Linear Algebraic Groups

BY

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This is a review of some of the notions and facts pertaining to linear algebraic groups. From §2 on, the word linear will usually be dropped, since more general algebraic groups will not be considered here.

1. **The notion of linear algebraic group.** According to one's taste about naturality and algebraic geometry, it is possible to give several definitions of linear algebraic groups. The first one is not intrinsic at all but suffices for what follows.

1.1. *Algebraic matrix group.* Let Ω be an algebraically closed field. We shall denote by $M(n, \Omega)$ the group of all $n \times n$ matrices with entries in Ω and by $GL(n, \Omega)$ the group of all $n \times n$ invertible matrices. $GL(n, \Omega)$ is an affine subvariety of Ω^{n^2+1} through the identification

$$g = (g_{ij}) \mapsto (g_{11}, g_{12}, \cdots, g_{nn}, (\det g)^{-1}).$$

The set $M(n, \Omega)$ carries a topology—the Zariski topology—the closed sets being the algebraic subsets of $M(n, \Omega) = \Omega^{n^2}$. The ring $GL(n, \Omega)$ is an open subset of $M(n, \Omega)$ and carries the induced topology.

A subgroup of G of $GL(n, \Omega)$ is called an algebraic matrix group if G is a closed subset of $GL(n, \Omega)$, i.e., if there exist polynomials $p_\alpha \in \Omega[X_{11}, X_{12}, \cdots, X_{nn}]$ $(\alpha \in J)$ such that

$$G = \{g = (g_{ij}) \in GL(n, \Omega) | p_\alpha(g_{ij}) = 0, \ (\alpha \in J)\}.$$

The coordinate ring $\Omega[G]$ of G, i.e., the ring of all regular functions on G, is the Ω-algebra generated by the coefficients g_{ij} and $(\det g)^{-1}$. It is the quotient ring $\Omega[X_{ij}, Z]/I$, where I is the ideal of polynomials in the $n^2 + 1$ letters X_{ij}, Z vanishing on G, considered as a subset of Ω^{n^2+1}, via the above imbedding of $GL(n, \Omega)$ in Ω^{n^2+1}.

When B is a subring of Ω, we shall denote by $GL(n, B)$ the set of $n \times n$ matrices g with entries in B, such that $\det g$ is a unit in B, and by G_B the intersection $G \cap GL(n, B)$.

Let k be a subfield of Ω. The algebraic matrix group G is *defined over* k or is a k-group if the ideal I of polynomials annihilated by G has a set of generators in $k[X_{ij}, Z]$. If I_k denotes the ideal of all polynomials with coefficients in k vanishing on G, the quotient ring $k[X_{ij}, Z]/I_k = k[G]$ is the coordinate ring of G over k.

REMARK. If the field k is not perfect, it is not enough to assume that G is k-closed (i.e., that G is defined by a set of equations with coefficients in k) to conclude that G is defined over k; one can only infer that G is defined over a purely inseparable extension k' of k.

The following variant of the definition eliminates the choice of a basis.

1.2. *Algebraic groups of automorphisms of a vector space.* Let V be an n-dimensional vector space over Ω, and $GL(V)$ the group of all automorphisms of V. Every base of V defines an isomorphism of $GL(V)$ with $GL(n, \Omega)$. A subgroup G of $GL(V)$ is called an algebraic group of automorphisms of V if any such isomorphism maps G onto an algebraic matrix group.

Let k be a subfield of Ω. Assume that V has a k-structure, i.e., that we are given a vector subspace V_k over k of V such that $V = V_k \otimes_k \Omega$. The subgroup of G of $GL(V)$ is then defined over k if there exists a basis of V_k such that the corresponding isomorphism $\beta : GL(V) \to GL(n, \Omega)$ maps G onto a k-group in the previous sense.

1.3. *Affine algebraic group.* Let G be an affine algebraic set. It is an affine algebraic group if there are given morphisms

$$\mu : G \times G \to G, \qquad \mu(a, b) = ab,$$
$$\rho : G \to G, \qquad \rho(a) = a^{-1},$$

of affine sets, with the usual properties. G is an affine algebraic group defined over k if G, μ and ρ are defined over k. One can prove that every affine algebraic group defined over k is isomorphic to an algebraic matrix group defined over k.

1.4. *Functorial definition of affine algebraic groups.* Sometimes one would like not to emphasize a particular algebraically closed extension of the field k. For instance in the case of adèle groups, an algebraically closed field containing every p-adic completion of a number field k is a cumbersome object. Let G be a k-group in the sense of §1.1. Then for any k-algebra A, we may consider the set G_A of elements of $GL(n, A)$ whose coefficients annihilate the polynomials in I_k. It is a group, which may be identified to the group $\mathrm{Hom}_k(k[G], A)$ of k-homomorphisms of $k[G]$ into A. Furthermore, to any homomorphism $\rho : A \to B$ of k-algebras corresponds canonically a homomorphism $G_A \to G_B$. Thus we may say that a k-group is a functor from k-algebras to groups, which is representable by a k-algebra $k[G]$ of finite type, such that $\bar{k} \otimes k[G]$ has no nilpotent element, $\bar{k}$ being an algebraic closure of k. (The last requirement stands for the condition that $I_k \otimes \Omega$ is the ideal of all polynomials vanishing on G, it would be left out in a more general context.) This definition was introduced by Cartier as a short cut to the notion of (absolutely reduced) "affine scheme of groups over k." The functors corresponding to the general linear group and the special linear group, will be denoted GL_n and SL_n, and $(GL_n)_A$ by $GL_n(A)$ or $GL(n, A)$.

Usually the more down to earth point of view of algebraic matrix groups will be sufficient.

1.5. *Connected component of the identity.* An algebraic set is reducible if it is the union of two proper closed subsets; it is nonconnected if it is the union of two proper disjoint closed subsets. An algebraic group is irreducible if and only if it is connected. To avoid confusion with the irreducibility of a linear group, we shall usually speak of connected algebraic groups. The connected component of the identity of G will be denoted by G^0. The index of G^0 in G is finite.

If $\Omega = \boldsymbol{C}$, every affine algebraic group G can be viewed as a complex Lie group; then G is connected as an algebraic group, if and only if G is connected as a Lie group. When G is defined over $\boldsymbol{R}$, $G_{\boldsymbol{R}}$ is a closed subgroup of $\boldsymbol{GL(n, R)}$ and hence a real Lie group. It is not true that for a connected algebraic $\boldsymbol{R}$-group, the Lie group $G_{\boldsymbol{R}}$ is also connected, but in any case it has only finitely many connected components. The connected component of the identity for the usual topology will be denoted $G_{\boldsymbol{R}}^0$.

