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Classical group

In mathematics, the **classical groups** are defined as the special linear groups over the reals **R**, the complex numbers **C** and the quaternions **H** together with special[1] automorphism groups of symmetric or skew-symmetric bilinear forms and Hermitian or skew-Hermitian sesquilinear forms defined on real, complex and quaternionic finite-dimensional vector spaces.[2] Of these, the **complex classical Lie groups** are four infinite families of Lie groups that together with the exceptional groups exhaust the classification of simple Lie groups. The **compact classical groups** are compact real forms of the complex classical groups. The finite analogues of the classical groups are the **classical groups of Lie type**. The term "classical group" was coined by Hermann Weyl, it being the title of his 1939 monograph *The Classical Groups*.[3]

The classical groups form the deepest and most useful part of the subject of linear Lie groups.[4] Most types of classical groups find application in classical and modern physics. A few examples are the following. The rotation group SO(3) is a symmetry of Euclidean space and all fundamental laws of physics, the Lorentz group O(3,1) is a symmetry group of spacetime of special relativity. The special unitary group SU(3) is the symmetry group of quantum chromodynamics and the symplectic group Sp(*m*) finds application in hamiltonian mechanics and quantum mechanical versions of it.

Contents

The classical groups

The **classical groups** are exactly the general linear groups over **R**, **C** and **H** together with the automorphism groups of non-degenerate forms discussed below.[5] These groups are usually additionally restricted to the subgroups whose elements have determinant 1, so that their centers are discrete. The classical groups, with the determinant 1 condition, are listed in the table below. In the sequel, the determinant 1 condition is *not* used consistently in the interest of greater generality.

The **complex classical groups** are SL(*n*, **C**), SO(*n*, **C**) and Sp(*m*, **C**). A group is complex according to whether its Lie algebra is complex. The **real classical groups** refers to all of the classical groups since any Lie algebra is a real algebra. The **compact classical groups** are the compact real forms of the complex classical groups. These are, in turn, SU(*n*), SO(*n*) and Sp(*m*). One characterization of the compact real form is in terms of the Lie algebra **g**. If **g** = **u** + *i***u**, the complexification of **u**, then if the connected group *K* generated by exp(*X*): *X* ∈ **u** is a compact, *K* is a compact real form.[6]

Name	Group	Field	Form	Maximal compact subgroup	Lie algebra	Root system
Special linear	SL(*n*, **R**)	**R**	-	SO(*n*)		
Complex special linear	SL(*n*, **C**)	**C**	-	**SU(*n*)**	Complex	A_{n-1}
Quaternionic special linear	SL(*n*, **H**) = SU*(2*n*)	**H**	-	Sp(*n*)		
(Indefinite) special orthogonal	SO(*p*, *q*)	**R**	Symmetric	S(O(*p*) × O(*q*))		
Complex special orthogonal	SO(*n*, **C**)	**C**	Symmetric	**SO(*n*)**	Complex	$\begin{cases} B_m, & n=2m+1 \\ D_n, & n=2m \end{cases}$
Symplectic	Sp(*m*, **R**)	**R**	Skew-symmetric	U(*m*)		
Complex symplectic	Sp(*m*, **C**)	**C**	Skew-symmetric	**Sp(*m*)**	Complex	C_n
(Indefinite) special unitary	SU(*p*, *q*)	**C**	Hermitian	S(U(*p*) × U(*q*))		
(Indefinite) quaternionic unitary	Sp(*p*, *q*)	**H**	Hermitian	Sp(*p*) × Sp(*q*)		
Quaternionic orthogonal	SO*(2*n*)	**H**	Skew-Hermitian	SO(2*n*)		

The classical groups can uniformly be characterized in a different way using real forms. The classical groups (here with the determinant 1 condition, but this is not necessary) are the following:

> The complex linear algebraic groups SL(*n*, **C**), SO(*n*, **C**), and Sp(*n*, **C**) together with their real forms.[7]

For instance, SO*(2*n*) is a real form of SO(2*n*, **C**), SU(*p*, *q*) is a real form of Sl(*n*, **C**), and Sl(*n*, **H**) is a real form of SO(2*n*, **C**). Without the determinant 1 condition, replace the special linear groups with the corresponding general linear groups in the characterization. The algebraic groups in question are Lie groups, but the "algebraic" qualifier is needed to get the right notion of "real form".

Bilinear and sesquilinear forms

The classical groups are defined in terms of forms defined on $\mathbf{R}^n$, $\mathbf{C}^n$, and $\mathbf{H}^n$, where **R** and **C** are the fields of the real and complex numbers. The quaternions, **H**, do not constitute a field because multiplication does not commute; they form a division ring or a **skew field** or **non-commutative field**. However, it is still possible to define matrix quaternionic groups. For this reason, a vector space *V* is allowed to be defined over

R, **C**, as well as **H** below. In the case of **H**, V is a *right* vector space to make possible the representation of the group action as matrix multiplication from the *left*, just as for **R** and **C**.[8]

A form φ: $V \times V \to F$ on some finite-dimensional right vector space over F = **R**, **C**, or **H** is bilinear if

$$\varphi(x\alpha, y\beta) = \alpha\varphi(x, y)\beta, \quad \forall x, y \in V, \forall\alpha, \beta \in F.$$

It is called **sesquilinear** if

$$\varphi(x\alpha, y\beta) = \bar{\alpha}\varphi(x, y)\beta, \quad \forall x, y \in V, \forall\alpha, \beta \in F.$$

