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On p-adic L-functions Attached to Motives over Q

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To K. Iwasawa

Introduction

It is a pleasure to dedicate this paper to K. Iwasawa in recognition of his great work on the mysterious connexion between special values of the Riemann zeta function and certain Galois modules, which we now call Iwasawa modules. His ideas have enormously influenced our own work, and the work of many others. If one seeks to generalize Iwasawa's work, one of the first problems which arises is that of constructing p-adic analogues of the complex L-functions of number theory. It is this question which we address in this highly conjectural paper. We have been tempted into making such broad and sweeping conjectures because of the success and usefulness of similar general conjectures about complex L-functions, which were made by Serre [10], and refined by Deligne [2], in 1970. The framework in which we work, following [10], is that of motives over Q—roughly speaking, such a motive V consists of a compatible system of l-adic representations of the Galois group $G=G(\bar{Q}/Q)$, together with a suitable Hodge structure at the infinite prime (see § 2 for more details). Let $L(V, s)$ be the complex L-function attached to V by Serre. The definition of $L(V, s)$ as an Euler product of factors determined by purely local data is simply not applicable in the p-adic case, and so, like all previous authors, we are forced to use p-adic interpolation to define the p-adic analogue of $L(V, s)$. Our conjectures are subject to two important restrictions on V and the prime p. Firstly, we must assume that $L(V, s)$ admits at least one critical point in the sense of Deligne [3]. Secondly, we must assume that V is ordinary at p (see § 4). There is little doubt that p-adic analogues of $L(V, s)$ should exist without these restrictions, but we are unable to formulate precise conjectures at present. For simplicity, we have deliberately not treated the case of motives defined over a finite extension of Q in this paper. When $F\neq Q$, new features arise in the theory because we must take our p-adic L-functions

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to be measures on the Galois group over F of the maximum abelian extension of F unramified outside p, and pro-p-part of this Galois group then has Z_p-rank >1 when F is not totally real.

Notation. Let $\bar{Q}$ be the algebraic closure of the rational field Q in the complex field C, and let $G=G(\bar{Q}/Q)$ be the Galois group of $\bar{Q}$ over Q. For each prime p, let Q_p, Z_p be the field of p-adic numbers, and the ring of p-adic integers. We write $Z_p^\times$ for the group of units of Z_p. Let $\bar{Q}_p$ denote a fixed algebraic closure of Q_p, and C_p the completion of $\bar{Q}_p$. If χ is a Dirichlet character, we write $c(\chi)$ for the conductor of χ. If $c(\chi)$ is a power of p, we define the integer $r_\chi\geq 0$ by $c(\chi)=p^{r_\chi}$. As usual, if $c(\chi)$ is a power of p, we identify χ with character of $Z_p^\times$ by composing χ with the canonical surjection $Z_p^\times\to(Z/p^{r_\chi}Z)^\times$ (if $c(\chi)=1$, we take the trivial character). We extend χ to a function on the whole of Z_p by putting $\chi(x)=0$ when $x\in pZ_p$.

§ 1. Distributions attached to Euler products

In this section, we elaborate an idea of Panciskin [7], for attaching canonical distributions to Euler products. Recall that a distribution on Z_p with values in a C-vector space A is a finitely additive function from the set of open and closed subsets of Z_p with values in A. Throughout this section, we shall take A to be the ring of formal Dirichlet series with complex coefficients. If μ is such a distribution, and f is a locally constant function on Z_p with values in A, we define as usual

$$\int_{Z_p} f d\mu=\sum_{a \bmod p^m Z_p} f(a)\mu(a+p^m Z_p),$$

where a runs over a set of representatives of the cosets of $p^m Z_p$, and m is chosen so large that f is constant on these cosets.

Consider now an arbitrary Dirichlet series with complex coefficients

$$D(s)=\sum_{n=1}^{\infty}\frac{c_n}{n^s}. \tag{1.1}$$

If we define

$$\mu_D(a+p^m Z_p)=\sum_{\substack{n=1\\ n\equiv a \bmod p^m}}^{\infty}\frac{c_n}{n^s},$$

we clearly obtain a distribution on Z_p with values in A, which is called the partial zeta distribution of $D(s)$. To attach other canonical distributions to $D(s)$, we will use some elementary harmonic analysis. We must assume that $D(s)$ admits an Euler factor at p in the following sense:

Hypothesis on $D(s)$. There exists a non-constant polynomial $D_p(X)\in C[X]$, with $D_p(0)=1$, such that

$$D(s)=\Bigg(\sum_{\substack{n=1\\ (n,p)=1}}^{\infty}\frac{c_n}{n^s}\Bigg)\cdot(D_p(p^{-s}))^{-1}. \tag{1.2}$$

Now fix a non-zero $\alpha\in C$ such that α^{-1} is a root of $D_p(X)$. We then introduce the new Dirichlet series

$$\mathscr{D}_\alpha(s)=D(s)\cdot(D_p(p^{-s})/(1-\alpha p^{-s})). \tag{1.3}$$

Let $\mu_{\mathscr{D}_\alpha}$ be the partial zeta distribution of $\mathscr{D}_\alpha(s)$.

Lemma 1.1. *For every compact open set U in Z_p, we have*

$$\mu_{\mathscr{D}_\alpha}(pU)=\frac{\alpha}{p^s}\mu_{\mathscr{D}_\alpha}(U). \tag{1.4}$$

Proof. Write $\mathscr{D}_\alpha(s)=\sum_{n=1}^{\infty}(d_n/n^s)$. Since $\mathscr{D}_\alpha(s)$ admits as Euler factor at p the expression $(1-\alpha p^{-s})$, we deduce immediately that $d_{np^k}=d_n\alpha^k$ for all $n\geq 1$ and $k\geq 0$. Now it suffices to verify (1.4) when U is of the form $a+p^m Z_p$. For such a U, $\mu_{\mathscr{D}_\alpha}(pU)$ is given by

$$\sum_{\substack{n\equiv pa \bmod p^{m+1}\\ n\geq 1}} d_n/n^s.$$

This can plainly be rewritten as

$$\sum_{\substack{n_1\equiv a \bmod p^m\\ n_1\geq 1}} d_{n_1 p}/(n_1 p)^s,$$

and, since $d_{n_1 p}=d_{n_1}\alpha$, assertion (1.4) follows.

We now define the Panciskin distribution $\mu_{D,\alpha}$ on Z_p by

$$\mu_{D,\alpha}(a+p^m Z_p)=\left(\frac{p^{s-1}}{\alpha}\right)^m\int_{Z_p}\exp(2\pi iax/p^m)d\mu_{\mathscr{D}_\alpha}, \tag{1.5}$$

where we view $\exp(2\pi iax/p^m)$ as a complex valued function of x in Z_p by evaluating it with x replaced by any rational integer which is congruent to x modulo p^m.

Lemma 1.2. *$\mu_{D,\alpha}$ is a distribution on Z_p.*

Proof. Let a be a fixed integer modulo p^m, where $m\geq 0$. Let B denote a set of representatives of the residue classes modulo p^{m+1} which map to $a+p^m Z_p$. We must show that

$$\sum_{b\in B}\mu_{D,\alpha}(b+p^{m+1}Z_p)=\mu_{D,\alpha}(a+p^mZ_p).$$

Now the left hand side of this equation is equal to (on taking $b=a+kp^m$, with $k=0,\cdots,p-1$)

$$\left(\frac{p^{s-1}}{\alpha}\right)^{m+1}\int_{Z_p}\exp(2\pi iax/p^{m+1})S(x)d\mu_{\mathscr{D}_\alpha},$$

where $S(x)=\sum_{k=0}^{p-1}\exp(2\pi ikx/p)$. But $S(x)$ is equal to 0 or p, according as $x\notin pZ_p$ or $x\in pZ_p$. Hence the above expression is equal to

$$\left(\frac{p^{s-1}}{\alpha}\right)^{m+1}p\int_{pZ_p}\exp(2\pi iax/p^{m+1})d\mu_{\mathscr{D}_\alpha}.$$

Making the change of variable $x=py$, and using (1.4), we deduce that this last quantity is equal to $\mu_{D,\alpha}(a+p^mZ_p)$, as required.

If χ is a Dirichlet character, we define

$$D(\chi,s)=\sum_{n=1}^{\infty}\chi(n)c_n/n^s. \tag{1.6}$$

We write $c(\chi)$ for the conductor of χ, and let

$$G(\chi)=\sum_{a \bmod c(\chi)}\chi(a)\exp(2\pi ia/c(\chi)) \tag{1.7}$$

be the Gauss sum of χ (if $c(\chi)=1$, take $G(\chi)=1$). We shall assume for the rest of this section that the conductor of χ is a power of p, say $c(\chi)=p^{r_\chi}$. As explained earlier, we also write χ for the character of $Z_p^\times$ induced by χ, and we then extend χ by 0 to the whole of Z_p. Let the Euler factor $D_p(X)$ of $D(s)$ be given explicitly by

$$D_p(X)=(1-\alpha_1X)\cdots(1-\alpha_dX)\qquad(\alpha_i\in C) \tag{1.8}$$

with $\alpha=\alpha_1$. The calculations of the next lemma suggest the following modification of the Euler factor at p, which turns out to be of the utmost importance in all that follows. Put $\alpha^*=p/\alpha$, and define

$$\Phi_{p,\alpha}(\chi^{-1},s)=(1-\alpha^*\chi(p)p^{s-2})^{-1}\prod_{i=2}^{d}(1-\alpha_i\chi^{-1}(p)p^{-s})^{-1}. \tag{1.9}$$

Lemma 1.3. *For all Dirichlet characters χ with $c(\chi)=p^{r_\chi}$, we have*

$$\int_{Z_p}\chi d\mu_{D,\alpha}=\frac{G(\chi)D(\chi^{-1},s)}{\Phi_{p,\alpha}(\chi^{-1},s)}\cdot(\alpha^*p^{s-2})^{r_\chi} \tag{1.10}$$

where $\alpha=\alpha_1$, $\alpha^*=p/\alpha$.

