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AUTOMORPHIC *L*-FUNCTIONS

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This paper is mainly devoted to the L-functions attached by Langlands [**35**] to an irreducible admissible automorphic representation π of a reductive group G over a global field k and to local and global problems pertaining to them. In the context of this Institute, it is meant to be complementary to various seminars, in particular to the $\mathbf{GL}_2$-seminars, and to stress the general case. We shall therefore start directly with the latter, and refer for background and motivation to other seminars, or to some expository articles on this topic in general [**3**] or on some aspects of it [**7**], [**14**], [**15**], [**23**].

The representation π is a tensor product $\pi = \bigotimes_v \pi_v$ over the places of k, where π_v is an irreducible admissible representation of $G(k_v)$ [**11**]. Accordingly the L-functions associated to π will be Euler products of local factors associated to the π_v's. The definition of those uses the notion of the L-group ${}^LG$ of, or associated to, G. This is the subject matter of Chapter I, whose presentation has been much influenced by a letter of Deligne to the author. The L-function will then be an Euler product $L(s, \pi, r)$ assigned to π and to a finite dimensional representation r of ${}^LG$. (If $G = \mathbf{GL}_n$, then the L-group is essentially $\mathbf{GL}_n(C)$, and we may tacitly take for r the standard representation r_n of $\mathbf{GL}_n(C)$, so that the discussion of $\mathbf{GL}_n$ can be carried out without any explicit mention of the L-group, as is done in the first six sections of [**3**].) The local L- and ε-factors are defined at all places where G and π are "unramified" in a suitable sense, a condition which excludes at most finitely many places. Chapter II is devoted to this case. The main point is to express the Satake isomorphism in terms of certain semisimple conjugacy classes in ${}^LG$ (7.1). At this time, the definition of the local factors at the ramified places is not known in general. For $\mathbf{GL}_n$ and r_n, however, there is a direct definition [**19**], [**25**]. In the general case, the most ambitious scheme is to associate canonically to an irreducible admissible representation of a reductive group H over a local field E a representation of the Weil-Deligne group W_E' of E into ${}^LH$, and then use L- and ε-factors associated to representations of W_E' [**60**]. This problem is the main topic of Chapter III.

The L-function $L(s, \pi, r)$ associated to π and r as above is introduced in §13. In fact, it is defined in general as a product of local factors indexed by almost all places of k. It converges absolutely in some right half-plane (13.3; 14.2). Some of the main

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conjectural analytic properties (meromorphic continuation, functional equation), and the evidence known so far, are discussed in §14.

From the point of view of **[35]**, a great many problems on automorphic representations and their L-functions are special cases of one, the so-called lifting problem or problem of functoriality with respect to L-groups. It is discussed in Chapter V. It is closely connected with Artin's conjecture (see §17 and the base-change seminar **[17]**). In §18 brief mention is made of some known or conjectured relations between automorphic L-functions and the Hasse-Weil zeta-function of certain varieties, to be discussed in more detail in the seminars on Shimura varieties **[8]**, **[40]**.

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CONTENTS

CHAPTER I. L-GROUPS.

k is a field, $\bar{k}$ an algebraic closure of k, k_s the separable closure of k in $\bar{k}$, and Γ_k the Galois group of k_s over k. G is a connected reductive group, over $\bar{k}$ in 1.1, 1.2, 2.1, 2.2, over k otherwise.

§§1, 2 will be used throughout, §3 from Chapter III on. The reader willing to take on faith various statements about restriction of scalars need not read §§4, 5.

1. Classification. We recall first some facts discussed in [58].

1.1. There is a canonical bijection between isomorphism classes of connected reductive $\bar{k}$-groups and isomorphism classes of root systems. It is defined by associating to G the root datum $\psi(G) = (X^*(T), \varphi, X_*(T), \varphi^v)$ where T is a maximal torus of G, $X^*(T)$ ($X_*(T)$) the group of characters (1-parameter subgroups) of T and Φ (Φ^v) the set of roots (coroots) of G with respect to T.

1.2. The choice of a Borel subgroup $B \supset T$ is equivalent to that of a basis Δ of $\Phi(G, T)$. The previous bijection yields one between isomorphism classes of triples (G, B, T) and isomorphism classes of based root data $\psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^v)$. There is a split exact sequence

$$(1) \qquad (1) \longrightarrow \text{Int } G \longrightarrow \text{Aut } G \longrightarrow \text{Aut } \psi_0(G) \longrightarrow (1).$$

To get a splitting, we may choose $x_\alpha \in G_\alpha$ ($\alpha \in \Delta$) and then have a canonical bijection

$$(2) \qquad \text{Aut } \psi_0(G) \xrightarrow{\sim} \text{Aut } (G, B, T, \{x_\alpha\}_{\alpha \in \Delta}).$$

Two such splittings differ by an inner automorphism Int t ($t \in T$).

1.3. Given $\gamma \in \Gamma_k$ there is $g \in G(k_s)$ such that $g \cdot {}^\gamma T \cdot g^{-1} = T$, $g \cdot {}^\gamma B \cdot g^{-1} = B$, whence an automorphism of $\psi_0(G)$, which depends only on γ. We let $\mu_G : \Gamma_k \to \text{Aut } \psi_0(G)$ be the homomorphism so defined. If G' is a k-group which is isomorphic to G over $\bar{k}$ (hence over k_s), then $\mu_G = \mu_{G'} \Leftrightarrow G, G'$ are inner forms of each other.

1.4. Let $f: G \to G'$ be a k-morphism, whose image is a normal subgroup. Then f induces a map $\psi(f): \psi(G) \to \psi(G')$ (contravariant (resp. covariant) in the first (last) two arguments). Given $B, T \subset G$ as above, there exists a Borel subgroup B' (resp. a maximal torus T') of G' such that $f(B) \subset B', f(T) \subset T'$, whence also a map $\psi_0(f)$: $\psi_0(G) \to \psi_0(G')$.

2. Definition of the L-group.

2.1. The inverse system $\Psi_0^\vee$ to the based root datum $\Psi_0 = (M, \Delta, M^*, \Delta^\vee)$ is $\psi_0^\vee = (M^*, \Delta^\vee, M, \Delta)$. To the $\bar{k}$-group G we first associate the group ${}^LG^\circ$ over $\boldsymbol{C}$ such that $\psi_0({}^LG^\circ) = \psi_0(G)^\vee$. We let ${}^LT^\circ$, ${}^LB^\circ$ be the maximal torus and Borel subgroup defined by $\psi_0^\vee$, and say they define the canonical splitting of ${}^LG^\circ$.

Let f be as in 1.4. Then f also induces a map $\psi_0^\vee(f) : \psi_0(G')^\vee \to \psi_0(G)^\vee$. An algebraic group morphism of ${}^LG'^\circ$ into ${}^LG^\circ$ associated to it will be denoted ${}^Lf^\circ$. Given one, any other is of the form Int $t \circ {}^Lf^\circ \circ$Int t' ($t \in {}^LT^\circ$, $t' \in {}^LT'^\circ$), and maps ${}^LT'^\circ$ (resp. ${}^LB'^\circ$) into ${}^LT^\circ$ (resp. ${}^LB^\circ$).

2.2. Examples. (1) Let $G = \mathbf{GL}_n$. Then ${}^LG^\circ = \mathbf{GL}_n$. In fact, let $M = \boldsymbol{Z}^n$ with $\{x_i\}$ its canonical basis. Let $\{e_i\}$ be the dual basis of $M^* = \boldsymbol{Z}^n$. Then $\Psi_0(\mathbf{GL}_n) = (M, \Delta, M^*, \Delta^\vee)$ with $\Delta = \{(x_i - x_{i+1}), 1 \leqq i < n\}$, $\Delta^\vee = \{(e_i - e_{i+1}), 1 \leqq i < n\}$, hence $\psi_0 = \psi_0^\vee$.

(2) Let G be semisimple and $\Psi_0(G) = (M, \Phi, M^*, \Phi^\vee)$. As usual, let $P(\Phi) \subset M \otimes \boldsymbol{Q}$ be the lattice of weights of Φ and $Q(\Phi)$ the group generated by Φ in M. Define $P(\Phi^\vee)$ and $Q(\Phi^\vee)$ similarly.

As is known G is simply connected (resp. of adjoint type) if and only if $P(\phi) = M$ (resp. $Q(\phi) = M$). Moreover

$$P(\Phi) = \{\lambda \in M \otimes \boldsymbol{Q} | \langle \lambda, \Phi^\vee \rangle \subset \boldsymbol{Z}\}, \qquad P(\Phi^\vee) = \{\lambda \in M^* \otimes \boldsymbol{Q} | \langle \lambda, \Phi \rangle \in \boldsymbol{Z}\}.$$

Therefore:

G simply connected $\Leftrightarrow {}^LG^\circ$ of adjoint type;

G of adjoint type $\Leftrightarrow {}^LG^\circ$ simply connected.

(3) Let G be simple. Up to central isogeny, it is characterized by one of the types A_n, ..., G_2 of the Killing-Cartan classification. It is well known that the map $\psi_0(G) \to \psi_0(G)^v$ permutes B_n and C_n and leaves all other types stable. Thus if $G = \mathbf{Sp}_{2n}$ (resp. $G = \mathbf{PSp}_{2n}$), then ${}^LG^\circ = SO_{2n+1}$ (resp. ${}^LG^\circ = \mathbf{Spin}_{2n+1}$). In all other cases, $G \mapsto {}^LG^\circ$ preserves the type (but goes from simply connected group to adjoint group, and vice versa).

(4) Let again G be reductive and let $f : G \to G'$ be a central isogeny. Let

$$N = \operatorname{coker} f_*: X_*(T) \longrightarrow X_*(T') \qquad (T' = f(T)),$$
$$N' = \operatorname{coker} f^*: X^*(T') \longrightarrow X^*(T).$$

Then N and N' are isomorphic and $\ker {}^Lf^\circ = \operatorname{Hom}(N, C^*) \xrightarrow{\sim} N$. In particular, ${}^Lf^\circ$ is an isomorphism if and only if f is one.

(5) Let $f : G \to G'$ be a central surjective morphism, $Q = \ker f$, and Q° the identity component of Q. Then $\ker {}^Lf^\circ \simeq Q/Q^\circ$.

If Q is connected, then $T'' = T \cap Q$ is a maximal torus of Q, and the injectivity of ${}^Lf^\circ$ follows from the fact that the exact sequence $1 \to T'' \to T \to T' \to 1$ necessarily splits. If Q is not connected, then $r : H = G/Q^\circ \to G'$ is a nontrivial separable isogeny, with kernel Q/Q°. ${}^Lf^\circ$ factors through ${}^Lr^\circ$ and, by the first part and (4), $\ker {}^Lf^\circ = \ker {}^Lr^\circ \cong Q/Q^\circ$. In particular, if we apply this to the case where $G' = G_{ad}$ is the adjoint group of G, and use (2), we see that the derived group of ${}^LG^\circ$ is simply connected if and only if the center of G is connected. As an example, let $G = \mathbf{GSp}_{2n}$ be the group of symplectic similitudes on a $2n$-dimensional space. Then the derived group of ${}^LG^\circ$ is isomorphic to $\mathbf{Spin}_{2n+1}$. In fact, we have ${}^LG^\circ = (\mathbf{GL}_1 \times \mathbf{Spin}_{2n+1})/A$ where $A = \{1, a\}$ and $a = (a_1, a_2)$, with a_1 of order two in $\mathbf{GL}_1$ and a_2 the nontrivial central element of $\mathbf{Spin}_{2n+1}$. If $n = 2$, then $\mathbf{Spin}_{2n+1} = \mathbf{Sp}_{2n}$. It follows that if $G = \mathbf{GSp}_4$, then ${}^LG^\circ = \mathbf{GSp}_4(C)$.

2.3. We have canonically $\operatorname{Aut} \Psi_0 = \operatorname{Aut} \Psi_0^\vee$. Therefore we may view μ_G as a homomorphism of Γ_k into $\operatorname{Aut} \psi_0^\vee$. Choose a monomorphism

$$\text{(1)} \qquad \operatorname{Aut} \psi_0^\vee \longrightarrow \operatorname{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ)$$

as in 1.2(2). We have then a homomorphism

$$\mu_G': \Gamma_k \longrightarrow \operatorname{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ).$$

The associated group to, or L-group of, G is then by definition the semidirect product

$$\text{(2)} \qquad {}^L(G/k) = {}^LG = {}^LG^\circ \rtimes \Gamma_k,$$

with respect to μ_G'. We note that μ_G' is well defined up to an inner automorphism by an element of ${}^LT^\circ$. The group ${}^LG$ is viewed as a topological group in the obvious way. The canonical splitting of ${}^LG^\circ$ (2.1) is stable under Γ_k.

We have a canonical projection ${}^LG \to \Gamma_k$ with kernel ${}^LG^\circ$. The splittings of the exact sequence

$$\text{(3)} \qquad 1 \longrightarrow {}^LG^\circ \longrightarrow {}^LG \xrightarrow{\nu_G} \Gamma_k \longrightarrow 1$$

defined as in 1.2 via an isomorphism $\mathrm{Aut}\,\Psi_0^\vee \xrightarrow{\sim} \mathrm{Aut}({}^LG^\circ, {}^LB^\circ, {}^LT^\circ, \{x_\alpha\})$ are called admissible. They differ by inner automorphisms $\mathrm{Int}\, t$ ($t \in {}^LT^\circ$). Note that if G splits over k, then Γ_k acts trivially on ${}^LG^\circ$ and ${}^LG$ is simply the direct product of ${}^LG^\circ$ and Γ_k.

2.4. REMARKS. (1) So far, we can in this definition take ${}^LG^\circ$ over any field. We have chosen $\boldsymbol{C}$ since this is the most important case at present, but it is occasionally useful to use other local fields.