These conventions are chosen because they work in all cases considered. An automorphism of φ is a map A in the set of linear operators on V such that

$$\varphi(Ax, Ay) = \varphi(x, y), \quad \forall x, y \in V. \qquad (1)$$

The set of all automorphisms of φ form a group, it is called the automorphism group of φ, denoted Aut(φ). This leads to a preliminary definition of a classical group:

> ***A classical group is a group that preserves a bilinear or sesquilinear form on finite-dimensional vector spaces over* R, C *or* H.**

This definition has some redundancy. In the case of F = **R**, bilinear is equivalent to sesquilinear. In the case of F = **H**, there are no non-zero bilinear forms.[9]

Symmetric, skew-symmetric, Hermitian, and skew-Hermitian forms

A form is **symmetric** if

$$\varphi(x, y) = \varphi(y, x).$$

It is **skew-symmetric** if

$$\varphi(x, y) = -\varphi(y, x).$$

It is **Hermitian** if

$$\varphi(x, y) = \overline{\varphi(y, x)}$$

Finally, it is **skew-Hermitian** if

$$\varphi(x, y) = -\overline{\varphi(y, x)}.$$

A bilinear form φ is uniquely a sum of a symmetric form and a skew-symmetric form. A transformation preserving φ preserves both parts separately. The groups preserving symmetric and skew-symmetric forms can thus be studied separately. The same applies, mutatis mutandis, to Hermitian and skew-Hermitian forms. For this reason, for the purposes of classification, only purely symmetric, skew-symmetric, Hermitian, or skew-Hermitian forms are considered. The **normal forms** of the forms correspond to specific suitable choices of bases. These are bases giving the following normal forms in coordinates:

$$\begin{aligned}
\text{Bilinear symmetric form in (pseudo-)orthonormal basis:} \quad & \varphi(x, y) = \pm\xi_1\eta_1 \pm \xi_2\eta_2 \pm \cdots \pm \xi_n\eta_n, \ (\mathbf{R}) \\
\text{Bilinear symmetric form in orthonormal basis:} \quad & \varphi(x, y) = \xi_1\eta_1 + \xi_2\eta_2 + \cdots + \xi_n\eta_n, \ (\mathbf{C}) \\
\text{Bilinear skew-symmetric in symplectic basis:} \quad & \varphi(x, y) = \xi_1\eta_{m+1} + \xi_2\eta_{m+2} + \cdots + \xi_m\eta_{2m=n} \\
& \qquad - \xi_{m+1}\eta_1 - \xi_{m+2}\eta_2 - \cdots - \xi_{2m=n}\eta_m, \ (\mathbf{R}, \mathbf{C}) \\
\text{Sesquilinear Hermitian:} \quad & \varphi(x, y) = \pm\bar{\xi}_1\eta_1 \pm \bar{\xi}_2\eta_2 \pm \cdots \pm \bar{\xi}_n\eta_n, \ (\mathbf{C}, \mathbf{H}) \\
\text{Sesquilinear skew-Hermitian:} \quad & \varphi(x, y) = \bar{\xi}_1\mathbf{j}\eta_1 + \bar{\xi}_2\mathbf{j}\eta_2 + \cdots + \bar{\xi}_n\mathbf{j}\eta_n, \ (\mathbf{H}).
\end{aligned}$$

The **j** in the skew-Hermitian form is the third basis element in the basis (**1**, **i**, **j**, **k**) for **H**. Proof of existence of these bases and Sylvester's law of inertia, the independence of the number of plus- and minus-signs, p and q, in the symmetric and Hermitian forms, as well as the presence or absence of the fields in each expression, can be found in Rossmann (2002) or Goodman & Wallach (2009). The pair (p, q), and sometimes $p - q$, is called the **signature** of the form.

Explanation of occurrence of the fields R, C, H: There are no nontrivial bilinear forms over **H**. In the symmetric bilinear case, only forms over **R** have a signature. In other words, a complex bilinear form with "signature" (p, q) can, by a change of basis, be reduced to a form where all signs are "+" in the above expression, whereas this is impossible in the real case, in which $p - q$ is independent of the basis when put into this form. However, Hermitian forms have basis-independent signature in both the complex and the quaternionic case. (The real case reduces to the symmetric case.) A skew-Hermitian form on a complex vector space is rendered Hermitian by multiplication by i, so in this case, only **H** is interesting.

Automorphism groups

The first section presents the general framework. The other sections exhaust the qualitatively different cases that arise as automorphism groups of bilinear and sesquilinear forms on finite-dimensional vector spaces over **R**, **C** and **H**.

Hermann Weyl, the author of The Classical Groups. Weyl made substantial contributions to the representation theory of the classical groups.

