Complex and p-adic L functions on classical groups. Admissible measures, special values.

("Fonctions L p-adiques et complexes sur les groupes classiques : mesures admissibles, valeurs spéciales").

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Summer School on the Theory of Motives and the Theory of Numbers (L'École d'été à LAMA)

Fonctions zetas, polyzetas, séries arithmétiques : applications aux motifs et à la théorie des nombres.

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This Summer School on Theory of Motives and Number Theory

at the crossroad of automorphic L functions (complex and p-adic), zeta functions, polyzeta functions and dynamical zeta function, was conceived as a continuation of a seris of Conferences "Zeta-functions I-VI", held in J.-V.Poncelet Laboratory UMI 2615 du CNRS, The Higher School of Economics, and Independent University in Moscow, and mainly organized by Alexey ZYKIN, professor of the French Polynesia University in Tahiti, who tragically dissapeared in April 2017 together with his wife Tatiana MAKAROVA.

The Summer Schools «Algebra and Geometry» (Yaroslavl, RUSSIA, July 2012-2016), were also largely organized by Alexey ZYKIN together with Fyodor BOGOMOLOV and Courant Institute (New York, USA), see the video and pdf of lectures at
`http://bogomolov-lab.ru/SHKOLA2012/talks/panchishkin.html`

Alexey ZYKIN near Grenoble on June 22, 2012

Figure: Climbing the mountain Chamchaude with Siegfried BOECHERER

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Lecture N°2. Classical and Hermitian modular forms. Automorphic complex L-functions on classical groups. Hecke algebras. The Rankin-Selberg method
Lecture N°3. Distributions, measures, Kummer congruences. Kubota-Leopoldt p-adic zeta function and Iwasawa algebra.
Lecture N°4. p-adic L-functions on classical groups. Ordinary case. Admissible measures, special values.

(L'intervention d'Alexei Pantchichkine:
1) Groupes classiques, le cas $\mathrm{GL}(n)$, le cas symplectique et unitaires. Formes modulaires et formes automorphes, exemples.
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Lecture N°1. Classical groups, the case of $\mathrm{GL}(n)$

The sympectic and unitary cases. Modular forms and automorphic forms. ("Groupes classiques, le cas $\mathrm{GL}(n)$ le cas symplectique et unitaires. Formes modulaires et formes automorphes, exemples").

- Linear Algebraic Groups. §1-6 of [Bor66]
- Radical. Parabolic subgroups. Reductive groups.
- Structure theorems for reductive groups.
  §6.5 of[MaPa], Automorphic Forms and The Langlands Program
- §6.5.1 A Relation Between Classical Modular Forms and Representation Theory
- §I-II of [Bor79]: I. Definition of the L-group
- II. Quasi-split groups *

Reductive groups

Recall: an algebraic group is irreducible if and only if it is connected. The connected component of the identity of G will be denoted by G^0. The index of G^0 in G is finite.

Definition

Let G be an algebraic group over a field k. The radical $R(G)$ of G is the greatest connected normal subgroup of G; the unipotent radical $R_u(G)$ is the greatest connected unipotent normal subgroup of G. The group G is semisimple (resp. reductive) if $R(G) = \{e\}$ (resp. $R_u(G) = \{e\}$).

The definitions of $R(G)$ and $R_u(G)$ make sense, because if H, H' are connected normal and solvable (resp. unipotent) subgroups, then so is $H \cdot H'$. Both radicals are k-closed if G is a k-group. Clearly, $R(G) = R(G^0)$ and $R_u(G) = R_u(G^0)$. The quotient G/R(G) is semisimple, and $G/R_u(G)$ is reductive. In characteristic zero, the unipotent radical has a complement; more precisely: Let G be defined over k. There exists a maximal reductive k-subgroup H of G such that $G = H \cdot R_u(G)$, the product being a semidirect product of algebraic groups. If H' is a reductive subgroup of G defined over k, then H' is conjugate over k to a subgroup of H of G such that $G = H \cdot R_u(G)$, the product being a semidirect product of algebraic groups. If H' is a reductive subgroup of G defined over k, then H' is conjugate over k to a subgroup of H.

Theorem (5.2 of [Bor66])

Let G be an algebraic group. The following conditions are equivalent :
(1) G^0 is reductive,
(2) $G^0 = S \cdot G'$, where S is a central torus and G' is semisimple,

Theorem (5.3 of [Bor66])

Let G be a connected algebraic group.
(1) All maximal tori of G are conjugate. Every semisimple element is contained in a torus. The centralizer of any subtorus is connected.
(2) All maximal connected solvable subgroups are conjugate. Every element of G belongs to one such group.
(3) If P is a closed subgroup of G, then G/P is a projective variety if and only if P contains a maximal connected solvable subgroup.

Characters and roots.

A character of G is a rational representation of degree 1; $\chi : G \to \mathrm{GL}_1$. The set of characters of G is a commutative group, denoted by $X(G)$ or $\hat{G}$. The group $\hat{G}$ is finitely generated; it is free if G is connected [Bor66], p.6. If one wants to write the composition-law in $\hat{G}$ multiplicatively, the value at $g \in G$ of $\chi \in \hat{G}$ should be noted $\chi(g)$. But since one is accustomed to add roots of Lie algebras, it is also natural to write the composition in G additively. The value of χ at g will then be denoted by g^χ. To see the similarity between roots and characters take $\Omega = \mathbb{C}$; if $X \in \mathfrak{g}$, the Lie algebra of G, $(e^x)^\chi = ed_\chi(X)$, where d_χ is the differential at e; d_χ is a linear form over $\mathfrak{g}$. In the sequel, we not make any notational distinction between a character and its differential at e.

Let $g \in \mathrm{GL}(n, \Omega)$, g can be written uniquely as the product $g = g_s \cdot g_n$, where g_s is a semisimple matrix (i.e., g. can be made diagonal) and g_n is a unipotent matrix (i.e., the only eigenvalue of g_n is 1, or equivalently $g_n - I$ is nilpotent) and $g_s \cdot g_n = g_n \cdot g_s$.

Example: the case of $\mathrm{GL}(n)$

The rank of G is the common dimension of the maximal tori, (notation rk(G)). A closed subgroup P of G is called parabolic, if G/P is a projective variety. A maximal connected closed solvable subgroup is called Borel subgroup.

Exemple. $G = \mathrm{GL}_n$. A flag $\mathcal{F}$ in a vector space V is a properly increasing sequence of subspaces

$$\mathcal{F} : 0 \neq V_1 \subset \cdots \subset V_t \subset V_{t+1} = V.$$

The sequence (d_i) $(d_i = \dim v_i, i = 1, \cdots t))$ describes the type of the flag. If $d_i = i$ and $t = \dim V - 1$, we speak of a full flag.
A parabolic subgroup of GL_n is the stability group of a flag $\mathcal{F}$ in Ω^n. G/P is the manifold of flags of the same type as F, and is well known to be a projective variety. A Borel subgroup is the stability group of a full flag. In a suitable basis, it is the group of all upper triangular matrices.

The case of orthogonal group $G = SO(F)$

of a nondegenerate quadratic form F on a vector space V_k (where, to be safe, one takes char $k \neq 2$). In a suitable basis

$$F(x_1, \cdots, x_n) = x_1x_n + x_2x_{n-1} + \cdots + x_qx_{n-q+1} + F_0(x_{q+1}, \cdots, x_{n-q})$$

where F_0 does not represent zero rationally. The index of F, the dimension of the maximal isotropic subspaces in V, is equal to q. A maximal k-split torus S is given by the set of following diagonal matrices:

$$\begin{pmatrix} s^{\lambda_1} & & & & & & & & & & \\ & s^{\lambda_2} & & & & & & & & & \\ & & \ddots & & & & & & & & \\ & & & s^{\lambda_q} & & & & & & & \\ & & & & 1 & & & & & & \\ & & & & & \ddots & & & & & \\ & & & & & & 1 & & & & \\ & & & & & & & s^{-\lambda_q} & & & \\ & & 0 & & & & & & s^{-\lambda_{q-1}} & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & s^{-\lambda_1} \end{pmatrix}$$

Example: Unitary group

Let us review some background and set up standard notation. Let E be a quadratic imaginary field, embedded in $\mathbb{C}$; $0 \leq m \leq n$ and $\Lambda = \mathcal{O}_E^{n+m}$. Let

$$I_{n,m} = \begin{pmatrix} & & I_m \\ & I_{n-m} & \\ I_m & & \end{pmatrix}$$

where I_ℓ is the unit matrix of size ℓ, and introduce the perfect hermitian pairing

$$(u, v) = {}^t\bar{u} I_{n,m} v$$

on Λ. Let $G = GU(\Lambda, (,))$ be the group of unitary similitudes of Λ, regarded as a group scheme over $\mathbb{Z}$; and denote by $\nu : G \to \mathbf{G}_m$ the similitude character. For any commutative ring R

$$G(R) = \{g \in GL_{n+m}(\mathcal{O}_E \otimes R) | \forall u, v \in \Lambda \otimes R,\ (gu, gv) = \nu(g)(u, v)\}.$$

Then $G(R) = GU(n, m)$ is the *general unitary group of signature* (n, m), and $G(\mathbb{C}) \cong GL_{n+m}(\mathbb{C}) \otimes \mathbb{C}$.

Geometric algebra (see [Dieudonné], [Garrett])

- $\mathrm{GL}(n)$ (geometric study).
- Bilinear and Hermitian forms, classical groups
- Witt theorem and extensions of isometries *

This section is based on notions of geometric algebra.
Concerning matrix notation, for a rectangular matrix $A = (a_{ij})$ let ${}^tA$ denote the transpose of A. If the entries of A belong to a ring D with involution muni d'involution σ, let A^σ given by $A^\sigma_{ij} = a^\sigma_{ij}$.
Geometric study of $\mathrm{GL}(n)$ and its subgroups. The group $\mathrm{GL}(n)$ is a basic classical group showing the most interesting phenomena used in many other situations. The general linear group $\mathrm{GL}(n, k)$ is the group of all invertible $n \times n$ matrices with entries in a commutative field k.The special linear group $\mathrm{SL}(n, k)$ is its subgroup of all $n \times n$ of determinant 1.
For an approach less dependant of coordonnates fix a k-vector space V of dimension n and let $\mathrm{GL}_k(V)$ be the group of all k-linear automorphisms of V . Any choice of a base in V gives an isomorphism $\mathrm{GL}_k(V) \to \mathrm{GL}(n, k)$ using the matrix of linear mapping in the chosen base. Let $e_1, \cdots, e_n$ the standard bases of k^n giving the isomorphhism $\mathrm{GL}_k(k^n) \to \mathrm{GL}(n, k)$.

Conjugation of parabolic subgroups.