(2) There are various variants of this notion, which may be more convenient in certain contexts. For instance we can divide Γ_k by a closed normal subgroup which acts trivially on $\Psi^\vee$, hence on ${}^LG^\circ$, e.g., by $\Gamma_{k'}$ if k' is a Galois extension of k over which G splits. Then Γ_k is replaced by $\mathrm{Gal}(k'/k)$, and ${}^LG$ is a complex reductive Lie group.

We can also define a semidirect product ${}^LG^\circ \rtimes \Sigma$, for any group Σ endowed with a homomorphism into Γ_k, e.g., the Weil group of k, if k is a local or global field. In that case, we get the "Weil form" of ${}^LG$.

(3) Let G' be a k-group which is isomorphic to G over $\bar{k}$. Then G and G' are inner forms of each other if and only if ${}^LG$ is isomorphic to ${}^LG'$ over Γ_k. In fact, the first condition is equivalent to $\mu_G = \mu_{G'}$, and the latter is easily seen to be equivalent to the second condition. In particular, since two quasi-split groups over k which are inner forms of each other are isomorphic over k, it follows that if G, G' are quasi-split and ${}^LG \xrightarrow{\sim} {}^LG'$ over Γ_k then G and G' are k-isomorphic.

2.5. *Functoriality.* Let $f\colon G \to G'$ be a k-morphism whose image is a normal subgroup. Then $f_{\psi_0}\colon \psi_0(G) \to \psi_0(G')$ clearly commutes with Γ_k, hence so does $f_{\psi_0}^\vee : \psi_0(G')^\vee \to \psi_0(G)^\vee$ and ${}^Lf^\circ\colon {}^LG'^\circ \to {}^LG^\circ$. We get therefore a continuous homomorphism ${}^Lf : {}^LG' \to {}^LG$ such that

$$\begin{array}{ccc} {}^LG' & \xrightarrow{\ {}^Lf\ } & {}^LG \\ & {\scriptstyle \nu_{G'}} \searrow \quad \swarrow {\scriptstyle \nu_G} & \\ & \Gamma_k & \end{array}$$

is commutative, which extends ${}^Lf^\circ$.

2.6. *Representations.* For brevity, by *representation* of ${}^LG$ we shall mean a continuous homomorphism $r\colon {}^LG \to \mathbf{GL}_m(\boldsymbol{C})$ whose restriction to ${}^LG^\circ$ is a morphism of complex Lie groups.

Clearly, $\ker r$ always contains an open subgroup of Γ_k, hence r factors through ${}^LG^\circ \rtimes \Gamma_{k'/k}$, where k' is a finite Galois extension of k over which G splits. The group ${}^LG^\circ \rtimes \Gamma_{k'/k}$ is canonically a complex algebraic group and r is a morphism of complex algebraic groups.

3. Parabolic subgroups.

3.1. *Notation.* We let $\mathscr{P}(G/k)$ denote the set of parabolic k-subgroups of G, and write $\mathscr{P}(G)$ for $\mathscr{P}(G/\bar{k})$. Let $p(G/k)$ be the set of conjugacy classes (with respect to $G(\bar{k})$ or $G(k)$, it is the same) of parabolic k-subgroups, and $p(G) = p(G/\bar{k})$. Let $p(G)_k$ be the set of conjugacy classes of parabolic subgroups which are defined over k (i.e., if $P \in \sigma \in p(G)_k$, then ${}^\gamma P \in \sigma$ for all $\gamma \in \Gamma_k$). In particular $p(G/k) \hookrightarrow p(G)_k$. There is equality if G is quasi-split$/k$.

3.2. We recall there is a canonical bijection between $p(G)$ and the subsets of Δ.

Then $p(G)_k$ corresponds to the Γ_k-stable subsets of Δ and $p(G/k)$ to those Γ_k-stable subsets which contain the set Δ_0 of simple roots of a Levi subgroup of a minimal parabolic k-group. In particular we have $p(G/k) = p(G)_k$ if G is quasi-split over k. Given $P \in \mathscr{P}(G)$, we let $J(P)$ be the subset of Δ assigned to the class of P

Since two conjugate parabolic subgroups whose intersection is a parabolic subgroup are identical, we see in particular that if P is defined over k, $P' \supset P$, and the class of P' is defined over k, then P' is defined over k.

3.3. *Parabolic subgroups of* ${}^LG$. A closed subgroup P of ${}^LG$ is parabolic if $\gamma_G(P) = \Gamma_k$ and $P^\circ = {}^LG^\circ \cap P$ is a parabolic subgroup of ${}^LG^\circ$. Then $P = N_{{}^LG}(P^\circ)$. In other words, a parabolic subgroup is the normalizer of a parabolic subgroup P° of ${}^LG^\circ$, provided the normalizer meets every class modulo ${}^LG^\circ$. We say P is standard if it contains ${}^LB$. The standard parabolic subgroups are the subgroups

(1) $$ {}^LP^\circ \rtimes \Gamma_k, $$

where ${}^LP^\circ$ runs through the standard parabolic subgroup of ${}^LG^\circ$ such that $J({}^LP^\circ) \subset \Delta^\vee$ is stable under Γ_k.

Every parabolic subgroup of ${}^LG$ is conjugate (under ${}^LG$ or, equivalently, ${}^LG^\circ$) to one and only one standard parabolic subgroup.

We let $\mathscr{P}({}^LG)$ be the set of parabolic subgroups of ${}^LG$ and $p({}^LG)$ the set of their conjugacy classes.

The given bijection $\Delta \leftrightarrow \Delta^\vee$ yields then, in view of 3.2, a bijection

(2) $$ p(G)_k \leftrightarrow p({}^LG). $$

We shall say that a parabolic subgroup of ${}^LG$ is *relevant* if its class corresponds to one of $p(G/k)$ under this map. We let ${}^L\mathscr{P}({}^LG)$ be the set of relevant parabolic subgroups and ${}^Lp({}^LG)$ the set of their conjugacy classes, the *relevant conjugacy classes* of parabolic subgroups. Thus, by definition

(3) $$ p(G/k) \leftrightarrow {}^Lp({}^LG). $$

Thus, if G and G' are inner forms of each other, $p({}^LG)$ and $p({}^LG')$ are the same, but ${}^Lp({}^LG)$ and ${}^Lp({}^LG')$ are not. If G' is quasi-split, then ${}^Lp({}^LG') \doteq p({}^LG')$; hence we have an injection

(4) $$ {}^Lp({}^LG) \subset {}^Lp({}^LG') = p({}^LG'). $$

If $\mathscr{D}G$ is anisotropic over k, then ${}^Lp({}^LG)$ consists of G alone.

3.4. *Levi subgroups.* Let P be a parabolic subgroup of ${}^LG$. The unipotent radical N of P° is normal in P and will also be called the unipotent radical of P. Then $P/N \xrightarrow{\sim} P^\circ/N \rtimes \Gamma_k$. In fact, it follows from (1) that P is a split extension of N, and is the semidirect product of N by the normalizer in P of any Levi subgroup M° of P°. Those normalizers will be called the Levi subgroups of P.

Let $P \in \mathscr{P}(G/k)$, M a Levi k-subgroup of P. Let ${}^LP$ be the standard parabolic subgroup in the class associated to that of P (see (3)). Then ${}^LM$ may be identified to a Levi subgroup of ${}^LP$. In fact if M corresponds to $(X^*(T), J, X_*(T), J^v)$, then ${}^LM^\circ$ corresponds to $(X_*(T), J^v, X^*(T), J)$ and ${}^LM^\circ \ltimes \Gamma_k$ is equal to ${}^LM$ by definition and is a Levi subgroup of ${}^LP$, as defined above.

A Levi subgroup of a parabolic subgroup P of ${}^LG$ is *relevant* if P is.

For the sake of brevity, we shall sometimes say "Levi subgroup in G" for "Levi subgroup of a parabolic subgroup of G." Similarly for ${}^LG$.

3.5. LEMMA. *The proper Levi subgroups in ${}^LG$ are the centralizers in ${}^LG$ of tori in $\mathscr{D}({}^LG^\circ)$, which project onto Γ_k.*

Let M be a proper Levi subgroup in ${}^LG$. It is conjugate to a subgroup $\mathscr{Z}(S)^\circ \rtimes \Gamma_k$, where $S \subset {}^LT^\circ$ is the identity component of the kernel of a subset $J \subsetneqq \Delta^\vee$ stable under Γ_k. Let then S' be the one-dimensional subtorus of $S \cap \mathscr{D}({}^LG^\circ)$ on which the remaining simple roots are all equal. It is clear that $\mathscr{Z}(S')^\circ = \mathscr{Z}(S)$, and that S' is pointwise fixed under Γ_k. We have then $M = \mathscr{Z}(S')$.

Let now S be a nontrivial torus in $\mathscr{D}({}^LG^\circ)$ such that $\mathscr{Z}(S)$ meets every connected component of ${}^LG$. Fix an ordering on $X^*(S)$. There is a proper parabolic subgroup P° of ${}^LG^\circ$ of the form $\mathscr{Z}(S)^\circ \cdot U$, such that the weights of S in the unipotent radical U of P° are the roots of ${}^LG^\circ$ with respect to S which are positive for this ordering. U is normalized by $\mathscr{Z}(S)$; hence $\mathscr{Z}(S)\cdot U$ is a proper parabolic subgroup P of ${}^LG$, and then $\mathscr{Z}(S)$ is a Levi subgroup of P.

3.6. PROPOSITION. *Let H be a subgroup of ${}^LG$ whose projection on Γ_k is dense in Γ_k. Then the Levi subgroups in ${}^LG$ which contain H minimally form one conjugacy class with respect to the centralizer of H in ${}^LG^\circ$.*

Let C be the identity component of the centralizer of H in $\mathscr{D}({}^LG^\circ)$, and D a maximal torus of H. If $D = \{1\}$, then, by 3.5, H is not contained in any proper Levi subgroup in ${}^LG$, and there is nothing to prove. So assume $D \neq \{1\}$.

Let Γ' be a normal open subgroup of Γ_k which acts trivially on ${}^LG^\circ$. It is then normal in ${}^LG$, and $H\cdot\Gamma'$ projects onto Γ_k. Since $\mathscr{Z}(D)$ contains $H\cdot\Gamma'$, it projects onto Γ_k, hence is a proper Levi subgroup by 3.5. Let M be a Levi subgroup containing H. By 3.5, $M = Z(S)$, where S is a torus in $\mathscr{D}({}^LG^\circ)$. Then $S \subset C$, there exists $c \in C$ such that $c\cdot S\cdot c^{-1} \subset D$, hence $c\cdot M\cdot c^{-1} = \mathscr{Z}(S') \supset \mathscr{Z}(D)$.

4. Remarks on induced groups. (To be used mainly to discuss restriction of scalars in §5 and 6.4.)

4.1. Let A be a group, A' a subgroup of finite index of A and E a group on which A' operates by automorphisms. Then we let

$$\text{(1)}\qquad \mathrm{Ind}_{A'}^{A}(E) = I_{A'}^{A}(E) = \{f : A \longrightarrow E \mid f(a'a) = a'\cdot f(a)\ (a \in A;\ a' \in A')\}.$$

It is a group (composition being defined by taking products of values). It is viewed as an A-group by right translations:

$$\text{(2)}\qquad r_a f(x) = f(xa) \qquad (x, a \in A).$$

For $s \in A'\backslash A$, let

$$\text{(3)}\qquad E_s = \{f \in I_{A'}^{A}(E) \mid f(a) = 0 \text{ if } a \notin s\}.$$

Then E_s is a subgroup, $I_{A'}^{A}(E)$ is the direct product of the E_s's ($s \in A'\backslash A$), and these subgroups are permuted by A. The subgroup $E_{\bar{e}}$ is stable under A' and is isomorphic to E as an A' module under the map $f \mapsto f(e)$. The product of the E_s's ($s \in A'\backslash A$, $s \neq e$) is also stable under A'. We have therefore canonical homomorphisms

(4) $$E \rtimes A' \longrightarrow I(E) \rtimes A' \longrightarrow E \rtimes A'$$

whose composition is the identity.

4.2. Let B be a group, $\mu: B \to A$ a homomorphism. Let $B' = \mu^{-1}(A')$ and assume that μ induces a bijection: $B' \backslash B \xrightarrow{\sim} A' \backslash A$. Let E be a group on which A' operates by automorphisms, also viewed as a B'-group via μ. Then $f \mapsto \mu \circ f$ induces an isomorphism

(1) $$\mu' : I_{A'}^{A}(E) \xrightarrow{\sim} I_{B'}^{B}(E),$$

whose inverse is μ-equivariant.

This follows immediately from the definitions.

4.3. Let A, E be as before, C a group and $\nu: C \to A$ a homomorphism. Let $\varphi: C \to E \rtimes A$ be a homomorphism over A. The map $\phi: C \to E$ such that $\varphi(c) = (\phi(c), \nu(c))$ $(c \in C)$ is a 1-cocycle of C in E and $\varphi \mapsto \phi$ induces a bijection

$$H^1(C; E) \xrightarrow{\sim} \varphi_A(C, E),$$

where, by definition, $\varphi_A(C, E)$ denotes the set of homomorphisms $\varphi: C \to E \rtimes A$ over A, modulo inner automorphisms by elements of E.

4.4. Let A, A', B, B' and E be as in 4.2. We have a commutative diagram with exact rows

(1) $$\begin{array}{ccccccccc} 1 & \longrightarrow & I_{A'}^{A}(E) & \longrightarrow & I_{A'}^{A}(E) \rtimes A & \longrightarrow & A & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & E & \longrightarrow & E \ltimes A' & \longrightarrow & A' & \longrightarrow & 1 \end{array}$$

where the vertical maps are natural inclusions (4.1).

