Pure and Applied Mathematics Quarterly
Volume 4, Number 4
(*Special Issue: In honor of
Jean–Pierre Serre, Part 1 of 2*)
1—32, 2008

p-adic Banach Modules of Arithmetical Modular Forms and Triple Products of Coleman's Families

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To dear Jean-Pierre Serre for his eightieth birthday with admiration

Abstract: For a prime number $p \geq 5$, we consider three classical cusp eigenforms

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} e(nz) \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3)$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters $\psi_j \bmod N_j$.
According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j $(j = 1, 2, 3)$ (under suitable assumptions on p and on f_j)

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j}) q^n\}$$

into a p-adic analytic family of cusp eigenforms f_{j,k_j} of weights k_j in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain p-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.
The conductor of the members of the families divides the prime-to-p part of the level $N = \mathrm{LCM}(N_1, N_2, N_3)$.
The purpose of this paper is to describe a four variable p-adic L function attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k) q^n \right\}$$

Received July 26, 2006.
Based on a talk for the French-German Seminar in Lille on November 9, 2005

of cusp eigenforms of three fixed slopes $\sigma_j = v_p(\alpha^{(1)}_{p,j}(k_j)) \geq 0$ where $\alpha^{(1)}_{p,j} = \alpha^{(1)}_{p,j}(k_j)$ is an eigenvalue (which depends on k_j) of Atkin's operator $U = U_p$ acting on Fourier expansions by $U(\sum_{n\geq 0}^{\infty} a_n q^n) = \sum_{n\geq 0}^{\infty} a_{np} q^n$.
Let us consider the product of three eigenvalues:

$$\lambda = \lambda(k_1, k_2, k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$$

and assume that the slope of this product

$$\sigma = v_p(\lambda(k_1, k_2, k_3)) = \sigma(k_1, k_2, k_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets (k_1, k_2, k_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1, k_2, k_3). The each value σ_j is fixed.
We consider the p-adic weight space X containing all (k_j, ψ_j). Our p-adic L-functions are Mellin transforms of certain measures with values in $\mathcal{A}$, where $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denotes an affinoid algebra associated with an affinoid space $\mathcal{B}$ as in [CoPB], where $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integers k_j and fixed Dirichlet characters $\psi_j \bmod N$).
We construct such a measure from higher twists of classical Siegel-Eisenstein series, which produce distributions with values in certain Banach $\mathcal{A}$-modules $\mathcal{M} = \mathcal{M}(N; \mathcal{A})$ of triple modular forms with coefficients in the algebra $\mathcal{A}$.

Acknowledgement. A part of these results was exposed in a talk at the French-German Seminar in Lille and the author is grateful to V.Gritsenko for the invitation.

It is a great pleasure for me to thank S.Boecherer, C.-G.Schmidt and R.Schulze-Pillot for valuable discussions.

Details of computations and proofs will appear elsewhere (a joint article with S.Boecherer in preparation for a special volume in the Contemporary Math. series of the AMS).

Contents

1. Introduction

Why study L-values attached to modular forms? A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$ $\rightsquigarrow$ Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$ $\rightsquigarrow$ A number (solution)

Example 1 [Chand70]: (Hardy-Ramanujan)

$$p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3),$$
$$\lambda_n = \sqrt{n-1/24},$$

$\uparrow$ Good bases, finite dimensions, many relations and identities

$\uparrow$ Values of L-functions, periods, congruences, ...

Other examples: Birch and Swinnerton-Dyer conjecture, ... L-values attached to modular forms

Our data: three primitive cusp eigenforms.

$$f_j(z) = \sum_{n=1}^{\infty} a_{n,j} q^n \in \mathcal{S}_{k_j}(N_j, \psi_j), \ (j = 1, 2, 3) \tag{1.1}$$

of weights k_1, k_2, k_3, of conductors N_1, N_2, N_3, and of nebentypus characters ψ_j mod N_j, $N = \mathrm{LCM}(N_1, N_2, N_3)$.

Let p be a prime, $p \nmid N$. We view $f_j \in \overline{\mathbb{Q}}[\![q]\!] \overset{i_p}{\hookrightarrow} \mathbb{C}_p[\![q]\!]$ via a fixed embedding $\overline{\mathbb{Q}} \overset{i_p}{\hookrightarrow} \mathbb{C}_p$, $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ is Tate's field.

Let χ denote a variable Dirichlet character $\bmod Np^v, v \geq 0$.

We view k_j as a variable weight in the weight space $X = X_{Np^v} = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$, $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^* \ni (y_0, y_p)$.

The space X is a p-adic analytic space first used in Serre's [Se73] *"Formes modulaires et fonctions zêta p-adiques"*. Denote by $(k, \chi) \in X$ the homomorphism $(y_0, y_p) \mapsto \chi(y_0)\chi(y_p \bmod p^v) y_p^k$. We write simply k_j for the couple $(k_j, \psi_j) \in X$.

The purpose of this paper is to describe a four variable p-adic L function attached to Garrett's triple product of three Coleman's families.

$$k_j \mapsto \left\{ f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k_j) q^n \right\}$$

of cusp eigenforms of three constant slopes $\sigma_j = \mathrm{ord}_p(\alpha_{p,j}^{(1)}(k_j)) \geq 0$ where $\alpha_{p,j}^{(1)}(k_j)$, $\alpha_{p,j}^{(2)}(k_j)$ are the Satake parameters given as inverse roots of the Hecke p–polynomial $1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1 - \alpha_{p,j}^{(1)}(p)X)(1 - \alpha_{p,j}^{(2)}(p)X)$. Each family of the associated primitive cusp eigenforms have conductors C_{k_j} which divide N if $k_j > 2\sigma_j + 2$ (see [CoM]).

We assume that $\mathrm{ord}_p(\alpha_{p,j}^{(1)}(k_j)) \leq \mathrm{ord}_p(\alpha_{p,j}^{(2)}(k_j))$ and $\alpha_{p,j}^{(1)}(k_j) \neq \alpha_{p,j}^{(2)}(k_j)$

Remark *of the referee: In the elliptic modular case, an equality is expected to*

never happen if the weight > 1 (and this fact is proven when the weight k is 2 and 3 by Coleman- Edixhoven and Ulmer under different conditions). However it happens in the Hilbert modular case, even when $k = 2$. Anyway once we assume that $\alpha_{p,j}^{(1)} \neq \alpha_{p,j}^{(2)}$ for the initial weight, this holds in an open neighborhood of the weight.

The present paper extends a previous result: (see [PaTV]) where a two variable p-adic L-function was constructed interpolating on all k a function $(k, s) \mapsto L^*(f_k, s, \chi)$ $(s = 1, \cdots, k-1)$ for such a family.

We use the theory of p-adic integration with values in spaces of nearly holomorphic modular forms (in the sense of Shimura, see [ShiAr]).

A family of slope $\sigma > 0$ of cusp eigenforms f_k of weight $k \geq 2$:

$k \mapsto f_k = \sum_{n=1}^{\infty} a_n(k) q^n$
$\in \overline{\mathbb{Q}}[\![q]\!] \subset \mathbb{C}_p[\![q]\!]$
A model example
of a p-adic family
(not cusp and $\sigma = 0$):
Eisenstein series
$a_n(k) = \sum_{d|n} d^{k-1}, f_k = E_k$

1) the Fourier coefficients $a_n(k)$ of f_k
and one of the Satake
p-parameters $\alpha(k) := \alpha_p^{(1)}(k)$
are given by certain p-adic analytic
functions $k \mapsto a_n(k)$ for $(n,p) = 1$

2) the slope is constant and positive:
$\mathrm{ord}(\alpha(k)) = \sigma > 0$

The existence of families of slope $\sigma > 0$ was established in [CoPB]

R.Coleman gave an example with
$p = 7$, $f = \Delta$, $k = 12$
$a_7 = \tau(7) = -7 \cdot 2392, \sigma = 1.$

A program in PARI for computing
such families is discussed in [CST98]
(see also the Web-page of W.Stein,
`http://modular.fas.harvard.edu/`)

It was established by Coleman that:

- The operator U acts as a completely continuous operator on each $\mathcal{A}$-submodule $\mathcal{M}^{\dagger}(Np^v; \mathcal{A}) \subset \mathcal{A}[\![q]\!]$ (i.e. U is a limit of finite-dimensional operators)

$\Longrightarrow$ there exists the Fredholm determinant $P_U(T) = \det(Id - T \cdot U) \in \mathcal{A}[\![T]\!]$

- there is a version of the Riesz theory: for any inverse root $\alpha \in \mathcal{A}^*$ of $P_U(T)$ there exists an eigenfunction g, $Ug = \alpha g$

such that $ev_k(g) \in \mathbb{C}_p[\![q]\!]$ are classical cusp eigenforms for all k in a neigbourhood $\mathcal{B} \subset X$ (see in [CoPB])

We refer to [CoPB], part B and [PaTV], section 1, for generalities on rigid analytic p-adic families of modular forms.

2. Generalities on triple products

The triple product with a Dirichlet character χ is defined as the following complex L-function (an Euler product of degree eight):

$$L(f_1 \otimes f_2 \otimes f_3, s, \chi) = \prod_{p \nmid N} L((f_1 \otimes f_2 \otimes f_3)_p, \chi(p) p^{-s}), \tag{2.2}$$

(2.3) where $L((f_1\otimes f_2\otimes f_3)_p, X)^{-1} =$

$$\det\left(1_8 - X\begin{pmatrix}\alpha_{p,1}^{(1)} & 0\\ 0 & \alpha_{p,1}^{(2)}\end{pmatrix}\otimes\begin{pmatrix}\alpha_{p,2}^{(1)} & 0\\ 0 & \alpha_{p,2}^{(2)}\end{pmatrix}\otimes\begin{pmatrix}\alpha_{p,3}^{(1)} & 0\\ 0 & \alpha_{p,3}^{(2)}\end{pmatrix}\right)$$

$$= \prod_\eta (1-\alpha_{p,1}^{(\eta(1))}\alpha_{p,2}^{(\eta(2))}\alpha_{p,3}^{(\eta(3))}X)$$

$$= (1-\alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(1)}X)(1-\alpha_{p,1}^{(1)}\alpha_{p,2}^{(1)}\alpha_{p,3}^{(2)}X)\cdot\ldots\cdot(1-\alpha_{p,1}^{(2)}\alpha_{p,2}^{(2)}\alpha_{p,3}^{(2)}X),$$

product taken over all 8 maps $\eta : \{1,2,3\}\to\{1,2\}$.

