chi)E(z, s; \mathbf{f}, \chi)$ can be continued to the whole $s$-plane as a meromorphic function with finitely many poles.

Though the above results concern holomorphic forms, our method is applicable to the unitary group of a totally definite hermitian form over a CM field. In this case, we can give an explicit class number formula for such a hermitian form, which is the fourth main result of this paper.

## SECTION 2.

Go to:

For an associative ring $R$ with identity element, we denote by $R^{\times}$ the group of all its invertible elements and by $R_n^m$ the $R$-module of all $m \times n$ matrices with entries in $R$. To indicate that a union $X = \cup_{i \in I} Y_i$ is disjoint, we write $X = \sqcup_{i \in I} Y_i$.

Let $K$ be an associative ring with identity element and an involution $\rho$. For a matrix $x$ with entries in $K$, we put $x^* = {}^t x^{\rho}$, and $\widehat{x} = (x^*)^{-1}$ if $x$ is square and invertible. Given a finitely generated left $K$-module $V$, we denote by $GL(V)$ the group of all $K$-linear automorphisms of $V$. We let $GL(V)$ act on $V$ on the right; namely we denote by $w\alpha$ the image of $w \in V$ under $\alpha \in GL(V)$. Given $\varepsilon = \pm 1$, by an $\varepsilon$-hermitian form on $V$, we understand a biadditive map $\phi: V \times V \to K$ such that $\phi(x, y)^{\rho} = \varepsilon\phi(y, x)$ and $\phi(ax, by) = a\phi(x, y)b^{\rho}$ for every $a, b \in K$. Assuming that $\phi$ is nondegenerate, we put

$$G^{\varphi} = G(\varphi) = G(V, \varphi) = \{\gamma \in GL(V) | \varphi(x\gamma, y\gamma) = \varphi(x, y)\}. \tag{2.1}$$

Given $(V, \phi)$ and $(W, \psi)$, we can define an $\varepsilon$-hermitian form $\phi \oplus \psi$ on $V \oplus W$ by

$$(\varphi \oplus \psi)(x + y, x' + y') = \varphi(x, x') + \psi(y, y')$$
$$(x, x' \in V; y, y' \in W). \tag{2.2}$$

We then write $(V \oplus W, \phi \oplus \psi) = (V, \phi) \oplus (W, \psi)$. If both $\phi$ and $\psi$ are nondegenerate, we can view $G^{\phi} \times G^{\psi}$ as a subgroup of $G^{\phi \oplus \psi}$. The element $(\alpha, \beta)$ of $G^{\phi} \times G^{\psi}$ viewed as an element of $G^{\phi \oplus \psi}$ will be denoted by $\alpha \times \beta$ or by $(\alpha, \beta)$. Given a positive integer $r$, we put $H_r = I'_r \oplus I_r$, $I_r = I'_r = K_r^1$ and

$$\eta_r(x + u, y + v) = u \cdot {}^t y^{\rho} + \varepsilon x \cdot {}^t v^{\rho} \quad (x, y \in I'_r; u, v \in I_r). \tag{2.3}$$

We shall always use $H_r$, $I'_r$, $I_r$, and $\eta_r$ in this sense. We understand that $H_0 = \{0\}$ and $\eta_0 = 0$.

Hereafter we fix $V$ and a nondegenerate $\phi$ on $V$, assuming that $K$ is a division ring whose characteristic is different from 2. Let $J$ be a $K$-submodule of $V$ which is totally $\phi$-isotropic, by which we mean that $\phi(J, J) = 0$. Then we can find a decomposition $(V, \phi) = (Z, \zeta) \oplus (H, \eta)$ and an isomorphism $f$ of $(H, \eta)$ onto $(H_r, \eta_r)$ such that $f(J) = I_r$. In this setting, we define the parabolic subgroup $P_J^{\varphi}$ of $G^{\phi}$ relative to $J$ by

$$P_J^{\varphi} = \{\pi \in G^{\varphi} | J\pi = J\}, \quad 2.4$$

and define homomorphisms $\pi_{\zeta}^{\varphi} : P_J^{\varphi} \to G^{\zeta}$ and $\lambda_J^{\varphi} : P_J^{\varphi} \to GL(J)$ such that $z\alpha - z\pi_{\zeta}^{\phi}(\alpha) \in J$ and $w\alpha = w\lambda_J^{\phi}(\alpha)$ if $z \in Z$, $w \in J$, and $\alpha \in P_J^{\varphi}$.

Taking a fixed nonnegative integer $m$, we put

$$(W, \psi) = (V, \varphi) \oplus (H_m, \eta_m), \quad (X, \omega) = (W, \psi) \oplus (V, -\varphi). \quad 2.5$$

We can naturally view $G^{\psi} \times G^{\phi}$ as a subgroup of $G^{\omega}$. Since $W = V \oplus H_m$, we can put $X = V \oplus H_m \oplus V$ with the first summand $V$ in $W$, and write every element of $X$ in the form $(u, h, v)$ with $(u, h) \in V \oplus H_m = W$ and $v \in V$. Put

$$U = \{(v, i, v) | v \in V, i \in I_m\}.$$

Observing that $U$ is totally ω-isotropic, we can define $P_U^{\omega}$.

PROPOSITION 1. *Let $\lambda(\phi)$ be the maximum dimension of totally $\phi$-isotropic K-submodules of V. Then*

$$P_U^{\omega} / G^{\omega} / [G^{\psi} \times G^{\varphi}] \quad 2.6$$

*has exactly $\lambda(\phi)$ orbits. Moreover,*

$$P_U^{\omega}[G^{\psi} \times G^{\varphi}] = \bigsqcup_{\beta, \xi} P_U^{\omega}((\xi \times 1_H)\beta, 1_V), \quad 2.7$$

*with $\xi$ running over $G^{\phi}$ and $\beta$ over $P_I^{\psi} G^{\psi}$, where $H = H_m$ and $I = I_m$.*

In fact, we can give an explicit set of representatives $\{\tau_e\}_{e=1}^{\lambda(\phi)}$ for Eq. 2.6 and also an explicit set of representatives for $P_U^{\omega} / P_U^{\omega} \tau_e [G^{\psi} \times G^{\phi}]$ in the same manner as in Eq. 2.7. This proposition plays an essential role in the analysis of our Eisenstein series $E(z, s; \mathbf{f}, \chi)$.