Proof. This divides into two cases, according as χ is trivial or not. Suppose first that χ is trivial. By (1.5), we have

$$\mu_{D,\alpha}(Z_p)=\mu_{\mathscr{D}_\alpha}(Z_p)=\mathscr{D}_\alpha(s),$$
$$\mu_{D,\alpha}(pZ_p)=\frac{p^{s-1}}{\alpha}\mu_{\mathscr{D}_\alpha}(Z_p)=\frac{p^{s-1}}{\alpha}\mathscr{D}_\alpha(s).$$

Formula (1.10) follows, because the integral on the left is equal to $\mu_{D,\alpha}(Z_p)-\mu_{D,\alpha}(pZ_p)$ (recall that we extend χ from $Z_p^\times$ to Z_p by 0). Suppose next that χ is not the trivial character. In particular, we have $\Phi_{p,\alpha}(\chi^{-1},s)=1$. Put

$$H(x)=\sum_{a \bmod p^{r_\chi}}\chi(a)\exp(2\pi iax/p^{r_\chi}).$$

If $x\in Z_p^\times$, we clearly have $H(x)=\chi^{-1}(x)G(\chi)$. On the other hand, if $x\in pZ_p$, we claim that $H(x)=0$. Indeed, let B denote a set of representatives of the residue classes modulo p^{r_χ} which map to the residue class of 1 modulo $p^{r_\chi-1}$, and let C denote a set of representatives of the relatively prime residue classes modulo $p^{r_\chi-1}$. Writing $x=px_1$, with $x_1\in Z_p$, we plainly have

$$H(x)=\sum_{b\in B}\sum_{c\in C}\chi(bc)\exp(2\pi icx_1/p^{r_\chi-1}).$$

Changing the order of summation, and noting that $\sum_{b\in B}\chi(b)=0$ because χ has conductor p^{r_χ}, it follows that $H(x)=0$. Now the integral on the left of (1.10) is equal to

$$\left(\frac{p^{s-1}}{\alpha}\right)^{r_\chi}\int_{Z_p}H(x)d\mu_{\mathscr{D}_\alpha}.$$

By the above remark on $H(x)$, this last integral is equal to

$$(p^{s-1}/\alpha)^{r_\chi}\int_{Z_p^\times}\chi^{-1}(x)G(\chi)d\mu_{\mathscr{D}_\alpha}=G(\chi)\mathscr{D}_\alpha(\chi^{-1},s)(\alpha^*p^{s-2})^{r_\chi}.$$

Since $\chi(p)=0$, we see that $\mathscr{D}_\alpha(\chi^{-1},s)=D(\chi^{-1},s)$, and the proof of the lemma is now complete.

Our aim now is to construct an analogue of the distribution $\mu_{D,\alpha}$, in which we replace the subset $\{\alpha\}$ of the inverse roots of $D_p(X)$ by an arbitrary non-empty subset of these inverse roots. In fact, we shall only be interested in finding such a distribution on $Z_p^\times$. Recall that a distribution on $Z_p^\times$ is a finitely additive function on the set of open and closed subsets of $Z_p^\times$, and the integral of a locally constant function against it is defined in an exactly parallel manner to that for $\boldsymbol{Z}_p$. Moreover, if μ is a distribution on $\boldsymbol{Z}_p$, its restriction to $\boldsymbol{Z}_p^\times$ is again a distribution, which we denote by $\mu^\times=\mu|\boldsymbol{Z}_p^\times$. Note that, when χ is a Dirichlet character of p-power conductor, we plainly have

$$\int_{\boldsymbol{Z}_p^\times}\chi d\mu^\times=\int_{\boldsymbol{Z}_p}\chi d\mu. \tag{1.11}$$

We shall need the notion of the convolution of two distributions ρ_1, ρ_2 on $\boldsymbol{Z}_p^\times$. Let R_m denote the multiplicative group of $\boldsymbol{Z}_p/p^m\boldsymbol{Z}_p$. We define the convolution $\rho_1*\rho_2$ by

$$(\rho_1*\rho_2)(\gamma)=\sum_{\beta\in R_m}\rho_1(\beta^{-1}\gamma)\rho_2(\beta)$$

for each $\gamma\in R_m$, with $m\geq 1$. One verifies immediately that $\rho_1*\rho_2$ is again a distribution on $\boldsymbol{Z}_p^\times$ with values in the algebra A.

Lemma 1.4. *Let ρ_1, ρ_2 be distributions on $\boldsymbol{Z}_p^\times$ with values in A. For each Dirichlet character χ of p-power conductor, we have*

$$\int_{\boldsymbol{Z}_p^\times}\chi d(\rho_1*\rho_2)=\left(\int_{\boldsymbol{Z}_p^\times}\chi d\rho_1\right)\cdot\left(\int_{\boldsymbol{Z}_p^\times}\chi d\rho_2\right). \tag{1.12}$$

Proof. Let m be any integer $\geq r_\chi$. The integral on the left is equal to

$$\sum_{\gamma\in R_m}\chi(\gamma)(\rho_1*\rho_2)(\gamma),$$

which becomes

$$\sum_{\beta\in R_m}\chi(\beta)\rho_2(\beta)\sum_{\gamma\in R_m}\chi(\beta^{-1}\gamma)\rho_1(\beta^{-1}\gamma),$$

and this last expression is equal to the right hand side of (1.12).

For each inverse root of $D(X)$, put

$$E_\alpha(s)=(1-\alpha p^{-s})^{-1}=\sum_{m=0}^{\infty}(\alpha/p^s)^m.$$

We can regard $E_\alpha(s)$ as a Dirichlet series in its own right. Write μ_α for the distribution on $\boldsymbol{Z}_p$ which is given by (1.5) with $D(s)$ replaced by the Dirichlet series $E_\alpha(s)$. Now let k be any integer with $1\leq k\leq d$. Put

$$\alpha_i^*=p/\alpha_i \qquad (1\leq i\leq k). \tag{1.13}$$

$$\Phi_{p,K}(\chi^{-1},s)=\prod_{i=1}^{k}(1-\alpha_i^*\chi(p)p^{s-2})^{-1}\prod_{i=k+1}^{d}(1-\alpha_i\chi^{-1}(p)p^{-s})^{-1}, \tag{1.14}$$

where K denotes $\{\alpha_1,\cdots,\alpha_k\}$. We then define the distribution $\rho_D(K,s)$ on $\boldsymbol{Z}_p^\times$ by

$$\rho_D(K,s)=\mu_{D,\alpha_1}^\times*\mu_{\alpha_2}^\times*\cdots*\mu_{\alpha_k}^\times. \tag{1.15}$$

Theorem 1.5. *For each non-empty subset $K=\{\alpha_1,\cdots,\alpha_k\}$ of the set $\{\alpha_1,\cdots,\alpha_d\}$ of inverse roots of $D_p(X)$, we have*

$$\int_{\boldsymbol{Z}_p^\times}\chi d\rho_D(K,s)=\frac{G(\chi)^kD(\chi^{-1},s)}{\Phi_{p,K}(\chi^{-1},s)}\cdot\left(\prod_{i=1}^{k}\alpha_i^*p^{s-2}\right)^{r_\chi}, \tag{1.16}$$

for all Dirichlet characters χ of conductors $c(\chi)=p^{r_\chi}$.

Proof. Noting (1.11) and Lemma 1.4, the assertion follows immediately from applying Lemma 1.3 to the distributions μ_{D,α_1} and μ_{α_i} $(i=2,\cdots,k)$.

The distribution $\rho_D(K,s)$, for a suitable choice of K, will play an important role in our later conjectures. Of course, it is obvious that a distribution exists on $\boldsymbol{Z}_p^\times$ satisfying (1.16) (we can trivially find a distribution μ on $\boldsymbol{Z}_p^\times$ such that the integrals of μ against Dirichlet characters of p-power conductor have any prescribed set of values). The interest of Theorem 1.5 is that it gives a natural construction of $\rho_D(k,s)$ using elementary harmonic analysis.

§ 2. Motives over $\boldsymbol{Q}$

The aim of this section is to recall and slightly reformulate the basic conjectures made in Deligne's paper [3] about the algebraic nature of the values at critical points of the L-functions attached to motives over $\boldsymbol{Q}$. We give this reformulation because it seems better adapted to the problem of p-adic interpolation of these special values, which is discussed in § 4. Because we work with the general formalism of motives, nearly all the theory discussed here is conjectural in this generality—nevertheless, we believe it is useful to place our conjectures in this general framework. Wherever possible, we have followed the notation and terminology of [3]. However, we shall simply view motives in the naive sense as being defined by the collection of their realisations.

The archetype of a motive arises as follows. Let X be a smooth projective variety defined over $\boldsymbol{Q}$, and let m be a fixed integer ≥ 0. Let $X(\boldsymbol{C})$ be the set of complex points of X. Classical algebraic and analytic geometry provides us with three different types of cohomology groups for X, namely the Betti cohomology with coefficients in $\boldsymbol{Q}$ of the complex manifold $X(\boldsymbol{C})$, the algebraic de Rham cohomology of the algebraic variety $X/\boldsymbol{Q}$, and, for each prime number l, the l-adic cohomology of X viewed over $\bar{\boldsymbol{Q}}$. We denote these cohomology groups respectively by

$$H_B^m(X(\boldsymbol{C})), \quad H_{DR}^m(X), \quad H_l^m(X). \tag{2.1}$$

Moreover, these cohomology groups are endowed with the following additional structures:

(i) The action of complex conjugation on $X(\boldsymbol{C})$ induces an involution of $H_B^m(X(\boldsymbol{C}))$, which we denote by ρ_B;

(ii) The Galois group $G=G(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ has a natural action on $H_l^m(X)$;

(iii) For each prime number l, there is a comparison isomorphism

$$\psi_l: H_l^m(X) \xrightarrow{\sim} H_B^m(X(\boldsymbol{C}))\otimes \boldsymbol{Q}_l,$$

which transforms the complex conjugation into ρ_B;

(iv) There exists a canonical decreasing filtration $\{F^k H_{DR}^m(X): k \in \boldsymbol{Z}\}$ on $H_{DR}^m(X)$;

(v) We have the Hodge decomposition into $\boldsymbol{C}$-vector spaces

$$H_B^m(X(\boldsymbol{C}))\otimes \boldsymbol{C} = \bigoplus_{i+j=m} \mathscr{H}^{j,i}(X),$$

where, letting ρ_B act on the vector space on the left via the first factor in the tensor product, we have

$$\rho_B(\mathscr{H}^{i,j}(X)) = \mathscr{H}^{j,i}(X); \tag{2.2}$$

(vi) Writing

$$H_m^B(X(\boldsymbol{C})) = \mathrm{Hom}_{\boldsymbol{Q}}(H_B^m(X(\boldsymbol{C})), \boldsymbol{Q})$$

for the m-th homology group of $X(\boldsymbol{C})$ with coefficients in $\boldsymbol{Q}$, we have the integration pairing

$$\langle\ ,\ \rangle_X: H_{DR}^m(X)\times H_m^B(X(\boldsymbol{C})) \longrightarrow \boldsymbol{C} \tag{2.3}$$

given by $\langle \omega, \xi\rangle_X = \int_\xi \omega$. This pairing induces an isomorphism

$$\psi_\infty: H_{DR}^m(X)\otimes \boldsymbol{C} \xrightarrow{\sim} \mathrm{Hom}_{\boldsymbol{Q}}(H_m^B(X(\boldsymbol{C})), \boldsymbol{C}) = H_B^m(X(\boldsymbol{C}))\otimes \boldsymbol{C}.$$

Note that there is an action of $D_\infty = G(\boldsymbol{C}/\boldsymbol{R})$ on the vector spaces appearing here. Namely, the complex conjugation in D_∞ acts on $H_{DR}^m(X)\otimes \boldsymbol{C}$ via the trivial action on the first factor and its natural action on the second, and on $H_B^m(X(\boldsymbol{C}))\otimes \boldsymbol{C}$ via ρ_B in the first factor and its natural action on the second. Then ψ_∞ is an isomorphism of D_∞-modules.