The Satake parameters and Hecke p–polynomials of forms f_j: Here the Satake parameters $\alpha_{p,j}^{(1)}$, $\alpha_{p,j}^{(2)}$ are given as inverse roots of the Hecke p–polynomials

$$1 - a_{p,j}X - \psi_j(p)p^{k_j-1}X^2 = (1-\alpha_{p,j}^{(1)}(p)X)(1-\alpha_{p,j}^{(2)}(p)X).$$

We always assume that the weights are “balanced”:

(2.4) $$k_1 \ge k_2 \ge k_3 \ge 2, \text{ and } k_1 \le k_2 + k_3 - 2$$

The non-balanced case (i.e. $k_1 > k_2 + k_3 - 2$) was treated in [HaTi], where p-adic measures for the square roots of special values of triple product L-functions, were constructed in the ordinary case. This construction can be probably extended to Coleman’s case by techniques similar to the present paper.

Critical values and functional equation. We use the corresponding normalized L function (see [De79], [Co], [Co-PeRi]), which has the form:

(2.5) $\Lambda(f_1\otimes f_2\otimes f_3, s, \chi) =$

$$\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-k_3+1)\Gamma_{\mathbb{C}}(s-k_2+1)\Gamma_{\mathbb{C}}(s-k_1+1)L(f_1\otimes f_2\otimes f_3, s, \chi),$$

where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

The Gamma-factor determines the critical values $s = k_1, \cdots, k_2+k_3-2$ of $\Lambda(s)$, which we explicitely evaluate (like in the classical formula $\zeta(2) = \frac{\pi^2}{6}$). A functional equation of $\Lambda(s)$ has the form:

$$s \mapsto k_1 + k_2 + k_3 - 2 - s.$$

. According to H.Hida [Hi86] and R.Coleman [CoPB], one can include each f_j ($j = 1,2,3$) (under suitable assumptions on p and on f_j) into a p-adic analytic family

$$\mathbf{f}_j : k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_n(f_{j,k_j})q^n\}$$

of cusp eigenforms f_{j,k_j} of weights k_j in such a way that $f_{j,k_j} = f_j$, and that all their Fourier coefficients $a_n(f_{j,k_j})$ are given by certain p-adic analytic functions $k_j \mapsto a_{n,j}(k_j)$.

3. Statement of the problem

Given three p-adic analytic families $\mathbf{f}_j$ of slope $\sigma_j \geq 0$, to construct a four-variable p-adic L-function attached to Garrett's triple product of these families. We show that this function interpolates the special values

$$(s, k_1, k_2, k_2) \longmapsto \Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, s, \chi)$$

at critical points $s = k_1, \cdots, k_2 + k_3 - 2$ for balanced weights $k_1 \leq k_2 + k_3 - 2$; we prove that these values are algebraic numbers after dividing out certain "periods".

However the construction uses directly modular forms, and not the L-values in question, and a comparison of special values of two functions is done after the construction.

Consider the product of the Satake parmeters

$$\lambda_p = \alpha_{p,1}^{(1)} \alpha_{p,2}^{(1)} \alpha_{p,3}^{(1)} = \lambda_p(k_1, k_2, k_3)$$

We assume that $\mathrm{ord}_p \alpha_{p,j}^{(1)} \leq \mathrm{ord}_p \alpha_{p,j}^{(2)}$, and that the slope $\sigma = \mathrm{ord}_p(\lambda_p(k_1, k_2, k_3))$ is constant and positive for all triplets (k_1, k_2, k_3) in a p-adic neighbourhood $\mathcal{B} \subset X^3$ of the fixed triplet of weights (k_1, k_2, k_3).

Our method includes: • a version of Garrett's integral representation for the triple L-functions of the form: for $r = 0, \cdots, k_2 + k_3 - k_1 - 2$,

$$\Lambda(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - r, \chi) = \int\int\int_{(\Gamma_0(N^2p^{2v})\backslash \mathbb{H})^3} \overline{\tilde{f}_{1,k_1}(z_1)\tilde{f}_{2,k_2}(z_2)\tilde{f}_{3,k_3}(z_3)} \mathcal{E}(z_1, z_2, z_3; -r, \chi) \prod_j \left(\frac{dx_j dy_j}{y_j^2}\right)$$

where $\tilde{f}_{j,k_j} =: f^0_{j,k_j}$ is an eigenfunction of U_p^* in $\mathcal{M}_{k_j}(Np, \psi_j)$, $f_{j,k_j,0}$ is the corresponding eigenfunction of U_p, and

$$\mathcal{E}(z_1, z_2, z_3; -r, \chi) \in \mathcal{M}_T(N^2p^{2v}) = \mathcal{M}_{k_1,r^*}(N^2p^{2v}, \psi_1) \otimes \mathcal{M}_{k_2,r^*}(N^2p^{2v}, \psi_2) \otimes \mathcal{M}_{k_3,r^*}(N^2p^{2v}, \psi_3)$$

is a (classical) nearly holomorphic triple modular form of triple weight (k_1, k_2, k_3) of some type $r^* \geq 0$ (see [ShiAr]), and of fixed triple Nebentupus character (ψ_1, ψ_2, ψ_3), obtained from a nearly holomorphic Siegel-Eisenstein series $F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi})$, of degree 3, of weight $k = k_2 + k_3 - k_1$, and the Nebentypus character $\boldsymbol{\psi} = \chi^2 \psi_1 \psi_2 \overline{\psi}_3$ ([Sh83]).

We obtain $\mathcal{E}(z_1, z_2, z_3; -r, \chi)$ from a Siegel-Eisenstein series by applying to $F_{\chi,r}$ Boecherer's higher twist (see (11.22)) and Ibukiyama's differential operator (see (11.23)).

These operations act explicitely on the Fourier expansions. Then one uses:

• The theory of p-adic integration with values in Serre's type $\mathcal{A}$-modules $\mathcal{M}_T(\mathcal{A})$ of triple arithmetical nearly holomorphic modular forms over p-adic Banach algebras $\mathcal{A}$. We shall use the notation

$$\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A}(\mathcal{B}_1))\hat{\otimes}\mathcal{M}(\mathcal{A}(\mathcal{B}_2))\hat{\otimes}\mathcal{M}(\mathcal{A}(\mathcal{B}_3))$$

for certain $\mathcal{A}$-modules of p-adic families of triple modular forms.

Explicit Fourier coefficients $a_{\chi,r}(R,\mathcal{T}) \in \overline{\mathbb{Q}}[R,T]$ of $\mathcal{E}(-r,\chi)$ are given by special polynomials of matricies $\mathcal{T} = (t_{ij})$, $R = (R_{ij})$ and of $\chi(\beta)\beta^r$ (with $\beta \in \mathbb{Z}_p^* \cap \mathbb{Q}$) i.e. the coefficients of $a_{\chi,r}$ by some elementry p-adic measures $\int_Y \chi y^r \mathrm{d}\mu_{\mathcal{T}} \in \mathcal{A}$. Here $\mathcal{A} = \mathcal{A}(\mathcal{B})$ is a certain p-adic Banach algebra of functions on an open analytic subspace $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$ in the product of three copies of the weight space $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^*)$.

These measures on the group $Y = (\mathbb{Z}/N\mathbb{Z})^* \times \mathbb{Z}_p^*$ produce the coefficients of $a_{\chi,r}$ of $\mathcal{E}(-r,\chi)$ of $\mathcal{M}_T(\mathcal{A})$ for all p-adic weights $x \in X$, given by $\int_Y x(y)\mathrm{d}\mu_{\mathcal{T}} \in \mathcal{A}$ (*an interpolation from* $x = \chi y_p^r$ *to all* $x \in X$).

• *The* spectral theory of triple Atkin's operator $U = U_{p,T}$. allows to evaluate the integral using at each weight (k_1,k_2,k_3) the equality $\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\rangle = \langle \mathbf{f}^0, \pi_\lambda(\mathcal{E}(-r,\chi))\rangle$ with the projection π_λ of $\mathcal{M}_T(\mathcal{A})$ to the λ-part $\mathcal{M}_T(\mathcal{A})^\lambda$, defined by:

$$\mathrm{Ker}\,\pi_\lambda := \bigcap_{n\ge 1} \mathrm{Im}\,(U_T - \lambda I)^n, \quad \mathrm{Im}\,\pi_\lambda := \bigcup_{n\ge 1} \mathrm{Ker}\,(U_T - \lambda I)^n.$$

Note that

$$\mathbf{f}^0 = f^0_{k_1,1} \otimes f^0_{k_2,2} \otimes f^0_{k_3,3}, \quad f^0_{k_j,j} = f^\rho_{k_j,j,0}|_{k_j} \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix} \text{(for } j = 1,2,3)$$

is a classical form, defined at triple weghts (k_1,k_2,k_3) as the Weil-involution image of $\mathbf{f}_0 = f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}$ as in [PaTV], at p.555. Note that the functions f^0 are eigenfunctions of the adjoint triple operator U_p^*. We consider also the sequences of triple modular forms:

$$\mathbf{f}_0 = \{f_{k_1,1,0} \otimes f_{k_2,2,0} \otimes f_{k_3,3,0}\}_{(k_1,k_2,k_3)}, \ \mathbf{f}^0 = \{f^0_{k_1,1} \otimes f^0_{k_2,2} \otimes f^0_{k_3,3}\}_{(k_1,k_2,k_3)}.$$

The functions $\mathbf{f}_0$ form a p-adic family. The evaluation at triple weight $k = (k_1,k_2,k_3)$ of the p-adic coordinate with respect to p-adic family $\mathbf{f}_0$ is expressed through the triple Petersson scalar product with $\mathbf{f}^0$, which is algebraically orthogonal to $\mathbf{f}_0$ in the sense of Hida [Hi90] for classical weights (and in [PaTV] for Coleman's families).

We prove that U is a completely continuos $\mathcal{A}$-linear operator on a certain Coleman's submodule $\mathcal{M}(\mathcal{A})^\dagger$ of Serre's type module $\mathcal{M}(\mathcal{A})$. Then the projection π_λ exists (on this submodule) due to general results of Serre and Coleman, see [CoPB], [SePB].

We show that there exists an element $\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}(\mathcal{A})^\dagger$ such that at each weight (k_1,k_2,k_3) the equality holds: $\left\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\right\rangle = \left\langle \mathbf{f}^0, \pi_\lambda(\tilde{\mathcal{E}}(-r,\chi))\right\rangle$, and the product can be expressed through certain coefficients the series $\tilde{\mathcal{E}}(-r,\chi)$ which are the same as those of $\mathcal{E}(-r,\chi)$.