Let $V = k^n$ and $\mathcal{F}$ the standard flag of type $(d_1, \cdots, d_m)$, the parabolic subgroup $P_{\mathcal{F}}$ is represented by blocs

$$\begin{pmatrix} d_1 \times d_1 & * & * & * \\ & (d_2 - d_1) \times (d_2 - d_1) & * & * \\ & & \ddots & * \\ & \cdots & \cdots & \\ 0 & 0 & 0 & (n - d_m) \times (n - d_m) \end{pmatrix}$$

Any $g \in P = P_{\mathcal{F}}$ induce a natural mapping on the quotients $Vd_i/V_{d_{i-1}}$, where $V_{d_0} = 0$ and $V_{d_{m+1}} = V$).
Then the unipotent radical $R_u P =$
$\{p \in P_{\mathcal{F}} \mid p = id$ on $Vd_i/V_{d_{i-1}}$ and $V/V_{d_m}\}$ is represnted by

$$\begin{pmatrix} 1_{d_1} & * & * & * \\ & 1_{d_2 - d_1} & * & * \\ & & \ddots & * \\ & \cdots & \cdots & \\ 0 & 0 & 0 & 1_{n-d_m} \end{pmatrix}.$$

Levi components and conjugation

Choose a complement V'_{n-d_i} of V_{d_i} in V with the property $V'_{n-d_m} \subset \cdots \subset V'_{n-d_1}$ (an opposit flag $\mathcal{F}'$ of $\mathcal{F}$) with the opposit parabolic $P' = P_{\mathcal{F}'}$. Then $M = P \cap P'$ is called a complementary Levi component in $P = M \ltimes R_u P$, a standard semi-direct product. Then the standard Levi component is the group of matrices of the form

$$\begin{pmatrix} d_1 \times d_1 & 0 & 0 & 0 \\ & (d_2 - d_1) \times (d_2 - d_1) & 0 & 0 \\ & & \ddots & 0 \\ & \cdots & \cdots & \\ 0 & 0 & 0 & (n - d_m) \times (n - d_m) \end{pmatrix}.$$

Proposition

a) All the parabolic subgroups of given type are conjugate in $\mathrm{GL}_k(V)$

b) All the Levi components of parabolic subgroup P are conjugate by elements of P

c) All the maximal k-split tori are conjugate in $\mathrm{GL}_k(V)$.

∎

Extension to modules over a scew field

This section applies unchanged when k is replaced by a scew field (a division ring) D. Without coordinates, define a vector space V of finite dimension over a scew field (a division ring) D as a finitely generated (left or right) module.
If D is not commutative, there is a modification in viewing at D-lineair endomorphisms.The the ring $\mathrm{End}_D(V)$ of all D-lineair endomorphisms does not contain D naturallurally. Then a choice of D-bases for a vector space D of given dimension gives an isomorphism $\mathrm{End}_D(V)$ to $n \times n$ matrices with coeffcients in D^{opp}, where D^{opp} is the opposite ring to D, i.e.with the same additive group D but with the multiplication $*$, given by $x * y = yx$ where yx is the multiplication in D. The linear group $\mathrm{GL}(n, D)$ over D is the group of all the invertible $n \times n$ matrices over D. A version without coordinates is $GL_D(V)$,and a choice of D-bases of V gives an isomorphism $\mathrm{GL}_D(V) \to \mathrm{GL}(n, D^{opp})$. Definitions concerning flags and parabolics are identical to the commutative case. A flag $\mathcal{F}$ in V is a chain $\mathcal{F} = (V_{d_1} \subset V_{d_2} \subset \cdots \subset V_{d_m})$ of subspaces.

Proposition

a) All the parabolic subgroups of a given type are conjugate in $\mathrm{GL}_D(V)$
b) All the Levi components of parabolic subgroup P are conjugate by elements of P.

∎

Bilinear, sesquilinear and Hermitian forms; classical groups

The classical groups are defined as certain isomtries or similitudes of "formes" on the vector spaces. First, orthogonal and symplectic groups are defined. These can be included into more general families.

Bilinear forms, symmetric and symplectic forms

Let $Q(v) = \langle v, v \rangle$ be the quadratic form attached to a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on a k-vector space V.
The associated orthogonal group $O(Q)$ is the group of isometries of Q (or of $\langle \cdot, \cdot \rangle$), defined as

$$O(Q) = O(\langle \cdot, \cdot \rangle) = \{g \in GL_k(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle\},$$

and the group of orthogonal similitudes is $GO(Q) = GO(\langle \cdot, \cdot \rangle)$

$$= \{g \in GL_k(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \nu(g) \langle v_1, v_2 \rangle\}$$

If $\forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = -\langle v_1, v_2 \rangle$, then the bilinear form $f : V \times V \to k, f(v_1, v_2) = \langle v_1, v_2 \rangle$ is said symplectic.The symplectic group attached to f is the group of isometries of the form $f = \langle v_1, v_2 \rangle$ defined by
$\mathrm{Sp}(f) = \{g \in GL_k(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle\}$, puis the group of symplectis symilitudes $\mathrm{GSp}(f) =$
$\{g \in GL_k(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \nu(g) \langle v_1, v_2 \rangle\}$.

Structure theorems for reductive groups.

Root systems. Let V be a finite dimensional real vector space endowed with a positive nondegenerate scalar product. A subset Φ of V is a root system when
(1) Φ consists of a finite number of nonzero vectors that generate V, and is symmetric ($\Phi = -\Phi$). (2) for every $\alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$, where s_α denotes reflection with respect to the hyperplane perpendicular to α. (3) if $\alpha, \beta \in \Phi$, then

$$2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}.$$

The group generated by the symmetries $s_\alpha(\alpha \in \Phi)$ is called the Weyl group of Φ (notation $W(\Phi)$ is finite. The integers $2(\alpha, \beta)/(\alpha, \alpha)$ are called the Cartan integers of Φ. The integrity condition means that for every α and β of Φ, $(s_\alpha(\beta) - \beta)$ is an integral multiple of α, since

$$s_\alpha(\beta) = \beta - 2\alpha(\alpha, \beta)/(\alpha, \alpha).$$

For the theory of reductive groups we shall have to enlarge slightly the notion of root system: if M is a subspace of V, we say that Φ is a root system in (N, M) if it generates a subspace P supplementary to M, and is a root system in P. The Weyl group $W(\Phi))$ is then understood to act trivially on M. A root system Φ in V is the direct sum of $\Phi' \subset V'$ and $\Phi" \subset V"$, if $V = V' \oplus V"$, and $\Phi = \Phi' \cup \Phi"$. The root system is called irreducible if it is not the direct sum of two subsystems.

Properties of root systems.

(1) Every root system is direct sum of irreducible root systems.
(2) If α and $\lambda\alpha \in \Phi$, then $\lambda = \pm 1, \pm(1/2)$, or ± 2.
The root system Φ, is called reduced when for every $\alpha \in \Phi$, the only multiples of α belonging to Φ are $\pm\alpha$. To every root system Φ, there belongs two natural reduced systems by removing for every $\alpha \in \Phi$, the longer (or the shorter) multiple of α :
(3) The only reduced irreducible root systems are the usual ones (The first four are those belonging to the classical series, , with the simple roots as follows: take $\mathbb{R}^n$ with the standard metric and basis $\{\lambda_1, \cdots, \lambda_n\}$)
$A_n (n \geq 1)$, $\mathfrak{sl}_{n+1}$ $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n, \lambda_n - \lambda_{n+1}$
$B_n (n \geq 2)$, $\mathfrak{so}_{2n+1}$, $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n, \lambda_n$,
$C_n (n \geq 3)$, $\mathfrak{sp}_{2n}$, $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n, 2\lambda_n$
$D_n (n \geq 4)$, $\mathfrak{so}_{2n}$ $\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n, \lambda_{n-1} + \lambda_n$
G_2, F_4, E_6, E_7, E_8 (the exceptional root systems, see [FuHa91], §21).
EXAMPLES : B_n: Take $\mathbb{R}^n$ with the standard metric and basis $\{\lambda_1, \cdots, \lambda_n\}$,
$B_n = \{\pm(\lambda_i \pm \lambda_i)(i < j)$ and $\pm\lambda_i (1 \leq i \leq n)\}$.
$W(B_n) = \{s \in \mathrm{GL}(n, R)$ is a product of a permutation matrix with a symmetry with respect to a coordinate subspace$\}$
$C_n = \{\pm(\lambda_i \pm \lambda_j)\ \ (i < j)$ and $\pm 2\lambda_i (1 \leq i \leq n\}$,
$W(C_n) = W(B_n)$,

Example of $G = GL(n)$

$S =$ group of diagonal mattices $= \{\mathrm{diag}(s^{\lambda_1}, s^{\lambda_2}, .., s^{\lambda_n})\}$ where $\lambda_i \in \hat{S}$ is such that $s^{\lambda_i} = s_{ii}$. S is a split torus and is maximal.
A minimal parabolic k-subgroup P is given by the upper triangular matrices, which is in this case a Borel subgroup. The unipotent radical U of P is given by the group of upper triangular matrices with ones in the diagonal. The Lie algebra $\mathfrak{g}l_n = \mathrm{M}_{n\times n}$ with a basis $\{e_{ij}\}_{i,j}$, where e_{ij} is the matrix having all components zero except that with index (i,j) equal to 1,

$$\mathrm{Ad}_G s(e_{ij}) = (s^{\lambda_i}/s^{\lambda_j})e_{ij}.$$

So the positive roots are $\lambda_i - \lambda_j (i < j)$ since the Lie algebra of U is generated by $e_{ij}(i < j)$. The simple roots are

$(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{n-1} - \lambda_n)$ (the root system of type A_{n-1})

The Weyl group is generated by s_α, where α is a positive root; since for $\alpha = \lambda_i - \lambda_j$, s_α permutes the i and j axis, ${}_kW = \mathfrak{S}_n$, the group of permutations of the basis elements. The parabolic subgroups are the stability groups of flags.

Matrix description of the symplectic group, see [An87]

Let $G = \mathrm{GSp}_{2n}$ be the algebraic subgroup of GL_{2n} defined by
$G_A = \{\gamma \in \mathrm{GL}_{2n}(A) \mid {}^t\gamma J_n \gamma = \nu(\gamma)J_n, \ \ \nu(\gamma) \in A^\times\}$, for any commutative ring A, where $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$. The elements of G_A are characterized by the conditions
$b^ta - a^tb = d^tc - c^td = 0_n, d^ta - c^tb = 1_n$, and if
$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_A$ then $\gamma^{-1} = \nu(\gamma)^{-1} \begin{pmatrix} {}^td & -{}^tb \\ -{}^tc & {}^ta \end{pmatrix}$.
The multiplier ν defines a homomorphism $\nu : G_A \to A^\times$ so that $\nu(\gamma)^{2n} = \det(\gamma)^2$ and $\ker(\nu)$ is denoted by $\mathrm{Sp}_n(A)$.

Matrix description of the symplectic Lie algebra $\mathfrak{sp}_{2n}$

$\mathfrak{sp}_{2n} = \{X \in \mathfrak{gl}_{2n} | {}^tXJ_n + J_nX = 0\}$. Writing elements of $\mathfrak{g} = \mathfrak{sp}_{2n}$ in block form: $X = \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix}$, where $A, B, C \in \mathrm{M}_{n\times n}$ and $B = {}^tB$, $C = {}^tC$. Note that the dimension of $\mathfrak{sp}_{2n}$ is $n(2n+1)$. A maximal

torus S is $\begin{pmatrix} s^{\lambda_1} & & & & & & & \\ & s^{\lambda_2} & & & & & & \\ & & \ddots & & & & & \\ & & & s^{\lambda_n} & & & & \\ & & & & s^{-\lambda_n} & & & \\ & & 0 & & & s^{-\lambda_{n-1}} & & \\ & & & & & & \ddots & \\ & & & & & & & s^{-\lambda_1} \end{pmatrix}$

Automorphic complex L-functions on classical groups.

- §6.5. of [MaPa] Automorphic Forms and The Langlands Program
- Automorphic L-Functions
- Analytic properties of automorphic L-functions
- Hecke algebras.
- Section IV of [Bor79] The L-function of an automorphic representation.