Aut(φ) – the automorphism group

Assume that φ is a non-degenerate form on a finite-dimensional vector space V over **R**, **C** or **H**. The automorphism group is defined, based on condition (1), as

$$\mathrm{Aut}(\varphi) = \{A \in \mathrm{GL}(V) : \varphi(Ax, Ay) = \varphi(x, y), \quad \forall x, y \in V\}.$$

Every $A \in M_n(V)$ has an adjoint A^φ with respect to φ defined by

$$\varphi(Ax, y) = \varphi(x, A^\varphi y), \qquad x, y \in V. \qquad (2)$$

Using this definition in condition (1), the automorphism group is seen to be given by

$$\mathrm{Aut}(\varphi) = \{A \in \mathrm{GL}(V) : A^\varphi A = 1\}. \qquad (3)$$ [10]

Fix a basis for V. In terms of this basis, put

$$\varphi(x, y) = \sum \xi_i\varphi_{ij}\eta_j$$

where ξ_i, η_j are the components of x, y. This is appropriate for the bilinear forms. Sesquilinear forms have similar expressions and are treated separately later. In matrix notation one finds

$$\varphi(x,y) = x^{\mathrm{T}}\Phi y$$

and

$$A^\varphi = \Phi^{-1}A^{\mathrm{T}}\Phi \quad (4)$$

[11]

from (**2**) where Φ is the matrix (φ_{ij}). The non-degeneracy condition means precisely that Φ is invertible, so the adjoint always exists. $\mathrm{Aut}(\varphi)$ expressed with this becomes

$$\mathrm{Aut}(\varphi) = \{A \in \mathrm{GL}(V) : \Phi^{-1}A^{\mathrm{T}}\Phi A = 1\}.$$

The Lie algebra $\mathbf{aut}(\varphi)$ of the automorphism groups can be written down immediately. Abstractly, $X \in \mathbf{aut}(\varphi)$ if and only if

$$(e^{tX})^\varphi e^{tX} = 1$$

for all t, corresponding to the condition in (**3**) under the exponential mapping of Lie algebras, so that

$$\mathfrak{aut}(\varphi) = \{X \in M_n(V) : X^\varphi = -X\},$$

or in a basis

$$\mathfrak{aut}(\varphi) = \{X \in M_n(V) : \Phi^{-1}X^{\mathrm{T}}\Phi = -X\} \quad (5)$$

as is seen using the power series expansion of the exponential mapping and the linearity of the involved operations. Conversely, suppose that $X \in \mathbf{aut}(\varphi)$. Then, using the above result, $\varphi(Xx, y) = \varphi(x, X^\varphi y) = -\varphi(x, Xy)$. Thus the Lie algebra can be characterized without reference to a basis, or the adjoint, as

$$\mathfrak{aut}(\varphi) = \{X \in M_n(V) : \varphi(Xx,y) = -\varphi(x,Xy), \quad \forall x,y \in V\}.$$

The normal form for φ will be given for each classical group below. From that normal form, the matrix Φ can be read off directly. Consequently, expressions for the adjoint and the Lie algebras can be obtained using formulas (**4**) and (**5**). This is demonstrated below in most of the non-trivial cases.

Bilinear case

When the form is symmetric, $\mathrm{Aut}(\varphi)$ is called $\mathrm{O}(\varphi)$. When it is skew-symmetric then $\mathrm{Aut}(\varphi)$ is called $\mathrm{Sp}(\varphi)$. This applies to the real and the complex cases. The quaternionic case is empty since no nonzero bilinear forms exists on quaternionic vector spaces.[12]

Real case

The real case breaks up into two cases, the symmetric and the antisymmetric forms that should be treated separately.

O(*p*, *q*) and O(*n*) – the orthogonal groups

If φ is symmetric and the vector space is real, a basis may be chosen so that

$$\varphi(x,y) = \pm\xi_1\eta_1 \pm \xi_1\eta_1 \cdots \pm \xi_n\eta_n.$$

The number of plus and minus-signs are independent of the particular basis.[13] In the case $V = \mathbf{R}^n$ one writes $\mathrm{O}(\varphi) = \mathrm{O}(p, q)$ where p is the number of plus signs and q is the number of minus-signs, $p + q = n$. If $q = 0$ the notation is $\mathrm{O}(n)$. The matrix Φ is in this case

$$\Phi = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \equiv I_{p,q}$$

after reordering the basis if necessary. The adjoint operation (**4**) then becomes

$$A^\varphi = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \begin{pmatrix} A_{11} & \cdots \\ \cdots & A_{nn} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix},$$

which reduces to the usual transpose when p or q is 0. The Lie algebra is found using equation (**5**) and a suitable ansatz (this is detailed for the case of $\mathrm{Sp}(m, \mathbf{R})$ below),

$$\mathfrak{o}(p,q) = \left\{ \begin{pmatrix} X_{p\times p} & Y_{p\times q} \\ Y^{\mathrm{T}} & W_{q\times q} \end{pmatrix} \middle| X^{\mathrm{T}} = -X, \quad W^{\mathrm{T}} = -W \right\},$$

and the group according to (**3**) is given by

$$\mathrm{O}(p,q) = \{g \in \mathrm{GL}(n,\mathbb{R}) | I_{p,q}^{-1} g^{\mathrm{T}} I_{p,q} g = I\}.$$

The groups $\mathrm{O}(p, q)$ and $\mathrm{O}(q, p)$ are isomorphic through the map

$$\mathrm{O}(p,q) \to \mathrm{O}(q,p), \quad g \to \sigma g \sigma^{-1}, \quad \sigma = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

For example, the Lie algebra of the Lorentz group could be written as

$$\mathfrak{o}(3,1) = \mathrm{span}\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}.$$

Naturally, it is possible to rearrange so that the q-block is the upper left (or any other block). Here the "time component" end up as the fourth coordinate in a physical interpretation, and not the first as may be more common.