• Key point: modular admissible measures. Let us write for simplicity: $\mathcal{E}(-r,\chi)$ for $\tilde{\mathcal{E}}(-r,\chi)$

$\mathcal{M}_T(\mathcal{A})$ instead of $\mathcal{M}_T(\mathcal{A})^\dagger$ (Coleman's submodule)

One defines admissible p-adic measures $\tilde{\Phi}^\lambda$ with values in Banach $\mathcal{A}$-modules $\mathcal{M}_T^\lambda(\mathcal{A})$ which are locally free of finite rank, using the test functions: $\int_Y \chi y_p^r \tilde{\Phi}^\lambda = \pi_\lambda(\mathcal{E}(-r,\chi))$.

Consider the evaluation maps $ev_{\mathbf{s}} : \mathcal{A} \to \mathbb{C}_p$ for any p-adic triple weights $\mathbf{s} = (s_1,s_2,s_3) \in \mathcal{B}$.

• Passage from values in modular forms to scalar values: apply an algebraic $\mathcal{A}$-linear form $\mathcal{M}_T^\lambda(\mathcal{A}) \stackrel{\ell_T}{\to} \mathcal{A}$ to the constructed measure $\tilde{\Phi}^\lambda$ (in modular forms), and the evaluation maps $\mathcal{A} \stackrel{ev_{\mathbf{s}}}{\to} \mathbb{C}_p$ for any p-adic triple weights $\mathbf{s} \in X^3$.

The linear form ℓ_T is an algebraic version of the Petersson product (a geometric meaning of ℓ_T: the first coordinate in an (orthogonal) $\mathcal{A}$-basis of eigenfunctions of all Hecke operators T_q for $q \nmid Np$, with the first basis element $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})$).

Using the evaluation map and the Mellin transform. We obtain the measure $\mu = \ell_T(\tilde{\Phi}^\lambda)$ with values in $\mathcal{A}$ on the profinite group Y.

• Construct an analytic function $\mathcal{L}_\mu : X \to \mathcal{A} = \mathcal{A}(\mathcal{B})$ as the p-adic Mellin transform $\mathcal{L}_\mu(x) = \int_Y x(y)\, d\mu(y) \in \mathcal{A}$, $x \in X$.

• Solution: the function in question $\mathcal{L}_\mu(x,\mathbf{s})$ is given by evaluation of $\mathcal{L}_\mu(x)$ at $\mathbf{s} = (s_1,s_2,s_3) \in \mathcal{B}$: this is a p-adic analytic function in four variables

$$(x,\mathbf{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$$

$\mathcal{L}_\mu(x,\mathbf{s}) := ev_{\mathbf{s}}(\mathcal{L}_\mu(x)) \quad (x \in X,\ \mathbf{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_\mu(x) \in \mathcal{A})$.

Final step: comparison between $\mathbb{C}$ and $\mathbb{C}_p$. • We check an equality relating the values of the constructed analytic function $\mathcal{L}_\mu(x, \mathsf{s})$ at the arithmetical characters $x = y_p^r\chi \in X$, and at triple weights $\mathsf{s} = (k_1, k_2, k_3) \in \mathcal{B}$, with the normalized critical special values

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \cdots, k_2 + k_3 - k_1 - 2),$$

for certain Dirichlet characters χ mod $Np^v, v \geq 1$. These are algebraic numbers, embedded into $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ (the Tate field of p-adic numbers). The normalisation of L^* includes at the same time Gauss sums, Petersson scalar products, powers of π, the product $\lambda_p(k_1, k_2, k_3)$, and a certain finite Euler product.

We refer to Theorem 0.3 at p.556 of [PaTV], where the explicit form of such finite Euler product is given in the case of the L-function of one Coleman's family.

4. Arithmetical nearly holomorphic modular forms

Arithmetical nearly holomorphic modular forms (the elliptic case). Let $\mathcal{A}$ be a commutative ring (a subring of $\mathbb{C}$ or $\mathbb{C}_p$)

Arithmetical nearly holomorphic modular forms (in the sense of Shimura, [ShiAr] are certain formal series

$$g = \sum_{n=0}^{\infty} a(n; R)q^n \in \mathcal{A}[\![q]\!][R], \text{ with the property}$$

that for $\mathcal{A} = \mathbb{C}$, $z = x + iy \in \mathbb{H}$, $R = (4\pi y)^{-1}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}$ of a given weight k and Dirichlet character ψ. The coefficients $a(n; R)$ are polynomials in $\mathcal{A}[R]$. If $\deg_R a(n; R) \leq r$ for all n, we call g nearly holomorphic of type r (it is annihilated by $(\frac{\partial}{\partial \bar z})^{r+1}$, see [ShiAr]).

. We use the notation $\mathcal{M}_{k,r}(N, \psi, \mathcal{A})$ or $\tilde{\mathcal{M}}(N, \psi, \mathcal{A})$ for $\mathcal{A}$-modules of such forms (In our constructions the weight k varies).

A known example (see the introduction to [ShiAr]) is given by the series

$$\begin{aligned} &-12R + E_2 := -12R + 1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n \\ &= \frac{3}{\pi^2}\lim_{s\to 0} y^s \sum_{m_1, m_2 \in \mathbb{Z}}{}' (m_1 + m_2 z)^{-2}|m_1 + m_2 z|^{-2s}, (R = (4\pi y)^{-1}) \end{aligned}$$

where $\sigma_1(n) = \sum_{d|n} d$.

The action of the Shimura differential operator

$$\delta_k : \mathcal{M}_{k,r}(N, \psi, \mathcal{A}) \to \mathcal{M}_{k+2,r+1}(N, \psi, \mathcal{A}),$$

is given over $\mathbb{C}$ by $\delta_k(f) = (\frac{1}{2\pi i}\frac{\partial}{\partial z} - \frac{k}{4\pi y})f$.

This operator is a correction of the Ramanujan operator

$$\theta(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} n a_n q^n = \frac{1}{2\pi i}\frac{\partial}{\partial z}(\sum_{n=0}^{\infty} a_n q^n) = q\frac{\partial}{\partial q}(\sum_{n=0}^{\infty} a_n q^n),$$

which does not preserve the modularity. For example $\theta\Delta = E_2\Delta$, where E_2 is a quasimodular form (in the sense of Kaneko and Zagier, see [Ka-Za]).

Notice that $\delta_k f = (\theta - kR)f$, and that Serre's operator $f \mapsto \theta f - \frac{k}{12}E_2 f$ takes $\mathcal{M}_k$ to $\mathcal{M}_{k+2}$.

Note that that the arithmetical twist operator

$$\theta_\chi(\sum_{n=0}^{\infty} a_n q^n) = \sum_{n=1}^{\infty} \chi(n) a_n q^n$$

is a natural analog of the Ramanujan operator.

Triple arithmetical modular forms. Let $\mathcal{A}$ be a commutative ring. The tensor product over $\mathcal{A}$

$$\mathcal{M}_{\mathbf{k},r,T}(N,\psi,\mathcal{A}) := \mathcal{M}_{k_1,r}(N,\psi_1,\mathcal{A}) \otimes \mathcal{M}_{k_2,r}(N,\psi_2,\mathcal{A}) \otimes \mathcal{M}_{k_3,r}(N,\psi_3,\mathcal{A})$$

consists of triple arithmetical modular forms as certain formal series of the form

$$\begin{aligned} g = &\sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3} \\ &\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j + iy_j \in \mathbb{H},\ R_j = (4\pi y_j)^{-1}, \end{aligned}$$

with the property that for $\mathcal{A} = \mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1,k_2,k_3) and character (ψ_1,ψ_2,ψ_3), $j = 1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$. Examples of such modular forms come from the restriction to the diagonal of Siegel modular forms of degree 3.

5. Siegel-Eisenstein series

Siegel modular groups. Let $J_{2m} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix}$. The symplectic group

$$\mathrm{Sp}_m(\mathbb{R}) = \left\{ g \in \mathrm{GL}_{2m}(\mathbb{R}) | {}^t g \cdot J_{2m} g = J_{2m} \right\},$$

acts on the Siegel upper half plane

$$\mathbb{H}_m = \left\{ z = {}^t z \in M_m(\mathbb{C}) | \operatorname{Im} z > 0 \right\}$$

by $g(z)=(az+b)(cz+d)^{-1}$, where we use the bloc notation $g=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{Sp}_{2m}(\mathbb{R})$. We use the congruence subgroup $\Gamma_0^m(N)=\{\gamma\in\mathrm{Sp}_m(\mathbb{Z})\mid \gamma\equiv\begin{pmatrix} * & *\\ 0 & *\end{pmatrix} \bmod N\}\subset \mathrm{Sp}_m(\mathbb{Z})$.

A Siegel modular form. $f\in\mathcal{M}_k(\Gamma_0^m(N),\chi)$ of degree $m>1$, weight k and a Dirichlet chracter $\chi \bmod N$ is a holomorphic function $f\colon \mathbb{H}_m\to\mathbb{C}$ such that for every $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0^m(N)$ one has

$$f(\gamma(z))=\chi(\det d)\det(cz+d)^k f(z).$$

The Fourier expansion of f uses the symbol

$$q^{\mathcal{T}}=\exp(2\pi i\mathrm{tr}(\mathcal{T}z))=\prod_{i=1}^m q_{ii}^{\mathcal{T}_{ii}}\prod_{i<j}q_{ij}^{2\mathcal{T}_{ij}}\in\mathbb{C}[q_{11},\cdots,q_{mm},q_{ij},q_{ij}^{-1}]_{1\le i<j\le m}$$

for each Fourier coefficient, where $q_{ij}=\exp(2\pi(\sqrt{-1}z_{i,j})$, and $\mathcal{T}$ in the semi-group $B_m=\{\mathcal{T}={}^t\mathcal{T}\ge 0|\mathcal{T}\text{ half-integral}\}$. We may consider $f(z)=\sum_{\mathcal{T}\in B_m}a(\mathcal{T})q^{\mathcal{T}}\in\mathbb{C}[\![q^{B_m}]\!]$ as a formal q-expansion in $\mathbb{C}[\![q^{B_m}]\!]$ (the subring of $\mathbb{C}[\![q_{11},...,q_{mm}]\!][q_{ij},q_{ij}^{-1}]$) generated by all $q^{\mathcal{T}}$).

Siegel-Eisenstein series.

Example 5.1 ([Nag2], p.408).

$$\begin{aligned}E_4^{(2)}(z)=&1+240q_{11}+240q_{22}+2160q_{11}^2+(240q_{12}^{-2}+13440q_{12}^{-1}\\&+30240+13440q_{12}+240q_{12}^2)q_{11}q_{22}+2160q_{22}^2+\dots\\E_6^{(2)}(z)=&1-504q_{11}-504q_{22}-16632q_{11}^2+(-540q_{12}^{-2}+44352q_{12}^{-1}\\&+166320+44352q_{12}-504q_{12}^2)q_{11}q_{22}-16632q_{22}^2+\dots.\end{aligned}$$

Arithmetical nearly holomorphic Siegel modular forms.