6.5 Automorphic Forms and The Langlands Program

6.5.1 A Relation Between Classical Modular Forms and Representation Theory

(cf. [Bor79], [PSh79]). The domain of definition of the classical modular forms (the upper half plane) is a homogeneous space $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$ of the reductive group $G(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$:

$$\mathbb{H} = \mathrm{GL}_2(\mathbb{R})/\mathcal{O}(2) \cdot Z,$$

where $Z = \{\left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \ |x \in \mathbb{R}^\times\}$ is the center of $G(\mathbb{R})$ and $\mathcal{O}(2)$ is the orthogonal group, see (6.3.1). Therefore each modular form

$$f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in \mathcal{M}_k(N, \psi) \subset \mathcal{M}_k(\Gamma_N) \tag{6.5.1}$$

can be lifted to a function $\tilde{f}$ on the group $\mathrm{GL}_2(\mathbb{R})$ with the invariance condition

$$\tilde{f}(\gamma g) = \tilde{f}(g) \quad \text{for all} \quad \gamma \in \Gamma_N \subset \mathrm{GL}_2(\mathbb{R}).$$

In order to do this let us consider the function

$$\tilde{f}(g) = \begin{cases} f(g(i))j(g,i)^{-k} & \text{if } \det g > 0, \\ f(g(-i))j(g,-i)^{-k} & \text{if } \det g < 0, \end{cases} \tag{6.5.2}$$

where $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and $j(g,i) = |\det g|^{-1/2}(cz+d)$ is the factor of automorphy.

One has $\tilde{f}(xg) = \exp(-ik\theta)\tilde{f}(g)$ if $x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is the rotation through the angle θ.

Consider the group $\mathrm{GL}_2(\mathbb{A})$ of non-degenerate matrices with coefficients in the adele ring $\mathbb{A}$ and its subgroup

$$U(N) = \left\{ g = 1 \times \prod_p g_p \in \mathrm{GL}_2(\mathbb{A}) \mid g_p \in \mathrm{GL}_2(\mathbb{Z}_p), g_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N\mathbb{Z}_p \right\}. \tag{6.5.3}$$

From the *chinese remainder theorem* (the *approximation theorem*) one obtains the following coset decomposition:

$$\Gamma_N \backslash \mathrm{GL}_2(\mathbb{R}) \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A})/U(N), \tag{6.5.4}$$

using which we may consider $\tilde{f}$ as a function on the homogeneous space (6.5.4), or even on the adele group $\mathrm{GL}_2(\mathbb{A})$.

The action of $\mathrm{GL}_2(\mathbb{A})$ on $\tilde{f}$ by group shifts defines a representation $\pi = \pi_f$ of the group $\mathrm{GL}_2(\mathbb{A})$ in the space of smooth complex valued functions on $\mathrm{GL}_2(\mathbb{A})$, for which

$$\left(\pi(h)\tilde{f}\right)(g) = \tilde{f}(gh) \quad (g, h \in \mathrm{GL}_2(\mathbb{A})).$$

The condition that the representation π_f be irreducible has a remarkable arithmetical interpretation: it is equivalent to f being an eigenfunction of the Hecke operators for almost all p. If this is the case then one has an infinite tensor product decomposition

$$\pi = \bigotimes_v \pi_v, \tag{6.5.5}$$

where the π_v are representations of the local groups $\mathrm{GL}_2(\mathbb{Q}_v)$ with $v = p$ or ∞.

Jacquet and Langlands chose irreducible representations of groups such as $\mathrm{GL}_2(\mathbb{Q}_v)$ as a starting point for the construction of L–functions (cf. [JL70], [Bor79]). These representations can be classified and explicitly described. Thus for the representations π_v in (6.5.5) one can verify for almost all $v = v_p$ that the representation π_v has the form of an induced representation $\pi_v = \mathrm{Ind}(\mu_1 \otimes \mu_2)$ from a one dimensional representation of the subgroup of diagonal matrices

$$(\mu_1 \otimes \mu_2) \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \mu_2(x)\mu_1(y),$$

where $\mu_i : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ are unramified quasicharacters (see §6.2.4). This classification makes it possible to define for almost all p an element $h_p = \begin{pmatrix} \mu_1(p) & 0 \\ 0 & \mu_2(p) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$. From this one can construct the following Euler product (the L–function of the automorphic representation π)

$$L(\pi, s) = \prod_{p \notin S} L(\pi_p, s) = \prod_{p \notin S} \det(1_2 - p^{-s} h_p)^{-1} \tag{6.5.6}$$

in which the product is extended over all but a finite number of primes.

It turns out that the function $L(\pi, s)$ coincides essentially with the Mellin transform of the modular form f:

$$L(s, f) = L(\pi_f, s + (k-1)/2).$$

The notion of a primitive form f also takes on a new meaning: the corresponding function $\tilde{f}$ from the representation space of an irreducible representation π must have a maximal stabilizer. The theory of Atkin–Lehner can be reformulated as saying that the representation π_f occurs with multiplicity one in the regular representation of the group $\mathrm{GL}_2(\mathbb{A})$ (the space of all square integrable functions).

More generally, an *automorphic representation* is defined as an irreducible representation of an adele reductive group $G(\mathbb{A})$ in the space of functions on $G(\mathbb{A})$ with some growth and smoothness conditions.

Jacquet and Langlands constructed for irreducible admissible automorphic representations π of the group $\mathrm{GL}_2(\mathbb{A})$ analytic continuations of the corresponding L–functions $L(\pi, s)$, and established functional equations relating $L(\pi, s)$ to $L(\tilde{\pi}, 1-s)$, where $\tilde{\pi}$ is the dual representation. For the functions $L(\pi_f, s)$ this functional equation is exactly Hecke's functional equation (see (6.3.44)).

Note that the notion of an automorphic representation includes as special cases: 1) the classical elliptic modular forms, 2) the real analytic wave modular forms of Maass, 3) Hilbert modular forms, 4) real analytic Eisenstein series of type $\sum' \dfrac{y^s}{|cz+d|^{2s}}$, 5) Hecke L-series with Grössen–characters (or rather their inverse Melin transforms), 6) automorphic forms on *quaternion algebras* etc.

Interesting classes of Euler products are related to finite dimensional complex representations

$$r : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C}).$$

Let us consider the Euler product

$$L(\pi, r, s) = \prod_p L(\pi_p, r, s), \tag{6.5.7}$$

where

$$L(\pi_p, r, s) = \det(1_m - p^{-s} r(h_p))^{-1}.$$

These products converge absolutely for $\mathrm{Re}(s) \gg 0$, and, conjecturally, admit analytic continuations to the entire complex plane and satisfy functional equations (cf. [Bor79], [BoCa79], [L71a], [Del79], [Se68a]).

This conjecture has been proved in some special cases, for example when $r = \mathrm{Sym}^i\mathrm{St}$ is the i^{th} symmetric power of the standard representation $\mathrm{St} : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$ for $i = 2, 3, 4, 5$ (cf. [Sh88]).

The Ramanujan–Petersson conjecture, proved by Deligne, can be formulated as saying that the absolute values of the eigenvalues of $h_p \in \mathrm{GL}_2(\mathbb{C})$ for a cusp form f are all equal to 1.

As a consequence of the conjectured analytic properties of the functions (6.5.7) one could deduce the following conjecture of Sato and Tate about the distribution of the arguments of the Frobenius elements: let $\alpha(p) = e^{i\varphi_p}$ $(0 \le \varphi_p \le \pi)$ be an eigenvalue of the matrix h_p defined above. Then for cusp forms f without complex multiplication (i.e. the Mellin transform of f is not the L–function of a Hecke Grössencharacter (see §6.2.4) of an imaginary quadratic field) the arguments φ_p are conjecturally uniformly distributed in the segment $[0, \pi]$ with respect to the measure $\dfrac{2}{\pi}\sin^2\varphi \, d\varphi$ (cf. [Se68a]).

In the case of complex multiplication the analytic properties of the L-functions are reduced to the corresponding properties of the L-functions of Hecke Grössencharacters (see §6.2.4), which imply the uniform distribution of the arguments φ_p with respect to the usual Lebesgue measure.

The arithmetical nature of the numbers $e^{i\varphi_p}$ is close to that of the signs of Gauss sums $\alpha(p) = g(\chi)/\sqrt{p}$ where $g(\chi) = \sum_{u=1}^{p-1} \chi(u)e(u/p)$, χ being a primitive Dirichlet character modulo p. Even if χ is a quadratic character, the precise evaluation of the sign $\alpha(p) = \pm 1$ is rather delicate (see [BS85]). If χ is a cubic character, i.e. if $\chi^3 = 1$ then $p = 6t + 1$, and the sums lie inside the 1st, the 3rd or the 5th sextant of the complex plane. Using methods from the theory of automorphic forms S.J.Patterson and D.R.Heath–Brown solved the problem of Kummer on the distribution of the arguments of cubic Gauss sums by means of a cubic analogue of the theta series, which is a certain automorphic form on the threefold covering of the group GL_2 ([Del80a], [HBP79], [Kub69]).

6.5.2 Automorphic L-Functions

The approach of Jacquet–Langlands made it possible to extend the whole series of notions and results concerning L-functions to the general case of automorphic representations of reductive groups over a global field K. Let G be a linear group over K, $G_{\mathbb{A}} = G(\mathbb{A})$ its group of points with coefficients in the adele ring of the field K. Automorphic representations are often defined as representations belonging to the regular smooth representation of the group $G_{\mathbb{A}}$, and one denotes by the symbol $\mathfrak{A}(G/K)$ the set of equivalence classes of irreducible admissible automorphic representations of $G_{\mathbb{A}}$. A representation π from this class admits a decomposition $\pi = \otimes_v \pi_v$ where $v \in \Sigma_K$ runs through the places of K and the π_v are representations of the groups $G_v = G(K_v)$. In order to construct L-functions, the L-group ${}^LG$ of G is introduced. Consider the tuple of root data (cf. [Bor79], [Spr81])

$$\psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^\vee) \tag{6.5.8}$$

of the group G; here T is a maximal torus of G (over a separable closure of the ground field K); $X^*(T)$ is the group of characters of T; $X_*(T)$ the group of one parameter subgroups of T and Δ (resp. $\Delta^\vee$) is a basis of the root system (resp. the dual basis of the system of coroots). The connected component of the Langlands L-group ${}^LG^0$ is defined to be the complex reductive group obtained by inversion $\psi_0 \mapsto \psi_0^\vee$, whose root data is isomorphic to the inverse

$$\psi_0(G)^\vee = (X_*(T), \Delta^\vee, X^*(T), \Delta). \tag{6.5.9}$$

If G is a simple group, then the group ${}^LG(\mathbb{C})$ can be characterized upto a central isogeny by one of the types A_n, B_n, $\ldots$, G_2 of the Cartan–Killing classification. It is known that the map $\psi_0 \mapsto \psi_0^\vee$ interchanges the types B_n and C_n, and leaves all other types fixed. Thus if $G = \mathrm{Sp}_n$ (respectively

GSp_n), then ${}^LG^0 = \mathrm{SO}_{2n+1}(\mathbb{C})$ (resp. ${}^LG^0 = \mathrm{Spin}_{2n+1}(\mathbb{C})$). The whole group ${}^LG$ is then defined as the semi–direct product of ${}^LG^0$ with the Galois group $\mathrm{Gal}(K^s/K)$ of an extension K^s of the ground field K over which G splits (i.e. its maximal torus T becomes isomorphic to GL_1^r). This semi-direct product is determined by the action of the Galois group $\Gamma_K = \mathrm{Gal}(K^s/K)$ on the set of maximal tori defined over K^s.

The most important classification result of the Langlands theory states that if

$$\pi = \bigotimes_v \pi_v \in \mathfrak{A}(G/K)$$

then for almost all v the local component π_v corresponds to a unique conjugacy class of an element h_v in the group ${}^LG$.