Let $\varphi: B \to I_{A'}^{A}(E) \rtimes A$ be a homomorphism over A. Using 4.1(4), we get by restriction a homomorphsim $\tilde{\varphi}: B' \to E \rtimes A'$ over A'.

4.5. Lemma. *The map* $\varphi \mapsto \tilde{\varphi}$ *of* 4.4 *induces a bijection* $\varphi_A(B, I_{A'}^{A}(E)) \xrightarrow{\sim} \varphi_{A'}(B', E)$.

We have, using 4.2, 4.3:

(1) $$\varphi_A(B, I_{A'}^{A}(E)) = H^1(B; I_{A'}^{A}(E)) = H^1(B; I_{B'}^{B}(E)),$$

(2) $$\Phi_{A'}(B', E) = H^1(B'; E).$$

By a variant of Shapiro's lemma, contained, e.g., in [**4**, 1.29]:

$$H^1(B; I_{B'}^{B}(E)) \xrightarrow{\sim} H^1(B'; E),$$

and it is clear that the isomorphisms (1), (2) carry this isomorphism over to $\varphi \mapsto \tilde{\varphi}$.

5. Restriction of scalars. *In this section,* k' *is a finite extension of* k *in* k_s, G' *is a connected* k'*-group, and* $G = \mathbf{R}_{k'/k}\, G'$.

5.1. The Galois group $\Gamma_{k'}$ of k_s over k' is an open subgroup (of finite index) of Γ_k and $\Sigma_{k',k} = \Gamma_{k'} \backslash \Gamma_k$ may be identified with the set of k-monomorphisms of k' into k_s. We have, in the notation of 4.1 (with $A = \Gamma_k$, $A' = \Gamma_{k'}$)

(1) $$G(\bar{k}) = I_{\Gamma_{k'}}^{\Gamma_k}(G'(\bar{k})) = \prod_{\sigma \in \Gamma_{k'} \backslash \Gamma_k} {}^{\sigma}G'(\bar{k}).$$

Assume G' to be reductive. Then we see easily that $\psi(G) = (M, \varphi, M^*, \varphi^\vee)$ is related to $\psi(G') = (M', \varphi', M'^*, \varphi'^\vee)$ by

$$(2) \qquad M = I^{\Gamma_k}_{\Gamma_{k'}}(M'), \qquad \varphi = \bigcup_{a \in A' \backslash A} \varphi' \cdot a.$$

Similarly, if Δ' is a basis of φ', then

$$(3) \qquad \Delta = \bigcup_a \Delta' \cdot a$$

is one for φ.

From this it follows that we have a natural isomorphism

$$(4) \qquad {}^LG^\circ \xrightarrow{\sim} I^{\Gamma_k}_{\Gamma_{k'}}({}^LG'^\circ).$$

We have then a commutative diagram

$$(5) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & {}^LG'^\circ & \longrightarrow & {}^LG' = {}^LG'^\circ \rtimes \Gamma_{k'} & \longrightarrow & \Gamma_{k'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & {}^LG^\circ = I^{\Gamma_k}_{\Gamma_{k'}}({}^LG'^\circ) & \longrightarrow & {}^LG = {}^LG^\circ \rtimes \Gamma_k & \longrightarrow & \Gamma_k & \longrightarrow & 1 \end{array}$$

5.2. The map $P' \mapsto R_{k'/k} P'$ induces a bijection between $\mathscr{P}(G'/k')$ and $\mathscr{P}(G/k)$. Moreover P' is a Borel subgroup of G' if and only if $R_{k'/k} P'$ is one of G. Hence G' is quasi-split/k' if and only if G is quasi-split over k (see [5, §6]).

Since $G(k) \xrightarrow{\sim} G'(k')$, we also get a bijection $p(G'/k') \xrightarrow{\sim} p(G/k)$.

If $J' \subset \Delta'$ is stable under $\Gamma_{k'}$ then $J = \bigcup_{a \in A' \backslash A} r_a(J')$ is stable under Γ_k. This map is easily seen to yield a bijection between $\Gamma_{k'}$-stable subsets of Δ' and Γ_k-stable subsets of Δ, whence also canonical bijections

$$p({}^LG') \xrightarrow{\sim} p({}^LG), \qquad {}^Lp({}^LG') \xrightarrow{\sim} {}^Lp({}^LG).$$

CHAPTER II. QUASI-SPLIT GROUPS. THE UNRAMIFIED CASE.

In this chapter, G is a connected reductive quasi-split k-group. From 6.2 *on, G is assumed to split over a cyclic extension k' of k, and σ denotes a generator of* $\mathrm{Gal}(k'/k)$.

6. Semisimple conjugacy classes in ${}^LG$.

6.1. Assume B and T to be defined over k. Then the action of Γ_k on $X^*(T)$ or $X_*(T)$ given by μ_G coincides with the ordinary action. The greatest k-split subtorus T_d of T is maximal k-split in G, and its centralizer is T; in particular, T_d contains regular elements of G. Hence any element $w \in W$ which leaves T_d stable is completely determined by its restriction to T_d. It follows that ${}_kW$ may be identified with the subgroup of the elements of W which leave T_d stable or, equivalently, with the fixed-point set of Γ_k in W. If we go over to the L-group and identify canonically W with $W({}^LG^\circ, {}^LT^\circ)$, then ${}_kW$ is also the fixed-point set of Γ_k in W, and it operates on the greatest subtorus S of ${}^LT^\circ$ which is pointwise fixed under Γ_k. The group S always contains regular elements; hence any element of ${}_kW$ is determined by its restriction to S. We let ${}_kN$ be the inverse image of ${}_kW$ in the normalizer N of ${}^LT^\circ$ in ${}^LG^\circ$.

6.2. LEMMA. *Every element $w \in {}_kW$ has a representative in ${}_kN$ which is fixed under σ.*

Write $\Delta^\vee = D_1 \cup \cdots \cup D_m$, where the D_i's are the distinct orbits of $\Gamma(k'/k)$ in $\Delta^\vee$. Let δ_i be the common restriction to S of the elements of D_i and S_i the identity component of the kernel of δ_i. Then ${}_kW$, viewed as a group of automorphisms of S, is generated by the reflections s_i to the S_i $(1 \leqq i \leqq m)$, and it suffices to prove the lemma for $w = s_i$ $(1 \leqq i \leqq m)$. [The "reflection" s_i is the unique element $\neq 1$ of W which leaves S stable, fixes S_i pointwise and is of order two.]

We let Lie(M) denote the Lie algebra of the complex Lie group M. For $\check{\alpha} \in \Delta^\vee$, let, as usual

(1) $$\mathfrak{g}_{\check{\alpha}} = \{X \in \mathrm{Lie}({}^LG^\circ) \mid \mathrm{Ad}\, t(X) = \check{\alpha}(t) \cdot X \ (t \in {}^LT^\circ)\}.$$

It is one-dimensional. Fix i between 1 and m. By construction of ${}^LG$, we can find nonzero elements $e_{\check{\alpha}} \in \mathfrak{g}_{\check{\alpha}}$ (resp. $e_{-\check{\alpha}} \in \mathfrak{g}_{-\check{\alpha}}$ $(\check{\alpha} \in D_i)$) which are permuted by σ. We have then

(2) $$[e_{\check{\alpha}}, e_{-\check{\alpha}}] = c \cdot \alpha,$$

where c is $\neq 0$, and independent of $\alpha \in D_i$ since $\Gamma(k'/k)$ is transitive on D_i. Here $X_*({}^LT^\circ) \otimes \boldsymbol{C}$ is identified with Lie(${}^LT^\circ$), and $\check{\alpha}$ with $\check{\alpha} \otimes 1$. The element

(3) $$f_{\pm i} = \sum_{\check{\alpha} \in D_i} e_{\pm\check{\alpha}},$$

is fixed under σ. Moreover, since the difference of two simple roots is not a root, we have

(4) $$h_i = [f_i, f_{-i}] = \sum_{\check{\alpha} \in D_i} [e_{\check{\alpha}}, e_{-\check{\alpha}}] = c \cdot \sum_{\check{\alpha} \in D_i} \alpha.$$

Using (3) and (4), we get

(5) $$[h_i, f_{\pm i}] = c \sum_{\check{\alpha}, \check{\beta} \in D_i} \langle \alpha, \beta \rangle\, e_{\pm\beta}.$$

By the transitivity of Gal(k'/k) on D_i, the number

(6) $$d = \sum_{\check{\alpha} \in D_i} \langle \alpha, \check{\beta} \rangle$$

is also independent of $\check{\beta} \in D_i$; therefore

(7) $$[h, f_{\pm i}] = c \cdot d \cdot f_{\pm i}.$$

We claim that $d \neq 0$, in fact that $d = 1, 2$. The irreducible components of D_i are permuted transitively by Gal(k'/k) and have a transitive group of automorphisms. Therefore they are of type A_1 or A_2. Then, accordingly, $d = 2$ or $d = 1$. It follows that h_i, f_i and f_{-i} span a three-dimensional simple algebra pointwise fixed under σ. Then so is the corresponding analytic subgroup G_i of ${}^LG^\circ$. The group G_i centralizes S_i and $S \cap G_i$ is a maximal torus of G_i, with Lie algebra spanned by h_i. Then the nontrivial element of $W(G_i, S \cap G_i)$ is the required element.

REMARK. An equivalent statement is proved, in a different manner, in [**35**, pp. 19–22].

6.3. We let $Y = {}^L(T_d)^\circ$. The group $X_*(T_d)$ may be identified to the fixed-point set of Γ_k in $X_*(T)$. The inclusion of $X_*(T_d) = X^*(Y)$ into $X_*(T) = X^*({}^LT^\circ)$ induces a surjective morphism ${}^LT^\circ \to Y$, to be denoted ν.

The map $A : t \mapsto t^{-1} \cdot {}^\sigma t$ is an endomorphism of ${}^LT^\circ$, whose differential dA at 1 is $(d\sigma - \mathrm{Id})$. Let

(1) $$U = (\ker A)^\circ, \qquad V = \operatorname{im} A.$$

Then U is pointwise fixed under σ, the Lie algebra of U (resp. V) is the kernel (resp. image) of dA. Since dA is semisimple, they are transversal to each other; hence

(2) $${}^LT^\circ = U\cdot V, \quad \text{and } U \cap V \text{ is finite.}$$

Moreover,

(3) $$V = \ker \nu, \qquad \nu(U) = Y.$$

In the rest of this chapter, we let ${}^LG$ stand for the "finite Galois form" ${}^LG^\circ \rtimes \operatorname{Gal}(k'/k)$ of the L-group. We now want to discuss the semisimple conjugacy classes in ${}^LG^\circ \rtimes \sigma$ with respect to ${}^LG^\circ$. We have

(4) $$g^{-1}\cdot(h \rtimes \sigma)\cdot g = g^{-1}\cdot h\cdot{}^\sigma g \rtimes \sigma \qquad (g, h \in {}^LG^\circ);$$

therefore ${}^LG^\circ$-conjugacy in ${}^LG^\circ \rtimes \sigma$ is equivalent to σ-conjugacy in ${}^LG^\circ$.

6.4. Lemma. *Let $\nu' : {}^LT^\circ \rtimes \sigma \to Y$ be defined by $\nu'(t \times \sigma) = \nu(t)$ $(t \in {}^LT^\circ)$. Then ν' induces a bijection*

(1) $$\bar{\nu} : ({}^LT^\circ \rtimes \sigma)/\operatorname{Int}{}_kN \xrightarrow{\sim} Y/{}_kW.$$

Let $n \in {}_kN$. By 6.2, we may write $n = w\cdot s$ with $w = {}^\sigma w$ and $s \in {}^LT^\circ$. Then the ${}^LT^\circ$-component of $n^{-1}(t \rtimes \sigma)\, n$ is

$$s^{-1}\cdot w^{-1}\cdot t\cdot w^\sigma s = s^{-1}\cdot{}^\sigma s\cdot(w^{-1}\cdot t\cdot w) \in V\cdot w^{-1}\cdot t\cdot w;$$

hence

$$\nu'(n^{-1}\cdot(t \rtimes \sigma)\cdot n) = \nu(w^{-1}\cdot t\cdot w) = w^{-1}\cdot\nu(t)\cdot w = w^{-1}\cdot\nu'(t \rtimes \sigma)\cdot w.$$

Thus ν' is equivariant with respect to the projection ${}_kN \to {}_kW$ and therefore induces a map of the left-hand side of (1) into the right-hand side of (1), which is obviously surjective. Let $t, t' \in {}^LT^\circ$ and assume that $\nu'(t \rtimes \sigma) = w^{-1}\cdot\nu'(t' \rtimes \sigma)\cdot w$ for some $w \in {}_kW$. Then we have $\nu(t) = \nu(w^{-1}\cdot t'\cdot w)$, where w is a representative of w fixed under σ, whence $t = v\cdot(w^{-1}\cdot t'\cdot w)$, with $v \in V$. We can write $v = s^{-1}\cdot{}^\sigma s$ for some $s \in {}^LT^\circ$, and get $t \rtimes \sigma = n^{-1}(t' \rtimes \sigma)n$, with $n = ws$.

6.5. Lemma. *Let $({}^LG^\circ \rtimes \sigma)_{ss}$ be the set of semisimple elements in ${}^LG^\circ \rtimes \sigma$. Then the map*

$$\bar{\mu} : ({}^LT^\circ \rtimes \sigma)/\operatorname{Int}_kN \longrightarrow ({}^LG^\circ \times \sigma)_{ss}/\operatorname{Int}{}^LG^\circ,$$

induced by inclusion is a bijection.