Sp(*m*, R) – the real symplectic group

If φ is skew-symmetric and the vector space is real, there is a basis giving

$$\varphi(x,y)=\xi_1\eta_{m+1}+\xi_2\eta_{m+2}\cdots+\xi_m\eta_{2m=n}-\xi_{m+1}\eta_1-\xi_{m+2}\eta_2\cdots-\xi_{2m=n}\eta_m,$$

where $n = 2m$. For Aut(φ) one writes Sp(φ) = Sp(V) In case $V = \mathbf{R}^n = \mathbf{R}^{2m}$ one writes Sp(m, **R**) or Sp($2m$, **R**). From the normal form one reads off

$$\Phi=\begin{pmatrix}0_m & I_m\\ -I_m & 0_m\end{pmatrix}=J_m.$$

By making the ansatz

$$V=\begin{pmatrix}X & Y\\ Z & W\end{pmatrix},$$

where X, Y, Z, W are m-dimensional matrices and considering (**5**),

$$\begin{pmatrix}0_m & -I_m\\ I_m & 0_m\end{pmatrix}\begin{pmatrix}X & Y\\ Z & W\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0_m & I_m\\ -I_m & 0_m\end{pmatrix}=-\begin{pmatrix}X & Y\\ Z & W\end{pmatrix}$$

one finds the Lie algebra of Sp(m, **R**),

$$\mathfrak{sp}(m,\mathbb{R})=\{X\in M_n(\mathbb{R}):J_mX+X^{\mathrm{T}}J_m=0\}=\left\{\begin{pmatrix}X & Y\\ Z & -X^{\mathrm{T}}\end{pmatrix}\middle|Y^{\mathrm{T}}=Y,Z^{\mathrm{T}}=Z\right\},$$

and the group is given by

$$\mathrm{Sp}(m,\mathbb{R})=\{g\in M_n(\mathbb{R})|g^{\mathrm{T}}J_mg=J_m\}.$$

Complex case

Like in the real case, there are two cases, the symmetric and the antisymmetric case that each yield a family of classical groups.

O(n, C) – the complex orthogonal group

If case φ is symmetric and the vector space is complex, a basis

$$\varphi(x,y)=\xi_1\eta_1+\xi_1\eta_1\cdots+\xi_n\eta_n$$

with only plus-signs can be used. The automorphism group is in the case of $V = \mathbf{C}^n$ called O(n, **C**). The lie algebra is simply a special case of that for **o**(p, q),

$$\mathfrak{o}(n,\mathbb{C})=\mathfrak{so}(n,\mathbb{C})=\{X|X^{\mathrm{T}}=-X\},$$

and the group is given by

$$\mathrm{O}(n,\mathbb{C})=\{g|g^{\mathrm{T}}g=I_n\}.$$

In terms of classification of simple Lie algebras, the **so**(n) are split into two classes, those with n odd with root system B_n and n even with root system D_n.

Sp(m, C) – the complex symplectic group

For φ skew-symmetric and the vector space complex, the same formula,

$$\varphi(x,y)=\xi_1\eta_{m+1}+\xi_2\eta_{m+2}\cdots+\xi_m\eta_{2m=n}-\xi_{m+1}\eta_1-\xi_{m+2}\eta_2\cdots-\xi_{2m=n}\eta_m,$$

applies as in the real case. For Aut(φ) one writes Sp(φ) = Sp(V) In case $V = \mathbb{C}^n = \mathbb{C}^{2m}$ one writes Sp(m, $\mathbb{C}$) or Sp($2m$, $\mathbb{C}$). The Lie algebra parallels that of **sp**(m, $\mathbb{R}$),

$$\mathfrak{sp}(m,\mathbb{C})=\{X\in M_n(\mathbb{C}):J_mX+X^{\mathrm{T}}J_m=0\}=\left\{\begin{pmatrix}X & Y\\ Z & -X^{\mathrm{T}}\end{pmatrix}\middle|Y^{\mathrm{T}}=Y,Z^{\mathrm{T}}=Z\right\},$$

and the group is given by

$$\mathrm{Sp}(m,\mathbb{C})=\{g\in M_n(\mathbb{C})|g^{\mathrm{T}}J_mg=J_m\}.$$

Sesquilinear case

In the sequilinear case, one makes a slightly different ansatz for the form in terms of a basis,

$$\varphi(x,y)=\sum\bar{\xi}_i\varphi_{ij}\eta_j.$$

The other expressions that get modified are

$$\varphi(x,y)=x^*\Phi y,\qquad A^\varphi=\Phi^{-1}A^*\Phi,$$ [14]

$$\mathrm{Aut}(\varphi)=\{A\in\mathrm{GL}(V):\Phi^{-1}A^*\Phi A=1\},$$

$$\mathfrak{aut}(\varphi)=\{X\in M_n(V):\Phi^{-1}X^*\Phi=-X\}.\qquad(6)$$

The real case, of course, provides nothing new. The complex and the quaternionic case will be considered below.

Complex case

From a qualitative point of view, consideration of skew-Hermitian forms (up to isomorphism) provide no new groups; multiplication by i renders a skew-Hermitian form Hermitian, and vice versa. Thus only the Hermitian case needs to be considered.