Let us consider the Euler product

$$L(\pi, r, s) = \prod_{v \notin S} L(\pi_v, r, s), \tag{6.5.10}$$

where S is a finite set of places of K,

$$L(\pi_v, r, s) = \det(1_m - \mathrm{N}v^{-s} r(h_v))^{-1}.$$

Langlands has shown that if $\pi \in \mathfrak{A}(G/K)$ then the product in (6.5.10) converges absolutely for all s with sufficiently large real part $\mathrm{Re}(s)$ (cf. [L71a]). The product (6.5.10) defines an automorphic L-function only up to a finite number of Euler factors. Although this is sufficient for certain questions related to analytic continuation of these functions, the precise form of these missing factors is very important in the study of the functional equations. A list of standard conjectures on the analytic properties of the L-functions (6.5.10) can be found in A.Borel's paper [Bor79]

We refer to recent introductory texts to the theory of automorphic L-functions and the Langlands program: [BCSGKK3], [Bum97], [Iw97],

For the group $G = \mathrm{GL}_n$ and the standard representation $r = r_n = \mathrm{St} : {}^LG^0 \xrightarrow{\sim} \mathrm{GL}_n(\mathbb{C})$ the main analytic properties of the L-functions (6.5.10) are proved in [JPShS], [GPShR87], [Sh88], [JSh] (see also [Bum97], [BCSGKK3], [CoPSh94]).

Also in the case $G = \mathrm{GL}_n$ the multiplicity one theorem (an analogue of the theorem of Atkin–Lehner) (cf. [AL70], [Mi89], [Li75]) has been extended (cf. [Gel75], [Gel76]). This is closely related to the non-vanishing theorem: for a cuspidal representation π one has $L(\pi, r_n, 1) \neq 0$.

For GL_3 an analogue of Weil's inverse theorem (see §6.3.8) has been proved: if all the L-functions of type $L(\pi \otimes \chi, r_3, s)$ (where χ is a Hecke character and π is an irreducible admissible representation) can be holomorphically continued to the entire complex plane, then the representation π can be realized in the space of cusp forms ([CoPSh94], [JPShS]). More recent results on the case of GL_n, cf. [CoPSh02].

Interesting classes of L-functions attached to Siegel modular forms were introduced and studied in [An74], [An79a], [AK78]. These modular forms and their L-functions have deep arithmetical significance and are closely related to the classical problem on the number of representations of a positive definite integral quadratic form by a given integral quadratic form (as generating functions, or theta–series). These numbers arise in Siegel's general formula considered above (5.3.71). From the point of view of the theory of automorphic representations, Siegel modular forms correspond to automorphic forms on the symplectic group $G = \mathrm{GSp}_n$. In this case the dual Langlands group coincides with the universal covering $\mathrm{Spin}_{2n+1}(\mathbb{C})$ of the orthogonal group $\mathrm{SO}_{2n+1}(\mathbb{C})$. To construct L-functions one uses the following two kinds of representation of the L-group ${}^LG = \mathrm{Spin}_{2n+1} \rtimes \mathrm{Gal}(K^s/K)$: ρ_{2n+1} and r_n, where ρ_{2n+1} is the standard representation of the orthogonal group, and r_n is the spinor representation of dimension 2^n. It is convenient to consider the following matrix realization of the orthogonal group:

$$\mathrm{SO}_{2n+1}(\mathbb{C}) = \left\{ g \in \mathrm{SL}_{2n+1}(\mathbb{C}) \mid {}^t g G_n g = G_n \right\},$$

with a quadratic form defined by the matrix

$$G_n = \begin{pmatrix} 0_n & 1_n & 0 \\ \cdots & \cdots & \cdots \\ 1_n & 0_n & 0 \\ 0 & \cdots & 1 \end{pmatrix}, \quad 1_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

If $\pi = \otimes_v \pi_v \in \mathfrak{A}(\mathrm{GSp}_n/K)$ then for almost all v the representation π_v corresponds to a conjugacy class h_v in ${}^LG$ whose image in the standard representation is given by a diagonal matrix of the type

$$\rho_{2n+1}(h_v) = \left\{ \alpha_{1,v}, \cdots, \alpha_{n,v}, \alpha_{1,v}^{-1}, , \alpha_{n,v}^{-1}, 1 \right\},$$

and in the spinor representation it becomes

$$r_n(h_v) = \{\beta_{0,v}, \beta_{0,v}\alpha_{1,v}, \cdots, \beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v} \cdots \alpha_{i_m,v}, \cdots\},$$

where for every $m \leq n$ all possible products of the type

$$\beta_{0,v}\alpha_{i_1,v}\alpha_{i_2,v} \cdots \alpha_{i_m,v}, \quad 1 \leq i_1 < i_2 < \cdots < i_m \leq n$$

arise.

The element h_v is uniquely defined upto the action of the Weyl group W_n generated by the substitutions

$$\beta_{0,v} \mapsto \beta_{0,v}\alpha_{i,v}, \quad \alpha_{i,v} \mapsto \alpha_{i,v}^{-1}, \quad \alpha_{j,v} \mapsto \alpha_{j,v} \quad (j \neq i)$$

and by all possible substitutions of the coordinates

$$\alpha_{i_1,v}, \alpha_{i_2,v}, \cdots \alpha_{i_n,v}.$$

A.N.Andrianov has established meromorphic continuations and functional equations for automorphic L-functions of the type $L(\pi_f, r_n, s)$ where π_f is the automorphic representation of $\mathrm{GSp}_n(\mathbb{A})$ over $\mathbb{Q}$ attached to a Siegel modular form f with respect to $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$, $n = 2$. He has also studied the holomorphy properties of these *spinor L-functions* for various classes of Siegel modular forms f, cf. [An74], [An79a] . Analytic properties of such functions are related to versions of the theory of new forms in the Siegel modular case for $n = 2$, cf. [AP2000]. A.N. Andrianov and V.L.Kalinin in [AK78] have studied the analytic properties of the *standard L-functions* $L(\pi_f, \rho_{2n+1}, s)$, where π_f is the automorphic representation of $\mathrm{GSp}_n(\mathbb{A})$ over $\mathbb{Q}$ attached to a Siegel modular form f with respect to the congruence subgroup $\Gamma_0^n(N) \subset \mathrm{Sp}_n(\mathbb{Z})$. For $n = 1$ these L-functions coincide with the symmetric squares of Hecke series, previously studied by Shimura.

A general *doubling method* giving explicit constructions of many automorphic L-functions, was developed in [Boe85] and [GPShR87].

Further analytic properties of automorphic L-functions

We refer to Sarnak's plenary lecture [Sar98] to ICI–1998, and to the related papers [IwSa99], [KS99], [KS99a], [LRS99], [KiSha99].

In [IwSa99], four fundamental conjectures were discussed: (A) Grand Riemann hypothesis; (B) Subconvexity problem; (C) Generalized Ramanujan conjecture; (D) Birch and Swinnerton-Dyer conjecture. Another problem which is related to (D) is a special value problem. Namely, the question as to whether an L-function vanishes at a special point on the critical line.

From the classical point of view, analytic and arithmetic properties of new classes of automorphic L-functions where studied in new Shimura's books [Shi2000], [Shi04], using a developed theory of Eisenstein series on reductive groups.

6.5.3 The Langlands Functoriality Principle

(cf. [Bor79], [BoCa79], [Gel75], [Pan84] , and for recent developments, [Lau02], [Hen01], [Car2000], [Li2000], [BCSGKK3], [CKPShSh]). This important principle establishes ties between automorphic representations of different reductive groups H and G. A homomorphism of the L-groups $u : {}^LH \to {}^LG$ attached to G and H is called an L-homomorphism if the restriction of u to ${}^LH^0(\mathbb{C})$ is a complex analytic homomorphism to ${}^LG^0(\mathbb{C})$, and u induces the identity map on the Galois group G_K. The functoriality principle is formulated in terms of the conjugacy classes of the matrices h_v corresponding to the local components π_v of an irreducible admissible representation $\pi = \otimes_v \pi_v$ of the group $H(\mathbb{A}_K)$. It includes the following statements:

1) *locally*: for almost all v there exists an irreducible admissible representation $u_*(\pi_v)$ of the group $G_v = G(K_v)$ which corresponds to the conjugacy class of the element $u(h_v)$ in ${}^LG$;

Lecture N°2. Classical and Hermitian modular forms

- Classical modular forms: the case of GL_2
- The Siegel modular case: $G = \mathrm{GSp}_n$
- Geometric algebra (see [Dieudonné], [Garrett])
- Sesquilinear formes, Hermitian and antihermitian forms
- Automorphic complex L-functions on classical groups.
- Hermitian modular forms and L functions.
- Hecke algebras.
- The Rankin-Selberg method.

Why study L-values attached to modular forms?

A popular proceedure in Number Theory is the following:

Construct a generating function $f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ of an arithmetical function $n \mapsto a_n$, for example $a_n = p(n)$ $\rightsquigarrow$ Compute f via modular forms, for example $\sum_{n=0}^{\infty} p(n) q^n = (\Delta/q)^{-1/24}$ $\rightsquigarrow$ A number (solution)

Example 1 [Chand70]: (Hardy-Ramanujan)

$$p(n) = \frac{e^{\pi\sqrt{2/3(n-1/24)}}}{4\sqrt{3}\lambda_n^2} + O(e^{\pi\sqrt{2/3(n-1/24)}}/\lambda_n^3),$$
$$\lambda_n = \sqrt{n - 1/24},$$

$\uparrow$ Good bases, finite dimensions, many relations and identities

$\uparrow$ Values of L-functions, periods, congruences, .

Other examples: Birch and Swinnerton-Dyer conjecture, ... L-values attached to modular forms

Modular forms, zeta functions, L-functions

Eisenstein series $E_k = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n \in \mathcal{M}_k$, a modular forms for even weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z}), q = e^{2\pi i z}$), and $E_2 \in \mathcal{QM}$ a quasimodular form. The ring of quasimodular forms, closed under differential operator $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, used in arithmetic, $\zeta(s)$ is the Riemann zeta function, $\zeta(-1) = -\frac{1}{12}$, $E_2 = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n$ is also a p-adic modular form (due to J.-P.Serre,[Se73], p.211)

Elliptic curves $E : y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, A.Wiles's modular forms $f_E = \sum_{n=1}^{\infty} a_n q^n$ with $a_p = p - CardE(\mathbb{F}_p)$ $(p \not| 4a^3 + 27b^2)$, and the L-function $L(E, s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Zeta-functions or L-functions

They are attached to various mathematical objects as certain Euler products.

- L-functions link such objects to each other (a general form of functoriality);
- Special L-values answer fundamental questions about these objects in the form of a number (complex or p-adic).

Computing these numbers use integration theory of Dirichlet-Hecke characters along p-adic and complex valued measures.
This approach originates in the Dirichlet class number formula using the L-values in order to compute class numbers of algebraic number fields through Dirichlet's L-series $L(s, \chi)$: for an imaginary quadratic field K of discriminant $-D < -4$, $\chi_D(n) = \left(\frac{-D}{n}\right)$

$$h_D = \frac{\sqrt{D}L(1, \chi_D)}{2\pi} = L(0, \chi) = -\frac{1}{D}\sum_{a=1}^{D-1} \chi_D(a)a.$$

(Example: $\mathrm{disc}(\mathbb{Q}(\sqrt{-5}))) = -20$, $h_{20} = 2$; in PARI/GP $\chi_{20}(n) =$ kronecker(-20,n), `gp > -sum(x=1,19,x*kronecker(-20,x))/20`
`% 29 = 2`
Another famous example: the Millenium BSD Conjecture gives the rank of an elliptic curve E as the order of $L(E, s)$ at s=1 (i.e. the residue of its logarithmic derivative, see [MaPa], Ch.6).