By results of F. Gantmacher [**12**, Theorem 14], $\bar{\mu}$ is surjective. Let now $s, t \in {}^LT^\circ$ and $g \in {}^LG^\circ$ be such that $g^{-1}\cdot(s \rtimes \sigma)\cdot g = t \rtimes \sigma$, i.e., such that $g^{-1}\cdot s\cdot{}^\sigma g = t$. Using the Bruhat decomposition of ${}^LG^\circ$ with respect to ${}^LB^\circ$, we can write uniquely $g = u\cdot n\cdot v$, with u, v in the unipotent radical of ${}^LB^\circ$ and n in the normalizer N of ${}^LT^\circ$. These groups are stable under σ, and normalized by ${}^LT^\circ$. We have then

$$s\cdot{}^\sigma u\cdot{}^\sigma n\cdot{}^\sigma v = u\cdot n\cdot v\cdot t, \qquad (s\cdot{}^\sigma n\cdot s^{-1})\cdot s\cdot{}^\sigma n\cdot{}^\sigma v = u\cdot n\cdot t\cdot(t^{-1}\cdot v\cdot t);$$

hence ${}^\sigma n\cdot s^{-1} = n\cdot t$. Therefore the connected component of n in N is stable under

σ, i.e., n represents an element of ${}_kW$; hence $n \in {}_kN$, and $(t \rtimes \sigma)$ and $(s \rtimes \sigma)$ are conjugate under ${}_kN$.

REMARK. This proof was suggested to me by T. Springer.

6.6. If M is a complex affine variety, we let $C[M]$ denote its coordinate algebra. The algebra $C[Y]$ may be identified with the group algebra of $X^*(Y) = X_*(T_d)$. The quotient $Y/{}_kW$ is also an affine variety (in fact isomorphic to an affine space) and $C[Y/{}_kW] = C[Y]^{{}_kW}$.

Let $\mathrm{Rep}({}^LG) \subset C[{}^LG]$ be the subalgebra generated by the characters of finite dimensional holomorphic representations. Its elements are constant on conjugacy classes. In particular, they define by restriction functions on $({}^LG^\circ \rtimes \sigma)_{ss}/\mathrm{Int}\, {}^LG^\circ$.

6.7. PROPOSITION. *The map*

$$\alpha = \bar{\mu} \circ \bar{\nu}^{-1} : Y/{}_kW \longrightarrow ({}^LG^\circ \rtimes \sigma)_{ss}/\mathrm{Int}\, {}^LG^\circ$$

is a bijection, which induces an isomorphism of $C[Y/{}_kW]$ *onto the algebra* A *of restrictions of elements of* $\mathrm{Rep}({}^LG)$.

REMARK. We shall use 6.7 only when k is a nonarchimedean local field. In that case 6.7 is proved in [35, pp. 18–24].

PROOF. That α is bijective follows from 6.4, 6.5. We prove the second assertion as in [35]. Let ρ be a finite dimensional holomorphic representation of ${}^LG$ and f_ρ the function on ${}^LT^\circ$ defined by $f_\rho(t) = \mathrm{tr}\ \rho(t \rtimes \sigma)$. It can be written as a finite linear combination $f = \sum c_\lambda \lambda$ of characters $\lambda \in X^*({}^LT^\circ)$. Since $\mathrm{tr}\ \rho$ is a class function on ${}^LG$, we have $f_\rho(s^{-1} \cdot t \cdot {}^\sigma s) = f_\rho(t)$ for all $s, t \in {}^LT^\circ$. By the linear independence of characters, it follows that if $c_\lambda \neq 0$, then λ is trivial on V (cf. 6.3(1)), hence is fixed under σ, i.e., may be identified to an element of $X^*(Y)$. Thus we may view f_ρ as an element of $C[Y]$. But invariance by conjugation and 6.4 imply that $f \in C[Y/{}_kW]$, whence a map $\beta : A \to C[Y/{}_kW]$, which is obviously induced by α. There remains to see that β is surjective. Note that $C[Y/{}_kW]$ is spanned, as a vector space, by the functions

$$(1) \qquad \varphi_\lambda = \sum_{w \in {}_kW} w \cdot \lambda,$$

where λ runs through a fundamental domain C of ${}_kW$ on $X_*(T_d)$. But it is standard that we may take for C the intersection of $X_*(T_d)$ with the Weyl chamber of W in $X_*(T)$ defined by B. Therefore every $\lambda \in C$ is a dominant weight for ${}^LG^\circ$ with respect to ${}^LT^\circ$. It is then the highest weight of an irreducible representation π_λ of ${}^LG^\circ$. Since it is fixed under σ, the representation ${}^\sigma\pi_\lambda : g \mapsto \pi_\lambda({}^\sigma g)$ is equivalent to π_λ. From this it is elementary that π_λ extends to an irreducible representation $\tilde{\pi}_\lambda$ of ${}^LG$ of the same degree as π_λ. The highest weight space is one-dimensional, stable under σ. Let c be the eigenvalue of σ on it. Then the trace gives rise to a function equal to $c \cdot \varphi_\lambda$ modulo a linear combination of functions φ_μ, with $\mu < \lambda$, in the usual ordering. That im β contains φ_λ $(\lambda \in C)$ is then proved by induction on the ordering.

7. The Satake isomorphism and the L-group. Local factors.

7.1. We keep the previous notation and conventions. We assume moreover k to be a nonarchimedean local field, k' to be unramified over k, and σ to be the image of a Frobenius element Fr in Γ_k.

Let Q be a special maximal compact subgroup of $G(k)$ [61]. We assume $Q \cap T$ is the greatest compact subgroup of $T(k)$ and Q contains representatives of ${}_kW$. Let

$H(G(k), Q)$ be the Hecke algebra of locally constant, Q-bi-invariant, and compactly supported complex valued functions on $G(k)$. The Satake isomorphism provides a canonical identification $H \xrightarrow{\sim} \boldsymbol{C}[Y/_k W]$, hence also one of $Y/_k W$ with the characters of H **[6]**.

By 6.7, we have now a canonical isomorphism of H with the algebra A of restrictions of characters of finite dimensional representations of ${}^LG$ to semisimple ${}^LG^\circ$-conjugacy classes in $({}^LG^\circ \rtimes \sigma)$, hence also a canonical bijection between characters of $H(G(k), Q)$ and semisimple classes in ${}^LG^\circ \rtimes \sigma$. Furthermore, each such class can be represented by an element of the form (t, σ), with $t \in {}^LT^\circ$ fixed under σ (and is determined modulo the finite group $U \cap V$, in the notation of 6.3).

7.2. *Local factors.* Assume now that U is *hyperspecial* **[61]**. Let ψ be an additive character of k. Let (π, U_π) be an irreducible admissible representation of $G(k)$ of class 1 for Q and r a representation of ${}^LG$. Then the space of fixed vectors of Q in U_π is one-dimensional, acted upon by H via a character χ_π. To the latter is assigned by 7.1 a semisimple class S_χ in ${}^LG^\circ \rtimes \sigma$. We then put

$$(1) \qquad L(s, \pi, r) = \det(1 - r((g \rtimes \sigma)), q^{-s})^{-1}, \qquad \varepsilon(s, \pi, r, \psi) = 1,$$

where q is the order of the residue field, and (g, σ) any element of S_χ.

Chapter III. Weil Groups and Representations. Local Factors.

In this section, k is a local field, W_k (resp. W_k') the absolute Weil group (resp. Weil-Deligne group) of k. If H is a reductive k-group, then $\Pi(H(k))$ is the set of infinitesimal equivalence classes of irreducible admissible representations of $H(k)$.

G denotes a connected reductive k-group.

The main local problem is to define a *partition* of $\Pi(G(k))$ into finite sets $\Pi_{\varphi,G}$ or Π_φ indexed by the set $\Phi(G)$ of admissible homomorphisms of W_k' into ${}^LG$, modulo inner automorphisms (see §8 for $\Phi(G)$), and satisfying a certain number of conditions. So far, this has been carried out for any G if $k = \boldsymbol{R}, \boldsymbol{C}$ **[37]**, for tori over any k **[34]** and (essentially) for $G = \mathbf{GL}_2$ (cf. 12.2). §9 recalls the results for tori; §10 describes some of the conditions to be imposed on this parametrization; §11 summarizes the construction over $\boldsymbol{R}$ or $\boldsymbol{C}$. Such a parametrization would allow one to assign canonically local L- and ε-factors to any $\pi \in \Pi(G(k))$ and any complex representation of ${}^LG$. Two elements π, π' in the same set Π_φ would always have the same local factors, and are hence called *L-indistinguishable*. In the case of $\mathbf{GL}_n$ however, local factors have been defined in an a priori quite different way, so that the parametrization problem becomes subordinated to one concerning L- and ε-factors. This is discussed in §12.

8. Definition of $\Phi(G)$.

8.1. *Jordan decomposition in W_k'.* If $k = \boldsymbol{R}, \boldsymbol{C}$, then $W_k' = W_k$ and, by definition, every element of W_k is semisimple.

Let k be nonarchimedean. Then $x \in W_k'$ is said to be unipotent if and only if it belongs to G_a; the element x is semisimple if either $\varepsilon(x) \neq 0$ or x is in the inertia group. Here ε: $W_k' \to \boldsymbol{Z}$ is the canonical homomorphism $W_k' \to W_k \to k^* \to \boldsymbol{Z}$. Every element $x \in W_k'$ admits a unique Jordan decomposition $x = x_s \cdot x_u$ with x_s semisimple, x_u unipotent and $x_s x_u = x_u x_s$ **[60]**.

8.2. *The set* $\Phi(G)$. We consider homomorphisms $\alpha : W'_k \to {}^LG$ over Γ_k, i.e., such that the diagram

$$\begin{array}{ccc} W'_k & \xrightarrow{\alpha} & {}^LG \\ & \searrow \quad \swarrow & \\ & \Gamma_k & \end{array}$$

is commutative, and which satisfy moreover the following conditions:

(i) α is continuous, $\alpha(G_a)$ is unipotent, in ${}^LG^\circ$, and α maps semisimple elements into semisimple elements (in ${}^LG$: $x = (u, \gamma)$ is said to be semisimple if its image under any representation (2.6) is so).

(ii) If $\alpha(W'_k)$ is contained in a Levi subgroup of a parabolic subgroup P of ${}^LG$, then P is relevant (3.3).

Such α's are called admissible. We let $\Phi(G)$ be the set of their equivalence classes modulo inner automorphisms by elements of ${}^LG^\circ$.

If we write $\alpha(w) = (a(w), \nu(w))$ with $a(w) \in {}^LG^\circ$ then $w \mapsto a(w)$ is a 1-cocycle of W'_k (acting on ${}^LG^\circ$ via $W'_k \to \Gamma_k$) in ${}^LG^\circ$. It follows that

(1) $$\Phi(G) \subset H^1(W'_k; {}^LG^\circ).$$

Let H be a subgroup of W'_k. Then α: $W'_k \to {}^LG$ is said to be trivial on H if $\nu(H)$ acts trivially on ${}^LG^\circ$ and $\alpha(H) = \{1\}$. Note that if $\nu(H)$ acts trivially on ${}^LG^\circ$, then $a|_H$ is a homomorphism.

8.3. Assume G' is an inner quasi-split form of G. Then

(1) $$\Phi(G) \subset \Phi(G').$$

In fact ${}^LG \cong {}^LG'$ and ${}^Lp({}^LG') \supset {}^Lp({}^LG)$; therefore $\alpha \in \Phi(G) \Rightarrow \alpha \in \Phi(G')$.

8.4. PROPOSITION. *Let k' be a finite separable extension of k; let G' be a connected reductive k-group and $G = R_{k'/k}G'$. Then there is a canonical bijection $\Phi(G) \xrightarrow{\sim} \Phi(G')$.*

We consider the situation of 5.2, 5.4 with $A = \Gamma_k$, $A' = \Gamma_{k'}$, $B = W'_k$, $B' = W'_{k'}$, $E = {}^LG'^\circ$. We have the injections (8.2):

$$\Phi(G) \subset H^1(W'_k; {}^LG^\circ), \qquad \Phi(G') \subset H^1(W'_{k'}; {}^LG'^\circ).$$

Moreover ${}^LG^\circ = I_{\Gamma_{k'}}^{\Gamma_k}({}^LG'^\circ)$ (see 5.1); whence, by Shapiro's lemma and 5.2:

$$H^1(W'_k;\ {}^LG^\circ) \xrightarrow{\sim} H^1(W'_{k'};\ {}^LG'^\circ).$$

But it is clear that this isomorphism maps $\Phi(G)$ onto $\Phi(G')$.

8.5. Let $Z_L = C({}^LG^\circ)$. If a: $W_k \to Z_L$ and b: $W'_k \to {}^LG^\circ$ are 1-cocycles, then ab: $w \mapsto a(w)b(w)$ is again a 1-cocycle of W'_k in ${}^LG^\circ$. If a is continuous and b corresponds to $\varphi \in \Phi(G)$, then ab corresponds to an element of $\Phi(G)$. We get therefore maps

(1) $$H^1(W_k; Z_L) \times H^1(W'_k\,; {}^LG^\circ) \longrightarrow H^1(W'_k\,; {}^LG^\circ),$$

(2) $$H^1(W_k\,; Z_L) \times \Phi(G) \longrightarrow \Phi\,(G),$$

which define actions of the group $H^1(W_k; Z_L)$ on the sets $H^1(W'_k; {}^LG^\circ)$ and $\Phi(G)$.

8.6. PROPOSITION. *Let $\varphi: W_k' \to {}^LG$ be an admissible homomorphism. Then the Levi subgroups in ${}^LG$ which contain $\varphi(W_k')$ minimally form one conjugacy class with respect to the centralizer of $\varphi(W_k')$ in ${}^LG^\circ$.*

Since $\varphi(W_k')$ projects onto a dense subgroup of Γ_k by definition, this follows from 3.6.

REMARK. Formally, this also applies to the archimedean case, but the proof in that case is simpler [**37**, pp. 78–79]. In fact, the argument there applies in all cases to admissible homomorphisms of the Weil (rather than Weil-Deligne) group because $\varphi(W_k)$ is always fully reducible. In this case, the Levi subgroups which contain $\varphi(W_k)$ minimally are those of the parabolic subgroups which contain $\varphi(W_k)$ minimally. Those parabolic subgroups form therefore one class of associated groups.