## SECTION 3.

Go to:

In this section, $K$ is a locally compact field of characteristic 0 with respect to a discrete valuation. Our aim is to establish the Euler factor $W_p$ of Eq. 1.2. We denote by r and q the valuation ring and its maximal ideal; we put $q = [r:q]$ and $|x| = q^{-\nu}$ if $x \in K$ and $x \in \pi^{\nu} r^{\times}$ with $\nu \in \mathbf{Z}$. We assume that $K$ has an automorphism $\rho$ such that $\rho^2 = 1$, and put $F = \{x \in K \mid x^{\rho} = x\}$, $g = F \cap r$, and $d^{-1} = \{x \in K \mid \mathrm{Tr}_{K/F}(xr) \subset g\}$ if $K \neq F$. We consider $(V, \phi)$ as in Section 2 with $V = K_n^1$ and $\phi$ defined by $\phi(x, y) = x\phi y^*$ for $x, y \in V$ with a matrix $\phi$ of the form

$$\varphi = \begin{bmatrix} 0 & 0 & \varepsilon\delta^{-\rho} 1_r \\ 0 & \theta & 0 \\ \delta^{-1} 1_r & 0 & 0 \end{bmatrix}, \theta = \varepsilon\theta * \in GL_t(K), \delta \in K^{\times}, \quad 3.1$$

where $t = n - 2r$. We assume that $\theta$ is anisotropic and also that

$$\varepsilon = \pm 1 \text{ and } \delta = 2 \text{ if } K = F, \quad 3.2a$$

$$\varepsilon = 1, \delta r = d, \text{ and } \delta^{\rho} = -\delta \text{ if } K \neq F. \quad 3.2b$$

Thus our group $G^{\phi}$ is orthogonal, symplectic, or unitary. The element $\delta$ of Eq. 3.2b can be obtained by putting $\delta = u - u^{\rho}$ with $u$ such that $\mathfrak{r} = \mathfrak{g}[u]$. We include the case $rt = 0$ in our discussion. If $t = 0$, we simply ignore $\theta$; this is always so if $K = F$ and $\varepsilon = -1$. We have $\phi = \theta$ if $r = 0$.

Denoting by $\{e_i\}$ the standard basis of $K_n^1$, we put

$$J = \sum_{i=1}^{r} Ke_{r+t+i}, T = \sum_{i=1}^{t} Ke_{r+i},$$

$$M = \sum_{i=1}^{r} (\mathfrak{r}e_i + \mathfrak{r}e_{r+t+i}) + N, N = \{u \in T | \varphi(u, u) \in \mathfrak{g}\},$$

$$C = \{\gamma \in G^{\varphi} | M\gamma = M\}, E = GL_r(\mathfrak{r}).$$

Then $G^{\phi} = P_J^{\phi}C$. We choose $\{e_{r+i}\}_{i=1}^{t}$ so that $N = \sum_{i=1}^{t}\mathfrak{r}e_{r+i}$. Then we can find an element $\lambda$ of $\mathfrak{r}_t^t$ such that

$$\theta = \delta^{-1}\lambda + \varepsilon(\delta^{-1}\lambda) * . \qquad 3.3$$

Put

$$S = S^r = \{h \in K_r^r | h* = -\varepsilon(\delta^{\rho}/\delta)h\}. \qquad 3.4$$

We can write every element of $P_J^{\phi}$ in the form

$$\xi = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & d \end{bmatrix}, \hat{a} = d \in GL_r(K), e \in G^{\theta},$$

$$b \in K_t^r, f = -\delta e\theta b * d, c = (s - b\lambda b*)d, s \in S. \qquad 3.5$$

If $t = 0$, we simply ignore $b$, $e$, and $f$, so that $\xi = \begin{bmatrix} a & sd \\ 0 & d \end{bmatrix}$; we have $\xi = e$ if $r = 0$.

We consider the Hecke algebra $\Re(E, GL_r(K))$ consisting of all formal finite sums $\sum c_x ExE$ with $c_x \in \mathbf{Q}$ and $x \in GL_r(K)$, with the law of multiplication defined as in ref. 1. Taking $r$ indeterminates $t_1, \ldots, t_r$, we define a $\mathbf{Q}$-linear map

$$\omega_0 : \Re(E, GL_r(K)) \to \mathbf{Q}[t_1, \ldots, t_r, t_1^{-1}, \ldots, t_r^{-1}] \qquad 3.6$$

as follows; given $ExE$ with $x \in GL_r(K)$, we can put $ExE = \sqcup_y Ey$ with upper triangular $y$ whose diagonal entries are $\pi^{e_1}, \ldots, \pi^{e_r}$ with $e_i \in \mathbf{Z}$. Then we put

$$\omega_0(ExE) = \sum_{y} \omega_0(Ey), \quad \omega_0(Ey) = \prod_{i=1}^{r} (q^{-i}t_i)^{e_i}. \qquad 3.7$$

Next we consider the Hecke algebra $\Re(C, G^{\phi})$ consisting of all formal finite sums $\sum c_{\tau} C\tau C$ with $c_{\tau} \in \mathbf{Q}$ and $\tau \in G^{\phi}$. We then define a $\mathbf{Q}$-linear map

$$\omega : \Re(C, G^{\varphi}) \to \mathbf{Q}[t_1, \ldots, t_r, t_1^{-1}, t \ldots, t_r^{-1}] \qquad 3.8$$

as follows; given $C\tau C$ with $\tau \in G^{\phi}$, we can put $C\tau C = \sqcup_{\xi} C\xi$ with $\xi \in P$ of form Eq. 3.5. We then put

$$\omega(C\tau C) = \sum_{\xi} \omega(C\xi), \quad \omega(C\xi) = \omega_0(Ed_{\xi}), \tag{3.9}$$

where $\omega_0$ is given by Eq. 3.6 and $d_{\xi}$ is the $d$-block in Eq. 3.5. We can prove that this is well defined and gives a ring-injection.

