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EULER PRODUCTS CORRESPONDING TO SIEGEL MODULAR FORMS OF GENUS 2

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In this article we construct a theory of Dirichlet series with Euler product expansions corresponding to analytic automorphic forms for the integral symplectic group in genus 2; in Chapter 2 we establish a connection between the eigenvalues of the Hecke operators on the spaces of such forms with the Fourier coefficients of the eigenfunctions (Theorem 2.4.1); in Chapter 3 we demonstrate the possibility of analytic continuation to the entire complex plane and derive a functional equation for Euler products corresponding to the eigenfunctions of the Hecke operators (Theorem 3.1.1). Chapter 1 contains a survey of the present state of the theory of Euler products for Siegel modular forms of arbitrary genus n, including a sketch of the classical Hecke theory for the case $n = 1$.

Contents

Introduction

In the numerous applications of zeta-functions to arithmetic it frequently turns out that a decisive role is played by the properties (zeros, poles, values taken, . . .) of zeta-functions outside, or on the boundary of, the domain of convergence of the Dirichlet series which define them. On the other hand, these applications themselves, especially in problems concerned with primes, usually turn out to be possible only for those zeta-functions that have expansions as products of Euler type or can be expressed in terms of such products. The problems of analytic continuation and of Euler product expansion of zeta-functions are different in character, but apparently deeply connected.

At the moment, the only general approach to the problem of analytic continuation is based on the theory of automorphic forms for discrete transformation groups (the equivalent language of the theory of induced representations is often used): if one succeeds in associating in a natural way a zeta-function with a suitable automorphic form, say, by means of an integral transformation, then its analytic continuability to the whole complex plane usually follows easily, and at the same time a functional equation, describing a symmetry of the zeta-function with respect to some vertical line. This is how the general Hecke L-functions of algebraic number fields, the zeta-functions of quadratic forms, the zeta-functions of simple algebras over global fields, and so on, have been considered. Naturally, by and by it was just the zeta-functions corresponding to various classes of automorphic forms that were discussed, often without any visible aim.

The theory of Euler product expansions of zeta-functions begins with the celebrated paper of Hecke in 1937 [1]. Up till then, apart from a few isolated examples,[1]) the possibility of expanding zeta-functions as Euler products either followed from general theorems of arithmetic such as the uniqueness of prime factorization, or was postulated as part of the definition (Artin L-functions); or else nothing could be said about the possibility of such an expansion. Hecke considered zeta-functions corresponding to classical modular forms, that is, automorphic forms for the modular group $Sp_1(Z) = SL_2(Z)$ and its congruence subgroups. This class includes, in particular, the zeta-functions of positive integral quadratic forms in an even number of variables. It turned out that although the Dirichlet

[1] In 1917 Mordell proved that Ramanujan's τ-function and some similar functions are multiplicative, from which the Euler product expansions for the corresponding Dirichlet series follow. In Mordell's proof the idea of Hecke operators was present, but was not developed at the time.

series corresponding to a specific modular form need not have an Euler product, the space of all such series, corresponding to forms of fixed integral weight for a fixed group (which is finite-dimensional) has a basis consisting of series that have Euler products of a special type. The modular forms whose associated Dirichlet series have an Euler product can be characterized in an invariant way as the eigenfunctions of certain operators the Hecke operators.

In the post-war years the algebraic aspects of Hecke's theory have been the subject of intensive generalization. Abstract Hecke rings were defined, the structure of Hecke rings was studied for many classes of algebraic groups, and their representations were classified; explicit formulae were obtained for the spherical functions giving one-dimensional representations of the Hecke rings of p-adic groups. Finally, Hecke rings came to play an important part in the theory of finite groups and in the theory of representations of algebraic groups.

Conversely, the analytical side of Hecke's theory turned out to be very difficult to generalize to higher dimensions. We have in mind the problem of analytic continuation and the functional equation for the Euler products that correspond naturally to representations of the Hecke rings of algebraic groups on spaces of automorphic forms (see Langlands' lectures [2] and [3], where the most general conjectures in this direction are stated). At the moment, comparatively general results have only been obtained for the groups GL_n (see [4] and the references listed there). The present situation is that to one and the same automorphic form there correspond both a Dirichlet series, for which the possibility of analytic continuation can be proved, although there is no way of deducing an Euler product expansion, and an Euler product, for which it is not clear how to prove analytic continuation. (In the case considered by Hecke, the Dirichlet series corresponding to the eigenfunctions of all the Hecke operators coincide with the Euler products, and the two parts of the theory merge.)

What has been said is entirely applicable to the most natural and important (from the analytic and arithmetical points of view) higher-dimensional analogue of modular forms – the case of analytic automorphic forms for the Siegel modular group $Sp_n(\mathbf{Z})$ and its congruence subgroups (Siegel modular forms). These were discovered by Siegel in 1935 in connection with his classical investigations on the problem of integral representations of quadratic forms, and they have numerous links with arithmetic. Since then, the theory of Siegel modular forms has reached about the same level as the theory of classical modular forms (see, for example, [5]). Much has also been done towards the construction of the analogue of Hecke's theory. By means of integral transformations generalizing Mellin transforms, Maass has associated with Siegel modular forms Dirichlet series admitting analytic continuation to the whole complex plane, and satisfying functional equations (see [6], where Maass's investigations in this direction are summarized). He also began [7] the development of the theory of Hecke operators on Siegel forms, he proved that they are Hermitian, and obtained interesting induction formulae. Shimura [8] determined the structure of the Hecke rings for the groups $\mathrm{Sp}_n(\mathbf{Z})$ and found generators for them. Satake [9] classified the one-dimensional representations of the p-adic symplectic group (spherical functions). Finally, the rationality of the local zeta-functions of the symplectic group (the p-factor in the Euler products) has been proved, and their structure has been determined (Shimura [8] for $n = 2$, and Andrianov [10], [11] for any n). In no case of $n > 1$ has the possibility of analytic continuation of the Euler products been proved, nor has their arithmetical meaning been clarified.

In the present article we construct a theory, to a certain extent complete, of Euler products for the next case after that analysed by Hecke – the case of modular forms for the full Siegel modular group of genus 2. The main part of the paper, Chapters 2 and 3, are devoted to this. The basic results of the theory are Theorem 2.4.1 (the relation between the eigenvalues of the Hecke operators and the Fourier coefficients of the eigenfunctions) and Theorem 3.1.1 (analytic continuability and the functional equation for Euler products). Chapter 1 contains a survey of the present state of the theory in arbitrary genus n, including a sketch of Hecke's theory.

The conceptual plan for the study of Euler products for modular forms in genus 2 was first realized by the author in [12], under some restrictions, which greatly simplify the technical side of the proofs. The interest that this article met among the specialists has prompted me to write this exposition of complete proofs of the general assertions. The results of the present article were announced in [13].

NOTATION. **Z**, **Q**, **R** and **C** denote the rings of rational integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively.

If **A** is a commutative ring with a unit element, then $M_n(\mathbf{A})$ is the ring of $n \times n$ matrices with entries in **A**; $\mathrm{GL}_n(\mathbf{A})$ and $\mathrm{SL}_n(\mathbf{A})$ denote the general linear group and the special linear group of order n over **A**. For a square matrix $X = (x_{ij})$, ${}^tX = (x_{ji})$ denotes the transpose, $\sigma(X) = \sum x_{ii}$ the trace of X, and $\det X$ the determinant of X. E_n is the $n \times n$ identity matrix, and

$$J_n = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

For a commutative ring **A** with a unit element

$$\mathrm{Sp}_n(\mathbf{A}) = \{M \in M_{2n}(\mathbf{A});\ MJ_n{}^tM = J_n\}$$

is the symplectic group of genus n over **A**.

The inequality $A > B$ (respectively, $A \geqslant B$) for real symmetric matrices A and B means that all the eigenvalues of the matrix $A - B$ are positive (respectively, non-negative).

Chapter 1

MODULAR FORMS AND EULER PRODUCTS

§ 1.1. Siegel modular forms

For further details on the facts and definitions set out below, see [5]–[7].

The Siegel upper half-plane of genus $n \geqslant 1$ is the set of symmetric $n \times n$ complex matrices having positive definite imaginary part:

$$H_n = \{Z = X + iY \in M_n(\mathbf{C});\quad {}^tZ = Z,\ Y > 0\}.$$

H_n is a complex manifold of complex dimension $n(n+1)/2$.

The real symplectic group of genus n $\mathrm{Sp}_n(\mathbf{R})$ acts on H_n: if $M \in \mathrm{Sp}_n(\mathbf{R})$, $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (M split up into $n \times n$ blocks), then the map

$$(1.1.1) \qquad Z \to M\langle Z\rangle = (AZ + B)(CZ + D)^{-1} \quad (Z \in H_n)$$

is an analytic automorphism of H_n. It is easy to see that this action is transitive (the subgroup of matrices with $C = 0$ already acting transitively), and that the composition of transformations corresponds to matrix multiplication:

$$(1.1.2) \qquad M_1\langle M_2\langle Z\rangle\rangle = M_1M_2\langle Z\rangle.$$

This allows us to identify H_n with a certain homogeneous space of the group $\mathrm{Sp}_n(\mathbf{R})$: we denote by K_n, say, the stabilizer of the point $iE_n \in H_n$:

$$K_n = \{M \in \mathrm{Sp}_n(\mathbf{R});\quad M\langle iE_n\rangle = iE_n\};$$

then the map $M \to M\langle iE_n\rangle$ defines a bijection

$$(1.1.3) \qquad \mathrm{Sp}_n(\mathbf{R})/K_n \to H_n,$$

which is compatible with the action of $\mathrm{Sp}_n(\mathbf{R})$ if it acts by right multiplication on the left-hand side. K_n is a (maximal) compact subgroup of the Lie group $\mathrm{Sp}_n(\mathbf{R})$, and the map $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \to A + iB$ identifies K_n with the group of $n \times n$ unitary matrices $U(n)$.

The image under the map $\mathrm{Sp}_n(\mathbf{R}) \to H_n$ of the Haar measure on $\mathrm{Sp}_n(\mathbf{R})$ defines an element of volume on H_n, unique up to a constant factor, that is invariant under $\mathrm{Sp}_n(\mathbf{R})$, and can be written in the form

$$(1.1.4)\qquad \widetilde{dZ}=(\det Y)^{-(n+1)}\prod_{\alpha\leqslant\beta}dx_{\alpha\beta}\prod_{\alpha\leqslant\beta}dy_{\alpha\beta}\qquad (Z=X+iY).$$

The most important discrete subgroups of $\mathrm{Sp}_n(\mathbf{R})$ for arithmetical applications are the subgroup of integral matrices

$$\Gamma_n=\mathrm{Sp}_n(\mathbf{Z})=\mathrm{Sp}_n(\mathbf{R})\cap M_{2n}(\mathbf{Z}),$$

which is called the Siegel modular group of genus n, and its subgroups of finite index. Each such subgroup acts discretely on H_n. As Siegel has proved, there exists a fundamental domain for the action of Γ_n on H_n, that is, a closed subset $D_n\subset H_n$ such that every orbit of Γ_n on H_n meets D_n, but that no two interior points of D_n are in the same orbit of Γ_n. Hence, there also exists a fundamental domain for any subgroup of finite index in Γ_n. The choice of fundamental domain is not unique. No fundamental domain for a subgroup of finite index of the Siegel modular group is compact, but it has finite invariant volume. We only consider the case of the full modular group Γ_n.

A modular form of genus n and weight k, where n and k are natural numbers, is any function $F(Z)$ that is holomorphic on H_n and satisfies the following two conditions:

(1.1.5) *for every* $M=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in\Gamma_n$ *we have the functional equation* $\det(CZ+D)^{-k}F(M\langle Z\rangle)=F(Z)\quad(Z\in H_n)$;

(1.1.6) *the function* $F(Z)$ *is bounded in every domain of the form* $\{Z=X+iY\in H_n;\ \ Y\geqslant cE_n,\ c>0\}$.

For $n>1$ the condition (1.1.6) is a consequence of holomorphism and (1.1.5) (the Koecher effect).

Modular forms occur in the following simple manner in arithmetic: let A be a symmetric positive definite integral matrix with even diagonal entries and $\det A=1$. The order m of such a matrix is necessarily divisible by 8. Then for every $n=1,\dots,m$, the theta-series of A,

$$(1.1.7)\qquad \Theta_A^{(n)}(Z)=\sum_{X\in M_{m,n}(\mathbf{Z})}\exp\left(\pi i\sigma\left({}^tXAXZ\right)\right)=\sum_B r_A(B)\exp\left(\pi i\sigma(BZ)\right),$$

where $z\in H_n$, B ranges over all the integral $n\times n$ matrices with even diagonal entries satisfying ${}^tB=B$, $B\geqslant 0$, and $r_A(B)$ denotes the number of integral representation of the quadratic form with the matrix $\frac{1}{2}B$ by the quadratic form with the matrix $\frac{1}{2}A$, is a modular form of genus n and weight $m/2$ [15]. Similarly, if we do not impose the condition $\det A=1$, we obtain modular forms with respect to subgroups of finite index of the Siegel modular group. This is the basis of the application of modular forms to the arithmetic of quadratic forms. For details see [14]–[17].

All modular forms of genus n and weight k form a vector space over $\mathbf{C}$, which we denote by $\mathfrak{M}_k^n$. It is one of the basic facts of the theory that for any n, $k = 1, 2, \ldots$, the space $\mathfrak{M}_k^n$ is finite-dimensional.

Every modular form $F \in \mathfrak{M}_k^n$ has a Fourier expansion

$$F(Z) = \sum_{N \in \mathfrak{N}_n,\ N \geqslant 0} a(N) \exp(2\pi i \sigma(NZ)), \tag{1.1.8}$$

where

$$\mathfrak{N}_n = \{N = (n_{ij}) \in M_n(\mathbf{Q});\ {}^tN = N,\ n_{ii},\ 2n_{ij} \in \mathbf{Z}\} \tag{1.1.9}$$

is the set of symmetric semi-integral $n \times n$ matrices.

We have seen above that important number-theoretical functions can turn up as the Fourier coefficients of modular forms. This, in part, explains why they are of constant interest to us. For the time being, we note just two general properties of the Fourier coefficients.

From (1.1.5) we obtain, for matrices of the form $\begin{pmatrix} {}^tV & 0 \\ 0 & V^{-1} \end{pmatrix}$, where $V \in \mathrm{GL}_n(\mathbf{Z})$, the relation $(\det V)^k F({}^tVZV) = F(Z)$, and hence for the Fourier coefficients the relation

$$a(VN{}^tV) = (\det V)^k a(N) \quad (N \in \mathfrak{N}_n,\ V \in \mathrm{GL}_n(\mathbf{Z})). \tag{1.1.10}$$

Let $N \in \mathfrak{N}_n$, $N \geqslant 0$ and let r be the rank of N. Then there exist matrices $V \in \mathrm{SL}_n(\mathbf{Z})$ and $N' \in \mathfrak{N}_r$, $N' > 0$, such that ${}^tVNV = \begin{pmatrix} N' & 0 \\ 0 & 0 \end{pmatrix}$.

Now $\det N'$ only depends on N; we denote it $\delta(N)$. Then for any modular form $F \in \mathfrak{M}_k^n$ we have the following bound on its Fourier coefficients [14]:

$$|a(N)| = O(\delta(N)^k), \tag{1.1.11}$$

where the O depends only on F.

The spaces $\mathfrak{M}_k^n$ for different n are connected by the Siegel operator Φ. If $Z' \in H_{n-1}$ and $\lambda > 0$, then $\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix} \in H_n$. It follows from (1.1.8) that for any $F \in \mathfrak{M}_k^n$ the following limit exists:

$$(\Phi F)(Z') = \lim_{\lambda \to \infty} F\left(\begin{pmatrix} Z' & 0 \\ 0 & i\lambda \end{pmatrix}\right). \tag{1.1.12}$$

It is easily checked that $\Phi F \in \mathfrak{M}_k^{n-1}$. The linear operator Φ: $\mathfrak{M}_k^n \to \mathfrak{M}_k^{n-1}$ $(n > 1)$ so constructed is called the Siegel operator, and in a number of cases it allows us to reduce certain problems about forms of genus n to analogous problems about forms of smaller genus (see § 1.3).

Let us consider the kernel of the operator Φ:

$$\mathfrak{N}_k^n = \{F \in \mathfrak{M}_k^n;\ \Phi F = 0\}. \tag{1.1.13}$$

Forms in $\mathfrak{N}_k^n$ are called parabolic of genus n and weight k, and can be characterized by the condition that in their Fourier expansion (1.1.8) all the coefficients $a(N)$ with $\det N = 0$ vanish. For $n = 1$ parabolic forms are determined by the condition that $F(i\lambda) \to 0$ as $\lambda \to +\infty$, which is equivalent to $a(0) = 0$. For any parabolic form $F \in \mathfrak{N}_k^n$ we have the following bound [14]:

$$(1.1.14) \qquad |F(Z)| = O((\det Y)^{-k/2}) \qquad (Z = X + iY \in H_n),$$

and hence for the Fourier coefficients of such a form the bound

$$(1.1.15) \qquad |a(N)| = O((\det N)^{k/2}),$$

where in both cases the O depends only on F.

We conclude this section with the construction of an invariant complement to the subspace of parabolic forms. It is easy to see that for every pair F, $F_1 \in \mathfrak{M}_k^n$ of modular forms, the measure on H_n

$$F(Z)\overline{F_1(Z)}(\det Y)^k d\tilde{Z}, \quad (Z = X + iY),$$

where $d\tilde{Z}$ is the invariant element of volume of (1.1.4), is Γ_n-invariant, so that the integral

$$(1.1.16) \qquad (F, F_1) = \int_{D_n} F(Z)\overline{F_1(Z)}(\det Y)^k d\tilde{Z},$$

where D_n is some fundamental domain for Γ_n, provided that it converges absolutely, does not depend on the choice of fundamental domain and defines a non-degenerate Hermitian pairing. The integral (1.1.16) converges absolutely provided that at least one of the forms F or F_1 is parabolic. In this case (F, F_1) is called the scalar product of F and F_1.

We denote by $\mathfrak{E}_k^n$ the orthogonal complement in $\mathfrak{M}_k^n$ to the subspace $\mathfrak{N}_k^n$ of parabolic forms. Then we have the direct decomposition

$$(1.1.17) \qquad \mathfrak{M}_k^n = \mathfrak{E}_k^n \oplus \mathfrak{N}_k^n.$$

Under Φ the space $\mathfrak{E}_k^n$ is embedded into $\mathfrak{M}_k^{n-1}$. The forms in $\mathfrak{E}_k^n$ are called Eisenstein series of genus n and weight k. In many respects the study of Eisenstein series often reduces to that of forms of smaller genus.

§1.2. Hecke's theory

In this section we give an account of the foundations of Hecke's theory [1] of Euler products corresponding to modular forms of genus 1. A more detailed treatment of all the relevant matters can be found in Ogg's lectures [17].

The spaces $\mathfrak{M}_k^1$ of modular forms of genus 1 and weight k is different from $\{0\}$ only for $k = 4,6,8,\ldots$. For all such k the subspace $\mathfrak{E}_k^1$ of Eisenstein series is one-dimensional and is spanned by the series

$$E_k(z) = \sum_{(n,m) \in \mathbf{Z}\times\mathbf{Z}-(0,0)} (nz+m)^{-k} \qquad (z = x + iy,\ y > 0).$$

The series $E_k(z)$ has the Fourier expansion

$$E_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \Big(\sum_{d \mid n} d^{k-1}\Big) \exp(2\pi i n z). \tag{1.2.1}$$

The example of the Eisenstein series (1.2.1), as well as the numerous examples of theta-series of positive definite integral quadratic forms in an even number of variables, which were analysed by the number theorists of the classical period, demonstrate an interesting phenomenon: it often happens that the Fourier coefficients $a(n)$ of modular forms of genus 1 are multiplicative number-theoretical functions or linear combinations of such functions. The reason for this is to be found in Hecke's theory.

One could argue as follows: let

$$f(z) = \sum_{n=0}^{\infty} a(n) \exp(2\pi i n z) \in \mathfrak{M}_k^1.$$

In the widest sense, the "multiplicativity" of the function $a(n)$ should mean that there is a regular connection between $a(n)$ and $a(nm)$ for any fixed m. The numbers $a(nm)$ $(n = 1, 2, \dots)$ are the Fourier coefficients of the function

$$f_m(z) = \frac{1}{m} \sum_{b=0}^{m-1} f\Big(\frac{z+b}{m}\Big) = m^{k-1} \sum_{b=0}^{m-1} f|_k \begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix}, \tag{1.2.2}$$

where for any function f defined on the upper half-plane H_1 and matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \{M \in \mathrm{GL}_2(\mathbf{R});\ \det M > 0\}$ we set

$$f|_k M = (cz+d)^{-k} f\Big(\frac{az+b}{cz+d}\Big). \tag{1.2.3}$$

If the operator $f \to f_m$ were to carry the space $\mathfrak{M}_k^1$ into itself, then one could hope to find its eigenfunctions in $\mathfrak{M}_k^1$; but for such functions $f_m = \lambda_m f$, that is, $a(nm) = \lambda_m a(n)$ $(n = 1, 2, \dots)$, and we would obtain the required multiplicativity. Now f_m belongs to $\mathfrak{M}_k^1$ if and only if $f_m|_k M = f_m$ for all $M \in \Gamma_1 = \mathrm{Sp}_1(\mathbf{Z})$. Since

$$f|_k M_1|_k M_2 = f|_k M_1 M_2, \tag{1.2.4}$$

and $f|_k M = f$ for $M \in \Gamma_1$, $f \in \mathfrak{M}_k^1$, the above condition holds, in particular, if for each matrix $M \in \Gamma_1$ the set of matrices $\left\{\begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix} M;\ b = 0, 1, \dots, m-1\right\}$ coincides (up to order) with a set of the form $\left\{M_b \begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix};\ b = 0, 1, \dots, m-1, \text{ with } M_b \in \Gamma_1\right\}$. Taking, for example, $M = J_1$, it is easy to check that this is false for any $m > 1$. To get out of this difficulty, it is natural to try to widen somewhat the class of matrices $\begin{pmatrix}1 & b\\ 0 & m\end{pmatrix}$ $(b=0, \ldots, m-1)$. All the matrices $\begin{pmatrix}1 & b\\ 0 & m\end{pmatrix}$ belong to the set S_m of all 2×2 integer matrices of determinant m. It is easy to see that for distinct $b = 0, \ldots, m-1$ the matrices $\begin{pmatrix}1 & b\\ 0 & m\end{pmatrix}$ belong to distinct left cosets of Γ_1 in S_m. Hence we can try to include the set $\left\{\begin{pmatrix}1 & b\\ 0 & m\end{pmatrix};\ b=0, \ldots, m-1\right\}$ in a left transversal to Γ_1 in S_m. It is easy to see that we can take as such a transversal the set

$$(1.2.5)\qquad \left\{\begin{pmatrix}a & b\\ 0 & d\end{pmatrix};\ a,\ d>0,\ ad=m,\ b=0,\ 1,\ \ldots,\ d-1\right\}.$$

In particular, there are only finitely many left cosets. Thus, in place of the operators $f \to f_m$ we arrive at the operator

$$(1.2.6)\qquad f \to T_k(m)f = m^{k-1} \sum_{\sigma\in\Gamma_1\setminus S_m} f|_k\sigma \qquad (f \in \mathfrak{M}_k^1).$$

THEOREM 1.2.1. *For every integer* $m \geqslant 1$ *and* $k > 0$ *the operator* $T_k(m)$ *does not depend on the left transversal to* Γ_1 *in* S_m *and maps* $\mathfrak{M}_k^1$ *into itself. The subspace* $\mathfrak{N}_k^1 \subset \mathfrak{M}_k^1$ *of parabolic forms is invariant under all the* $T_k(m)$.

PROOF. If $\{M_\sigma\sigma;\ M_\sigma \in \Gamma_1\}$ is another transversal, then bearing in mind (1.2.4) and the definition of modular forms, we obtain

$$\sum_\sigma f|_k M_\sigma\sigma = \sum_\sigma f|_k M_\sigma|_k\sigma = \sum_\sigma f|_k\sigma.$$

Using the transversal (1.2.5), we see that $T_k(m)f$ together with f also satisfies the analytic conditions in the definition of a modular (respectively, parabolic) forms. Finally, if $\{\sigma\}$ is a left transversal to Γ_1 in S_m, then for any $M \in \Gamma_1$ the set $\{\sigma M\}$ is another transversal. Hence, for any $M \in \Gamma_1$ we have $(T_k(m)f)|_kM = T_k(m)f$, and the theorem is proved.

The operators $T_k(m)$ were defined by Hecke and are named after him. The further development of the theory is merely a technical matter.

Multiplying together the transversals (1.2.5) for different m, it is not difficult to establish the multiplication table for the operators $T_k(m)$ on $\mathfrak{M}_k^1$:

$$(1.2.7)\qquad T_k(m)T_k(m_1) = \sum_{d\mid m,\, m_1} T_k\left(\frac{mm_1}{d^2}\right)d^{k-1} \qquad (m,\ m_1=1,\ 2,\ \ldots).$$

In particular, all the $T_k(m)$ commute.

By direct computation using the transversal (1.2.5) we find the Fourier expansion of the modular form $T_k(m)f$ (for $f \in \mathfrak{M}_k^1$):

$$(1.2.8)\qquad T_k(m)f = a(0)\sum_{d\mid m} d^{k-1} + \sum_{n=1}^{\infty}\left(\sum_{d\mid m,\, n} a\left(\frac{mn}{d^2}\right)d^{k-1}\right)\exp(2\pi inz),$$

where $a(n)$ $(n = 0, 1, \dots)$ are the Fourier coefficients of f.

Suppose that f is an eigenfunction for all the Hecke operators $T_k(m)$ $(m = 1, 2, \dots)$: $T_k(m)f = \lambda_f(m)f$. Equating the corresponding Fourier coefficients on the left- and right-hand sides of this relation, and using (1.2.8), we obtain

$$a(0) \sum_{d \mid m} d^{k-1} = \lambda_f(m)\, a(0) \qquad (m = 1, 2, \dots), \tag{1.2.9}$$

and

$$\sum_{d \mid m,\, n} a\left(\frac{mn}{d^2}\right) d^{k-1} = \lambda_f(m)\, a(n) \qquad (m, n = 1, 2, \dots). \tag{1.2.10}$$

Setting $n = 1$ in (1.2.10), we obtain

$$a(m) = \lambda_f(m) a(1) \qquad (m = 1, 2, \dots). \tag{1.2.11}$$

Thus, the Fourier coefficients of an eigenfunction are multiples of a multiplicative function $\lambda_f(m)$ with the multiplication table

$$\lambda_f(m)\, \lambda_f(m_1) = \sum_{d \mid m,\, m_1} \lambda_f\left(\frac{mm_1}{d^2}\right) d^{k-1} \qquad (m, m_1 = 1, 2, \dots) \tag{1.2.12}$$

(see (1.2.7)).

This has a particularly pretty formulation in the language of Dirichlet series (see [1]).

THEOREM 1.2.2. (I). *Let* $f \in \mathfrak{M}_k^1$ *be an eigenfunction of all the Hecke operators* $T_k(m)$:

$$T_k(m)f = \lambda_f(m)f \qquad (m = 1, 2, \dots).$$

Then the Dirichlet series

$$D_f(s) = \sum_{m=1}^{\infty} \frac{\lambda_f(m)}{m^s} \tag{1.2.13}$$

is absolutely convergent in the domain Re $s > k + 1$ (*respectively, in* Re $s > \frac{k}{2} + 1$, *if f is a parabolic form*), *and has in this domain an Euler product expansion of the form*

$$D_f(s) = \prod_p (1 - \lambda_f(p)\, p^{-s} + p^{k-1-2s})^{-1}, \tag{1.2.14}$$

where p ranges over all the prime numbers.

(II). *With any form*

$$f = \sum_{n=0}^{\infty} a(n) \exp(2\pi i n z) \in \mathfrak{M}_k^1$$

we associate the Dirichlet series

$$R_f(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{1.2.15}$$

Then the series $R_f(s)$ *is absolutely convergent in the domain* $\mathrm{Re}\, s > k + 1$ (*respectively*, $\mathrm{Re}\, s > k/2 + 1$ *if* $f \in \mathfrak{N}_k^1$), *and if f is an eigenfunction of all the Hecke operators* $T_k(m)$, *then we have the identity*

$$R_f(s) = a(1) D_f(s). \tag{1.2.16}$$

PROOF. The identity (1.2.16) follows from (1.2.11). The convergence in the relevant domains follows from the bounds (1.1.11) and (1.1.15). The Euler product expansion (1.2.14) follows from the multiplication table (1.2.12): since $\lambda_f(m)\lambda_f(m_1) = \lambda_f(mm_1)$ if $(m,m_1) = 1$, we have

$$D_f(s) = \prod_p \Big(\sum_{\delta=0}^{\infty} \lambda_f(p^\delta)\, p^{-\delta s} \Big);$$

next, each of the power series $\sum \lambda_f(p^\delta) t^\delta$ can be summed on the basis of the recurrence relation $\lambda_f(p)\lambda_f(p^\delta) = \lambda_f(p^{\delta+1}) + p^{k-1}\lambda_f(p^{\delta-1})$, $\delta \geqslant 1$. This proves the theorem.