Arithmetical Siegel modular forms. Consider a commutative ring $\mathcal{A}$, the formal variables $q=(q_{i,j})_{i,j=1,...,m}$, $R=(R_{i,j})_{i,j=1,...,m}$, and the ring of formal Fourier series

$$\mathcal{A}[\![q^{B_m}]\!][R_{i,j}]=\left\{f=\sum_{\mathcal{T}\in B_m}a(\mathcal{T},R)q^{\mathcal{T}}\;\middle|\;a(\mathcal{T},R)\in\mathcal{A}[R_{i,j}]\right\}\tag{5.6}$$

(over the complex numbers this notation corresponds to $q^{\mathcal{T}}=\exp(2\pi i\mathrm{tr}(\mathcal{T}z))$, $R=(4\pi\mathrm{Im}(z))^{-1}$).

The formal Fourier expansion of a nearly holomorphic Siegel modular form f with coefficients in $\mathcal{A}$ is a certain element of $\mathcal{A}[\![q^{B_m}]\!][R_{i,j}]$. We call f arithmetical in the sense of Shimura [ShiAr], if $\mathcal{A} = \overline{\mathbb{Q}}$.

5.1. **Algebraic differential operators of Maass and Shimura.**

Maass differential operator. Let us consider the Maass differential operator (see [Maa]) Δ_m of degree m, acting on complex $\mathcal{C}^\infty$-functions on $\mathbb{H}_m$ by:

$$\Delta_m = \det(\tilde{\partial}_{ij}), \qquad \tilde{\partial}_{ij} = 2^{-1}(1+\delta_{ij})\partial/\partial_{ij}, \tag{5.7}$$

its algebraic version is the Ramanujan operator of degree m:

$$\Theta_m := \det(\frac{1}{2\pi i}\tilde{\partial}_{ij}) = \det(\theta_{ij}) = \frac{1}{(2\pi i)^m}\Delta_m, \tag{5.8}$$

where $\Theta_m(q^{\mathcal{T}}) = \det(\mathcal{T})q^{\mathcal{T}}$.

Shimura differential operator. The Shimura differential operator (see [Shi76, ShiAr]):

$$\delta_k f(z) = \det(R)^{k+1-\varkappa}\Theta_m\left[\det(R)^{\varkappa-1-k}f\right], \text{ where } R = (4\pi y)^{-1},$$

acts on arithmetic nearly holomorphic Siegel modular forms, and the composition is defined

$$\delta_k^{(r)} = \delta_{k+2r-2}\circ\cdots\circ\delta_k : \widetilde{\mathcal{M}}_k^m(N,\psi;\overline{\mathbb{Q}}) \to \widetilde{\mathcal{M}}_{k+2rm}^m(N,\psi;\overline{\mathbb{Q}}), \tag{5.9}$$

where

$$\delta_k f(z) = \left(\frac{-1}{4\pi}\right)^m \det(y)^{-1}\det(z-\bar{z})^{\varkappa-k}\Delta_m\left[\det(z-\bar{z})^{k-\varkappa+1}f(z)\right].$$

Universal polynomials $Q(R,\mathcal{T};k,r)$. Let $f = \sum_{\mathcal{T}\in B_m} c(\mathcal{T})q^{\mathcal{T}} \in \mathcal{M}_k^m(N,\psi)$ be a formal holomorphic Fourier expansion. One shows that $\delta_k^{(r)}f$ is given by

$$\delta_k^{(r)}f = \sum_{\mathcal{T}\in B_m} Q(R,\mathcal{T};k,r)c(\mathcal{T})q^{\mathcal{T}}.$$

Here we use a universal polynomial (5.10) which can be defined for all $k\in\mathbb{C}$, and it expresses the action of the Shimura operator on the exponential (of degree m):

$$\delta_k^{(r)}(q^{\mathcal{T}}) = Q(R,\mathcal{T};k,r)q^{\mathcal{T}}.$$

If $m=1$, r arbitrary (see [Shi76]), $\delta_k^{(r)} = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\frac{\Gamma(k+r)}{\Gamma(k+j)}R^{r-j}\theta^j$,
$Q(R,n;k,r)=\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\frac{\Gamma(k+r)}{\Gamma(k+j)}R^{r-j}n^j.$

If $r=1$, m arbitrary, one has (see [Maa]):

$$\delta_k f(z) = \sum_{\mathcal{T}\in B_m} c(\mathcal{T})\sum_{l=0}^{m}(-1)^{m-l}c_{m-l}(k+1-\varkappa)\mathrm{tr}\left({}^t\rho_{m-l}(R)\cdot\rho_l^\star(\mathcal{T})\right)q^{\mathcal{T}}$$

where $R=(4\pi y)^{-1}=(R_{i,j})\in \mathrm{M}_m(\mathbb{R})$, $c_m(\alpha)=\dfrac{\Gamma_m(\alpha+\kappa)}{\Gamma_m(\alpha+\kappa-1)}$, $\Gamma_m(s)=\pi^{m(m-1)/4}\prod_{j=0}^{m-1}\Gamma(s-(j/2))$.

Here we use the natural representation $\rho_r : \mathrm{GL}_m(\mathbb{C}) \longrightarrow \mathrm{GL}(\wedge^r\mathbb{C}^m)$ $(0\le r\le m)$ of the group $\mathrm{GL}_m(\mathbb{C})$ on the vector space $\Lambda^r\mathbb{C}^m$. Thus $\rho_r(z)$ is a matrix of size $\binom{m}{r}\times\binom{m}{r}$ composed of the subdeterminants of z of degree r. Put $\rho_r^\star(z)=\det(z)\rho_{m-r}({}^tz)^{-1}$.

Then the representations ρ_r and $\rho_r^\star$ turn out to be polynomial representations.

. In general (see [CourPa], Theorem 3.14) one has:

$$\begin{aligned}(5.10)\qquad Q(R,\mathcal{T}) &= Q(R,\mathcal{T};k,r)\\ &=\sum_{t=0}^{r}\binom{r}{t}\det(\mathcal{T})^{r-t}\sum_{|L|\le mt-t}R_L(\kappa-k-r)Q_L(R,\mathcal{T}),\\ Q_L(R,\mathcal{T}) &= \mathrm{tr}\left({}^t\rho_{m-l_1}(R)\rho_{l_1}^\star(\mathcal{T})\right)\cdot\ldots\cdot\mathrm{tr}\left({}^t\rho_{m-l_t}(R)\rho_{l_t}^\star(\mathcal{T})\right)).\end{aligned}$$

In (5.10), L goes over all the multi-indices $0\le l_1\le\cdots\le l_t\le m$, such that $|L|=l_1+\cdots+l_t\le mt-t$, and $R_L(\beta)\in\mathbb{Z}[1/2][\beta]$ in (5.10) are polynomials in β of degree $(mt-|L|)$ (used with $\beta=\kappa-k-r$).

Note the differentiation rule of degree m (see [Sh83], p.466):
$\Delta(fg)=\sum_{r=0}^{m}\mathrm{tr}\left({}^t\rho_r(\tilde\partial/\partial z)f\cdot\rho_{m-r}^\star(\tilde\partial/\partial z)g\right)$. As in (5.8), we write here $\tilde\partial/\partial z = (\tilde\partial/\partial z_{ij})$ for the matrix with entries $\tilde\partial_{ij}=2^{-1}(1+\delta_{ij})\partial/\partial_{ij}$.

.

EXAMPLE 5.2 (Siegel-Eisenstein series of odd degree and higher level).

$$G^*(z,s;k,\boldsymbol{\psi},N) = \det(y)^s \sum_{c,d} \boldsymbol{\psi}(\det c)\det(cz+d)^{-k}|\det(cz+d)|^{-2s} \cdot \cdot \tilde{\Gamma}(k,s) L_N(k+2s,\boldsymbol{\psi}) \left(\prod_{i=1}^{[m/2]} L_N(2k+4s-2i,\boldsymbol{\psi}^2)\right), \quad (5.11)$$ *where*

(c,d) runs over all "non-associated coprime symmetric pairs" with $\det(c)$ *coprime to N, $\kappa = (m+1)/2$, and for m odd the Γ-factor has the form:*
$\tilde{\Gamma}(k,s) = i^{mk} 2^{-m(k+1)} \pi^{-m(s+k)} \Gamma_m(k+s)$.

We use this series with $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, $k = k_2+k_3-k_1 \geq 2$, $m = 3$, $\kappa = \dfrac{m+1}{2} = 2$, $[m/2] = 1$.

.

THEOREM 5.3 (Siegel, Shimura [Sh83], P. Feit [Fei86]). *Let m be an* odd integer *such that $2k > m$, and $N > 1$ be an integer, then:*

For an integer s such that $s = -r$, $0 \leq r \leq k - \kappa$, there is the following Fourier expansion

$$G^\star(z,-r) = G^\star(z,-r;k,\boldsymbol{\psi},N) = \sum_{A_m \ni \mathcal{T} \geq 0} a(\mathcal{T},R) q^{\mathcal{T}}, \quad (5.12)$$

where for $s > (m+2-2k)/4$ in (5.12) the only non-zero terms occur for positive definite $\mathcal{T} > 0$,

$$a(\mathcal{T},R) = M(\mathcal{T},\boldsymbol{\psi},k-2r) \cdot \det(\mathcal{T})^{k-2r-\kappa} Q(R,\mathcal{T};k-2r,r), \quad (5.13)$$

$$M(\mathcal{T},k-2r,\boldsymbol{\psi}) = \prod_{\ell | \det(2\mathcal{T})} M_\ell(\mathcal{T},\boldsymbol{\psi}(\ell)\ell^{-k+2r}) \quad (5.14)$$

polynomials $Q(R,\mathcal{T};k-2r,r)$ are given by (5.10), and for all $\mathcal{T} > 0$, $\mathcal{T} \in A_m$, is a finite Euler product, in which $M_\ell(\mathcal{T},x) \in \mathbb{Z}[x]$. □

6. STATEMENT OF THE MAIN RESULT

Main Theorem (on p-adic analytic function in four variables). The following Main Theorem corresponds to Theorem 0.3 at p.556 of [PaTV] in the situation of the L-function attached to one Coleman's family.