Given $x \in K_n^m$, we denote by $\nu_0(x)$ the ideal of r which is the inverse of the product of all the elementary divisor ideals of $x$ not contained in r; we put then $\nu(x) = [\mathrm{r}:\nu_0(x)]$. We call $x$ primitive if rank$(x)$ = Min$(m, n)$ and all the elementary divisor ideals of $x$ are r.

PROPOSITION 2. *Given* $\xi$ *as in Eq.* **3.5**, *suppose that both* $e$ *and* $(\delta\theta)^{-1}(e-1)$ *have coefficients in* r *if* $t > 0$. *Let* $a = g^{-1}h$ *with primitive* $[g\ h] \in \mathrm{r}_{2r}^{r}$ *and* $gb = j^{-1}k$ *with primitive* $[j\ k] \in \mathrm{r}_{r+t}^{r}$. *Then*

$$\nu_0((\delta\varphi)^{-1}(\xi - 1)) = \det(ghj^2)\nu_0(jgsg * j*),$$

*where we take* $j = 1_r$ *if* $t = 0$.

We now define a formal Dirichlet series $\mathfrak{T}$ by

$$\mathfrak{T}(s) = \sum_{\tau \in A} \omega(C\tau C)\nu(\tau)^{-s}, \quad A = C/G^{\varphi}/C. \tag{3.10}$$

This is a formal version of the Euler factor of Eq. 1.2 at a fixed nonarchimedean prime.

THEOREM 1. *Suppose that* $\delta\phi \in GL_n(r)$; *put* $p = [\mathrm{g}:\mathrm{g} \cap \mathrm{q}]$. (*Thus* $p = q$ *if* $K = F$.) *Then*

$$\mathfrak{T}(s) = \frac{1 - p^{-s}}{1 - p^{r-s}} \prod_{i=1}^{r} \frac{(1 - p^{2i-2s})}{(1 - p^{r-s}t_i)(1 - p^{r-s}t_i^{-1})}$$

$$(K = F, \varepsilon = -1),$$

$$\mathfrak{T}(s) = \prod_{i=1}^{r} \frac{(1 - p^{2i-2-2s})}{(1 - p^{r+t-2-s}t_i)(1 - p^{r-s}t_i^{-1})} \quad (K = F, \varepsilon = 1),$$

$$\mathfrak{T}(s) = \frac{\prod_{i=1}^{2r}(1 - \theta^{i-1}p^{i-1-2s})}{\prod_{i=1}^{r}(1 - q^{r+t-1-s}t_i)(1 - q^{r-s}t_i^{-1})} \quad (K \neq F).$$

*Here* $\theta^i = 1$ *if i is even; when i is odd,* $\theta^i$ *is* $-1$ *or* $0$ *according as* $d = r$ *or* $d \neq r$.

This can be proved in the same manner as in ref. 2 by means of *Proposition 2*.

Since we are going to take localizations of a global unitary group, we have to consider $G^{\phi} = G(V, \phi)$ of Eq. 2.1 with $V = K_n^1$, $K = F \times F$, and $\rho$ defined by $(x, y)^{\rho} = (y, x)$, where $F$ is a locally compact field of characteristic 0 with respect to a discrete valuation. Let g and p be the valution ring of $F$ and its maximal ideal; put $\mathrm{r} = \mathrm{g} \times \mathrm{g}$ and $p = [\mathrm{g}:\mathrm{p}]$. We consider $\mathfrak{R}(C, G^{\phi})$ with $C = G^{\phi} \cap GL_n(\mathrm{r})$. Then the projection map pr of $GL_n(K)$ onto $GL_n(F)$ gives an isomorphism $\eta:\mathfrak{R}(C, G^{\phi}) \rightarrow \mathfrak{R}(E_1, GL_n(F))$, where $E_1 = GL_n(\mathrm{g})$. To be explicit, we have $\eta(C(x, {}^t x^{-1})C) = E_1 x E_1$. Let $\omega_1$ denote the map of Eq. 3.6 defined with $n$, $E_1$, and $F$ in place of $r$, $E$, and $K$. Putting $\omega = \omega_1 \circ \eta$, we obtain a ring-injection

$$\omega : \mathfrak{R}(C, G^{\varphi}) \rightarrow \boldsymbol{Q}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]. \tag{3.11}$$

For $z = (x, y) \in K_n^n$ with $x, y \in F_n^n$ put $\nu_1(z) = \nu(x)$ and $\nu_2(z) = \nu(y)$, where ν is defined with respect to g instead of r. We then put

$$\mathfrak{T}(s,\, s') = \sum_{\tau \in R} \omega(C\tau C)\nu_1(\tau)^{-s}\nu_2(\tau)^{-s'}, \quad R = C/G^{\varphi}/C. \tag{3.12}$$

Then we obtain

$$\mathfrak{T}(s,\, s') = \prod_{i=1}^{n} \frac{1-p^{i-1-s-s'}}{(1-p^{n-s}t_i^{-1})(1-p^{-1-s'}t_i)}. \tag{3.13}$$

## SECTION 4.

Go to:

We now take a totally imaginary quadratic extension $K$ of a totally real algebraic number field $F$ of finite degree. We denote by **a** (resp. **h**) the set of archimedean (resp. nonarchimedean) primes of $F$; further we denote by g (resp. r) the maximal order of $F$ (resp. $K$). Let $V$ be a vector space over $K$ of dimension $n$. We take a $K$-valued nondegenerate ε-hermitian form $\phi$ on $V$ with $\varepsilon = 1$ with respect to the Galois involution of $K$ over $F$, and define $G^{\phi}$ as in Section 2. For every $v \in \mathbf{a} \cup \mathbf{h}$ and an object $X$, we denote by $X_v$ its localization at $v$. For $v \in \mathbf{h}$ not splitting in $K$ and for $v \in \mathbf{a}$, we take a decomposition

$$(V_v,\, \varphi_v) = (T_v,\, \theta'_v) \oplus (H_{r_v},\, \eta_{r_v}) \tag{4.1}$$

with anisotropic $\theta'_v$ and a nonnegative integer $r_v$. Put $t_v = \dim(T_v)$. Then $n = 2r_v + t_v$. If $n$ is odd, then $t_v = 1$ for every $v \in \mathbf{h}$. If $n$ is even, then $t_v = 0$ for almost all $v \in \mathbf{h}$ and $t_v = 2$ for the remaining $v \in \mathbf{h}$. If $n$ is odd, by replacing $\phi$ by $c\phi$ with a suitable $c \in F$, we may assume that $\phi$ is represented by a matrix whose determinant times $(-1)^{(n-1)/2}$ belongs to $N_{K/F}(K)$.

We take and fix an element κ of $K$ such that $\kappa^{\rho} = -\kappa$ and $i\kappa_v\phi_v$ has signature $(r_v + t_v, r_v)$ for every $v \in \mathbf{a}$. Then $G(i\kappa_v\phi_v)$ modulo a maximal compact subgroup is a hermitian symmetric space which we denote by $\mathfrak{Z}_v^{\phi}$. We take a suitable point $\mathbf{i}_v$ of $\mathfrak{Z}_v^{\phi}$ which plays the role of "origin" of the space. If $r_v = 0$, we understand that $\mathfrak{Z}_v^{\phi}$ consists of a single point $\mathbf{i}_v$. We put $\mathfrak{Z}^{\phi} = \prod_{v\in\mathbf{a}} \mathfrak{Z}_v^{\phi}$. To simplify our notation, for $x \in K_{\mathbf{A}}^{\times}$ or $x \in (\mathbf{C}^{\times})^{\mathbf{a}}$, $a \in \mathbf{Z}^{\mathbf{a}}$, and $c \in (\mathbf{C}^{\times})^{\mathbf{a}}$, we put

$$x^{a} = \prod_{v\in \boldsymbol{a}} x_v^{a_v}, \quad |x|^{c} = \prod_{v\in \boldsymbol{a}} (x_v\bar{x}_v)^{c_v/2}. \tag{4.2}$$

For $\xi \in G_v^{\phi}$ and $w \in \mathfrak{Z}_v^{\phi}$, we define $\xi w \in \mathfrak{Z}_v^{\phi}$ in a natural way and define also a scalar factor of automorphy $j_{\xi}(w)$ so that $\det(\xi)^{r_v} j_{\xi}(w)^{-n}$ is the jacobian of ξ. Given $k, \nu \in \mathbf{Z}^{\mathbf{a}}$, $z \in \mathfrak{Z}^{\phi}$, and $\alpha \in G_{\mathbf{A}}^{\phi}$, we put

$$\alpha z = (\alpha_v z_v)_{v\in\boldsymbol{a}}, \quad j_{\alpha}^{k,\nu}(z) = \det(\alpha)^{\nu} j_{\alpha}(z)^{k}. \tag{4.3}$$

Then, for a function $f{:}\mathfrak{Z}^{\phi} \to \mathbf{C}$, we define $f\|_{k,\nu}\alpha{:}\mathfrak{Z}^{\phi} \to \mathbf{C}$ by

$$(f\|_{k,\nu}\alpha)(z) = j_{\alpha}^{k,\nu}(z)^{-1} f(\alpha z) \qquad (z\in Z^{\varphi}). \tag{4.4}$$

Now, given a congruence subgroup Γ of $G^{\phi}$, we denote by $\mathfrak{M}_{k,\nu}^{\phi}(\Gamma)$ the vector space of all holomorphic functions $f$ on $\mathfrak{Z}^{\phi}$ which satisfy $f\|_{k,\nu}\gamma = f$ for every $\gamma \in \Gamma$ and also the cusp condition if $G^{\phi}$ is of the elliptic modular type. We then denote by $\mathfrak{S}_{k,\nu}^{\phi}(\Gamma)$ the set of all cusp forms belonging to $\mathfrak{M}_{k,\nu}^{\phi}(\Gamma)$.

Further, we denote by $\mathfrak{M}_{k,\nu}{}^{\phi}$ resp. $\mathfrak{S}_{k,\nu}{}^{\phi}$ the union of $\mathfrak{M}_{k,\nu}{}^{\phi}(\Gamma)$ resp. $\mathfrak{S}_{k,\nu}{}^{\phi}(\Gamma)$ for all congruence subgroups $\Gamma$ of $G$. If $\phi$ is anisotropic, we understand that $\mathfrak{S}_{0,\nu}{}^{\phi} = \mathbf{C}$.

Let $D$ be an open subgroup of $G_{\mathbf{A}}{}^{\phi}$ such that $D \cap G_{\mathbf{h}}{}^{\phi}$ is compact. We then denote by $\mathfrak{S}_{k,\nu}{}^{\phi}(D)$ the set of all functions $\mathbf{f}$: $G_{\mathbf{A}}{}^{\phi} \to \mathbf{C}$ satisfying the following conditions:

$$\boldsymbol{f}(\alpha x w) = \boldsymbol{f}(x) \text{ if } \alpha \in G^{\varphi} \text{ and } w \in D \cap G^{\varphi}_{\boldsymbol{h}}; \quad 4.5$$

for every $p \in G_{\mathbf{h}}{}^{\phi}$ there exists an element $f_p \in \mathfrak{S}_{k,\nu}{}^{\phi}$ such that

$$\boldsymbol{f}(py) = (f_p \|_{k,\nu} y)(\boldsymbol{i}^{\varphi}) \text{ for every } y \in G^{\varphi}_{\boldsymbol{a}}, \text{ where } \boldsymbol{i}^{\varphi} = (\boldsymbol{i}_v)_{v \in \boldsymbol{a}}. \quad 4.6$$