U(p, q) and U(n) – the unitary groups

A non-degenerate hermitian form has the normal form

$$\varphi(x,y) = \pm\bar{\xi_1}\eta_1 \pm \bar{\xi_2}\eta_2 \cdots \pm \bar{\xi_n}\eta_n.$$

As in the bilinear case, the signature (p, q) is independent of the basis. The automorphism group is denoted U(V), or, in the case of $V = \mathbf{C}^n$, U(p, q). If $q = 0$ the notation is U(n). In this case, Φ takes the form

$$\Phi = \begin{pmatrix} 1_p & 0 \\ 0 & -1_q \end{pmatrix} = I_{p,q},$$

and the Lie algebra is given by

$$\mathfrak{u}(p,q) = \left\{ \left. \begin{pmatrix} X_{p\times p} & Z_{p\times q} \\ \overline{Z}^{\mathrm{T}} & Y_{q\times q} \end{pmatrix} \right| \overline{X}^{\mathrm{T}} = -X, \quad \overline{Y}^{\mathrm{T}} = -Y \right\}.$$

The group is given by

$$\mathrm{U}(p,q) = \{g | I_{p,q}^{-1} g^* I_{p,q} g = I\}.$$

Quaternionic case

The space $\mathbf{H}^n$ is considered as a *right* vector space over **H**. This way, $A(vh) = (Av)h$ for a quaternion h, a quaternion column vector v and quaternion matrix A. If $\mathbf{H}^n$ was a *left* vector space over **H**, then matrix multiplication from the *right* on row vectors would be required to maintain linearity. This does not correspond to the usual linear operation of a group on a vector space when a basis is given, which is matrix multiplication from the *left* on column vectors. Thus V is henceforth a right vector space over **H**. Even so, care must be taken due to the non-commutative nature of **H**. The (mostly obvious) details are skipped because complex representations will be used.

When dealing with quaternionic groups it is convenient to represent quaternions using complex 2×2-matrices,

$$q = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = \alpha + j\beta \leftrightarrow \begin{bmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} = Q, \quad q \in \mathbb{H}, \quad a,b,c,d \in \mathbb{R}, \quad \alpha,\beta \in \mathbb{C}. \qquad (7)$$

[15]

With this representation, quaternionic multiplication becomes matrix multiplication and quaternionic conjugation becomes taking the Hermitian adjoint. Moreover, if a quaternion according to the complex encoding $q = x + \mathbf{j}y$ is given as a column vector $(x, y)^{\mathrm{T}}$, then multiplication from the left by a matrix representation of a quaternion produces a new column vector representing the correct quaternion. This representation differs slightly from a more common representation found in the quaternion article. The more common convention would force multiplication from the right on a row matrix to achieve the same thing.

Incidentally, the representation above makes it clear that the group of unit quaternions ($\alpha\overline{\alpha} + \beta\overline{\beta} = 1 = \det Q$) is isomorphic to SU(2).

Quaternionic $n\times n$-matrices can, by obvious extension, be represented by $2n\times 2n$ block-matrices of complex numbers.[16] If one agrees to represent a quaternionic $n\times 1$ column vector by a $2n\times 1$ column vector with complex numbers according to the encoding of above, with the upper n numbers being the α_i and the lower n the β_i, then a quaternionic $n\times n$-matrix becomes a complex $2n\times 2n$-matrix exactly of the form given above, but now with α and β $n\times n$-matrices. More formally

$$(Q)_{n\times n} = (X)_{n\times n} + \mathbf{j}(Y)_{n\times n} \leftrightarrow \begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{pmatrix}_{2n\times 2n}. \qquad (8)$$

A matrix $T \in \mathrm{GL}(2n, \mathbf{C})$ has the form displayed in (**8**) if and only if $J_n\overline{T} = TJ_n$. With these identifications,

$$\mathbb{H}^n \approx \mathbb{C}^{2n}, M_n(\mathbb{H}) \approx \left\{ T \in M_{2n}(\mathbb{C}) |\, J_nT = \overline{T}J_n, \quad J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

The space $M_n(\mathbf{H}) \subset M_{2n}(\mathbf{C})$ is a real algebra, but it is not a complex subspace of $M_{2n}(\mathbf{C})$. Multiplication (from the left) by **i** in $M_n(\mathbf{H})$ using entry-wise quaternionic multiplication and then mapping to the image in $M_{2n}(\mathbf{C})$ yields a different result than multiplying entry-wise by i directly in $M_{2n}(\mathbf{C})$. The quaternionic multiplication rules give $\mathbf{i}(X + \mathbf{j}Y) = (\mathbf{i}X) + \mathbf{j}(-\mathbf{i}Y)$ where the new X and Y are inside the parentheses.

The action of the quaternionic matrices on quaternionic vectors is now represented by complex quantities, but otherwise it is the same as for "ordinary" matrices and vectors. The quaternionic groups are thus embedded in $\mathrm{M}_{2n}(C)$ where n is the dimension of the quaternionic matrices.

The determinant of a quaternionic matrix is defined in this representation as being the ordinary complex determinant of its representative matrix. The non-commutative nature of quaternionic multiplication would, in the quaternionic representation of matrices, be ambiguous. The way $M_n(\mathbf{H})$ is embedded in $M_{2n}(\mathbf{C})$ is not unique, but all such embeddings are related through $g \mapsto AgA^{-1}$, $g \in \mathrm{GL}(2n, \mathbf{C})$ for $A \in \mathrm{O}(2n, \mathbf{C})$, leaving the determinant unaffected.[17] The name of SL(n, **H**) in this complex guise is $\mathrm{SU}^*(2n)$.