A short story of critical values, see [YS]

Euler discovered $\zeta(2) = \frac{\pi^2}{6}$, and $\frac{2\zeta(2n)}{(2\pi i)^{2n}} = -\frac{B_{2n}}{(2n)!} \in \mathbb{Q}, (n \geq 1)$.
These are examples of critical values (in the sense of Deligne): for a more general zeta function $\mathcal{D}(s)$ the critical values are defined using its gamma factor $\Gamma_{\mathcal{D}}(s)$ such that the product $\Gamma_{\mathcal{D}}(s)\mathcal{D}(s)$ satisfies a standard functional equation under the symmetry $s \mapsto v - s$. Then $\mathcal{D}(n)$, $n \in \mathbb{Z}$ is a critical value of $\mathcal{D}(s)$ if both $\Gamma_{\mathcal{D}}(n)$ and $\Gamma_{\mathcal{D}}(v-n)$ are finite.

Hurwitz [Hur1899] showed a striking analogy to Euler's theorem:

$$\frac{\sum'_{\alpha \in \mathbb{Z}[i]} \alpha^{-4m}}{\Omega^{4m}} = \frac{H_m}{(4m)!} \in \mathbb{Q}, \Omega = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.6220575542\cdots$$

for $1 \leq m \in \mathbb{Z}$, where $\alpha = a + ib$, $a, b \in \mathbb{Z}$ are non-zero Gaussian integers and H_m are Hurwitz numbers (recursively computed, [Sl]): $H_1, H_2, \cdots = \frac{1}{10}, \frac{3}{10}, \frac{567}{130}, \frac{43659}{170}, \frac{392931}{10}, \cdots$. Recall the formula:
Let $\wp$ be the Weierstrass $\wp$-function satisfying $\wp'^2 = 4\wp^3 - 4\wp$.
Then $\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2^{4n} H_n z^{4n-2}}{4n(4n-2)!}$. A rapid computation of these values: take the Fourier expansion of the Eisenstein series at $z = i$, $q = e^{-2\pi}$:

$$G_{4m}(z) = \sum_{a,b}{}'(az+b)^{-4m} = 2\zeta(4m) + \frac{2(2\pi)^{4m}}{(4m-1)!} \sum_{d \geq 1} \frac{d^{4m-1} q^d}{(1-q^d)},$$

$\frac{G_{4m}(i)}{\Omega^{4m}} = \frac{H_m}{(4m)!}$, π, Ω – periods of $\zeta(s)$ and of $E : y^2 = 4x^3 - 4x$.

Classical modular forms: definition;

Let Γ be a subgroup of finite index in the modular group $\mathrm{SL}_2(\mathbb{Z})$. A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is called a modular form of (integral) weight k with respect to Γ iff the conditions a) and b) are satisfied:

- a) *Automorphy condition*

$$f((a_\gamma z + b_\gamma)/(c_\gamma z + d_\gamma)) = (c_\gamma z + d_\gamma)^k f(z) \qquad (1)$$

  for all elements $\gamma \in \Gamma$;
- b) *Regularity at cusps*: f is regular at cusps $z \in \mathbb{Q} \cup i\infty$ (the cusps can be viewed as fixed points of parabolic elements of Γ); this means that for each element $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ the function $(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$ admits a Fourier expansion over non–negative powers of $q^{1/N} = e(z/N)$ for a natural number N. One writes traditionally

$$q = e(z) = \exp(2\pi i z).$$

  A modular form $f(z) = \sum_{n=0}^{\infty} a(n) e(nz/N)$ is called a cusp form if f vanishes at all cusps (i.e. if the above Fourier expansion contains only positive powers of $q^{1/N}$), see [LangMF], [MaPa]

The complex vector space of all modular (resp. cusp) forms

of weight k with respect to Γ is denoted by $\mathcal{M}_k(\Gamma)$ (resp. $\mathcal{S}_k(\Gamma)$). A basic fact from the theory of modular forms is that the spaces of modular forms are finite dimensional. Also, one has $\mathcal{M}_k(\Gamma)\mathcal{M}_l(\Gamma) \subset \mathcal{M}_{k+l}(\Gamma)$. The direct sum

$$\mathcal{M}(\Gamma) = \bigoplus_{k=0}^{\infty} \mathcal{M}_k(\Gamma)$$

turns out to be a graded algebra over $\mathbb{C}$ with a finite number of generators.
An example of a modular form with respect to $\mathrm{SL}_2(\mathbb{Z})$ of weight $k \geq 4$ is given by the *Eisenstein series*

$$G_k(z) = \sum_{m_1,m_2\in\mathbb{Z}}{}' (m_1 + m_2 z)^{-k} \qquad (2)$$

(prime denoting $(m_1, m_2) \neq (0, 0)$). For these series the automorphy condition (1) can be deduced straight from the definition. One has $G_k(z) \equiv 0$ for odd k and

$$G_k(z) = \frac{2(2\pi i)^k}{(k-1)!}\left[-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)e(nz)\right], \qquad (3)$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and B_k is the k^{th} *Bernoulli number*.
The graded algebra $\mathcal{M}(\mathrm{SL}_2(\mathbb{Z}))$ is isomorphic to the polynomial ring of the (independent) variables G_4 and G_6.

Examples

Recall that B_k denote the Bernoullli numbers defined by the development

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!},$$

One has for even $k \geq 2$, $2\zeta(k) = -\frac{(2\pi i)^k B_k}{k!}$, $G_k(z) = \frac{2(2\pi i)^k}{(k-1)!}\left[-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n\right]$, $E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n$

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n \in \mathcal{M}_4(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_6(z) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^n \in \mathcal{M}_6(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_8(z) = 1 + 480\sum_{n=1}^{\infty} \sigma_7(n)q^n \in \mathcal{M}_8(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{10}(z) = 1 - 264\sum_{n=1}^{\infty} \sigma_9(n)q^n \in \mathcal{M}_{10}(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{12}(z) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty} \sigma_{11}(n)q^n \in \mathcal{M}_{12}(\mathrm{SL}(2,\mathbb{Z})),$$

$$E_{14}(z) = 1 - 24\sum_{n=1}^{\infty} \sigma_{13}(n)q^n \in \mathcal{M}_{14}(\mathrm{SL}(2,\mathbb{Z})).$$ (Proof see in [Se70]).

```
with PARI/GP:
gp > k=14;Ek=1-(2*k)/bernfrac(k)*sum(d=1,20,d^(k-1)*q^d/(1-q^d)+O(q^4>
% = 1 - 24*q - 196632*q^2 - 38263776*q^3 + O(q^4)
```

Fast computation of the Ramanujan function $\tau(n)$

Put $h_k := \sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n = \sum_{d=1}^{\infty}\frac{d^{k-1}q^d}{1-q^d}$. The classical fact is that

$\Delta = \frac{(E_4^3 - E_6^2)}{1728} = q\prod_{n\geq 1}(1-q^n)^{24} = \sum_{n\geq 1}\tau(n)q^n$, ($E_4 = 1+240h_4$, $E_6 = 1-504h_6$).

Computing with PARI-GP see [BBBCO], The PARI/GP number theory system),

http://pari.math.u-bordeaux.fr $h_k := \sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n = \sum_{d=1}^{\infty}\frac{d^{k-1}q^d}{1-q^d} \Longrightarrow$

```
gp > h6=sum(d=1,20,d^5*q^d/(1-q^d)+O(q^20))
gp > h4=sum(d=1,20,d^3*q^d/(1-q^d)+O(q^20)
gp > Delta=((1+240*h4)^3-(1-504*h6)^2)/1728
```

Congruence of Ramanujan $\tau(n) \equiv \sum_{d|n} d^{11} \mod 691$:

```
gp > (Delta-h12)/691
% = -3*q^2 - 256*q^3 - 6075*q^4 - 70656*q^5 - 525300*q^6 + O(q^7)
```

More programs of computing $\tau(n)$ (see [Sl])

```
(MAGMA) M12:=ModularForms(Gamma0(1), 12); t1:=Basis(M12)[2];
PowerSeries(t1[1], 100); Coefficients($1);
(PARI) a(n)=if(n<1, 0, polcoeff(x*eta(x+x*O(x^n))^24, n))
(PARI) {tau(n)=if(n<1, 0, polcoeff(x*(sum(i=1, (sqrtint(8*n-7)+1)\2,
(-1)^i*(2*i-1)*x^((i^2-i)/2), O(x^n)))^8, n));}
gp > tau(6911)
%3 = -615012709514736031488
```


Multiplicativity and Hecke theory, [An87]

The example of the Eisenstein series, of the Ramanujan function $\tau(n)$ demonstrate an interesting phenomenon: often the Fourier coefficients $a(n)$ of modular forms for $\mathrm{SL}_2(\mathbb{Z})$ are multiplicative number-theoretical functions or linear combinations of such functions. The reason for this is to be found in Hecke's theory as follows. Let $f(z) = \sum_{n=0}^{\infty} a(n)e(nz) \in \mathcal{M}_k$ The "multiplicativity" of the function $a(n)$ should mean that there is a regular connection between $a(n)$ and $a(nm)$ for any fixed m. The numbers $a(nm) = (n = 1, 2, \dots)$ are the Fourier coefficients of the function

$f_m(z) = \sum_{n=0}^{\infty} a(nm)e(nz) = \frac{1}{m}\sum_{b=0}^{m-1} f\left(\frac{z+b}{m}\right) =$

$m^{k-1}\sum_{b=0}^{m-1} f|_k \begin{pmatrix} 1 & b \\ 0 & m\end{pmatrix}$ where $f|_k M = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$.

Defining Hecke operators through $U(m)$, $V(m)$

If the operator $f \to f_m := f|U(m)$ were to carry the space $\mathcal{M}_k$ into itself, then one could hope to find its eigenfunctions in $\mathcal{M}_k$ but for such functions $f_m = \lambda_m f$, that is, $a(nm) = \lambda_m a(n)(n = 1, 2, \dots)$, and the desired multiplicativity property would follow.

To develop this idea let us use for any positive integer m the operators $f \mapsto f|U(m)$, $f \mapsto f|V(m)$:

$$f|U(m)(z) = \sum_{n=0}^{\infty} a(mn)e(nz) = m^{k-1}\sum_{b=0}^{m-1} f|_k \begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix}$$

$$f|V(m)(z) = \sum_{n=0}^{\infty} a(n)e(mnz) = m^{k-1} f|_k \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$$

In the general case note that the matrices $\begin{pmatrix} 1 & b \\ 0 & m \end{pmatrix}$ in the definition of $U(m)$ form part of a complete system of right coset representatives for $SL_2(\mathbb{Z})\backslash\Delta_m \left\{\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} |ad = m, , b = 0 \dots, d-1\right\}$, where Δ_m denotes the set $\Delta_m = \left\{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \det \gamma = m\right\}$ which is invariant under right multiplication by elements of $\mathrm{SL}_2(\mathbb{Z})$. The action of this coset representatives produce the Hecke operator $f|T(m) = \sum_{a|m} a^{k-1} f|U(m/a)V(a)$.

The Siegel modular case: $G = \mathrm{GSp}_n$

Let $\mathbb{H}_n$ denote the Siegel upper half plane of genus n, $\mathbb{H}_n = \{z \in \mathrm{M}_n(\mathbb{C}) \quad | \quad {}^t z = z = x + iy, \quad y > 0\}$, so that $\mathbb{H}_n$ is a complex analytic variety whose dimension is denoted by $\langle n\rangle = n(n+1)/2$.