We now take $D$ in a special form. We take a maximal r-lattice $M$ in $V$ whose norm is g in the sense of ref. 3 (p. 375) and put

$$C = \{\alpha \in G^{\varphi}_{\mathbf{A}} | M_v \alpha_v = M_v \text{ for every } v \in \boldsymbol{h}\}, \quad 4.7$$

$$\tilde{M} = \{x \in V | \varphi(x, M) \subset d^{-1}\}, \quad 4.8$$

$$D = D^{\varphi} = \{\gamma \in C | \tilde{M}_v(\gamma_v - 1) \subset c_v M_v \text{ for every } v \in \boldsymbol{h}\}, \quad 4.9$$

where d is the different of $K$ relative to $F$ and c is a fixed integral g-ideal. Clearly $\widetilde{M}$ is an r-lattice in $V$ containing $M$, and we easily see that $D^{\phi}$ is an open subgroup of $G_{\mathbf{A}}{}^{\phi}$. We assume that

$$v | c \text{ if } \tilde{M}_v \neq M_v. \quad 4.10$$

Define a subgroup $\mathfrak{X}$ of $G_{\mathbf{A}}{}^{\phi}$ by

$$X = \{y \in G^{\varphi}_{\mathbf{A}} | y_v \in D \text{ for every } v | c\}. \quad 4.11$$

We then consider the algebra $\Re(D, \mathfrak{X})$ consisting of all the finite linear combinations of $D\tau D$ with $\tau \in \mathfrak{X}$ and define its action on $\mathfrak{S}_{k,\nu}{}^{\phi}(D)$ as follows. Given $\tau \in \mathfrak{X}$ and $\mathbf{f} \in \mathfrak{S}_{k,\nu}{}^{\phi}(D)$, take a finite subset $Y$ of $G_{\mathbf{h}}{}^{\phi}$ so that $D\tau D = \sqcup_{\eta \in Y} D\eta$ and define $\mathbf{f}|D\tau D: G_{\mathbf{A}}{}^{\phi} \to \mathbf{C}$ by

$$(\boldsymbol{f}|D\tau D)(x) = \sum_{\eta \in Y} \boldsymbol{f}(x\eta^{-1}) \quad (x \in G^{\varphi}_{\boldsymbol{A}}). \quad 4.12$$

These operators form a commutative ring of normal operators on $\mathfrak{S}_{k,\nu}{}^{\phi}(D)$.

For $x \in G_{\mathbf{A}}{}^{\phi}$, we define an ideal $\nu_0(x)$ of r by

$$\nu_0(x) = \prod_{v \in \boldsymbol{h}} \nu_0(x_v), \quad 4.13$$

where $\nu_0(x_v)$ is defined as in Section 3 with respect to an $r_v$-basis of $M_v$. Clearly $\nu_0(x)$ depends only on $CxC$.

Let $\mathbf{f}$ be an element of $\mathfrak{S}_{k,\nu}{}^{\phi}(D)$ that is a common eigenfunction of all the $D\tau D$ with $\tau \in \mathfrak{X}$, and let $\mathbf{f}|D\tau D = \lambda(\tau)\mathbf{f}$ with $\lambda(\tau) \in \mathbf{C}$. Given a Hecke ideal character $\chi$ of $K$ such that $|\chi| = 1$, define a Dirichlet series $\mathfrak{T}(s, \mathbf{f}, \chi)$ by

$$\mathfrak{T}(s, \boldsymbol{f}, \chi) = \sum_{\tau \in D/X/D} \lambda(\tau)\chi * (\nu_0(\tau))N(\nu_0(\tau))^{-s}, \tag{4.14}$$

where $\chi^*$ is the ideal character associated with $\chi$ and $N(\mathfrak{a})$ is the norm of an ideal $\mathfrak{a}$. Denote by $\chi_1$ the restriction of $\chi$ to $F_{\mathbf{A}}^{\times}$, and by $\theta$ the Hecke character of $F$ corresponding to the quadratic extension $K/F$. For any Hecke character $\xi$ of $F$, put

$$L_c(s, \xi) = \prod_{p \nmid c} [1 - \xi * (p)N(p)^{-s}]^{-1}. \tag{4.15}$$

From *Theorem 1* and Eq. 3.13, we see that

$$\mathfrak{T}(s, \boldsymbol{f}, \chi) \prod_{i=1}^{n} L_c(2s - i + 1, \chi_1\theta^{i-1})$$

$$= \prod_{q \nmid c} W_q[\chi * (q)N(q)^{-s}]^{-1} \tag{4.16}$$

with a polynomial $W_q$ *of degree n* whose constant term is 1, where $q$ runs over all the prime ideals of $K$ prime to $c$. Let $Z(s, \mathbf{f}, \chi)$ denote the function of Eq. 4.16. Put

$$\Gamma_m(s) = \pi^{m(m-1)/2} \prod_{k=0}^{m-1} \Gamma(s - k). \tag{4.17}$$

THEOREM 2. *Suppose that* $\chi_{\mathbf{a}}(b) = b^{\mu}|b|^{i\kappa-\mu}$ *with* $\mu \in \mathbf{Z}^{\mathbf{a}}$ *and* $\kappa \in \mathbf{R}^{\mathbf{a}}$ *such that* $\sum_{v\in\mathbf{a}} \kappa_v = 0$. *Put* $m = k + 2\nu - \mu$ *and*

$$\Re(s, \boldsymbol{f}, \chi) = \prod_{v\in\boldsymbol{a}} \gamma_v(s + (i\kappa_v/2))\cdot Z(s, \boldsymbol{f}, \chi)$$