As opposed to in the case of **C**, both the Hermitian and the skew-Hermitian case bring in something new when **H** is considered, so these cases are considered separately.

GL(*n*, H) and SL(*n*, H)

Under the identification above,

$$\mathrm{GL}(n,\mathbb{H}) = \{g \in \mathrm{GL}(2n,\mathbb{C}) | Jg = \overline{g}J, \det \quad g \neq 0\} \equiv \mathrm{U}^*(2n).$$

Its Lie algebra $\mathbf{gl}(n, \mathbf{H})$ is the set of all matrices in the image of the mapping $M_n(\mathbf{H}) \leftrightarrow M_{2n}(\mathbf{C})$ of above,

$$\mathfrak{gl}(n,\mathbb{H}) = \left\{ \left. \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \right| X, Y \in \mathfrak{gl}(n,\mathbb{C}) \right\} \equiv \mathfrak{u}^*(2n).$$

The quaternionic special linear group is given by

$$\mathrm{SL}(n,\mathbb{H}) = \{g \in \mathrm{GL}(n,\mathbb{H}) | \det g = 1\} \equiv \mathrm{SU}^*(2n),$$

where the determinant is taken on the matrices in $\mathbf{C}^{2n}$. The Lie algebra is

$$\mathfrak{sl}(n,\mathbb{H}) = \left\{ \left. \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \right| \mathrm{Tr}\, X = 0 \right\} \equiv \mathfrak{su}^*(2n).$$

Sp(*p*, *q*) – the quaternionic unitary group

As above in the complex case, the normal form is

$$\varphi(x,y) = \pm\bar{\xi_1}\eta_1 \pm \bar{\xi_2}\eta_2 \cdots \pm \bar{\xi_n}\eta_n$$

and the number of plus-signs is independent of basis. When $V = \mathbf{H}^n$ with this form, $\mathrm{Sp}(\varphi) = \mathrm{Sp}(p, q)$. The reason for the notation is that the group can be represented, using the above prescription, as a subgroup of

Sp(n, **C**) preserving a complex-hermitian form of signature $(2p, 2q)$[18] If p or q = 0 the group is denoted U(n, **H**). It is sometimes called the **hyperunitary group**.

In quaternionic notation,

$$\Phi = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} = I_{p,q}$$

meaning that *quaternionic* matrices of the form

$$\mathcal{Q} = \begin{pmatrix} \mathcal{X}_{p\times p} & \mathcal{Z}_{p\times q} \\ \mathcal{Z}^* & \mathcal{Y}_{q\times q} \end{pmatrix}, \qquad \mathcal{X}^* = -\mathcal{X}, \mathcal{Y}^* = -\mathcal{Y} \qquad (9)$$

will satisfy

$$\Phi^{-1}\mathcal{Q}^*\Phi = -\mathcal{Q},$$

see the section about **u**(p, q). Caution needs to be exercised when dealing with quaternionic matrix multiplication, but here only I and $-I$ are involved and these commute with every quaternion matrix. Now apply prescription (**8**) to each block,

$$\mathcal{X} = \begin{pmatrix} X_{1(p\times p)} & -\overline{X}_2 \\ X_2 & \overline{X}_1 \end{pmatrix}, \mathcal{Y} = \begin{pmatrix} Y_{1(q\times q)} & -\overline{Y}_2 \\ Y_2 & \overline{Y}_1 \end{pmatrix}, \mathcal{Z} = \begin{pmatrix} Z_{1(p\times q)} & -\overline{Z}_2 \\ Z_2 & \overline{Z}_1 \end{pmatrix},$$

and the relations in (**9**) will be satisfied if

$$X_1^* = -X, Y_1^* = -Y.$$

The Lie algebra becomes

$$\mathfrak{sp}(p,q) = \left\{ \left(\begin{matrix} \begin{bmatrix} X_{1(p\times p)} & -\overline{X}_2 \\ X_2 & \overline{X}_1 \end{bmatrix} & \begin{bmatrix} Z_{1(p\times q)} & -\overline{Z}_2 \\ Z_2 & \overline{Z}_1 \end{bmatrix} \\ \begin{bmatrix} Z_{1(p\times q)} & -\overline{Z}_2 \\ Z_2 & \overline{Z}_1 \end{bmatrix}^* & \begin{bmatrix} Y_{1(q\times q)} & -\overline{Y}_2 \\ Y_2 & \overline{Y}_1 \end{bmatrix} \end{matrix} \right) \middle| X_1^* = -X, Y_1^* = -Y \right\}.$$

The group is given by

$$\mathrm{Sp}(p,q) = \{g \in \mathrm{GL}(n,\mathbb{H}) | I_{p,q}^{-1} g^* I_{p,q} g = I_{p+q}\} = \{g \in \mathrm{GL}(2n,\mathbb{C}) | K_{p,q}^{-1} g^* K_{p,q} g = I_{2(p+q)}, \qquad K = \mathrm{diag}(I_{p,q}, I_{p,q})\}.$$