In connection with these results there arises the question how many eigenfunctions of all the Hecke operators $T_k(m)$ there are in $\mathfrak{M}_k^1$. This question was answered by Petersson soon after Hecke's paper. Petersson defined the inner product (1.1.16) (for $n = 1$ and for congruence subgroups of Γ_1) and showed that for any two modular forms $f, f_1 \in \mathfrak{M}_k^1$, of which at least one is parabolic

$$(T_k(m)f, f_1) = (f, T_k(m)f_1) \qquad (m = 1, 2, \dots). \tag{1.2.17}$$

From this we obtain the following theorem.

THEOREM 1.2.3. $\mathfrak{M}_k^1$ *has a basis consisting of eigenfunctions for all the Hecke operators* $T_k(m)$. *All the eigenvalues of all the* $T_k(m)$ *are real.*

PROOF. It follows from (1.2.17) and the fact that $\mathfrak{E}_k^1$ is one-dimensional that the Eisenstein series (1.2.1) is an eigenfunction of all the $T_k(m)$. It therefore suffices to prove the theorem for the subspace $\mathfrak{N}_k^1$ of parabolic forms. The relations (1.2.17) show that each of the operators $T_k(m)$ on $\mathfrak{N}_k^1$ is Hermitian. Since they all commute (see (1.2.7)), the theorem follows from a standard result in linear algebra.

It follows from Theorems 1.2.2 and 1.2.3 that the space of Dirichlet series $R_f(s)$ associated as in (1.2.15) with modular forms $f \in \mathfrak{M}_k^1$ has a basis consisting of Dirichlet series having expansions as Euler products of the form (1.2.14).

Finally, the space of Dirichlet series $R_f(s)$ (with $f \in \mathfrak{M}_k^1$) can be characterized by simple analytic properties.

THEOREM 1.2.4. (I) *Let* $f \in \mathfrak{M}_k^1$. *Then the Dirichlet series* $R_f(s)$ (*see* (1.2.15)) *has the following properties*:

(1.2.18) $R_f(s)$ *can be continued as a meromorphic function to the whole s-plane.*

(1.2.19) *The function*

$$\psi_f(s) + \frac{a(0)}{s} + \frac{(-1)^{\frac{k}{2}} a(0)}{k-s},$$

where $\psi_f(s) = (2\pi)^{-s}\,\Gamma(s)R_f(s)$, $\Gamma(s)$ *is the gamma-function, and* $a(0)$ *the zeroth coefficient of* f, *is entire.*

(1.2.20) *The following functional equation holds:*

$$\psi_f(k-s) = (-1)^{\frac{k}{2}}\,\psi_f(s).$$

(II). *Conversely, every Dirichlet series with coefficients of no more than polynomial growth, satisfying the conditions* (1.2.18), (1.2.19) *and* (1.2.20) *is of the form* $R_f(s)$ *for some* $f \in \mathfrak{M}_k^1$.

PROOF. Using the Mellin transform we obtain the integral representation

$$\text{(1.2.21)} \quad \psi_f(s) = \int_0^\infty (f(it) - a(0))\, t^{s-1}\, dt \qquad (\operatorname{Re} s > k+1).$$

Since $f(-z^{-1}) = z^k f(z)$ (for $z \in H_1$), we obtain

$$\psi_f(s) = \int_1^\infty (f(it) - a(0))\, t^{s-1}\, dt - \frac{a(0)}{s} + \int_1^\infty f\left(-\frac{1}{it}\right) t^{-1-s}\, dt =$$

$$= \int_1^\infty (f(it) - a(0))\,(t^{s-1} + i^k t^{k-s-1})\, dt - \frac{a(0)}{s} - \frac{i^k a(0)}{k-s}.$$

The function $f(it) - a(0)$ tends to zero exponentially as $t \to +\infty$. Thus, the last integral above is absolutely convergent for all s and is a holomorphic function of s. This proves (1.2.18) and (1.2.19). The last expression for $\psi_f(s)$ is multiplied by i^k under the substitution $s \to k - s$, and this proves the functional equation.

The assertion (II) is easily obtained if we use the inverse Mellin transform and the fact that the modular group $\Gamma_1 = SL_2(\mathbf{Z})$ is generated by the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. This proves the theorem.

Here we have the classical theory for the full modular group $\Gamma_1 = Sp_1(\mathbf{Z})$. The basic features of this theory carry over, with natural technical complications, to modular forms of integral weight for congruence subgroups of Γ_1 (see [1], [17]). We shall be concerned with the case $n > 1$ later.

§ 1.3. Euler products corresponding to forms of genus n.

For the definition of the Hecke operators, only the following three properties of the set S_m were essential: the transformations in S_m take the upper half-plane H_1 into itself; $\Gamma_1 S_m \Gamma_1 = S_m$; and S_m consists of finitely many left cosets of Γ_1. This remark makes it straightforward to define analogous operators for any genus n.

THEOREM 1.3.1. *For $n \geqslant 1$ we write*

$$S^{(n)} = \{g \in M_{2n}(\mathbf{Z});\ {}^t g J_n g = r(g) J_n,\ r(g) = 1, 2, \ldots\} \tag{1.3.1}$$

(see notation, p. 48). *Then every double coset $\Gamma_n g \Gamma_n$, with $g \in S^{(n)}$, splits into finitely many left cosets of Γ_n:*

$$\Gamma_n g \Gamma_n = \bigcup_{i=1}^{\mu} \Gamma_n \sigma_i. \tag{1.3.2}$$

For each such double coset and each modular form $F \in \mathfrak{M}_k^n$ we set

$$T_k(\Gamma_n g \Gamma_n) F = r(g)^{nk - \frac{n(n+1)}{2}} \sum_{i=1}^{\mu} f|_k \sigma_i, \tag{1.3.3}$$

where for $\sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in S^{(n)}$

$$F|_k \sigma = \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1}). \tag{1.3.4}$$

Then the operator $T_k(\Gamma_n g \Gamma_n)$ does not depend on the choice of the transversal and maps the space $\mathfrak{M}_k^n$ into itself; the subspace $\mathfrak{N}_k^n$ of parabolic forms is invariant under all the T_k.

PROOF. It is easy to see that the number of left cosets of Γ_n in $\Gamma_n g \Gamma_n$ is equal to the index of $g^{-1} \Gamma_n g \cap \Gamma_n$ in Γ_n. This index is finite, since the subgroup contains a subgroup of Γ_n of finite index, namely the subgroup of matrices congruent to the identity matrix modulo $r(g)$. It is easy to see that for $g, g_1 \in S^{(n)}$

$$F|_k g|_k g_1 = F|_k g g_1. \tag{1.3.5}$$

By the definition of modular forms, $F|_k M = F$ for $M \in \Gamma_n$. Thus, if $\{M_i \sigma_i\}$, with $M_i \in \Gamma_n$, is another left transversal, then for each i we have $F|_k M_i \sigma_i = F|_k \sigma_i$, and the operator $T_k(\Gamma_n g \Gamma_n)$ is indeed independent of the choice of transversal.

If $M \in \Gamma_n$, then the set $\{\sigma_i M\}$ is another left transversal so that $(T_k(\Gamma_n g \Gamma_n) F)|_k M = T_k(\Gamma_n g \Gamma_n) F$. It is not hard to check that for $g \in S^{(n)}$ each left coset $\Gamma_n g$ has a "triangular" representative, that is, one of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$. Taking a transversal consisting of upper-triangular matrices, we verify that the function $T_k(\Gamma_n g \Gamma_n) F$ also satisfies, together with F, the analytic conditions in the definition of modular forms (respectively, parabolic forms). This proves the theorem.

The operators $T_k(\Gamma_n g \Gamma_n)$ (for $g \in S^{(n)}$) generate a ring of operators on each space $\mathfrak{M}_k^n$ (or $\mathfrak{N}_k^n$). To study general properties of all these rings for a given n one introduces an abstract ring such that all the relevant rings of operators are representations of it. The definition below is due to

Shimura [8].

Let $L = L^{(n)}$ be the free **C**-module generated by finite linear combinations of the double cosets $(\Gamma_n g \Gamma_n)$ for all $g \in S^{(n)}$. We define a multiplication law in L starting from the requirement that the product of two double cosets should correspond to the product of the corresponding operators. Let $g_1, g_2 \in S^{(n)}$ and let $\Gamma_n g_1 \Gamma_n = \cup \Gamma_n g_{1i}$, $\Gamma_n g_2 \Gamma_n = \cup \Gamma_n g_{2j}$ be decompositions into disjoint left cosets. It is easy to check that for every $g \in S^{(n)}$ the number of pairs (i, j) of indices such that $\Gamma_n g_{1i} g_{2j} = \Gamma_n g$ depends only on the double cosets $\Gamma_n g_1 \Gamma_n$, $\Gamma_n g_2 \Gamma_n$ and $\Gamma_n g \Gamma_n$. We denote this number by $\mu \Gamma_n g_1 \Gamma_n . \Gamma_n g_2 \Gamma_n ; \Gamma_n g \Gamma_n)$ and define the product of the double cosets $\Gamma_n g_1 \Gamma_n$ and $\Gamma_n g_2 \Gamma_n$ by the formula

$$(\Gamma_n g_1 \Gamma_n)(\Gamma_n g_2 \Gamma_n) = \sum \mu(\Gamma_n g_1 \Gamma_n \cdot \Gamma_n g_2 \Gamma_n;\ \Gamma_n g \Gamma_n)(\Gamma_n g \Gamma_n),$$

where the summation extends over all the double cosets $\Gamma_n g \Gamma_n$ contained in $\Gamma_n g_1 \Gamma_n g_2 \Gamma_n$ (there are only finitely many of these). Extending this multiplication law linearly to the whole of $L^{(n)}$ we obtain an associative ring, which is called the (abstract) Hecke ring of Γ_n. As Shimura has shown, $L^{(n)}$ is a commutative integral domain. The map $(\Gamma_n g \Gamma_n) \to T_k(\Gamma_n g \Gamma_n)$ obviously defines a representation of the Hecke ring $L^{(n)}$ on the space $\mathfrak{M}_k^n$ of modular forms.

Just as in the case $n = 1$, for the discussion of Euler products we need a certain amount of information about the multiplication rules for the operators T_k. We obtain this by considering the multiplication of the corresponding elements of $L^{(n)}$. As Shimura [18] has shown, each double coset $\Gamma_n g \Gamma_n$ (with $g \in S^{(n)}$)) contains a unique diagonal representative having diagonal entries $d_1, \ldots, d_n, e_1, \ldots, e_n$ such that

$$(1.3.6) \qquad d_i | d_{i+1}, \quad d_n | e_n, \quad e_{i+1} | e_i, \quad d_i e_i = r(g).$$

We denote by $T(d_1, \ldots, d_n; e_1, \ldots, e_n)$ the corresponding double coset, regarded as an element of $L^{(n)}$. One sees easily that

$$(1.3.7) \quad T(d_1, \ldots, d_n;\ e_1, \ldots, e_n) T(d_1', \ldots, d_n';\ e_1', \ldots, e_n') = T(d_1 d_1', \ldots, e_n e_n'),$$

provided that $(e_1, e_1') = 1$. In particular, if for every $m = 1, 2, \ldots$ we let $T(m)$ denote the sum of all the double cosets $\Gamma_n g \Gamma_n$ with $r(g) = m$ (by (1.3.6) these are finite in number):

$$(1.3.8) \quad T(m) = \sum_{r(g)=m} (\Gamma_n g \Gamma_n) = \sum_{d_1 e_1 = m} T(d_1, \ldots, d_n;\ e_1, \ldots, e_n),$$

then

$$(1.3.9) \qquad T(m) T(m_1) = T(m m_1) \text{ for } (m, m_1) = 1.$$

We consider the formal Dirichlet series

$$D(s) = \sum_{m=1}^{\infty} T(m)\, m^{-s}. \tag{1.3.10}$$

By (1.3.9), $D(s)$ has a (formal) expansion as an Euler product of the form

$$D(s) = \prod_p D_p(s), \quad \text{where} \quad D_p(s) = \sum_{\delta=0}^{\infty} T(p^\delta)\, p^{-\delta s} \tag{1.3.11}$$

(p ranges over all prime numbers).

To compute the p-component $D_p(s)$ we need some information on the product of the double cosets $\Gamma_n g \Gamma_n$ for which $r(g)$ is a power of a fixed prime p. All such double cosets generate a subring $L_p^{(n)}$ of $L^{(n)}$. Shimura [8] has shown that $L_p^{(n)}$ is a polynomial ring over $\mathbf{C}$ in $n + 1$ independent variables, which we can take to be the double cosets

$$T(\underbrace{1, \ldots, 1}_{n}, \underbrace{p, \ldots, p}_{n}); \; T(\underbrace{1, \ldots, 1}_{n-i}, \underbrace{p, \ldots, p}_{i}, \underbrace{p^2, \ldots, p^2}_{n-i}, \underbrace{p, \ldots, p}_{i})$$
$$(1 \leqslant i \leqslant n).$$

Thus, $D_p(s)$ can be regarded as a power series in $t = p^{-s}$ with polynomial coefficients, and the question arises, say, whether or not it is formally equal to some rational function in t with polynomial coefficients. For small n one can carry out direct computations. For instance, it follows from Hecke's computations [1] that for $n = 1$

$$D_p(s) = (1 - T(1, p)p^{-s} + pT(p, p)p^{-2s})^{-1}.$$

For $n = 2$ and 3 the series $D_p(s)$ were computed by Shimura [8] and Andrianov [19], respectively. For arbitrary $n \geqslant 1$, Shimura [8] has conjectured that for any prime p the series $D_p(s)$ is formally equal to a rational function in p^{-s} with coefficients in $L_p^{(n)}$, whose numerator and denominator are of degrees 2^n-2 and 2^n, respectively. This conjecture was proved by Andrianov [10], [11].[1)] We give a detailed description of the corresponding result in the language of one-dimensional representations of Hecke rings.

The set of all $\mathbf{C}$-linear homomorphisms of $L_p^{(n)}$ into $\mathbf{C}$ that take the unit element into 1 has a simple parametrization. Each double coset $\Gamma_n g \Gamma_n$, with $g \in S^{(n)}$ and $r(g) = p^\delta$, has a left transversal Γ_n consisting of matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \text{ with } D = \begin{pmatrix} p^{d_1} & * & * & \ldots & * \\ 0 & p^{d_2} & * & \ldots & * \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & p^{d_n} \end{pmatrix}. \tag{1.3.12}$$

Let $A = (\alpha_0, \alpha_1, \ldots, \alpha_n)$ be an arbitrary $(n + 1)$-tuple of non-zero complex numbers. With $(\Gamma_n g \Gamma_n)$ and the $(n + 1)$-tuple of parameters A we associate the number

1) As I. G. Macdonald has pointed out to the author, Shimura had an unpublished proof of his conjecture at about the same time, but by a different method. He has subsequently obtained considerably more general results.

$$(1.3.13) \qquad (\Gamma_n g \Gamma_n)_A = \alpha_0^\delta \sum \prod_{i=1}^{n} (\alpha_i p^{-i})^{d_i},$$

where the summation extends over a left transversal of $\Gamma_n g \Gamma_n$ in Γ_n, written in the form (1.3.12). It follows easily from the definition of multiplication in $L_p^{(n)}$ that the map $(\Gamma_n g \Gamma_n) \to (\Gamma_n g \Gamma_n)_A$ defines a **C**-linear homomorphism of $L_p^{(n)}$ into **C** taking the unit element into 1. Conversely, every such homomorphism φ: $L_p^{(n)} \to \mathbf{C}$ has the form $\varphi(\Gamma_n g \Gamma_n) = (\Gamma_n g \Gamma_n)_A$ for some $(n + 1)$-tuple $A = (\alpha_0, \ldots, \alpha_n) \in \mathbf{C}^{n+1}$, $\alpha_0 \ldots \alpha_n \neq 0$. The homomorphisms corresponding to two $(n + 1)$-tuples $(\alpha_0, \ldots, \alpha_n)$ and $(\alpha_0', \ldots, \alpha_n')$ coincide if and only if $\alpha_0 = \alpha_0'$, and $\alpha_i' = \alpha_{\sigma(i)}$ $(i = 1, \ldots, n)$, for some permutation σ. In these terms we have the following result on the series $D_p(s)$ (see [11]).

For any $(n + 1)$-tuple of non-zero complex numbers $A = (\alpha_0, \ldots, \alpha_n)$ there is a formal identity

$$\sum_{\delta=0}^{\infty} T_A(p^\delta)\, t^\delta = P_{p,A}(t)\, Q_{p,A}(t)^{-1},$$

where

$$T_A(p^\delta) = \sum_{r(g)=p^\delta} (\Gamma_n g \Gamma_n)_A, \quad P_{p,A}(t) = \sum_{i=0}^{2^n-2} \varphi_i(\alpha_1, \ldots, \alpha_n)\, \alpha_0^i t^i,$$

all the φ_i are symmetric polynomials, $\varphi_0 \equiv 1$, $\varphi_{2^n-2}(\alpha_1, \ldots, \alpha_n) =$
$= p^{-\frac{(n-1)n}{2}} (\alpha_1 \ldots \alpha_n)^{2^{n-1}-1}$, and

$$Q_{p,A}(t) = (1 - \alpha_0 t) \prod_{r=1}^{n} \prod_{1 \leq i_1 < \ldots < i_r \leq n} (1 - \alpha_0 \alpha_{i_1} \ldots \alpha_{i_r} t);$$

The power series on the left-hand side of the identity is convergent in some neighbourhood of zero, and is equal to the right-hand side in this neighbourhood.

We now turn our attention to the representations of Hecke rings on spaces of modular forms. Let $F \in \mathfrak{M}_k^n$ be a modular form of genus n and weight k. Suppose that F is an eigenfunction of all the operators $T_k(\Gamma_n g \Gamma_n)$, for $g \in S^{(n)}$:

$$T_k(\Gamma_n g \Gamma_n) F = \lambda_F(\Gamma_n g \Gamma_n) F.$$

It is not hard to show that the eigenvalues $\lambda_F(\Gamma_n g \Gamma_n)$ are of no more than polynomial growth:

$$(1.3.14) \qquad |\lambda_F(\Gamma_n g \Gamma_n)| = O(r(g)^c),$$

where the constant c depends on n and k only (the idea of the proof: it is easy to give an estimate of the form $O(r(g)^{c_1})$ for the number of left coset in the double cosets $\Gamma_n g \Gamma_n$ (with $g \in S^{(n)}$), where c_1 depends on n only; now we take a left transversal consisting of matrices of the form (1.3.12) and find the Fourier coefficients of the form $T_k(\Gamma_n g \Gamma_n)F$; on the other hand, these coefficients are equal to the coefficients of F, multiplied by $\lambda_F(\Gamma_n g\Gamma_n)$; by equating the corresponding expressions and using the estimate (1.1.11), we obtain the required result). In particular, for the eigenvalues of the operators $T_k(m)$ ($m = 1, 2, \ldots$) corresponding to the elements $T(m)$ of the Hecke ring (see (1.3.8)),

$$T_k(m)F = m^{nk-\frac{n(n+1)}{2}} \sum_{\sigma\in\Gamma_n\setminus S_m^{(n)}} F|_k\sigma = \lambda_F(m)F, \tag{1.3.15}$$

where $S_m^{(n)} = \{g \in S^{(n)};\ r(g)=m\}$, we have the estimate

$$|\lambda_F(m)| = O(m^c), \tag{1.3.16}$$

with c depending only on n and k.

We consider the Dirichlet series

$$D_F(s) = \sum_{m=1}^{\infty} \frac{\lambda_F(m)}{m^s}. \tag{1.3.17}$$

From the theory of the formal series $D(s)$ treated above and the estimate (1.3.16) we deduce the following theorem on the series $D_F(s)$, which is analogous to the first part of Theorem 1.2.2.

THEOREM 1.3.2. *Let $F \in \mathfrak{M}_k^n$ be an eigenfunction of all the Hecke operators $T_k(\Gamma_n g\Gamma_n)$ (for $g \in S^{(n)}$). In particular,*

$$T_k(m)F = \lambda_F(m)F \text{ (for } m = 1, 2, \ldots)$$

(see (1.3.15)). *Then the Dirichlet series* (1.3.17) *is absolutely convergent in a right half-plane* $\operatorname{Re} s > \sigma$ *and has there an Euler product expansion*

$$D_F(s) = \prod_p \left(\sum_{\delta=0}^{\infty} \lambda_F(p^\delta)p^{-\delta s}\right) = \prod_p D_{p,F}(s); \tag{1.3.18}$$

each of the p-components $D_{p,F}(s)$ is a rational fraction of p^{-s}:

$$D_{p,F}(s) = P_{p,F}(p^{-s})Q_{p,F}(p^{-s})^{-1}, \tag{1.3.19}$$

where $P_{p,F}(t)$ and $Q_{p,F}(t)$ are polynomials with real coefficients of degree 2^n-2 and 2^n, respectively; the polynomial $Q_{p,F}(t)$ has the form

$$Q_{p,F}(t) = 1 - \lambda_F(p)t + \ldots + p^{2^{n-1}\left(nk-\frac{n(n+1)}{2}\right)}t^{2^n} = \tag{1.3.20}$$

$$= (1-\alpha_0^F(p)t)\prod_{r=1}^{n}\prod_{1\leq i_1<\ldots<i_r\leq n}(1-\alpha_0^F(p)\alpha_{i_1}^F(p)\ldots\alpha_{i_r}^F(p)t),$$

where $A^F(p) = (\alpha_0^F(p), \ldots, \alpha_n^F(p))$ are the parameters that by (1.3.13) *correspond to the one-dimensional representation $T(\Gamma_n g\Gamma_n) \to \lambda_F(\Gamma_n g\Gamma_n)$ of the ring $L_p^{(n)}$.*

PROOF. All the assertions of the theorem, apart from the fact that the coefficients of $P_{p,F}$ and $Q_{p,F}$ are real, follow from what has been said above. It is shown in [11] that the coefficients of $P_{p,F}$ and $Q_{p,F}$ are linear combinations with rational coefficients of the eigenvalues $\lambda_F(\Gamma_n g\Gamma_n)$. We shall see later that all these eigenvalues are real, and the assertion follows.

DEFINITION. Let $F \in \mathfrak{M}_k^n$ be an eigenfunction of all the Hecke operators $T_k(\Gamma_n g \Gamma_n)$ (for $g \in S^{(n)}$). We define *the zeta-function of the form* F to be the Euler product

$$Z_F(s) = \prod_p Q_{p,F}(p^{-s})^{-1}, \tag{1.3.21}$$

where $Q_{p,F}(t)$ are the polynomials defined in Theorem 1.3.2.

It follows from the estimate (1.3.14) that the parameters $\alpha_i^F(p)$ satisfy

$$|\alpha_i^F(p)| = O(p^{c'}),$$

where c' depends only on n and k. Hence the product (1.3.21) is absolutely convergent in a half-plane Re $s > \sigma'$ and is an analytic function of s in this half-plane.

For $n = 1$, $Z_F(s) = D_F(s)$. For $n \geqslant 2$ these functions do not coincide. That $Z_F(s)$, rather than $D_F(s)$, can reasonably be regarded as the "right" zeta-function of F is shown by the analysis of the case $n = 2$ (see §§2.4 and 3.1), as well as by the relation due to Zharkovskaya, which is introduced below. General conjectures about the analytic properties of $Z_F(s)$ are given in conclusion at the end of this article.

Recently Zharkovskaya [20] has proved a theorem that reduces the study of the zeta-functions $Z_F(s)$ of arbitrary modular forms to that of parabolic forms. For $n = 2$ this theorem, as shown in [12], follows from results of Maass [7].

THEOREM 1.3.3. *Let* $F \in \mathfrak{M}_k^n$ $(n > 1,\ k > 0)$ *be an eigenfunction of all operators of the Hecke ring* $L^{(n)}$. *Then the form* $\Phi F \in \mathfrak{M}_k^{n-1}$, *where* Φ *is the Siegel operator* (1.1.12), *is an eigenfunction of all the operators of the Hecke ring* $L^{(n-1)}$, *and if* F *is not parabolic* (*that is*, $\Phi F \neq 0$), *then in the domain of absolute convergence the following relation holds between the zeta-functions of* F *and* ΦF:

$$Z_F(s) = Z_{\Phi F}(s) Z_{\Phi F}(s - k + n). \tag{1.3.22}$$

Finally, we consider the question of the existence of eigenfunctions of all the Hecke operators.

THEOREM 1.3.4. *For all n and $k \geqslant 1$ the space* $\mathfrak{M}_k^n$, *has a basis consisting of eigenfunction of all the Hecke operators* $T_k(\Gamma_n g \Gamma_n)$ (*for* $g \in S^{(n)}$). *All the eigenvalues of all the Hecke operators on* $\mathfrak{M}_k^n$ *are real.*

PROOF. First of all, one checks that for any two modular forms F, $F_1 \in \mathfrak{M}_k^n$, of which at least one is parabolic, and for any $g \in S^{(n)}$

$$(T_k(\Gamma_n g \Gamma_n)F, F_1) = (F, T_k(\Gamma_n g \Gamma_n)F_1), \tag{1.3.23}$$

where $(\,,\,)$ is the scalar product (1.1.16). This relation was proved by Maass [7] for the operators $T_k(m)$; for an arbitrary double coset the proof is analogous. By (1.2.23), the Hecke operators on the space of parabolic forms are all Hermitian. Since they commute, the theorem is true for $\mathfrak{N}_k^n$. By definition (see 1.1.17) and (1.3.23)), the subspace $\mathfrak{E}_k^n \subset \mathfrak{M}_k^n$ of

Eisenstein series is invariant under all the T_k. As was shown by Zharkovskaya [20], there exists an epimorphism $T \to T^*$ of $L^{(n)}$ onto $L^{(n-1)}$ such that for any $F \in \mathfrak{M}_k^n$ we have the relation $\Phi T_k F = T_k{}^* \Phi F$. Hence the subspace $\Phi \mathfrak{E}_k^n \subset \mathfrak{M}_k^{n-1}$ is invariant under all the operators of $L^{(n-1)}$. Using induction on n and Theorem 1.2.3, we arrive at the required result.

With this we come to an end of the general facts known about the Euler products corresponding to Siegel modular forms of arbitrary genus. The principal gap in the theory as compared with Hecke's theory for $n = 1$ is the lack of any analytical connection between the Euler product and the corresponding modular form. In the following chapters we shall see the form this connection takes in the case $n = 2$.

Chapter 2

EIGENVALUES OF THE HECKE OPERATORS AND THE FOURIER COEFFICIENTS OF EIGENFUNCTIONS IN GENUS 2.

§2.1. Action of the Hecke operators on the Fourier coefficients

Let $F \in \mathfrak{M}_k^2$ be a modular form of genus 2 and weight k. In this section we begin the study of the Fourier coefficients of the form $T_k(m)F$, where $T_k(m)$ is the mth Hecke operator (see (1.3.15)). According to (1.3.9), it suffices to restrict ourselves to the case when $m = p^\delta$ is a power of a fixed prime number. With this in mind, we begin by constructing a special left transversal $V(p^\delta)$ to $S_{p^\delta}^{(2)}$ in Γ_2, where

$$S_{p^\delta}^{(2)} = \{g \in M_4(\mathbf{Z});\ {}^t g J_2 g = p^\delta J_2\}. \tag{2.1.1}$$

As was pointed out in §1.3, each left coset of $S_{p^\delta}^{(2)}$ in Γ_2 has a representative of the form $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, with $A, B, D \in M_2(\mathbf{Z})$. A matrix of the shape indicated belong to $S_{p^\delta}^{(2)}$ if and only if

$${}^t A D = p^\delta E_2, \quad {}^t B D = {}^t D B. \tag{2.1.2}$$

It is easily seen that two integral matrices $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ and $\begin{pmatrix} A_1 & B_1 \\ 0 & D_1 \end{pmatrix}$ satisfying (2.1.2) belong to the same left coset of Γ_2 if and only if there are matrices $V \in GL_2(\mathbf{Z})$ and $T \in M_2(\mathbf{Z})$, with ${}^t T V = {}^t V T$, such that

$$D_1 = VD, \quad A_1 = {}^t V^{-1} A, \quad B_1 = {}^t V^{-1} B + TD.$$

For a given D the matrix A is uniquely determined by the first relation in (2.1.2). Clearly, for an integral D the corresponding A is again integral if and only if

$$D \in \mathrm{GL}_2(\mathbf{Z}) \begin{pmatrix} p^\alpha & 0 \\ 0 & p^{\alpha+\beta} \end{pmatrix} \mathrm{GL}_2(\mathbf{Z}), \tag{2.1.3}$$

where $\alpha, \beta \geqslant 0$, $\alpha + \beta \leqslant \delta$. The double coset (2.1.3) splits into finitely many left cosets of $GL_2(\mathbf{Z})$. It is easy to see that a transversal can be taken to be the set

$$\begin{pmatrix} p^{\alpha} & 0 \\ 0 & p^{\alpha+\beta} \end{pmatrix} R(p^{\beta}),$$

where

$$(2.1.4) \qquad R(p^{\beta}) = \left\{ \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \in SL_2(\mathbf{Z});\ (u_1, u_2) \bmod p^{\beta} \right\}$$

is any set of 2×2 integral matrices $\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}$, whose first rows range over a complete system of representatives of the equivalence classes of pairs of coprime integers under the equivalence relation

$$(2.1.5) \qquad (u_1, u_2) \sim (u_1', u_2') (\bmod\ p^{\beta}) \Leftrightarrow au_1 \equiv u_1',\ au_2 \equiv u_2' \pmod{p^{\beta}}$$

(that is, the "projective line mod p^{β}"), and whose second rows are chosen so that $u_1 v_2 - u_2 v_1 = 1$.