Let G be connected. Then $\Omega[G]$ is an integral domain. Its field of fractions $\Omega(G)$ is the field of rational functions on G. The quotient field of $k[G]$ is a subfield of $\Omega(G)$, consisting of those rational functions which are defined over k.

1.6. *The Lie algebra of an algebraic group.* A group variety is nonsingular, so that the tangent space at every point is well defined. The tangent space $\mathfrak{g}$ at e can be identified with the set of Ω-derivations of $\Omega[G]$ which commute with right translations. $\mathfrak{g}$, endowed with the Lie algebra structure defined by the usual vector space structure and bracket operations on derivations, is the *Lie algebra of* $\mathfrak{g}$. Of course, G and G^0 have the same Lie algebra. If G is connected, $\mathfrak{g}$ could alternatively be defined as the Lie algebra of Ω-derivations of the field $\Omega(G)$, which commute with right translations (and the definition would then be valid for any algebraic group linear or not). If G is defined over k, we have $\mathfrak{g} = \mathfrak{g}_k \otimes_k \Omega$, where $\mathfrak{g}_k$ is the set of derivations which leave $k[G]$ stable. If the characteristic p of k is $\neq 0$, then $\mathfrak{g}$ and $\mathfrak{g}_k$ are restricted Lie algebras, in the sense of Jacobson. However the connection between an algebraic group and its Lie algebra in characteristic $p \neq 0$ is weaker than for a Lie group; for instance, there does not correspond a subgroup to every restricted Lie subalgebra of $\mathfrak{g}$, and it may happen that several algebraic subgroups have the same Lie algebra.

The group G operates on itself by inner automorphisms. The differential of $\operatorname{Int} g : x \mapsto g \cdot x \cdot g^{-1}$ $(x, g \in G)$ at e is denoted $\operatorname{Ad}_{\mathfrak{g}} g$. The map $g \mapsto \operatorname{Ad}_{\mathfrak{g}} g$ is a k-morphism (in the sense of 2.1) of G into $GL(\mathfrak{g})$, called the *adjoint representation of* G.

1.7. *Algebraic transformation group.* If G is an algebraic group and V is an algebraic set, G operates morphically on V (or G is an algebraic transformation group) when there is given a morphism $\tau : G \times V \to V$ with the usual properties of transformation groups. It operates k-morphically if G, V and τ are defined over k.

An elementary, but basic, property of algebraic transformation groups is the existence of at least one closed orbit (e.g. an orbit of smallest possible dimension [**1**, §16]).

2. Homomorphism, characters, subgroups and quotient groups of algebraic groups.

2.1. *Homomorphisms of algebraic groups.* Let ρ, G, G' be algebraic groups and $\rho : G_\Omega \to G'_\Omega$ be a map. It is a morphism of algebraic groups if:

(1) ρ is a group homomorphism from G_Ω to G'_Ω;

(2) the transposed map ρ^0 of ρ is a homomorphism of $\Omega[G']$ into $\Omega[G]$ (if $f \in \Omega[G'], f$ is a map from G'_Ω to Ω and $\rho^0(f) = f \circ \rho$). In case G and G' are defined over k, the map ρ is a k-morphism if moreover ρ^0 maps $k[G']$ into $k[G]$. The differential $d\rho$ at the identity element of the morphism $\rho: G \to G'$ defines a homomorphism $d\rho: \mathfrak{g} \to \mathfrak{g}'$ of the corresponding Lie algebras.

A rational representation of G is a morphism $\rho: G \to \boldsymbol{GL}_m$. Let G be considered as a matrix group so that $\Omega[G] = \Omega[g_{11}, \cdots, g_{nn}, (\det g)^{-1}]$. Each coefficient of the matrix $\rho(g), g \in G$, is then a polynomial in $g_{11}, g_{12}, \cdots, g_{nn}, (\det g)^{-1}$.

2.2. *Characters.* A *character* of G is a rational representation of degree 1; $\chi: G \to \boldsymbol{GL}_1$. The set of characters of G is a commutative group, denoted by $X(G)$ or $\hat{G}$. The group $\hat{G}$ is finitely generated; it is free if G is connected **[8]**. If one wants to write the composition-law in $\hat{G}$ multiplicatively, the value at $g \in G$ of $\chi \in G$ should be noted $\chi(g)$. But since one is accustomed to add roots of Lie algebras, it is also natural to write the composition in $\hat{G}$ additively. The value of χ at g will then be denoted by g^χ. To see the similarity between roots and characters take $\Omega = \boldsymbol{C}$; if $X \in \mathfrak{g}$, the Lie algebra of G, $(e^X)^\chi = e^{d\chi(X)}$, where $d\chi$ is the differential of $d\chi$ at e; $d\chi$ is a linear form over $\mathfrak{g}$. In the sequel, we shall often not make any notational distinction between a character and its differential at e.

2.3. *Subgroups, quotients* **[4, 7]**. Let G be an algebraic group defined over k, H a closed subgroup of G; it is a k-subgroup of G if it is defined over k as an algebraic group. H is in particular k-closed. The converse need not be true. The homogeneous space G/H can be given in a natural way a structure of quasi-projective algebraic set defined over k. (A quasi-projective algebraic set is an algebraic set isomorphic to an open subset of a projective set.) The projection $\pi: G \to G/H$ is a k-morphism of algebraic sets which is "separable" ($d\pi$ is surjective everywhere) such that every morphism $\phi: G \to V$, constant along the cosets of H, can be factored through π. Moreover, G acts on G/H as an algebraic group of transformation; if H is a normal k-subgroup of G, then G/H is an algebraic group defined over k.

Assume that in G there exist a subgroup H and a normal subgroup N such that

(1) G is the semidirect product of H and N as abstract group,

(2) the map $\mu: H \times N \to G$, with $\mu(h, n) = hn$, is an isomorphism of algebraic varieties.

Then G is called the semidirect product of the algebraic groups H and N. In characteristic zero the condition (2) follows from (1). In characteristic $p > 0$, it is equivalent with the transversality of the Lie algebras of H and N or with the regularity of $d\mu$ at the origin, but does not follow from (1).