Moreover, ψ_∞ preserves filtrations, in the sense that, for all $k \in \boldsymbol{Z}$, we have

$$\psi_\infty(F^k H_{DR}^m(X)\otimes \boldsymbol{C}) = \bigoplus_{i\geq k} \mathscr{H}^{i,j}(X). \tag{2.4}$$

Finally, we call the integer m the weight of this motive, and the dimension of the motive will be the common dimension of the vector spaces (2.1).

We shall simply define a homogeneous motive V over $\boldsymbol{Q}$ of weight $m(V) \in \boldsymbol{Z}$ and dimension $d(V)\geq 1$ to be an abstraction of the above. Thus V will be defined by specifying Betti, de Rham and l-adic realisations (for each prime l)

$$H_B(V), \quad H_{DR}(V), \quad H_l(V)$$

which are, respectively, vector spaces over $\boldsymbol{Q}$, $\boldsymbol{Q}$, and $\boldsymbol{Q}_l$, of common dimension $d(V)$, and which are endowed with the following structures:

(i) $H_B(V)$ admits an involution ρ_B;

(ii) The Galois group G has a natural action on $H_l(V)$;

(iii) For each prime l, there is an isomorphism

$$\psi_l: H_l(V) \xrightarrow{\sim} H_B(V)\otimes \boldsymbol{Q}_l,$$

which transforms the complex conjugation to ρ_B;

(iv) There is a decreasing filtration $\{F^k H_{DR}(V): k \in \boldsymbol{Z}\}$ on $H_{DR}(V)$;

(v) There is a Hodge decomposition into $\boldsymbol{C}$-vector spaces

$$H_B(V)\otimes \boldsymbol{C} = \bigoplus_{i+j=m(V)} \mathscr{H}^{i,j}(V),$$

where, letting ρ_B act on the vector space on the left via the first factor in the tensor product, we have

$$\rho_B(\mathscr{H}^{i,j}(V)) = \mathscr{H}^{j,i}(V); \tag{2.5}$$

(vi) Define the Betti homology $H^B(V)$ of V by

$$H^B(V) = \mathrm{Hom}\,(H_B(V), \boldsymbol{Q}). \tag{2.6}$$

Then there is a pairing

$$\langle\ ,\ \rangle_V: H_{DR}(V)\times H^B(V) \longrightarrow \boldsymbol{C},$$

which induces an isomorphism

$$\psi_\infty: H_{DR}(V)\otimes \boldsymbol{C} \xrightarrow{\sim} \mathrm{Hom}_{\boldsymbol{Q}}(H^B(V), \boldsymbol{C}) = H_B(V)\otimes \boldsymbol{C}.$$

Moreover, ψ_∞ is a $D_\infty = G(\boldsymbol{C}/\boldsymbol{R})$-homomorphism, where D_∞ acts on these spaces as in (vi) above. Finally, for all $k \in \boldsymbol{Z}$, we impose that

(2.7) $$\psi_\infty(F^k H_{DR}(V)\otimes C)=\bigoplus_{i\geq k}\mathscr{H}^{i,j}(V).$$

Example. *The Tate motive* $Q(n)$ $(n\in Z)$. This motive has weight $-2n$ and dimension 1. Its realisations are as follows. Let μ_{l^k} denote the group of l^k-th roots of unity in $\bar{Q}$, and put

$$V_l(\mu)=(\varprojlim \mu_{l^k})\otimes_{Z_l} Q_l.$$

Let $V_l(\mu)^{\otimes n}$ denote the n-th tensor power of $V_l(\mu)$ (as usual, if $n<0$, this means the $|n|$-th tensor power of $\mathrm{Hom}_{Q_l}(V_l(\mu), Q_l)$). Then

$$H_B(Q(n))=Q, \qquad \rho_B=(-1)^n,$$
$$H_{DR}(Q(n))=Q, \qquad H_l(Q(n))=V_l(\mu)^{\otimes n}.$$

The action of G on $H_l(Q(n))$ is the natural one. The Hodge decomposition of $H_B(Q(n))\otimes C$ is simply given by taking $\mathscr{H}^{-n,-n}(Q(n))=C$. The filtration of $H_{DR}(Q(n))$ is specified by letting $F^k H_{DR}(Q(n))$ be Q or 0, according as $k\leq -n$ or $k>-n$. The isomorphism ψ_l is given by

$$\psi_l(\eta_l^{\otimes n}\otimes\alpha)=1\otimes\alpha,$$

where $\eta_l^{\otimes n}$ is the n-th tensor of $\eta_l\in V_l(\mu)$ defined by $\eta_l=(\exp(2\pi i/l^k))\otimes 1$. Finally, the pairing $\langle\ ,\ \rangle_{Q(n)}$ is given by

(2.8) $$\langle\alpha,\beta\rangle_{Q(n)}=\alpha\beta(2\pi i)^{-n}.$$

Let V be any homogeneous motive over Q as above. Then we can construct the following motives from V:

(a) *The twists* $V(n)$ $(n\in Z)$. We define $V(n)=V\otimes Q(n)$. The realisations of $V(n)$ are simply the tensor products of the corresponding realisations of V and $Q(n)$.

(b) *The dual* $\check{V}$. The realisations of $\check{V}$ are simply the dual vector spaces of the realisations of V.

(c) *The Kummer dual* V^*. We define $V^*=\mathrm{Hom}(V, Q(-1))$. In other words, the realisations of V^* are the spaces of homomorphisms between the realisations of V and the realisations of $Q(-1)$.

Clearly, we have

(2.9) $$V^*=\check{V}(-1).$$

We note the decomposition

(2.10) $$H_B(V)=H_B(V)^+\oplus H_B(V)^-,$$

where ρ_B acts on plus space by $+1$, and on the minus space by -1. We define

$$d^\pm(V)=\dim_Q H_B(V)^\pm.$$

We impose henceforth the following condition on V. Put

(2.11) $$W(V)=\mathscr{H}^{m(V)/2,m(V)/2}(V),$$

when $m(V)$ is even.

Hypothesis M. If $m(V)$ is even and $W(V)\neq 0$, the involution ρ_B acts on $W(V)$ by a scalar $\eta(V)=\pm 1$.

If either $m(V)$ is odd or $W(V)=0$, $F^\pm H_{DR}(V)=F^k H_{DR}(V)$, where $k=1+[m(V)/2]$. Otherwise, let

(2.12) $$F^\alpha H_{DR}(V)=F^{(m(V)+1-\alpha\eta(V))/2}H_{DR}(V),$$

where α denotes either the sign $+$ or the sign $-$. Using (2.5) and (2.7), one verifies easily that

(2.13) $$d^\alpha(V)=\dim_Q(F^\alpha H_{DR}(V)).$$

We now recall the definitions of the periods attached to V. We have chosen a slightly different point of view to that in [3]. If u, v are complex numbers, we write $u\sim v$ if there exists a non-zero $\lambda\in Q$ such that $u=\lambda v$. Our definitions of periods are, as in [3], only up to the equivalence relation $\sim$, i.e. our periods are only well defined in $C^\times/Q^\times$. Suppose $A\subset H_{DR}(V)$ and $A'\subset H^B(V)=\mathrm{Hom}(H_B(V), Q)$ are Q-vector subspaces of the same dimension, say r. Let $\{a_i\}$ and $\{b_j\}$ be arbitrary Q-bases of A and A'. Define

$$\mathrm{disc}\langle A, A'\rangle_V=\det(\langle a_i, b_j\rangle_V) \qquad (1\leq i,j\leq r),$$

where $\langle\ ,\ \rangle_V$ is the bilinear form in axiom (vi) above. If $A=A'=0$, we define $\mathrm{disc}\langle A, A'\rangle=1$. Note that $\mathrm{disc}\langle A, A'\rangle_V$ is only defined up to $\sim$. As above, let α denote either $+$ or $-$. Write $H^B(V)^\alpha$ for the subspace of $H^B(V)$ on which ρ_B acts by 1 with the sign α attached to it. We now define

(2.14) $$\xi^\alpha(V)=\mathrm{disc}\langle F^\alpha H_{DR}(V), H^B(V)^\alpha\rangle_V,$$

(2.15) $$\rho(V)=\mathrm{disc}\langle H_{DR}(V), H^B(V)\rangle_V.$$

One verifies easily from our axioms that $\xi^\alpha(V)$ and $\rho(V)$ are both non-zero. Here are the relationship between these periods, and the periods $c^{\alpha}(V)$, $\delta(V)$ of Deligne [3]. We have

(2.16) $$\xi^{\alpha}(V) \sim c^{\alpha}(\check{V}), \qquad \rho(V) \sim \delta(\check{V}) \sim \delta(V)^{-1}$$

(see the remark at the end of § 1 on p. 320 of [3]). Moreover, formula 5.1.7 on p. 328 of [3] shows that

$$\delta(V) \sim c^{+}(V)/c^{-}(\check{V}).$$

From these results, we conclude that

(2.17) $$c^{\alpha}(V) \sim \xi^{-\alpha}(V)/\rho(V).$$

One also deduces from the definitions the following behaviour of the periods under twisting by roots of unity. For each $n \in \boldsymbol{Z}$, we have

(2.18) $$\xi^{\alpha(-)^n}(V(n)) = (2\pi i)^{-d^{\alpha}(V)n} \xi^{\alpha}(V),$$

(2.19) $$\rho(V(n)) = (2\pi i)^{-d(V)n} \rho(V).$$

We next recall the description, due to Serre [10], of the complex L-function attached to V, and its conjectural functional equation. We begin with the Euler factor at ∞. Put

$$\Gamma_{\boldsymbol{R}}(s) = \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\boldsymbol{C}}(s) = 2(2\pi)^{-s}\Gamma(s),$$
$$h(i,j) = \dim_{\boldsymbol{C}} \mathscr{H}^{i,j}(V).$$

If $m(V)$ is odd, put $r^{\alpha}(V)=0$. If $m(V)$ is even, recall that $W(V)=\mathscr{H}^{m(V)/2, m(V)/2}(V)$, and put

$$r^{\alpha}(V) = \dim_{\boldsymbol{C}}\{x \in W(V) : \rho_B(x) = \alpha(-1)^{m(V)/2}x\}.$$

In view of Hypothesis M, one of the two $r^{\alpha}(V)$ is equal to 0, and the other is equal to $h(m(V)/2, m(V)/2)$. The Euler factor at ∞ is then defined by

(2.20) $$L_{\infty}(V,s) = L_{\infty,\boldsymbol{R}}(V,s) L_{\infty,\boldsymbol{C}}(V,s),$$

where

(2.21) $$L_{\infty,\boldsymbol{C}}(V,s) = \prod_{i<j} \Gamma_{\boldsymbol{C}}(s-i)^{h(i,j)}$$

(2.22) $$L_{\infty,\boldsymbol{R}}(V,s) = \Gamma_{\boldsymbol{R}}(s-m(V)/2)^{r^{+}(V)} \Gamma_{\boldsymbol{R}}(s+1-m(V)/2)^{r^{-}(V)}.$$