Returning to the normal form of $\varphi(w, z)$ for Sp(p, q), make the substitutions $w \to u + jv$ and $z \to x + jy$ with u, v, x, y ∈ $\mathbf{C}^n$. Then

$$\varphi(w,z) = [u^* \quad v^*] K_{p,q} \begin{bmatrix} x \\ y \end{bmatrix} + j[u \quad -v] K_{p,q} \begin{bmatrix} y \\ x \end{bmatrix} = \varphi_1(w,z) + \mathbf{j}\varphi_2(w,z), \qquad K_{p,q} = \mathrm{diag}(I_{p,q}, I_{p,q})$$

viewed as a **H**-valued form on $\mathbf{C}^{2n}$.[19] Thus the elements of Sp(p, q), viewed as linear transformations of $\mathbf{C}^{2n}$, preserve both a Hermitian form of signature $(2p, 2q)$and a non-degenerate skew-symmetric form. Both forms take purely complex values and due to the prefactor of **j** of the second form, they are separately conserved. This means that

$$\mathrm{Sp}(p,q) = \mathrm{U}(\mathbb{C}^{2n}, \varphi_1) \cap \mathrm{Sp}(\mathbb{C}^{2n}, \varphi_2)$$

and this explains both the name of the group and the notation.

O*(2*n*)= O(*n*, H)- quaternionic orthogonal group

The normal form for a skew-hermitian form is given by

$$\varphi(x,y) = \bar{\xi_1}\mathbf{j}\eta_1 + \bar{\xi_2}\mathbf{j}\eta_2 \cdots + \bar{\xi_n}\mathbf{j}\eta_n,$$

where **j** is the third basis quaternion in the ordered listing (**1**, **i**, **j**, **k**). In this case, Aut(φ) = $\mathrm{O}^*(2n)$ may be realized, using the complex matrix encoding of above, as a subgroup of O($2n$, **C**) which preserves a non-degenerate complex skew-hermitian form of signature (n, n).[20] From the normal form one sees that in quaternionic notation

$$\Phi = \begin{pmatrix} \mathrm{j} & 0 & \cdots & 0 \\ 0 & \mathrm{j} & \cdots & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \mathrm{j} \end{pmatrix} \equiv \mathrm{j}_n$$

and from (**6**) follows that

$$-\Phi V^*\Phi = -V \Leftrightarrow V^* = \mathrm{j}_n V \mathrm{j}_n. \qquad (9)$$

for $V \in \mathbf{o}(2n)$. Now put

$$V = X + \mathbf{j}Y \leftrightarrow \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix}$$

according to prescription (**8**). The same prescription yields for Φ,

$$\Phi \leftrightarrow \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \equiv J_n.$$

Now the last condition in (**9**) in complex notation reads

$$\begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix}^* = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X} \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \Leftrightarrow X^{\mathrm{T}} = -X, \quad \overline{Y}^{\mathrm{T}} = Y.$$

The Lie algebra becomes

$$\mathfrak{o}^*(2n) = \left\{\left.\begin{pmatrix} X & -\overline{Y} \\ Y & \overline{X}\end{pmatrix}\right| X^\mathrm{T} = -X, \quad \overline{Y}^\mathrm{T} = Y\right\},$$

and the group is given by

$$\mathrm{O}^*(2n) = \{g \in \mathrm{GL}(n, \mathbb{H}) | \mathrm{j}_n^{-1}g^*\mathrm{j}_n g = I_n\} = \{g \in \mathrm{GL}(2n, \mathbb{C}) | J_n^{-1}g^*J_n g = I_{2n}\}.$$

The group $\mathrm{SO}^*(2n)$ can be characterized as

$$\mathrm{O}^*(2n) = \{g \in \mathrm{O}(2n, \mathbb{C}) | \theta(\overline{g}) = g\},$$ [21]

where the map θ: $\mathrm{GL}(2n, \mathbf{C}) \to \mathrm{GL}(2n, \mathbf{C})$ is defined by $g \mapsto -J_{2n}gJ_{2n}$. Also, the form determining the group can be viewed as a $\mathbf{H}$-valued form on $\mathbf{C}^{2n}$.[22] Make the substitutions $x \to w_1 + iw_2$ and $y \to z_1 + iz_2$ in the expression for the form. Then

$$\varphi(x, y) = \overline{w}_2 I_n z_1 - \overline{w}_1 I_n z_2 + \mathbf{j}(w_1 I_n z_1 + w_2 I_n z_2) = \overline{\varphi_1(w, z)} + \mathbf{j}\varphi_2(w, z).$$

The form φ_1 is Hermitian (while the first form on the left hand side is skew-Hermitian) of signature (n, n). The signature is made evident by a change of basis from $(\mathbf{e}, \mathbf{f})$ to $((\mathbf{e} + i\mathbf{f})/\sqrt{2}, (\mathbf{e} - i\mathbf{f})/\sqrt{2})$ where $\mathbf{e}$, $\mathbf{f}$ are the first and last n basis vectors respectively. The second form, φ_2 is symmetric positive definite. Thus, due to the factor $\mathbf{j}$, $\mathbf{O}^*(2n)$ preserves both separately and it may be concluded that

$$\mathrm{O}^*(2n) = \mathrm{O}(2n, \mathbb{C}) \cap \mathrm{U}(\mathbb{C}^{2n}, \varphi_1),$$

and the notation "O" is explained.

Classical groups over general fields or algebras

Classical groups, more broadly considered in algebra, provide particularly interesting matrix groups. When the field F of coefficients of the matrix group is either real number or complex numbers, these groups are just the classical Lie groups. When the ground field is a finite field, then the classical groups are groups of Lie type. These groups play an important role in the classification of finite simple groups. Also, one may consider classical groups over a unital associative algebra R over F; where $R = \mathbf{H}$ (an algebra over reals) represents an important case. For the sake of generality the article will refer to groups over R, where R may be the ground field F itself.