*with* $\gamma_v$ *defined by*

$$\gamma_v(s) = p_v(s)q_v(s)\Gamma_{r_v}\left(s - n + r_v + \frac{k_v + |m_v|}{2}\right)$$

$$\cdot\Gamma_{n-r_v}\left(s - r_v + \frac{|\mu_v - 2\nu_v|}{2}\right),$$

$$p_v(s) = \begin{cases} \Gamma_{r_v}\left(s + \frac{|k_v - m_v|}{2}\right)\Gamma_{r_v}\left(s + \frac{k_v - m_v}{2}\right)^{-1} & if\ m_v \geq 0, \\ \Gamma_{r_v}\left(s - \frac{k_v + m_v}{2}\right)\Gamma_{r_v}\left(s - \frac{k_v - m_v}{2}\right)^{-1} & if\ m_v < 0, \end{cases}$$

$$q_v(s) = \prod_{i=1}^{n-\ell-1} \Gamma\left(s - \frac{\ell}{2} - \left[\frac{i}{2}\right]\right)$$

$$\cdot\Gamma\left(s - \frac{\ell}{2 - i}\right)^{-1}, \quad \ell = |\mu_v - 2\nu_v|.$$

*Then* $\mathfrak{R}(s, \mathbf{f}, \chi)$ *can be continued to the whole s-plane as a meromorphic function with finitely many poles, which are all simple. It is entire if* $\chi_1 \neq \theta^{\nu}$ *for* $\nu = 0, 1$.

We can give an explicitly defined finite set of points in which the possible poles of $\mathfrak{R}$ belong. Notice that $p_v$ and $q_v$ are polynomials; in particular, $p_v = 1$ if $0 \leq m_v \leq k_v$ and $q_v = 1$ if $|\mu_v - 2\nu_v| \geq n - 1$.

The results of the above type and also of the type of *Theorem 3* below were obtained in refs. 2, 4, and 5 for the forms on the symplectic and metaplectic groups over a totally real number field. The Euler product of type $Z$, its analytic continuation, and its relationship with the Fourier coefficients of **f** have been obtained by Oh (6) for the group $G^{\phi}$ as above when $\phi = \eta_r$.

## SECTION 5.

Go to:

We now put $(W, \psi) = (V, \phi) \oplus (H_m, \eta_m)$ as in Eq. 2.5 with $(V, \phi)$ of Section 4 and $m \geq 0$. Writing simply $I = I_m$, we can consider the parabolic subgroup $P_I^{\psi}$ of $G^{\psi}$. We put $P^{\psi} = P_I^{\psi}$ for simplicity, $\lambda_0(\alpha) = \det(\lambda_I^{\psi}(p))$ for $p \in P^{\psi}$, and

$$L = \sum_{i=1}^{m} (r\varepsilon_i + d^{-1}\varepsilon_{m+n+i}) + M, \qquad 5.1$$

with $M$ of Section 4 and the standard basis $\{\varepsilon_i, \varepsilon_{m+n+i}\}_{i=1}^{m}$ of $H_m$. We can define the space $\mathfrak{Z}^{\psi}$ and its origin $\mathbf{i}^{\psi}$ in the same manner as for $G^{\phi}$. We then put

$$C^{\psi} = \{x \in G_{\mathbf{A}}^{\psi} | Lx = L\}, \quad C_0^{\psi} = \{x \in C^{\psi} | x(\boldsymbol{i}^{\psi}) = \boldsymbol{i}^{\psi}\}, \qquad 5.2$$

$$D^{\psi} = \{x \in C^{\psi} | \tilde{M}_v(e_v - 1) \subset c_v M_v \text{ for every } v \in \boldsymbol{h}\}. \qquad 5.3$$

Here $e_v$ is the element of $\mathrm{End}(V_v)$ defined for $x_v$ by $wx_v - we_v \in (H_m)_v$ for $w \in V_v$. We define an $\mathbf{R}$-valued function $h$ on $G_{\mathbf{A}}^{\psi}$ by

$$h(x) = |\lambda_0(p)|_{\mathbf{A}} \text{ if } x \in pC_0^{\psi} \text{ with } p \in P_{\mathbf{A}}. \qquad 5.4$$

Taking $\mathbf{f} \in \mathfrak{S}_{k,\nu}^{\phi}(D^{\phi})$ and $\chi$ as in Section 4, we define $\mu: G_{\mathbf{A}}^{\psi} \to \mathbf{C}$ as follows: $\mu(x) = 0$ if $x \notin P_{\mathbf{A}}^{\psi} D^{\psi}$; if $x = pw$ with $p \in P_{\mathbf{A}}^{\psi}$ and $w \in D^{\psi} \cap C_0^{\psi}$, then we put

$$\mu(x) = \chi(\lambda_0(p))^{-1}\chi_c(\lambda_0(w))^{-1} j_w^{k,\nu}(\boldsymbol{i}^{\psi})^{-1} \boldsymbol{f}(\pi_{\varphi}^{\psi}(p)), \qquad 5.5$$

where $\chi_c = \prod_{v|c} \chi v$. Then we define $E(x, s)$ for $x \in G_{\mathbf{A}}^{\psi}$ and $s \in \mathbf{C}$ by

$$E(x, s) = E(x, s;\, \boldsymbol{f}, \chi, \mathrm{D}^{\psi}) = \sum_{\alpha \in A} \mu(\alpha x) h(\alpha x)^{-s},$$

$$A = P_I^{\psi} / G^{\psi}. \qquad 5.6$$

This is meaningful if $\chi_{\mathbf{a}}(b) = b^{k+2\nu}|b|^{i\kappa-k-2\nu}$ with $\kappa \in \mathbf{R}^{\mathbf{a}}$, $\sum_{v \in \mathbf{a}} \kappa_v = 0$, and the conductor of $\chi$ divides c. We take such a $\chi$ in the following theorem. The series of Eq. 5.6 is the adelized version of a collection of several series of the type in Eq. 1.3.