Turning to the finite Euler factors, let r be a prime number, and write $D_r \supset I_r$ for the decomposition and inertia groups of some fixed prime of $\bar{\boldsymbol{Q}}$ lying above r. We write Frob_r for the element of D_r/I_r which operates on $\bar{\boldsymbol{F}}_r$ by $x \mapsto x^r$, i.e. Frob_r is the arithmetic Frobenius automorphism. For each prime $l \neq r$, we put

(2.23) $$L_r(V,X) = \det(1 - \mathrm{Frob}_r^{-1} \cdot X \mid H_l(V)^{I_r})^{-1}.$$

We impose on V the standard hypothesis that the polynomial $L_r(V,X)$ has coefficients in $\boldsymbol{Q}$ which are independent of the particular prime $l \neq r$. We then define

$$L(V,s) = \prod_r L_r(V, r^{-s})$$

$$\Lambda(V,s) = L_{\infty}(V,s) L(V,s).$$

Since Frob_r acts on $V_l(\mu)$ $(l \neq r)$ by raising to the r-th power, we plainly have

(2.24) $$L(V(n),s) = L(V, s+n) \qquad (n \in \boldsymbol{Z}).$$

To proceed further, we must impose additional standard hypothesis (see [10]). Namely, we assume the existence of a finite set of primes S such that, for each prime l, and each $r \notin S \cup \{l\}$, the inertia group I_r operates trivially on $H_l(V)$. In addition, for each $r \in S$, and each $l \neq r$, we assume that the conductor of $H_l(V)$ as a representation of the local Galois group D_r does not depend on l (see [10]). This enables us to define the global conductor $c(V)$ of V as the product of the local conductors at the primes $r \in S$. Finally, for each $r \notin S$, we assume that the reciprocal complex roots of the polynomial $L_r(V,X)$ have absolute value equal to $r^{m(V)/2}$. This last assumption shows that the Euler product $L(V,s)$ converges in the half plane $R(s) > 1 + m(V)/2$. Note that if V satisfies these conditions, then it is plain that its Kummer dual $V^* = \check{V}(-1)$ also satisfies them. The next conjecture is the standard one for the functional equation of $L(V,s)$, except that we use the Kummer dual V^* rather than the ordinary dual $\check{V}$.

Conjecture 2.1. *The function $\Lambda(V,s)$ has a meromorphic continuation over the whole complex plane, and satisfies the functional equation*

(2.25) $$\Lambda(V,s) = \varepsilon(V) c(V)^{(m(V)+1)/2-s} \Lambda(V^*, 2-s),$$

where $\varepsilon(V)$ is a complex number which is independent of s. Moreover, $L(V,s)$ is holomorphic unless both $m(V)$ is even and the motive $\boldsymbol{Q}(-m(V)/2)$

is a direct summand of V, in which case $L(V, s)$ may possibly have a pole at the point $s=1+m(V)/2$.

We next discuss the notion of the *critical points* of the motive V (see [3]).

Definition 2.2. An integer $s=n$ is said to be critical for V if both the infinite Euler factors $L_\infty(V, s)$ and $L_\infty(V^*, 2-s)$ are holomorphic at $s=n$.

The following characterization of critical values, which will be used in § 4, has been pointed out by Bloch.

Lemma 2.3. *Assume $n\in \mathbf{Z}$, and put $\alpha_n=+$ if n is odd, and $\alpha_n=-$ if n is even. Then $s=n$ is critical for V if and only if*

$$(2.26)\qquad \psi_\infty(F^{\alpha_n}H_{DR}(V)\otimes \mathbf{C})=\bigoplus_{i\geq n}\mathscr{H}^{i,j}(V),$$

where $F^{\alpha_n}H_{DR}(V)$ is given by (2.12).

Proof. We start by establishing the lemma for $n=1$ — this is the main part of the proof. We claim that the following are necessary and sufficient conditions for $s=1$ to be critical for V:

Case 1. Assume $m(V)>0$. Then (a) $h(i, m(V)-i)=0$ for $1\leq i< m(V)/2$, and (b) if $W(V)\neq 0$, we must have $\eta(V)=1$.

Case 2. Assume $m(V)\leq 0$. Then (a) $h(i, m(V)-i)=0$ for $m(V)/2 <i\leq 0$, and (b) if $W(V)\neq 0$, we have $\eta(V)=-1$.

To prove this, we note that $m(V^*)=2-m(V)$, and

$$h(i, j)=h_{V^*}(1-i, 1-j)\qquad (i+j=m(V)),$$

where, as indicated, the Hodge numbers on the right are those of V^*. Hence

$$(2.27)\qquad L_{\infty,\mathbf{C}}(V^*, 2-s)=\prod_{i<j}\Gamma_{\mathbf{C}}(1+m(V)-i-s)^{h(i,j)},$$

$$(2.28)\qquad L_{\infty,\mathbf{R}}(V^*, 2-s)=\Gamma_{\mathbf{R}}(1-s+m(V)/2)^{r^+(V)}\Gamma_{\mathbf{R}}(2-s+m(V)/2)^{r^-(V)}.$$

We then simply make a direct analysis of when $s=1$ is not a pole of the Γ-factors (2.21), (2.22), (2.27), and (2.28). In each case, part (a) above gives the necessary and sufficient conditions for $L_{\infty,\mathbf{C}}(V, s)$ and $L_{\infty,\mathbf{C}}(V^*, 2-s)$ to be holomorphic at $s=1$, and part (b) gives the necessary and sufficient conditions for $L_{\infty,\mathbf{R}}(V, s)$ and $L_{\infty,\mathbf{R}}(V^*, 2-s)$ to be holomorphic at $s=1$.

We next verify that the above provide necessary and sufficient conditions for (2.26) to hold when $n=1$. By virtue of (2.7), we have

$$\psi_\infty(F^+H_{DR}(V)\otimes \mathbf{C})=\bigoplus_{i\geq b(V)}\mathscr{H}^{i,m(V)-i}(V),$$

where we have put $b(V)=(m(V)+1-\eta(V))/2$, and $\eta(V)=0$ if $W(V)=0$ or $m(V)$ is odd. Assume first that $m(V)>0$, so that $b(V)\geq 1$. Hence necessary and sufficient conditions for (2.26) to hold for $n=1$ are given by

$$(2.29)\qquad \mathscr{H}^{i,m(V)-i}(V)=0\qquad \text{for } 1\leq i<b(V).$$

But (2.29) is plainly equivalent to the conditions (a) and (b) of Case 1 above. Next assume that $m(V)\leq 0$, so that $b(V)\leq 1$. Hence necessary and sufficient conditions for (2.26) to hold for $n=1$ are given by

$$(2.30)\qquad \mathscr{H}^{i,m(V)-i}(V)=0\qquad \text{for } b(V)\leq i\leq 0.$$

But again (2.30) is equivalent to conditions (a) and (b) of Case 2 above. This completes the proof for $n=1$.

Now suppose that $s=n$ is an arbitrary critical point for V. Plainly, $s=1$ is then a critical point for the motive $V(n-1)$. Moreover, we have $\eta(V(n-1))=\alpha_n\eta(V)$, and $m(V(n-1))=m(V)+2-2n$. It is then easy to see that the truth of (2.26) for $s=1$ and $V(n-1)$ is equivalent to the truth of (2.26) for $s=n$ and V. This completes the proof of Lemma 2.3.

Prior to discussing the algebraicity of $L(V, s)$ at critical points, we introduce a modification of the Euler factor at ∞, which seems particularly useful for questions of p-adic interpolation. Define

$$(2.31)\qquad \Phi_\infty(V, s)=L_{\infty,\mathbf{R}}(V^*, 2-s)\qquad \text{or } L_{\infty,\mathbf{R}}(V, s),$$

according as $m(V)\leq 0$ or $m(V)>0$ (of course, we interpret these Euler factors to be 1 if either $m(V)$ is odd, or $W(V)=0$). Since $m(V^*)=2-m(V)$, it is plain that $\Phi_\infty(V, s)$ satisfies the functional equation

$$(2.32)\qquad \Phi_\infty(V, s)=\Phi_\infty(V^*, 2-s).$$

Now define the modified Euler factor at ∞ by

$$(2.33)\qquad \tilde{L}_\infty(V, s)=L_\infty(V, s)/\Phi_\infty(V, s).$$

The next lemma is useful in studying algebraicity problems when one has several critical points.

Lemma 2.4. *Assume that $s=1$ and $s=n\in \boldsymbol{Z}$ are both critical points for V, and that $n\equiv 1 \bmod 2$. Then*

$$\tilde{L}_\infty(V,n)\sim(2\pi i)^{d^-(V)(1-n)}\tilde{L}_\infty(V,1). \tag{2.34}$$

Proof. We begin by settling the easy contribution from the complex Γ-factors. For $s>0$ in $\boldsymbol{Z}$, we have

$$\Gamma_C(s)\sim(2\pi)^{-s}.$$

Hence (2.21) implies that, for each critical interger $s=t$,

$$L_{\infty,C}(V,t)\sim(2\pi)^{\sum_{i<m(V)/2}(i-t)h(i,m(V)-i)}.$$

In particular, we conclude that

$$L_{\infty,C}(V,n)\sim(2\pi)^{(1-n)q(V)}L_{\infty,C}(V,1), \tag{2.35}$$

where we have put $q(V)=\sum_{i<m(V)/2}h(i,m(V)-i)$. Note that if no real Γ-factors actually occur for the motive V, then (2.34) just boils down to (2.35), since in this situation we have $d^+(V)=d^-(V)=q(V)$.

The whole delicacy of Lemma 2.4 lies in the behaviour of the real Γ-factors. Thus we suppose now that these actually occur, i.e. that $m(V)$ is even and precisely one of $r^+(V)$ and $r^-(V)$ is positive. If $s\in\boldsymbol{Z}$, recall that

$$\Gamma_R(s)\sim(2\pi)^{-s/2} \qquad \text{for } s \text{ even and } >0,$$
$$\Gamma_R(s)\sim(2\pi)^{(1-s)/2} \qquad \text{for } s \text{ odd}.$$

The argument now breaks up into two cases.

Case 1. Assume that $m(V)\leq 0$, so that $\Phi_\infty(V,s)=L_{\infty,R}(V^*,2-s)$. Suppose first that $r^+(V)>0$. As 1 is critical, it follows from (2.28) that $m(V)/2$ is odd. Hence $\eta(V)=-1$, and

$$d^-(V)=r^+(V)+q(V). \tag{2.36}$$

Now suppose that t is any odd integer, which is critical for V. We deduce from (2.22) that $t-m(V)/2$ is necessarily positive, because it is even. The above formulae then give

$$L_{\infty,R}(V,t)\sim(2\pi)^{r^+(V)(m(V)/4-t/2)},$$
$$L_{\infty,R}(V^*,2-t)\sim(2\pi)^{r^+(V)(t/2-m(V)/4)}.$$

Taking $t=1$ and $t=n$, and using (2.35) and (2.36), we deduce (2.34), as

desired. Suppose next that $r^-(V)>0$. As 1 is critical, it follows from (2.28) that $1+m(V)/2\leq 1$ is odd, whence $m(V)/2$ is even. Hence $\eta(V)=-1$, and we now have

$$d^-(V)=r^-(V)+q(V). \tag{2.37}$$

Again take t to be an odd integer, which is critical for V. We deduce from (2.22) that $t+1-m(V)/2$ is necessarily positive, because it is even, and so

$$L_{\infty,R}(V,t)\sim(2\pi)^{r^-(V)(m(V)/4-t/2-1/2)},$$
$$L_{\infty,R}(V^*,2-t)\sim(2\pi)^{r^-(V)(-m(V)/4+t/2-1/2)}.$$

Taking $t=1$ and $t=n$, and using (2.35) and (2.37), we again obtain (2.34).