9. The correspondence for tori.

9.1. Let T be a complex torus. A continuous homomorphism $\varphi: T \to \boldsymbol{C}^*$ is described by a pair of elements $\lambda, \mu \in X^*(T) \otimes \boldsymbol{C}$ such that $\lambda - \mu \in X^*(T)$, by the rule $\varphi(t) = t^\lambda \bar{t}^\mu$.

Similarly, a continuous homomorphism $\varphi: \boldsymbol{C}^* \to T$ is given by $\mu, \nu \in X_*(T) \otimes \boldsymbol{C}$ such that $\mu - \nu \in X_*(T)$; we have $\varphi(z) = z^\mu \bar{z}^\nu$, meaning that, for any $\lambda \in X^*(T)$, $\lambda \circ \varphi: \boldsymbol{C}^* \to \boldsymbol{C}^*$ is given by

$$\lambda(\varphi(z)) = z^{\langle \lambda, \mu \rangle} \bar{z}^{\langle \lambda, \nu \rangle}.$$

This can also be interpreted in the following way: identify $X_*(T) \otimes \boldsymbol{C}$ with the Lie algebra $\mathrm{Lie}(T(\boldsymbol{C}))$. Then the exponential map yields an isomorphism $(X_*(T) \otimes \boldsymbol{C})/2\pi i X_*(T) = T(\boldsymbol{C})$. Then $\mu, \nu \in \mathrm{Lie}(T(\boldsymbol{C}))$ are such that $\varphi(e^h) = e^{h \cdot \mu + \bar{h} \cdot \mu}$ $(h \in \boldsymbol{C})$.

9.2. Let $G = T$ be a k-torus, and $l = \dim T$.

Any $\varphi \in \Phi(G)$ is trivial on G_a; hence

(1) $$\Phi(G) = H^1_{\mathrm{ct}}(W_k; {}^LT^\circ) = H^1_{\mathrm{ct}}(W_k; X^*(T) \otimes \boldsymbol{C}^*),$$

where H^1_{ct} refers to continuous cocycles.

On the other hand

(2) $$\Pi(G) = \mathrm{Hom}((X_*(T) \otimes k_s^*)^{\Gamma_k}, \boldsymbol{C}^*).$$

We have canonically [**34**, Theorem 1]

(3) $$\Pi(G) = \Phi(G).$$

In fact, ${}^LT$ and W_k are replaced in [**34**] by a finite Galois form ${}^LT^\circ \rtimes \Gamma_{k'/k}$ and a relative Weil group $W_{k'/k}$, where k' is a finite Galois extension of k whose Galois group acts trivially on ${}^LT^\circ$; this is easily seen not to change $\Phi(G)$. The proof then consists in showing first that the transfer from $W_{k'/k}$ to k'^* yields an isomorphism

(4) $$H_1(W_{k'/k}; X_*(T)) \xrightarrow{\sim} H_1(k'^*; X_*(T))^{\Gamma_{k'/k}} = (k'^* \otimes X_*(T))^{\Gamma_{k'/k}},$$

and second that the pairing

(5) $$H^1_{\mathrm{ct}}(W_{k'/k}; {}^LT^\circ) \times H_1(W_{k'/k}; X_*(T)) \longrightarrow \boldsymbol{C}^*,$$

associated to the evaluation map $(t, \lambda) \mapsto \lambda(t)$ $(t \in {}^LT^\circ; \lambda \in X_*(T))$ yields an isomorphism of the first group onto the group of characters of the second group, which is then (3) by definition.

For illustrations, we discuss some simple cases.

9.3. $k = \mathbf{C}$. Then $W_k = \mathbf{C}^*$ and $\Phi(G) = \mathrm{Hom}(\mathbf{C}^*, {}^LT^\circ)$. The correspondence follows from 9.1 since both $\mathrm{Hom}(\mathbf{C}^*, {}^LT^\circ)$ and $\mathrm{Hom}(T, \mathbf{C}^*)$ are canonically identified with $\{(\lambda, \mu)|\lambda, \mu \in X^*(T) \otimes \mathbf{C}, \lambda - \mu \in X^*(T)\}$.

9.4. $k = \mathbf{R}$. We have

$$W_{\mathbf{R}} = \mathbf{C}^* \rtimes \{\tau\} \quad \text{with } \tau^2 = -1,\ \tau \cdot z \cdot \tau^{-1} = \bar{z}\ (z \in \mathbf{C}^*). \tag{1}$$

Put $\mathbf{C}^* = S \times \mathbf{R}^+$, with $S = \{z \in \mathbf{C}^*, z \cdot \bar{z} = 1\}$. Then $\mathrm{Int}\, \tau$ is the identity on $\mathbf{R}^+$, the inversion on S.

Write $\varphi(\tau) = (a, \sigma)$, where a is determined modulo σ-conjugacy, hence may be assumed to be fixed under σ (6.3(3)). We have then $\varphi(-1) = a^2$. Let μ, ν be the elements of $X^*(T) \otimes \mathbf{C}$ such that

$$\varphi(z) = z^\mu \cdot \bar{z}^\nu \qquad (z \in \mathbf{C}^*),\ \mu - \nu \in X^*(T), \tag{2}$$

(see 9.1). We have $\varphi(\bar{z}) = \sigma(\varphi(z))$ $(z \in \mathbf{C}^*)$; hence $\nu = \sigma(\mu)$. Fix $h \in X^*(T) \otimes \mathbf{C}$ such that $a = \exp 2\pi i h$. Then the character π associated to φ is given by

$$\pi(e^x) = \exp(\langle h, x - \sigma \cdot \bar{x}\rangle) \cdot \exp(\langle \mu, x + \sigma \cdot x\rangle)/2 \quad (x \in X_*(T) \otimes \mathbf{C}) \tag{3}$$

[37, p. 27]. Here $^-$ denotes the complex conjugation of $\mathrm{Lie}(T(\mathbf{C})) = X_*(T) \otimes \mathbf{C}$ with respect to $X_*(T) \otimes \mathbf{R}$; hence $x \mapsto \sigma \cdot \bar{x}$ is the complex conjugation with respect to $\mathrm{Lie}(T(\mathbf{R}))$. It follows that $e^x \in T(\mathbf{R})$ if and only if $x - \sigma \cdot \bar{x} \in 2\pi i \cdot X_*(T)$.

EXAMPLES. (a) Let T be anisotropic over $\mathbf{R}$. Then $\sigma = -1$ and we may assume $a = 1, h = 0$. We have $e^x \in T(\mathbf{R})$ if x is purely imaginary and then (3) yields $\pi = \mu$. The fact that $\varphi(-1) = 1$ shows that $\mu \in X^*(T)$, confirming that $\Pi(T(\mathbf{R})) = X^*(T)$.

(b) Let T be split over $\mathbf{R}$. Then $\sigma = 1$, $\mu = \nu$, $\varphi(z) = (z \cdot \bar{z})^\mu$, $a^2 = 1$ and $h \in X^*(T)/2$. We have $e^x \in T(\mathbf{R})$ if and only if $x - \bar{x} \in 2\pi i \cdot X^*(T)$. It is then easily checked that π is given by μ on the connected identity component of $T(\mathbf{R})$, while its restriction to the torsion subgroup of $T(\mathbf{R})$ is the character naturally defined by h.

9.5. *The unramified case.* Let k be nonarchimedean, and assume T to split over an unramified extension k' of k. A character χ of $T(k)$ is said to be unramified if it is trivial on the greatest compact subgroup ${}^0T(k)$ of $T(k)$. On the other hand, $\varphi \in \Phi(T)$ is unramified if it is trivial (see 6.2) on the inertia group. The bijection $\Phi(T) \xrightarrow{\sim} \Pi(T)$ induces a bijection between the sets $\Phi_{\mathrm{unr}}(T)$ and $\Pi_{\mathrm{unr}}(T)$ of unramified elements [34]. In view of its importance, we describe it in more detail.

Given $t \in T(k')$, let $v(t) \in \mathrm{Hom}(X^*(T), \mathbf{Z})$ be defined by $v(t)(m) = \mathrm{ord}\ m(t)$ $(m \in X^*(T))$. It is well known, and easily deduced from Hilbert's Theorem 90, that $H^1(\Gamma_{k'/k}; o_{k'}^*) = 0$, where $o_{k'}^*$ is the group of units in the ring $o_{k'}$ of integers of k'. Since T splits over k', it follows that $H^1(\Gamma_{k'/k}; {}^0T(k')) = 0$. By Galois cohomology, this implies that $(T(k')/{}^0T(k'))^{\Gamma_k} = T(k)/{}^0T(k)$, therefore $t \mapsto v(t)$ yields a bijection

$$T(k)/{}^0T(k) \xrightarrow{\sim} \mathrm{Hom}(X^*(T), \mathbf{Z})^{\Gamma_k} = X_*(T)^{\Gamma_k} = X_*(T_d), \tag{1}$$

where T_d is the greatest k-split torus of T (this can also be expressed by saying that the inclusion $T_d \subset T$ induces an isomorphism $T_d(k)/{}^0T_d(k) \xrightarrow{\sim} T(k)/{}^0T(k)$, cf. [6]). We have then

$$(2) \qquad \Pi_{\text{unr}}(T(k)) = \text{Hom}(X_*(T)^{\Gamma_k}, C^*) = Y.$$

The group Γ_k operates on ${}^LT^\circ$ via the cyclic group $\Gamma_{k'/k}$ which is generated by the image σ of a Frobenius element Fr. An unramified φ is completely determined by $\varphi(\text{Fr})$, which can be written $\varphi(\text{Fr}) = (t, \text{Fr})$, where $t \in {}^LT^\circ$ is determined up to conjugacy by ${}^LT^\circ$. Thus $\Phi_{\text{unr}}(T) = ({}^LT^\circ \rtimes \sigma)/\text{Int}\ {}^LT^\circ$; and elementary special case of 6.4 provides a canonical isomorphsim of the latter set onto Y, whence the desired isomorphism.

10. Desiderata. In order to formulate them, we need two preliminary constructions.

10.1. *The character* χ_φ *of* $C(G)$ *associated to* $\varphi \in \Phi(G)$ (*cf.* [37, *pp.* 20–34]). We want to associate canonically to $\varphi \in \Phi(G)$ a character of the center $C(G)$ of G. Let $\mathfrak{G}_{\text{rad}}$ be the greatest central torus of $\mathfrak{G}$. Then $G_{\text{rad}} \to G$ yields a surjective homomorphism ${}^LG \to {}^LG_{\text{rad}}$, whence a map $\Phi(G) \to \Phi(G_{\text{rad}})$. In view of 7.2, this allows us to associate to $\varphi \in \Phi(G)$ a character χ_φ of G_{rad}. Thus, if $C(G(k)) \subset G_{\text{rad}}(k)$, our problem is solved.

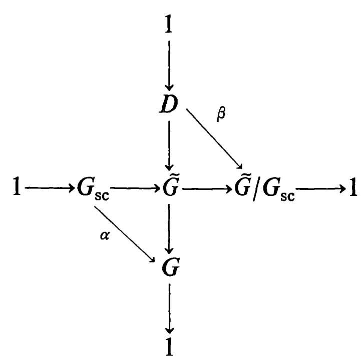
In the general case, G is enlarged to a bigger connected reductive G_1 generated by G and a central torus, whose center is a torus. One shows that $\Phi(G_1) \to \Phi(G)$ is surjective. Using the previous step, we get a character of $C(G_1)$, hence one of $C(G)$ by restriction. It is shown to be independent of the choice of G_1 (loc. cit.), and is χ_φ by definition.

The map $\varphi \mapsto \chi_\varphi$ is compatible with restriction of scalars [37, 2.11].

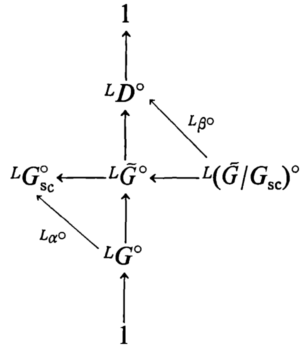
10.2. *The character* π_α *associated to* $\alpha \in H^1(W'_k; Z_L)$ [37, *pp.* 34–36]. We recall that Z_L denotes the center of ${}^LG^\circ$ (8.5). We can always find a k-torus D such that $H^1(\Gamma_k; D) = 0$, and a k-group $\tilde{G}$ isogeneous to $G \times D$ such that there is an exact sequence

$$(1) \qquad 1 \longrightarrow D \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.$$

Since $H^1(\Gamma_k; D) = 0$, the map $\mu\colon \tilde{G}(k) \to G(k)$ is surjective. Let G_{sc} be the universal covering of the derived group $\mathscr{D}G$ of G. We have a commutative diagram



Going over to L-groups, we get



Since ${}^LG^\circ_{sc}$ is of adjoint type (2.2), we see that $Z_L = \ker {}^L\alpha^\circ$. Moreover, it is easily seen that

$$(2) \qquad Z_L \cong \ker {}^L\beta^\circ.$$

This yields a map

$$(3) \qquad H^1(W_k'; Z_L) \longrightarrow \ker\{H^1(W_k'; {}^L(\tilde{G}/G_{sc})^\circ) \longrightarrow H^1(W_k'; {}^LD^\circ)\}.$$

This allows us to associate to $\alpha \in H^1(W_k'; Z_L)$ a character σ_α of $(\tilde{G}/G_{sc})(k)$ which is trivial on $D(k)$, hence a character π_α of $G(k) = \tilde{G}(k)/D(k)$. It can be shown to be independent of the choice of D. The map $\alpha \mapsto \pi_\alpha$ is compatible with restriction of scalars [**37**, 2.12] and satisfies:

$$(4) \qquad \chi_{\alpha\varphi} = \pi_\alpha \cdot \chi_\varphi \qquad (\alpha \in H^1(W_k'; Z_L),\ \varphi \in \Phi(G)).$$

10.3. *Conditions on the sets Π_φ.* (1) If $\pi \in \Pi_\varphi$, then $\pi(z) = \chi_\varphi(z)\cdot \mathrm{Id}$ $(z \in C(G))$.