Main Theorem 6.1. 1) *The function* $\mathcal{L}_{\boldsymbol{f}} : (s, k_1, k_2, k_3) \mapsto \frac{\langle \boldsymbol{f}^0, \mathcal{E}(-r,\chi)\rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0\rangle}$ *extends to a* p*-adic analytic function on four variables* $(\chi \cdot y_p^r, k_1, k_2, k_3) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$*;*

2) *Comparison of complex and* p*-adic values: for all* (k_1, k_2, k_3) *in an affinoid neighborhood* $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X^3$*, satisfying* $k_1 \le k_2 + k_3 - 2$*: the values at* $s = k_2 + k_3 - 2 - r$ *coincide with the normalized critical special values*

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \cdots, k_2 + k_3 - k_1 - 2), \tag{6.15}$$

for Dirichlet characters $\chi \bmod Np^v, v \ge 1$*, such that all three corresponding Dirichlet characters* χ_j *have* Np*-complete conductors:*

$$\chi_1 \bmod Np^v = \chi,\ \ \chi_2 \bmod Np^v = \psi_2\bar{\psi}_3\chi, \quad \chi_3 \bmod Np^v = \psi_1\bar{\psi}_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3. \tag{6.16}$$

The normalisation of L^* *in (6.15) is the same as in Theorem C below.*

3) *Dependence on* $x \in X$*: let* $H = [2\mathrm{ord}_p(\lambda)] + 1$*. For any fixed* $(k_1, k_2, k_3) \in \mathcal{B}$ *and* $x = \chi \cdot y_p^r$ *the function*

$$x \longmapsto \frac{\langle \boldsymbol{f}^0, \mathcal{E}(-r,\chi)\rangle}{\langle \boldsymbol{f}^0, \boldsymbol{f}_0\rangle}$$

extends to a p*-adic analytic function of type* $o(\log^H(\cdot))$ *of the variable* $x \in X$*.*

Remark. The function $\mathcal{L}_{\mathbf{f}}$ depends on the variables (s, k_1, k_2, k_3) in a different way: it is a mixture of the p-adic Mellin transform (in s), and of a rigid analytic function (in k_1, k_2, k_3).

Outline of the proof. The proof follows the lines given in Sections 5-7 in [PaTV] (the case of the L-function of one Coleman's family).

1) • (compare with Section 5 in [PaTV]). At each classical weight (k_1, k_2, k_3) let us use the equality

$$\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\rangle = \langle \mathbf{f}^0, \pi_\lambda(\mathcal{E}(-r,\chi))\rangle$$

which is deduced from the definition of the projector π_λ: $\mathrm{Ker}\,\pi_\lambda := \bigcap_{n\ge1} \mathrm{Im}\,(U_T - \lambda I)^n$, $\mathrm{Im}\,\pi_\lambda := \bigcup_{n\ge1} \mathrm{Ker}\,(U_T - \lambda I)^n$.

Notice that the coefficients of $\mathcal{E}(-r,\chi) \in \mathcal{M}(\mathcal{A})$ depend p-adic analytically on $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, where $\mathcal{A} = \mathcal{A}(\mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3)$ is the p-adic Banach algebra of rigid-analytic functions on $\mathcal{B}$.

Interpolation to all p-adic weights: • At each classical weight (k_1, k_2, k_3) the scalar product $\langle \mathbf{f}^0, \mathcal{E}(-r,\chi)\rangle$ is given by the first coordinate of $\pi_\lambda(\mathcal{E}(-r,\chi))$ with respect to an orthogonal basis of $\mathcal{M}^\lambda(\mathcal{A})$ containing $\mathbf{f}_0$ with respect to Hida's algebraic Petersson product $\langle g, h\rangle_a := \left\langle g^\rho | \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix}, h \right\rangle$, see [Hi90].

Let us extend the linear form $\ell(h) = \frac{\langle \mathbf{f}^0, h\rangle}{\langle \mathbf{f}^0, \mathbf{f}_0\rangle}$ (defined firs only for classical weights), to Coleman's type submodule of overconvergent families $h \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}^\lambda(\mathcal{A})$ as the first coordinate of h with respect to some $\mathcal{A}$-basis of eigenfunctions of all (triple) Hecke operators T_q for $q \nmid Np$, having the first basis vector $\mathbf{f}_0 \in \mathcal{M}^\lambda(\mathcal{A})^\dagger$.

The linear form ℓ can be characterized as a normalized eigenfunction of the adjoint Atkin's operator, acting on the dual $\mathcal{A}$-module of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell(\mathbf{f}_0) = 1$.

In order to extend ℓ to $h = \mathcal{E}(-r,\chi)$, we need to choose a certain representative of $\mathcal{E}(-r,\chi)$ in the $\mathcal{A}$-submodule $\mathcal{M}^\lambda(\mathcal{A})^\dagger$, which is locally free of finite rank.

A representative of $\mathcal{E}(-r,\chi)$ in the (locally free of finite rank $\mathcal{A}$-submodule) $\mathcal{M}^\lambda(\mathcal{A})^\dagger$. (compare with Section 6 in [PaTV] in the case of the L-function of one Coleman's family). Choose a (local) basis $\ell^1, \cdots, \ell^n$ given by some *triple Fourier coefficients* of the dual (locally free of finite rank) $\mathcal{A}$-module $\mathcal{M}^\lambda(\mathcal{A})^{\dagger *}$.

Then define
$$\ell = \beta_1\ell^1 + \cdots + \beta_n\ell^n,$$
where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and ℓ_i denotes the dual basis of $\mathcal{M}^\lambda(\mathcal{A})^\dagger$: $\ell^j(\ell_i) = \delta_{ij}$.

At each p-adic weight $(k_1, k_2, k_3) \in \mathcal{B}$ let us define
$$\ell(\mathcal{E}(-r,\chi)) := \beta_1\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n\ell^n(\mathcal{E}(-r,\chi)) \text{ (belongs to } \mathcal{A}),$$
where $\beta_i = \ell(\ell_i) \in \mathcal{A}$, and $\ell^i(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ are certain Fourier coefficients of the seies $\mathcal{E}(-r,\chi)$.

Conclusion. There exists an element $\tilde{\mathcal{E}}(-r,\chi) \in \mathcal{M}^\lambda(\mathcal{A})^\dagger \subset \mathcal{M}(\mathcal{A})^\dagger$ such that
$$\ell(\mathcal{E}(-r,\chi)) = \ell(\tilde{\mathcal{E}}(-r,\chi))$$
(at each triple weight (k_1, k_2, k_3)). In fact, let us define
$$\tilde{\mathcal{E}}(-r,\chi) := \ell^1(\mathcal{E}(-r,\chi))\ell_1 + \cdots + \ell^n(\mathcal{E}(-r,\chi))\ell_n$$
$\Rightarrow \ell(\tilde{\mathcal{E}}(-r,\chi)) = \ell(\ell_1)\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \ell(\ell_n)\ell^n(\mathcal{E}(-r,\chi))$
$= \beta_1\ell^1(\mathcal{E}(-r,\chi)) + \cdots + \beta_n\ell^n(\mathcal{E}(-r,\chi))$
$= \ell(\mathcal{E}(-r,\chi))$ (at each weight (k_1, k_2, k_3)).

Thus, the dependence of $\ell(\mathcal{E}(-r,\chi)) \in \mathcal{A}$ on $(k_1, k_2, k_3) \in X^3$ is p-adic analytic.

In order to prove the remaining statements 2), 3), the dependence on $x = \chi \cdot y_p^r$ is studied in the next section.

7. Distributions and admissible measures

Distributions and measures with values in Banach modules. We refer to Section 4 of [PaTV] for similar constructions in the case of the L-function of one Coleman's family.

	$\mathcal{A}$	(a p-adic Banach algebra)
	V	(an $\mathcal{A}$-module)
Notation.	$\mathcal{C}(Y, \mathcal{A})$	(the $\mathcal{A}$-Banach algebra
	$\cup$	of continuous functions on Y)
	$\mathcal{C}^{loc-const}(Y, \mathcal{A})$	(the $\mathcal{A}$-algebra
		of locally constant functions on Y)

Definition 7.1 (Distributions and measures). *a) A* distribution $\mathcal{D}$ *on* Y *with values in* V *is an* $\mathcal{A}$*-linear form*

$$\mathcal{D} : \mathcal{C}^{loc-const}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \mathcal{D}(\varphi) = \int_Y \varphi d\mathcal{D}.$$

b) A measure μ *on* Y *with values in* V *is a continuous* $\mathcal{A}$*-linear form*

$$\mu : \mathcal{C}(Y, \mathcal{A}) \to V, \quad \varphi \mapsto \int_Y \varphi d\mu.$$

The integral $\int_Y \varphi d\mu$ can be defined for any continuous function φ, and any bounded distribution μ, using the Riemann sums.

Admissible measures of Amice-Vélu.

Admissible measures. Let h be a positive integer. A more delicate notion of an h-admissible measure was introduced in [Am-V] by Y. Amice, J. Vélu (see also [MTT], [V]):

Definition 7.2.

a) For $h \in \mathbb{N}, h \geq 1$ *let* $\mathcal{P}^h(Y, \mathcal{A})$ *denote the* $\mathcal{A}$*-module of* locally polynomial functions *of degree* $< h$ *of the variable* $y_p : Y \to \mathbb{Z}_p^\times \hookrightarrow \mathcal{A}^\times$*; in particular,*

$$\mathcal{P}^1(Y, \mathcal{A}) = \mathcal{C}^{loc-const}(Y, \mathcal{A})$$

(the $\mathcal{A}$-submodule of locally constant functions*). Let also denote $\mathcal{C}^{loc-an}(Y,\mathcal{A})$ the $\mathcal{A}$-module of* locally analytic functions*, so that*

$$\mathcal{P}^1(Y,\mathcal{A}) \subset \mathcal{P}^h(Y,\mathcal{A}) \subset \mathcal{C}^{loc-an}(Y,\mathcal{A}) \subset \mathcal{C}(Y,\mathcal{A}).$$

b) Let V be a normed $\mathcal{A}$-module with the norm $|\cdot|_{p,V}$. For a given positive integer h an h-admissible measure on Y with values in V is an $\mathcal{A}$-module homomorphism

$$\tilde{\Phi} : \mathcal{P}^h(Y,\mathcal{A}) \to V$$

such that for fixed $a \in Y$ and for $v \to \infty$ the following growth condition *is satisfied:*

$$\left| \int_{a+(Np^v)} (y_p - a_p)^{h'} d\tilde{\Phi} \right|_{p,V} = o(p^{-v(h'-h)}) \tag{7.17}$$

$$\text{for all} \quad h' = 0, 1, \ldots, h-1, a_p := y_p(a)$$

The condition (7.17) allows to integrate the locally-analytic functions on Y along $\tilde{\Phi}$ using Taylor's expansions! This means: there exists a unique extension of $\tilde{\Phi}$ to $\mathcal{C}^{loc-an}(Y,\mathcal{A}) \to V$.

7.1. U_p–**Operator and the method of canonical projection.** We refer to Section 5 of [PaTV] for similar constructions in the case of the L-function of one Coleman's family.