Case 2. Assume now that $m(V)>0$, so that $\Phi_\infty(V,s)=L_{\infty,R}(V,s)$. Now we have already seen (cf. Case 1 in the proof of Lemma 2.3) that the fact that 1 is critical for V implies that $\eta(V)=1$, and so

$$d^-(V)=q(V). \tag{2.38}$$

But, in this case, $\tilde{L}_\infty(V,s)=L_{\infty,C}(V,s)$, and so (2.34) follows from (2.35) and (2.38). This completes the proof of Lemma 2.4.

We can now give the basic conjectures about the algebraic nature of the values of $L(V,s)$ at critical points. We begin with the conjecture of Deligne in [3].

Conjecture 2.5. *Assume $s=n$ is any critical point for V. Then*

$$L(V,n)/c^+(V(n))\in\boldsymbol{Q}.$$

As n ranges over the odd integers, the variation of the period $c^+(V(n))$ is given by (see p. 328 of [3])

$$c^+(V(n))=c^-(V)(2\pi i)^{nd^-(V)}. \tag{2.39}$$

This variation with n is not well adapted for questions of p-adic interpolation, and we use a different approach. We fix one point which is critical for V, which we take to be $s=1$ (if V admits a critical point, we can always twist V so that this point becomes $s=1$). We then define

$$\Omega(V)=\xi^+(V)\cdot\rho(V)^{-1}\cdot(2\pi i)^{d^-(V)}\tilde{L}_\infty(V,1), \tag{2.40}$$

where $\xi^+(V)$ and $\rho(V)$ are given by (2.14) and (2.15). Using (2.17) and (2.39), we see that an alternative expression for $\Omega(V)$ is given by

$$\Omega(V)=c^+(V(1))\cdot\tilde{L}_\infty(V,1).$$

Define

(2.41) $$\tilde{\Lambda}(V,s)=\tilde{L}_\infty(V,s)L(V,s)=L_\infty(V,s)L(V,s)/\Phi_\infty(V,s).$$

Conjecture 2.6. *Assume $s=1$ is critical for V, and let $\Omega(V)$ be given by* (2.40). *Then for all odd critical integers n, we have*

(2.42) $$\tilde{\Lambda}(V,n)/\Omega(V)\in\mathbf{Q}.$$

We hope that the advantage of Conjecture 2.6 is apparent. We can use the same period $\Omega(V)$ for all odd critical points of V. This is vital for formulating the principal conjecture of § 4.

Lemma 2.7. *Assume $s=1$ is a critical point for V. Then Conjecture* 2.6 *is equivalent to the validity of Conjecture* 2.5 *for all odd integers n which are critical for V.*

Proof. Since, as remarked above, we have $\Omega(V)=c^+(V(1))\cdot\tilde{L}_\infty(V,1)^{-1}$, the assertion of the lemma follows from (2.39) and Lemma 2.4.

We end this section by a remark on the compatibility of Conjecture 2.6 with the complex functional equation (2.25). In view of (2.32), we see that (2.25) is equivalent to

(2.43) $$\tilde{\Lambda}(V,s)=\varepsilon(V)\,c(V)^{(m(V)+1)/2-s}\tilde{\Lambda}(V^*,2-s).$$

Now suppose that n is an integer which is critical for V, whence $2-n$ is also critical for V. We deduce from (2.43) that

(2.44) $$\tilde{\Lambda}(V,n)=\varepsilon(V)c(V)^{(m(V)+1)/2-n}\tilde{\Lambda}(V^*,2-n).$$

Assuming $s=1$ is critical for V (and so also for V^*), this last equation strongly suggests, in view of Conjecture 2.6, that

(2.45) $$\Omega(V^*)\sim\Omega(V)/(\varepsilon(V)c(V)^{(m(V)-1)/2}).$$

Using the results of [3] (see Theorem 5.6, but note that this is contingent on the additional Conjecture 6.6), one can in fact derive (2.45) from the definition of $\Omega(V^*)$. We omit the details.

§ 3. Twisting by Dirichlet characters

In order to uniquely determine the distributions of § 4, it is essential to also consider the critical values of the twists of $L(V,s)$ by Dirichlet characters. It is a little long to discuss the full theory of twisting of motives by Dirichlet characters, and so we only give several key proofs, leaving other details to the reader.

Let χ be a Dirichlet character of $\mathbf{Q}$. Let $c(\chi)$ be the conductor of χ. We write χ also for the corresponding character of $G=G(\bar{\mathbf{Q}}/\mathbf{Q})$. This character is unramified outside $c(\chi)$, and satisfies $\chi(\mathrm{Frob}_r)=\chi(r)$ for all primes r with $(r,c(\chi))=1$. Let F be a fixed finite extension of $\mathbf{Q}$ containing all values of χ. We write $[\chi]$ for the 1-dimensional vector space over F on which G acts via χ. As is explained in § 6 of [3], we can identify $[\chi]$ with a motive over $\mathbf{Q}$ with coefficients in F, which we denote by the same symbol $[\chi]$. For each finite prime λ of F, let F_λ denote the completion of F at λ, and let $[\chi]_\lambda$ be the 1-dimensional vector space over F_λ on which G acts via χ. The λ-adic realisation of $[\chi]$ is then simply $[\chi]_\lambda$. We refer the reader to [3] for a description of the Betti and de Rham realisations of $[\chi]$. Suppose that V is a motive over $\mathbf{Q}$ as in § 2. We define $V(\chi)$ to be the motive over $\mathbf{Q}$, with coefficients in F, which is tensor product over F of the motives $V\otimes_{\mathbf{Q}}F$ and $[\chi]$-here tensor product has the obvious meaning in terms of realisations.

Throughout this section, we shall impose the following hypothesis on the Dirichlet characters χ which we consider:

Hypothesis C. The conductor of χ is prime to the conductor of V, and $\chi(-1)=1$.

Thanks to this hypothesis, it is easy to write down the L-function attached to $V(\chi)$, and its functional equation. Recall that the Gauss sum $G(\chi)$ is given by (1.7).

Lemma 3.1. *Under Hypothesis C, we have:*

(1) $L_\infty(V(\chi),s)=L_\infty(V,s)$;

(2) *For each finite prime r*, $L_r(V(\chi),X)=L_r(V,\chi^{-1}(r)X)$;

(3) $c(V(\chi))=c(V)\cdot c(\chi)^{d(V)}$;

(4) *For each prime l with* $(l,c(\chi))=1$,

(3.1) $$\frac{\varepsilon(V(\chi))}{\varepsilon(V)}=G(\chi^{-1})^{d(V)}\chi^{-1}(c(V))c(\chi)^{-d(V)(1+m(V))/2}\times\prod_{r\mid c(\chi)}\det\rho_l(\mathrm{Frob}_r^{-1})^{\mathrm{ord}_r(c(\chi))},$$

where $\rho_l\colon G\to\mathrm{Aut}(H_l(V))$ is the homomorphism giving the action of G on $H_l(V)$.

Remark. Our hypothesis that $(c(\chi),c(V))=1$ shows that ρ_l is unramified at all primes r dividing $c(\chi)$. Hence $\rho_l(\mathrm{Frob}_r^{-1})\in\mathrm{Aut}(H_l(V))$ is well defined, and is an element of $\boldsymbol{Q}$ which does not depend on l with $(l, c(\chi))=1$, because we have assumed the stronger conjecture that this is true for all coefficients of $L_r(V, X)$.

Proof. The first assertion is plain, since the fact that $\chi(-1)=1$ implies that the Betti realisation of $V(\chi)$ is the same as the Betti realisation of $V\otimes F$. For the remaining assertions, we use the λ-adic representations attached to $V(\chi)$, where throughout λ will denote a prime of F which does not divide the rational prime r. Now the λ-adic representation attached to $V(\chi)$ is $V_\lambda \otimes_{F_\lambda} [\chi]_\lambda$, where

$$V_\lambda = H_l(V) \underset{\boldsymbol{Q}_l}{\otimes} F_\lambda \qquad (\lambda \mid l). \tag{3.2}$$

By definition,

$$L_r(V(\chi), X) = \det(1 - \mathrm{Frob}_r^{-1}\cdot X \mid V_\lambda \underset{F_\lambda}{\otimes} [\chi]_\lambda),$$

whence (2) is clear since $\chi(\mathrm{Frob}_r^{-1}) = \chi^{-1}(r)$. To prove (3) and (4), we shall need the theory of Artin conductors and ε-factors attached to the λ-adic representations of the local Weil group $\Sigma_r \subset D_r = G(\bar{\boldsymbol{Q}}_r/\boldsymbol{Q}_r)$—see [9], § 4, for a very readable account of this theory. If U_λ is a λ-adic representation of Σ_r, we write U_λ^{ss} for its semi-simplification. The essential point (see [9], p. 22) is that we can view U_λ^{ss} as an ordinary complex representation of Σ_r, and so use the theory of Artin conductors and ε-factors attached to complex representations of Σ_r (however, we recall that it is yet another conjecture that the complex representation of Σ_r obtained in this manner is independent, up to isomorphism, of the choice of embedding of F_λ into $\boldsymbol{C}$). Clearly

$$(V_\lambda \underset{F_\lambda}{\otimes} [\chi]_\lambda)^{ss} = V_\lambda^{ss} \underset{F_\lambda}{\otimes} [\chi]_\lambda. \tag{3.3}$$

We now consider (3). If U is any complex representation of Σ_r, we follow [9] and write $a(U)$ for the exponent of its conductor. One then defines (see [9], p. 22) the exponent $a(U_\lambda)$ of the conductor of the λ-adic representation U_λ by the formula

$$a(U_\lambda) = a(U_\lambda^{ss}) + \dim_{F_\lambda}(U_\lambda^{ss})^{I_r} - \dim_{F_\lambda}(U_\lambda)^{I_r}, \tag{3.4}$$

where $I_r \subset \Sigma_r$ denotes the inertia subgroup. Writing Y_λ for the representation (3.3), we therefore have

$$a_r(V(\chi)) = a_r(Y_\lambda) + \dim_{F_\lambda}(Y_\lambda)^{I_r} - \dim_{F_\lambda}(V_\lambda \otimes_{F_\lambda} [\chi]_\lambda)^{I_r}. \tag{3.5}$$