THEOREM 3. *Define* $\gamma_v$ *as in* Theorem 2 *with m = 0. Put*

$$\gamma_v'(s) = q'(s, |k_v|)\gamma_v(s) q_v(s)^{-1} \Gamma_m(s - n + (k_v/2)),$$

$$q'(s,\ell) = \prod_{i=1}^{m+n-\ell-1} \Gamma\left(s - \frac{\ell}{2} - \left[\frac{i}{2}\right]\right) \Gamma\left(s - \frac{\ell}{2} - i\right)^{-1}.$$

*Then the product*

$$\prod_{v\in\boldsymbol{a}} \gamma'_v(s + (i\kappa_v/2)) \prod_{j=n}^{m+n-1} \cdot L_c(2s - j, \chi_1\theta^j)$$

$$\cdot Z(s, \boldsymbol{f}, \chi) E(x, s; \boldsymbol{f}, \chi, D^{\psi})$$

*can be continued to the whole* s*-plane as a meromorphic function with finitely many poles, which are all simple.*

We can give an explicitly defined finite set of points in which the possible poles of the above product belong.

## SECTION 6.

Go to:

Let $G$ be an arbitrary reductive algebraic group over $\mathbf{Q}$. Given an open subgroup $U$ of $G_{\mathbf{A}}$ containing $G_{\mathbf{a}}$ and such that $U \cap G_{\mathbf{h}}$ is compact, we put $U^a = aUa^{-1}$ and $\Gamma^a = G \cap U^a$ for every $a \in G_{\mathbf{A}}$. We assume that $G_{\mathbf{a}}$ acts on a symmetric space $\mathfrak{W}$, and we let $G$ act on $\mathfrak{W}$ via its projection to $G_{\mathbf{a}}$. We also assume that $\Gamma^a/\mathfrak{W}$ has finite measure, written $\mathrm{vol}(\Gamma^a/\mathfrak{W})$, with respect to a fixed $G_{\mathbf{a}}$-invariant measure on $\mathfrak{W}$. Taking a complete set of representatives $\mathfrak{B}$ for $G/G_{\mathbf{A}}/U$, we put

$$\sigma(G, U) = \sigma(U) = \sum_{a\in\mathrm{B}} [\Gamma^a \cap T : 1]^{-1} \mathrm{vol}(\Gamma^a/W), \qquad 6.1$$

where $T$ is the set of elements of $G$ which act trivially on $\mathfrak{W}$, and we assume that $[\Gamma^a \cap T{:}1]$ is finite. Clearly $\sigma(U)$ does not depend on the choice of $\mathfrak{B}$. We call $\sigma(G, U)$ the mass of $G$ with respect to $U$. If $G_{\mathbf{a}}$ is compact, we take $\mathfrak{W}$ to be a single point of measure 1 on which $G_{\mathbf{a}}$ acts trivially. Then we have

$$\sigma(G, U) = \sigma(U) = \sum_{a\in\mathrm{B}} [\Gamma^a : 1]^{-1}. \qquad 6.2$$

We can show that $\sigma(U') = [U{:}U']\sigma(U')$ if $U'$ is a subgroup of $U$. If strong approximation holds for the semisimple factor of $G$, then it often happens that both $[\Gamma^a \cap T{:}1]$ and $\mathrm{vol}(\Gamma^a/\mathfrak{W})$ depend only on $U$, so that

$$\sigma(G, U) = \sigma(U) = \#(G\backslash G_{\boldsymbol{A}}/U)[\Gamma^1 \cap T : 1]^{-1} \mathrm{vol}(\Gamma^1/W). \qquad 6.3$$

If $G_{\mathbf{a}}$ is compact and $U$ is sufficiently small, then $\Gamma^a = \{1\}$ for every $a$, in which case we have $\sigma(U) = \#(G/G_{\mathbf{A}}/U)$. If $U$ is the stabilizer of a lattice $L$ in a vector space on which $G$ acts, then $\#(G/G_{\mathbf{A}}/U)$ is the number of classes in the genus of $L$. Therefore, $\sigma(U)$ may be viewed as a refined version of the class number in this sense.

Coming back to the unitary group $G^{\phi}$ of Section 4, we can prove the following theorem.

THEOREM 4. *Suppose that* $G_{\mathbf{a}}^{\phi}$ *is compact. Let* $M$ *be a* $\mathfrak{g}$*-maximal lattice in* $V$ *of norm* $\mathfrak{g}$ *and let* $\mathfrak{d}$ *be the different of* $K$ *relative to* $F$*. Define an open subgroup* $D$ *of* $G_{\mathbf{A}}^{\phi}$ *by Eq.* **4.9** *with an integral ideal* $\mathfrak{c}$*. If* $n$ *is odd, assume that* $\phi$ *is represented by a matrix whose determinant times* $(-1)^{(n-1)/2}$ *belongs to* $N_{K/F}(K)$*; if* $n$ *is even, assume that* $\mathfrak{c}$ *is divisible by the product* $\mathfrak{e}$ *of all prime ideals for which* $t_v = 2$*. Then*

$$\sigma(G^{\varphi}, D) = 2\cdot\left\{\prod_{k=1}^{n}(n-k)\right\}^{d} D_F^{(n^2-n)/2} N(c)^{n^2}$$

$$\cdot A \prod_{k=1}^{n}\{N(d)^{k/2} D_F^{1/2}(2\pi)^{-kd} L_c(k, \theta^k)\},$$

*where* $d = [F{:}\mathbf{Q}]$, $D_F$ *is the discriminant of* $F$, *and* $A = 1$ *or* $A = N(e)^n N(d)^{-n/2}$ *according as* $n$ *is odd or even*.

If $n$ is odd, we can also consider $\sigma(D')$ for

$$D' = \{\gamma \in C \mid M_v(\gamma_v - 1) \subset c_v M_v \text{ for every } v \in \boldsymbol{h}\} \qquad 6.4$$

with an arbitrary integral ideal c. Then $\sigma(D') = 2^{-\tau}\sigma(D)$, where $\tau$ is the number of primes in $F$ ramified in $K$.