Using the canonical projection π_λ. We construct our H-admissible measure $\widetilde{\Phi}^\lambda$: $\mathcal{P}^H(Y,\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ out of a sequence of distributions $\Phi_r : \mathcal{P}^1(Y,\mathcal{A}) \to \mathcal{M}(\mathcal{A})$ defined on local monomials y_p^r of each degree r by the rule

$$\int_Y \chi y_p^r \, d\widetilde{\Phi}^\lambda = \pi_\lambda(\tilde{\mathcal{E}}(-r,\chi)), \text{ where } \tilde{\mathcal{E}}(-r,\chi) \in M = \mathcal{M}(\mathcal{A}).$$

Here $\tilde{\mathcal{E}}(-r,\chi)$ takes values in an $\mathcal{A}$-module

$$M = \mathcal{M}(\mathcal{A}) \subset \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3]$$

of nearly holomorphic (overconvergent) triple modular forms over $\mathcal{A}$ (for $0 \le r \le H-1$, $H = [2\mathrm{ord}_p\lambda_p]+1$), and the formal series $\tilde{\mathcal{E}}(-r,\chi)$ was constructed in the proof of 1) of Main Theorem.

Definition of the canonical projection π_λ. Here $\mathcal{A}$ is an $\mathbb{C}_p$-algebra, and $\lambda \in \mathcal{A}^\times$ is a fixed non-zero eigenvalue of triple Atkin's operator $U_T = U_{T,p}$, acting on $\mathcal{M}(\mathcal{A})$,

$$\pi_\lambda : \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})^\lambda$$

is the canonical projection operator onto the maximal $\mathcal{A}$-submodule $\mathcal{M}(\mathcal{A})^\lambda$ over which the operator $U_T - \lambda I$ is nilpotent (we call $\mathcal{M}(\mathcal{A})^\lambda$ the λ-characteristic

submodule of $\mathcal{M}(\mathcal{A})$).
The projector π_λ is defined by its kernel:

$$\operatorname{Ker}\pi_\lambda := \bigcap_{n\geq 1} \operatorname{Im}(U_T-\lambda I)^n, \quad \operatorname{Im}\pi_\lambda := \bigcup_{n\geq 1} \operatorname{Ker}(U_T-\lambda I)^n.$$

8. Triple modular forms

Triple modular forms are certain formal series

$$g = \sum_{n_1,n_2,n_3=0}^{\infty} a(n_1,n_2,n_3;R_1,R_2,R_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}$$
$$\in \mathcal{A}[\![q_1,q_2,q_3]\!][R_1,R_2,R_3], \text{ where } z_j = x_j+iy_j \in \mathbb{H},\ R_j = (4\pi y_j)^{-1},$$

with the property that for $\mathcal{A}=\mathbb{C}$, the series converges to a $\mathcal{C}^\infty$-modular form on $\mathbb{H}^3$ of a given weight (k_1,k_2,k_3) and character (ψ_1,ψ_2,ψ_3), $j=1,2,3$. The coefficients $a(n_1,n_2,n_3;R_1,R_2,R_3)$ are polynomials in $\mathcal{A}[R_1,R_2,R_3]$, and the triple Atkin's operator is given by

$$U_T(g) = \sum_{n_1,n_2,n_3=0}^{\infty} a(pn_1,pn_2,pn_3;pR_1,pR_2,pR_3) q_1^{n_1} q_2^{n_2} q_3^{n_3}.$$

Eigenfunctions of U_p and of U_p^*.

Functions $f_{j,0}$ and f_j^0. Recall that for any primitive cusp eigenform $f_j = \sum_{n=1}^\infty a_n(f)q^n$, there is an eigenfunction $f_{j,0} = \sum_{n=1}^\infty a_n(f_{j,0})q^n \in \overline{\mathbb{Q}}[\![q]\!]$ of $U=U_p$ with the eigenvalue $\alpha = \alpha_{p,j}^{(1)} \in \overline{\mathbb{Q}}$ $(U(f_0)=\alpha f_0)$ given by

$$f_{j,0} = f_j - \alpha_{p,j}^{(2)} f_j|V_p = f_j - \alpha_{p,j}^{(2)} p^{-k/2} f_j| \begin{pmatrix} p & 0 \\ 0 & 1\end{pmatrix} \tag{8.18}$$
$$\sum_{n=1}^{\infty} a_n(f_{j,0}) n^{-s} = \sum_{\substack{n=1\\ p\nmid n}}^{\infty} a_n(f_j) n^{-s} (1-\alpha_{p,j}^{(1)} p^{-s})^{-1}.$$

Moreover, there is an eigenfunction f_j^0 of U_p^* given by

$$f_j^0 = f_{j,0}^\rho\Big|_k \begin{pmatrix} 0 & -1 \\ Np & 0\end{pmatrix}, \text{ where } f_{j,0}^\rho = \sum_{n=1}^\infty \overline{a(n,f_0)} q^n. \tag{8.19}$$

Therefore, $U_T(f_{1,0}\otimes f_{2,0}\otimes f_{3,0}) = \lambda(f_{1,0}\otimes f_{2,0}\otimes f_{3,0})$.

9. Critical values of the L function $L(f_1\otimes f_2\otimes f_3, s, \chi)$

(compare with Section 7 in [PaTV]).

Choice of Dirichlet characters. For an arbitrary Dirichlet character χ mod Np^v consider the following Dirichlet characters:

$$\begin{aligned}(9.20)\qquad &\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi,\\ &\chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3;\end{aligned}$$

later on we impose the condition that the conductors of the corresponding primitive characters $\chi_{0,1}, \chi_{0,2}, \chi_{0,3}$ are Np-completes (i.e. have the same prime divisors as resp. those of Np).

Theorem A (algebraic properties of the triple product). *Assume that $k_2 + k_3 - k_1 \geq 2$, then for all pairs (χ, r) such that the corresonding Dirichlet characters χ_j have Np-complete conductors, and $0 \leq r \leq k_2 + k_3 - k_1 - 2$, we have that*

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)}{\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T} \in \overline{\mathbb{Q}}$$

where

$$\begin{aligned}&\langle f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, f_1^\rho \otimes f_2^\rho \otimes f_3^\rho\rangle_T := \langle f_1^\rho, f_1^\rho\rangle_N \langle f_2^\rho, f_2^\rho\rangle_N \langle f_3^\rho, f_3^\rho\rangle_N\\ &\quad = \langle f_1, f_1\rangle_N \langle f_2, f_2\rangle_N \langle f_3, f_3\rangle_N.\end{aligned}$$

10. Theorems B-D

Recall: the p-adic weight space and the Mellin transform. (for generalities we also refer to the introduction of [PaTV] in the case of the L-function of one Coleman's family). The p-adic weight space is the group $X = \mathrm{Hom}_{cont}(Y, \mathbb{C}_p^\times)$ of (continuouos) p-adic characters of the commutative profinite group $Y = \varprojlim_v (\mathbb{Z}/Np^v\mathbb{Z})^*$

The group X is isomorphic to a finite union of discs $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$.

A p-adic L-function $L_{(p)} : X \to \mathbb{C}_p$ is a certain meromorphic function on X. Such a function usually come from a p-adic measure μ on Y (*bounded* or *admissible* in the sense of Amice-Vélu, see [Am-V]). The p-adic Mellin transform of μ is given for all $x \in X$ by

$$L_{(p)}(x) = \int_{Y_{N,p}} x(y)\mathrm{d}\mu(y), L_{(p)} : X \to \mathbb{C}_p$$

Theorem B (on admissible measures attached to the triple product:fixed balanced weights case). Under the assumptions as above there exist a $\mathbb{C}_p$-valued measure $\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}$ on $Y_{N,p}$, and a $\mathbb{C}_p$-analytic function $\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) : X_p \to \mathbb{C}_p$, given for all $x \in X_{N,p}$ by the integral $\mathcal{D}_{(p)}(x, f_1 \otimes f_2 \otimes f_3) = \int_{Y_{N,p}} x(y)\mathrm{d}\tilde\mu^\lambda_{f_1\otimes f_2\otimes f_3}(y)$, and having the following properties:

(i) *for all pairs* (r,χ) *such that* $\chi \in X_{N,p}^{\mathrm{tors}}$*, and all three corresponding Dirichlet characters* χ_j *have* Np*-complete conductor* $(j = 1,2,3)$*, and* $r \in \mathbb{Z}$ *is an integer with* $0 \le r \le k_2 + k_3 - k_1 - 2$*, the following equality holds:*

$$\mathcal{D}_{(p)}(\chi x_p^r, f_1 \otimes f_2 \otimes f_3) = i_p\Big(\frac{(\psi_1\psi_2)(2)C_\chi^{4(k_2+k_3-2-r)}}{G(\chi_1)G(\chi_2)G(\chi_3)G(\psi_1\psi_2\chi_1)\lambda_p^{2v}}$$

$$\frac{\Lambda(f_1^\rho \otimes f_2^\rho \otimes f_3^\rho, k_2+k_3-2-r, \psi_1\psi_2\chi)}{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, f_{1,0} \otimes f_{2,0} \otimes f_{3,0}\rangle_{T,Np}}\Big)$$

where $v = \mathrm{ord}_p(C_\chi)$*,* $G(\chi)$ *denotes the Gauß sum of a primitive Dirichlet character* χ_0 *attached to* χ *(modulo the conductor of* χ_0*),*

(ii) *if* $\mathrm{ord}_p\lambda_p = 0$ *then the holomorphic function in (i) is a* bounded $\mathbb{C}_p$-analytic function*;*

(iii) *in the general case (but assuming that* $\lambda_p \ne 0$*) the holomorphic function in (i) belongs to the* type $o(\log(x_p^H))$ with $H = [2\mathrm{ord}_p\lambda_p]+1$ *and it can be represented as the Mellin transform of the* H*-admissible* $\mathbb{C}_p$*-valued measure* $\tilde\mu^{\lambda}_{f_1\otimes f_2\otimes f_3}$ *(in the sense of Amice-Vélu) on* Y

(iv) *Let* $k = k_2 + k_3 - k_1 \ge 2$*. If* $H \le k-2$ *then the function* $\mathcal{D}_{(p)}$ *is uniquely determined by the above conditions (i).* Let us describe now p-adic measures attached to Garrett's triple product of three Coleman's families

$$k_j \mapsto \{f_{j,k_j} = \sum_{n=1}^{\infty} a_{n,j}(k)q^n\}(j = 1,2,3). \tag{10.21}$$

Consider the product of three eigenvalues:

$$\lambda = \lambda_p(k_1,k_2,k_3) = \alpha^{(1)}_{p,1}(k_1)\alpha^{(1)}_{p,2}(k_2)\alpha^{(1)}_{p,3}(k_3)$$

and assume that the slope of this product

$$\sigma = \mathrm{ord}_p(\lambda(k_1,k_2,k_3)) = \sigma(k_1,k_2,k_3) = \sigma_1 + \sigma_2 + \sigma_3$$

is constant and positive for all triplets (k_1,k_2,k_3) in an appropriate p-adic neighbourhood of the fixed triplet of weights (k_1,k_2,k_3).