The argument now breaks up into two cases. Suppose first that $(r, c(V))=1$. Thus V_λ is unramified, and so breaks up over $\boldsymbol{C}$ into a sum of $d(V)$ 1-dimensional representations. In particular, $V_\lambda^{ss} = V_\lambda$, whence the two terms on the extreme right of (3.5) cancel out. Also, as the Artin conductor is additive, it is plain that the remaining term on the right of (3.5) yields

$$a_r(V(\chi)) = d(V) a_r(\chi) \qquad (r, c(V))=1. \tag{3.6}$$

Suppose next that $(r, c(\chi))=1$. Hence $[\chi]_\lambda$ is unramified at r, and so the right hand side of (3.5) simplifies to

$$a_r(V_\lambda^{ss}) + \dim_{F_\lambda}(V_\lambda^{ss})^{I_r} - \dim_{F_\lambda}(V_\lambda)^{I_r} = a_r(V) \qquad (r, c(\chi))=1. \tag{3.7}$$

Since $(c(V), c(\chi))=1$, assertion (3) now follows from (3.6), (3.7), and the product decompositions

$$c(V(\chi)) = \prod_r r^{a_r(V(\chi))}, \qquad c(\chi) = \prod_r r^{a_r(\chi)}.$$

Now consider (4). Our normalisation of ε-factors is slightly different from that of [9], so we shall add a subscript T to denote the ε-factors as normalized in [9]. Also, we recall that Tate uses the inverse of the local reciprocity map used by us. It is easy to see that

$$\varepsilon_T(V(\chi)) = \varepsilon(V(\chi)) \cdot c(V(\chi))^{(m(V)+1)/2}.$$

Thus, in view of part (3), formula (3.1) is equivalent to

$$\varepsilon_T(V(\chi))/\varepsilon_T(V) = G(\chi^{-1})^{d(V)} \chi^{-1}(c(V)) \prod_{r \mid c(\chi)} \det \rho_l(\mathrm{Frob}_r^{-a_r(\chi)}). \tag{3.8}$$

We now calculate the left hand side of (3.8) by using the product formula

$$\varepsilon_T(V(\chi))/\varepsilon_T(V) = \prod_r (\varepsilon_{T,r}(V(\chi))/\varepsilon_{T,r}(V)). \tag{3.9}$$

Here we have already eliminated the contribution from the infinite prime, because our hypothesis that $\chi(-1)=1$ implies that it is 1. To calculate the local factor on the right, we appeal to its definition (see p. 22 of [9])

$$\varepsilon_{T,r}(V(\chi)) = \varepsilon_{T,r}(V_\lambda^{ss} \otimes [\chi]_\lambda) \frac{\det(-\varphi_r \mid (V_\lambda^{ss} \otimes [\chi]_\lambda)^{I_r})}{\det(-\varphi_r \mid (V_\lambda \otimes [\chi]_\lambda)^{I_r})}, \qquad (r, \lambda)=1, \tag{3.10}$$

where we have put $\varphi_r = \mathrm{Frob}_r^{-1}$. Suppose first that $(r, c(V))=1$, so that V_λ is unramified at r. Thus $V_\lambda^{ss} = V_\lambda$, and (3.10) yields

$$\varepsilon_{T,r}(V(\chi)) = \varepsilon_{T,r}(V_\lambda \otimes [\chi]_\lambda).$$

Recalling that Tate's local reciprocity map is the inverse of ours, formula (3.4.6) on p. 15 of [9] gives

$$(3.11)\qquad \varepsilon_{T,r}(V(\chi))=\varepsilon_{T,r}(\chi)^{d(V)}\det\rho_l(\mathrm{Frob}_r^{-a_r(\chi)}).$$

Note that $\varepsilon_{T,r}(V)=1$ in this case, because V_λ breaks up over C into a sum of unramified characters. Suppose next that $(r, c(\chi))=1$. Hence, if U_λ denotes either V_λ^{ss} or V_λ, we have

$$(U_\lambda\otimes[\chi]_\lambda)^{I_r}=(U_\lambda)^{I_r}\otimes[\chi]_\lambda.$$

Thus, in this case, the ratio on the extreme right of (3.10) is equal to

$$\chi(r)^{b_r(V_\lambda)}\cdot\det(-\varphi_r\,|\,(V_\lambda^{ss})^{I_r})/\det(-\varphi_r\,|\,V_\lambda^{I_r}),$$

where

$$b_r(V_\lambda)=\dim_{F_\lambda}(V_\lambda)^{I_r}-\dim_{F_\lambda}(V_\lambda^{ss})^{I_r}.$$

Note that, by the definition of the Artin conductor, we have

$$b_r(V_\lambda)=a_r(V_\lambda^{ss})-a_r(V_\lambda).$$

On the other hand, since χ is unramified at r, formula (3.4.6) on p. 15 of [9] gives

$$\varepsilon_{T,r}(V_\lambda^{ss}\otimes[\chi]_\lambda)=\varepsilon_{T,r}(V_\lambda^{ss})\chi(\mathrm{Frob}_r^{-a_r(V_\lambda^{ss})}).$$

Putting all these into (3.10), and making use of (3.10) for χ the trivial character, we obtain

$$(3.12)\qquad \varepsilon_{T,r}(V(\chi))=\varepsilon_{T,r}(V)\chi^{-1}(r^{a_r(V_\lambda)}).$$

Also $\varepsilon_{T,r}(\chi)=1$ because χ is unramified. Now, taking the product over all r, we obtain (3.8) from (3.9), (3.11), and (3.12), and the fact that

$$G(\chi^{-1})=\prod_r\varepsilon_{T,r}(\chi).$$

This completes the proof of Lemma 3.1.

We can now write down $L(V(\chi), s)$ and its conjectural functional equation. Part (1) of Lemma 3.1 shows that $L_\infty(V(\chi), s)=L_\infty(V, s)$, and part (2) shows that, in the notation of (1.6), we have

$$(3.13)\qquad L(V(\chi), s)=L(V, \chi^{-1}, s).$$

Put

$$(3.14)\qquad \Lambda(V(\chi), s)=L_\infty(V, s)L(V(\chi), s).$$

Finally, we note that we plainly have $(V(\chi))^*=V^*(\chi^{-1})$.

Conjecture 3.2. *Let χ be a Dirichlet character satisfying Hypothesis C. Then $L(V(\chi), s)$ has a meromorphic continuation over the whole complex plane, and satisfies*

$$(3.15)\qquad \Lambda(V(\chi), s)=\varepsilon(V(\chi))c(V(\chi))^{(m(V)+1)/2-s}\Lambda(V^*(\chi^{-1}), 2-s),$$

where $c(V(\chi))$ and $\varepsilon(V(\chi))$ are given by (3) *and* (4) *of Lemma* 3.1.

Note that an integer $s=n$ is critical for $V(\chi)$ if and only if it is critical for V, because the two L-functions have the same Euler factors at infinity. The analogue of Conjecture 2.6 is as follows. Put

$$(3.16)\qquad \tilde{\Lambda}(V(\chi), s)=L_\infty(V, s)L(V(\chi), s)/\Phi_\infty(V, s).$$

Conjecture 3.3. *Assume that $s=1$ is critical for V, and let $\Omega(V)$ be defined by* (2.39). *Then for all odd integers n which are critical for V, and all Dirichlet characters satisfying Hypothesis C, we have*

$$(3.17)\qquad \tilde{\Lambda}(V(\chi), n)\Omega(V)^{-1}G(\chi)^{d^-(V)}\in\bar{Q}.$$

Moreover, the effect of an automorphism σ in $G=G(\bar{Q}/Q)$ on the expression (3.17) *is to replace χ by χ^σ.*

Conjecture 3.3 is compatible with the functional equation (3.15) in the following sense. If we assume (3.15), and (2.45), then Conjecture 3.3 is true for V if and only if it is true for V^*. Moreover, Conjecture 3.3 is also what one would expect from the motivic theory, because

$$\Omega(V(\chi))\sim\Omega(V)G(\chi)^{-d^-(V)},$$

a fact which was pointed out to us by B. Gross.

§ 4. *p*-adic *L*-functions

We shall express our conjectures in terms of the existence of certain p-adic measures on $Z_p^\times$ (the reader interested in generalisations to motives over a number field should identify $Z_p^\times$, via the cyclotomic character, with the Galois group over Q of the maximum abelian extension of Q unramified outisde p and ∞). Our conjectures will depend on the hypothesis that V is ordinary at p, which we now explain. Let

$$(4.1)\qquad \phi_p\colon D_p=G(\bar{Q}_p/Q_p)\longrightarrow Z_p^\times$$

be the homomorphism given by the action of D_p on $V_p(\mu)$.

Definition 4.1. We say that V is ordinary at p if the following two conditions are satisfied:

(i) I_p acts trivially on $H_l(V)$ for all primes $l \neq p$;

(ii) There exists a filtration of $H_p(V)$

$$(4.2) \qquad W_1(V)=H_p(V)\supsetneqq W_2(V)\supsetneqq\cdots\supsetneqq W_{w+1}(V)=0$$

by Q_p-vector spaces which are stable under the action of D_p, and which are such that the inertia subgroup I_p operates on $W_i(V)/W_{i+1}(V)$ by some power of ϕ_p, say $\phi_p^{-e_i(V)}$ $(1\leq i\leq w)$. Moreover, these integers $e_i(V)$ satisfy

$$(4.3) \qquad e_1(V)\geq e_2(V)\geq\cdots\geq e_w(V).$$

We first note that, if V is ordinary at p, then V^* is also ordinary at p, and

$$(4.4) \qquad e_i(V^*)=1-e_{w+1-i}(V) \qquad (1\leq i\leq w).$$

Indeed, recalling that $V^*=\check{V}(-1)$, it is plain that a suitable filtration of $H_p(V^*)$ is given by

$$W_i(V^*)=X_{w+2-i}\underset{Q_p}{\otimes} V_p(\mu)^{\otimes(-1)}$$

where X_i denotes the orthogonal complement of $W_i(V)$ in the dual pairing of $H_p(V)$ and $H_p(\check{V})$ into Q_p. Since

$$W_i(V^*)/W_{i+1}(V^*) \text{ is dual to } (W_{w+1-i}(V)/W_{w+2-i}(V))\underset{Q_p}{\otimes} V_p(\mu),$$

the equation (4.4) is clear.

Roughly speaking, if V is ordinary at p, we expect a close connexion between the Galois module $H_p(V)$ and the Hodge decomposition of $H_B(V)\otimes C$. Here is the precise conjecture we will need. Assuming V is ordinary at p, condition (i) ensures that the p-Euler factor $L_p(V, X)$ has exact degree $d(V)$. Hence, factorising it in $\bar{Q}_p$, we have

$$(4.5) \qquad L_p(V, X)^{-1}=(1-\alpha_1X)\cdots(1-\alpha_{d(V)}X) \qquad (\alpha_i\in\bar{Q}_p),$$

where none of the α_i is equal to 0. Let ord_p denote the order valuation of $\bar{Q}_p$, normalized so that $\operatorname{ord}_p(p)=1$. We suppose that we have chosen our indices so that

$$(4.6) \qquad \operatorname{ord}_p(\alpha_1)\geq\operatorname{ord}_p(\alpha_2)\geq\cdots\geq\operatorname{ord}_p(\alpha_{d(V)}).$$

Finally, we recall the definition of the complex Hodge numbers $h(i, m(V)-i)=\dim_C\mathscr{H}^{i,m(V)-i}(V)$. Also, put $e_i=e_i(V)$.