2.4. *Jordan decomposition of an element of an algebraic group* **[1, 4]**. Let

$$g \in \boldsymbol{GL}(n, \Omega),$$

g can be written uniquely as the product $g = g_s \cdot g_n$, where g_s is a semisimple matrix (i.e., g_s can be made diagonal) and g_n is a unipotent matrix (i.e., the only eigenvalue of g_n is 1, or equivalently $g_n - I$ is nilpotent) and $g_s \cdot g_n = g_n \cdot g_s$. If G is an algebraic matrix group and $g \in G$, one proves that g_n and g_s belong also to G and that the decomposition of g in a semisimple and an unipotent part does not depend on the representation of G as a matrix group. More generally, if $\phi: G \to G'$ is a morphism of algebraic groups and $g \in G$, then $\phi(g_s) = [\phi(g)]_s$ and $\phi(g_n) = [\phi(g)]_n$. If $g \in G_k$, g_s and g_n are rational over a purely inseparable extension of k.

3. **Algebraic tori [1, 3, 4]**. An algebraic group G is an *algebraic torus* if G is isomorphic to a product of d copies of $\boldsymbol{GL}_1$ (where $d = \dim G$).

If $\Omega = \boldsymbol{C}$, an algebraic torus is isomorphic to $(\boldsymbol{C}^*)^d$, and so is not an ordinary torus. However, the algebraic tori have many properties analogous to those of usual tori in compact real Lie groups. Since in what follows the tori in the topological sense will occur rarely, the adjective "algebraic" will be dropped.

3.1. THEOREM. *For a connected algebraic group G the following conditions are equivalent*:

(1) *G is a torus*;

(2) *G consists only of semisimple elements*;

(3) *G, considered as matrix group, can be made diagonal.*

Property (3) means that there always exists a basis of Ω^n such that G is represented by diagonal matrices with respect to that basis. Each diagonal element of the matrix, considered as a function on G, is then a character.

Let T be a torus of dimension d. Every element $x \in T$ can be represented by $(x_1, \cdots, x_d)$, with $x_i \in \Omega^*$. A character χ of T can then be written

$$\chi(x) = x_1^{n_1} x_2^{n_2} \cdots x_d^{n_d}$$

with $n_i \in \boldsymbol{Z}$ hence $\hat{T} \cong \boldsymbol{Z}^d$.

3.2. THEOREM. *Let T be a torus defined over k. The following conditions are equivalent*:

(1) *All characters of T are defined over k: $\hat{T} = \hat{T}_k$.*

(2) *T has a diagonal realization over k.*

(3) *For every representation $\rho: T \to \boldsymbol{GL}_m$, defined over k, the group $\rho(T)$ is diagonalizable over k.*

DEFINITION. If T satisfies these three equivalent conditions, T is called a *split k-torus*, and is said to *split over k*.

If T splits over k, so does every subtorus and quotient of T. There always exists a finite separable Galois-extension k'/k such that T splits over k'. The Galois-group operates on $\hat{T}$. This action determines completely the k-structure of T. The subgroup $\hat{T}_k$ is the set of characters left fixed by the Galois group.

DEFINITION. A torus T is called *anisotropic over k* if $\hat{T}_k = \{1\}$. The anisotropic tori are very close to the usual compact tori. Let $k = \boldsymbol{R}$. If $\dim T = 1$, there are two possibilities; either T splits over k, and then $T_{\boldsymbol{R}} \cong \boldsymbol{R}^*$, or T is anisotropic over k; then T is isomorphic over k with $\boldsymbol{SO}_2$, and $T_{\boldsymbol{R}} = \boldsymbol{SO}(2, \boldsymbol{R})$ is the circle group. In the general case $T_{\boldsymbol{R}}$ is compact if and only if T is anisotropic over $\boldsymbol{R}$

(this is also true if $\boldsymbol{R}$ is replaced by a p-adic field). In this case, $T_{\boldsymbol{R}}$ is a topological torus (product of circle groups).

3.3. THEOREM. *Let T be a k-torus. There exist two uniquely defined k-subtori T_d and T_a, such that*

(1) *T_d splits over k,*
(2) *T_a is anisotropic over k,*
(3) *$T_d \cap T_a$ is finite and $T = T_d \cdot T_a$.*

This decomposition is compatible with morphisms of algebraic groups. (Property 3 will be abbreviated by saying that T is the *almost direct product of* T_d and T_a.)

If S is a k-subtorus of T, then there exists a k-subtorus S' such that T is almost direct product of S and S'.

EXAMPLE. If $k = \boldsymbol{R}$, $T = T_1 \cdot T_2 \cdot \cdots \cdot T_d$ where every T_i is one dimensional. The product is an almost direct product.

4. Solvable, nilpotent and unipotent groups.

4.1. DEFINITION. The algebraic group G is *unipotent* if every element of G is unipotent.

EXAMPLE. If $\dim G = 1$ and G is connected unipotent then G is isomorphic to the additive group of the field;

$$G \cong \boldsymbol{G}_a = \left\{ g \in \boldsymbol{GL}_2 | g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}.$$

A connected and unipotent matrix group is conjugate to a group of upper-triangular matrices with ones in the diagonal. Hence it is nilpotent; more precisely there exists a central series

$$G = G_0 \supset G_1 \supset \cdots \supset G_i \supset G_{i+1} \supset \cdots \supset G_n = \{e\}$$

such that $G_i/G_{i+1} \cong \boldsymbol{G}_a$. Conversely, if there exists a normal series ending with $\{e\}$ such that $G_i/G_{i+1} \cong \boldsymbol{G}_a$, where G_i is an algebraic subgroup of G, then G is unipotent.

In characteristic 0, a unipotent algebraic group is connected, and the exponential is a bijective polynomial mapping from the Lie algebra $\mathfrak{g}$ to G; the inverse map is the logarithm. In characteristic $p > 0$, this is no more true; in that case, G is a p-group.

DEFINITION. G is a *solvable* (resp. *nilpotent*) algebraic group, if it is solvable (resp. nilpotent) as an abstract group.

4.2. We now state some basic properties of a connected solvable group G.

(1) (Theorem of Lie–Kolchin): If G is represented as a matrix group, it is conjugate (over Ω) to a group of triangular matrices **[1]**.

(2) If G operates on a complete algebraic variety (in particular on a projective variety) then G has a fixed point **[1]**.