Put $G_\infty = G_\mathbb{R}$, $G_\infty^+ = \{\gamma \in G_\infty \mid \nu(\gamma) > 0\}$, $G_\mathbb{Q}^+ = G_\infty^+ \cap G_\mathbb{Q}$.

The group G_∞^+ acts transitively on the Siegel upper half plane $\mathbb{H}_n$ by $z \longmapsto \gamma(z) = (az+b)(cz+d)^{-1}, z \in \mathbb{H}_n$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_\infty^+$. so that the scalar matrices act trivially, and $\mathbb{H}_n$ can be identified with a homogeneous space of the group $\mathrm{Sp}_n(\mathbb{R})$. Let K_n denote the stabilizer of the point $i1_n \in \mathbb{H}_n$ in the group $\mathrm{Sp}_n(\mathbb{R})$, $K_n = \{\gamma \in \mathrm{Sp}_n(\mathbb{R}) \mid \gamma(i1_n) = i1_n\}$, then there is a bijection $\mathrm{Sp}_n(\mathbb{R})/K_n \simeq \mathbb{H}_n$ and $K_n = \mathrm{Sp}_n(\mathbb{R}) \cap \mathrm{SO}_{2n}$. The group K_n is a maximal compact subgroup of the Lie group $\mathrm{Sp}_n(\mathbb{R})$ and it can be identified with the group $U(n)$ of all unitary $n \times n$ -matrices via the map $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a + ib$.

The Siegel modular group $\Gamma^n = \mathrm{Sp}_n(\mathbb{Z})$

Let $\Gamma \subset G^+_{\mathbb{Q}}$ be its arbitrary congruence subgroup. The general definition of modular forms is given for a rational representation $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_r(\mathbb{C})$ a rational representation also denoted by ρ. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+_\infty$ and for any complex valued function $f : \mathbb{H}_n \to \mathbb{C}^r$ we use the notation $f|_\rho\gamma(z) = \rho(cz+d)^{-1} f(\gamma(z))$.

Definition A function $f : \mathbb{H}_n \to \mathbb{C}^r$ is called a holomorphic modular form of weight ρ on Γ if the following conditions are satisfied:

$f|_\rho = f$,
f is holomorphic on $\mathbb{H}_n$,
if $n = 1$ then f is holomorphic at cusps of Γ.

Let $\mathcal{M}_\rho(\Gamma)$ be the complex vector space of functions satisfying the above conditions. For each $f \in \mathcal{M}_\rho(\Gamma)$ there is the following Fourier expansion $f(z) = \sum_\xi c(\xi) e_n(\xi z)$, where $c(\xi) \in \mathbb{C}^r$, ξ run over all $\xi = {}^t\xi \in \mathrm{M}_n(\mathbb{Q})$, $\xi \geq 0$ (for $n > 1$ the last condition automatically follows by the Koecher principle).

The Fourier expansions of Siegel modular forms

Let M be the smallest integer such that
$\Gamma \supset \left\{ \begin{pmatrix} 1_m & Mu \\ 0_m & 1_m \end{pmatrix} \mid u \in \mathrm{M}_m(\mathbb{Z}), {}^tu = u \right\}$ and we put

$$\begin{array}{lllll} A & = & A_m & = & \{\xi = (\xi_{ij}) \in \mathrm{M}_m(\mathbb{R}) \mid \xi = {}^t\xi, \xi_{ij}, 2\xi_{ii} \in \mathbb{Z}\}, \\ B & = & B_m & = & \{\xi \in A \mid \xi \geq 0\}, \\ C & = & C_m & = & \{\xi \in A \mid \xi > 0\}. \end{array}$$

Then A_m is a lattice of half-integral matrices in the $\mathbb{R}$-vector space of symmetric matrices $V = \{x \in \mathrm{M}_m(\mathbb{R}) \mid {}^tx = x\}$ dual to the lattice $L = \mathrm{M}_m(\mathbb{Z}) \cap V$ with respect to the action $(\xi, x) \longmapsto e_m(\xi x)$ and for each $f \in \mathcal{M}_\rho(\Gamma)$ there is the following Fourier expansion

$$f(z) = \sum_{\xi \in M^{-1}B} c(\xi) e_m(\xi z), \tag{4}$$

Moreover, for each $\gamma \in G^+_{\mathbb{Q}}$ we have that $f|_\rho\gamma \in \mathcal{M}_\rho(\Gamma(\gamma))$, where $\Gamma(\gamma)$ is a congruence subgroup,

$$(f|_\rho\gamma)(z) = \sum_{\xi \in M_\gamma^{-1}B} c_\gamma(\xi) e_m(\xi z),$$

with $c_\gamma(\xi) \in \mathbb{C}^r$, $M_\gamma \in \mathbb{N}$. A form f is called a cusp form if for all ξ with $\det(\xi) = 0$ in expansion (4) one has $c_\gamma(\xi) = 0$ for all $\gamma \in G^+_{\mathbb{Q}}$ that is

$$(f|_\rho\gamma)(z) = \sum_{\xi \in M_\gamma^{-1}C} c_\gamma(\xi) e_m(\xi z).$$

We denote by $\mathcal{S}_\rho(\Gamma) \subset \mathcal{M}_\rho(\Gamma)$ the subspace of cusp forms.

The Petersson scalar product

For $f \in \mathcal{S}_n^k(N,\psi)$ and $h \in \mathcal{M}_n^k(N,\psi)$ the Petersson scalar product is defined by

$$\langle f, h\rangle_N = \int_{\Phi_0(N)} \overline{f(z)}h(z)\det(y)^k \mathrm{d}^\times z, \qquad (5)$$

where $\Phi_0(N) = \Gamma_0^n(N)\backslash\mathbb{H}_n$ is a fundamental domain for the group $\Gamma_0^n(N)$ with the notations

$\mathrm{d}x = \prod_{i\le j} \mathrm{d}x_{ij}, \quad \mathrm{d}y = \prod_{i\le j} y_{ij}, \quad \mathrm{d}z = \mathrm{d}x\mathrm{d}y,$

$\mathrm{d}^\times y = \det(y)^{-\kappa}\mathrm{d}y, \quad \mathrm{d}^\times z = \det(y)^{-\kappa}\mathrm{d}z$, where $z = x + iy$,

$x = (x_{ij}) = {}^t x$, $y = (y_{ij}) = {}^t y > 0$, $\kappa = \dfrac{n(n+1)}{2}$. Then $\mathrm{d}^\times z$ is a differential on $\mathbb{H}_n$ invariant under the action of the group G_∞^+, and the measure $\mathrm{d}^\times y$ is invariant under the action of elements $a \in \mathrm{GL}_n(\mathbb{R})$ on $Y = \{y \in \mathrm{M}_n(\mathbb{R}) \mid {}^t y = y > 0\}$ defined by the rule $y \longmapsto {}^t aya$.

Sesquilinear formes, Hermitian and antihermitian forms

Let K be a quadratic extension of k, its subfield fixed be the involution σ.

Definition

a) A k-bilinear form $f : V \times V \to K, f(v_1, v_2) = \langle v_1, v_2\rangle$ on a K-vectorspace V of finite dimension is said sesquilinear (with an implicite reference to σ) if $\langle \alpha v_1, \beta v_2\rangle = \alpha^\sigma\beta\langle v_1, v_2\rangle$ $(\forall \alpha, \beta \in K$ and $v_1, v_2 \in V$).

b) A sesquilinear form $f(v_1, v_2) = \langle v_1, v_2\rangle$ on a K-vector space V of finite dimension is said hermitian if $\forall v_1, v_2 \in V, \langle v_2, v_1\rangle = \langle v_1, v_2\rangle^\sigma$.

The unitary group $U(f)$ is the group of isometries f (or of $\langle\cdot,\cdot\rangle$), defined as

$U(f) = U(\langle\cdot,\cdot\rangle) = \{g \in GL_K(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \langle v_1, v_2\rangle\}$,

and the group of unitary similitudes is $GU(f) = GU(\langle\cdot,\cdot\rangle)$
$= \{g \in GL_K(V), \exists\nu(g) \in K^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2\rangle = \nu(g)\langle v_1, v_2\rangle\}$

Simultaneuos treatment of general isometries groups

Over a division algebra D with an anti-involution σ. Note that $\sigma : D \to D$ satisfies the properties

$$\forall \alpha, \beta \in D, \alpha^{\sigma\sigma} = \alpha, \ (\alpha+\beta)^\sigma = \alpha^\sigma + \beta^\sigma \text{ and } (\alpha\beta)^\sigma = \beta^\sigma \alpha^\sigma$$

Let Z be the center of D. Suppose that D is of finite dimension over Z, and that $k = \{x \in Z | x^\sigma = x\}$. Let V be a D-vector space of finite dimension, and fix $\varepsilon = \pm 1$. Let $f = \langle \cdot, \cdot \rangle, f : V \times V \to D$ a k-bilinear form with values in D on V such that $\forall \alpha, \beta \in D$, $\forall v_1, v_2 \in V, \langle v_2, v_1 \rangle = \varepsilon \langle v_1, v_2 \rangle^\sigma, \ \langle \alpha v_1, \beta v_2 \rangle = \alpha^\sigma \langle v_1, v_2 \rangle \beta$. Such a form is said ε-hermitian on V, and such space V (endowed with $\langle \cdot, \cdot \rangle$) is called a (D, σ, ε)-space.

The group of isometries $U(f)$ of f (or of $\langle \cdot, \cdot \rangle$), is defined as

$$U(f) = U(\langle \cdot, \cdot \rangle) = \{g \in GL_D(V) \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle\},$$

and the group of isometry similitudes is $GU(f) = GU(\langle \cdot, \cdot \rangle)$

$$= \{g \in GL_D(V), \exists \nu(g) \in k^* \mid \forall v_1, v_2 \in V, \langle gv_1, gv_2 \rangle = \nu(g) \langle v_1, v_2 \rangle\}$$

Orthogonalisation and isotropy vectors

A D - vector subspace U in a (D, ε, σ) - vector space admits an orthogonal complement $U^\perp = \{u' \in V \langle u', u \rangle = 0, \forall u \in U\}$. Note that $U \cap U^\perp = 0$ is not valid in general. The kernel V is denoted $V^\perp$. The form is called non degenerate if $V^\perp = 0$. Suppose for simplicity that the space V is non-degenerate.

If V_1, V_2 are two (D, ε, σ) - vector spaces endowed with forms, respectively, $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$, then the direct sum $V_1 \oplus V_2$ of D- vector spaces is a (D, ε, σ) - vector space with the form

$$\langle v_1 + v_2, v_1' + v_2' \rangle = \langle v_1, v_1' \rangle_1 \langle v_2, v_2' \rangle_2$$

called the orthogonal sum. In general, two subspaces V_1, V_2 of a (D, ε, σ) - vector space V are orthogonal if $V_1 \subset V_2^\perp$ or equivalently, if $V_2 \subset V_1^\perp$.

If $\langle v, v \rangle = 0$ for $v \in V$, then v is called an isotropic vecteur. If $\langle v, v' \rangle = 0$ for all $v, v' \in U$ for a subspace U of V then U is a (totally) isotropic. If there is no isotropic non zero vector in U, then U is said anisotropic.

Proposition

Let V be a (D, ε, σ)- non degenerate vector space with a subspace U. Then U is non degenerate iff $V = U \oplus U^\perp$, with $U^\perp$ non degenerate.