Conjecture 4.2. *Assume that V is ordinary at p. Then the sequence of p-adic orders (4.6) consists of e_1 repeated $h(e_1, m(V)-e_1)$ times, followed by e_2 repeated $h(e_2, m(V)-e_2)$ times, $\cdots$, followed finally by e_w repeated $h(e_w, m(V)-e_w)$ times.*

We take this as a conjecture because we do not know in what generality it has been proven by the algebraic geometers (see Bloch-Kato [1]).

We shall impose the following hypothesis on V for the rest of this section:

Hypothesis O. V is ordinary at p, and satisfies Conjecture 4.2. In addition, $s=1$ is critical for V.

We first note that the Kummer dual V^* then also satisfies Hypothesis O. The only point that is not immediate is that V^* satisfies Conjecture 4.2. Suppose that

$$(4.7) \qquad L_p(V^*, X)^{-1}=(1-\alpha_1^*X)\cdots(1-\alpha_{d(V)}^*X) \qquad (\alpha_i^*\in\bar{Q}_p).$$

Since $H_l(V^*)=H_l(\check{V})\otimes V_l(\mu)^{\otimes(-1)}$, we have

$$(4.8) \qquad \alpha_i^*=p/\alpha_{d(V)+1-i} \qquad (1\leq i\leq d(V)),$$

whence the desired assertion follows from Conjecture 4.2 for V, in view of (4.4) and the fact that $h_{V^*}(i, j)=h_V(1-i, 1-j)$.

Recall that C_p denotes the completion of $\bar{Q}_p$, and that $\bar{Q}$ was taken to be the algebraic closure of Q in C. Even though we shall not indicate it explicitly in what follows, we fix once and for all an embedding

$$j: \bar{Q}\hookrightarrow C_p.$$

We also suppose that the roots α_i, α_i^* $(1\leq i\leq d(V))$ appearing in (4.5) and (4.7) now lie in $\bar{Q}$, by identifying them with their images under j.

We now define

$$(4.9) \qquad J=\{i: 1\leq i\leq d(V),\ \operatorname{ord}_p(\alpha_i)\geq 1\},$$

$$(4.10) \qquad J^*=\{i: 1\leq i\leq d(V),\ \operatorname{ord}_p(\alpha_i^*)\geq 1\}.$$

In view of (4.8), we have

$$d(V)=\#(J)+\#(J^*).$$

The next lemma is crucial for the conjectures which follow.

Lemma 4.3. *We have*

$$\#(J)=d^+(V), \qquad \#(J^*)=d^-(V).$$

Proof. By Conjecture 4.2, we have

$$\#(J)=\sum_{k\geq 1} h(k, m(V)-k).$$

But, since $s=1$ is critical for V, Lemma 2.3 shows that

$$\psi_\infty(F^+H_{DR}(V)\otimes C)=\bigoplus_{k\geq 1} \mathscr{H}^{k,j}(V).$$

Equating C-dimensions, we obtain

$$d^+(V)=\sum_{k\geq 1} h(k, m(V)-k),$$

and so the first assertion follows. The second assertion is then a consequence of the remark made just before Lemma 4.3.

Let χ be a Dirichlet character satisfying Hypothesis C. The desire to find one period which gives the transcendental part of all odd critical values of $L(V(\chi), s)$ led us earlier to modify the Euler factor at infinity, and introduce the new function $\tilde{\Lambda}(V(\chi), s)$. The desire to p-adically interpolate these critical values now forces a similar modification of the Euler factor at p. The analogy between these two modifications of Euler factors (especially the fact that the modifying factors satisfy the functional equation) seems to us rather striking. Following a suggestion of R. Greenberg, we define

$$\Phi_p(V(\chi), s)=\prod_{i\in J}(1-\alpha_i\chi^{-1}(p)p^{-s})^{-1}\prod_{i\in J^*}(1-\alpha_i^*\chi(p)p^{s-2})^{-1}. \tag{4.11}$$

Note that we plainly have the functional equation

$$\Phi_p(V(\chi), s)=\Phi_p(V^*(\chi^{-1}), 2-s). \tag{4.12}$$

Key Definition. Put

$$\tilde{\Lambda}_p(V(\chi), s)=\frac{L_\infty(V, s)L(V(\chi), s)}{\Phi_\infty(V, s)\Phi_p(V(\chi), s)}. \tag{4.13}$$

Note that, in view of (2.32) and (4.12), the function $\tilde{\Lambda}_p(V(\chi), s)$ satisfies (conjecturally) exactly the the same functional equation (3.15) as $\Lambda(V(\chi), s)$. Also (3.17) will hold for $\tilde{\Lambda}_p(V(\chi), n)$ in place of $\tilde{\Lambda}(V(\chi), n)$ if we assume Conjecture 3.3.

We can at last state the principal conjecture of this paper about the existence of canonical p-adic measures attached to motives. Recall that a measure on $Z_p^\times$ with values in C_p is simply a distribution on $Z_p^\times$ which is p-adically bounded (see for example, [11]; but note that we allow our measures to have values in C_p which are bounded, rather than necessarily integral). Of course, we can integrate any continuous function on $Z_p^\times$ against such a measure. We say such a measure μ is *even* if the integral against it of every Dirichlet character χ, of p-power conductor, with $\chi(-1)=-1$, is equal to 0.

Definition. Let M denote the set of all Dirichlet characters χ of p-power conductor with $\chi(-1)=1$. For $\chi\in M$, we let the conductor $c(\chi)$ be given by $c(\chi)=p^{r_\chi}$.

Finally, we briefly recall the main hypotheses we impose on V.

Assumptions on V. V satisfies Conjectures 3.2 and 3.3 for χ ranging over M. In addition, V is ordinary at p, and satisfies Conjecture 4.2.

We put

$$q(V)=\sum_{i<j} h(i, j), \qquad I(V)=\{i: \mathrm{ord}_p(\alpha_i)\leq 0\}.$$

Principal Conjecture. *In addition to the above assumptions on V, suppose that* (i) $s=1$ *is a critical point for* V, *and* (ii) $Q(-m(V)/2)$ *is not a direct summand of V when $m(V)$ is even. Then, for each choice of $\Omega(V)$ given by* (2.40), *there exists a unique even measure $\mu_{V,\Omega(V)}$ on $Z_p^\times$, satisfying*

$$\int_{Z_p^\times}\chi x^{n-1}d\mu_{V,\Omega(V)}=\frac{\Lambda(V(\chi), n)}{\Phi_\infty(V, n)\Phi_p(V(\chi), n)}\times\frac{G(\chi)^{d^-(V)}}{\Omega(V)} \times(-1)^{((1-n)/2)q(V)}\times\prod_{i\in I(V)}\left(\frac{p^{n-1}}{\alpha_i}\right)^{r_\chi}, \tag{4.14}$$

for all odd integers n which are critical for V, and all χ in M.

Remark. If $m(V)$ is even and $V=Q(-m(V)/2)$, the work of Kubota-Leopoldt-Iwasawa, which began the whole theory of p-adic L-functions, asserts that $\mu_{V,\Omega}$ is a pseudo-measure in the sense of Serre [11].

We first discuss the behaviour of the above conjecture under twisting by roots of unity. For each motive V satisfying the above bypotheses, let $\mu_{V,\Omega(V)}$ be the unique even distribution on $Z_p^\times$ satisfying (4.14) for $n=1$ and all $\chi\in M$. Suppose n is an odd integer which is also critical for V. Thus $s=1$ will also be critical for the motive $V(n-1)$. In view of Conjecture 3.3, it then makes sense to form the distribution $\mu_{V(n-1),\Omega(V)}$.

Lemma 4.4. *Assume $s=1$ and $s=n$ are both critical for V, where n is odd. Then, for all χ in M,*

$$\Phi_\infty(V(n-1),\,1)=\Phi_\infty(V,n),\qquad \Phi_p(V(n-1)(\chi),\,1)=\Phi_p(V(\chi),n). \tag{4.15}$$

Moreover, assuming the principal conjecture, we have

$$\mu_{V(n-1),\Omega(V)}=x^{n-1}(-1)^{((n-1)/2)q(V)}\mu_{V,\Omega(V)}. \tag{4.16}$$

Proof Put $U=V(n-1)$, so that $m(U)=m(V)-2n+2$. We begin by proving the first assertion of (4.15). We can assume that $W(V)\neq 0$, since otherwise the statement is trivial. If $m(V)\leq 0$, we must have $\eta(V)=-1$, because $s=1$ is critical. But then the holomorphy of $L_{\infty,R}(V^*,2-s)$ at $s=n$ implies that $n\geq m(V)/2+1$ (cf. Case 1 in the proof of Lemma 2.4). If $m(V)>0$, we have $\eta(V)=1$, because $s=1$ is critical. In this case, the holomorphy of $L_{\infty,R}(V^*,2-s)$ at $s=n$ implies that $n<m(V)/2+1$. The first assertion of (4.15) is now plain since, by definition, $\Phi_\infty(U,1)$ is equal to $L_{\infty,R}(V^*,2-n)$ or $L_{\infty,R}(V,n)$, according as $n\geq m(V)/2+1$ or $n<m(V)/2+1$.

We now turn to the second assertion of (4.15). Clearly, the inverse roots of (4.5) for U (resp. U^*) are given by

$$\alpha_i(U)=\alpha_i p^{1-n},\qquad \alpha_i^*(U)=\alpha_i^* p^{1-n}.$$

Write $J(U)$, $J^*(U)$ for the sets of indices given by (4.9) and (4.10) for the motive U instead of V. The key observation is that we then have $J(U)=J$, $J^*(U)=J^*$. This is immediate from Lemma 2.3 and Conjecture 4.2, since both n and $2-n$ are odd, and critical for V and V^* respectively. This is precisely the point in the p-adic theory where Lemma 2.3 is vital. The second assertion of (4.15) now follows.

By (4.16), we mean that the integral of any continuous function $f(x)$ on $\boldsymbol{Z}_p^\times$ against $\mu_{U,\Omega(V)}$ is equal to the integral of $(-1)^{((n-1)/2)q(V)}x^{n-1}$ against $\mu_{V,\Omega(V)}$. To prove this, we note that in view of (4.15) and the above remarks about the sets $J(U)$, $J^*(U)$, the integral of $\chi\in M$ against $\mu_{U,\Omega(V)}$ is indeed the expression on the right hand side of (4.14). This completes the proof.

Next, we wish to point out (although we do not see how to exploit this fact at present) that the distributions discussed above are essentially the same as those constructed by elementary p-adic harmonic analysis in § 1. Given a distribution μ, we let μ^e be the unique even distribution given by

$$\int_{\boldsymbol{Z}_p^\times}\chi d\mu^e=\frac{1}{2}(\chi(-1)+1)\int_{\boldsymbol{Z}_p^\times}\chi d\mu$$

for all Dirichlet characters χ of p-power conductor.