(3) The set of unipotent elements in G is a normal connected subgroup U. If G is defined over k, it has a maximal torus defined over k; G is the semidirect

product, as algebraic group, of T and U; any two maximal tori defined over k are conjugate by an element of G_k (Rosenlicht, Annali di Mat. (iv), **61** (1963), 97–120; see also [**3**, §11]).

(4) G has a composition series

$$G = G_0 \supset G_1 \supset \cdots \supset G_i \supset G_{i+1} \supset \cdots \supset G_n = \{e\}$$

where the G_i are algebraic subgroups of G such that G_i/G_{i+1} is isomorphic with $\boldsymbol{G}_a$ or $\boldsymbol{GL}_1$.

(5) The group G is nilpotent if and only if it is the direct product of a maximal torus T and of a unipotent subgroup U. In this case T (resp. U) consists of all semisimple (resp. unipotent) elements of G.

Properties (1) and (2) are closely connected. In fact (2) implies (1): take the manifold of full flags (see §5.3) of the ambient vector space V, on which G acts in a natural way. Let F be a flag fixed by G; if one chooses a basis of V adapted to F, then G is triangular. On the other hand, for projective varieties, (2) follows immediately from (1). Property (4) is an immediate consequence of (1). In (3), one has to take care that, in contradistinction to the existence of a maximal torus defined over k, the normal subgroup U need not be defined over k, although it is k-closed [**8**].

4.3. Definition. Let G be a connected solvable group defined over k. G *splits over* k if there exists a composition series

$$G = G_0 \supset G_1 \supset \cdots \supset G_i \supset G_{i+1} \supset \cdots \supset G_m = \{e\}$$

consisting of connected k-subgroups of G such that G_i/G_{i+1} is isomorphic over k with $\boldsymbol{G}_a$ or $\boldsymbol{GL}_1$.

In particular, every torus T of G splits then over k. Conversely, when k is perfect, if the maximal tori of G which are defined over k split over k, then so does G.

Let G be a connected solvable k-group which splits over k, and V a k-variety on which G operates k-morphically. Then: (a) if V is complete and V_k is not empty, G has a fixed point in V_k [**8**]; (b) if G is transitive on V, the set V_k is not empty [**7**].

5. Radical. Parabolic subgroups. Reductive groups.

5.1. Definitions. Let G be an algebraic group. The *radical* $R(G)$ of G is the greatest connected normal subgroup of G; the *unipotent radical* $R_u(G)$ is the greatest connected unipotent normal subgroup of G. The group G is *semisimple* (resp. *reductive*) if $R(G) = \{e\}$ (resp. $R_u(G) = \{e\}$).

The definitions of $R(G)$ and $R_u(G)$ make sense, because if H, H' are connected normal and solvable (resp. unipotent) subgroups, then so is $H \cdot H'$. Both radicals are k-closed if G is a k-group. Clearly, $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$.

The quotient $G/R(G)$ is semisimple, and $G/R_u(G)$ is reductive. In characteristic zero, the unipotent radical has a complement; more precisely: Let G be defined over k. There exists a maximal reductive k-subgroup H of G such that

$$G = H \cdot R_u(G),$$

the product being a semidirect product of algebraic groups. If H' is a reductive subgroup of G defined over k, then H' is conjugate over k to a subgroup of H. In characteristic $p > 0$, this theorem is false (not just for questions of inseparability): according to Chevalley, there does not always exist a complement to the unipotent radical, moreover there are easy counter-examples to the conjugacy property [3].

5.2. THEOREM. *Let G be an algebraic group. The following conditions are equivalent*:

(1) *G^0 is reductive,*

(2) *$G^0 = S \cdot G'$, where S is a central torus and G' is semisimple,*

(3) *G^0 has a locally faithful fully reducible rational representation,*

(4) *If moreover the characteristic of Ω is 0, all rational representations of G are fully reducible.*

If G is a k-group, and $k = \boldsymbol{R}$, these conditions are also equivalent to the existence of a matrix realization of G such that $G_{\boldsymbol{R}}$ is "self-adjoint"

$$(g \in G_{\boldsymbol{R}} \Rightarrow {}^t g \in G_{\boldsymbol{R}}).$$

In property (2) G' is the commutator subgroup $\mathscr{D}(G)$ of G; it contains every semisimple subgroup of G. The group G is separably isogenous to $S \times G'$ and every torus T of G is separably isogenous to $(T \cap S) \times (T \cap G')$.

5.3. THEOREM [1]. *Let G be a connected algebraic group.*

(1) *All maximal tori of G are conjugate. Every semisimple element is contained in a torus. The centralizer of any subtorus is connected.*

(2) *All maximal connected solvable subgroups are conjugate. Every element of G belongs to one such group.*

(3) *If P is a closed subgroup of G, then G/P is a projective variety if and only if P contains a maximal connected solvable subgroup.*

The *rank* of G is the common dimension of the maximal tori, (notation $rk(G)$).

A closed subgroup P of G is called parabolic, if G/P is a projective variety. Following a rather usual practice, the speaker will sometimes allow himself to abbreviate "maximal connected closed solvable subgroup" by "Borel subgroup."

EXAMPLE. $G = \boldsymbol{GL}_n$. A *flag* in a vector space V is a properly increasing sequence of subspaces $0 \neq V_1 \subset \cdots \subset V_t \subset V_{t+1} = V$. The sequence (d_i)

$$(d_i = \dim V_i, i = 1, \cdots t))$$

describes the type of the flag. If $d_i = i$ and $t = \dim V - 1$, we speak of a *full flag*.

A parabolic subgroup of GL_n is the stability group of a flag F in Ω^n. G/P is the manifold of flags of the same type as F, and is well known to be a projective variety. A Borel subgroup is the stability group of a full flag. In a suitable basis, it is the group of all upper triangular matrices.

5.4. With respect to rationality question one can state that if G is a connected algebraic group defined over k, then

(1) G has a maximal torus defined over k (Grothendieck **[5]**, see also **[2]**). The centralizer of any k-subtorus is defined over k (**[5]**, **[3**, §10**]**).

(2) If k is infinite and G is reductive, G_k is Zariski dense in G (Grothendieck **[5]**, see also **[2]**).

(3) If k is infinite and perfect, G_k is Zariski dense in G (Rosenlicht **[8]**).