Orthogonalisation in (D, σ, ϵ)-spaces

This is used for classification of orthogonal and hermitians spaces

Proposition

Let V be a (D, ε, σ)- non degenerate vector space. Suppose that the case where $\varepsilon = -1$, $D = k$, and σ trivial is excluded. If the product $\langle \cdot, \cdot \rangle$ does not vanish identically then there exists $v \in V$ with $\langle v, v \rangle \neq 0$. If V is non degenerate then it has an orthogonal basis.

Proof. Suppose that $\langle v, v \rangle = 0$ for all $v \in V$. Then

$$0 = \langle x + y, x + y \rangle = \langle x, x \rangle + \langle y, y \rangle + \langle x, y \rangle + \varepsilon \langle x, y \rangle^{\sigma}.$$

If $\varepsilon = 1$ and the product $\langle x + y, x + y \rangle$ does not vanish identically, then there exist x, y such that $\langle x, y \rangle = 1$. Contradiction. Suppose that $\varepsilon = -1$ and σ non trivial on D. Then there exists $\alpha \in D$ such that $\alpha \neq \alpha^{\sigma}$, with $\omega = \alpha - \alpha^{\sigma}$, $\omega = -\omega^{\sigma}$. If $\langle x, y \rangle$ does not vanish identically then there existe x, y such that $\langle x, y \rangle = 1$. Then one has $0 = \langle \omega x, y \rangle + \varepsilon \langle \omega x, y \rangle^{\sigma} = \omega^{\sigma} \langle x, y \rangle + \langle x, y \rangle^{\sigma} \omega = -\omega + \varepsilon \omega = -2\omega$, Contradiction. ∎

In order to construct an orthogonal basis, one uses induction on dimension. If the dimension of a non degenerate vector space V is 1, then any non-zero forme admits an orthogonal basis. In general, one finds $v \in V$ such that $\langle v, v \rangle \neq 0$. Then $Dv^{\perp}$ is non degenerate and V is the orthogonal direct sum of Dv and$Dv^{\perp}$, by the previous proposition.

A Hermitian modular form of weight ℓ with character σ

is a holomorphic function F on $\mathcal{H}_n$ ($n \geq 2$) such that $F(g\langle Z \rangle) = \sigma(g) F(Z) j(g, Z)^{\ell}$ for any $g \in \Gamma_{n,K}$. Here σ be a character of $\Gamma_K^{(n)}$, trivial on $\left\{ \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \right\}$, and for $Z \in \mathcal{H}_n$, put $g\langle Z \rangle = (AZ + B)(CZ + D)^{-1}$, $j(g, Z) = \det(CZ + D)$.

Fourier expansions: a semi-integral Hermitian matrix is a Hermitian matrix $H \in (\sqrt{-D_K})^{-1} M_n(\mathcal{O})$ whose diagonal entries are integral. Denote the set of semi-integral Hermitian matrices by $\Lambda_n(\mathcal{O})$, the subset of its positive definite elements is $\Lambda_n(\mathcal{O})^{+}$.

A Hermitian modular form F is called a cusp form if it has a Fourier expansion of the form $F(Z) = \sum_{H \in \Lambda_n(\mathcal{O})^{+}} A(H) q^{H}$. Denote the space of cusp forms of weight ℓ with character σ by $\mathcal{S}_{\ell}(\Gamma_{n,K}, \sigma)$.

Hermitian modular forms and standard zeta functions.

Automorphic complex L-functions on classical groups. Hecke algebras. The Rankin-Selberg method.

Hermitian modular group $\Gamma_{n,K}$ *and the standard zeta function* $\mathcal{Z}(s;\mathbf{f})$ *(definitions)*

Let $K=\mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field, $\theta=\theta_K$ its quadratic character, $n\in\mathbb{N}, n'=\left[\frac{n}{2}\right]$. The Hermitian group

$$\Gamma_{n,K}=\left\{M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{GL}_{2n}(\mathcal{O}_K)|M\eta_n M^*=\eta_n\right\}, \eta_n=\begin{pmatrix}0_n & -I_n\\ I_n & 0_n\end{pmatrix}$$

$$\mathcal{Z}(s,\mathbf{f})=\left(\prod_{i=1}^{2n}L(2s-i+1,\theta^{i-1})\right)\sum_{\mathfrak{a}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

(via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a})=\lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a}\subset\mathcal{O}_K$)

$=\prod_{\mathfrak{q}}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q}\subset\mathcal{O}_K$,

with $\deg\mathcal{Z}_{\mathfrak{q}}(X)=2n,$ the Satake parameters $t_{i,\mathfrak{q}}, i=1,\cdots,n$),

$\mathcal{D}(s,\mathbf{f})=\mathcal{Z}(s-\frac{\ell}{2}+\frac{1}{2},\mathbf{f})$ (Motivically normalized standard zeta function with a functional equation $s\mapsto\ell-s;\mathrm{rk}=4n$)

The standard zeta function of a Hermitian modular form

Fix an integral ideal $\mathfrak{c}$ of $\mathcal{O}_K$. Denote by $C\subset\Gamma_{n,K}$ the congruence subgroup of level $\mathfrak{c}$; the group is essentially a principal congruence subgroup; it is an analogue of the group $\Gamma_0(N)$ in the elliptic modular case. Write $T(\mathfrak{a})$ for the Hecke operator associated to it as it is defined in [Shi00], page 162, using the action of double cosets $C\xi C$ with $\xi=\mathrm{diag}(\hat{D},D)$, $(\det(D))=(\alpha)$, $\hat{D}=(D^*)^{-1}$.

Consider a non-zero Hermitian modular form $\mathbf{f}\in\mathcal{M}_k(C,\psi)$ and assume $\mathbf{f}|T(\mathfrak{a})=\lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a})\in\mathbb{C}$ for all integral ideals $\mathfrak{a}\in\mathcal{O}$. Then

$$\mathcal{Z}(s,\mathbf{f})=\left(\prod_{i=1}^{2n}L_{\mathfrak{c}}(2s-i+1,\theta^{i-1})\right)\sum_{\mathfrak{a}}\lambda(\mathfrak{a})N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of $\mathcal{O}_K$.

This series has an Euler product representation $\mathcal{Z}(s,\mathbf{f})=\prod_{\mathfrak{q}}(\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$, where the product is over all prime ideals of $\mathcal{O}_K$, $\mathcal{Z}_{\mathfrak{q}}(X)$ is the numerator of the series $\sum_{r\geq 0}\lambda(\mathfrak{q}^r)X^r\in\mathbb{C}(X)$, computed by Shimura (see [Shi00], p. 171).

Recalls about Hecke algebras ([An87], [Shi71])

For any subgroup Γ of a semigroup S consider the $\mathbb{Q}$-vector space $L_{\mathbb{Q}}(\Gamma, S)$ generated by all left classes ΓM. If each double class $(M) = \Gamma M \Gamma \subset S$ is a finite union of left classes, that is $(M) = \underset{j}{\cup} \Gamma M_j$, one defines the Hecke algebra $\mathcal{D}_{\mathbb{Q}}(\Gamma, S) = L_{\mathbb{Q}}(\Gamma, S)^{\Gamma}$ as the $\mathbb{Q}$-espace vectiriel of elements in $L_{\mathbb{Q}}(\Gamma, S)^{\Gamma}$ fixed by Γ; the multiplication in $\mathcal{D}_{\mathbb{Q}}(\Gamma, S)$ being defined by:

$$(\sum_j a_j(\Gamma g_j))(\sum_{j'} a'_{j'}(\Gamma g'_{j'})) = \sum_{j,j'} a_j a'_{j'}(\Gamma g_j g'_{j'}).$$

For two subgroups Γ and Γ' write $\Gamma \sim \Gamma'$ if Γ and Γ' are c, i.e. $\Gamma \cap \Gamma'$ is of finite index in both Γ and Γ'. The subgroup $\tilde{\Gamma}=\{M \in S \mid M\Gamma M^{-1} \sim \Gamma\}$ is the commensurator of Γ in S. By Prop.3.1 [Shi71], for a family $\{\Gamma_\lambda\}_\lambda$ of subgroups commensurables to a fixed $\Gamma \subset S$ the following hold $\Gamma_\lambda M \Gamma_\mu = \overset{d}{\underset{i=1}{\cup}} \Gamma_\lambda M_i$ with $d = [\Gamma_\mu : \Gamma_\mu \cap M^{-1}\Gamma_\mu M]$. There is the decomposition $\Gamma_\mu = \overset{d}{\underset{i=1}{\cup}} (\Gamma_\mu \cap M^{-1}\Gamma_\lambda M)P_i$, $M_i = MP_i$.

The Hecke algebra in the symplectic Siegel modular case

For a prime q, $q \nmid N$, let $\Delta = \Delta_q^n(N) =$

$$\left\{\gamma = \begin{pmatrix} a\ b \\ c\ d \end{pmatrix} \in G_{\mathbb{Q}}^{+} \cap \mathrm{GL}_{2n}(\mathbb{Z}[q^{-1}]) \middle| \nu(\gamma)^{\pm} \in \mathbb{Z}[q^{-1}], c \equiv 0_n \bmod N\right\}$$

is a subgroup in $G_{\mathbb{Q}}^{+}$ containing $\Gamma = \Gamma_0^n(N)$. The local Hecke algebra

$$\mathcal{L} = \mathcal{L}_q^n(N) = \mathcal{D}_{\mathbb{Q}}(\Gamma, \Delta) \text{ over } \mathbb{Q}$$

is defined as a $\mathbb{Q}$-linear space generated by the double cosets $(g) = (\Gamma g \Gamma)$, $g \in \Delta$ of the group Δ, with respect to the subgroup Γ for which multiplication is defined by the standard rule (see [An87], [Shi71]). The operators of diagonal matrices $\mathbf{T}(q)$, $\mathbf{T}_i(q^2)$ $\mathbf{T}(q) = \mathbf{T}(\underbrace{1, \cdots, 1}_{n}, \underbrace{q, \cdots, q}_{n})$ generate it (see p.149 of [An87]),

$$\mathbf{T}_i(q^2) = \mathbf{T}(\underbrace{1, \cdots, 1}_{n-i}, \underbrace{q, \cdots, q}_{i}, \underbrace{q^2, \cdots, q^2}_{n-i}, \underbrace{q, \cdots, q}_{i}), i = 1, \cdots, n.$$

The structure of the local Hecke algebras via Satake isomorphism

For $\mathcal{L} = \mathcal{L}_q^n(N)$, $(q \nmid N)$ for each j, $1 \le j \le n$ us above let us denote by ω_j an automorphism of the algebra $\mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}]$ defined on its generators by the rule:

$$x_0 \longmapsto x_0 x_j, \quad x_j \longmapsto x_j^{-1}, \quad x_i \longmapsto x_i \qquad (\text{where } 1 \le i \le n, \quad i \ne j).$$

Then the automorphisms ω_j and the permutation group S_n of the variables x_i $(1 \le i \le n)$ generate together the Weyl group $W = W_n$, and there is the *Satake isomorphism* :

$$Sat : \mathcal{L} \xrightarrow{\sim} \mathbb{Q}[x_0^{\pm 1}, x_1^{\pm 1}, \cdots, x_n^{\pm 1}]^{W_n} \tag{6}$$

where W_n indicates the subalgebra of elements fixed by W_n. For any commutative $\mathbb{Q}$-algebra A the group W_n acts on the set $(A^\times)^{n+1}$, therefore any homomorphism of $\mathbb{Q}$-algebras $\lambda : \mathcal{L} \mapsto A$ can be identified with some element

$$(\alpha_0, \alpha_1, \cdots, \alpha_n) \in [(A^\times)^{n+1}]^{W_n}. \tag{7}$$

Explicit action of the Hecke operators

Any double coset $(g) = (\Gamma g \Gamma)$ $(g \in \Delta = \Delta_q^n(N))$ can be represented as a disjoint union of left cosets: $(g) = \bigcup_{i=1}^{t(g)} \Gamma g_i$, therefore any element $X \in \mathcal{L}$ of the Hecke algebra $\mathcal{L}$ takes the form of a finite linear combination $X = \sum_{i=1}^{t(X)} \mu_i(\Gamma g_i)$, $\mu_i \in \mathbb{Q}$, $g_i \in \Delta$.