Lemma 4.5. *Assume $J^*\neq\phi$. Then, for each odd integer n which is critical for V, we have*

$$\mu_{V(n-1),\Omega(V)}=\Omega(V)^{-1}\tilde{L}_\infty(V,n)\rho_{L(V,s)}(K,n)^e, \tag{4.17}$$

where K is the subset of the inverse roots of $L_p(V,X)^{-1}$ given by $K=\{\alpha_i\colon \operatorname{ord}_p(\alpha_i)\leq 0\}$.

Remark. If $J^*=\phi$, then we must replace V by V^* to obtain a suitable version of Lemma 4.5.

As for the proof of Lemma 4.5, we simply remark that Theorem 1.5 shows that, for each $\chi\in M$, the integral of χ against the two distributions in (4.17) is the same, being equal to the right hand side of (4.14).

We next discuss the compatibility of the principal conjecture with the complex functional equation (3.15). Put

$$U(V(\chi),s)=\prod_{i\in J^*}(\alpha_i^* p^{s-2})^{r_\chi}.$$

Lemma 4.6. *For each prime $l\neq p$, we have*

$$U(V(\chi),s)\det\rho_l(\mathrm{Frob}_p^{-r_\chi})=U(V^*(\chi^{-1}),2-s)c(\chi)^{sd(V)-d^-(V)}. \tag{4.18}$$

Proof. From the definition (2.23) of $L_p(V,X)$, we have

$$\det\rho_l(\mathrm{Frob}_p^{-1})=\prod_{i=1}^{d(V)}\alpha_i.$$

Using this expression, (4.8), and the fact that $\#(J^*)=d^-(V)$, we conclude that the left hand side of (4.18) is equal to

$$c(\chi)^{(s-1)d^-(V)}\prod_{i\in J}\alpha_i^{r_\chi}.$$

Since $\#(J)=d^+(V)$, this last expression is equal to the right hand side of (4.18), as required.

We have already remarked at the end of § 2 that $\Omega(V^*)$ is given by (2.45). Let ξ denote the non-zero rational number such that

$$\Omega(V^*)=\xi\Omega(V)/(\varepsilon(V)c(V)^{(m(V)-1)/2}). \tag{4.19}$$

Theorem 4.7. *The principal conjecture holds for V if and only if it holds for V^*. Assuming it does hold, we have*

$$\int_{\boldsymbol{Z}_p^\times}\chi x^{n-1}d\mu_{V,\Omega(V)}=\xi^{-1}c(V)^{1-n}\chi^{-1}(c(V))\int_{\boldsymbol{Z}_p^\times}\chi^{-1}x^{1-n}d\mu_{V^*,\Omega(V^*)}, \tag{4.20}$$

for all odd integers n which are critical for V, and all $\chi \in M$.

Proof. In view of the proof of Lemma 4.4, it suffices to show that

$$(4.21)\qquad \int_{\mathbf{Z}_p^\times} \chi d\mu_{V(n-1),\Omega(V)} = \xi^{-1} c(V)^{1-n} \chi^{-1}(c(V)) \int_{\mathbf{Z}_p^\times} \chi^{-1} d\mu_{V^*(1-n),\Omega(V^*)},$$

for all odd integers n which are critical for V, and all $\chi \in M$. By the proof of Lemma, 4.4, the integral on the left of (4.21) is equal to the expression on the right of (4.14). Since $\tilde{\Lambda}_p(V(\chi), s)$ satisfies the functional equation (3.15), we can rewrite the right hand side of (4.14) as

$$\Lambda_p(V^*(\chi^{-1}), 2-n)\varepsilon(V(\chi))c(V(\chi))^{(m(V)+1)/2-n} G(\chi)^{d-(V)} \Omega(V)^{-1} \prod_{i \in J^*} (\alpha_i^* p^{n-2})^{r_\chi}.$$

Now use (3) and (4) of Lemma 3.1, Lemma 4.6, and the fact that $G(\chi)G(\chi^{-1}) = c(\chi)$ because $\chi(-1)=1$, to deduce that this last expression simplifies to

$$\Lambda_p(V^*(\chi^{-1}), 2-n) G(\chi^{-1})^{d-(V^*)} \prod_{i \in J} (\alpha_i p^{-n})^{r_\chi} \Omega(V^*)^{-1} \xi^{-1} c(V)^{1-n} \chi^{-1}(c(V)),$$

which, again by the proof of Lemma 4.4, is equal to the right hand side of (4.21). This completes the proof of Theorem 4.7.

It is not our aim in this paper to discuss the conjectural relationship between these measures and algebraic Iwasawa theory (for a discussion of this latter theory for motives, see the papers by Greenberg and Schneider in this volume). However, we do want to mention one consequence of our principal conjecture, which turns out to be highly interesting for algebraic Iwasawa theory, as well as the analytic theory. We say that an odd integer n, which is a critical point for V, is a *trivial zero* of $\mu_{V,\Omega(V)}$ if the Euler factor

$$\Phi_p(V, s)^{-1} = \prod_{i \in J} (1 - \alpha_i p^{-s}) \prod_{i \in J^*} (1 - \alpha_i^* p^{s-2})$$

vanishes at $s=n$, i.e. either there exists a root α_i with $\mathrm{ord}_p(\alpha_i) \geq 1$ and $\alpha_i = p^n$, or there exists a root α_i^* with $\mathrm{ord}_p(\alpha_i^*) \geq 1$ and $\alpha_i^* = p^{2-n}$. If $s=n$ is a trivial zero, then the principal conjecture gives

$$\int_{\mathbf{Z}_p^\times} x^{n-1} d\mu_{V,\Omega(V)} = 0.$$

Finally, we mention that there is a more intrinsic way of writing our measures, which avoids a particular choice of $\Omega(V)$. Define

$$\mu_V = \Omega(V) \cdot \mu_{V,\Omega(V)},$$

and view μ_V as a distribution on $\mathbf{Z}_p^\times$, with values in the space $W = \Omega(V) \cdot \overline{\mathbf{Q}}$, which conjecturally does not depend on the choice of $\Omega(V)$. As W is itself not p-adically complete, we identify W with its image in $W \otimes_{\overline{\mathbf{Q}}} \mathbf{C}_p$ under the map $x \to x \otimes 1$, and view μ_V as a distribution on $\mathbf{Z}_p^\times$ with values in $W \otimes_{\overline{\mathbf{Q}}} \mathbf{C}_p$. It is plain that one can then formulate the principal conjecture, as well as the results of this section, in terms of μ_V.

§ 5. Examples

Our knowledge of both the complex and p-adic conjectures discussed in this paper is still very limited. The basic source of accessible examples is either the cylotomic theory, or modular forms and their symmetric powers.

We begin by recalling the very well known situation for the cyclotomic theory. Take $V = \mathbf{Q}(w)$, where w is any non-zero integer. Then $d(V) = 1$ and $m(V) = -2w$. We have

$$L_\infty(V, s) = \Gamma_{\mathbf{R}}(s+w), \qquad L(V, s) = \zeta(s+w).$$

Also, $V^* = \mathbf{Q}(-1-w)$. The point $s=1$ is critical for V if and only if either w is an odd integer ≥ 1, or w is an even integer <0. Conjectures 3.2 and 3.3 are valid. Every prime p is ordinary for V, and Conjecture 4.2 is true. Moreover, the work of Leopoldt-Kubota-Iwasawa establishes the slightly weaker form of the principal conjecture, in which $\mu_{V,\Omega(V)}$ is a pseudo-measure in the sense of [11]. This result is best possible in this case.

We next turn to modular forms. Let $f = \sum_{n=1}^\infty a_n q^n$, $q = e^{2\pi i z}$, be a primitive cusp form of weight $k \geq 2$ for $\Gamma_0(N)$, where N is any integer ≥ 1. For simplicity, assume that f has a trivial character. Suppose also that $a_1 = 1$, and that $a_n \in \mathbf{Q}$ $(n = 1, 2, \cdots)$. Then f defines a motive $V(f)$ over $\mathbf{Q}$ of dimension 2 and weight $k-1$ (see § 7 of [3]). The Hodge decomposition of $H_B(V(f)) \otimes \mathbf{C}$ consists of two 1-dimensional subspaces of type $(0, k-1)$ and $(k-1, 0)$. For each prime l, $H_l(V(f))$ is the l-adic representation of $G = G(\overline{\mathbf{Q}}/\mathbf{Q})$ attached to f by Deligne. By a theorem of Carayol, we have

$$L_\infty(V(f), s) = \Gamma_{\mathbf{C}}(s), \qquad L(V)(f), s) = \sum_{n=1}^\infty \frac{a_n}{n^s}.$$

Moreover, $V(f)^* = V(f)(k-2)$. It follows that the critical values for $V(f)$ are the integers $s = 1, \cdots, k-1$. Conjectures 3.2 and 3.3 are both known. The prime p is ordinary for $V(f)$ if and only if $(p, N) = 1$ and $a_p \not\equiv 0 \bmod p$, and Conjecture 4.2 is valid. Moreover, the ideas initially found by Mazur and Swinnerton-Dyer have yielded a proof of the principal conjecture in this case (see [5]).

Now let h be an integer ≥ 2, and let $\mathrm{Sym}^h(V(f))$ be the motive over $\boldsymbol{Q}$ whose realisations are the h-th symmetric powers of the realisations of $V(f)$. There would be great interest in establishing the complex and p-adic conjectures of this paper for $\mathrm{Sym}^h(V(f))$. If $h=2$, the complex conjectures were settled by the work of Jacquet-Gelbart [4], and recent work of Schmidt [8] proves most of the principal conjecture for the p-adic theory. If $h=3$, we understand that the complex theory has now been proven, and there would be much interest in now attachking the principal conjecture in this case (see [6]).

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Anderson-Ihara Theory: Gauss Sums and Circular Units

R. F. Coleman

Thrice the brinded cat hath mewed,
Thrice and once the hedge pig whined,
Harpier cries. 'tis time, 'tis time.

The Three Witches
Macbeth, Act IV Scene

Dedicated to Iwasawa on the occasion of his seventieth birthday

A few years ago, Ihara, [I], discovered a new sort of power series connected with the action of G_Q on the Tate-modules of Fermat curves of l-power degree. Since then Anderson, [A], refined and generalized these power series, interpreting them as analogues of the classical beta function. Moreover, once this analogy was made he naturally was forced to factor them into a product of three "gamma functions."

The previous paragraph is purposely vague and oversimplified. In this article I will attempt to make some of it a little less vague and indicate how the theory of these "gamma" and "beta" functions may be connected with and applied to other aspects of cyclotomy.

I. Ihara's "Beta" series

Let X_n denote the projective plane curve over $\boldsymbol{Q}$ determined by the homogeneous equation:

$$X^{l^n}+Y^{l^n}+Z^{l^n}=0.$$

Let J_n denote the Jacobian of X_n. We have natural maps, $X_{n+1}\to X_n$ and corresponding maps on the Jacobians. Hence we may define the G_Q-module

$$T=\varprojlim T_l(J_n(\bar{Q})).$$

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