Rosenlicht has constructed an example of a one dimensional unipotent group defined over a field k of characteristic 2 such that G is not isomorphic to G_a over k and G_k is not dense in G **[8]**. An analogous example exists for every positive characteristic (Cartier).

6. **Structure theorems for reductive groups.** The results stated below are proved in **[3]**. Over perfect fields, some of them are established in **[6]**, **[9]**.

6.1. *Root systems.* Let V be a finite dimensional real vector space endowed with a positive nondegenerate scalar product. A subset Φ of V is a root system when

(1) Φ consists of a finite number of nonzero vectors that generate V, and is symmetric ($\Phi = -\Phi$).

(2) for every $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$, where s_α denotes reflection with respect to the hyperplane perpendicular to α.

(3) if α, $\beta \in \Phi$, then $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbf{Z}$. The group generated by the symmetrics s_α ($\alpha \in \Phi$) is called the Weyl group of Φ (notation $W(\Phi)$). It is finite. The integers $2(\alpha, \beta)/(\alpha, \alpha)$ are called the Cartan integers of Φ. Condition (3) means that for every α and β of Φ, $(s_\alpha(\beta) - \beta)$ is an integral multiple of α, since

$$s_\alpha(\beta) = \beta - 2\alpha(\alpha, \beta)/(\alpha, \alpha).$$

For the theory of reductive groups we shall have to enlarge slightly the notion of root system: if M is a subspace of V, we say that Φ is a root system in (N, M) if it generates a subspace P supplementary to M, and is a root system in P. The Weyl group $W(\Phi)$ is then understood to act trivially on M.

A root system Φ in V is the direct sum of $\Phi' \subset V'$ and $\Phi'' \subset V''$, if $V = V' \oplus V''$ and $\Phi = \Phi' \cup \Phi''$. The root system is called *irreducible* if it is not the direct sum of two subsystems.

6.2. *Properties of root systems.*

(1) Every root system is direct sum of irreducible root systems.

(2) If α and $\lambda\alpha \in \Phi$, then $\lambda = \pm 1, \pm\frac{1}{2}$, or ± 2.

The root system Φ is called *reduced* when for every $\alpha \in \Phi$, the only multiples of α belonging to Φ are $\pm\alpha$. To every root system Φ, there belongs two natural reduced systems by removing for every $\alpha \in \Phi$ the longer (or the shorter) multiple of α:

$$\Phi_s = \{\alpha \in \Phi | \tfrac{1}{2}\alpha \notin \Phi\},$$
$$\Phi_e = \{\alpha \in \Phi | 2\alpha \notin \Phi\}.$$

(3) The only reduced irreducible root systems are the usual ones:

$$\boldsymbol{A}_n \quad (n \geqq 1), \quad \boldsymbol{B}_n \quad (n \geqq 2), \quad \boldsymbol{C}_n \quad (n \geqq 3), \quad \boldsymbol{D}_n \quad (n \geqq 4),$$
$$\boldsymbol{G}_2, \quad \boldsymbol{F}_4, \quad \boldsymbol{E}_6, \quad \boldsymbol{E}_7, \quad \boldsymbol{E}_8.$$

(4) For each dimension n, there exists one irreducible nonreduced system, denoted by $\boldsymbol{BC}_n$ (see below).

EXAMPLES. $\boldsymbol{B}_n$: Take $\boldsymbol{R}^n$ with the standard metric and basis $\{x_1, \cdots, x_n\}$.

$$\boldsymbol{B}_n = \{\pm(x_i \pm x_j) \quad (i < j) \text{ and } \pm x_i \quad (1 \leqq i \leqq n)\}.$$

$W(\boldsymbol{B}_n) = \{s \in \boldsymbol{GL}(n, \boldsymbol{R}) | s$ a product of a permutation matrix with a symmetry with respect to a coordinate subspace$\}$

$$\boldsymbol{C}_n = \{\pm(x_i \pm x_j) \quad (i < j) \text{ and } \pm 2x_i \quad (1 \leqq i \leqq n)\},$$
$$W(\boldsymbol{C}_n) = W(\boldsymbol{B}_n),$$
$$\boldsymbol{BC}_n = \{\pm(x_i \pm x_j) \quad (i < j), \ \pm x_i \text{ and } \pm 2x_i \quad (1 \leqq i \leqq n)\},$$
$$W(\boldsymbol{BC}_n) = W(\boldsymbol{B}_n).$$

DEFINITION. A hyperplane of V is called *singular* if it is orthogonal to a root $\alpha \in \Phi$. A Weyl-chamber C^0 is a connected component of the complement of the union of the singular hyperplanes.

To a Weyl-chamber, is associated an ordering of the roots defined by:

$$\alpha > 0, \text{ if } (\alpha, v) > 0 \text{ for every } v \text{ in } C^0.$$

The root α is *simple* (relative to the given ordering) if it is not the sum of two positive roots. The set of simple roots is denoted by Δ. Δ is connected if it cannot be written as the union of $\Delta' \cup \Delta''$ where Δ' is orthogonal to Δ''.

(5) The Weyl group acts simply transitively on the Weyl-chambers (i.e., there is exactly one element of the Weyl group mapping a given Weyl-chamber on to another one).

(6) Every root of Φ is the sum of simple roots with integral coefficients of the same sign.

(7) The root system Φ is irreducible if and only if Δ is connected.

6.3. *Roots of a reductive group, with reference to a torus.* Let G be a reductive group, and S a torus of G. It operates on the Lie-algebra $\mathfrak{g}$ of G by the adjoint representation. Since S consists of semisimple elements, $\mathrm{Ad}_{\mathfrak{g}} S$ is diagonalizable

$$\mathfrak{g} = \mathfrak{g}_0^{(S)} \oplus \coprod_\alpha \mathfrak{g}_\alpha^{(S)}$$

where

$$\mathfrak{g}_\alpha^{(S)} = \{X \in \mathfrak{g} | \mathrm{Ad}\, s(x) = s^\alpha \cdot X\} \qquad (\alpha \in \hat{S}\,;\ \alpha \neq 0).$$

The set $\Phi(G, S)$ of roots of G relative to the torus S is the set of nontrivial characters of S appearing in the above decomposition of the adjoint representation. If $T \supset S$, every root of G relative to T that is not trivial on S defines a root relative to S. If T is maximal $\Phi(G, T) = \Phi(G)$ is the set of roots of G in the usual sense.