(2) If $\varphi' = \alpha\cdot\varphi$ $(\varphi, \varphi' \in \Phi(G),\ \alpha \in H^1(W_k'; Z_L))$ (see 6.5), then $\Pi_{\varphi'} = \{\pi_\alpha \otimes \pi \mid \pi \in \Pi_\varphi\}$.

(3) The following conditions on a set Π_φ are equivalent:

(i) One element of Π_φ is square-integrable modulo $C(G)$.

(ii) All elements of Π_φ are square-integrable modulo $C(G)$.

(iii) $\varphi(W_k')$ is not contained in any proper Levi subgroup in ${}^LG$.

(4) Assume $\varphi(G_a) = \{1\}$. The following conditions on a set Π_φ are equivalent:

(i) One element of Π_φ is tempered.

(ii) All elements of Π_φ are tempered.

(iii) $\varphi(W_k)$ is bounded.

(5) Let H be a connected reductive k-group and $\eta\colon H \to G$ a k-morphism with commutative kernel and cokernel. Let $\varphi \in \Phi(G)$ and $\varphi' = {}^L\eta \circ \varphi$. Then any $\pi \in \Pi_\varphi$, viewed as an $H(k)$-module, is the direct sum of finitely many irreducible admissible representations belonging to $\Pi_{\varphi'}$.

10.4. *The unramified case.* We say that $\varphi \in \Phi(G)$ is unramified if it is trivial, in the sense of 6.2, on G_a and on the inertia group I. If so, $\mathrm{Im}\ \varphi$ may be assumed to be in ${}^LT$. Therefore, if $\Phi(G)$ contains an unramified element, then G is quasi-split (see 8.2 (ii)).

Assume now G to be quasi-split, to split over an unramified Galois extension

k' of k, and let $\varphi \in \Phi(G)$ be unramified. There exists $t \in ({}^LT^\circ)^{\Gamma_k}$ such that

(1) $$\varphi(\mathrm{Fr}) = (t, \mathrm{Fr}),$$

(9.5) and we have

(2) $$\varphi(w) = (t, \mathrm{Fr})^{\varepsilon(w)} \qquad (w \in W_k'),$$

where $\varepsilon: W_k' \to \boldsymbol{Z}$ is the canonical homomorphism. The element t defines an unramified character χ of a maximal k-torus T of a Borel k-subgroup B of G (9.5). It is then required that Π_φ consists of the constituents of the unramified normalized principal series $\mathrm{PS}(\chi)$ which have a nonzero vector fixed under some hyperspecial maximal compact subgroup. Conversely let (π, V) be an irreducible admissible representation with a nonzero vector fixed under some hyperspecial maximal compact subgroup. There exists then an unramified character χ of T such that (π, V) is a constituent of $\mathrm{PS}(\chi)$ (and χ is determined modulo the relative Weyl group). We have then $(\pi, V) \in \Pi_\varphi$, for the unramified φ which maps Fr to (t, Fr), where t represents χ (9.5). Note that if U is a special maximal compact subgroup of $G(k)$, then $G(k) = B(k) \cdot U$; hence the fixed-point set of U in $\mathrm{PS}(\chi)$ is at most one-dimensional. It follows that $\mathrm{PS}(\chi)$ has at most one irreducible constituent with nonzero fixed vectors under U.

This assignment is consistent with 7.2. Namely, if $\pi \in \Pi_\varphi$, then the semisimple class S_χ in ${}^LG^\circ \rtimes \sigma$ corresponding to the character of the Hecke algebra defined by π is indeed represented by $t \rtimes \sigma$. This follows from **[6]**.

REMARK. Originally, it was thought that Π_φ should consist of those constituents of $\mathrm{PS}(\chi)$ which had a nonzero fixed vector under some special maximal compact subgroup. However it was pointed out during the Institute by I. Macdonald that such representations may belong to the discrete series. If so, this condition would contradict 10.3(3). Upon a suggestion of J. Tits, this has led to the restriction to hyperspecial maximal compact subgroups made above. Those cannot belong to the discrete series, so that 10.3(3) and 10.4 are consistent.

10.5. EXAMPLE. Assume that $k = \boldsymbol{R}$ and that G is semisimple, possesses a Cartan subgroup T which is anisotropic over $\boldsymbol{R}$, and is an inner form of a split group. Then ${}^LG$ is the direct product of ${}^LG^\circ$ and Γ_k, the Weyl group W contains $-\mathrm{Id}$ and $G(\boldsymbol{R})$ has a discrete series. We want to describe the parametrization of the latter in terms of $\Phi(G)$. As the notation implies, we shall view ${}^LT$ as the L-group of T. Let $\varphi \in \Phi(G)$. It is given by a continuous homomorphism $\varphi': W_{\boldsymbol{R}} \to {}^LG^\circ$. We may assume that Im φ' is contained in the normalizer of ${}^LT^\circ$. Let $n = \varphi(\tau)$ and let $w \in W$ be the element of W represented by n. Then $w^2 = 1$. Let $\mu, \nu \in X^*(T) \otimes \boldsymbol{C}$ be such that

(1) $$\varphi(z) = z^\mu \cdot \bar{z}^\nu \qquad (z \in \boldsymbol{C}^*),\ \mu - \nu \in X^*(T)$$

(see 9.1). We have

(2) $$\varphi(\bar{z}) = n \cdot \varphi(z) \cdot n^{-1} = z^{w\cdot\mu} \cdot \bar{z}^{w\cdot\nu};$$

hence $\mu = w \cdot \nu$, $\nu = w \cdot \mu$. Assume now that Im φ is not contained in any proper Levi subgroup in ${}^LG$ or, equivalently, that Im φ' is not contained in any proper Levi subgroup in ${}^LG^\circ$. Then $w = -\mathrm{Id}$ and μ is regular: in fact, the proper Levi subgroups in ${}^LG^\circ$ are the centralizers of nontrivial tori. This implies first that w does not fix pointwise any nontrivial torus in ${}^LT^\circ$, hence $w = -\mathrm{Id}$; if now μ were singular, then the centralizer of $\mu(\boldsymbol{C}^*)$ would contain a semisimple subgroup $H \neq \{1\}$ stable under Int n, the latter would leave pointwise fixed a torus $S \neq \{1\}$ of H, and Im φ' would be contained in the centralizer $Z(S)$ of S, a contradiction. Since $\nu = w \cdot \mu = -\mu$, we have

$$\lambda(\varphi(-1)) = (-1)^{\langle 2\mu, \lambda \rangle}, \quad \text{for all } \lambda \in X_*(T). \tag{3}$$

Let δ be half the sum of the roots α of G with respect to T such that $\langle \mu, \check{\alpha} \rangle > 0$. Then, Lemma 3.2 of **[37]** implies in particular

$$\lambda(\varphi'(-1)) = (-1)^{\langle 2\delta, \lambda \rangle}, \quad \text{for all } \lambda \in X_*(T). \tag{4}$$

It follows that

$$\mu \in \delta + X^*(T). \tag{5}$$

Therefore μ is among the elements of $X^*(T) \otimes \boldsymbol{Q}$ which parametrize the discrete series in Harish-Chandra's theorem. We then let Π_φ be the set of discrete series representations of $G(\boldsymbol{R})$ with infinitesimal character χ_μ. If $G(\boldsymbol{R})$ is compact, then Π_φ consists of the irreducible finite dimensional representation with dominant weight $\mu - \delta$. In that case, no proper parabolic subgroup of ${}^LG$ is relevant; hence $\Phi(G)$ consists of the φ considered here.

10.6. *Let* $G = \mathbf{GL}_n$, *k nonarchimedean.* Let ψ be an admissible representation of W_k'. If it is irreducible, then $\psi(G_a) = 1$. If it is indecomposable, then it is a tensor product $\rho \otimes \mathrm{sp}(m)$, where m divides n, ρ is irreducible of degree n/m, and $\mathrm{sp}(m)$ is m-dimensional, trivial on 1, maps a generator of the Lie algebra of G_a onto the nilpotent matrix with ones above the diagonal, zero elsewhere, and $w \in W_k'$ onto the diagonal matrix with entries $a(w)^i$ $(0 \leqq i < n)$ **[9**, 3.1.3]. If χ is a character of W_k (hence of k^*), and $\varphi = \chi \otimes \mathrm{sp}(n)$, then Π_φ consists of the special representation with central character determined by χ. In fact, the Weil-Deligne group came up for the first time precisely to fit the special representations of $\mathbf{GL}_2$ into the general scheme (see **[9]**).

11. Outline of the construction over $\boldsymbol{R}$, $\boldsymbol{C}$. We sketch here the various steps which yield the sets Π_φ when $k = \boldsymbol{R}$. For the proofs see **[37]**.

We note first that we may always assume $\varphi(W_k) \subset N({}^LT^\circ)$, and we can write (9.1)

$$\varphi(z) = z^\mu \bar{z}^\nu \qquad (z \in \boldsymbol{C}^*;\ \mu, \nu \in X^*(T) \otimes \boldsymbol{C},\ \mu - \nu \in X^*(T)).$$

11.1. LEMMA. *Let* $\varphi \in \Phi(G)$. *Assume* $\varphi(W_{\boldsymbol{R}})$ *is not contained in any proper Levi subgroup in* ${}^LG$. *Then*

(i) *G has a Cartan k-subgroup C such that* $(\mathscr{D}G \cap S)(\boldsymbol{R})$ *is compact* **[28**, 3.1].

(ii) μ *is regular*; $\varphi(C^*)$ *contains regular elements* **[37**, 3.3].

The group ${}^LC^\circ$ may be viewed as a maximal torus of ${}^LG^\circ$; hence there is an isomorphism ${}^LC \to^\sim {}^LT$ defined modulo an element of W. Therefore φ defines an orbit of W in $\Phi(C)$, hence, by 9.2, an orbit X_φ of W in $X(C(\boldsymbol{R}))$. [Note that W, which is defined in $G(\boldsymbol{C})$, operates on $C(\boldsymbol{R})$, since $C(\boldsymbol{R}) \cap \mathscr{D}G$ is compact, hence on $X(C(\boldsymbol{R}))$.]

11.2. Let $G_0 = C(\boldsymbol{R})((\mathscr{D}G)(\boldsymbol{R}))^\circ$. Let A_0 be the set of representations of G_0 which are square-integrable modulo the center, and have infinitesimal character χ_λ ($\lambda \in X_\varphi$). The induced representations $\pi = I_{G_0(\boldsymbol{R})}^{G(\boldsymbol{R})}(\pi_0)$ ($\pi_0 \in A_0$) are irreducible [37, p. 50]. By definition, Π_φ is the set of equivalence classes of these representations [37, p. 54].

11.3. Let $\varphi \in \Phi(G)$. Let ${}^LM$ be a minimal relevant Levi subgroup containing Im φ. It is essentially unique (8.6). We assume ${}^LM \neq {}^LG$; we may view φ as an element of $\Phi(M)$. By 11.2, there is associated to it a finite set of $\Pi_{\varphi,M}$ of discrete series representations of M.

We may assume ${}^LM$ to be a Levi subgroup of a relevant parabolic subgroup ${}^LP$ corresponding to $P \in \mathscr{P}(G/k)$. Then $U = X^*(T) \otimes \boldsymbol{R} = X_*({}^LT^\circ) \otimes \boldsymbol{R}$. Let V be the subspace of elements of U which are orthogonal to roots of ${}^LM$, and fixed under Γ_k. It may be identified with the dual $\mathfrak{a}_P^*$ of the Lie algebra of a split component A of P.

Let ξ be the character of $C(M)$ defined by the elements of $\Pi_{\varphi,M}$. We may assume that $|\xi| \in \mathrm{Cl}(\mathfrak{a}_P^{*+})$. Let P_1 be the smallest parabolic k-subgroup containing P such that $|\xi|$, when restricted to $\mathfrak{a}_{P_1}$, is an element of the Weyl chamber $\mathfrak{a}_{P_1}^{*+}$. Let $M_1 = z(\mathfrak{a}_{P_1})$ and $P' = P \cap M_1$. Then P' is a parabolic subgroup of M_1. Moreover the restriction of $|\xi|$ to the split component $M_1 \cap A_P$ of P' is one; therefore, for each $\rho \in \Pi_{\varphi,M}$, the induced representation $\mathrm{Ind}_{P_1}^{M_1}(\rho)$ is tempered. Let Π'_φ be the set of all constituents of such representations. Then by definition, Π_φ is the set of Langlands quotients $J(P_1, \sigma)$ with $\sigma \in \Pi'_\varphi$ (cf. [37, p. 82]).

11.4. *Complex groups.* Assume now $k = \boldsymbol{C}$. Then $W_k = \boldsymbol{C}^*$, and $\Phi(G)$ may be identified to the set of homomorphisms of $\boldsymbol{C}^*$ into ${}^LT^\circ$, modulo the Weyl group W, i.e., to

$$\{(\lambda, \mu), \text{ where } \lambda, \mu \in X^*(T) \otimes \boldsymbol{C}, \lambda - \mu \in X^*(T)\} \tag{1}$$

modulo the (diagonal) action of W. In this case Im φ is in the Levi subgroup ${}^LT$ of ${}^LB$, which is the ${}^LP$ of 11.3. The set $\Pi_{\varphi,M}$ consists of one character of T (cf. 9.1). Choose P_1, M, as in 11.3. Since the unitary principal series of a complex group are irreducible (N. Wallach), the set Π'_φ consists of one element. Hence so does Π_φ. Thus each Π_φ is a singleton. The classification thus obtained is equivalent to that of Zelovenko.