## SECTION 7.

Go to:

Let us now sketch the proof of the above theorems. The full details will be given in ref. 7. We first take $\mathfrak{B} \subset G_{\boldsymbol{h}}^{\varphi}$ so that $G_{\boldsymbol{A}}^{\varphi} = \sqcup_{b\in\mathfrak{B}} G^{\phi} b D^{\phi}$. Given $E(x, s)$ as in Eq. 5.6, for each $q \in G_{\mathbf{h}}^{\Psi}$ we can define a function $E_q(z, s)$ of $(z, s) \in \mathfrak{Z}^{\Psi} \times \mathbf{C}$ by

$$E(qy, s) = E_q(y(\boldsymbol{i}^{\psi}), s) j_y^{k,\nu}(\boldsymbol{i}^{\psi})^{-1} \text{ for every } y \in G_{\boldsymbol{a}}^{\psi}. \qquad 7.1$$

The principle is the same as in Eq. 4.6, and so it is sufficient to prove the assertion of *Theorem 3* with $E_q(z, s)$ in place of $E(x, s)$. In particular, we can take $q$ to be $q = b \times 1_{2m}$ with $b \in \mathfrak{B}$. Define $(X, \omega)$ as in Eq. 2.5. Then there is an isomorphism of $(X, \omega)$ to $(H_{m+n}, \eta_{m+n})$ which maps $P_U^{\omega}$ of *Proposition 1* to the standard parabolic subgroup $P$ of $G(\eta_{m+n})$. Therefore, we can identify $\mathfrak{Z}^{\omega}$ with the space $\mathrm{h}^{\mathbf{a}}$ with

$$h = \{z \in \boldsymbol{C}_{m+n}^{m+n} \mid i(z* - z) \text{ is positive definite}\}. \qquad 7.2$$

We can also define an Eisenstein series $E'(x, s; \chi)$ for $x \in G_{\mathbf{A}}^{\omega}$ and $s \in \mathbf{C}$, which is defined by Eq. 5.6 with $(G(\eta_{m+n})_{\mathbf{A}}, P, 1)$ in place of $(G_{\mathbf{A}}^{\Psi}, P^{\Psi}, \mathbf{f})$. Taking $E'$ and $(q, a) \in G_{\mathbf{h}}^{\omega}$ (with $a \in \mathfrak{B}$) in place of $E(x, s)$ and $q$, we can define a function $E'_{q,a}(z, s)$ of $(z, s) \in \mathrm{h}^{\mathbf{a}} \times \mathbf{C}$ in the same manner as in Eq. 7.1. There is also an injection $\iota$ of $\mathfrak{Z}^{\Psi} \times \mathfrak{Z}^{\phi}$ into $\mathrm{h}^{\mathbf{a}}$ compatible with the embedding $G^{\Psi} \times G^{\phi} \to G(\eta_{m+n})$. We put then

$$g^{\circ}(z, w) = \delta(w, z)^{-k} g(\iota(z, w)) \quad (z \in Z^{\psi}, w \in Z^{\varphi}) \qquad 7.3$$

for every function $g$ on $\mathrm{h}^{\mathbf{a}}$, where $\delta(w, z)$ is a natural factor of automorphy associated with the embedding $\iota$. Take a Hecke eigenform $\mathbf{f}$ as in Section 4 and define $f_a$ by the principle of Eq. 4.6. Then, employing *Proposition 1*, we can prove

$$A(s)\mathfrak{T}(s, \boldsymbol{f}, \chi) E_q(z, s)$$

$$= \sum_{a\in\mathbf{B}} \int_{\Phi_a} (E'_{q,a})^{o}(z, w; s) f_a(w) \delta(w)^k dw, \qquad 7.4$$

where $q = b \times 1_{2m}$, $A$ is a certain gamma factor, and $\Phi_a = \Gamma^a/\mathfrak{Z}^{\phi}$. The computation is similar to, but more involved than, that of ref. 4 (Section 4). Since the analytic nature of $E'_{q,a}$ can be seen from the results of ref. 8, we can derive *Theorem 3* from Eq. 7.4.

Take $m = 0$. Then $\psi = \phi$ and $E_q(z, s) = f_b(z)$. Then the analytic nature of $\mathfrak{T}(s, \mathbf{f}, \chi)$, and consequently that of $Z(s, \mathbf{f}, \chi)$, can be derived from Eq. 7.4. However, here we have to assume that $\chi_{\mathbf{a}}(b) = b^{k+2\nu}|b|^{i\kappa-k-2\nu}$ with $\kappa \in \mathbf{R}^{\mathbf{a}}$, $\sum_{v\in\mathbf{a}} \kappa_v = 0$, and the conductor of $\chi$ divides c. The latter condition on c is a minor matter, but the condition on $\chi_{\mathbf{a}}$ is essential. To obtain $Z(s, \mathbf{f}, \chi)$ with an arbitrary $\chi$, we have to replace $E'_{q,a}$ by $\mathfrak{D}E''_{q,a}$, where $E''$ is a series of type $E'$ with $2\nu - \mu$ in place of $k$ and $\mathfrak{D}$ is a certain differential operator on $\mathrm{h}^{\mathbf{a}}$.

As for *Theorem 4*, we take again $\psi = \phi$ and observe that a constant function can be taken as $\mathbf{f}$ if $G_{\mathbf{a}}{}^{\phi}$ is compact. The space $\mathfrak{Z}^{\phi}$ consists of a single point. The integral on the right-hand side of Eq. 7.4 is merely the value $(E'_{q,a})^{\circ}(z, w; s)$. We can compute its residue at $s = n$ explicitly. Comparing it with the residue on the left-hand side, we obtain *Theorem 4* when c satisfies Eq. 4.10. If $n$ is odd, we can remove this condition by computing a group index of type $[U:U']$.

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