6.4. *Anisotropic reductive groups.* A connected reductive group G defined over k is called *anisotropic* over k, if it has no k-split torus $S \neq \{e\}$.

EXAMPLES. (1) Let F be a nondegenerate quadratic form on a k-vector space V with coefficients in k. Let $G = O(F)$ be the orthogonal group of F. The group G is anisotropic over k if and only if F does not represent 0 over k, i.e., if V_k has no nonzero isotropic vector.

PROOF. Assume v is an isotropic vector. Then there exists a hyperbolic plane through v and in a suitable basis the quadratic form becomes

$$F(x_1, \cdots, x_n) = x_1 x_2 + F'(x_3, \cdots, x_n).$$

If $\lambda \in \Omega^*$, the set of transformations

$$x_1' = \lambda x_1, \qquad x_2' = \lambda^{-1} \cdot x_2, \qquad x_i' = x_i \qquad (i \geqq 3)$$

is a torus of G split over k. Conversely if there exists a torus S of G which splits over k, diagonalize S. There is a vector $v \in V_k - \{0\}$ and a nontrivial character $\chi \in \hat{S}$, such that $s(v) \equiv s^\chi v$. Since $s^\chi \neq \pm 1$ for some s, and $F(v) = F(s(v))$, one has $F(v) = 0$ and v is isotropic.

(2) If $k = \boldsymbol{R}$ or is a $\mathfrak{p}$-adic field, G is anisotropic over k if and only if G_k is compact. If k is an arbitrary field of characteristic 0, G is anisotropic over k if and only if G_k has no unipotent element $\neq e$ and $\hat{G}_k = \{1\}$.

6.5. *Properties of reductive k-groups.* Let G be a connected reductive group defined over k.

(1) The maximal k-split tori of G are conjugate over k (i.e., by elements of G_k). If S is such a maximal k-split torus, the dimension of S is called the k-rank of G (notation: $rk_k(G)$). $Z(S)$ is the connected component of $N(S)$. The finite group $N(S)/Z(S)$ is called the Weyl group of G relative to k (notation: ${}_kW(G)$). Every coset of $N(S)/Z(S)$ is represented by an element rational over k: $N(S) = N(S)_k Z(S)$.

(2) The elements of $\Phi(G, S)$, where S is a maximal k-split torus are called the k-roots, or roots relative to k. We write ${}_k\Phi$ or ${}_k\Phi(G)$ for $\Phi(G, S)$. This is a root system in $(\hat{S} \otimes \boldsymbol{R}, M)$ where M is the vector space over $\boldsymbol{R}$ generated by the characters which are trivial on $S \cap \mathscr{D}(G)$. The Weyl group of G relative to k and the Weyl group of ${}_k\Phi$ are isomorphic:

$$W({}_k\Phi) \cong {}_kW(G).$$

If G is simple over k, ${}_k\Phi$ is irreducible.

(3) The minimal parabolic k-subgroups P of G are conjugate over k. Furthermore there exists a k-split torus S such that

$$P = Z(S) \cdot R_u(P)$$

where the semidirect product is algebraic and everything is defined over k. If P and P' are minimal parabolic k-subgroups containing a maximal k-split torus S, then $P \cap P'$ contains the centralizer of S. The minimal parabolic k-subgroups containing $Z(S)$ are in (1, 1) correspondence with the Weyl chambers: P corresponds to the Weyl chamber C if the Lie algebra of $R_u(P)$ is $\sum_{\alpha>0} \mathfrak{g}_\alpha^{(S)}$, where the ordering of the roots is associated to the Weyl-chamber C. The Weyl group ${}_kW(G)$ permutes in a simply transitive way the minimal parabolic k-subgroups containing $Z(S)$. The unipotent radical of a minimal parabolic k-subgroup is a maximal unipotent k-subgroup, at least for a field of characteristic 0.

(4) Bruhat decomposition of G_k. Put $V = R_u(P)$, where P is a minimal parabolic k-subgroup. Then

$$G_k = U_k \cdot N(S)_k \cdot U_k,$$

and different elements of $N(S)_k$ define different double cosets; more generally if $n, n' \in N(S)$: $UnU = Un'U \Leftrightarrow n = n'$. Choose for every $w \in {}_kW$ a representative $n_w \in N(S)_k$; then the above equality can be written as

$$G_k = \bigcup_{w \in {}_kW} U_k \cdot n_w \cdot P_k,$$

the union being disjoint.

One can phrase this decomposition in a more precise way. If we fix $w \in {}_kW$, then there exist two k-subgroups U'_w and U''_w of U, such that $U = U'_w \times U''_w$ as an algebraic variety and such that the map of $U'_w \times P$ onto Un_wP sending (x, y) onto xn_wy is a biregular map defined over k. This decomposition gives rise to a cellular decomposition of G_k/P_k. Let π be the projection of G onto G/P. Then

$$(G/P)_k = G_k/P_k = \bigcup_{w \in {}_kW} \pi(U'_{w,k}).$$

If k is algebraically closed, $U'_{w''}$ as a unipotent group is isomorphic to an affine space. So one gets a cellular decomposition of G/P.

(5) *Standard parabolic k-subgroups* (with respect to a choice of S and P). Let ${}_k\Phi$ be the root system of G relative to k defined by the torus S. The choice of the minimal parabolic k-subgroup P determines a Weyl chamber of ${}_k\Phi$ and so a set of positive roots. Let ${}_k\Delta$ be the set of simple k-roots for this ordering. If Θ is a subset of ${}_k\Delta$, denote by S_Θ the identity component of $\bigcap_{\alpha \in \Theta} \ker \alpha$. S_Θ is a k-split torus, the dimension of which is $\dim S_\Theta = rk_k(G) - \text{card}\, \Theta$. The standard parabolic k-subgroup defined by Θ is then the subgroup ${}_kP_\Theta$ generated by $Z(S_\Theta)$ and U. That subgroup can be written as the semidirect product $Z(S_\Theta)$. U_Θ, where $U_\Theta = R_u(P_\Theta)$. The Lie algebra of U_Θ is $\sum \mathfrak{g}_\alpha$, the sum going over all positive roots that are not linear combination of elements in Θ.

(6) Every parabolic k-subgroup is conjugate over k to one and only one standard parabolic k-subgroup. In particular, if two parabolic k-subgroups are conjugate over Ω, they are already conjugate over k.