11.5. Let $G = \mathbf{GL}_n$, $k = \boldsymbol{R}$. In this case, it is also true that the tempered representations induced from discrete series are irreducible [22]; therefore each set Π'_φ (cf. 9.3) consists of only one element, hence so does Π_φ and we get a bijection between $\Phi(G)$ and $\Pi G(\boldsymbol{R})$.

Let $n = 2$. If φ is reducible, then Im φ is commutative; hence φ factors through $(W_{\boldsymbol{R}})^{\mathrm{ab}} = \boldsymbol{R}^*$ and is described by two characters μ, ν of $\boldsymbol{R}^*$. Then Π_φ consists of a principal series representation $\pi(\mu, \nu)$ (including finite dimensional representations, as usual). In particular there are three φ's with kernel $\boldsymbol{C}^*$, to which correspond respectively $\pi(1, 1)$, $\pi(\mathrm{sgn}, \mathrm{sgn})$ and $\pi(1, \mathrm{sgn})$, where sgn is the sign character. If φ is irreducible, then $\varphi(\tau)$ may be assumed to be equal to (s_0, τ), where s_0 is a fixed element of the normalizer of ${}^LT^\circ$ inducing the inversion on it. $\varphi(\boldsymbol{R}^+)$ belongs to the center of ${}^LG^\circ$, and $\varphi(S)$ is sum of two characters, described by two integers. Then Π_φ consists of a discrete series representation, twisted by a one-dimensional representation.

11.6. As is clear from these two examples, the main point to get explicit knowledge of the sets Π_φ is the decomposition of representations induced from tempered representations of parabolic subgroups. This last problem has been solved by A. Knapp and G. Zuckerman **[29]**, **[30]**.

11.7. *Remark on the nonarchimedean case*. Langlands' classification **[37]** is also valid over p-adic fields **[57]**. In view of 8.6, it is then clear that the last step (11.3) of the previous construction can also be carried out in the nonarchimedean case. Thus, besides the decomposition of tempered representations, the main unsolved problem in the p-adic case is the construction and parametrization of the discrete series.

12. Local factors.

12.1. Let $\pi \in \Pi(G(k))$ and r be a representation of ${}^LG$ (2.6). Assume that $\pi \in \Pi_\varphi$ for some $\varphi \in \Phi(G)$. For a nontrivial additive character ψ of k, we let

(1) $$L(s, \pi, r) = L(s, r \circ \varphi), \qquad \varepsilon(s, \pi, r) = \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \varphi, \psi),$$

where on the right-hand sides we have the L- and ε-factors assigned to the representation $r \circ \varphi$ of W_k' **[60]**. In the unramified situation of 10.4, this coincides with the definition given in 7.2.

In view of what has been recalled so far, these local factors are defined if k is archimedean, or if k is nonarchimedean in the unramified case, or if G is a torus.

12.2. Let now $G = \mathbf{GL}_n$. In this case there are associated to $\pi \in \Pi(G(k))$ local factors $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ defined by a generalization of Tate's method, in **[25]** for $n = 2$, in **[19]** for any n, which play a considerable role in the parametrization problem and in the local lifting. A natural question is then whether these factors can be viewed as special cases of 12.1, where $r = r_n$ is the standard representation of $\mathbf{GL}_n$, i.e., whether we have equalities

(1) $$L(s, \pi) = L(s, \pi, r_n), \qquad \varepsilon(s, \pi, \psi) = \varepsilon(s, \pi, r_n, \psi),$$

with the right-hand side defined by the rule of 12.1.

(a) Let $n = 2$. It has been shown in **[25]** that the equivalence class of π is characterized by the functions $L(s, \pi \otimes \chi)$, $\varepsilon(s, \pi \otimes \chi, \psi)$, where χ varies through the characters of k^*. In this case, the parametrization problem and the proof of (1) are part of the following problem:

(*) Given $\sigma \in \Phi(G)$, find $\pi = \pi(\sigma)$ such that

(2) $$L(s, \sigma \otimes \chi) = L(s, \pi \otimes \chi), \qquad \varepsilon(s, \sigma \otimes \chi, \psi) = \varepsilon(s, \pi \otimes \chi, \psi)$$

for all χ's, and prove that $\sigma \mapsto \pi(\sigma)$ establishes a bijection between $\Phi(G)$ and $\Pi(G(k))$.

This problem was stated and partially solved in **[25]**. The most recent and most complete results in preprint form are in **[62]**; they still leave out some cases of even residual characteristic, although some arguments sketched by Deligne might take care of them (see **[63]** for a survey).

As stated, the problem is local, but, except at infinity, progress was achieved first mostly by global methods: one uses a global field E whose completion at some place v is k, a reductive E-group H isomorphic to G over k, an element $\rho \in \Phi(H/k)$ whose restriction to ${}^L(H/k_v) = {}^LG$ is σ, chosen so that there exists an automorphic representation $\pi(\rho)$ with the L-series $L(s, \rho)$ (see §14 for the latter). This construction relies, among other things, on Artin's conjecture in some cases, and **[38]**. In fact, it was already shown in **[25]** that (*) for odd residual characteristics follows from Artin's conjecture, leading to a proof in the equal characteristic case. At present, there are in principle purely local proofs in the odd residue characteristic case **[63]**. Note also that the injectivity assertion is a statement on two-dimensional admissible representations of W_k', namely, whether such a representation σ is determined, up to equivalence, by the factors $L(s, \sigma \circ \chi)$ and $\varepsilon(s, \sigma \circ \chi, \psi)$. But, so far, the known proofs all use admissible representations of reductive groups **[63]**.

(b) For arbitrary n, (1) has been proved in the unramified case, for special representations, and by H. Jacquet for $k = \boldsymbol{R}, \boldsymbol{C}$ **[24]**.

(c) Local L- and ε-factors are also introduced for $G = \mathbf{GL}_2 \times \mathbf{GL}_2$ in **[21]**, at any rate for products $\pi \times \pi'$ of infinite dimensional irreducible representations. Partial extensions of this to $\mathbf{GL}_m \times \mathbf{GL}_n$ for other values of m, n are known to experts.

(d) For $n = 3$, $\pi \in \Pi(G(k))$ is again characterized uniquely by the factors $L(s, \pi \otimes \chi)$ and $\varepsilon(s, \pi \otimes \chi, \psi)$ **[27]**, **[46]**. For $n \geqq 4$ on, this is false **[46]**. However, it may be there are still such characterizations if χ is allowed to run through suitable elements of $\Pi(\mathbf{GL}_{n-1}(k))$ or maybe just $\Pi(\mathbf{GL}_{n-2}(k))$.

12.3. Local factors have also been defined directly for some other classical groups, in particular for $\mathbf{GSp}_4$ by F. Rodier **[48]**, extending earlier work of M. E. Novodvorsky and I. Piatetskii-Shapiro, for split orthogonal groups, in an odd number $2n + 1$ of variables by M. E. Novodvorsky **[41]**. In the latter case ${}^LG^\circ = \mathbf{Sp}_{2n}$, and in the unramified case, the local factors coincide (up to a translation in s) with those associated by 7.2 to the standard $2n$-dimensional representation of the L-group. See also **[42]**.

CHAPTER IV. THE L-FUNCTION OF AN AUTOMORPHIC REPRESENTATION.

From now on, k is a global field, $\mathfrak{o} = \mathfrak{o}_k$ the ring of integers of k, A_k or A the ring of adeles of k, V (resp. V_∞, resp. V_f) the set of places (resp. infinite places, resp. finite places) of V. For $v \in V$, k_v, o_v and Nv have the usual meaning. Unless otherwise stated, G is a connected reductive k-group.

13. The L-function of an irreducible admissible representation of G_A.

13.1. Let π be an irreducible admissible representation of G_A and r a representation of ${}^LG$. There exists a finite Galois extension k' of k over which G splits and such that r factors through ${}^LG^\circ \rtimes \Gamma_{k'/k}$. We want to associate to π and r infinite Euler products $L(s, \pi, r)$ and $\varepsilon(s, \pi, r)$, whose factors are defined (at least) for almost all places of k.

Let $v \in V$. By restriction, r defines a representation r_v of $L(G/k_v) = {}^LG^\circ \rtimes \Gamma_k$. On the other hand, $\pi = \bigotimes_v \pi_v$, with $\pi_v \in \Pi(G(k_v))$ **[11]**. Assume the parametrization problem of Chapter III solved. Then there is a unique $\varphi_v \in \Phi(G/k_v)$ such that $\pi_v \in \Pi_{\varphi_v}$. Then we let

$$L(s, \pi, r) = \Pi_v L(s, \pi_v, r_v), \tag{1}$$

$$\varepsilon(s, \pi, r) = \Pi_v \varepsilon(s, \pi_v, r_v, \psi_v), \tag{2}$$

where ψ_v is an additive character of k_v associated to a given nontrivial additive character of k, and the factors on the right are given by 12.1(1).

The local problem is solved for archimedean v's, and for almost all finite v's (see below) so that the factors on the right are defined except for at most finitely many $v \in V_f$. For questions of convergence or meromorphic analytic continuation this does not matter, and we shall also denote such partial products by $L(s, \pi, r)$.

By 10.4, φ_v is well defined if the following conditions are fulfilled: G is quasi-split over k_v, $G(o_v)$ is a very special maximal compact subgroup of $G(k_v)$, k' is unramified over k, and π_v is of class one with respect to $G(o_v)$. All but finitely many $v \in V_f$ satisfy those conditions **[61]**.

13.2. THEOREM **[35]**. *Let π be an irreducible admissible unitarizable representation of G_A and r be a representation of ${}^LG$ (2.6). Then $L(s, \pi, r)$ converges absolutely for* Re s *sufficiently large.*

We may and do view r as a complex analytic representation of ${}^LG^\circ \rtimes \Gamma_{k'/k}$, where k' is a finite Galois extension of k over which G splits (2.7). We let V_1 be the set of $v \in V_f$ satisfying the conditions listed at the end of 13.1. We have to show that

$$L' = \prod_{v \in V_1} L(s, \pi_v, r_v), \tag{1}$$

converges in some right half-plane.

Let Fr_v be the Frobenius element of $\Gamma_{k'_{v'}/k_v}$, where $v' \in V_{k'}$ lies over $v \in V_1$. We have

$$\varphi_v(\mathrm{Fr}_v) = (t_v, \mathrm{Fr}_v), \quad \text{with } t_v \in {}^LT^\circ \tag{2}$$

and

$$L(s, \pi_v, r_v) = (\det(1 - r((t_v, \mathrm{Fr}_v))N_v^{-s}))^{-1}. \tag{3}$$

To prove the theorem, it suffices therefore to show the existence of a constant $a > 0$ such that

$$|\mu| \leqq (Nv)^a \quad \text{for every } v \in V_1 \text{ and eigenvalue } \mu \text{ of } r((t_v, \mathrm{Fr}_v)). \tag{4}$$

Let $n = [k' : k]$. Since we may assume t_v fixed under Γ_{k_v} (6.3), we have $t_v^n = (t_v, \mathrm{Fr}_v)^n$; hence it is equivalent to show (4) for all eigenvalues μ of $r(t_v)$. These are of the form t_v^λ, where λ runs through the set P_r of weights of r, restricted to ${}^LG^\circ$. Thus we have to show the existence of $a > 0$ such that

$$|t_v|^{\mathrm{Re}\,\lambda} \leqq (Nv)^a \quad \text{for all } v \in V_1 \text{ and } \lambda \in P_r. \tag{5}$$

Let G' be a quasi-split inner k-form of G. Then ${}^LG = {}^LG'$, and G is isomorphic to G' over k_v for all $v \in V_1$. We may therefore replace G by G'; changing the notation slightly, we may (and do) assume G to be quasi-split over k. We then fix a Borel k-subgroup B of G and view ${}^LT$ as the L-group of a maximal k-torus T of G.

For a cyclic subgroup D of $\Gamma_{k'/k}$, let V_D be the set of $v \in V_1$ for which Γ_{k_v} is equal to the inverse image of D in Γ_k. The group $U = X_*(T)^D$ is then the group of one-parameter subgroups of a subtorus S of T such that S/k_v is a maximal k_v-split torus of G/k_v for all $v \in V_D$. The group

$$Y = \mathrm{Hom}(U, C^*) = \mathrm{Hom}(X_*(T)^D, C^*) \qquad (v \in V_D), \tag{6}$$

is independent of v, and is the Y of §6 for G/k_v. The root datum $\psi(G/k_v)$, which is determined by the action of D, is also independent of $v \in V_D$.

Given $y \in Y$, let y_0 be a "logarithm" of y, i.e., an element of $\mathrm{Hom}(X_*(T)^D, \boldsymbol{C})$ such that

$$y(u) = Nv^{y_0(u)} = Nv^{\langle y_0, u\rangle}, \quad \text{for } u \in X_*(T)^D. \tag{7}$$

This element is determined modulo a lattice, but its real part $\mathrm{Re}\, y_0 \in \mathrm{Hom}(U, \boldsymbol{R})$, defined by

$$y(u) = Nv^{\langle \mathrm{Re}\, y_0, u\rangle} \tag{8}$$

is well defined. If y has values in $\boldsymbol{R}_+^*$, then we choose y_0 to be equal to its real part. The space $\mathfrak{a}^*$ is the dual of $\mathfrak{a} = U \otimes \boldsymbol{R}$ (the so-called real Lie algebra of S/k_v), and is acted upon canonically by ${}_kW$ as a reflection group. We let $\mathfrak{a}^{*+}$ be the positive Weyl chamber defined by $\boldsymbol{B}$.