Summing up what we have said, we obtain the following proposition.

PROPOSITION 2.1.1. *For every prime p and every integer $\delta \geqslant 0$ we can take as a left transversal to $S^{(2)}_{p^{\delta}}$* in Γ_2 (see (2.1.1)) *the set*

$$(2.1.6)\ V(p^{\delta}) = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \middle| D \in \begin{pmatrix} p^{\alpha} & 0 \\ 0 & p^{\alpha+\beta} \end{pmatrix} R(p^{\beta}),\ \alpha \geqslant 0,\ \beta \geqslant 0, \right.$$
$$\left. \alpha + \beta \leqslant \delta,\ A = p^{\delta}\, {}^tD^{-1}, {}^tBD = {}^tDB,\ B \bmod D \right\},$$

where the sets $R(p^{\beta})$ are defined above (see (2.1.4), *and B* mod *D means that B ranges over a complete system of representatives of the equivalence classes of* 2×2 *integral matrices (satisfying the previous condition) under the equivalence relation*

$$B \equiv B_1 (\bmod\ D) \Leftrightarrow (B - B_1)D^{-1} \in M_2(\mathbf{Z}).$$

Let

$$F(Z) = \sum_{N \in \mathfrak{N}_2,\ N \geqslant 0} a(N) \exp(2\pi i \sigma(NZ)) \in \mathfrak{M}_k^2$$

be the Fourier expansion of some modular form of genus 2 and weight k (see (1.1.8), (1.1.9)). By definition (see (1.3.3) and (1.3.8)) and by Proposition 2.1.1. we have

$$T_k(p^{\delta}) F = p^{(2k-3)\delta} \sum_{g \in V(p^{\delta})} F|_k g =$$
$$= p^{(2k-3)\delta} \sum_{N} \sum_{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in V(p^{\delta})} a(N) (\det D)^{-k} \exp(2\pi i \sigma(N(AZ+B)D^{-1})) =$$
$$= p^{(2k-3)\delta} \sum_{N} \sum_{D} a(N) (\det D)^{-k} \exp(2\pi i \sigma(p^{\delta} D^{-1} N\, {}^tD^{-1} Z))\, l_D(N),$$

where

$$l_D(N) = \sum_{{}^tBD={}^tDB,\ B \bmod D} \exp(2\pi i\sigma(NBD^{-1})).$$

Collecting together the terms with the same power in the exponent, we obtain

$$T_k(p^\delta)F = \sum_{N\in\mathfrak{N}_2,\ N\geqslant 0} a(p^\delta; N)\exp(2\pi i\sigma(NZ)),$$

where

$$(2.1.7)\quad a(p^\delta; N) = \\ = p^{(2k-3)\delta} \sum_{\substack{D\in\begin{pmatrix}p^\alpha & 0\\ 0 & p^{\alpha+\beta}\end{pmatrix}R(p^\beta),\\ \alpha,\ \beta\geqslant 0,\ \alpha+\beta\leqslant\delta}} a(p^{-\delta}DN^tD)(\det D)^{-k}\, l_D(p^{-\delta}DN^tD).$$

It is easy to see that for $U\in \mathrm{SL}_2(\mathbf{Z})$ and half-integral N we have the relation

$$(2.1.8)\qquad l_{DU}(N) = l_D(N),$$

and that for $D=\begin{pmatrix}p^\alpha & 0\\ 0 & p^{\alpha+\beta}\end{pmatrix}$, $N=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}$ we have

$$(2.1.9)\quad l_D(N) = \begin{cases} p^{3\alpha+\beta}, & \text{if } a\equiv b\equiv 0 \pmod{p^\alpha},\ c\equiv 0\pmod{p^{\alpha+\beta}},\\ 0 & \text{otherwise.}\end{cases}$$

We set $\delta-(\alpha+\beta)=\gamma\geqslant 0$. Using (2.1.8) and the fact that $R(p^\beta)\subset \mathrm{SL}_2(\mathbf{Z})$, the relation (2.1.7) can be rewritten in the form

$$(2.1.10)\quad a(p^\delta; N) = \sum_{\substack{\alpha+\beta+\gamma=\delta,\\ \alpha,\ \beta,\ \gamma\geqslant 0}} p^{(k-2)\beta+(2k-3)\gamma}\times \\ \times \sum_{U\in R(p^\beta)} a(p^{-\delta}D_{\alpha\beta}UN^tUD_{\alpha\beta})\, p^{-(3\alpha+\beta)} l_{D_{\alpha\beta}}(p^{-\delta}D_{\alpha\beta}UN^tUD_{\alpha\beta}),$$

where $D_{\alpha\beta}=\begin{pmatrix}p^\alpha & 0\\ 0 & p^{\alpha+\beta}\end{pmatrix}$. We set $N=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}$, and $UN^tU=\begin{pmatrix}a_u & b_u/2\\ b_u/2 & c_u\end{pmatrix}$ (for $U\in \mathrm{SL}_2(\mathbf{Z})$). Then

$$p^{-\delta}D_{\alpha\beta}UN^tUD_{\alpha\beta} = \begin{pmatrix} a_u p^{\alpha-\beta-\gamma} & \frac{1}{2}b_u p^{\alpha-\gamma}\\ \frac{1}{2}b_u p^{\alpha-\gamma} & c_u p^{\alpha+\beta-\gamma}\end{pmatrix},$$

and hence, bearing (2.1.9) in mind, we obtain

$$(2.1.11)\quad a(p^\delta; N) = \\ = \sum_{\substack{\alpha+\beta+\gamma=\delta,\\ \alpha,\ \beta,\ \gamma\geqslant 0}} p^{(k-2)\beta+(2k-3)\gamma} \sum_{\substack{U\in R(p^\beta),\\ a_u\equiv 0(p^{\beta+\gamma}),\\ b_u\equiv c_u\equiv 0(p^\gamma)}} a\left(p^\alpha\begin{pmatrix} a_u p^{-\beta-\gamma} & \frac{1}{2}b_u p^{-\gamma}\\ \frac{1}{2}b_u p^{-\gamma} & c_u p^{\beta-\gamma}\end{pmatrix}\right).$$

We now introduce some notation, to put this formula in a more convenient form.

Denote $\mathfrak{A}$ the set of all complex-valued functions on the set $\{N \in \mathfrak{N}_2, N \geqslant 0\}$ that are constant on the classes of equivalent matrices in the narrow sense:

$$(2.1.12) \qquad \mathfrak{A} = \{\varphi: \{N \in \mathfrak{N}_2,\ N \geqslant 0\} \to \mathbf{C} \mid \varphi(UN{}^tU) = \varphi(N)(U \in \mathrm{SL}_2(\mathbf{Z}))\}.$$

According to (1.1.10), the Fourier coefficients $a(N)$ of any modular form $F \in \mathfrak{M}_k^2$ can be regarded as the values taken by some function in $\mathfrak{A}$.

We define a representation of the Hecke ring $L^{(1)}$ of the group $\Gamma_1 = \mathrm{SL}_2(\mathbf{Z})$ on $\mathfrak{A}$. Let $g \in S^{(1)}$ (see (1.3.1)), and let

$$\Gamma_1 g \Gamma_1 = \bigcup_i \Gamma_1 \sigma_i$$

be a decomposition of the double coset $\Gamma_1 g \Gamma_1$ into a disjoint union of left cosets of Γ_1. For each $\varphi \in \mathfrak{A}$ we set

$$(2.1.13) \qquad (T_a(\Gamma_1 g \Gamma_1)\varphi)(N) = \sum_i \varphi(\sigma_i N {}^t\sigma_i).$$

It is easily seen that all the operators T_a are independent of the system of representatives $\{\sigma_i\}$ and map $\mathfrak{A}$ into itself. We extend this action linearly to the whole of $L^{(1)}$. It follows easily from the definition of multiplication in $L^{(1)}$ and the fact that $L^{(1)}$ is commutative that the map $T \to T_a$ is a representation of $L^{(1)}$ on $\mathfrak{A}$.

We also define operators $\Delta^+(m)$ and $\Delta^-(m)$ on $\mathfrak{A}$ for every natural number m:

$$(2.1.14) \qquad \begin{cases} (\Delta^+(m)\varphi(N) = \varphi(mN), \\ (\Delta^-(m)\varphi)(N) = \begin{cases} \varphi(m^{-1}N) & \text{if } m^{-1}N \in \mathfrak{N}_2 \\ 0 & \text{if } m^{-1}N \notin \mathfrak{N}_2. \end{cases} \end{cases}$$

As an immediate consequence of the definition we have the formulae

$$(2.1.15) \qquad \begin{cases} \Delta^+(m)\Delta^+(m_1) = \Delta^+(mm_1) \quad (m, m_1 = 1, 2, \ldots), \\ \Delta^-(m)\Delta^-(m_1) = \Delta^-(mm_1) \quad (m, m_1 = 1, 2, \ldots), \\ \Delta^+(m) T_a(\Gamma_1 g \Gamma_1)\Delta^-(m) = T_a(\Gamma_1 g \Gamma_1) \ (m = 1, 2, \ldots, \ g \in S^{(1)}). \end{cases}$$

Finally, for each $m = 1, 2, \ldots$ we set

$$(2.1.16) \qquad \Pi(m) = T_a\left(\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \Gamma_1\right) \Delta^-(m).$$

Since $\begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix} R(p^\beta)$, where $R(p^\beta)$ is the set (2.1.4)), is a left transversal to Γ_1 in $\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix} \Gamma_1$, for any prime power $m = p^\beta$ we have

$$(2.1.17) \qquad (\Pi(p^\beta)\varphi)(N) = \sum_{U \in R(p^\beta)} (\Delta^-(p^\beta)\varphi)\left(\begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix} UN{}^tU \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix}\right) =$$

$$= \sum_{\substack{U\in R(p^\beta),\\ a_u\equiv 0 \pmod{p^\beta}}} \varphi\left(\begin{pmatrix} p^{-\beta}a_u & b_u/2 \\ b_u/2 & p^\beta c_u\end{pmatrix}\right),$$

where $\begin{pmatrix} a_u & b_u/2 \\ b_u/2 & c_u\end{pmatrix} = UN{}^tU.$

We now write out (2.1.11) in terms of the notation just introduced.

PROPOSITION 2.1.2. *Let*

$$F(Z) = \sum_{N\in\mathfrak{N}_2,\ N\geqslant 0} a(N)\exp(2\pi i\sigma(NZ)) \in \mathfrak{M}_k^2,$$

and let p be a prime number, and $\delta \geqslant 0$. *Then*

$$(T_k(p^\delta)F)(Z) = \sum_{N\in\mathfrak{N}_2,\ N\geqslant 0} a(p^\delta; N)\exp(2\pi i\sigma(NZ)),$$

where

$$a(p^\delta; N) = \sum_{\substack{\alpha+\beta+\gamma=\delta,\\ \alpha,\beta,\gamma\geqslant 0}} p^{(k-2)\beta+(2k-3)\gamma}(\Delta^-(p^\gamma)\Pi(p^\beta)\Delta^+(p^\alpha)a)(N)$$

(and $a(\ )$ *is regarded as a function in* $\mathfrak{A}$*).*

REMARK. In the corresponding formula in [12] the operators are written down in the reverse order, a common error in such cases. However, the formula is used correctly.

PROOF. Since $U\in SL_2(\mathbf{Z})$, the conditions $a_u\equiv b_u\equiv c_u\equiv 0 \pmod{p^\gamma}$ are equivalent to $a\equiv b\equiv c\equiv 0 \pmod{p^\gamma}$. Then keeping (2.1.14) and (2.1.17) in mind we find that the term in (2.1.11) corresponding to the partition $\delta = \alpha+\beta+\gamma$ is equal to

$$p^{(k-2)\beta+(2k-3)\gamma} \sum_{\substack{U\in R(p^\beta),\\ a_u\equiv 0 \pmod{p^\beta}}} (\Delta^+(p^\alpha)a)\left(p^{-\gamma}\begin{pmatrix} a_u p^{-\beta} & b_u/2 \\ b_u/2 & c_u p^\beta\end{pmatrix}\right) =$$

$$= p^{(k-2)\beta+(2k-3)\gamma}(\Pi(p^\beta)\Delta^+(p^\alpha)a)(p^{-\gamma}N),$$

when $p^{-\gamma}N\in\mathfrak{N}_2$, and is zero otherwise. The proposition is now proved.

§2.2. Summation of the series $\sum_{\delta=0}^{\infty} a(p^\delta N)t^\delta$

In this section we establish that for every prime p and positive definite $N = \begin{pmatrix} a & b/2 \\ b/2 & c\end{pmatrix} \in \mathfrak{N}_2$, with $(a, b, c, p) = 1$, the series of the section heading, where $a(\ldots)$ are the Fourier coefficients of some eigenfunction $F\in\mathfrak{M}_k^2$ of all the Hecke operators $T_k(p^\delta)$) is closely connected with the generating series for the corresponding eigenvalues, and is, in particular, a rational fraction with the same denominator.

First of all we introduce, in a form that will be convenient in what follows, a result of Shimura [8] on the summation of generating series for eigenvalues.

THEOREM 2.2.1. *Let p be a prime number, and let $F \in \mathfrak{M}_k^2$ be an eigenfunction of all the Hecke operators $T_k(p^\delta)$ (with $\delta \geqslant 0$):*

$$T_k(p^\delta)F = \lambda_F(p^\delta)F \quad (\delta = 0, 1, \ldots).$$

Then we have the identity

$$(2.2.1) \qquad \sum_{\delta=0}^{\infty} \lambda_F(p^\delta)\, t^\delta = (1 - p^{2k-4}t^2)\, Q_{p,F}(t)^{-1},$$

where

$$(2.2.2)\quad Q_{p,F}(t) = 1 - \lambda_F(p)\, t + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})\, t^2 - \lambda_F(p)\, p^{2k-3}t^3 + p^{4k-6}t^4.$$

COROLLARY. *If $F \in \mathfrak{M}_k^2$ is an eigenfunction of all the Hecke operators $T_k(\Gamma_2 g \Gamma_2)$ (for $g \in S^{(2)}$), then in the domain of absolute convergence the following relation holds between the Dirichlet series $D_F(s)$ and $Z_F(s)$ (see* (1.3.17) *and* (1.3.21))

$$D_F(s) = \zeta(2s - 2k + 4)^{-1} Z_F(s),$$

where ζ is the Riemann zeta-function.

The main result of this section is the following proposition.

PROPOSITION 2.2.1. *Suppose that the modular form*

$$F(Z) = \sum_{N \in \mathfrak{N}_2,\, N \geqslant 0} a(N) \exp(2\pi i \sigma(NZ)) \in \mathfrak{M}_k^2$$

is an eigenfucntion of all the Hecke operators $T_k(p^\delta)$ for some fixed prime p and $\delta \geqslant 0$:

$$T_k(p^\delta)F = \lambda_F(p^\delta)F \quad (\delta = 0, 1, \ldots).$$

Then for any positive definite matrix $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathfrak{N}_2$ such that $(a, b, c, p) = 1$ we have the identity

$$(2.2.3) \quad \Big\{\sum_{\delta=0}^{\infty} a(p^\delta N)\, t^\delta\Big\}\, Q_{p,F}(t) = a(N) - p^{k-2}(\Pi(p)\, a)(N)\, t + [p^{2k-4}((\Pi(p)^2 - \Pi(p^2) - 1)\, a)(N) + p^{3k-5}(\Pi(p)\, \Delta^-(p)\, a)(N)]\, t^2,$$

where $Q_{p,F}(t)$ is the polynomial (2.2.2), *and Δ^- and Π are the operators* (2.1.14) *and* (2.1.16).

PROOF. Let us set

$$\Big\{\sum_{\delta=0}^{\infty} a(p^\delta N)\, t^\delta\Big\}\, Q_{p,F}(t) = \sum_{\nu=0}^{\infty} b_\nu t^\nu.$$

Multiplying out the expression on the left-hand side, and using (2.2.2), we obtain

$$(2.2.4)\; b_\nu = a(p^\nu N) - \lambda_F(p)\, a(p^{\nu-1}N) + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})\, a(p^{\nu-2}N) - p^{2k-3}\lambda_F(p)\, a(p^{\nu-3}N) + p^{4k-6}a(p^{\nu-4}N),$$

where $a(p^{\nu-i}N) = 0$ if $\nu - i < 0$. By Proposition 2.1.2 we have

$$(2.2.5)\quad \lambda_F(p^\delta)\, a(p^\sigma N) = \\ = \sum_{\alpha+\beta+\gamma=\delta} p^{(k-2)\beta+(2k-3)\gamma} (\Delta^+(p^\sigma)\,\Delta^-(p^\gamma)\,\Pi(p^\beta)\,\Delta^+(p^\alpha)\,a)(N).$$

Carrying out a purely formal computation using (2.1.15), we come to the relation

$$(2.2.6)\quad (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})\, a(p^\sigma N) = \\ = (\Delta^+(p^\sigma)\,(p^{2k-4}(\Pi(p)^2 - \Pi(p^2) + p - 1) + p^{k-2}\Delta^+(p)\,\Pi(p) + \\ + p^{3k-5}\Pi(p)\,\Delta^-(p))\,a)(N).$$

If $\nu \geqslant 4$, the formulae obtained then give us

$$b_\nu = \{[\Delta^+(p^\nu) - \Delta^+(p^{\nu-1})\,(\Delta^+(p) + p^{k-2}\Pi(p) + p^{2k-3}\Delta^-(p)) + \\ + \Delta^+(p^{\nu-2})\,(p^{2k-4}(\Pi(p)^2 - \Pi(p^2) + p - 1) + p^{k-2}\Delta^+(p)\,\Pi(p) + p^{3k-5}\Pi(p)\,\Delta^-(p)) - \\ - p^{2k-3}\Delta^+(p^{\nu-3})\,(\Delta^+(p) + p^{k-2}\Pi(p) + p^{2k-3}\Delta^-(p)) + p^{4k-6}\Delta^+(p^{\nu-4})]\,a\}(N),$$

hence, using (2.1.15), we obtain

$$b_\nu = p^{2k-4}(\Delta^+(p^{\nu-2})\,(\Pi(p)^2 - \Pi(p^2) - (p+1))\,a\}(N).$$

Similarly, since $(\Delta^-(p)a)(N) = 0$ we find

$$b_3 = p^{2k-4}(\Delta^+(p)\,(\Pi(p)^2 - \Pi(p^2) - (p+1))\,a)(N),$$
$$b_2 = ((p^{2k-4}(\Pi(p)^2 - \Pi(p^2) - 1) + p^{3k-5}\Pi(p)\,\Delta^-(p))\,a)(N),$$
$$b_1 = -p^{k-2}(\Pi(p)\,a)(N),$$
$$b_0 = 1.$$

Thus, to complete the proof of Proposition 2.2.1 it suffices to prove the following lemma.

LEMMA 2.2.1. *For every prime p the following operator identity holds:*

$$\Delta^+(p)(\Pi(p)^2 - \Pi(p^2) - (p+1)) = 0.$$

PROOF. By (2.1.16) and (2.1.15) we have

$$\Delta^+(p)\,\Pi(p)^2 = \Delta^+(p)\,T_a\left(\Gamma_1\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1\right)\Delta^-(p)\,T_a\left(\Gamma_1\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1\right)\Delta^-(p) = \\ = T_a\left(\Gamma_1\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1\right)^2\Delta^-(p).$$

The following identity in the Hecke ring $L^{(1)}$ is well known and readily verified:

$$T\left(\Gamma_1\begin{pmatrix}1&0\\0&p\end{pmatrix}\Gamma_1\right)^2 = T\left(\Gamma_1\begin{pmatrix}1&0\\0&p^2\end{pmatrix}\Gamma_1\right) + (p+1)\,T\left(\Gamma_1\begin{pmatrix}p&0\\0&p\end{pmatrix}\Gamma_1\right).$$

Since the mapping $T \to T_a$ is a representation of $L^{(1)}$, we have

$$\Delta^+(p)\,\Pi(p)^2 = T_a\left(\Gamma_1\begin{pmatrix}1&0\\0&p^2\end{pmatrix}\Gamma_1\right)\Delta^-(p) + (p+1)\,T\left(\Gamma_1\begin{pmatrix}p&0\\0&p\end{pmatrix}\Gamma_1\right)\Delta^-(p) = \\ = \Delta^+(p)\,\Pi(p^2) + (p+1)\,\Delta^+(p).$$

This proves the lemma and the proposition.

§2.3. The operators $\Pi(p^\beta)$ and the composition of quadratic modules

In this section we show that the operators $\Pi(p^\beta)$ on $\mathfrak{A}$ have a simple interpretation in terms of the composition of modules in imaginary quadratic fields.

We start by recalling the basic definitions and facts of the theory of modules in quadratic fields. More detail can be found in [21], Ch. 2.

Let K be an algebraic number field of finite degree over the field $\mathbf{Q}$ of rational numbers. A *module*[1] in K is any $\mathbf{Z}$-module $M \subset K$. Every module $M \subset K$ has a $\mathbf{Z}$*-basis*, that is, a finite collection $\omega_1, \ldots, \omega_m$ of elements of M such that every element of M has a unique expression as an linear combination of them with coefficients in $\mathbf{Z}$.

Two modules M_1 and M_2 in K are said to be *similar* if $M_1 = \alpha M_2$ for some $\alpha \neq 0$ in K.

A module M of K is said to be *full* if $\mathbf{Q}M = K$. Every full module has a $\mathbf{Z}$-basis of $[K : \mathbf{Q}]$ elements. A full module K that contains 1 and is itself a ring is called an *order* of K.

If M is a module of K, the ring $\mathfrak{O}_M = \{\alpha \in K;\ \alpha M \subset M\}$ is called the *coefficient ring of the module M*. The coefficient rings of similar modules coincide. For every full module there exists a similar module that is contained in its coefficient ring.

The coefficient ring of any full module of K is an order of K. Every order of K is contained in the maximal order $\mathfrak{O}$ of all the integers of K.

Let M be a full module with basis $\omega_1, \ldots, \omega_n$, where $n = [K : \mathbf{Q}]$. The number $D(M) = \det(\mathrm{Sp}_{K/\mathbf{Q}}(\omega_i\omega_j))$ is independent of the choice of basis and is called the *discriminant of M*. The discriminant of the maximal order $\mathfrak{O}$,

$$d = d_K = d(\mathfrak{O})$$

is called the *discriminant of K*.

Let M be a full module of a field K of degree n over $\mathbf{Q}$, and let $\mathfrak{O}_M$ be its coefficient ring. In $\mathfrak{O}_M$ we choose a basis $\alpha_1, \ldots, \alpha_n$, and $\omega_1, \ldots, \omega_n$ in M. Then the absolute value of the determinant of the transition matrix from the first basis to the second,

$$N(M) = |\det(a_{ij})|, \text{ where } \omega_i = \sum a_{ij}\alpha_j, \tag{2.3.1}$$

is independent of the choices of bases, and is called the *norm of M*. If a full module M is contained in its coefficient ring $\mathfrak{O}_M$, then $N(M) = [\mathfrak{O}_M : M]$ (the index of M in $\mathfrak{O}_M$).

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[1] We only consider modules of finite type.

Now let K be a quadratic extension of $\mathbf{Q}$. Every such field K is of the form $K = \mathbf{Q}(\sqrt{d_0})$, where $d_0 \neq 0, 1$ is a square-free rational integer. As a basis of the maximal order $\mathfrak{O}$ of $\mathbf{Q}(\sqrt{d_0})$ we can take the numbers 1 and ω, where $\omega = \frac{1+\sqrt{d_0}}{2}$ if $d_0 \equiv 1 \pmod 4$, and $\omega = \sqrt{d_0}$ if $d_0 \equiv 2$ or $3 \pmod 4$. The discriminant d of $\mathfrak{O}$ (in other words, the discriminant of the field $\mathbf{Q}(\sqrt{d_0})$) is equal to d_0 in the first case, and to $4d_0$ in the second. Any order $\mathfrak{O}'$ of $\mathbf{Q}(\sqrt{d_0})$ has the form

$$\mathfrak{O}_f = \mathbf{Z} + \mathbf{Z}f\omega, \tag{2.3.2}$$

where f in the index $[\mathfrak{O} : \mathfrak{O}_f]$. The discriminant of $\mathfrak{O}_f$ is equal to df^2.

Let M_1 and M_2 be two full modules of $K = \mathbf{Q}(\sqrt{d})$. Then the set M_1M_2 is again a full module in K and is called the product of M_1 and M_2. If $\mathfrak{O}_{M_1} = \mathfrak{O}_{f_1}$ and $\mathfrak{O}_{M_2} = \mathfrak{O}_{f_2}$, then

$$\mathfrak{O}_{M_1M_2} = \mathfrak{O}_f, \tag{2.3.3}$$

where f is the greatest common divisor of f_1 and f_2. For any two full modules M_1 and M_2 of K the following relation for the norms holds:

$$N(M_1M_2) = N(M_1)N(M_2). \tag{2.3.4}$$

For every full module M we denote by $\overline{M}$ the module that consists of the conjugates $\bar{\alpha}$ over $\mathbf{Q}$ of the elements α of M. $\overline{M}$ is a full module with the same coefficient ring as M, and we have the formula

$$M\overline{M} = N(M)\,\mathfrak{O}_M. \tag{2.3.5}$$

$\overline{M}$ is said to be *conjugate* to M.

Let $\mathfrak{O}'$ be a fixed order of K. It follows from (2.3.3) and (2.3.5) that all the full modules of K having $\mathfrak{O}'$ as coefficient ring form a commutative group under multiplication of modules. The quotient group of this group by the subgroup of modules similar to $\mathfrak{O}'$ is called the *class group* of modules of the ring $\mathfrak{O}'$, and is denoted by

$$H(\mathfrak{O}') = H(D),$$

where D is the discriminant of $\mathfrak{O}'$. The group $H(\mathfrak{O}')$ is finite for any order $\mathfrak{O}'$. For the order $h(D)$ of the group $H(D)$ we have the formula

$$h(df^2) = h(d)\,\frac{\Phi(f)}{e_f\varphi(f)}, \tag{2.3.6}$$

where Φ and φ are the Euler functions of K and $\mathbf{Q}$, respectively, and e_f is the index of the group of units of $\mathfrak{O}_f$ in the group of units of the maximal order $\mathfrak{O}$.

Suppose that f' divides f. Then the map

$$M \to \mathfrak{O}_{f'}\,M \tag{2.3.7}$$

induces an epimorphism of the class group of modules of $\mathfrak{O}_f$ onto the class group of modules of $\mathfrak{O}_{f'}$, which we denote by $\nu(f, f')$:

$$(2.3.8) \qquad \nu(f, f'):H(df^2)\to H(df'^2) \qquad (f' \mid f).$$

Let us consider the relation between quadratic modules and prime numbers.

PROPOSITION 2.3.1. *Let $K = \mathbf{Q}(\sqrt{d})$ be a quadratic field with discriminant d, and $\mathfrak{O}_f$ an order of K with discriminant df^2. Suppose that p is a prime number not dividing f. Then the existence of a full module M in K satisfying the conditions*

$$(2.3.9) \qquad \mathfrak{O}_M = \mathfrak{O}_f,\ M \subset \mathfrak{O}_f,\ N(M) = p$$

is equivalent to the solubility of the congruence $x^2 \equiv d \pmod{4p}$. If this congruence is soluble, then there are precisely two modules M_1 and M_2 having the properties (2.3.9) if p does not divide d, and then $M_2 = \overline{M}_1$, or precisely one such module M if p divides d.

PROOF. We denote by $\{\alpha, \beta\}$ the full module M with the basis α, β. We use the following lemma, which is proved in [21], Ch. 2, §7, Lemma 1.

LEMMA 2.3.1. *Let $\gamma \in K$, $\gamma \notin \mathbf{Q}$; let $a\gamma^2 + b\gamma + c = 0$, where a, b, c relatively prime rational integers with $a > 0$ (such a, b and c obviously exist, and are uniquely determined by γ). We set $M = \{1, \gamma\}$. Then*

$$(2.3.10) \qquad N(M) = 1/a,\ \mathfrak{O}_M = \{1, a\gamma\},\ D(\mathfrak{O}_M) = b^2 - 4ac.$$

Let M satisfy the conditions (2.3.9). Since the index of M in $\mathfrak{O}_f$ is p, the smallest natural number contained in M is p. Hence M has a basis of the form $p, p\gamma : M = \{p, p\gamma\}$, with $\gamma \in K$. Let $a\gamma^2 + b\gamma + c = 0$, where a, b and c are integers with $(a, b, c) = 1$ and $a > 0$. Then by Lemma 2.3.1, $p = N(M) = N(p)N\{1, \gamma\} = p^2/a$, hence $a = p$. On the other hand, $b^2 - 4ac = D = df^2$. Thus, $b^2 \equiv df^2 \pmod{4p}$. Since $d \equiv 1$ or $0 \pmod 4$ and $(f, p) = 1$, it follows that the congruence $x^2 = d \pmod{4p}$ is soluble. Conversely, if this congruence is soluble, then $b^2 - 4pc = d$ has a solution in integers (b, c). Since d is the discriminant, $(p, b, c) = 1$. Consequently, $(p, bf, cf^2) = 1$. Let γ be a root of the equation $p\gamma^2 + bf\gamma + cf^2 = 0$. Then it follows from Lemma 2.3.1 that the module $M = \{p, p\gamma\}$ satisfies the conditions (2.3.9).