(7) Let W_Θ be the subgroup of the Weyl group ${}_kW$ generated by the reflections s_α for $\alpha \in \Theta$. Then if Θ and Θ' are two subsets of ${}_k\Delta$,

$$ {}_kP_{\Theta,k}\backslash G_k/{}_kP_{\Theta',k} \cong W_\Theta\backslash{}_kW/W_{\Theta'}. $$

6.6. EXAMPLES. (1) $G = GL(n)$,

$$ S = \text{group of diagonal matrices} = \left\{\begin{pmatrix} s^{\lambda_1} & & & 0 \\ & s^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & s^{\lambda_n}\end{pmatrix}\right\} $$

where $\lambda_i \in \hat{S}$ is such that $s^{\lambda_i} = s_{ii}$. S is obviously a split torus and is maximal. A minimal parabolic k-subgroup P is given by the upper triangular matrices, which is in this case a Borel subgroup. The unipotent radical U of P is given by the group of upper triangular matrices with ones in the diagonal. If e_{ij} is the matrix having all components zero except that with index (i,j) equal to 1, $\mathrm{Ad}_G\, s(e_{ij}) = (s^{\lambda_i}/s^{\lambda_j})e_{ij}$. So the positive roots are $\lambda_i - \lambda_j$ $(i < j)$ since the Lie algebra of U is generated by e_{ij} $(i < j)$. The simple roots are $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n)$. The Weyl group is generated by s_α, where α is a positive root; since for $\alpha = \lambda_i - \lambda_j$, s_α permutes the i and j axis, ${}_kW \cong \mathfrak{S}_n$, the group of permutations of the basis elements. The parabolic subgroups are the stability groups of flags.

(2) G "splits over k" (i.e., G has a maximal torus which splits over k). Example (1) enters in this category. The k-roots are just the usual roots. A minimal parabolic k-subgroup is a maximal connected solvable subgroup. If k is algebraically closed G always splits over k and this gives just the usual properties of semi-simple or reductive linear groups.

(3) G is the orthogonal group $SO(F)$ of a nondegenerate quadratic form F on a vector space V_k (where, to be safe, one takes char $k \neq 2$). In a suitable basis

$$ F(x_1, \cdots, x_n) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + F_0(x_{q+1}, \cdots, x_{n-q}) $$

where F_0 does not represent zero rationally. The index of F, the dimension of the maximal isotropic subspaces in V_k, is equal to q. A maximal k-split torus S is given by the set of following diagonal matrices:

$$ \begin{pmatrix} s^{\lambda_1} & & & & & & & & \\ & s^{\lambda_2} & & & & & & & \\ & & \ddots & & & & & & \\ & & & s^{\lambda_q} & & & & & \\ & & & & 1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & 1 & & \\ & & & & & & & s^{-\lambda_q} & & & \\ & 0 & & & & & & & s^{-\lambda_{q-1}} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & s^{-\lambda_1} \end{pmatrix} $$

Let $SO(F_0)$ denote the proper orthogonal group of the quadratic form F_0, imbedded in $SO(F)$ by acting trivially on $x_1, \cdots, x_q$, $x_{n-q+1}, \ldots, x_n$. Then $Z(S) = S \times SO(F_0)$. The minimal parabolic k-subgroups are the stability groups of the full isotropic flags. For the above choice of S, and ordering of the coordinates, the standard full isotropic flag is

$$[e_1] \subset [e_1, e_2] \subset \cdots \subset [e_1, \cdots, e_q].$$

The corresponding minimal parabolic k-subgroup takes then the form

$$P = \left\{ \begin{pmatrix} A_0 & A_1 & A_2 \\ 0 & B & A_3 \\ 0 & 0 & A_4 \end{pmatrix} \right\}$$

where A_0 and A_4 are upper triangular $q \times q$ matrices, $B \in SO(F_0)$, with additional relations that insure that $P \subset SO(F)$. The unipotent radical U of P is the set of matrices in P, where $B = I$, A_0, A_4 are unipotent, and

$$A_4 = {}^{\sigma}A_0^{-1}; \qquad Q \cdot A_3 + {}^{t}A_1 \cdot J \cdot A_4 = 0,$$

$${}^{t}A_4 \cdot J \cdot A_2 + {}^{t}A_3 \cdot Q \cdot A_3 + {}^{t}A_2 \cdot J \cdot A_4 = 0,$$

where Q is the matrix of the quadratic form F_0, J is the $q \times q$ matrix with one's in the nonprincipal diagonal and zeros elsewhere, and σ is the transposition with respect to the same diagonal, (${}^{\sigma}M = J{}^{t}MJ$). To determine the positive roots, one has to let S operate on U. To compute the root spaces it is easier to diagonalize Q: $q_{ij} = d_i \cdot \delta_{ij}$. Three cases are to be considered.

$i < j \leqq q$; $\lambda_i - \lambda_j$ is a root; the corresponding root space is generated by $e_{ij} - e_{n-j+1,n-i+1}$; the multiplicity of the root is 1.

$i \leqq q < j \leqq n - q$; λ_i is a root with multiplicity $n - 2q$; the corresponding root space is generated by

$$e_{ij} - d_j^{-1} e_{j,n-i+1} \qquad (q + 1 \leqq j < n - q).$$

$i < j \leqq q$; $\lambda_i + \lambda_j$ is a root with multiplicity one; the corresponding root space is generated by $e_{i,n-j+1} - e_{j,n-i+1}$. The simple roots are

$$\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{q-1} - \lambda_q,$$

and λ_q if $n \neq 2q$, $\lambda_{q-1} + \lambda_q$ if $n = 2q$. The Weyl group consists of all products of permutation matrices with symmetries with respect to a coordinate subspace (of any dimension if $n \neq 2q$ of even dimension if $n = 2q$). The group $SO(F)$ splits if and only if $q = [n/2]$. If it does not split, there exist roots with multiplicity > 1. The parabolic k-subgroups are the stability subgroups of rational isotropic flags. The parabolic k-subgroups are conjugate over k if and only if there exists an element of G_k mapping one flag onto the other; by Witt's theorem this is possible if and only if the two flags have the same type.

(4) When one starts with a hermitian form, the same considerations apply, except that one gets a root system of type BC_q.