Considering their abstract group theory, many linear groups have a "**special**" subgroup, usually consisting of the elements of determinant 1 over the ground field, and most of them have associated "**projective**" quotients, which are the quotients by the center of the group. For orthogonal groups in characteristic 2 "S" has a different meaning.

The word "**general**" in front of a group name usually means that the group is allowed to multiply some sort of form by a constant, rather than leaving it fixed. The subscript n usually indicates the dimension of the module on which the group is acting; it is a vector space if $R = F$. Caveat: this notation clashes somewhat with the n of Dynkin diagrams, which is the rank.

General and special linear groups

The general linear group $\mathrm{GL}_n(R)$ is the group of all R-linear automorphisms of R^n. There is a subgroup: the special linear group $\mathrm{SL}_n(R)$, and their quotients: the projective general linear group $\mathrm{PGL}_n(R) = \mathrm{GL}_n(R)/\mathrm{Z}(\mathrm{GL}_n(R))$ and the projective special linear group $\mathrm{PSL}_n(R) = \mathrm{SL}_n(R)/\mathrm{Z}(\mathrm{SL}_n(R))$. The projective special linear group $\mathrm{PSL}_n(F)$ over a field F is simple for $n \geq 2$, except for the two cases when $n = 2$ and the field has order 2 or 3.

Unitary groups

The unitary group $\mathrm{U}_n(R)$ is a group preserving a sesquilinear form on a module. There is a subgroup, the special unitary group $\mathrm{SU}_n(R)$ and their quotients the projective unitary group $\mathrm{PU}_n(R) = \mathrm{U}_n(R)/\mathrm{Z}(\mathrm{U}_n(R))$ and the projective special unitary group $\mathrm{PSU}_n(R) = \mathrm{SU}_n(R)/\mathrm{Z}(\mathrm{SU}_n(R))$

Symplectic groups

The symplectic group $\mathrm{Sp}_{2n}(R)$ preserves a skew symmetric form on a module. It has a quotient, the projective symplectic group $\mathrm{PSp}_{2n}(R)$. The general symplectic group $\mathrm{GSp}_{2n}(R)$ consists of the automorphisms of a module multiplying a skew symmetric form by some invertible scalar. The projective symplectic group $\mathrm{PSp}_{2n}(\mathbf{F}_q)$ over a finite field is simple for $n \geq 1$, except for the cases of PSp_2 over the fields of two and three elements.

Orthogonal groups

The orthogonal group $\mathrm{O}_n(R)$ preserves a non-degenerate quadratic form on a module. There is a subgroup, the special orthogonal group $\mathrm{SO}_n(R)$ and quotients, the projective orthogonal group $\mathrm{PO}_n(R)$, and the projective special orthogonal group $\mathrm{PSO}_n(R)$. In characteristic 2 the determinant is always 1, so the special orthogonal group is often defined as the subgroup of elements of Dickson invariant 1.

There is a nameless group often denoted by $\Omega_n(R)$ consisting of the elements of the orthogonal group of elements of spinor norm 1, with corresponding subgroup and quotient groups $\mathrm{S}\Omega_n(R)$, $\mathrm{P}\Omega_n(R)$, $\mathrm{PS}\Omega_n(R)$. (For positive definite quadratic forms over the reals, the group Ω happens to be the same as the orthogonal group, but in general it is smaller.) There is also a double cover of $\Omega_n(R)$, called the pin group $\mathrm{Pin}_n(R)$, and it has a subgroup called the spin group $\mathrm{Spin}_n(R)$. The general orthogonal group $\mathrm{GO}_n(R)$ consists of the automorphisms of a module multiplying a quadratic form by some invertible scalar.

Notational conventions

Contrast with exceptional Lie groups

Contrasting with the classical Lie groups are the exceptional Lie groups, G_2, F_4, E_6, E_7, E_8, which share their abstract properties, but not their familiarity.[23] These were only discovered around 1890 in the classification of the simple Lie algebras over the complex numbers by Wilhelm Killing and Élie Cartan.

Notes

1. Here, *special* means the subgroup of the full automorphism group whose elements have determinant 1.
2. Rossmann 2002 p. 94.
3. Weyl 1939
4. Rossmann 2002 p. 91.
5. Rossmann 2002 p, 94
6. Rossmann 2002 p. 103.
7. Goodman & Wallach 2009 See end of chapter 1.
8. Rossmann 2002p. 93.
9. Rossmann 2002 p. 105
10. Rossmann 2002 p. 91
11. Rossmann 2002 p. 92
12. Rossmann 2002 p. 105
13. Rossmann 2002 p. 107.
14. Rossmann 2002 p. 93
15. Rossmann 2002 p. 95.
16. Rossmann 2002 p. 94.
17. Goodman & Wallach 2009 Exercise 14, Section 1.1.

18. Rossmann 2002 p. 94.
19. Goodman & Wallach 2009Exercise 11, Chapter 1.
20. Rossmann 2002 p. 94.
21. Goodman & Wallach 2009 p.11.
22. Goodman & Wallach 2009 Exercise 12 Chapter 1.
23. Wybourne, B. G. (1974). *Classical Groups for Physicists*, Wiley-Interscience. ISBN 0471965057.

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