Let $M_i = \{p, p\gamma_i\}$ $(i = 1, 2)$ be two modules satisfying (2.3.9). As we have seen, the γ_i satisfy equations of the form $p\gamma_i^2 + b_i\gamma_i + c_i = 0$ $(i = 1, 2)$, with $(p, b_i, c_i) = 1$ and $b_i^2 - 4pc_i = df^2$. Since $b_i^2 \equiv df^2 \pmod{4p}$, we see that $b_1 \equiv \pm b_2 \pmod{2p}$. In the first case $b_2 = b_1 + 2lp$ and

$$M_2 = \left\{p, \frac{-b_2 \pm f\sqrt{d}}{2}\right\} = \left\{p, \frac{-b_1 \pm f\sqrt{d}}{2} - lp\right\} = \left\{p, \frac{-b_1 \pm f\sqrt{d}}{2}\right\}.$$

The last module is equal to either M_1 or $\overline{M_1}$. Similarly, in the second case $b_2 = -b_1 + 2lp$ and

$$M_2=\left\{p, \frac{b_1 \pm f\sqrt{d}}{2}-lp\right\}=\left\{p, \frac{-b_1 \mp f\sqrt{d}}{2}\right\}=M_1 \text{ or } \overline{M}_1.$$

This completes the proof of the proposition, because if p divides d, then $M_1 = \overline{M}_1$.

We conclude the survey part of this section with a description of the correspondence between modules in quadratic fields and binary quadratic forms. For definiteness, we restrict ourselves to imaginary quadratic fields, since we do not need any others.

First we recall some definitions. Let

$$F(x, y)=ax^2+bxy+cy^2$$

be a binary quadratic form. The *discriminant* of F is the number

$$D=d(F)=b^2-4ac.$$

The form is called *integral* if a, b and c are integers. In this case the greatest common divisor

$$e(F)=(a, b, c)$$

is called the *divisor* of F; F is said to be *primitive* if $e(f) = 1$. With each binary form we associate the matrix

$$N=N(F)=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

If F is positive definite then $N>0$, and if F is integral, $N \in \mathfrak{N}_2$. Two forms F and F_1 are said to be *equivalent* (in the narrow sense) if

$$N(F_1)={}^tUN(F)U, \quad U \in \mathrm{SL}_2(\mathbf{Z}). \tag{2.3.11}$$

This means that F_1 can be obtained from F by means of an integral linear change of variables with determinant 1. If F and F_1 are two equivalent forms, then $d(F) = d(F_1)$ and $e(F) = e(F_1)$. The set of all integral forms with a fixed discriminant $D \neq 0$ splits into finitely many classes of equivalent forms.

Let $K = \mathbf{Q}(\sqrt{d})$ be an imaginary quadratic field with discriminant d. Let M be a full module in K. With every ordered basis α, β of M, starting from the condition

$$\frac{1}{i}\det\begin{pmatrix} \alpha & \beta \\ \bar{\alpha} & \bar{\beta} \end{pmatrix}>0, \tag{2.3.12}$$

we associate the binary quadratic form

$$F=F_M(x, y)=ax^2+bxy+cy^2= \tag{2.3.13}$$
$$=\frac{1}{N(M)}N(\alpha x+\beta y)=\frac{1}{N(M)}(\alpha x+\beta y)(\bar{\alpha}x+\bar{\beta}y).$$

Clearly, F is positive definite. It follows easily from Lemma 2.3.1 that F is integral, primitive, and that $d(F) = b^2 - 4ac = df^2$, where df^2 is the discriminant of the coefficient ring $\mathfrak{O}_M = \mathfrak{O}_f$ of M. Conversely, if

$F(x, y) = ax^2 + bxy + cy^2$ is a positive definite integral primitive form of discriminant $df^2 = b^2 - 4ac > 0$, then

$$M = M(F) = \left\{ a, \frac{b - f\sqrt{d}}{2} \right\} \tag{2.3.14}$$

is a full module with coefficient ring $\mathfrak{O}_f$.

It is easy to check (see [21], Ch. 2, §7, Theorem 4), that the indicated correspondence defines a bijection between the set of all classes of similar modules of $\mathbf{Q}(\sqrt{d})$ with the coefficient ring $\mathfrak{O}_f$ and the set of all classes of equivalent (in the narrow sense) positive definite integral primitive binary quadratic forms of discriminant df^2. As the first of these sets is a group, the second one can also be given a group structure. The group law on the classes of binary forms was first introduced and studied by Gauss and is called the composition of forms. We again denote by $H(D)$ the group of all classes of equivalent (in the narrow sense) positive definite integral primitive binary quadratic forms of discriminant D.

We now turn to the operators $\Pi(p^\beta)$ (see §2.1). First of all, using the correspondence just described between binary quadratic forms and modules, we give another realization of the space $\mathfrak{A}$ (see (2.1.12)), or to be more precise, of one of its subspaces.

Each matrix $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathfrak{N}_2$, with $N > 0$ can be regarded as the matrix of the positive definite integral binary quadratic form

$$F = F_N = ax^2 + bxy + cy^2, \tag{2.3.15}$$

and vice versa. For matrices $N \in \mathfrak{N}_2$ we use the same definitions and notation as for the corresponding forms F_N. In particular, $e(N) = e(F_N)$, $d(N) = d(F_N)$, and $M(N) = M(F_N)$ (see (2.3.14)).

We denote by $\widetilde{\mathfrak{A}}$ the space of all complex-valued functions $\widetilde{\varphi}$ on the product $\{1, 2, \ldots\} \times \mathfrak{M}$, where $\mathfrak{M}$ is the set of all full modules of all imaginary quadratic extensions of $\mathbf{Q}$ satisfying the condition $\widetilde{\varphi}(m; M) = \widetilde{\varphi}(m; M_1)$ if M and M_1 lie in one and the same quadratic field and are similar. On the other hand, we denote by $\mathfrak{A}^*$ the subset of $\mathfrak{A}$ consisting of functions that vanish on $\{N \in \mathfrak{N}_2;\ \det N = 0\}$.

If $N \in \mathfrak{N}_2$, $N > 0$, then $N = e(N)N'$, where N' is primitive. We associate with $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$ a function $\widetilde{\varphi} \in \mathfrak{A}^*$, setting

$$\varphi(N) = \varphi(e(N)N') = \widetilde{\varphi}(e(N); M(N')). \tag{2.3.16}$$

Obviously, the map $\widetilde{\varphi} \to \varphi$ is an isomorphism of the spaces $\widetilde{\mathfrak{A}}$ and $\mathfrak{A}^*$. Therefore, we can regard any operator on $\mathfrak{A}^*$ as an operator on $\widetilde{\mathfrak{A}}$, and conversely. In particular, carrying over to $\widetilde{\mathfrak{A}}$ the operators (2.1.14), we arrive at the operators

$$(2.3.17)\qquad \begin{cases} (\Delta^+(m)\widetilde{\varphi})(m_1; M) = \widetilde{\varphi}(mm_1; M), \\ (\Delta^-(m)\widetilde{\varphi})(m_1; M) = \begin{cases} \widetilde{\varphi}(m_1/m; M) & \text{if } m \mid m_1, \\ 0 & \text{if } m \nmid m_1, \end{cases} \end{cases}$$

for $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$, and $m = 1, 2, \ldots$. We now turn to the operators $\Pi(p^\beta)$.

THEOREM 2.3.1. *Let $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$, and let m be a natural number. Let $K = \mathbf{Q}(\sqrt{d})$ be an imaginary quadratic field of discriminant d, and M a full module in K with coefficient ring $\mathfrak{O}_f$. Suppose that p is a prime number with $(p, m) = 1$. Then*

(I) *if $(p, f) = 1$, if the congruence $x^2 \equiv d \pmod{4p}$ is soluble and p does not divide, d, then*

$$(\Pi(p^\beta)\widetilde{\varphi})(m; M) = \widetilde{\varphi}(m; \mathfrak{P}^\beta M) + \widetilde{\varphi}(m; \overline{\mathfrak{P}}^\beta M) \qquad (\beta = 1, 2, \ldots),$$

where $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ are the modules of index p in $\mathfrak{O}_f$ with coefficient ring $\mathfrak{O}_f$ (see Proposition 2.3.1);

(II) *if $(p, f) = 1$, if the congruence $x^2 \equiv d \pmod{4p}$ is soluble, and p divides d, then*

$$(\Pi(p^\beta)\widetilde{\varphi})(m; M) = \begin{cases} \widetilde{\varphi}(m; \mathfrak{P}M), & \text{if } \beta = 1 \\ 0, & \text{if } \beta > 1, \end{cases}$$

where $\mathfrak{P}$ is the submodule of $\mathfrak{O}_f$ of index p with coefficient ring $\mathfrak{O}_f$ (see Proposition 2.3.1);

(III) *if $(p, f) = 1$ and the congruence $x^2 \equiv d \pmod{4p}$ is soluble, then*

$$(\Pi(p^\beta)\widetilde{\varphi})(m; M) = 0 \qquad (\beta = 1, 2, \ldots);$$

(IV) *if p divides f, then*

$$(\Pi(p)\widetilde{\varphi})(m; M) = \widetilde{\varphi}(pm; \mathfrak{O}_{f/p}M),$$

where $\mathfrak{O}_{f/p}$ is the order of K with discriminant $d(f/p)^2$ (compare (2.3.7)).

PROOF. Let

$$F(x, y) = F_M(x, y) = ax^2 + bxy + cy^2, \quad (a, b, c) = 1, \quad b^2 - 4ac = df^2,$$

be the binary quadratic form corresponding to the module M under (2.3.13). We consider the congruence

$$(2.3.18)\qquad F(x, y) = ax^2 + bxy + cy^2 \equiv 0 \pmod{p^\beta}.$$

Let us call a solution (u_1, u_2) of (2.3.18) *primitive* if $(u_1, u_2) = 1$. We introduce the equivalence relation (2.1.5) in the set of all primitive solutions of the congruence (2.3.18). It is easy to verify that in case (I) (2.3.18) has precisely two inequivalent primitive solutions, in case (II) there are no primitive solutions if $\beta > 1$, and if $\beta = 1$, then all primitive solutions are equivalent; in case (III) there are no primitive solutions, and in case (IV) all primitive solutions are equivalent if $\beta = 1$. Hence and from (2.1.17) it follows that the theorem holds in case (II) for $\beta > 1$ and in case (III)

$\left(\text{for } U = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}, \ a_u = au_1^2 + bu_1u_2 + cu_2^2\right).$

CASE (I). Let (u_1, u_2) and (u_1', u_2') be two inequivalent primitive solutions of (2.3.18). Since $b^2 - 4ac \not\equiv 0 \pmod p$, we may assume that $F(u_1, u_2) = p^\beta a_1$, $F(u_1', u_2') = p^\beta a_2$, where $(a_1, p) = (a_2, p) = 1$. We choose integers v_1, v_2, v_1', v_2' such that

$$U_1 = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}), \quad U_2 = \begin{pmatrix} u_1' & u_2' \\ v_1' & v_2' \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

We denote by $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ the matrix of F and set

$$N_1 = U_1 N {}^tU_1 = \begin{pmatrix} a_1 p^\beta & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}, \quad N_1^* = \begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & p^\beta c_1 \end{pmatrix},$$

$$N_2 = U_2 N {}^tU_2 = \begin{pmatrix} a_2 p^\beta & b_2/2 \\ b_2/2 & c_2 \end{pmatrix}, \quad N_2^* = \begin{pmatrix} a_2 & b_2/2 \\ b_2/2 & p^\beta c_2 \end{pmatrix}.$$

Then, by definition (see (2.1.17)),

$$(\Pi(p^\beta)\widetilde{\varphi})(m, M) = \varphi(mN_1^*) + \varphi(mN_2^*),$$

where φ is the function in $\mathfrak{A}^*$ corresponding to $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$ (see (2.3.16)). We consider the modules

$$\mathfrak{Q}_1 = \left\{p^\beta, \frac{b_1 - f\sqrt{d}}{2}\right\}, \quad \mathfrak{Q}_2 = \left\{p^\beta, \frac{b_2 - f\sqrt{d}}{2}\right\}$$

of the field $\mathbf{Q}(\sqrt{d})$. It follows from Lemma 2.3.1 that the coefficient ring of each of these modules is $\mathfrak{O}_f$, that they are both contained in $\mathfrak{O}_f$, and that $N(\mathfrak{Q}_1) = N(\mathfrak{Q}_2) = p^\beta$. We claim that $\mathfrak{Q}_1 + \mathfrak{Q}_2 = \mathfrak{O}_f$. To see this it is obviously sufficient to check that $\left(\frac{b_2 - b_1}{2}, p\right) = 1$. We set

$$U_2 U_1^{-1} = T = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Since the solutions (u_1, u_2) and (u_1', u_2') are inequivalent, $t_2 = u_1 u_2' - u_2 u_1' \not\equiv 0 \pmod p$. From the fact that $TN_1 {}^tT = N_2$ it follows that $a_1 p^\beta t_1^2 + b_1 t_1 t_2 + c_1 t_2^2 = a_2 p^\beta$ and $2a_1 p^\beta t_1 t_3 + b_1(t_1 t_4 + t_2 t_4) + 2c_1 t_2 t_4 = b_2$. The congruence $b_1 t_1 + c_1 t_2 \equiv 0 \pmod p$ follows from the first equality; from the second it follows that $\frac{b_2 - b_1}{2} \equiv t_2(b_1 t_3 + c_1 t_4) \pmod{p^\beta}$. If $\frac{b_2 - b_1}{2}$ were divisible by p, we would have the system of congruences $b_1 t_1 + c_1 t_4 \equiv b_1 t_3 + c_1 t_4 \equiv 0 \pmod p$, which would imply that $b_1 \equiv c_1 \equiv 0 \pmod p$, which is impossible, because $b_1^2 - p^\beta a_1 c_1 = df^2$ is not divisible by p. From what we have proved it follows that $\mathfrak{Q}_1 = \mathfrak{P}^\beta$, $\mathfrak{Q}_2 = \overline{\mathfrak{P}}^\beta$, where $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ are the unique pair of submodules of $\mathfrak{O}_f$ of index p with coefficient ring $\mathfrak{O}_f$. Let M_1, M_1^*, M_2, M_2^* be the modules of $\mathbf{Q}(\sqrt{d})$ corresponding to the matrices N_1, N_1^*, N_2, N_2^*, respectively. Then since $(a_1, p) = 1$, we have

$$M_1^*\mathfrak{P}^\beta = \left\{a_1, \frac{b_1 - f\sqrt{d}}{2}\right\}\left\{p^\beta, \frac{b_1 - f\sqrt{d}}{2}\right\} = \left\{a_1 p^\beta, \frac{b_1 - f\sqrt{d}}{2}\right\} = M_1,$$

from which, using (2.3.5) we deduce that the module M_1^* is similar to $M_1\overline{\mathfrak{P}}^\beta$. Similarly one checks that M_2^* is similar to $M_2\mathfrak{P}^\beta$. Thus, since M_1 and M_2 are similar to M, we have

$$(\Pi(p^\beta)\widetilde{\varphi})(m;\ M) = \widetilde{\varphi}(m;\ M_1^*) + \widetilde{\varphi}(m;\ M_2^*) = \widetilde{\varphi}(m;\ M\overline{\mathfrak{P}}^\beta) + \widetilde{\varphi}(m;\ M\mathfrak{P}^\beta).$$

This proves (I).

CASE (II). As was pointed out above, we need only consider the case $\beta = 1$. Let (u_1, u_2) be a primitive solution of (2.3.18). Since this congruence does not have any primitive solutions modulo p^2, we have $F(u_1, u_2) = a_1 p$, with $(a_1, p) = 1$. We choose integers v_1, v_2 such that

$$U = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

Let

$$N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \quad N_1 = UN{}^tU = \begin{pmatrix} a_1 p & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}; \quad N_1^* = \begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & pc_1 \end{pmatrix},$$

and let M, M_1 and M_1^* be the modules corresponding to the matrices N, N_1 and N_1^*, respectively. According to the definition, in this case $(\Pi(p)\widetilde{\varphi})(m;\ M) = \widetilde{\varphi}\ (m;\ M_1^*)$, so that we need only check that M_1^* is similar to $M\mathfrak{P}$. It follows from Lemma 2.3.1 that $\mathfrak{P} = \left\{p, \frac{b_1 - f\sqrt{d}}{2}\right\}$, hence

$$M_1^*\mathfrak{P} = \left\{a_1, \tfrac{1}{2}(b_1 - f\sqrt{d})\right\}\left\{p, \tfrac{1}{2}(b_1 - f\sqrt{d})\right\} = \left\{a_1 p, \tfrac{1}{2}(b_1 - f\sqrt{d})\right\} = M_1.$$

Since M_1 is similar to M and $\mathfrak{P}^2 = \mathfrak{P}\overline{\mathfrak{P}}$ is similar to $\mathfrak{O}_f$, we see that M_1^* is similar to $M\mathfrak{P}$, and (II) is proved.

CASE (IV). Let (u_1, u_2) be the unique primitive solution (up to equivalence) of the congruence (2.3.18) modulo p; let v_1, v_2 be integer such that

$$U = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}).$$

We set

$$N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \quad N_1 = UN{}^tU = \begin{pmatrix} pa_1 & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}, \quad N_1^* = \begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & pc_1 \end{pmatrix}.$$

Since $b_1^2 - 4pa_1c_1 = df^2 \equiv 0 \pmod{p^2}$, we have $b_1 \equiv 0 \pmod{p}$. Since the matrix N_1 is primitive together with N, we have $(c_1, p) = 1$. It follows that $a_1 \equiv 0 \pmod p$ (this is trivial for $p \neq 2$; for $p = 2$ we have to take into account that $d \equiv 1$ or $0 \pmod 4$). We set $N_1^* = pN_2$. From what we have just said it follows that N_2 is semi-integral; it is easy to see that N_2 is primitive. According to (2.1.17) we have $(\Pi(p)\widetilde{\varphi}(m; M) = \varphi(mpN_2)$, where φ is the function in $\mathfrak{A}^*$ corresponding to $\widetilde{\varphi}$. Let M_1 and M_2 be the modules corresponding to the matrices N_1 and N_2, respectively. Since M_1 is similar to M, we need only check that M_2 is similar to $\mathfrak{O}_{f/p}M_1$. By Lemma 2.3.1,

$$\mathfrak{O}_{f/p} = \left\{1, \frac{b_1 - f\sqrt{d}}{2p}\right\}.$$

Thus,

$$\mathfrak{O}_{f/p}M_1 = \left\{1, \frac{b_1 - f\sqrt{d}}{2p}\right\}\left\{pa_1, \frac{b_1 - f\sqrt{d}}{2}\right\} =$$

$$= \left\{pa_1, a_1c_1, \frac{b_1 - f\sqrt{d}}{2}, a_1\frac{b_1 - f\sqrt{d}}{2}\right\} = \left\{a_1, \frac{b_1 - f\sqrt{d}}{2}\right\} = pM_2$$

(we have used the fact that $(c_1, p) = 1$). This proves (IV).

Below we need the following lemma, which is not a formal consequence of Theorem 2.3.1.

LEMMA 2.3.2. *In the notation and under the hypotheses of case* (IV) *of Theorem* 2.3.1, *the following identity holds for every* $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$:

$$((\Pi(p)^2 - \Pi(p^2) - 1)\widetilde{\varphi})(m, M) = 0.$$

PROOF. In the notation introduced in the proof of case (IV) of Theorem 2.3.1, we have $(\Pi(p)\varphi(mM) = \varphi(mpN_2)$. Since the matrices $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}$ (for $l = 0, 1, \ldots, p-1$) form a left transversal to Γ_1 in $\Gamma_1 \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_1$, by definition of the operators $\Pi(p)$ (see (2.1.13) and (2.1.16)) we have

$$(\Pi(p)^2\varphi)(mN) = \sum_{l=0}^{p-1} \varphi\left(m\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix} N_2 {}^t\begin{pmatrix} 1 & l \\ 0 & p \end{pmatrix}\right) + \varphi\left(m\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} N_2 {}^t\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\right) =$$

$$= \sum_{l=0}^{p-1} \varphi\left(m\begin{pmatrix} \frac{a_1}{p} + \frac{lb_1}{p} + l^2c_1 & \frac{b_1}{2} + plc_1 \\ \frac{b_1}{2} + plc_1 & p^2c_1 \end{pmatrix}\right) + \varphi(mN_1).$$

On the other hand, since $a_1 \equiv 0 \pmod p$, any primitive solution of the congruence (2.3.18) modulo p is also a solution mod p^2. It then follows that the set $(u_1 + lv_1, u_2 + lv_2)$ (for $l = 0, 1, \ldots, p-1$) is a complete set of mod p^2 inequivalent primitive solutions of the congruence (2.3.18) mod p^2, and hence (see 2.1.17))

$$(\Pi(p^2)\varphi)(mN) =$$

$$= \sum_{l=0}^{p-1} \varphi\left(\frac{m}{p^2}\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}\begin{pmatrix}u_1+lv_1 & u_2+lv_2\\ v_1 & v_2\end{pmatrix} N \, {}^t\begin{pmatrix}u_1+lv_1 & u_2+lv_2\\ v_1 & v_2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & p^2\end{pmatrix}\right) =$$

$$= \sum_{l=0}^{p-1} \varphi\left(m\begin{pmatrix}\frac{a_1}{p}+\frac{lb_1}{p}+l^2c_1 & \frac{b_1}{2}+plc_1\\ \frac{b_1}{2}+plc_1 & p^2c_1\end{pmatrix}\right).$$

Comparing the expressions obtained and noting that $\varphi(mN_1) = \varphi(mN)$, we came to the assertion of the lemma.

The following theorem, or more precisely a corollary of it, is needed in the proof of the functional equation for Euler products. This is an analogue to Theorem 2.3.1 for the case when m is divisible by p and $\beta = 1$.

THEOREM 2.3.2. *Let $\widetilde{\varphi} \in \widetilde{\mathfrak{A}}$, let $K = \mathbf{Q}(\sqrt{d})$ be an imaginary quadratic field with discriminant d, and let M be a full module in K with coefficient ring $\mathfrak{O}_f$. Then for any prime number p and any natural number m the following formula holds for the action of the operator $\Pi(p)$;*

$$(\Pi(p)\widetilde{\varphi})(pm;\ M) = (A) + \frac{e_{pf}}{e_f} \sum_{\substack{\{M_i\}\in H(d(pf)^2),\\ \nu(pf,\,f)\{M_i\}=\{M\}}} \widetilde{\varphi}(m;\ M_i),$$

where

$$(A) = \widetilde{\varphi}(pm;\ \mathfrak{P}M) + \widetilde{\varphi}(pm;\ \overline{\mathfrak{P}}M) \tag{2.3.19}$$

if $(p, f) = 1$, if the congruence $x^2 \equiv d \pmod{4p}$ is soluble and p does not divide d; here $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ are the submodules of $\mathfrak{O}_f$ of index p with coefficient ring $\mathfrak{O}_f$;

$$(A) = \widetilde{\varphi}(pm;\ \mathfrak{P}M) \tag{2.3.20}$$

if $(p, f) = 1$, $x^2 \equiv d \pmod{4p}$ is soluble and p divides d; here $\mathfrak{P}$ is the submodule of index p in $\mathfrak{O}_f$ with coefficient ring $\mathfrak{O}_f$;

$$(A) = 0 \tag{2.3.21}$$

if $(p, f) = 1$, $x^2 \equiv d \pmod{4p}$ is insoluble, and

$$(A) = \widetilde{\varphi}(p^2m;\ \mathfrak{O}_{f/p}M) \tag{2.3.22}$$

if p divides f (see Theorem 2.3.1 and Proposition 2.3.1); in the sum on the right-hand side e_δ denotes the index of the group of units of $\mathfrak{O}_\delta$ in the group of units of $\mathfrak{O}$, $\{M'\}$ the similarity class of the module M', and $\nu(f', f)$ the epimorphism (2.3.8).

PROOF. In the computations to follow we go over wherever convenient from the language of modules to the language of matrices or quadratic forms and vice versa. In particular, let

$$N = \begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}, \quad b^2 - 4ac = df^2,$$

be the semi-integral primitive positive definite matrix corresponding to the module M, and let $\varphi \in \mathfrak{A}^*$ be the function corresponding to $\widetilde{\varphi}$ (see (2.3.13) and (2.3.16)). According to (2.1.17) we have

$$(\Pi(p)\,\widetilde{\varphi})(pm;\ M) = \sum_{U \in R(p)} \varphi\left(m \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} UN^tU \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right) = (A) + (B),$$

where

$$(A) = \sum_{\substack{U = \begin{pmatrix} u_1 u_2 \\ v_1 v_2 \end{pmatrix} \in R(p), \\ au_1^2 + bu_1u_2 + cu_2^2 \equiv 0 \pmod p}} \varphi\left(pm \cdot \frac{1}{p} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} UN^tU \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right),$$

$$(B) = \sum_{\substack{U = \begin{pmatrix} u_1 u_2 \\ v_1 v_2 \end{pmatrix} \in R(p), \\ au_1^2 + bu_1u_2 + cu_2^2 \not\equiv 0 \pmod p}} \varphi\left(m \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} UN^tU \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}\right).$$

As far as the sum (A) is concerned, it follows from the proof of Theorem 2.3.1 that one of the formulae (2.3.19)–(2.3.22) holds for it, according as to which of the clauses are satisfied. Thus, to prove the theorem it remains to check that (B) is equal to

$$(B) = \frac{e_{pf}}{e_f} \sum_{\substack{\{M_i\} \in H(d(pf)^2), \\ \nu(pf,\, f)\{M_i\} = \{M\}}} \widetilde{\varphi}(m;\ M_i). \tag{2.3.23}$$

Obviously the summation in (B) is precisely over those $U \in R(p)$ for which the matrix $N_u = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} UN^tU \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ is primitive.

LEMMA 2.3.3. *Let* $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ *be a semi-integral primitive positive definite matrix of discriminant* $D = b^2 - 4ac = df^2$, *and* p *a prime number. Then for every* $U \in R(p)$ *for which the matrix* N_u *is primitive, we have* $\nu(pf, f)\{N_u\} = \{N\}$, *and conversely, every primitive matrix* N' *of discriminant* $d(pf)^2$ *such that* $\nu(pf, f)\{N'\} = \{N\}$, *is equivalent to some matrix* N_u *(with* $U \in R(p)$*) (we denote by* $\{N'\}$ *the equivalence class in the narrow sense of the matrix* N', *and by* $\nu(,)$ *the map on groups of equivalence classes of matrices that corresponds to the map (2.3.8) on module class groups).*

PROOF OF THE LEMMA. Replacing N by UN^tU, we see that the first assertion need only be checked in the particular case when $U = E_2$ and $a \not\equiv 0 \pmod p$. We set $\omega = \frac{1}{2}(b - \sqrt{D})$. Then in the case under discussion to the matrix N there corresponds the module $\{a, \omega\}$, and to $N_u = N_{E_2}$ the module $\{a, p\omega\}$. It is required to prove that $\mathfrak{O}_f\{a, p\omega\}$ is similar to

$\{a, \omega\}$. Since $\mathfrak{O}_f = \{1, \omega\}$ and $(a, p) = 1$, we have

$$\mathfrak{O}_f\{a, p\omega\} = \{a, \omega p, \omega a, pb\omega - pac\} = \{a, \omega\}.$$

Conversely, let $N' = \begin{pmatrix} a' & b'/2 \\ b'/2 & c' \end{pmatrix}$. At the expense of passing to an equivalent matrix we may suppose that $c' \equiv 0 \pmod p$. Since $(b')^2 - 4a'c' = d(pf)^2 \equiv 0 \pmod{p^2}$ and N' is primitive, we then have $b' \equiv 0 \pmod p$ and $c' \equiv 0 \pmod{p^2}$ (the latter is obvious if $p \neq 2$, and if $p = 2$, we must take into account that $d \equiv 0$ or $1 \pmod 4$)). Thus, $N' = \begin{pmatrix} a_1 & pb_1/2 \\ pb_1/2 & p^2c_1 \end{pmatrix}$, and $(a_1, p) = 1$. We have to check that the matrix $N_1 = \begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}$ is equivalent to N. We set $\omega_1 = \frac{1}{2}(b_1 - \sqrt{D})$ and $\omega = \frac{1}{2}(b - \sqrt{D})$. Then $M(N') = \{a_1, p\omega_1\}$, $M(N_1) = \{a_1, \omega_1\}$, and $M(N) = \{a, \omega\}$. By assumption, $\mathfrak{O}_f\{a_1, p\omega_1\} \sim \{a, \omega\}$, but $\mathfrak{O}_f\{a_1, p\omega_1\} = \{1, \omega_1\}\{a_1, p\omega_1\} = \{a_1, \omega_1\}$. Thus, the modules $M(N_1)$ and $M(N)$ are similar, and hence N_1 is equivalent to N, which proves the lemma.

Since

$$(B) = \sum_{\substack{U \in R(p), \\ N_u \text{ primitive}}} \varphi(mN_u), \tag{2.3.24}$$

by Lemma 2.3.3, to prove (2.3.33) it suffices to check that in this sum each class $\{N_u\}$ occurs precisely e_{pf}/e_f times.