Let $\mathcal{A} = \mathcal{A}(\mathcal{B})$ denote an affinoid algebra $\mathcal{A}$ associated with an analytic space $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, a neighbourhood around $(k_1,k_2,k_3) \in X^3$ (with a given k and $\psi \bmod N$).

Theorem C (on p-adic measures for families of triple products). Put $H = [2\mathrm{ord}_p(\lambda)] + 1$*. There exists a sequence of distributions* Φ_r *on* Y *with values in* $\mathcal{M} = \mathcal{M}(\mathcal{A})$ *giving an* H*-admissible measure* $\tilde\Phi^\lambda$ *with values in* $\mathcal{M}^\lambda \subset \mathcal{M}$ *with the following properties:*

There exists an $\mathcal{A}$-linear form $\ell = \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}$ (given by (11.24)), such that the composition

$$\tilde{\mu} = \tilde{\mu}_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda} := \ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda}(\tilde{\Phi}^\lambda)$$

is an H-admissible measure with values in $\mathcal{A}$, and for all (k_1, k_2, k_3) in the affinoid neighborhood $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, as above, satisfying $k_1 \le k_2 + k_3 - 2$ we have that the evaluated integrals

$$ev_{(k_1,k_2,k_3)}\Big((\ell_{\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3,\lambda})(\tilde{\Phi}^\lambda)(y_p^r\chi)\Big)$$

on the arithmetical chracters $y_p^r\chi$ coincide with the critical special values

$$\Lambda^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi)$$

for $r = 0, \cdots, k_2+k_3-k_1-2$, and for all Dirichlet characters $\chi \bmod Np^v, v \ge 1$, with all three corresonding Dirichlet characters χ_j given by (6.16), having Np-complete conductors. Here the normalisation of Λ^* includes at the same time certain Gauss sums, Petersson scalar products, powers of π and of $\lambda(k_1, k_2, k_3)$, and a certain finite Euler product. The precise form of the Euler-like p-factor is given by a general motivic setting as in [Pa94], [Co], [Co-PeRi]; we also refer to Section 7 of [PaTV] in the case of the L-function of one Coleman's family. However, our modular construction of the admissible measures of Theorem C does not use these explicit formulae. Moreover, these measures are uniquely determined by general unicity properties by all but a finite number of values on characters $y_p^r\chi$.

The p-adic Mellin transform and four variable p-adic analytic functions. Any h-admissible measure $\tilde{\mu}$ on Y with values in a p-adic Banach algebra $\mathcal{A}$ can be caracterized its *Mellin transform* $\mathcal{L}_{\tilde{\mu}}(x)$ $\mathcal{L}_{\tilde{\mu}} : X \to \mathcal{A}$, defined by $\mathcal{L}_{\tilde{\mu}}(x) = \int_Y x(y)d\tilde{\mu}(y)$, where $x \in X$, $\mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}$,

Key property of h-admissible measures $\tilde{\mu}$: its Mellin transform $\mathcal{L}_{\tilde{\mu}}$ is analytic with values in $\mathcal{A}$.

Let $\mathcal{A} = \mathcal{A}(\mathcal{B}) = \mathcal{A}_1\hat{\otimes}\mathcal{A}_2\hat{\otimes}\mathcal{A}_3 = \mathcal{A}(\mathcal{B}_1)\hat{\otimes}\mathcal{A}(\mathcal{B}_2)\hat{\otimes}\mathcal{A}(\mathcal{B}_3)$ denote again the Banach algebra $\mathcal{A}$ where $\mathcal{B}$ is an affinoid neighbourhood around $(k_1, k_2, k_3) \in X^3$ (with a given integer k and Dirichlet character $\psi \bmod N$).

Theorem D (on p-adic analytic function in four variables). Put $H = [2\mathrm{ord}_p(\lambda)]+1$. There exists a p-adic analytic function in four variables $(x, \boldsymbol{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde{\mu}} : (x, \boldsymbol{s}) \longmapsto ev_{\boldsymbol{s}}(\mathcal{L}_{\tilde{\mu}(x)}) \quad (x \in X, \quad \mathcal{L}_{\tilde{\mu}}(x) \in \mathcal{A}).$$

with values in $\mathbb{C}_p$, such that for all (k_1, k_2, k_3) in the affinoid neighborhood as above $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, satisfying $k_1 \le k_2 + k_3 - 2$, we have that the special *values* $\mathcal{L}_{\tilde\mu}(x, \boldsymbol{s})$ *at the arithmetical chracters* $x = y_p^r\chi$, *and* $\boldsymbol{s} = (k_1, k_2, k_3) \in \mathcal{B}$ *coincide with the normalized critical special values*

$$L^*(f_{1,k_1} \otimes f_{2,k_2} \otimes f_{3,k_3}, k_2 + k_3 - 2 - r, \chi) \quad (r = 0, \cdots, k_2 + k_3 - k_1 - 2),$$

for Dirichlet characters $\chi \bmod Np^v, v \geq 1$, *such that all three corresonding Dirichlet characters* χ_j *given by (6.16), have* Np*-complete conductors where the same normalisation of* L^* *as in Theorem C.*

Moreover, for any fixed $\boldsymbol{s} = (k_1, k_2, k_3) \in \mathcal{B}$ *the function*

$$x \longmapsto \mathcal{L}_{\tilde\mu}(x, \boldsymbol{s})$$

is p*-adic analytic of type* $o(\log^H(\cdot))$.

Indeed, we obtain the function in question $\mathcal{L}_\mu(x, \mathrm{s})$ by evaluation at

$$\mathrm{s} = ((s_1, \psi_1), (s_2, \psi_2), (s_3, \psi_3)) \in \mathcal{B} :$$

this is a p-adic analytic function in four variables $(x, \mathrm{s}) \in X \times \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3 \subset X \times X \times X \times X$:

$$\mathcal{L}_{\tilde\mu}(x, \mathrm{s}) := ev_{\mathrm{s}}(\mathcal{L}_{\tilde\mu})(x) \quad (x \in X,\ \mathrm{s} \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3,\ \mathcal{L}_{\tilde\mu}(x) \in \mathcal{A}).$$

This is a joint work in progress with S.Boecherer, we use:

1) the higher twists of the Siegel-Eisenstein series, introduced in [Boe-Schm],

2) Ibukiyama's differential operators (see [Ibu], [BSY]).

11. Scheme of the Proof

11.1. Boecherer's higher twist.

Boecherer's Higher Twist. 1) We define the higher twist of the series $F_{\chi,r} = \sum_{\mathcal{T}} a_{\chi,r}(R, \mathcal{T})q^{\mathcal{T}}$ by some Dirichlet characters $\bar\chi_1, \bar\chi_2, \bar\chi_3$ as the following formal nearly holomorphic Fourier expansion:

$$F_{\chi,r} = \sum_{\mathcal{T}} \bar\chi_1(t_{12})\bar\chi_2(t_{13})\bar\chi_3(t_{23})a_{\chi,r}(R, \mathcal{T})q^{\mathcal{T}}. \tag{11.22}$$

The seies (11.22) is a Siegel modular form of some higher level, but it has additional symmetries with respect to symplectic embedding $\iota_3 : \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \times \Gamma_0(Np^{2v}) \to \mathrm{Sp}_3$: its triple Nebentypus character does not depend on $\chi \bmod Np^v$, and is equal to (ψ_1, ψ_2, ψ_3), if we choose Dirichlet characters as in (6.16):

$$\chi_1 \bmod Np^v = \chi,\ \chi_2 \bmod Np^v = \psi_2\bar\psi_3\chi,$$
$$\chi_3 \bmod Np^v = \psi_1\bar\psi_3\chi, \boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3.$$

We use the Siegel-Eisenstein series $F_{\chi,r}$ which depends on the character χ, but its precise nebentypus character is $\boldsymbol{\psi} = \chi^2\psi_1\psi_2\overline{\psi}_3$, and it is defined by $F_{\chi,r} = G^\star(z, -r; k, (Np^v)^2, \boldsymbol{\psi})$, where z denotes a variable in the Siegel upper half space $\mathbb{H}_3$, and the normalized series $G^\star(z, s; k, (Np^v)^2, \boldsymbol{\psi})$ is given by (5.11).

This series depends on $s = -r$, and for the critical values at integral points $s \in \mathbb{Z}$ such that $2-k \le s \le 0$, it represents a (*nearly*) *holomorphic* Siegel modular form in the sense of Shimura [ShiAr]:

$$F_{\chi,r} = \sum_{\mathcal{T}} \det(\mathcal{T})^{k-2r-\kappa} Q(R, \mathcal{T}; k-2r, r) a_{\chi,r}(\mathcal{T}) q^{\mathcal{T}}.$$

11.2. **Ibukiyama's differential operator.**

Ibukiyama's differential operator. 2) We use an algebraic version of Ibukiyama's differential operator, which generalizes the algebraic "pull-back": it transforms a nearly holomorphic Siegel modular form of weight k

to a nearly holomorphic triple modular form of weight (k_1, k_2, k_3) $(k = k_2 + k_3 - k_1)$.

On a holomorphic Siegel modular form $F = \sum_{\mathcal{T}} a(\mathcal{T}) q^{\mathcal{T}}$, this operator has the form

$$\mathcal{L}_k^{\lambda,\nu}(F) = \sum_{\mathcal{T}} \mathcal{P}(k_1, k_2, k_3, 0, \mathcal{T}) a(\mathcal{T}) q_1^{t_{11}} q_2^{t_{22}} q_3^{t_{33}}, \tag{11.23}$$

where $\lambda = k_1 - k_3 \ge \mu = k_1 - k_2 \ge 0$, and $\mathcal{P}(k_1, k_2, k_3; r; \mathcal{T})$ is certain Ibukiyama's polynomial, equal to $(t_{11}t_{22}t_{33})^\lambda (t_{12}t_{13}t_{23})^\mu$, if $r = 0$.