Let ρ_v be the unramified character of $T(k_v)$, given by $t \mapsto |\delta(t)|_v$, where $|\ |_v$ is the normalized valuation at v and δ half the sum of the positive roots. Then its real logarithm ρ_0 *is independent of* $v \in V_D$. In fact, it is a positive integral power of Nv whose exponent is determined by the k_v-roots, their multiplicities, and the indices q_α of the Bruhat-Tits theory [**61**]. But those are determined by the previous data and the action of Γ_{k_v} on the completed Dynkin diagram [**61**], which is also independent of $v \in V_D$. We write ρ_0 instead of $\rho_{v,0}$. We have $\rho_0 \in \mathfrak{a}^{*+}$.

The representation π_v is a constituent of an unramified principal series $\mathrm{PS}(\chi_v)$, where χ_v is an unramified character of $T(k_v)$, or, equivalently, of $S(k_v)$, determined up to a transformation by an element of ${}_kW$. Thus we may assume $\chi_{v,0}$ to be contained in the closure $\mathscr{C}\ell(\mathfrak{a}^{*+})$ of $\mathfrak{a}^{*+}$. Since π_v is unitary, the associated spherical function is bounded, and hence $\mathrm{Re}\,\chi_{v,0}$ is contained in the convex hull of ${}_kW(\rho_0)$, i.e., we have

$$\langle \rho_0 - \chi_{v,0}, \lambda\rangle \geqq 0, \quad \text{for all } \lambda \in \mathfrak{a}^{*+}. \tag{9}$$

(See remark following the proof.)

For $\lambda \in X^*({}^LT^\circ)$, let λ' be the restriction of λ to $X_*(T^D)$. In view of 10.4 and our conventions, we have then

$$|\lambda(t_v)| = Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \lambda'\rangle}. \tag{10}$$

Let $\bar\lambda = {}_kW(\lambda') \cap \mathscr{C}\ell(\mathfrak{a}^{*+})$. Since $\mathrm{Re}\,\chi_{v,0} \in \mathscr{C}\ell(\mathfrak{a}^{*+})$, we have

$$Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \lambda'\rangle} \leqq Nv^{\langle \mathrm{Re}\,\chi_{v,0}, \bar\lambda\rangle}. \tag{11}$$

Combined with (9), this implies

$$|\lambda(t_v)| \leqq Nv^{\langle \rho_0, \bar\lambda\rangle}. \tag{12}$$

If now λ runs through P_r, there are only finitely many possibilities for $\bar\lambda$, whence (4), with $a = \sup\langle \rho_0, \bar\lambda\rangle$ $(\lambda \in P_r)$, for $v \in V_D$. Since V_1 is a finite union of such sets, this proves (4).

REMARK. The relation (9) is proved in [**35**, pp. 27–29] for the split case. For a general semisimple simply connected group, see I. Macdonald, *Spherical functions on a group of p-adic type*, Publ. Ramanujan Institute 2, Madras, Theorem 4.7.1, or H. Matsumoto, Lecture Notes in Math., vol. 590, Springer-Verlag, Berlin and New York, Proposition 4.4.11. In fact, we have used it for a general connected reductive group but the reduction to the case of simply connected semisimple groups is easily carried out by going over to the universal covering of the derived group.

13.3. COROLLARY. *Let P be a parabolic k-subgroup of G, $P = M \cdot N$ a Levi decomposition over k of P. Assume that π is a constituent of a representation* Ind $_{P_A}^{G_A}(\sigma)$ *induced from a unitarizable irreducible admissible representation σ of M_A, viewed as a representation of P_A trivial on N_A. Then $L(s, \pi, r)$ is absolutely convergent in some right half-plane.*

We view ${}^LM$ as a subgroup of ${}^LG$ (3.3). Let r' be the restriction of r to ${}^LM$.

Let $v \in V_f$ be such that the conditions listed at the end of 13.1 are satisfied by M, G, σ_v and π_v. Then, by the transitivity of induction, it follows that there exists χ_v as in the above proof such that σ_v (resp. π_v) is the constituent of class 1 with respect to $M(o_v)$ (resp. $G(o_v)$) of the principal series PS(χ_v) for $M(k_v)$ (resp. $G(k_v)$). Then $L(s, \pi_v, r) = L(s, \sigma_v, r')$ (7.2, 10.4). This being true for almost all v's, we are reduced to 13.2.

14. The L-function of an automorphic representation.

14.1. A smooth representation of G_A is *automorphic* if it is a subquotient of the regular representation of G_A in $G_k \backslash G_A$. It is cuspidal if it consists of cusp forms. If so, it is unitary modulo the center. We let $\mathfrak{A}(G/k)$ denote the set of equivalence classes of irreducible admissible automorphic representations of G_A. By Proposition 2 of **[39]**, every $\pi \in \mathfrak{A}(G/k)$ is a constituent of a representation induced from some cuspidal $\sigma \in \mathfrak{A}(M/k)$, where M is a Levi k-subgroup of a parabolic k-subgroup of G. Combined with 13.3 this yields the

14.2. THEOREM (LANGLANDS). *Let $\pi \in \mathfrak{A}(G/k)$ and r be a representation of ${}^LG$. Then $L(s, \pi, r)$ is absolutely convergent in some right half-plane.*

The L-function of an irreducible admissible automorphic representation will also be called an automorphic L-function.

14.3. There are several conjectures on the analytic character of $L(s, \pi, r)$ for automorphic π, all checked in some special cases, going back to the work of Hecke on L-series attached to Grössencharaktere and to modular forms.

(a) If $\pi \in \mathfrak{A}(G/k)$, then $L(s, \pi, r)$ admits a meromorphic continuation to the whole complex plane.

(b) Assume that π and G are such that the local solution to the local problem yields factors L and ε at all places. It is then conjectured that there is a functional equation $L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1 - s, \tilde{\pi}, r)$, where $\tilde{\pi}$ is the contragredient representation to π.

(c) In a number of cases, it has been shown that:

(*) If π is cuspidal, r irreducible nontrivial, then $L(s, \pi, r)$ is entire.

Here and there, conjectures to the effect that this should be a general phenomenon have been stated. However, there are counterexamples. Heuristically, one sees this is likely to happen if π is lifted from a cuspidal representation of a reductive group H (in the sense of V below) and the restriction of r to ${}^LH$ contains the trivial representation.

14.4. (a) Let $G = \mathbf{GL}_n$ and $r = r_n$ be the standard representation of $\mathbf{GL}_n(\mathbf{C})$. Then 14.3(b), (c) are proved in **[25]** for $n = 2$, in **[19]** for $n \geqq 2$, if L and ε are defined to be the products of the L- and ε-factors mentioned in 12.4. As recalled in 12.4, these are the same as those considered here at almost all places, and for $n = 2$, at all places.

(b) If $G = \mathbf{GL}_2 \times \mathbf{GL}_2$ and $r = r_2 \otimes r_2$, similar results are established by Jacquet in **[21]**.

(c) Let $G = \mathbf{GL}_2$. If $r: \mathbf{GL}_2(C) \to \mathbf{GL}_3(C)$ is the adjoint representation, then 14.3(b), (c) are announced in **[16]**. This extends results of Shimura **[54]**. If $r = \mathrm{Sym}^3(r_2)$, $\mathrm{Sym}^4(r_2)$, then 14.3(b) is stated in **[15]**, in the context of the global lifting (see V); for $\mathrm{Sym}^3(r_2)$, it is also proved in **[51]**, in the framework of 14.5 below.

(d) Let k be a function field, $G = \mathbf{GL}_m \times \mathbf{GL}_n$ and $r = r_m \otimes r_n$. Let π (resp. π') be a cuspidal automorphic representation of the first (resp. second) factor. By the methods of **[19]**, **[26]**, **[27]**, one can define L and ε, and (Jacquet dixit) show 14.3(b), and also the holomorphy, except when $m = n$ and π is contragredient to π'. These methods also yield further examples for other groups and for other representations. It is expected that similar results hold over number fields.

(e) 14.3(a) has also been checked when $G = \mathbf{PSp}(4)$ in some cases in **[1]**, and, in general, in **[42]**. A functional equation is also established. 14.3(a), (b) are announced in **[41]** for orthogonal groups in an odd number of variables over functional fields, for the local factors mentioned in 12.3. For a survey and earlier references, see **[43]**. See also **[44]**.

14.5. We describe some cases in which 14.3(a) has been verified in **[33]** (see also **[18]** for a survey). Let C be a split k-group, of adjoint type, endowed with its canonical $\mathfrak{o}$-structure. Fix a Borel subgroup B of C and a maximal torus T of B defined over $\mathfrak{o}$. Let P be a maximal proper standard parabolic subgroup and $P = M \cdot N$ its standard Levi decomposition. Since C is adjoint, it is easily seen that $C(M)$ is a torus. The group $M/C(M)$ is semisimple, split over k, of adjoint type, of rank equal to $\mathrm{rk}(C) - 1$. We let $G = M/C(M)$. The group ${}^LG^\circ$ is simply connected (2.2(2)). We have a natural inclusion ${}^LG \to {}^LM$, and ${}^LM$ is the Levi subgroup of a standard parabolic subgroup ${}^LP = {}^LM \cdot U$ with unipotent radical U (3.3). Let A be the split component of P in T, and ${}^LA^\circ$ the split component of ${}^LP^\circ$ in ${}^LT^\circ$. The group ${}^LA^\circ$ acts on the Lie algebra $\mathfrak{u}$ of U and its eigenspaces are irreducible ${}^LG^\circ$-modules. We let F_P denote the set of contragredient representations to these ${}^LG^\circ$-modules. The L-functions considered in **[33]** are of the form $L(s, \pi, r)$ with $r \in F_P$ and π an irreducible cuspidal automorphic representation of G. A number of examples are given in which $L(s, \pi, r)$ admits a meromorphic continuation. This is deduced from the results of **[32]**: let m be the length of a composition series of $\mathfrak{u}$ with respect to M. Then, for suitable numbering of the elements of F_P and strictly positive integers a_i, there is a relation

$$(1) \qquad M(s) = \prod_{1 \leq i \leq m} L(a_i s, \pi, r_i) \cdot L(sa_i + 1, \pi, r_i)^{-1},$$

where $M(s)$ is the intertwining operator occurring in the theory of Eisenstein series with respect to P, and is known to have a meromorphic continuation to the complex plane **[32]**. If $r = 1$, this and 13.2 yield the meromorphic continuation. In general, if we have the analytic continuation for all r_i's except one, (1) gives it for the remaining one.

14.6. The *converse problem* is to what extent automorphic representations can be characterized by analytic properties of their L-functions, or to give analytic conditions on a given L-function which will insure that it is automorphic. The first main result was Hecke's characterization of the Mellin transform of a parabolic modular form. Then came Weil's extension of this theorem to congruence subgroups **[64]**, **[65]**, its generalization in the context of representations in **[25]**, and the extension to $\mathbf{GL}_3$ **[46]**, **[27]**. In those results, conditions are imposed on the L-functions of π and of the twists $\pi \otimes \chi$ of π by characters. However, the analogous statement is false from $n = 4$ on **[46]**. It may remain true if one imposes conditions on the twist $\pi \otimes \rho$ of π by representations of $\mathbf{GL}_{n-1}$ or only of $\mathbf{GL}_{n-2}$. For results in that direction, over function fields, see **[45]**.

Note however that in the general problem outlined here, one wishes rather to turn things around and deduce the analytical properties of some given L-series by showing directly that it is automorphic (see the seminars on base change and on zeta-functions of Shimura varieties **[17]**, **[8]**, **[40]**).

14.7. *Other problems.* (1) One "representation theoretic" form of "Ramanujan's conjecture" is the following: if $\pi = \bigotimes \pi_v$ is an irreducible nontrivial admissible cuspidal automorphic representation (and G is simple), then each π_v is tempered. It is now well known to be false for certain orthogonal or unitary groups, and even for one split group **[20]**.

(2) Let π be a unitary irreducible representation of G_A. If $G = \mathbf{GL}_2$, then its multiplicity in the space of cusp forms ${}^0L_2(G(k)\backslash G(A))$ is at most one, "multiplicity one theorem" **[25]**. In fact there is even a "strong multiplicity one theorem" **[38]**: given π_v for almost all v's, there is at most one constituent π of the space of cusp-forms with those local factors.

The multiplicity one theorem has been proved for $\mathbf{GL}_n$ **[52]** and the strong form for $\mathbf{GL}_3$ **[28]**. It is unknown whether it is true for $\mathbf{SL}_2$. On the other hand, there are counterexamples for some inner forms of $\mathbf{SL}_2$ **[31]**.

Chapter V. Lifting Problems.

Although the problems on automorphic L-functions discussed in §14 are only partially solved, the solutions provide practically all cases in which an L-series (automorphic or not) has been proved to have meromorphic or holomorphic analytic continuation with functional equation. This suggests trying, given an L-series and a reductive group G, to see whether G has an automorphic representation with the given L-series. Many instances of such questions can be viewed more precisely as special cases of the "lifting problem" or of the "problem of functoriality with respect to morphisms of L-groups." There is also a local version. For the sake of exposition, we shall start with the latter, but it should be borne in mind that the motivation and requirements stem from the global one, and that local and global are at present inextricably linked in many proofs. These questions were raised by Langlands in **[35]**.

15. L-homomorphisms of L-groups.

15.1. Let E be a field and H, G connected reductive E-groups. A homomorphism $u\colon {}^LH \to {}^LG$ over Γ_k is said to be an L-homomorphism if it is continuous and if its restriction to ${}^LH^\circ$ is a complex analytic homomorphism of ${}^LH^\circ$ into ${}^LG^\circ$. Let E be local and G quasi-split. If $\varphi \in \Phi(H)$, then $u \circ \varphi \in \Phi(G)$. In fact, condition 8.2(i) is clearly satisfied, by $u \circ \varphi$, and so is 8.2(ii) because every parabolic subgroup of ${}^LG$