Bearing in mind the fact that there are precisely two units in the order of an imaginary quadratic field, apart from the maximal orders of the fields $\mathbf{Q}(\sqrt{-4})$ and $\mathbf{Q}(\sqrt{-3})$, where there are, respectively, 4 and 6 units, we find that

$$\frac{e_{pf}}{e_f} = \begin{cases} 2 & \text{if } d = -4,\ f = 1 \\ 3 & \text{if } d = -3,\ f = 1 \\ 1 & \text{otherwise} \end{cases} \tag{2.3.25}$$

We consider the following four cases:

(I) $(p, f) = (p, d) = 1$, and the congruence $x^2 \equiv d \pmod{4p}$ is soluble;

(II) $(p, f) = 1$, $(p, d) = p$, and the congruence $x^2 \equiv d \pmod{4p}$ is soluble;

(III) $(p, f) = 1$, and the congruence $x^2 \equiv d \pmod{4p}$ is insoluble;

(IV) $(p, f) = p$.

Since the number of elements of the set $R(p)$ is $p + 1$, and the number of mod p inequivalent primitive solutions of the congruence $ax^2 + bxy + cy^2 \equiv 0 \pmod p$ (see (2.1.5)) is equal in the cases (I), (II), (III), and (IV) to 2, 1, 0, and 1, respectively (see the proof of Theorem 2.3.1), the number of terms in the sum (2.3.24) is $p - 1$, p, $p + 1$, and p in the cases (I), (II), (III) and (IV), respectively.

On the other hand, by (2.3.6) the number of distinct classes $\{N'\} \in H(d(pf)^2)$ such that $\nu(pf, f)\{N'\} = \{N\}$ that is, the order of the kernel of the epimorphism $\nu(pf, f)$ is equal to

$$\frac{h(d(pf)^2)}{h(df^2)} = \frac{e_f}{e_{pf}} \cdot \frac{\Phi(pf)}{\Phi(f)} \cdot \frac{\varphi(f)}{\varphi(pf)}.$$

It is easy to verify that this number in the four cases (I) – (IV), respectively, is equal to

$$\text{(2.3.26)} \qquad \frac{e_f}{e_{pf}}(p-1), \quad \frac{e_f}{e_{pf}}p, \quad \frac{e_f}{e_{pf}}(p+1), \quad \frac{e_f}{e_{pf}}p.$$

Comparing these numbers with the number of terms in the sum (2.3.24) and taking Lemma 2.3.3 into account, we see that (2.3.23) holds if $e_f = e_{pf}$.

According to (2.3.25) it remains to consider two cases: $d = -4$, $f = 1$ and $d = -3$, $f = 1$. Let $d = -4$, $f = 1$; in this case we can take for N the identity matrix E_2. By definition (see §2.1), the elements of $R(p)$ can be enumerated by the points (u_1, u_2) of the projective line mod p. We consider the automorphism σ of this line defined by $\sigma(u_1, u_2) = (-u_1, u_2)$. This automorphism is of order 2, and it is easy to see that its fixed points are precisely the equivalence classes mod p (see (2.1.5)) of primitive solutions of the congruence $u_1^2 + u_2^2 \equiv 0 \pmod p$. Thus, on the set of classes (u_1, u_2) for which the matrix N_u is primitive $\left(U = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\right)$, σ acts without fixed points. Since

$$\begin{pmatrix} -u_2 & u_1 \\ -v_2 & v_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^t \begin{pmatrix} -u_2 & u_1 \\ -v_2 & v_1 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^t \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix},$$

in this case every class $\{N_u\}$ occurs at least twice in the sum (2.3.24), and (2.3.23) follows from Lemma 2.3.3 if we compare the number of terms in (2.3.24) with the numbers (2.3.26).

The discussion of the case $d = -3$, $f = 1$ is similar. Here we can take N to be $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and consider the automorphism τ of order 3 of the projective line mod p given by $\tau(u_1, u_2) = (u_1 + u_2, -u_1)$. The theorem is now proved.

§2.4. Euler products

In this section, using the formulae obtained above, we establish explicit relations between the eigenvalues of the Hecke operators on the space $\mathfrak{M}_k^2$ of modular forms of genus 2 and weight k and the Fourier coefficients of the eigenfunctions. As we have seen in §1.2, in the case of modular forms of genus 1, these connections look equally simple in the language of the coefficients themselves and in the language of the corresponding Dirichlet series (see (1.2.11) and (1.2.16)). For genus 2 the language of Dirichlet series turns out to be more natural. Here, instead of the single identity (1.2.16) there arises a infinite series of identities, numbered by the equivalence classes in the narrow sense of positive definite integral primitive binary quadratic forms.

Let

$$F(Z) = \sum_{N \in \mathfrak{N}_2,\ N \geqslant 0} a(N) \exp(2\pi i \sigma(NZ)) \in \mathfrak{M}_k^2$$

be a modular form of genus 2 and weight k. Suppose that F is an eigenfunction of all the Hecke operators $T_k(m)$ (for $m = 1, 2, \ldots$):

$$T_k(m)F = \lambda_F(m)F \quad (m = 1, 2, \ldots).$$

As in §2.2, we define for every prime p the polynomial

$$Q_{p,F}(t) = 1 - \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})t^2 - \lambda_F(p)p^{2k-3}t^3 + p^{4k-6}t^4.$$

As we have seen in §2.2 (see the Corollary of Theorem 2.2.1), in some right half-plane Re $s > \sigma$ we have the identity

$$Z_F(s) = \prod_p Q_{p,F}(p^{-s})^{-1} = \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \frac{\lambda_F(m)}{m^s},$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, let us fix some integer $D < 0$. We denote by d the discriminant of the imaginary quadratic field $K = \mathbf{Q}(\sqrt{D})$, so that $D = df^2$ For every character χ of the class group $H(\mathfrak{O}_f) = H(D)$ of similar modules of K with coefficient ring $\mathfrak{O}_f$, we denote by

$$L_D(s, \chi) = \prod_{\mathfrak{P}} \left(1 - \frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^s}\right)^{-1} \quad (\mathrm{Re}\, s > 1),$$

the L-series of $\mathfrak{O}_f$ with character χ, where in the product $\mathfrak{P}$ ranges over all the prime ideals of $\mathfrak{O}_f$ whose coefficient ring is $\mathfrak{O}_f$ and whose norms are coprime to f (every ideal is also a full module). In §2.3 we have defined a bijection between the set of equivalence classes in the narrow sense of positive definite matrices $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \in \mathfrak{N}_2$ with discriminant $b^2 - 4ac = D$ and the set of similarity classes of modules of $K = \mathbf{Q}(\sqrt{D})$ with coefficient ring $\mathfrak{O}_f$. This bijection allows us to introduce on the set of equivalence classes of matrices, the structure of an Abelian group which we continue to denote $H(D)$.

In this notation, the main result of the present chapter can be summarized as follows:

THEOREM 2.4.1. *For every integer* $D = df^2 < 0$ *and every character* χ *of the group* $H(D)$, *the following identity holds in some right half-plane* Re $s > \sigma$:

$$L_D(s-k+2,\ \chi)\sum_{i=1}^{h}\chi(N_i)\sum_{m=1}^{\infty}\frac{a(mN_i)}{m^s}=\Phi_F(s,\ \chi)\prod_p Q_{p,F}(p^{-s})^{-1},$$

where N_i *(for* $i = 1, 2, \ldots, h = h(D)$*) is a complete system of representatives of the equivalence classes in the narrow sense of positive definite primitive matrices* $N \in \mathfrak{N}_2$ *with discriminant* D *and*

$$\Phi_F(s,\ \chi)=\sum_{i=1}^{h}\chi(N_i)\left\{\prod_{p\mid f}\left(1-\frac{\Pi(p)}{p^{s-k+2}}\right)\left(1-\frac{\Delta^-(p)}{p^{s-2k+3}}\right)a\right\}(N_i);$$

here p *ranges over the prime factors of* f, *the Fourier coefficients* $a(N)$ *of* F *are regarded as the values of some function* $a \in \mathfrak{A}$ *(see* (2.1.12)), *and* $\Pi(p)$ *and* $\Delta^-(p)$ *are the operators on* $\mathfrak{A}$ *defined in* §2.1 *(see* (2.1.14) *and* (2.1.16)).

REMARK. The function $\Phi_F(s, \chi)$ can be computed explicitly on the basis of the formulae in §2.3 (see (2.3.17) and the formula in case (IV) of Theorem 2.3.1). Let M_i be modules corresponding to the binary forms with matrices N_i (see (2.3.14)), let $\widetilde{a}(m; M)$ be the function in $\widetilde{\mathfrak{A}}$ corresponding to $a(N)$ (see (2.3.16)). Then

$$(2.4.1)\quad \Phi_F(s,\ \chi)=\sum_{\delta_1\mid\delta\mid f}\frac{\mu(\delta_1)\,\mu(\delta)}{\delta^{s-k+2}\delta_1^{s-2k+3}}\sum_{i=1}^{h}\chi(M_i)\,\widetilde{a}\left(\frac{\delta}{\delta_1};\ \mathfrak{O}_{f/\delta}M_i\right),$$

where μ is the Möbius function.

PROOF OF THE THEOREM. The absolute convergence of the left- and right-hand sides of the identity in some right half-plane follows from (1.1.11) and the results of §1.3.

Let $M_i = M(F_{N_i})$ (see (2.3.15) and (2.3.14)), and let $\widetilde{a}(m; M)$ be the function in $\widetilde{\mathfrak{A}}$ corresponding to $a(n)$ (see (2.3.16)). For every prime p and every natural number m such that $(m, p) = 1$ we now compute the series

$$\sum_{\delta=0}^{\infty}a(mp^{\delta};\ \chi)\,p^{-\delta s},\ \text{where}\ a(n,\ \chi)=\sum_{i=1}^{h}a(nN_i)\,\chi(N_i)=\sum_{i=1}^{h}\widetilde{a}(n;\ M_i)\,\chi(M_i).$$

For brevity we call an ideal of $\mathfrak{O}_f$ regular if its coefficient ring coincides with $\mathfrak{O}_f$. According to Proposition 2.3.1 and Theorem 2.3.1, the following cases are possible.

(I) p does not divide f and $\mathfrak{O}_f$ has precisely two regular prime ideals $\mathfrak{P}$ and $\overline{\mathfrak{P}}$ of norm p. In this case we have, by the first part of Theorem 2.3.1,

$$(\Pi(p)\,\widetilde{a})(m;\ M_i)=\widetilde{a}(m;\ \mathfrak{P}M_i)+\widetilde{a}(m;\ \overline{\mathfrak{P}}M_i),$$

$$((\Pi(p)^2-\Pi(p^2)-1)\,\widetilde{a})(m;\ M_i)=\widetilde{a}(m;\ \mathfrak{P}^2M_i)+2\widetilde{a}(m;\ \mathfrak{P}\overline{\mathfrak{P}}M_i)+$$
$$+\widetilde{a}(m;\ \overline{\mathfrak{P}}^2M_i)-\widetilde{a}(m;\ \mathfrak{P}^2M_i)-\widetilde{a}(m;\ \overline{\mathfrak{P}}^2M_i)-\widetilde{a}(m;\ M_i)=\widetilde{a}(m;\ M_i),$$

since the module $\mathfrak{P}\overline{\mathfrak{P}}M_i=pM_i$ is similar to M_i and

$$(\Pi(p)\,\Delta^-(p)\,\widetilde{a})\,(m;\; M_i) = (\Delta^-(p)\,\widetilde{a})\,(m;\; \mathfrak{P}M_i) + (\Delta^-(p)\,\widetilde{a})\,(m;\; \overline{\mathfrak{P}}M_i) = 0.$$

Thus, by Proposition 2.2.1 we obtain:

$$(2.4.2) \quad \left\{ \sum_{\delta=0}^{\infty} a\,(mp^{\delta},\; \chi)\, p^{-\delta s} \right\} Q_{p,\,F}(p^{-s}) =$$

$$= \sum_{i=1}^{h} \chi(M_i) \left\{ \widetilde{a}\,(m;\; M_i) - \frac{1}{p^{s-k+2}}\,(\widetilde{a}\,(m;\; \mathfrak{P}M_i) + \widetilde{a}\,(m;\; \overline{\mathfrak{P}}M_i)) + \frac{\widetilde{a}\,(m;\; M_i)}{p^{2s-2k+4}} \right\} =$$

$$= a\,(m;\; \chi) - \frac{\chi(\mathfrak{P}) + \chi(\overline{\mathfrak{P}})}{p^{s-k+2}}\, a\,(m,\; \chi) + \frac{1}{p^{2s-2k+4}}\, a\,(m,\; \chi) =$$

$$= \left(1 - \frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^{s-k+2}}\right) \left(1 - \frac{\chi(\overline{\mathfrak{P}})}{(N\overline{\mathfrak{P}})^{s-k+2}}\right) a\,(m,\; \chi).$$

(II) p does not divide f and $\mathfrak{O}_f$ has just one regular prime ideal $\mathfrak{P} = \overline{\mathfrak{P}}$ of norm p. In this case we have, by the second part of Theorem 2.3.1,

$$(\Pi(p)\,\widetilde{a})\,(m;\, M_i) = \widetilde{a}\,(m;\, \mathfrak{P}M_i),$$

$$((\Pi(p)^2 - \Pi(p^2) - 1)\,\widetilde{a})\,(m;\, M_i) = \widetilde{a}\,(m;\, \mathfrak{P}^2 M_i) - a\,(m;\; M_i) = 0,$$

since the module

$$\mathfrak{P}^2 M_i = \mathfrak{P}\overline{\mathfrak{P}} M_i = pM_i$$

is similar to M_i, and

$$(\Pi(p)\,\Delta^-(p)\,\widetilde{a})\,(m;\; M_i) = (\Delta^-(p)\,\widetilde{a})\,(m;\; \mathfrak{P}M_i) = 0.$$

Thus, by Proposition 2.2.1 we obtain:

$$(2.4.3) \quad \left\{ \sum_{\delta=0}^{\infty} a\,(mp^{\delta},\; \chi)\, p^{-\delta s} \right\} Q_{p,\,F}(p^{-s}) =$$

$$= \sum_{i=1}^{h} \chi(M_i) \left\{ \widetilde{a}\,(m;\; M_i) - \frac{1}{p^{s-k+2}}\, \widetilde{a}\,(m;\, \mathfrak{P}M_i) \right\} =$$

$$= \left(1 - \frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^{s-k+2}}\right) a\,(m,\; \chi).$$

(III) p does not divide f and $\mathfrak{O}_f$ has no prime ideal of norm p. In this case $\mathfrak{P} = p\mathfrak{O}_f$ is a regular prime ideal of norm p^2.

By the third part of Theorem 2.3.1, we obtain:

$$(\Pi(p)\,\widetilde{a})\,(m;\; M_i) = 0,$$

$$((\Pi(p)^2 - \Pi(p^2) - 1)\,\widetilde{a})\,(m;\; M_i) = -\widetilde{a}\,(m;\; M_i),$$

$$(\Pi(p)\,\Delta^-(p)\,\widetilde{a})\,(m;\; M_i) = 0,$$

and hence, by Proposition 2.2.1,

$$(2.4.4)\quad \left\{\sum_{\delta=0}^{\infty} a(mp^{\delta},\chi)\,p^{-\delta s}\right\} Q_{p,F}(p^{-s}) =$$
$$= \sum_{i=1}^{h} \chi(M_i)\left\{\tilde{a}(m;M_i) - \frac{1}{p^{2s-2k+4}}\tilde{a}(m;M_i)\right\} =$$
$$= \left(1-\frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^{s-k+2}}\right) a(m,\chi)$$
$$(\chi(\mathfrak{P}) = \chi(p\mathfrak{O}_f) = 1).$$

(IV) p divides f. In this case, using Proposition 2.2.1 and Lemma 2.3.2, we obtain:

$$(2.4.5)\quad \left\{\sum_{\delta=0}^{\infty} a(mp^{\delta},\chi)\,p^{-\delta s}\right\} Q_{p,F}(p^{-s}) =$$
$$= \sum_{i=1}^{h} \chi(M_i)\left\{\tilde{a}(m;M_i) - \frac{1}{p^{s-k+2}}(\Pi(p)\tilde{a})(m;M_i) + \right.$$
$$\left. + \frac{1}{p^{2s-3k+5}}(\Pi(p)\Delta^{-}(p)\tilde{a})(m;M_i)\right\} =$$
$$= \sum_{i=1}^{h} \chi(M_i)\left\{\left(1-\frac{\Pi(p)}{p^{s-k+2}}\right)\left(1-\frac{\Delta^{-}(p)}{p^{s-2k+3}}\right)\tilde{a}\right\}(m;M_i).$$

The assertion of the theorem is a formal consequence of the identities (2.4.2)–(2.4.5): applying for each p the corresponding identity (it is convenient to consider first the primes p that are coprime to f, and then the p dividing f), we obtain

$$\left\{\sum_{m=1}^{\infty}\frac{a(m,\chi)}{m^s}\right\}\prod_{p^*} Q_{p,F}(p^{-s}) =$$
$$= \sum_{i=1}^{h}\chi(M_i)\left\{\prod_{p\mid f}\left(1-\frac{\Pi(p)}{p^{s-k+2}}\right)\left(1-\frac{\Delta^{-}(p)}{p^{s-2k+3}}\right)\tilde{a}\right\}(1,M_i)\prod_{\mathfrak{P}}\left(1-\frac{\chi(\mathfrak{P})}{(N\mathfrak{P})^{s-k+2}}\right),$$

where $\mathfrak{P}$ ranges over all the regular prime ideals of $\mathfrak{O}_f$ of norm coprime to f.

The theorem is now proved.

Chapter 3

ANALYTIC CONTINUABILITY AND THE FUNCTIONAL EQUATION FOR GENUS 2

§3.1. The main theorem

This chapter is devoted to a proof of the main theorem, which we now state.

THEOREM 3.1.1. *Let* $F \in \mathfrak{M}_k^2$ *be a modular form of genus* 2 *and* *weight k (for integral* $k \geqslant 0$*). Suppose that F is an eigenfunction of all the Hecke operators* $T_k(m)$ *(see (1.3.15)):* $T_k(m)F = \lambda_F(m)F$ *(for* $m = 1, 2, \dots$*). For every prime number p, let*

$$Q_{p,F}(t) = 1 - \lambda_F(p)t + (\lambda_F(p)^2 - \lambda_F(p^2) - p^{2k-4})t^2 - \lambda_F(p)p^{2k-3}t^3 + p^{4k-6}t^4,$$

and for Re $s > \sigma$, *let*

$$Z_F(s) = \prod_p Q_{p,F}(p^{-s})^{-1} \quad \left(= \zeta(2s - 2k + 4) \sum_{m=1}^{\infty} \frac{\lambda_F(m)}{m^s} \right)$$

be the zeta-function of F (see §§1.3 and 2.2). We set

$$\Psi_F(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s - k + 2)Z_F(s),$$

where $\Gamma(s)$ *is the gamma-function. Then the following assertions hold:*

(I) *the function* $\Psi_F(s)$ *can be continued analytically to the whole s-plane as a meromorphic function having at most finitely many poles;*

(II) $\Psi_F(s)$ *satisfies the functional equation* $\Psi_F(2k - 2 - s) = (-1)^k \Psi_F(s)$;

(III) *if* $\Phi F \neq 0$, *where* Φ *is the Siegel operator* (1.1.12), *then* $\Psi_F(s) = c\psi_{\Phi F}(s)\psi_{\Phi F}(s - k + 2)$, *where* $\psi_{\Phi F}(s)$ *is the function of Theorem* 1.2.4 *corresponding to the modular form* $\Phi F \in \mathfrak{M}_k^1$, *and c is a certain constant; in particular,* $\Psi_F(s)$ *has four simple poles at the points* $s = 0, k-2, k, 2k-2$ *if* ΦF *is an Eisenstein series, and is an entire function if* ΦF *is a parabolic form;*

(IV) *if* $\Phi F = 0$, *that is, if F is a parabolic form, then* $\Psi_F(s)$ *has at most two simple poles at the points* $s = k-2, k$; *if k is odd, then* $\Psi_F(s)$ *is an entire function.*

§3.2. Reduction to the case of parabolic forms

In this section we show that Theorem 3.1.1 holds when F is not a parabolic form, that is, when $\Phi F \neq 0$.

In this case, either by Theorem 1.3.3. or by direct computation based on the relations of Maass [7], as in [12], we obtain the relation

$$Z_F(s) = Z_{\Phi F}(s)Z_{\Phi F}(s - k + 2) \tag{3.2.1}$$

(for Re s sufficiently large), where $Z_{\Phi F}(s)$ is the zeta-function of the form $\Phi F \in \mathfrak{M}_k^1$. By Theorem 1.2.2, the Euler product $Z_{\Phi F}(s) = D_{\Phi F}(s)$ is a multiple of the Dirichlet series $R_{\Phi F}(s)$ corresponding to ΦF. Then $\psi_{\Phi F}(s)$ is a multiple of the function $(2\pi)^{-s}\Gamma(s)Z_{\Phi F}(s)$, from which the assertion (III) of Theorem 3.1.1 follows. The remaining assertions follow from (III) and the properties of $\psi_{\Phi F}(s)$ listed in the first part of Theorem 1.2.4.

§3.3. The integral representation (I)

To prove Theorem 3.1.1 for parabolic forms we set off from the identities of Theorem 2.4.1. The left-hand sides of these identities admit an integral representation, from which the theorem follows. Here we take the first step towards setting up this integral representation.

Let $F \in \mathfrak{M}_k^2$ be a parabolic form with Fourier coefficients $a(N)$ (for $N \in \mathfrak{N}_2$, $N > 0$). The Dirichlet series on the left-hand sides of the identities of Theorem 2.4.1 can be expressed in terms of series of the form

$$R_N(s) = R_{N,F}(s) = \sum_{m=1}^{\infty} \frac{a(mN)}{m^s}, \tag{3.3.1}$$

where $N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$ is an arbitrary positive definite semi-integral primitive matrix. It follows from (1.1.15) that the series $R_N(s)$ is absolutely convergent in the domain Re $s > k + 1$. We set

$$\begin{cases} X_N(\mathbf{R}) = \{X \in M_2(\mathbf{R});\ {}^tX = X,\ \sigma(XN) = 0\}, \\ X_N(\mathbf{Z}) = X_N(\mathbf{R}) \cap M_2(\mathbf{Z}), \end{cases} \tag{3.3.2}$$

where σ, as usual, denotes the trace. For every $M \in \mathfrak{N}_2$, the function $\exp(2\pi i\sigma(MX))$ is a character of the compact group $X_N(\mathbf{R})/X_N(\mathbf{Z})$. This character is trivial if and only if M is a multiple of N : $M = mN$ (for some $m \in \mathbf{Z}$). Thus,

$$\int_{X_N(\mathbf{R})/X_N(\mathbf{Z})} \exp(2\pi i\sigma(MX))\,dX = \begin{cases} 1 & \text{if } M = mN, \\ 0 & \text{otherwise} \end{cases} \tag{3.3.3}$$

where dX is the normalized Haar measure on $X_N(\mathbf{R})/X_N(\mathbf{Z})$. We set $\widetilde{N} = \begin{pmatrix} c & -b/2 \\ -b/2 & a \end{pmatrix}$ and let $v > 0$ be a real number. Integrating term-by-term the Fourier expansion of F over $X_N(\mathbf{R})/X_N(\mathbf{Z})$ and using (3.3.3) we obtain

$$\int_{X_N(\mathbf{R})/X_N(\mathbf{Z})} F\left(X + \frac{iv}{\sqrt{\det N}}\widetilde{N}\right) dX = \tag{3.3.4}$$

$$= \sum_{m=1}^{\infty} a(mN) \exp\left(-\frac{2\pi v m\sigma(N\widetilde{N})}{\sqrt{\det N}}\right) =$$

$$= \sum_{m=1}^{\infty} a(mN) \exp(-4\pi\sqrt{\det N}\ vm)$$

($a(0N) = 0$, since F is a parabolic form). Using the Mellin integral

$$(3.3.5) \quad \int_0^\infty \exp(-\alpha v)\, v^{s-1}\, dv = \Gamma(s)\, \alpha^{-s} \quad (\alpha > 0,\ \operatorname{Re} s > 0),$$

we obtain

$$(3.3.6) \quad (4\pi \sqrt{\det N})^{-s}\, \Gamma(s)\, R_N(s) = \int_0^\infty \left\{ \int_{X_N(\mathbf{R})/X_N(\mathbf{Z})} F\left(X + \frac{iv}{\sqrt{\det N}} \widetilde{N}\right) dX \right\} v^{s-1}\, dv,$$

where $R_N(s)$ is the series (3.3.1). As we have noted, the left-hand side of this identity is absolutely convergent in the domain $\operatorname{Re} s > k + 1$. It follows from (3.3.4) that the inner integral on the right-hand side tends to zero exponentially as $v \to +\infty$. It follows from this and the estimate (1.1.14) that the right-hand side converges absolutely in the domain $\operatorname{Re} s > k$.

§ 3.4. Hyperbolic geometry

The integral representation (3.3.6) shows that the Dirichlet series $R_N(s)$ in which we are interested can be obtained by some sort of integration of the restriction of F to a (real) three-dimensional domain $H_N \subset H_2$. The subsequent transformations of this representation are essentially based on the fortunate circumstance that the Siegel modular group $\Gamma_2 = Sp_2(\mathbf{Z})$ has a rather large subgroup Γ_N which acts on H_N as an automorphism group (for instance, $\Gamma_N \backslash H_N$ has finite invariant volume). In this section we consider the geometrical side of this situation.

We introduce some notation. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we set

$$(3.4.1) \quad \widetilde{A} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} = J_1 A J_1^{-1} \quad \left(J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right).$$

Obviously, we have

$$(3.4.2) \quad \widetilde{AA_1} = \widetilde{A}\widetilde{A}_1, \quad A^t \widetilde{A} = \det A \cdot E_2.$$

If N is a real symmetric positive definite 2×2 matrix, we set

$$(3.4.3) \quad \begin{cases} X_N = X_N(\mathbf{R}) = \{X \in M_2(\mathbf{R});\ {}^tX = X,\ \sigma(XN) = 0\}, \\ Y_N = Y_N(\mathbf{R}) = \{Y \in M_2(\mathbf{R});\ YN{}^tY = \det Y \cdot N,\ \det Y \geqslant 0\}, \\ H_N = \left\{X + \dfrac{iv}{\sqrt{\det N}} \widetilde{N};\ X \in X_N,\ v > 0\right\} \subset H_2. \end{cases}$$

THEOREM 3.4.1. *Let $N \in M_2(\mathbf{R})$, with ${}^tN = N$ and $N > 0$. We set*

$$(3.4.4) \quad G_N = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix};\ A \in Y_{\widetilde{N}},\ B \in X_N,\ C \in X_{\widetilde{N}},\ D \in Y_N,\ A^tD - B^tC = E_2 \right\}.$$

Then G_N *is a subgroup of the real symplectic group* $\mathrm{Sp}_2(\mathbf{R})$ *of genus* 2. *For every* $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in G_N$ *the transformation*

$$Z\to M\langle Z\rangle=(AZ+B)(CZ+D)^{-1}\qquad (Z\in H_N) \tag{3.4.5}$$

is an analytic automorphism of the domain $H_N\subset H_2$; *the group* G_N *acts transitively on* H_N.

PROOF. It follows at once from the definition that for any $g\in\mathrm{GL}_2(\mathbf{R})$, the following sets are equal:

$$X_N={}^tgX_{gN^tg}g,\quad Y_N=g^{-1}Y_{gN^tg}g, \tag{3.4.6}$$

$$H_N={}^tgH_{gN^tg}g. \tag{3.4.7}$$

It follows from (3.4.6) and (3.4.2) that

$$\begin{cases} X_{\widetilde{N}}={}^t\widetilde{g}X_{\widetilde{gN^tg}}\widetilde{g}=g^{-1}X_{\widetilde{gN^tg}}{}^tg^{-1},\\ Y_{\widetilde{N}}=\widetilde{g}^{-1}Y_{\widetilde{gN^tg}}\widetilde{g}={}^tgY_{\widetilde{gN^tg}}{}^tg^{-1}.\end{cases} \tag{3.4.8}$$

It follows from (3.4.4), (3.4.6), and (3.4.8) that

$$G_N=M_gG_{gN^tg}M_g^{-1},\ \text{where}\ M_g=\begin{pmatrix}{}^tg & 0\\ 0 & g^{-1}\end{pmatrix}. \tag{3.4.9}$$

Suppose that the theorem holds for the matrix gN^tg. Since $M_g\in\mathrm{Sp}_2(\mathbf{R})$, it then follows from (3.4.9) that G_N is a group and $G_N\subset\mathrm{Sp}_2(\mathbf{R})$. Let $M\in G_N$ and $Z\in H_N$. Then $M\langle Z\rangle=(M_g(M_g^{-1}MM_g)M_g^{-1})\langle Z\rangle$. Since, by (3.4.9) and (3.4.7), $M_g^{-1}MM_g\in G_{gN^tg}$ and $M_g^{-1}\langle Z\rangle\in H_{gN^tg}$, we have $M\langle Z\rangle\in M_g\langle H_{gN^tg}\rangle=H_N$.