. As a result we obtain a sequence of triple modular distributions $\Phi_r(\chi)$ with values in the tensor product $\mathcal{M}_T(\mathcal{A}) = \mathcal{M}(\mathcal{A}) \widehat{\otimes}_{\mathcal{A}} \mathcal{M}(\mathcal{A}) \widehat{\otimes}_{\mathcal{A}} \mathcal{M}(\mathcal{A})$ of three Banach $\mathcal{A}$-modules of arithmetical nearly holomorphic modular forms (the normalizing factor 2^r is neeeded in order to prove certain congruences between Φ_r). Note that $\mathcal{M}_T(\mathcal{A})$ is again a Banach $\mathcal{A}$-module on which U_T acts as a completely continuous operator.

The important property of the triple modular forms $\Phi_r(\chi)$: the nebentypus character is fixed and is equal to (ψ_1, ψ_2, ψ_3) (for all (k_1, k_2, k_3) and χ in question).

Using this property we compute the canonical projection $\pi_\lambda(\Phi_r(\chi))$ of the triple modular form $\Phi_r(\chi)$ onto the λ-characteristic $\mathcal{A}$-submodule $\mathcal{M}_T{}^\lambda(\mathcal{A})$ of the triple Atkin's operator $U_{T,p}$:

$$\pi_\lambda : \mathcal{M}_T(\mathcal{A}) \to \mathcal{M}_T^\lambda(\mathcal{A}).$$

. We prove that the resulting sequence of modular distributions $\pi_\lambda(\Phi_r)$ on the profinite group Y produces a certain p-adic admissible measure $\tilde{\Phi}^\lambda$ (in the sense of Amice-Vélu, [Am-V]) with values in a certain locally free $\mathcal{A}$-submodule of finite rank

$$\mathcal{M}_T^\lambda(\mathcal{A}) \subset \mathcal{M}_T(\mathcal{A}) \subset \bigcup_{v\geq 0} \mathcal{M}_T(Np^v, \psi_1, \psi_2, \psi_3; \mathcal{A})$$

of formal nearly holomorphic triple modular forms of all levels Np^v and the fixed nebentypus characters (ψ_1, ψ_2, ψ_3). We use congruences between triple modular forms $\Phi_r(\chi) \in \mathcal{M}_T(\mathcal{A})$ (they have explicit formal Fourier coefficients).

Then we use a general admissibility criterion saying that these congruences imply H-admissibility for their projections in $\mathcal{M}_T^\lambda(\mathcal{A})$, where $H = [2\mathrm{ord}_p(\lambda)] + 1$.

11.3. **Algebraic linear form.** We refer here to Section 6 of [PaTV] for similar constructions in the case of the L-function of one Coleman's family.

. 3) From $\mathcal{M}_T^\lambda(\mathcal{A})$ to $\mathcal{A}$: we use a $\overline{\mathbb{Q}}$-valued linear forms of type

$$\mathcal{L}\ : h \longmapsto \frac{\langle f_1^0 \otimes f_2^0 \otimes f_3^0, h\rangle}{\langle f_1^0, f_{1,0}\rangle \langle f_2^0, f_{2,0}\rangle \langle f_3^0, f_{3,0}\rangle}$$

where f_j^0 is the eigenfunction (8.18) of the conjugate Atkin's operator U_p^*, and $f_{j,0}$ is the eigenfunction (8.19) of U_p. The linear form $\mathcal{L}$ is defined on the finite dimensional $\overline{\mathbb{Q}}$-vector characteristic subspace

$$h \in \mathcal{M}_{\mathbf{k}}(\overline{\mathbb{Q}})^{\lambda(\mathbf{k})} \subset$$
$$\mathcal{M}_{k_1,r^*}(Np, \psi_1; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_2,r^*}(Np, \psi_2; \overline{\mathbb{Q}}) \otimes \mathcal{M}_{k_3,r^*}(Np, \psi_3; \overline{\mathbb{Q}}).$$

This map is then extended to an $\mathcal{A}$-linear map

$$\ell = \ell_{\mathbf{f}_1\otimes\mathbf{f}_2\otimes\mathbf{f}_3,\lambda} : \mathcal{M}(\mathcal{A})^\lambda \to \mathcal{A}; \tag{11.24}$$

on the locally free $\mathcal{A}$-module of finite rank $\mathcal{M}(\mathcal{A})^\lambda$.

. This map produces a sequence of $\mathcal{A}$-valued distributions $\mu_r^\lambda(\chi) \in \mathcal{A}$ in such a way that for all suitable weights $\mathbf{k} \in \mathcal{B}$ one has

$$ev_{\mathbf{k}}(\mu_r^\lambda(\chi)) = \mathcal{L}(ev_{\mathbf{k}}(\pi_\lambda(\Phi_r)(\chi))), \lambda \in \mathcal{A}^\times, \lambda(\mathbf{k}) \in \overline{\mathbb{Q}}^\times,$$

where $\mathbf{k} = (k_1, k_2, k_3) \in \mathcal{B}$, $ev_{\mathbf{k}} : \mathcal{B} \to \mathbb{C}_p$ denotes the evaluation map with the property

$$ev_{\mathbf{k}} : \mathcal{M}(\mathcal{A}) \to \mathcal{M}_{\mathbf{k}}(\mathbb{C}_p).$$

. More precisely, we consider three auxilliary families of modular forms

(11.25) $$\tilde f_{j,k_j}(z) = \sum_{n=1}^{\infty} \tilde a_{n,j,k_j} e(nz) \in S_{k_j}(\Gamma_0(N_j p^{\nu_j}), \psi_j),\ (1 \le j \le 3, \nu_j \ge 1),$$

with the same eigenvalues as those of (10.21),

for all Hecke operators T_q, with q prime to Np. In our construction we use as $\tilde f_{j,k_j}$ certain "easy transforms" of primitive cusp forms in (1.1). In particular, we choose as $\tilde f_j$ certain eigenfunctions $\tilde f_{j,k_j} = f^0_{j,k_j}$ of the adjoint Atkin's operator U_p^*, in this case we denote by $f_{j,k_j,0}$ the corresponding eigenfunctions of U_p.

The $\overline{\mathbb{Q}}$-linear form $\mathcal{L}$ produces a $\mathbb{C}_p$-valued admissible measure $\tilde\mu^\lambda = \ell(\tilde\Phi^\lambda)$ starting from the modular p-adic admissible measure $\tilde\Phi^\lambda$ of stage 3), where $\ell : \mathcal{M}_T(\mathbb{C}_p) \to \mathbb{C}_p$ denotes a $\mathbb{C}_p$-linear form, interpolating $\mathcal{L}$.

11.4. **Evaluation of p-adic integrals.** We refer to Section 7 of [PaTV] for similar constructions in the case of the L-function of one Coleman's family.

L-values and p-adic integrals. 4) We show that for any appropriate Dirichlet character χ mod Np^v the integral

$$\mu_r^\lambda(\chi) = \mathcal{L}(\pi_\lambda(\Phi_r(\chi))) \in \mathcal{A}$$

evaluated at $(k_1, k_2, k_3) \in \mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3$, coincides (up to a normalisation) with the special L-value

$$L^*(f^\rho_{1,k_1} \otimes f^\rho_{2,k_2} \otimes f^\rho_{3,k_3}, k_2 + k_3 - 2 - r, \psi_1\psi_2\chi)$$

under the above assumptions on χ and r).

A general integral representation of Garrett's type. The basic idea how a Dirichlet character χ is incorporated in the integral representation [Ga87, BoeSP] is somewhat similar to the one used in [Boe-Schm], but (surprisingly) more complicated to carry out.

Note however that the existence of a $\mathcal{A}$-valued admissible measure $\tilde\mu^\lambda = \ell(\tilde\Phi^\lambda)$ established at stage 4), does not depend on this technical computation.

In order to control the denominators of the modular forms

$$\pi_\lambda(\tilde{\mathcal{E}}(-r, \chi)) \in \mathcal{M}^\lambda(\mathcal{A}) =: \Phi_r(\chi),$$

used in the construction (the admissibility condition) we use the following result.

12. Criterion of admissibility

THEOREM 12.1 (Criterion of admissibility). *Let $\alpha \in \mathcal{A}^*$, $0 < |\alpha|_p < 1$ and suppose that there exists a positive integer $\varkappa$ such that the following conditions are satisfied:*

1) for all $r = 0, 1, \cdots, h-1$ with $h = [\varkappa \mathrm{ord}_p \alpha] + 1$, and $v \geq 1$,

$$\Phi_r(a + (Np^v)) \in \mathcal{M}(Np^{\varkappa v}) \qquad \text{(the level condition)} \tag{12.26}$$

2) the following congruence for the coefficients holds: for all $t = 0, 1, \cdots, h-1$

$$U^{\varkappa v} \sum_{r=0}^{t} \binom{t}{r} (-a_p)^{t-r} \Phi_r(a + (Np^v)) \equiv 0 \bmod p^{vt} \tag{12.27}$$

(the divisibility condition)

Then the linear form given by $\tilde{\Phi}^\alpha(\delta_{a+(Np^v)} y_p^r) := \pi_\alpha(\Phi_r(a + (Np^v))$ on local monomials (for all $r = 0, 1, \cdots, h-1$), is an h-admissible measure: $\tilde{\Phi}^\alpha : \mathcal{P}^h(Y, \overline{\mathbb{Q}}) \to \mathcal{M}^\alpha \subset \mathcal{M}$

Proof uses the commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(Np^{v+1}, \psi; \mathcal{A}) & \xrightarrow{\pi_{\alpha,v}} & \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}) \\ U^v \downarrow & & \downarrow \wr\, U^v \\ \mathcal{M}(Np, \psi; \mathcal{A}) & \underset{\pi_{\alpha,0}}{\longrightarrow} & \mathcal{M}^\alpha(Np, \psi; \mathcal{A}) \quad = \mathcal{M}^\alpha(Np^{v+1}, \psi; \mathcal{A}). \end{array}$$

The existence of the projectors $\pi_{\alpha,v}$ comes from Coleman's Theorem A.4.3 [CoPB].

On the right: U acts on the locally free $\mathcal{A}$-module $\mathcal{M}^\alpha(Np^{v+1}, \mathcal{A})$ via the matrix:

$$\begin{pmatrix} \alpha \cdots & \cdots & * \\ 0 & \alpha & \cdots & * \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & \cdots & \alpha \end{pmatrix} \text{ where } \alpha \in \mathcal{A}^\times$$

$\Longrightarrow U^v$ is an isomorphism over $\mathcal{A}$,

and one controls the denominators of the modular forms of all levels v by the relation:

$$\pi_{\alpha,v}(h) = U^{-v} \pi_{\alpha,0}(U^v h) =: \pi_\alpha(h) \tag{12.28}$$

The equality (12.28) can be used as the definition of π_α at any level Np^v.

The growth condition (see (7.17)) for $\pi_\alpha(\Phi_r)$ is deduced from the congruences (12.27) between modular forms, using the relation (12.28).

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