(5) For real Lie groups, this theory is closely connected with the Iwasawa and Cartan decompositions. If $\mathfrak{g}$ is the real Lie algebra of $G_{\mathbf{R}}$, G being a connected algebraic reductive group, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{g}$ is the Lie algebra of a maximal compact subgroup of $G_{\mathbf{R}}$. Then $G = K \cdot (\exp \mathfrak{p})$. Let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathfrak{p}$. Then $A = \exp \mathfrak{a}$ is the topological connected component of the groups of real points in a torus S which is maximal among $\mathbf{R}$-split tori. (On the Riemannian symmetric space $G_{\mathbf{R}}/K$, it represents a maximal totally geodesic flat subspace.)

$$N(A)_{\mathbf{R}} = N(S)_{\mathbf{R}} = [K \cap N(A)] \cdot A,$$

and

$$Z(S)_{\mathbf{R}} = (K \cap Z(A)) \cdot A;$$

the group $K \cap Z(A)$ is usually denoted by M. The Weyl group ${}_{\mathbf{R}}W(G, S)$ is isomorphic to $(K \cap N(A))/M$, i.e., to the Weyl group of the symmetric space G/K as introduced by E. Cartan. Similarly ${}_{\mathbf{R}}\Phi$ may be identified to the set of roots of the symmetric pair (G, K). Let $\mathfrak{n}$ be the Lie subalgebra generated by the root spaces corresponding to positive roots $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{p}_\alpha^{(S)}$, $\alpha \in {}_{\mathbf{R}}\Phi(G, S)$, for some ordering. Let $N = \exp \mathfrak{n}$. Then $G = K \cdot A \cdot N$ is an Iwasawa decomposition and $M \cdot A \cdot N$ is the group of real points of a minimal parabolic $\mathbf{R}$-group. Assume $G_{\mathbf{R}}$ simple and G/K to be a bounded symmetric domain. Then there are two possibilities for the root system ${}_{\mathbf{R}}\Phi$:

$$G_{\mathbf{R}}/K \text{ is a tube domain} \Leftrightarrow {}_{\mathbf{R}}\Phi \text{ is of type } C_t,$$

$$G_{\mathbf{R}}/K \text{ is not a tube domain} \Leftrightarrow {}_{\mathbf{R}}\Phi \text{ is of type } BC_t.$$

7. **Representations in characteristic zero** [3]. We assume here the ground field to be of characteristic zero, and G to be semisimple, connected. Let $P = Z(S) \cdot U$ be a minimal parabolic k-group, where $U = R_u(P)$, and S is a maximal k-split torus. We put on $X(S)$ an ordering such that u is the sum of the positive k-root spaces.

Assume first k to be algebraically closed. Let $\rho: G \to \mathrm{GL}(V)$ be an irreducible representation. It is well known that there is one and only one line $D_\rho \subset V$ which is stable under P. The character defined by the 1-dimensional representation of P in V is the highest weight λ_ρ of ρ. The orbit $G(D_\rho) = \mathscr{C}_\rho$ is a closed homogeneous cone (minus the origin). The stability group of D_ρ is a standard parabolic group $P_\rho \supset P$. The stability groups of the lines in $\mathscr{C}_\rho$ are conjugate to P_ρ, and these lines are the only ones to be stable under some parabolic subgroup of G. Every highest weight λ_ρ is a sum $\lambda_\rho = \sum_{\alpha \in \Delta} c_\alpha \cdot \Lambda_\alpha (c_\alpha \geqq 0, c_\alpha \in \mathbf{Z})$ of the fundamental highest weights Λ_α (and conversely if G is simply connected), where Λ_α is defined by $2(\Lambda_\alpha, \beta) \cdot (\beta, \beta)^{-1} = \delta_{\alpha\beta} (\alpha, \beta \in \Delta)$.

We want to indicate here a "relativization" of these facts for a nonnecessarily algebraically closed k.

Let T be a maximal torus of G, defined over k, containing S. We choose an ordering on $X(T)$ compatible with the given one on $X(S)$ (i.e., if $\alpha > 0$, and $r(\alpha) \neq 0$, then $r(\alpha) > 0$ where $r: X(T) \to X(S)$ is the restriction homomorphism. The k-weights of ρ are the restrictions to S of the weights of ρ with respect to T; the highest k-weight μ_ρ is the restriction of λ_ρ. It follows from standard facts that every k-weight μ is of the form

$$\mu = \mu_\rho - \sum m_\alpha(\mu)_\alpha, \qquad (\alpha \in \Delta),$$

with

$$m_\alpha(\mu) \in \boldsymbol{Z}, \qquad m_\alpha(\mu) \geqq 0.$$

Let

$$\Theta(\mu) = \{\alpha \in {}_k\Delta | m_\alpha(\mu) \neq 0\}.$$

Then $\Theta \subset {}_k\Delta$ is a $\Theta(\mu)$, for some k-weight μ, if and only if $\Theta(\mu) \cup \mu_\rho$ is connected.

Let us say that ρ is *strongly rational* over k if it is defined over k and if the cone $G(D_\rho)$ has a rational point over k. This is the case if and only if the above coefficients $\subset_\alpha$ satisfy the following conditions:

$$\subset_\alpha = 0 \text{ if } r(\alpha) = 0, \qquad \subset_\alpha = \subset_\beta \text{ if } r(\alpha) = r(\beta) \qquad (\alpha, \beta \in \Delta).$$

The highest weight of a strongly rational representation is a sum, with positive integral coefficients, of fundamental highest weights $M_\beta (\beta \in {}_k\Delta)$ where

$$M_\beta = \sum_{\alpha \in \Delta, r(\alpha) = \beta} r(\Lambda_\alpha)$$

(and conversely if G is simply connected). The $M_\beta (\beta \in {}_k\Delta)$ satisfy relations of the form $(M_{\beta,}\gamma) = d_\beta \cdot \delta_{\beta,\gamma}$, with $d_\beta > 0$. They will be called the *fundamental highest k-weights.*

Assume $k \subset \boldsymbol{C}$. Let ρ be strongly rational over k. Put on the representation space a Hilbert structure. Let $v \in D_\rho - 0$. Then the function $\phi: G \to \boldsymbol{R}^+$ defined by

$$\phi(g) = \|\rho(g) \cdot v\|,$$

satisfies

$$\phi(g \cdot p) = \phi(g)|p^{\mu_\rho}| \qquad (g \in G, \quad p \in P_\rho).$$

If in particular $k = \boldsymbol{Q}$, such functions appear in the discussion of fundamental sets and of Eisenstein series for arithmetic groups.

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