Now we choose g such that $gN^tg=E_2$. By the preceding argument it suffices to prove the assertion for $N=E_2$. In this case $N=\widetilde{N}=E_2$, and

$$\begin{cases} X_E=\left\{\begin{pmatrix}a & b\\ b & -a\end{pmatrix};\ a,\ b\in\mathbf{R}\right\},\quad Y_E=\left\{\begin{pmatrix}a & b\\ -b & a\end{pmatrix};\ a,\ b\in\mathbf{R}\right\},\\ H_E=\left\{\begin{pmatrix}a & b\\ b & -a\end{pmatrix}+ivE;\ a,\ b\in\mathbf{R},\ v>0\right\}.\end{cases} \tag{3.4.10}$$

We set

$$R=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix},\ A,\ D\in Y_E,\ B,\ C\in X_E\right\},$$

$$S=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in M_4(\mathbf{R});\ A^tD-B^tC=E\right\}.$$

Then $G_E=R\cap S$. The condition $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}_n(\mathbf{R})$ for a real $2n\times 2n$ matrix is equivalent to the relations.

$$A^tB=B^tA,\quad C^tD=D^tC,\quad A^tD-B^tC=E_n. \tag{3.4.11}$$

Since the matrix $\begin{pmatrix}a & b\\ -b & a\end{pmatrix}^t\begin{pmatrix}c & d\\ d & -c\end{pmatrix}$ is always symmetric, it follows from (3.4.11) that $G_E\subset\mathrm{Sp}_2(\mathbf{R})$. Let $M_1, M_2\in G_E$. Since $M_1M_2\in\mathrm{Sp}_2(\mathbf{R})$, we then have $M_1M_2 \in S$. From the obvious fact that X_E and Y_E are Abelian groups under addition, and from the easily checked inclusions

$$Y_EY_E \subset Y_E, \quad X_EX_E \subset Y_E, \quad X_EY_E \subset X_E, \quad Y_EX_E \subset X_E$$

it follows that $M_1M_2 \in R$. Thus, $M_1M_2 \in R \cap S = G_E$. Further, since for $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_2(\mathbf{R})$

$$M^{-1} = \begin{pmatrix} {}^tD & -{}^tB \\ -{}^tC & {}^tA \end{pmatrix}$$

and since ${}^tX_E = X_E$, ${}^tY_E = Y_E$, the matrix M^{-1} belongs to G_E together with M. Hence G_E is a subgroup of $\mathrm{Sp}_2(\mathbf{R})$.

We claim that

$$H_E = \{M\langle iE\rangle;\ M \in G_E\}. \tag{3.4.12}$$

Let $Z = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} + ivE \in H_E$. Obviously,

$$M = \begin{pmatrix} \sqrt{v}\,E & \frac{1}{\sqrt{v}}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \\ 0 & \frac{1}{\sqrt{v}}E \end{pmatrix} \in G_E$$

and $M\langle iE\rangle = Z$. Thus, the left-hand side of (3.4.12) is included in the right-hand side.

To prove the reverse inclusion we use the following well known and easily checked identity: if

$$Z = X + iY \in H_n, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbf{R}), \quad M\langle Z\rangle = X' + iY',$$

then

$$Y' = {}^t(C\overline{Z} + D)^{-1}\, Y\, (CZ + D)^{-1}. \tag{3.4.13}$$

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_E, A = \begin{pmatrix} a & a_1 \\ -a_1 & a \end{pmatrix}; \quad B = \begin{pmatrix} b & b_1 \\ b_1 & -b \end{pmatrix}, \quad C = \begin{pmatrix} c & c_1 \\ c_1 & -c \end{pmatrix}$, $D = \begin{pmatrix} d & d_1 \\ -d_1 & d \end{pmatrix}$, and $M\langle iE\rangle = X' + iY'$. Then by (3.4.13) and (3.4.11) we have

$$Y' = {}^t(D - iC)^{-1}(D + iC)^{-1} = [D\,{}^tD + C\,{}^tC]^{-1} = (d^2 + d_1^2 + c^2 + c_1^2)^{-1}E.$$

Thus, it is enough to check that $\sigma(X') = 0$. Since the determinant of $iC + D$ is $d^2 + d_1^2 + c^2 + c_1^2$,

$$M\langle iE\rangle = (d^2 + d_1^2 + c^2 + c_1^2)^{-1}\left[\begin{pmatrix} b + ia & b_1 + ia_1 \\ b_1 - ia_1 & -b + ia \end{pmatrix}\begin{pmatrix} d - ic & -d_1 - ic_1 \\ d_1 - ic_1 & d + ic \end{pmatrix}\right]$$

and $\sigma(X') = 0$ can be checked immediately. We have proved (3.4.12), from which the second and third assertions of the theorem obviously follow for

$N = E$. This completes the proof.

According to Theorem 3.3.1, for every real symmetric positive definite 2×2 matrix N we have a Lie group $G_N \subset \mathrm{Sp}_2(\mathbf{R})$ and its homogeneous space $H_N \subset H_2$. We now claim that every such pair (G_N, H_N) is isomorphic in the natural sense to the pair (G, H), where $G = \mathrm{SL}_2(\mathbf{C})$ and H is a three-dimensional hyperbolic space.

The group $G = \mathrm{SL}_2(\mathbf{C})$ acts on the three-dimensional hyperbolic space

$$H = \{u = (z, v);\ z \in \mathbf{C},\ v > 0\} \tag{3.4.14}$$

by the rule

$$\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : u = (z, v) \to \sigma(u) = \left(\frac{(\alpha z + \beta)(\overline{\gamma}\overline{z} + \overline{\delta})}{\Delta_\sigma(u)},\ \frac{v}{\Delta_\sigma(u)} \right), \tag{3.4.15}$$

where

$$\Delta_\sigma(u) = |\gamma z + \delta|^2 + |\gamma|^2 v^2. \tag{3.4.16}$$

To the product of elements of G there corresponds the composition of transformations, and the function $\Delta_\sigma(u)$ is an automorphy factor:

$$\Delta_{\sigma\tau}(u) = \Delta_\sigma(\tau(u))\Delta_\tau(u) \qquad (\sigma, \tau \in G,\ u \in H). \tag{3.4.17}$$

The action of G on H is transitive, and the stabilizer of the point $(0, 1)$ is the special unitary group $U = \mathrm{SU}(2)$. Thus, the map

$$\sigma \to \sigma((0, 1)) \tag{3.4.18}$$

identifies H with the homogeneous space G/U, on which G acts by left multiplication. Note, finally, that the invariant element of volume on H is

$$du = v^{-3}\, dx\, dy\, dv \qquad (u = (x + iy),\ v)). \tag{3.4.19}$$

Let us now go over to establishing an isomorphism of any one of the pairs (G_N, H_N) with (G, H). First let $N = E_2$. If

$$M = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & b_2 & -a_2 \\ a_3 & b_3 & a_4 & b_4 \\ b_3 & -a_3 & -b_4 & a_4 \end{pmatrix} \in G_E$$

(see (3.4.4), (3.4.10)), we set

$$\psi_E(M) = \begin{pmatrix} a_1 + ib_1 & b_2 + ia_2 \\ b_3 - ia_3 & a_4 - ib_4 \end{pmatrix} \in M_2(\mathbf{C}), \tag{3.4.20}$$

and for

$$Z = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} + ivE \in H_E$$

(see (3.4.10)) we set

$$h_E(Z) = (b + ia, v) \in H. \tag{3.4.21}$$

For an arbitrary real symmetric positive definite 2×2 matrix N, we choose a real g such that $gN{}^tg = \mu E$ (with $\mu > 0$). If $M \in G_N$ and $Z \in H_N$, then according to (3.4.9) and (3.4.7),

$$M_g^{-1}MM_g \in G_{\mu E} = G_E, \quad M_g^{-1}\langle Z\rangle \in H_{\mu E} = H_E \left(M_g = \begin{pmatrix} {}^tg & 0 \\ 0 & g^{-1}\end{pmatrix}\right).$$

We set

$$(3.4.22) \qquad \begin{cases} \psi_g(M) = \psi_E(M_g^{-1}MM_g) \in M_2(\mathbf{C}) & (M \in G_N), \\ h_g(Z) = h_E(M_g^{-1}\langle Z\rangle) \in H & (Z \in H_N). \end{cases}$$

THEOREM 3.4.2. *Let $N \in M_2(\mathbf{R})$ with ${}^tN = N$ and $N > 0$, let $g \in \mathrm{GL}_2(\mathbf{R})$ be such that $gN{}^tg = \mu E$ (with $\mu > 0$). Then the following assertions hold:*

(I) $\psi_g(G_N) = G = \mathrm{SL}_2(\mathbf{C})$, *and the map* $\psi_g : G_N \to \mathrm{SL}_2(\mathbf{C})$ *is an isomorphism of real Lie groups;*

(II) *the map* $h_g : H_N \to H = \{u = (z, v);\ z \in \mathbf{C},\ v > 0\}$ *is an analytic isomorphism;*

(III) *the map h_g is compatible with the actions of G_N and G, that is, for any $Z \in H_N$ and $M \in G_N$ we have the relation* $h_g(M\langle Z\rangle) = \psi_g(M)(h_g(Z))$;

(IV) *under the given maps the automorphy factor of the pair (G_N, H_N) goes over into the automorphy factor of the pair (G, H), that is, for any $Z \in H_N$ and any $M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in G_N$ we have the relation*

$\det(CZ + D) = \Delta_{\psi_g(M)}(h_g(Z))$ (see (3.4.16)).

PROOF. First let N and g be the identity matrix E. The assertions (I) and (IV) can be checked by direct computation, and (II) is trivial. Let us check (III). Since $h_E(iE) = (0, 1)$, it is enough to check that ψ_E realises an isomorphism of the stabilizer of iE in G_E with the stabilizer of $(0, 1)$ in G. It is easy to see that the stabilizer of iE in G_E consists of all the matrices of the form

$$M = \begin{pmatrix} a_1 & b_1 & a_2 & b_2 \\ -b_1 & a_1 & b_2 & -a_2 \\ -a_2 & -b_2 & a_1 & b_1 \\ -b_2 & a_2 & -b_1 & a_1 \end{pmatrix},$$

where the matrix $\begin{pmatrix} a_1 & b_1 \\ -b_1 & a_1\end{pmatrix} + i\begin{pmatrix} a_2 & b_2 \\ b_2 & -a_2\end{pmatrix}$ is unitary. The fact that this matrix is unitary is equivalent to $a_1^2 + a_2^2 + b_1^2 + b_2^2 = 1$. On the other hand, the image under ψ_E of M is the matrix $\begin{pmatrix} a_1 + ib_1 & b_2 + ia_2 \\ -b_2 + ia_2 & a_1 - ib_1\end{pmatrix}$, and this also is unitary if and only if $a_1^2 + b_1^2 + a_2^2 + b_2^2 = 1$. Hence the stabilizer of iE goes over into the group $\mathrm{SU}(2) = U$, the stabilizer of the point $(0, 1) \in H$, and (III) follows.

Now let N be any positive definite real 2×2 matrix, and let $gN{}^tg = \mu E$ (for $g \in \mathrm{GL}_2(\mathbf{R})$. Since the map $M \to M_g^{-1}MM_g$ (where $\left(M_g = \begin{pmatrix} {}^tg & 0 \\ 0 & g^{-1}\end{pmatrix}\right)$ is an analytic isomorphism of G_N onto $G_{\mu E} = G_E$ (see

(3.4.9)), and the map $Z \to M_g^{-1}\langle Z\rangle$ is an analytic bijection of H_N onto H_E (see (3.4.7)), the assertions (I) and (II) follow from the corresponding assertions for $N = E$. Let us check (III) and (IV), relying on the fact that they have already been checked for $N = E$. Let $Z \in H_N$ and $M \in G_N$; then

$$h_g(M\langle Z\rangle)=h_E(M_g^{-1}M\langle Z\rangle)=h_E(M_g^{-1}MM_g\langle M_g^{-1}\langle Z\rangle\rangle)=$$
$$=\psi_E(M_g^{-1}MM_g)(h_E(M_g^{-1}\langle Z\rangle))=\psi_g(M)(h_g(Z)).$$

Similarly, if $Z \in H_N$ and $M=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in G_N$, then since

$$M_g^{-1}MM_g=\begin{pmatrix} * & * \\ gC^tg & gDg^{-1}\end{pmatrix},$$

we have

$$\det(CZ+D)=\det(gC^tg^tg^{-1}Zg^{-1}+gDg^{-1})=$$
$$=\Delta_{\psi_E(M_g^{-1}MM_g)}(h_E(M_g^{-1}\langle Z\rangle))=\Delta_{\psi_g(M)}(h_g(Z)).$$

The theorem is now proved.

§3.5. Picard subgroups of the modular group

In this section we investigate the nature of the discrete subgroup

$$\Gamma_N = G_N \cap \mathrm{Sp}_2(\mathbf{Z}) \subset G_N \tag{3.5.1}$$

when N is semi-integral. For this purpose we consider the image of Γ_N under the map ψ_g constructed in the last section. It turns out that for a suitable choice of g this image is an arithmetic discrete subgroup of Picard type of the group $G = \mathrm{SL}_2(\mathbf{C})$.

We fix some notation. Let

$$N=\begin{pmatrix} a & b/2\\ b/2 & c\end{pmatrix},\quad a,\ b,\ c\in\mathbf{Z},\quad a>0,\quad D=D(N)=b^2-4ac<0. \tag{3.5.2}$$

We set

$$\begin{cases} g_N=\dfrac{1}{\sqrt{(2c\sqrt{-D})}}\begin{pmatrix} 2c & -b\\ 0 & \sqrt{-D}\end{pmatrix}\in \mathrm{SL}_2(\mathbf{R}),\\ M_{g_N}=\begin{pmatrix} {}^tg_N & 0\\ 0 & g_N^{-1}\end{pmatrix}\in \mathrm{Sp}_2(\mathbf{R}).\end{cases} \tag{3.5.3}$$

It is easy to see that

$$g_NN^tg_N=\sqrt{(\det N)}E. \tag{3.5.4}$$

For $M \in G_N$ and $Z \in H_N$ we set

$$\begin{cases} \psi^N(M)=\psi_{g_N}(M)=\psi_E(M_{g_N}^{-1}MM_{g_N})\in G,\\ h^N(Z)=h_{g_N}(Z)=h_E(M_{g_N}^{-1}\langle Z\rangle)\in H\end{cases} \tag{3.5.5}$$

(see §3.4).

On the other hand, let K be an imaginary quadratic field, $\mathfrak{O}$ the ring of integers of K, and $\mathfrak{O}_f$ the subring of index f. For any full module M in K with coefficient ring $\mathfrak{O}_f$ we set

(3.5.6) $\Gamma(\mathfrak{O}_f, M)=\left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}_2(\mathbf{C});\ \alpha,\ \delta\in\mathfrak{O}_f,\ \beta\in M,\ \gamma\in M^{-1}\right\}.$

The $\Gamma(\mathfrak{O}_f, M)$ is an arithmetic discrete subgroup of $\mathrm{SL}_2(\mathbf{C})$ with quotient space of finite volume.

The main result of the present section is the following:

THEOREM 3.5.1. *Let*

$$N=\begin{pmatrix}a & b/2\\ b/2 & c\end{pmatrix}>0,$$

with a, b and c relatively prime integers. Let $D = b^2 - 4ac = df^2$, *where d is the discriminant of the imaginary quadratic field* $K = \mathbf{Q}(\sqrt{D})$. *Then the restriction of* ψ^N *to the subgroup* $\Gamma_N \subset G_N$ *maps it isomorphically onto* $\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$:

$$\psi^N\colon \Gamma_N\to\Gamma(\mathfrak{O}_f,\ \mathfrak{A}_N),$$

where $\mathfrak{A}_N=\frac{c}{\omega^2}\{a,\ \omega\}^2$, $\omega=\frac{b-\sqrt{D}}{2}$ *and* $\{a, \omega\}$ *is the module of K having* a, ω *as basis.*

REMARK 1. The condition $(a, b, c) = 1$ does not reduce the generality, because $G_{tN} = G_N$ and $\Gamma_{tN} = \Gamma_N$ for $t \in \mathbf{Z}$.

REMARK 2. In §2.3 we have defined a correspondence between binary quadratic forms and modules in quadratic fields; under it the form with the matrix N corresponds precisely to the module $\{a, \omega\}$ (see (2.3.14)).

PROOF. We define the integral analogues of the sets X_N, Y_N, $X_{\tilde N}$ and $Y_{\tilde N}$ (see (3.4.3) and (3.4.1)), by setting

$$(3.5.7)\qquad \begin{cases} X_N(\mathbf{Z})=X_N\cap M_2(\mathbf{Z}), & Y_N(\mathbf{Z})=Y_N\cap M_2(\mathbf{Z}),\\ X_{\tilde N}(\mathbf{Z})=X_{\tilde N}\cap M_2(\mathbf{Z}), & Y_{\tilde N}(\mathbf{Z})=Y_{\tilde N}\cap M_2(\mathbf{Z}).\end{cases}$$

Then

$$\Gamma_N=\left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix};\ A\in Y_{\tilde N}(\mathbf{Z}),\ B\in X_N(\mathbf{Z}),\ C\in X_{\tilde N}(\mathbf{Z}),\ D\in Y_N(\mathbf{Z}),\ A^tD-B^tC=E_2\right\}.$$

We define $\mathbf{R}$-linear maps ψ_1, ψ_2, ψ_3 and ψ_4 of the sets $Y_{\tilde N}$, X_N, $X_{\tilde N}$ and Y_N, respectively, into $\mathbf{C}$. Let $A \in Y_{\tilde N}$. Then according to (3.4.8), ${}^tg_{\tilde N}^{-1}A^tg_N\in Y_{\tilde E}=Y_E$, that is ${}^tg_{\tilde N}^{-1}A^tg_N$ has the form $\begin{pmatrix}a_1 & b_1\\ -b_1 & a_1\end{pmatrix}$, (with $a_1, b_1 \in \mathbf{R}$). We then set

$$\psi_1(A)=a_1+ib_1\ \text{for}\ (A\in Y_{\tilde N}).$$

If $B \in X_N$, then ${}^tg_{\tilde N}^{-1}Bg_{\tilde N}^{-1}=\begin{pmatrix}a_2 & b_2\\ b_2 & -a_2\end{pmatrix}\in X_E$. We then set

$$(3.5.8)\qquad \psi_2(B)=b_2+ia_2\ \text{for}\ (B\in X_N).$$

If $C \in X_{\tilde N}$, then $g_NC^tg_N=\begin{pmatrix}a_3 & b_3\\ b_3 & -a_3\end{pmatrix}\in X_E$. We set

$$\psi_3(C)=b_3-ia_3 \quad \textit{for} \quad (C\in X_{\tilde{N}}).$$

If $D\in Y_N$, then $g_N D g_N^{-1}=\begin{pmatrix} a_4 & b_4 \\ -b_4 & a_4\end{pmatrix}\in Y_E$. We set

$$\psi_4(D)=a_4-ib_4 \textit{ for } (D\in Y_N).$$

In this notation, for $\begin{pmatrix} A & B \\ C & D\end{pmatrix}\in G_N$ we obviously have

$$\psi^N\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right)=\begin{pmatrix} \psi_1(A) & \psi_2(B) \\ \psi_3(C) & \psi_4(D)\end{pmatrix}.$$

Thus, to prove the theorem it suffices to check that

(3.5.9) $$\psi_1(Y_{\tilde{N}}(\mathbf{Z}))=\psi_4(Y_N(\mathbf{Z}))=\mathfrak{D}_f,$$

(3.5.10) $$\psi_2(X_N(\mathbf{Z}))=\frac{c}{\omega^2}\{a,\omega\}^2,$$

(3.5.11) $$(\psi_3(X_{\tilde{N}}(\mathbf{Z}))=\frac{c}{\overline{\omega}^2}\{a,\overline{\omega}\}^2.$$

It follows from Lemma 2.3.1 that $\{a,\omega\}\cdot\{a,\overline{\omega}\}=a\mathfrak{D}_f$ and since $\omega\overline{\omega}=ac$, we see that the modules $\frac{c}{\omega^2}\{a,\omega\}^2$ and $\frac{c}{\overline{\omega}^2}\{a,\overline{\omega}\}^2$ are mutually inverse. All the sets (3.5.7) are obviously free **Z**-modules of rank 2. Let us find **Z**-bases for them. Since $(a, b, c) = 1$, it is easy to see that as bases of the **Z**-modules $Y_{\tilde{N}}(\mathbf{Z})$ and $Y_N(\mathbf{Z})$ we can take, respectively,

$$\left(E_2, \begin{pmatrix} b & c \\ -a & 0\end{pmatrix}\right) \text{ and } \left(E_2, \begin{pmatrix} 0 & a \\ -c & b\end{pmatrix}\right).$$

Let $(a, c) = \delta$. We choose $\gamma, \alpha\in\mathbf{Z}$ such that

(3.5.12) $$\alpha a+\gamma c=-\delta.$$

Then it is not hard to verify that as **Z**-bases of the modules $X_N(\mathbf{Z})$ and $X_{\tilde{N}}(\mathbf{Z})$ we can take, respectively

(3.5.13) $$\left(\frac{1}{\delta}\begin{pmatrix} c & 0 \\ 0 & -a\end{pmatrix}, \begin{pmatrix} b\alpha & \delta \\ \delta & b\gamma\end{pmatrix}\right) \text{ and } \left(\frac{1}{\delta}\begin{pmatrix} -a & 0 \\ 0 & c\end{pmatrix}, \begin{pmatrix} b\gamma & -\delta \\ -\delta & b\alpha\end{pmatrix}\right).$$

Using these bases, let us check the equations (3.5.9)–(3.5.11). From the definition of the maps ψ_1 and ψ_4 we find easily that

$$\psi_1(E_2)=\psi_4(E_2)=1,\quad \psi_1\left(\begin{pmatrix} b & c \\ -a & 0\end{pmatrix}\right)=\overline{\omega},\quad \psi_4\left(\begin{pmatrix} 0 & a \\ -c & b\end{pmatrix}\right)=\omega.$$

Since, by Lemma 2.3.1, $\mathfrak{D}_f=\{1,\omega\}=\{1,\overline{\omega}\}$ this then proves (3.5.9). Using the definition of ψ_2 and (3.5.12), we obtain after the corresponding computations

$$\psi_2\left(\frac{1}{\delta}\begin{pmatrix} c & 0 \\ 0 & -a\end{pmatrix}\right)=\frac{1}{\delta}\overline{\omega},\quad \psi_2\left(\begin{pmatrix} b\alpha & \delta \\ \delta & b\gamma\end{pmatrix}\right)=\frac{1}{c}(c\delta+b\alpha\overline{\omega}).$$

On the other hand, since $\omega\bar\omega = ac$, $\bar\omega^2 = b\bar\omega - ac$ and $\begin{pmatrix} -\gamma & \alpha \\ a/\delta & c/\delta \end{pmatrix} \in SL_2(\mathbf{Z})$ (see (3.5.12)), we obtain

$$\frac{c}{\omega^2}\{a, \omega\}^2 = c\left\{\frac{\bar\omega}{c}, 1\right\}^2 = c\left\{1, \frac{\bar\omega}{c}, \frac{b\bar\omega - ac}{c^2}\right\} =$$

$$= \left\{\bar\omega, -\gamma c + \alpha\left(\frac{b\bar\omega - ac}{c}\right), \frac{a}{\delta}c + \frac{c}{\delta}\left(\frac{b\bar\omega - ac}{c}\right)\right\} =$$

$$= \left\{\bar\omega, \delta + \frac{\alpha b}{c}\bar\omega, \frac{b}{\delta}\bar\omega\right\} = \left\{\frac{1}{\delta}\bar\omega, \delta + \frac{b\alpha}{c}\bar\omega\right\}.$$

(Note that $(b, \delta) = 1$ because $(a, b, c) = 1$. This proves (3.5.10); (3.5.11) is proved similarly, and the theorem is proved.

Now we consider how the various groups $\Gamma(\mathfrak{O}_f, M)$ are related to one another.

LEMMA 3.5.1. *Let M be a full module of an imaginary quadratic field K. Then there exist numbers $\alpha, \gamma \in M$ and $\beta, \delta \in M^{-1}$ such that* $\alpha\delta - \gamma\beta = 1$.

PROOF. For (α, γ) we take any $\mathbf{Z}$-basis of the module M. Since $1 \in MM^{-1}$, there exist an x and a y in M^{-1} and integers n_1, n_2, m_1 and m_2 such that $(\alpha n_1 + \gamma n_2)x + (\alpha m_1 + \gamma m_2)y = 1$. Hence, setting $\delta = n_1 x + m_1 y$ and $\beta = -(n_2 x + m_2 y)$ we come to the required assertion, and the lemma is proved.

PROPOSITION 3.5.1. *Let M be a full module of an imaginary quadratic field K with coefficient ring $\mathfrak{O}_f$. If M is similar to the square of some $\mathfrak{O}_f$-module, then $\Gamma(\mathfrak{O}_f, M)$ is conjugate inside $SL_2(\mathbf{C})$ to $\Gamma(\mathfrak{O}_f, \mathfrak{O}_f) = SL_2(\mathfrak{O}_f)$. More precisely, let $M = \varepsilon M_1^2$. Let $\alpha, \gamma \in M_1^{-1}$ and $\beta, \delta \in M_1$ with $\alpha\delta - \gamma\beta = 1$ (see Lemma 3.5.1). Then*

$$\sigma\Gamma(\mathfrak{O}_f, M)\sigma^{-1} = SL_2(\mathfrak{O}_f), \quad \textit{where} \quad \sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} (\sqrt{\varepsilon})^{-1} & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix}.$$

PROOF. The proposition is proved by direct computation.

COROLLARY. *In the notation of Theorem 3.5.1, we suppose that $(a, c) = 1$ and set*

$$\text{(3.5.14)} \qquad \sigma_N = \begin{pmatrix} -\gamma_N \frac{\bar\omega}{a} & a \\ \alpha_N & \omega \end{pmatrix}\begin{pmatrix} \frac{\omega}{\sqrt{c}} & 0 \\ 0 & \frac{\bar\omega}{a\sqrt{c}} \end{pmatrix} \in SL_2(\mathbf{C}),$$

where α_N and γ_N are integers for which

$$\text{(3.5.15)} \qquad \alpha_N a + \gamma_N c = -1.$$

Then

$$\text{(3.5.16)} \qquad \sigma_N\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)\sigma_N^{-1} = SL_2(\mathfrak{O}_f).$$

PROOF. The corollary follows from Theorem 3.5.1 and Proposition 3.5.1 by taking into account that $\omega\bar\omega = ac$ and $\{a, \omega\}^{-1} = \frac{1}{a}\{a, \bar\omega\}$.

We conclude this section by comparing the groups Γ_N for different N having the same discriminant.

PROPOSITION 3.5.2. *Let*

$$N=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}, \text{ and } N_1=\begin{pmatrix} a_1 & b_1/2 \\ b_1/2 & c_1 \end{pmatrix}$$

be two semi-integral positive definite matrices. Suppose that $(a, c) = (a_1, c_1) = 1$ *and that they have the same discriminant* $b^2 - 4ac = b_1^2 - 4a_1c_1 = df^2$, *where* d *is the discriminant of the corresponding imaginary quadratic field. We set*

$$M = M_{g_N}\psi_E^{-1}(\sigma_N^{-1}\sigma_{N_1})M_{g_{N_1}}^{-1} \in \mathrm{Sp}_2(\mathbf{R}), \tag{3.5.17}$$

where the matrices M_{g_N} and $M_{g_{N_1}}$ *are defined by* (3.5.3), *the matrices* σ_N *and* σ_{N_1} *by* (3.5.14), *and* ψ_E *is the isomorphism of* G_E *onto* $\mathrm{SL}_2(\mathbf{C})$ *defined by* (3.4.20). *Then the following assertions hold*:

(I) $MG_{N_1}M^{-1} = G_N$;

(II) *for every* $u \in H$ *(see* (3.5.5))

$$M\langle (h^{N_1})^{-1}(\sigma_{N_1}^{-1}(u))\rangle = (h^N)^{-1}(\sigma_N^{-1}(u)),$$

in particular, $M\langle H_{N_1}\rangle = H_N$;

(III) $M\Gamma_{N_1}M^{-1} = \Gamma_N$;

(IV) $M \in \mathrm{Sp}_2(\mathbf{Z})$.

PROOF. (I) and (III) are proved similarly. We prove (III), say. From (3.5.16) we have

$$\psi^N(\Gamma_N) = \sigma_N^{-1}\sigma_{N_1}\psi^{N_1}(\Gamma_{N_1})\sigma_{N_1}^{-1}\sigma_N.$$

Recalling the definition of the map ψ^N (see (3.5.5)), this can be rewritten in the form

$$\psi_E(M_{g_N}^{-1}\Gamma_N M_{g_N}) = \sigma_N^{-1}\sigma_{N_1}\psi_E(M_{g_{N_1}}^{-1}\Gamma_{N_1}M_{g_{N_1}})\sigma_{N_1}^{-1}\sigma_N =$$
$$= \psi_E(\psi_E^{-1}(\sigma_N^{-1}\sigma_{N_1})M_{g_{N_1}}^{-1}\Gamma_{N_1}M_{g_{N_1}}\psi_E^{-1}(\sigma_{N_1}^{-1}\sigma_N)),$$

and hence, since ψ_E is an isomorphism, we obtain

$$M_{g_N}^{-1}\Gamma_N M_{g_N} = \psi_E^{-1}(\sigma_N^{-1}\sigma_{N_1})M_{g_{N_1}}^{-1}\Gamma_{N_1}M_{g_{N_1}}\psi_E^{-1}(\sigma_{N_1}^{-1}\sigma_N),$$

which proves (III).

Using the definition of the map h^N (see (3.5.5)), we find that

$$M\langle (h^{N_1})^{-1}(\sigma_{N_1}^{-1}(u))\rangle = M\langle M_{g_{N_1}}\langle h_E^{-1}(\sigma_{N_1}^{-1}(u))\rangle\rangle = M_{g_N}\psi_E^{-1}(\sigma_N^{-1}\sigma_{N_1})\langle h_E^{-1}(\sigma_{N_1}^{-1}(u))\rangle.$$

By part (III) of Theorem 3.4.2 this last expression can be rewritten as

$$M_{g_N}\psi_E^{-1}(\sigma_N^{-1})\langle h_E^{-1}(u)\rangle = M_{g_N}\langle h_E^{-1}(\sigma_N^{-1}(u))\rangle = (h^N)^{-1}(\sigma_N^{-1}(u)),$$

which proves (II).

As for the last assertion, it has to be proved by an explicit (and rather long) computation of the matrix M. Since there is nothing in this computation apart from matrix multiplication, we allow ourselves simply to state the final result:

$$M=\begin{pmatrix} -(a_1\alpha_1+\gamma_1 c) & -\frac{1}{2}\alpha_1(b_1-b) & (c-c_1) & \frac{1}{2}(b_1-b) \\ \frac{1}{2}\gamma_1(b-b_1) & -(c_1\gamma_1+\alpha_1 a) & \frac{1}{2}(b_1-b) & (a-a_1) \\ (\alpha\gamma_1 a-\gamma\alpha_1 a_1) & \frac{1}{2}\gamma\alpha_1(b-b_1) & -(\gamma c_1+\alpha a)\cdot & \frac{1}{2}\gamma(b_1-b) \\ \frac{1}{2}\alpha\gamma_1(b-b_1) & (\gamma\alpha_1 c-\alpha\gamma_1 c_1) & \frac{1}{2}\alpha(b_1-b) & -(\alpha a_1+\gamma c) \end{pmatrix},$$

where (α, γ) are integers satisfying (3.5.15), and (α_1, γ_1) are similar numbers for N_1. Since $b^2-4ac=b_1^2-4a_1c_1$, we have $b\equiv b_1 \pmod 2$. Hence M is an integral matrix. Since it is symplectic, by construction, this proves (IV) and with it the proposition.

§3.6. The integral representation (II)

We now concern ourselves with further transformations of the integral representation (3.3.6). We keep to the notation and assumptions of §3.3. We also fix some more notation and assumptions about N:

(3.6.1) $$N=\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\in\mathfrak{N}_2,\quad N>0,\quad (a, b, c)=1,\quad D=b^2-4ac;$$

$K=\mathbf{Q}(\sqrt{D})$, d is the discriminant of K and $D=df^2$; $\mathfrak{O}$ is the ring of integers of K, $\mathfrak{O}_f$ is the subring of index f, $\omega=\frac{b-\sqrt{D}}{2}$, $\mathfrak{A}_N=\frac{c}{\omega^2}\{a, \omega\}^2$, and

$$S_N=\{u=(z, v)\in H;\ z\in \mathbf{C}/\mathfrak{A}_N,\ v>0\}.$$

The integration on the right-hand side of (3.3.6) is over a certain subset of the domain $H_N\subset H_2$ (see (3.4.3)) isomorphic to a three-dimensional hyperbolic space H. The isomorphism is realized by means of the map h^N (see (3.5.5)). By definition,

$$h^N\left(X+\frac{iv}{\sqrt{\det N}}\widetilde{N}\right)=(\psi_2(X),\ v)\in H\qquad (X\in X_N(\mathbf{R}),\ v>0),$$

where $\psi_2\colon X_N(\mathbf{R})\to\mathbf{C}$ is the map (3.5.8). According to (3.5.10), the map h^N gives an analytic bijection between the domain of integration on the right-hand side of (3.3.6) and the set S_N. It is easy to see that the Jacobian of the change of variables is equal to $(\det N)^{-1/2}$. Thus,

(3.6.2) $$\int_0^\infty \left\{\int_{X_N(\mathbf{R})/X_N(\mathbf{Z})} F\left(X+\frac{iv}{\sqrt{\det N}}\widetilde{N}\right) dX\right\} v^{s-1}\,dv = (\det N)^{-1/2}\int_{S_N} F((h^N)^{-1}(u))\,v^{s-1}\,dz\,dv,$$

where $u=(x+iy, v)$, $dz=dx\,dy$.

PROPOSITION 3.6.1. *Let* $F\in\mathfrak{M}_k^2$. *We set* $\widetilde{F}(Z)=(\det Y)^{k/2}F(Z)$, *where* $Z = X + iY \in H_2$. *For every* N *satisfying* (3.6.1) *we set*

$$(3.6.3) \quad \widetilde{F}_N(u) = \widetilde{F}((h^N)^{-1}(u)) = v^k F((h^N)^{-1}(u)) \quad (\textit{for } u = (z, v) \in H)$$

Then the following assertions hold:

(I) *if* $\sigma \in \Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$ (*see* (3.5.6)), *then*

$$(3.6.4) \qquad \widetilde{F}_N(\sigma(u)) = \widetilde{F}_N(u) \quad (\textit{for } u \in H)$$

(II) *if* F *is a parabolic form, then* $\widetilde{F}_N(u)$ *is bounded on* H.

PROOF. It follows from (1.1.5) and (3.4.13) that

$$(3.6.5) \qquad \widetilde{F}(M\langle Z\rangle) = \widetilde{F}(Z) \quad (\textit{for } Z \in H_N,\ M \in \Gamma_N).$$

If $\sigma \in \Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$, then, by Theorem 3.5.1, $\sigma = \psi^N(M)$, with $M \in \Gamma_N \subset \Gamma_2$, and hence by part (III) of Theorem 3.4.2, we obtain

$$\widetilde{F}_N(\sigma(u)) = \widetilde{F}((h^N)^{-1}(\sigma(u))) = \widetilde{F}(M\langle (h^N)^{-1}(u)\rangle) = \widetilde{F}_N(u).$$

Now (II) follows from (1.1.14), and the proposition is proved.

From (3.3.6), (3.6.2) and (3.6.3) we see that if $F \in \mathfrak{M}_k^2$ and N satisfies (3.6.1), then

$$(3.6.6) \quad (4\pi)^{-s}(\det N)^{\frac{1-s}{2}}\Gamma(s) R_N(s) = \int_{S_N} \widetilde{F}_N(u) v^{s-k+2}\, du \quad (\mathrm{Re}\, s > k+1),$$

where du is the invariant measure (3.4.19). In the relevant domain both sides are absolutely convergent. The right-hand side converges absolutely in Re $s > k$.

Let

$$\Gamma_\infty(\mathfrak{O}_f, \mathfrak{A}_N) = \left\{ \begin{pmatrix} \pm 1 & \beta \\ 0 & \pm 1 \end{pmatrix};\ \beta \in \mathfrak{A}_N \right\}$$

be the subgroup of parallel translations in $\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$, and let

$$(3.6.7) \qquad \Gamma'(\mathfrak{O}_f, \mathfrak{A}_N) = \bigcup_i \Gamma'_\infty(\mathfrak{O}_f, \mathfrak{A}_N)\sigma_i,$$

(where G' denotes for every group $G \subset \mathrm{SL}_2(\mathbf{C})$ the corresponding group of transformations of H) be a decomposition into disjoint left cosets. Let D_N be a fundamental domain for $\Gamma'(\mathfrak{O}_f, \mathfrak{A}_N)$ on H. Then the set $S'_N = \bigcup_i \sigma_i D_N$ is, obviously, a fundamental domain for $\Gamma'_\infty(\mathfrak{O}_f, \mathfrak{A}_N)$. On the other hand, since the set S_N is also a fundamental domain for $\Gamma'_\infty(\mathfrak{O}_f, \mathfrak{A}_N)$ and the expression under the integral sign in (3.6.6) is invariant under transformations of $\Gamma'_\infty(\mathfrak{O}_f, \mathfrak{A}_N)$, we can write down (so far purely formally)

$$\int_{S_N} \widetilde{F}_N(u) v^{s-k+2}\, du = \int_{S'_N} \widetilde{F}_N(u) v^{s-k+2}\, du = \sum_i \int_{\sigma_i D_N} \widetilde{F}_N(u) v^{s-k+2}\, du =$$

$$= \sum_i \int_{D_N} \widetilde{F}_N(\sigma_i(u))\, v(\sigma_i(u))^{s-k+2}\, d\sigma_i(u) = \sum_i \int_{D_N} \widetilde{F}_N(u) \left(\frac{v}{\Delta_{\sigma_i}(u)}\right)^{s-k+2} du =$$

$$= \int_{D_N} E_N(u, s-k+2)\widetilde{F}_N(u)\,du,$$

where

$$(3.6.8)\qquad E_N(u, s) = v^s \sum_{\sigma\in\Gamma'_\infty(\mathfrak{O}_f, \mathfrak{A}_N)\backslash\Gamma'(\mathfrak{O}_f, \mathfrak{A}_N)} \Delta_\sigma(u)^{-s} \quad (u\in H).$$

(We have used Proposition 3.6.1, the fact that the measure du is invariant, and (3.4.15)).

The series (3.6.8) is an Eisenstein series for the group $\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$ (see [22]). As is well known, for arithmetic discrete subgroups of $SL_2(\mathbf{C})$ every such series converges uniformly and absolutely in any domain of the form $\operatorname{Re} s > 2 + \varepsilon$ (with $\varepsilon > 0$) (see [22] and the references quoted there; for the series (3.6.8) this can easily be deduced from the results of §3.7). It follows that the transformation carried out above is legitimate in the domain $\operatorname{Re} s > k$, where all the series and integrals converge absolutely. So we have proved the following proposition.

PROPOSITION 3.6.2. *For every $F \in \mathfrak{M}_k^2$ and every N satisfying* (3.6.1) *we have the identity*

$$(3.6.9)\qquad (4\pi)^{-s}(\det N)^{\frac{1-s}{2}}\Gamma(s)R_N(s) = \int_{D_N} E_N(u, s-k+2)\widetilde{F}_N(u)\,du,$$

where $E_N(u, s)$ is the series (3.6.8) *and $\widetilde{F}_N(u)$ the function* (3.6.3). *The left- and right-hand sides are absolutely convergent in the domains* $\operatorname{Re} s > k+1$ *and* $\operatorname{Re} s > k$, *respectively.*

To obtain a convenient integral representation for the linear combinations of series $R_N(s)$ that occur in the identity of Theorem 2.4.1 we need to put together the integral representations (3.6.9) for various N with fixed discriminant. To do this we use Propositions 3.5.1 and 3.5.2.

Suppose that the matrix N satisfies (3.6.1) and that $(a, c) = 1$; let σ_N be the matrix (3.5.14). Then by the corollary to Proposition 3.5.1,

$$(3.6.10)\qquad \sigma_N\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)\sigma_N^{-1} = SL_2(\mathfrak{O}_f).$$

It follows that for any fundamental domain D_N of $\Gamma(\mathfrak{O}_f, \mathfrak{A}_N)$ the set $D_f = \sigma_N(D_N)$ is a fundamental domain for $SL_2(\mathfrak{O}_f)$. Making the change of variables $u \to \sigma_N^{-1}(u)$ on the right-hand side of (3.6.9) we obtain

$$(3.6.11)\qquad \int_{D_N} E_N(u, s-k+2)\widetilde{F}_N(u)\,du =$$

$$= \int_{D_f} E_N(\sigma_N^{-1}(u), s-k+2)\widetilde{F}_N(\sigma_N^{-1}(u))\,du.$$

We now consider the nature of the function occurring in the second integral.

PROPOSITION 3.6.3. *Suppose that N satisfies* (3.6.1) *with* $(a, c) = 1$, *and let σ_N be the matrix* (3.5.14). *Then the following identity holds:*

$$(3.6.12)\qquad E_N(\sigma_N^{-1}(u), s) = \frac{1}{2}(av)^s \sum_{\substack{\gamma,\delta\in K\\ \gamma\mathfrak{D}_f+\delta\mathfrak{D}_f=\{a,\bar\omega\}}} \Delta_{(\gamma,\delta)}^{-s}(u) \quad (u=(z,v),\ \mathrm{Re}\, s>2),$$

where $\Delta_{(\gamma,\delta)}(u) = |\gamma z+\delta|^2 + |\gamma|^2 v^2$.

PROOF. It is easy to see that the transformations corresponding to two matrices in $\Gamma(\mathfrak{D}_f, \mathfrak{A}_N)$ having bottom rows (γ, δ) and (γ_1, δ_1) belong to the same left coset of $\Gamma'_\infty(\mathfrak{D}_f, \mathfrak{A}_N)$, if and only if $(\gamma, \delta) = \pm(\gamma_1, \delta_1)$. Furthermore, it is not hard to check that a pair (γ, δ) is the bottom row of some matrix of $\Gamma(\mathfrak{D}_f, \mathfrak{A}_N)$, if and only if $\gamma\mathfrak{A}_N + \delta\mathfrak{D}_f = \mathfrak{D}_f$. Since by (3.4.15)–(3.4.17) for $\sigma = \left(\begin{smallmatrix}\alpha&\beta\\ \gamma&\delta\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{C})$, and $u = (z, v) \in H$ we have

$$\frac{v(\sigma_N^{-1}(u))}{\Delta_\sigma(\sigma_N^{-1}(u))} = \frac{v}{\Delta_\sigma(\sigma_N^{-1}(u))\,\Delta_{\sigma_N^{-1}}(u)} = \frac{v}{\Delta_{\sigma\sigma_N^{-1}}(u)} = \frac{v}{\Delta_{(\gamma,\delta)\sigma_N^{-1}}(u)},$$

and $\Delta_{(\varepsilon\gamma,\varepsilon\delta)}(u) = |\varepsilon|^2 \Delta_{(\gamma,\delta)}(u)$, for $\varepsilon \in \mathbf{C}$, to prove the theorem it therefore suffices, by what we have said above, to prove the following lemma.

LEMMA 3.6.1. *Let* $\gamma, \delta \in K$. *We set*

$$(\gamma, \delta)\,\sigma_N^{-1} = \frac{\sqrt{c}}{\bar\omega}(\gamma_1, \delta_1).$$

Then the condition $\gamma\mathfrak{A}_N + \delta\mathfrak{D}_f = \mathfrak{D}_f$ *is equivalent to the condition* $\gamma_1\mathfrak{D}_f + \delta_1\mathfrak{D}_f = \{a, \bar\omega\}$.

PROOF. As we pointed out above, the condition $\gamma\mathfrak{A}_N + \delta\mathfrak{D}_f = \mathfrak{D}_f$ is equivalent to the existence of $\alpha, \beta \in K$ such that $\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix} \in \Gamma(\mathfrak{D}_f, \mathfrak{A}_N)$. By Proposition 3.5.1,

$$\sigma_N \begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix} \sigma_N^{-1} = \begin{pmatrix}\alpha_0&\beta_0\\ \gamma_0&\delta_0\end{pmatrix} \in \mathrm{SL}_2(\mathfrak{D}_f).$$

By (3.5.14), the matrix σ_N^{-1} has the form

$$\sigma_N^{-1} = \frac{1}{\sqrt{c}}\begin{pmatrix}\frac{\bar\omega}{a}&0\\ 0&\omega\end{pmatrix}\begin{pmatrix}\varepsilon_1&\varepsilon_2\\ \varepsilon_3&\varepsilon_4\end{pmatrix},$$

where $\varepsilon_1, \varepsilon_2 \in \{a, \omega\}$, $\varepsilon_3, \varepsilon_4 \in \{a, \omega\}^{-1} = \frac{1}{a}\{a, \bar\omega\}$, and $\varepsilon_1\varepsilon_4 - \varepsilon_2\varepsilon_3 = 1$. From these relations we obtain

$$\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\sigma_N^{-1} = \frac{1}{\sqrt{c}}\begin{pmatrix}\frac{\bar\omega}{a}&0\\ 0&\omega\end{pmatrix}\begin{pmatrix}\varepsilon_1&\varepsilon_2\\ \varepsilon_3&\varepsilon_4\end{pmatrix}\begin{pmatrix}\alpha_0&\beta_0\\ \gamma_0&\delta_0\end{pmatrix} = \begin{pmatrix}\frac{\bar\omega}{\sqrt{c}}&0\\ 0&\frac{\sqrt{c}}{\bar\omega}\end{pmatrix}\begin{pmatrix}\alpha_1&\beta_1\\ \gamma_1&\delta_1\end{pmatrix},$$

where

$$\alpha_1 = \frac{1}{a}(\varepsilon_1\alpha_0 + \varepsilon_2\gamma_0) \in \frac{1}{a}\{a, \omega\} = \{a, \bar\omega\}^{-1},\quad \beta_1 = \frac{1}{a}(\varepsilon_1\beta_0 + \varepsilon_2\delta_0) \in \{a, \bar\omega\}^{-1},$$

$$\gamma_1 = a(\varepsilon_3\alpha_0 + \varepsilon_4\gamma_0) \in \{a, \bar\omega\},\qquad \delta_1 = a(\varepsilon_3\beta_0 + \varepsilon_4\delta_0) \in \{a, \bar\omega\}.$$

It is easy to check that for any full module M of K with coefficient ring $\mathfrak{O}_f$ the condition $\gamma\mathfrak{O}_f+\delta\mathfrak{O}_f=M$ for some pair $\gamma, \delta \in K$ is equivalent to the condition that $\gamma, \delta \in M$, and that there exist $\alpha, \beta \in M^{-1}$ such that $\alpha\delta - \beta\gamma = 1$. The lemma now follows from this remark and from the computations made above, and with it the proposition is proved.

PROPOSITION 3.6.4. *Let $F \in \mathfrak{M}_k^2$ and let D be a negative integer. We choose some N satisfying* (3.6.1) *and with* $(a, c) = 1$, *and we set*

$$F_D(u)=\widetilde{F}_N(\sigma_N^{-1}(u)) \qquad (u\in H), \tag{3.6.13}$$

where $\widetilde{F}_N$ is the function (3.6.3) *and σ_N is the matrix* (3.5.14). *Then*

(I) *$F_D(u)$ does not depend on the choice of N with the conditions imposed*;

(II) *$F_D(\sigma(u)) = F_D(u)$ for every $\sigma \in \mathrm{SL}_2(\mathfrak{O}_f)$, and $u \in H$*;

(III) *if F is a parabolic form, then $F_D(u)$ is bounded on H.*

PROOF. Let N and N_1 be two matrices with these properties, and let M be the matrix (3.5.17). Using the definition of $\widetilde{F}_N(u)$, the assertions (II) and (IV) of Proposition 3.5.2, and the relation (3.6.5), we obtain

$$\widetilde{F}_N(\sigma_N^{-1}(u))=\widetilde{F}((h^N)^{-1}(\sigma_N^{-1}(u)))=\widetilde{F}(M\langle(h^{N_1})^{-1}(\sigma_{N_1}^{-1}(u))\rangle)=$$
$$=\widetilde{F}((h^{N_1})^{-1}(\sigma_{N_1}^{-1}(u))=\widetilde{F}_{N_1}(\sigma_{N_1}^{-1}(u)),$$

which proves (I); now (II) follows from (3.6.4) and (3.6.10); and (III) follows from part (II) of Proposition 3.6.1.

THEOREM 3.6.1. *Let $F \in \mathfrak{N}_k^2$ and let D be a negative integer. We represent D in the form $D = df^2$, where d is the discriminant of the field $K = \mathbf{Q}(\sqrt{D})$. We denote by (N_i) (for $i = 1, \dots, h = h(D)$) (respectively, (M_j), for $j = 1, \dots, h = h(D)$)) a complete system of representatives of the equivalence classes in the narrow sense of positive definite primitive matrices $N \in \mathfrak{N}_2$ with discriminant D (respectively, of the similarity classes of modules of K with coefficient ring $\mathfrak{O}_f$). Then for any character χ of the group $H(D)$ we have the identity*

$$(4\pi)^{-s}\left(\frac{|D|}{4}\right)^{\frac{1-s}{2}}\Gamma(s)\sum_{i=1}^{h}\chi(N_i)R_{N_i}(s)=$$
$$=\int_{D_f}\left\{\sum_{j=1}^{h}\overline{\chi(M_j)}\,E^{M_j}(u, s-k+2)\right\}F_D(u)\,du, \tag{3.6.14}$$

where $R_{N_i}(s)$ are the series (3.3.1), *$F_D(u)$ is the function* (3.6.13), *for every module M of K with coefficient ring $\mathfrak{O}_f$*

$$E^M(u, s)=\frac{1}{2}(N(M)v)^s\sum_{\substack{\gamma,\,\delta\in K,\\ \gamma\mathfrak{O}_f+\delta\mathfrak{O}_f=M}}\Delta_{(\gamma,\,\delta)}^{-s}(u) \qquad (u\in H,\ \mathrm{Re}\, s>2), \tag{3.6.15}$$

D_f is a fundamental domain of $\mathrm{SL}_2(\mathfrak{O}_f)$ on H, and du is the measure (3.4.19). *The left-hand side of* (3.6.15) *is absolutely convergent in the*

domain Re $s > k + 1$, *and the right-hand side in* Re $s > k$.

PROOF. If the representatives $N_i = \begin{pmatrix} a_i & b_i/2 \\ b_i/2 & c_i \end{pmatrix}$ satisfy the condition $(a_i, c_i) = 1$, then from Proposition 3.6.2, the relations (3.6.11) and (3.6.12), and Proposition 3.6.4, we infer that the left-hand side of (3.6.14) is equal to

$$\int_{D_f} \left\{ \sum_{i=1}^{h} \chi(N_i) E^{\overline{M(N_i)}}(u,\ s-k+2) \right\} F_D(u) \tag{3.6.16}$$

where $M(N_i) = \{a_i, \frac{1}{2}(b_i - \sqrt{D})\}$ is the module of K corresponding to the binary form with matrix N_i (see §2.3).

According to §2.3, the set of $(\overline{M(N_i)})$ (for $i = 1, \ldots, h(D)$) is a complete system of representatives for the classes of $H(D)$ and $\chi(N_i) = \chi(M(N_i)) = \bar{\chi}(\overline{M(N_i)})$. Since every series $E^M(u, s)$ obviously only depends on the class of the module M, the sum on the right-hand side of (3.6.14) does not depend on the choice of the system of representatives (M_i). Replacing the (M_j) by $(\overline{M(N_j)})$, we see that the right-hand side of (3.6.14) is equal to (3.6.16).

Let us now get rid of the condition $(a_i, c_i) = 1$. Since (by (1.1.10)) the left-hand side of (3.6.14) does not depend on the choice of the system of representatives (N_i), it is sufficient for this purpose to remark that each integral primitive non-degenerate binary quadratic form $ax^2 + bxy + cy^2$ is equivalent in the narrow sense to a form $a'x_1^2 + b'x_1y_1 + c'y_1^2$ with $(a', c') = 1$. It is easy to see that the required form can be obtained by a change of variables of the form $x = x_1 + l$, $y = y_1$, for a suitable integer l. This proves the theorem.

§3.7. Eisenstein series and theta-series

In this section we obtain an integral representation of the Eisenstein series of the previous section in terms of suitable theta-series.

Let K be an imaginary quadratic field. For every pair M_1, M_2 of full modules of K and every real $t > 0$ and $u = (z, v) \in H$ we set

$$\Theta_{M_1, M_2}(t, u) = \sum_{(\gamma, \delta) \in M_1 \times M_2} \exp\left(-\frac{\pi t}{v} \Delta_{(\gamma, \delta)}(u)\right), \tag{3.7.1}$$

where $\Delta_{(\gamma, \delta)}(u) = |\gamma z + \delta|^2 + |\gamma|^2 v^2$.

PROPOSITION 3.7.1. *The following properties hold for the theta-series* $\Theta_{M_1, M_2}(t, u)$:

(I) *for any fixed* $u = (z, v) \in H$ *the series* (3.7.1) *is absolutely and uniformly convergent in any domain of the form* $t \geqslant \varepsilon$ (*with* $\varepsilon > 0$);

(II) *for every* $t > 0$ *and* $u \in H$,

$$(3.7.2)\qquad \Theta_{M_1, M_2}(t, u) \leqslant \left[1 + c'\left(\frac{|z|^2+v^2+1}{tv}\right)^{1/2} \exp\left(-\frac{ctv}{(|z|^2+v^2+1)}\right)\right]^4,$$

where the positive constants c and c' depend only on M_1 *and* M_2.

(III) (the inversion formula) *for every* $t > 0$ *and* $u \in H$

$$(3.7.3)\qquad \Theta_{M_1, M_2}(t, u) = \frac{1}{t^2|\Delta(M_1)||\Delta(M_2)|}\Theta_{M_2^*, M_1^*}\left(\frac{1}{t}, u\right),$$

where for every module M with the **Z**-*basis* (α, β) *we set*

$$(3.7.4)\qquad \Delta(M) = \frac{1}{2}\det\begin{pmatrix}\alpha & \beta\\ \overline{\alpha} & \overline{\beta}\end{pmatrix}, \quad M^* = \frac{1}{\Delta(M)}\overline{M}.$$

The theorem is proved using standard methods in the theory of theta-series. We merely indicate the main ideas. (I) is obvious, since the series (3.7.1) is, in fact, the theta-series of some positive definite quaternionic quadratic form (depending on u). From the inequality

$$\Delta_{(\gamma,\delta)}(u) \geqslant \frac{v^2}{|z|^2+v^2+1}(|\gamma|^2+|\delta|^2)$$

it follows that the series (3.7.1) is majorized by the fourth power of the standard theta-function $\Theta(\alpha) = \sum \exp(-\pi\alpha n^2)$ with $\alpha = ctv(|z|^2+v^2+1)^{-1}$, and c only depending on M_1 and M_2. Combining this with the estimate $|\Theta(\alpha) - 1| < c_1\alpha^{-1/2}\exp\left(-\frac{\pi\alpha}{2}\right)$ (for $\alpha > 0$), which follows from the inversion formula for $\Theta(\alpha)$, we obtain (II). The inversion formula (3.7.3) can be proved directly on the basis of the Poisson summation formula, or can be deduced from the inversion formula for the theta-series of quadratic forms (see [17], Proposition 23).

Let us now compute the Mellin transform of the theta-series (3.7.1). For subsequent applications it is enough to restrict ourselves to the case $M_1 = M_2 = M$. We use the notation of §2.3: $K = \mathbf{Q}(\sqrt{D})$ is an imaginary quadratic field of discriminant d, $\mathfrak{O}_f$ is the subring of index f in the ring of integers $\mathfrak{O}$ of K; for a full module M of K we denote by $\mathfrak{O}_M$ its coefficient ring and by $N(M)$ its norm.

PROPOSITION 3.7.2. *Let M be a full module of K with* $\mathfrak{O}_M = \mathfrak{O}_f$. *Suppose that* $M \subset \mathfrak{O}_f$ *and* $(N(M), f) = 1$. *Then*

$$M \times M - (0, 0) = \bigcup_{f'|f} f' \bigcup_{\substack{M' \subset \mathfrak{O}_{f/f'}M \\ \mathfrak{O}_{M'} = \mathfrak{O}_{f/f'},\ (N(M'), f/f') = 1}} X^{f/f'}(M'),$$

where for every $l \geqslant 1$ *and every module M such that* $M \subset \mathfrak{O}_l$ *and* $\mathfrak{O}_l \subset \mathfrak{O}_M$ *we set*

$$X^l(M) = \{(\gamma, \delta) \in K \times K;\ \gamma\mathfrak{O}_l + \delta\mathfrak{O}_l = M\}.$$

We begin by proving two lemmas.

LEMMA 3.7.1. *Let M be a full module of K. Suppose that* $M \subset \mathfrak{O}_f$ *and* $\mathfrak{O}_f \subset \mathfrak{O}_M$. *Then the conditions*

$$(3.7.5)\qquad (N(M)\ f) = 1,$$

and

$$(3.7.6) \qquad M + f\mathfrak{O} = \mathfrak{O}_f$$

are equivalent, and if they are satisfied, then $\mathfrak{O}_M = \mathfrak{O}_f$.

PROOF. Let $\mathfrak{O} = \{1, \omega\}$. Then $\mathfrak{O}_f = \{1, f\omega\}$. Since $M\overline{M} = N(M)\mathfrak{O}_M$, we can find α_i, $\beta_i \in M$ such that $\sum \alpha_i \overline{\beta}_i = N(M)$. Let $\beta_i = a_i + b_i f\omega$ with a_i and $b_i \in \mathbf{Z}$). If $(N(M), f) = 1$, then we can find c and $d \in \mathbf{Z}$ such that $cN(M) + df = 1$, hence

$1 = c\sum \alpha_i \overline{\beta}_i + df = c\sum \alpha_i a_i + f(d + c\overline{\omega} \sum \alpha_i \beta_i) \in M + f\mathfrak{O}$, from which (3.7.6) follows. Conversely, from (3.7.6) it follows that we can find $\alpha \in M$ and $\beta \in \mathfrak{O}$ such that $\alpha + f\beta = 1$, hence $N(\alpha) \equiv 1 \pmod f$. In particular $(N(\alpha), f) = 1$. Since $N(\alpha) \equiv 0 \pmod{N(M)}$, this proves (3.7.5).

Suppose then that (3.7.5) and (3.7.6) are satisfied. Since $M + f\mathfrak{O} = \mathfrak{O}_f$, we have $\overline{M} + f\mathfrak{O} = \mathfrak{O}_f$, hence $M\overline{M} + f^2\mathfrak{O} + f(\mathfrak{O}M + \mathfrak{O}\overline{M}) = \mathfrak{O}_f$, so that $M\overline{M} \subset \mathfrak{O}_f$. But $M\overline{M} = N(M)\mathfrak{O}_M$. Since $(N(M), f) = 1$, it follows from $N(M)\mathfrak{O}_M \subset \mathfrak{O}_f$ that $\mathfrak{O}_M \subset \mathfrak{O}_f$ and the lemma is proved.

LEMMA 3.7.2. *Let M be a full module of K. Suppose that* $\mathfrak{O}_M = \mathfrak{O}_f$, $M \subset \mathfrak{O}_f$ *and that* $(N(M), f) = 1$. *Then for any divisor* f' *of* f *and any number* $\gamma \in \mathfrak{O}_{f/f'}$ *the following two conditions are equivalent*:

$$(3.7.7) \qquad f'\gamma \in M,$$

$$(3.7.8) \qquad \gamma \in \mathfrak{O}_{f/f'} M.$$

PROOF. By Lemma 3.7.1 there are $\alpha \in M$ and $\beta \in \mathfrak{O}$ such that $\alpha + f\beta = 1$, hence if (3.7.7) holds, we obtain

$\gamma = \gamma\alpha + (f/f')\beta \times f'\gamma \in \mathfrak{O}_{f/f'}M + \mathfrak{O}_{f/f'} M = \mathfrak{O}_{f/f'} M$. Conversely, since $f'\mathfrak{O}_{f/f'} \subset \mathfrak{O}_f$, from (3.7.8) we have $f'\gamma \in f'\mathfrak{O}_{f/f'} M \subset \mathfrak{O}_f M \subset M$, and the lemma is proved.

PROOF OF PROPOSITION 3.7.2. For every pair $(\gamma, \delta) \neq (0, 0)$ of integers of K and every natural number f we denote by (γ, δ, f) the greatest natural number that is a common divisor of γ, δ and f. Let

$$(\gamma, \delta) \in M \times M - (0, 0), \quad f' = (\gamma, \delta, f), \quad (\gamma', \delta') = \frac{1}{f'}(\gamma, \delta).$$

We set $M' = \mathfrak{O}_{f/f'}\gamma' + \mathfrak{O}_{f/f'}\delta'$. Obviously, $\mathfrak{O}_{f/f'} \subset \mathfrak{O}_{M'}$. By Lemma 3.7.2 $M' \subset \mathfrak{O}_{f/f'} M$. Since $(\gamma', \delta', f/f') = 1$ and $\gamma', \delta' \in \mathfrak{O}_{f/f'}$, we can find rational integers a, b and $\beta \in \mathfrak{O}$ such that $a\gamma' + b\delta' + (f/f')\beta = 1$. Hence $M' + (f/f')\mathfrak{O} = \mathfrak{O}_{f/f'}$. Then by Lemma 3.7.1 we see that $(N(M'), f/f') = 1$ and $\mathfrak{O}_{M'} = \mathfrak{O}_{f/f'}$. Thus, the left-hand side of the equality to be proved is contained in the right-hand side. Conversely, let $(\gamma', \delta') \in X^{f/f'}(M')$, with $M' \subset \mathfrak{O}_{f/f'} M$. We set $(\gamma, \delta) = (f'\gamma', f'\delta')$. Then by Lemma 3.7.3, $(\gamma, \delta) \in M \times M$. This proves the proposition.

PROPOSITION 3.7.3. *Let M be a full module of the imaginary quadratic field K, and* $\mathfrak{O}_M = \mathfrak{O}_f$. *Suppose that* $M \subset \mathfrak{O}_f$ *and* $(N(M), f) = 1$. *Then* *the Mellin integral of the function* $\Theta_{M,M}(t, u) - 1$:

$$\int_0^\infty (\Theta_{M,M}(t, u)-1)\, t^{s-1}\, dt$$

is absolutely convergent in the domain ($\operatorname{Re} s > 2$, $u \in H$) *and in this domain it is equal to the expression*

$$2\pi^{-s}\Gamma(s)\sum_{f'|f}(f')^{-2s}\sum_{\substack{M'\subset\mathfrak{O}_{f/f'}M,\\ \mathfrak{O}_{M'}=\mathfrak{O}_{f/f'},\,(N(M'),\,f/f')=1}} N(M')^{-s}E^{M'}(u, s),$$

where $\Gamma(s)$ *is the gamma-function and* $E^{M'}(u, s)$ *the Eisenstein series* (3.6.15).

PROOF. The convergence follows from the estimate (3.7.2). Since

$$\int_0^\infty \exp\left(-\frac{\pi t}{v}\Delta_{(\gamma,\delta)}(u)\right)t^{s-1}\,dt = \pi^{-s}\Gamma(s)\,v^s\Delta^{-s}_{(\gamma,\delta)}(u),$$

and, by definition

$$v^s\sum_{(\gamma,\delta)\in X^l(M')}\Delta^{-s}_{(\gamma,\delta)}(u) = 2N(M')^{-s}E^{M'}(u, s),$$

the formula follows from Proposition 3.7.2.

Before going further, let us convince ourselves that there is no loss of generality in assuming that $(N(M), f) = 1$.

LEMMA 3.7.3. *In each class of the group* $H(df^2)$ *there is a module* M *for which* $M \subset \mathfrak{O}_f$ *and* $(N(M), f) = 1$.

PROOF. In the language of quadratic forms (see §2.3) this means that every primitive integral positive definite binary quadratic form $ax^2 + bxy + cy^2$ of discriminant $b^2 - 4ac = df^2$ represents some number coprime to f, which is obvious.

THEOREM 3.7.1. *Let* K *be an imaginary quadratic form of discriminant* d, *let* f *be a natural number, and let* $M_1, \ldots, M_h$ (where $h = h(df^2)$) *be a complete set of representatives of the similarity classes of modules of* K *with coefficient ring* $\mathfrak{O}_f$. *Let* χ *be a character of the group* $H(df^2)$ *satisfying the following condition:*

(3.7.9) *for every* $f' > 1$ *such that* $f'|f$ *the character* χ *is non-trivial on the kernel of the epimorphism* $\nu(f, f/f')$: $H(df^2) \to H(d(f/f')^2)$.

Then in the domain ($\operatorname{Re} s > 2$, $u \in H$) *the following identity holds:*

$$(3.7.10)\quad 2\pi^{-s}\Gamma(s)\,L_{df^2}(s, \chi)\sum_{i=1}^h\bar{\chi}(M_i)\,E^{M_i}(u, s) =$$

$$= \int_0^\infty\left\{\sum_{i=1}^h\bar{\chi}(M_i)\,N(M_i)^s\,(\Theta_{M_i,M_i}(t, u)-1)\right\}t^{s-1}\,dt,$$

where the $E^{M_i}(u, s)$ *are the Eisenstein series* (3.6.15), $\Theta_{M_i,M_i}(t, u)$ *are the theta-series* (3.7.1), *and* $L_{df^2}(s, \chi)$ *is the L-series of the order* $\mathfrak{O}_f$ *with* *character* χ, *defined in* §2.4. *In the domain in question the left- and right-hand sides of the identity* (3.7.10) *are absolutely convergent.*

PROOF. First of all we note that both sides of the identity (3.7.10) are independent of the choice of the system of representatives (M_i). For the left-hand side this is obvious, and for the right-hand side it follows easily from the identity $\Theta_{\alpha M,\alpha M}(t, u) = \Theta_{M,M}(|\alpha|^2 t, u)$ (for $\alpha \in K$) by a change of variables in the corresponding integrals. Thus, we may prove the identity for any special system of representatives. We choose the (M_i) in such a way that for every $i = 1, \dots, h$ the modules M_i satisfy the conditions $M_i \subset \mathfrak{O}_f$, and $(N(M), f) = 1$. This can be done by Lemma 3.7.3.

The absolute convergence of the right-hand side of the identity in ($\mathrm{Re}\, s > 2$, $u \in H$) follows from Proposition 3.7.3; the convergence of the left-hand side follows from the properties of Eisenstein series mentioned above.

Let M_1 and M_2 be modules of K such that $M_1, M_2 \subset \mathfrak{O}_l$, $\mathfrak{O}_{M_1} = \mathfrak{O}_{M_2} = \mathfrak{O}_l$, and $(N(M_1), l) = (N(M_2), l) = 1$. Then the conditions $M' \subset M_1$, $M' \sim M_2$ and $(N(M'), l) = 1$ on a module M' are obviously equivalent to the following conditions on $M'' = N(M_1)^{-1}M'\overline{M}_1$: $M'' \subset \mathfrak{O}_l$, $M'' \sim M_2M_1^{-1}$ and $(N(M''), l) = 1$. Since $N(M') = N(M'')N(M_1)$ and the Eisenstein series $E^{M'}(u, s)$ depends only on the class of M', we now obtain the identity

$$\sum_{\substack{M'\subset M_1,\, M'\sim M_2,\\ (N(M'),\, l)=1}} N(M')^{-s} E^{M'}(u, s) = N(M_1)^{-s} \zeta_{dl^2}(s, M_2M_1^{-1}) E^{M_2}(u, s),$$

where $\zeta_{dl^2}(s, M) = \sum N(M'')^{-s}$ and the sum extends over all modules M'' for which $M'' \subset \mathfrak{O}_l$, $M'' \sim M$ and $(N(M''), l) = 1$.

Applying this identity to the inner sum on the right-hand side of the identity of Proposition 3.7.4 for $M = M_i$, we can rewrite it in the form

$$\int_0^\infty (\Theta_{M_i, M_i}(t, u) - 1) t^{s-1}\, dt =$$

$$= 2\pi^{-s}\Gamma(s) \sum_{f'|f} (f')^{-s} N(\mathfrak{O}_{f/f'}M_i)^{-s} \sum_{j=1}^{h'} \zeta_{d(f/f')^2}(s, M'_j(\mathfrak{O}_{f/f'}M_i)^{-1}) E^{M'_j}(u, s),$$

where the M'_j (for $j = 1, \dots, h' = h(d(f/f')^2)$) ranges over a complete set of representatives of the classes of $H(d(f/f')^2)$ chosen such that $M'_j \subset \mathfrak{O}_{f/f'}$ and $(N(M'_j), f/f') = 1$ for every j. Since $N(\mathfrak{O}_{f/f'} M_i) = N(M_i)$, after multiplying both sides of the last identity by $\overline{\chi}(M_i)N(M_i)^s$ and summing over i from 1 to h, we obtain

$$\int_0^\infty \left\{ \sum_{i=1}^{h} \overline{\chi}(M_i) N(M_i)^s (\Theta_{M_i, M_i}(t, u) - 1) \right\} t^{s-1}\, dt =$$

$$= 2\pi^{-s}\Gamma(s)\sum_{f'|f}(f')^{-2s}\sum_{k,j=1}^{h'}\Big(\sum_{\substack{1\leqslant i\leqslant h,\\ \mathfrak{O}_{f/f'}M_i\sim M'_k}}\bar{\chi}(M_i)\Big)\zeta_{d(f/f')^2}(s, M'_j(M'_k)^{-1})E^{M'_j}(u, s),$$

where $h' = h(d(f/f')^2)$. By the condition (3.7.9) on χ the inner summation over i on the right-hand side of the relation just obtained is zero if $f' > 1$. Thus, extending the equality, we have

$$2\pi^{-s}\Gamma(s)\sum_{i,j=1}^{h}\bar{\chi}(M_i)\zeta_{df^2}(s, M_jM_i^{-1})E^{M_j}(u, s) =$$

$$= 2\pi^{-s}\Gamma(s)\Big\{\sum_{i=1}^{h}\chi(M_i)\zeta_{df^2}(s, M_i)\Big\}\Big\{\sum_{j=1}^{h}\bar{\chi}(M_j)E^{M_j}(u, s)\Big\}.$$

Finally, we note that for integral ideals M of $\mathfrak{O}_f$ such that $\mathfrak{O}_M = \mathfrak{O}_f$ and $(N(M), f) = 1$ the theorem on unique factorization into prime ideals holds It follows that

$$\sum_{\substack{M\subset\mathfrak{O}_f,\\ \mathfrak{O}_M=\mathfrak{O}_f,\ (N(M),f)=1}}\frac{\chi(M)}{N(M)^s} = \prod_{\mathfrak{P}}\Big(1-\frac{\chi(\mathfrak{P})}{N(\mathfrak{P})^s}\Big)^{-1},$$

where $\mathfrak{P}$ ranges over all the prime ideals of $\mathfrak{O}_f$ such that $\mathfrak{O}_{\mathfrak{P}} = \mathfrak{O}_f$ and $(N(\mathfrak{P}), f)=1$. The theorem is now proved.

§3.8. Proof of the main theorem for parabolic forms

Keeping to the notation and the assumptions of Theorem 3.1.1, suppose that F is a parabolic form and let $a(N)$ (for $N \in \mathfrak{N}_2$, $N > 0$) be the coefficients of its Fourier expansion (1.1.8). For every primitive $N \in \mathfrak{N}_2$ with $N > 0$ we denote by $R_N(s) = R_{N,F}(s)$ the Dirichlet series (3.3.1).

Since $F \not\equiv 0$, we can find an integer $D < 0$ with the following two properties:

(3.8.1) there exists a primitive matrix $N \in \mathfrak{N}_2$ with $N > 0$ and $D(N) = D$ such that $R_N(s) \not\equiv 0$;

(3.8.2) represent D in the form $D = df^2$, where d is the discriminant of the field $K = \mathbf{Q}(\sqrt{D})$; then for any integer $f' > 1$ with $f' \mid f$ and for any primitive matrix $N' \in \mathfrak{N}_2$ with $N' > 0$ and of discriminant $d(f/f')^2$ the series $R_{N'}(s)$ is identically zero.

We fix a number D with the properties (3.8.1) and (3.8.2) up to the end of this section, and keep to the notation d, f, K introduced in (3.8.2).

Let $N_1, \ldots, N_h$ (with $h = h(D)$) be a complete set of representatives of the equivalence classes in the narrow sense of primitive matrices $N \in \mathfrak{N}_2$ with $N > 0$ and of discriminant D. It follows from (3.8.1) that there exists a character χ of the group $H(D)$ such that

$$\sum_{i=1}^{h}\chi(N_i)R_{N_i}(s) \not\equiv 0. \qquad (3.8.3)$$

Let us write down the identity of Theorem 2.4.1 for the form F and a pair (D, χ) with the properties (3.8.1), (3.8.2) and (3.8.3). From the theorem on the identity of absolutely convergent Dirichlet series and from the condition (3.8.2) it follows that the function $\Phi_F(s, \chi)$ of Theorem 2.4.1 is a constant:

$$(3.8.4) \qquad \Phi_F(s, \chi) = \sum_{i=1}^{h} \chi(N_i)\, a(N_i) = a(\chi).$$

Thus, in this case the identity of Theorem 2.4.1 has the form

$$(3.8.5) \qquad L_D(s-k+2, \chi) \sum_{i=1}^{h} \chi(N_i)\, R_{N_i}(s) = a(\chi)\, Z_F(s).$$

Using this identity and Theorem 3.6.1 we obtain the integral representation

$$(3.8.6) \quad a(\chi)(4\pi)^{-s}\Gamma(s) Z_F(s) =$$

$$= \left(\frac{|D|}{4}\right)^{\frac{s-1}{2}} \int_{D_f} L_D(s-k+2, \chi) \left\{ \sum_{j=1}^{h} \overline{\chi}(M_j) E^{M_j}(u, s-k+2) \right\} F_D(u)\, du,$$

where $\operatorname{Re} s$ is sufficiently large and the notation is as in Theorem 3.6.1.

As our next step we transform the expression under the integral sign on the right-hand side of (3.8.6), on the basis of the identity (3.7.10) of Theorem 3.7.1. First of all, let us make sure that we can do this.

LEMMA 3.8.1. *Suppose that D satisfies* (3.8.1) *and* (3.8.2) *and that χ is chosen so that* (3.8.3) *holds. Then χ satisfies the condition* (3.7.9) *of Theorem* 3.7.1.

PROOF. Suppose the contrary. Let χ be a character that is trivial on the kernel of the epimorphism $\nu(f, f/f')$ for some $f' > 1$ with $f' \mid f$. Since the same is then true for any prime divisor p of f', we may assume that $f' = p$ is a prime. Then χ is induced by some character χ' of $H(d(f/p)^2)$. Let $N_1', \ldots, N_{h'}'$ (with $h' = h(d(f/p)^2)$) be a complete set of representatives of the classes of primitive matrices of discriminant $d(f/p)^2$. Then for every $m \geqslant 1$ we have

$$(3.8.7) \quad \sum_{i=1}^{h} \chi(N_i)\, a(mN_i) = \sum_{j=1}^{h'} \chi'(N_j') \sum_{\substack{1 \leqslant i \leqslant h, \\ \nu(f, f/p)\{N_i\} = \{N_j'\}}} a(mN_i),$$

where $\{N\}$ denotes the class of N.

We claim that for every $j = 1, \ldots, h'$ and $m \geqslant 1$ the inner sum on the right-hand side of (3.8.7) is zero. To see this we compute in two ways the Fourier coefficient with the suffix pmN_j' of $T_k(p)F$, where $T_k(p)$ is the pth Hecke operator. On the one hand, since $T_k(p)F = \lambda_F(p)F$, by (3.8.2), this coefficient is equal to $\lambda_F(p)a(pmN_j') = 0$. On the other hand, using Proposition 2.1.2 and (3.8.2), we find that it is equal to

$$p^{2k-3}a(mN_j') + p^{k-2}(\Pi(p)a)(pmN_j') + a(p^2mN_j') = p^{k-2}(\Pi(p)a)(pmN_j').$$

We now write out this expression by the formula of Theorem 2.3.2 (this is the only place where we use this formula). It follows from (3.8.2) that the term (A) in this formula is zero, so that

$$(\Pi(p)a)(pmN_j') = \frac{e_f}{e_{f/p}} \sum_{\substack{1\leq i\leq h,\\ \nu(f,f/p)\{N_i\}=\{N_j'\}}} a(mN_i).$$

It follows from what we have said that the sum (3.8.7) is zero for all $m \geqslant 1$, which contradicts the condition (3.8.3) and proves the lemma.

Combining the identity (3.8.6) with the identity (3.7.10) of Theorem 3.7.1, we obtain

$$(3.8.8)\quad a(\chi)(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z_F(s) =$$

$$= \frac{\pi^{2-k}}{2}\left|\frac{D}{4}\right|^{\frac{s-1}{2}} \int_{D_f}\left\{\int_0^\infty \sum_{j=1}^{h} \overline{\chi}(M_j)N(M_j)^{s-k+2}(\Theta_{M_j,M_j}(t,u)-1)t^{s-k+1}dt\right\}F_D(u)\,du.$$

Now we transform the inner integral in the right-hand side of (3.8.8), using the inversion formula (3.7.3). For every full module M of K, let $\Delta(M)$ and M^* be defined by the equations (3.7.4). Then it easily follows from the definitions that

$$(3.8.9)\qquad |\Delta(M)| = N(M)\frac{\sqrt{|D|}}{2}, \quad |\Delta(M)|\,|\Delta(M^*)| = 1,$$

where D is the discriminant of the coefficient ring of the module M. Hence in the case $M_1 = M_2 = M$ the inversion formula (3.7.3) can be re-written in the form

$$|\Delta(M)|(\Theta_{M,M}(t,u)-1) = \frac{1}{t^2}|\Delta(M^*)|(\Theta_{M^*,M^*}(t,u)-1) + \frac{|\Delta(M^*)|}{t^2} - |\Delta(M)|.$$

Using this formula for $M = M_j$, we obtain

$$\int_0^\infty |\Delta(M_j)|(\Theta_{M_j,M_j}(t,u)-1)t^{s-k+1}\,dt =$$

$$= \int_1^\infty |\Delta(M_j)|(\Theta_{M_j,M_j}(t,u)-1)t^{s-k+1}\,dt +$$

$$+ \int_1^\infty |\Delta(M_j)|\left(\Theta_{M_j,M_j}\left(\frac{1}{t},u\right)-1\right)t^{k-s-3}\,dt =$$

$$= \int_1^\infty |\Delta(M_j)|(\Theta_{M_j,M_j}(t,u)-1)t^{s-k+1}\,dt +$$

$$+ \int_1^\infty |\Delta(M_j^*)|(\Theta_{M_j^*,M_j^*}(t,u)-1)t^{k-s-1}\,dt +$$

$$+ \int_1^\infty |\Delta(M_j^*)|t^{k-s-1}\,dt - \int_1^\infty |\Delta(M_j)|t^{k-s-3}\,dt.$$

Computing the last two integrals, substituting the expressions obtained in (3.8.8) taking account of (3.8.9) and the obvious relation $\overline{\chi}(M_j) = \chi(M_j^*)$ (since M and M^* belong to reciprocal classes), we finally find the following integral representation: if Re s is sufficiently large, then

(3.8.10) $$2\pi^{k-2}\left|\frac{D}{4}\right|^{\frac{3-k}{2}} a(\chi)\,\Psi_F(s) =$$

$$= \int_{D_f}\left\{\int_1^\infty \sum_{j=1}^h \overline{\chi}(M_j)\,|\Delta(M_j)|^{s-k+2}(\Theta_{M_j, M_j}(t, u)-1)\,t^{s-k+1}\,dt\right\} F_D(u)\,du +$$

$$+ \int_{D_f}\left\{\int_1^\infty \sum_{j=1}^h \chi(M_j^*)\,|\Delta(M_j^*)|^{k-s}(\Theta_{M_j^*, M_j^*}(t, u)-1)\,t^{k-s-1}\,dt\right\} F_D(u)\,du +$$

$$+\frac{1}{k-2-s}\left\{\sum_{j=1}^h \overline{\chi}(M_j)\,|\Delta(M_j)|^{s-k+2}\right\}\int_{D_f} F_D(u)\,du +$$

$$+\frac{1}{s-k}\left\{\sum_{j=1}^h \chi(M_j^*)\,|\Delta(M_j^*)|^{k-s}\right\}\int_{D_f} F_D(u)\,du.$$

We claim that the first two integrals in (3.8.10) are absolutely convergent for all s and are therefore entire functions of s. To see this it is enough to show that for every full module M of K with coefficient ring $\mathfrak{O}_f$ the integral

$$\int_{D_f}\left\{\int_1^\infty |\Theta_{M, M}(t, u)-1|\,t^s\,dt\right\}|F_D(u)|\,du$$

is finite for all s. From (3.7.2) it follows easily that the inner integral is finite for all s, and if $v \to 0$ (respectively, ∞) and $|z|$ is bounded, it tends to infinity not faster than v^{-c} (respectively, v^c), where c is a positive constant depending on M and s. The fundamental domain D_f is a union of a compact set and finitely many neighbourhoods of parabolic vertices, that is, points where D_f goes out to the boundary of H at inequivalent parabolic fixed points of the group $SL_2(\mathfrak{O}_f)$ (see [22]). Since F is a parabolic form, it easily follows from the definition of $F_D(u)$ that, as u tends from within D_f to one of its parabolic vertices, $F_D(u)$ tends to zero like $\exp\left(-\frac{c'}{v}\right)$ (respectively, $\exp(-c'v)$) if this vertex lies in the plane $v = 0$ (respectively at infinity). Thus, the function

$$|F_D(u)|\int_1^\infty |\Theta_{M, M}(t, u)-1|\,t^s dt$$

is bounded on D_f for all s. Since D_f has finite invariant volume, the relevant integral is indeed finite for all s.

Thus, the representation (3.8.10) gives a meromorphic analytic continuation of $\Psi_F(s)$ to the whole complex plane. This function is regular at all points, except possibly for simple poles at $s = k$ and $s = k - 2$. If at these points there are poles, then the residues there are obviously equal to

$$(3.8.11)\qquad \pm\frac{\pi^{2-k}}{2}\left|\frac{D}{4}\right|^{\frac{k-3}{2}} a(\chi)^{-1}\left\{\sum_{j=1}^{h}\chi(M_j)\right\}\int_{D_j} F_D(u)\,du\ .$$

for any pair D, χ satisfying (3.8.1)–(3.8.3).

Now we prove the functional equation. Since by (3.8.9) $(M_j^*)^* = M_j$, substituting into the right-hand side of (3.8.10) the system of representatives (M_j^*) for (M_j), $\bar{\chi}$ for χ and $2k-2-s$ for s, we see that this does not change it. Hence,

$$(3.8.12)\qquad a(\chi)\Psi_F(s) = a(\bar{\chi})\Psi_F(2k-2-s).$$

From the definition of the group structure on the set of classes of primitive positive definite matrices $N \in \mathfrak{N}_2$ with fixed discriminant (see §2.3) it follows that the matrices

$$N = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix} \text{ and } N' = \begin{pmatrix} a & -b/2 \\ -b/2 & c \end{pmatrix} = {}^tUNU \left(U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$

belong to reciprocal classes (since the modules corresponding to them are conjugate). From (1.1.10) we see that $a(N') = a({}^tUNU) = (-1)^k a(N)$, hence

$$(3.8.13)\qquad a(\bar{\chi}) = \sum_{i=1}^{h}\bar{\chi}(N_i)\,a(N_i) = \sum_{i=1}^{h}\bar{\chi}(N_i')\,a(N_i') = \\ = (-1)^k\sum_{i=1}^{h}\chi(N_i)\,a(N_i) = (-1)^k a(\chi).$$

Since by (3.8.3) and (3.8.5), $a(\chi) \neq 0$, we deduce from (3.8.12) and (3.8.13) the functional equation $\Psi_F(2k-2-s) = (-1)^k\Psi_F(s)$.

Finally, from $a(\chi) \neq 0$ and (3.8.13) we see that $\chi \neq \bar{\chi}$ for odd k, so that χ is not identity character, and then the residues (3.8.11) at the possible poles are zero and $\Psi_F(s)$ is an entire function. Theorem 3.1.1 is now proved.

Conclusion

We give here a list of some open problems that arise naturally in connection with the theory we have developed.

1. Let $F \in \mathfrak{N}_k^2$ be a parabolic form of genus 2 and weight k, which is an eigenfunction of all the Hecke operators $T_k(m)$ (for $m = 1, 2, \ldots$). Is F determined by its eigenvalues.

2. For some positive integer k we denote by $\mathfrak{Z}_k^2$ the space of Dirichlet series whose coefficients have not more than polynomial growth and which satisfy the two following conditions:

(I) for every $Z(s) \in \mathfrak{Z}_k^2$, the function $\Psi(s) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-k+2)Z(s)$ can be continued analytically as an entire function to the whole s-plane;

(II) the functional equation $\Psi(2k-2-s) = (-1)^k\Psi(s)$ holds.

Is it then true that dim $\mathfrak{Z}_k^2 < \infty$? Does $\mathfrak{Z}_k^2$ have a basis consisting of series with an Euler product expansion? Do these Euler products have the form $Z_F(s)$ for some $F \in \mathfrak{M}_k^2$?

3. Theorem 1.2.4 and Theorem 3.1.1 and the Zharkovskaya relation (Theorem 1.3.3) suggest the following conjecture on the analytical properties of the zeta-function $Z_F(s)$ of a modular form of genus n.

Let $F \in \mathfrak{M}_k^n$ (for n and $k \geqslant 1$) be an eigenfunction of all the Hecke operators. We set

$$\Psi_F(s)=(2\pi)^{-2^n s}\, \gamma_{n,k}(s) Z_F(s),$$

where $Z_F(s)$ is the zeta-function (1.3.21) and the $\gamma_{n,k}(s)$ are defined by the relations

$$\gamma_{1,k}(s)=\Gamma(s), \quad \gamma_{n,k}(s)=\gamma_{n-1,k}(s)\gamma_{n-1,k}(s-k+n) \quad (n>1).$$

Then $\Psi_F(s)$ can be continued analytically to the whole s-plane as a meromorphic function with finitely many poles; the functional equation

$$\Psi_F\left(nk-\frac{n(n+1)}{2}+1-s\right)=(-1)^{k\cdot 2^{n-2}}\, \Psi_F(s)$$

holds; if F is a parabolic form, then $\Psi_F(s)$ is entire.

From Theorem 3.1.1 and Theorem 1.3.3 it follows that these assertions hold if $\Phi^{n-2}F \neq 0$, where Φ is the Siegel operator. It follows from Theorem 1.3.3 that it is enough to prove them for parabolic forms.

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THE METHOD OF DIAGRAMS IN PERTURBATION THEORY

E. B. Gledzer and A. S. Monin

In this paper the mathematical methods of quantum field theory are applied to some problems that arise in the statistical description of mechanical systems with very many (in the idealized case, infinitely many) degrees of freedom.

This application is based on a graphical representation of the individual terms of the formal perturbation series in powers of the coupling constant in the form of Feynman diagrams.

A variety of properties of such diagrams makes it possible to sum partially the perturbation series with a view to obtaining closed integral equations that contain the required quantities as unknowns. The approach is treated in more detail in connection with the statistical hydrodynamics of a developed turbulent flow, which is similar to the theory of a quantum Bose field with strong interaction.

The functional formulation of statistical hydrodynamics makes it possible to obtain integral equations of turbulence theory, which can also be derived by means of diagram methods. At the end of the paper, some closed equations of statistical hydrodynamics are considered.

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