In order to define explicit action of the Hecke operators on the complex vector space $\mathcal{M}_n^k(N, \psi)$ $(q \nmid N)$ for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta$ $(f|_{k,\psi} g)(z) = \det(g)^{k-\kappa} \psi(\det(a)) \det(cz+d)^{-k} f(g(z))$ (the convenient notation by Petersson and Andrianov) With this notation the automorphy condition can be rewritten as follows

$$(f|_{k,\psi}\gamma)(z) = f \text{ for all } \gamma \in \Gamma = \Gamma_0^n(N).$$

In this case for any $X = \sum_{i=1}^{t(X)} \mu_i(\Gamma g_i) \in \mathcal{L}$ this explicit expression

$f|X = \sum_{i=1}^{t(X)} \mu_i f|_{k,\psi} g_i$, is well defined

Hecke polynomials in the Siegel modular case

Consider the polynomials $\tilde{Q}(z) \in \mathbb{Q}[x_0, \cdots, x_n][z]$ and $\tilde{R}(z) \in \mathbb{Q}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}][z]$:

$$\tilde{Q}(z) = (1 - x_0 z) \prod_{r=1}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - x_0 x_{i_1} \cdots x_{i_r} z),$$

$\tilde{R}(z) = \prod_{i=1}^{n} (1 - x_i^{-1} z)(1 - x_i z)$.

The coefficients of the powers of the variable z all belong to the subring $\mathbb{Q}[x_0^{\pm 1}, \cdots, x_n^{\pm 1}]^{W_n}$. Therefore under the Satake isomorphism (6) there exist polynomials over $\mathcal{L} = \mathcal{L}_q^n(N)$ $Q(z) = \sum_{i=0}^{2^n} (-1)^i T_i z^i$, $R(z) = \sum_{i=0}^{2n} (-1)^i R_i z^i \in \mathcal{L}[z]$, such that

$$\tilde{Q}(z) = \sum_{i=0}^{2^n} (-1)^i \tilde{T}_i z^i, \ \tilde{R}(z) = \sum_{i=0}^{2n} (-1)^i \tilde{R}_i z^i, \text{ and } \tilde{X} = SatX \in \mathcal{L}.$$

Moreover, the polynomials ${\Delta'_M}^{\pm 1}$, R_i $(1 \le i \le n-1)$ and T_1 with

$\tilde{\Delta}'^{\pm 1}_M = x_0^2 x_1 \cdots x_n$,

$\tilde{R}_i = S_i(x_1, \cdots, x_n; x_1^{-1}, \cdots, x_n^{-1})$,

$$\tilde{T}_1 = x_0 \sum_{i=1}^{n} S_i(x_1, \cdots, x_n) = x_0 \prod_{i=1}^{n} (1 + x_i),$$ generate of the *Hecke algebra* , where S_i denotes the elementary symmetric polynomial of degree i of the corresponding set of variables.

The spinor zeta function and the standard zeta function

Let $f \in \mathcal{M}_n^k(N, \psi)$ be an eigenfunction of all *Hecke operators* $f \longmapsto f|X$, $X \in \mathcal{L}_q^n(N)$ with q being prime numbers, $q \nmid N$, so that $f|X = \lambda_f(X) f$. Then the numbers $\lambda_f(X) \in \mathbb{C}$ define a homomorphism $\lambda_f : \mathcal{L} \longrightarrow \mathbb{C}$, determined by a $(n+1)$-tuple of numbers $(\alpha_0, \alpha_1, \cdots, \alpha_n) \left[(\mathbb{C}^\times)^{n+1}\right]^{W_n}$ (the Satake parameters). Now let the variables x_0, x_1, $\cdots$, x_n be Satake q-parameters $\alpha_{0,f}(q)$, $\alpha_{1,f}(q)$, $\cdots$, $\alpha_{n,f}(q)$, then

$Q_{f,q}(z) = (1 - \alpha_0 z) \prod_{r=1}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - \alpha_0 \alpha_{i_1} \cdots \alpha_{i_r} z)$,

$R_{f,q}(z) = \prod_{i=1}^{n} (1 - \alpha_i^{-1} z)(1 - \alpha_i z) \in \mathbb{Q}[\alpha_0^{\pm 1}, \cdots, \alpha_n^{\pm 1}][z]$.

It follows then that the coefficients of (49) can be expressed in terms of the eigenvalues $\lambda_f(X)$ of the Hecke operators $X = T_i$, R_i. Next let us put $Z^{(q)}(s, f) = Q_{f,q}(q^{-s})^{-1} =$

$$\left[(1 - \alpha_0(q) q^{-s}) \prod_{r=1}^{n} \prod_{1 \le i_1 < \cdots < i_r \le n} (1 - \alpha_0(q) \alpha_{i_1}(q) \cdots \alpha_{i_r}(q) q^{-s}\right]^{-1}$$

The spinor zeta function $Z(s, f)$ of $f \in \mathcal{M}_n^k(N, \psi)$ is the following product $Z(s, f) = \prod_{q \nmid N} Z^{(q)}(s, f)$.

The standard zeta function $\mathcal{D}(s, f, \chi)$ of $f \in \mathcal{M}_n^k(N, \psi)$ is the product $\mathcal{D}(s, f, \chi) = \prod_{d \nmid N} \mathcal{D}^{(q)}(s, f, \chi)$ with $\mathcal{D}^{(q)}(s, f, \chi) =$

$(1 - \chi(q)\psi(q) q^{-s})^{-1} R_{f,q}(\chi(q)\psi(q) q^{-s})^{-1}$ For a Dirichlet character χ modulo M, $\mathcal{D}(s, f, \chi) =$

$$\prod_q \left[\left(1 - \frac{\chi\psi(q)}{q^s}\right) \prod_{l=1}^{n} \left(1 - \frac{\chi\psi(q)\alpha_l(q)}{q^s}\right) \left(1 - \frac{\chi\psi(q)\alpha_l(q)^{-1}}{q^s}\right)\right]^{-1}.$$

Examples of subgroup families $\{\Gamma_\lambda\}_\lambda$ in p-adic constructions

Let us fix Γ and for $\lambda = (v), v \geq 1$ put

$$\Gamma_\lambda = \Gamma_{(v)} = \Gamma \cap M^{-v}\Gamma M^v.$$

For example if $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, $M = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, the congruence subgroups $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \cap \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathrm{SL}_2(\mathbb{Z}) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \right\}$ produce the family $\{\Gamma_{(v)}\}_{(v)}$ attached to M and a fixed $\Gamma_{(0)} = \Gamma_0(C) \subset \Gamma$:

$$\Gamma_{(v)} = \Gamma_{(0)} \cap M^{-v}\Gamma M^v = \Gamma_0(p^v).$$

Notice the acion of the lowering operator (generalized Atkin's operator): for $v' > v$,

$$U_M^v = (\Gamma_{(v')} M^v (\Gamma_{(v'-v)}) : \mathcal{M}_k(\Gamma_{(v')}) \to \mathcal{M}_k \Gamma_{(v'-v)}),$$

compare with §4 of [Hi85]. Indeed,
$\forall \gamma \in \Gamma_{(v')}, h|\gamma = h \Rightarrow \forall \gamma \in \Gamma_{(v'-v)} f|U_M^v|\gamma = f|U_M^v$

Examples of Hermitian cusp forms

The Hermitian Ikeda lift, [Ike08]. Assume $n = 2n'$ even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L-function is given by

$$L(f,s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \theta(p)p^{2k-2s})^{-1} \prod_{p|D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \theta(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \theta(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X)$$

For $p|D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_{2n}.$$

The first Hermitian lift (even case)

Theorem 5.1 (Case E) of [Ike08] Assume that $n = 2n'$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

In the case when n is odd, consider a similar lifting for a normalized Hecke eigenform $n = 2n' + 1$ is odd. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N$ $\in \mathcal{S}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p, define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_n.$$

The second Hermitian lift (odd case)

Theorem 5.2 (Case O) of [Ike08]. Assume that $n = 2n' + 1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

The lift $Lift^{(n)}(f)$ of f is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem 13.6).

Theorem 18.1 of [Ike08]. Let n, n', and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(n)}(f) \neq 0$. Let $L(s, Lift^{(n)}(f), st)$ be the L-function of $Lift^{(n)}(\mathbf{f})$ associated to $st : {}^L\mathcal{G} \to \mathrm{GL}_{4n}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(n)}(f), st)$ is equal to

$$\prod_{i=1}^{n} L(s + k + n' - i + \frac{1}{2}, f)L(s + k + n' - i + \frac{1}{2}, f, \theta).$$

Moreover, the $4n$ charcteristic roots of $L(s, Lift^{(n)}(f), st)$ given as follows: for $i = 1, \cdots, n$

$$\alpha_p p^{-k-n'+i-\frac{1}{2}}, \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}$$

Functional equation of the lift (Sho Takemori)

There are two cases [Ike08]: the even case (E) and the odd case (O):

$$\begin{cases} f\in S_{2k+1}(\Gamma_0(D),\theta),\ F=Lift^{(n)}(f) & (E)\\ \text{(the lift is of even degree } n=2n' \text{ and of weight } 2k+2n') & \\ f\in S_{2k}(\mathrm{SL}(\mathbb{Z})),\ F=Lift^{(n)}(f) & (O)\\ \text{(the lift is of odd degree } n=2n'+1 \text{ and of weight } 2k+2n'). & \end{cases}$$

Then, up to bad Euler factors, the standard L-function of $F=Lift^{(n)}(f)$ is given by

$$\prod_{i=1}^{n} L(s+k+n'-i+\tfrac{1}{2},f)L(s+k+n'-i+\tfrac{1}{2},f,\theta)$$

$$=\begin{cases} \prod_{i=1}^{2n'} L(s+k+n'-i+\frac{1}{2},f)L(s+k+n'-i+\frac{1}{2},f,\theta) & (E)\\ \prod_{i=1}^{n'} L(t(s,i),f)L(t(s,2n'+1-i),f) & \\ L(t(s,i),f,\theta)L(t(s,2n'+1-i),f,\theta) & \\ \prod_{i=1}^{2n'+1} L(s+k+n'-i+\frac{1}{2},f) & \\ \times L(s+k+n'-i+\frac{1}{2},f,\theta) & (O)\\ = L(s+k-\frac{1}{2},f)L(s+k-\frac{1}{2},f,\theta) & \\ \prod_{i=1}^{n'} L(t(s,i),f)L(t(s,2n'+2-i),f) & \\ L(t(s,i),f,\theta)L(t(s,2n'+2-i),f,\theta) & \end{cases}$$

where $t(s,i)=s+k+n'-i+\frac{1}{2}$.

The Gamma factor $\Gamma_{\mathcal{Z}}(s)$ of Ikeda's lift

In the even case since $(2k+1)-t(s,i)=t(1-s,2n'+1-i)$, using the Hecke functional equation in the symmetric terms of the product, gives the functional equation of the standard L function of the form $s\mapsto 1-s$, and the gamma factor is given by

$$\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(s+k+n'-i+1/2)^2=\Gamma_{\mathcal{D}}(s+n'+\frac{1}{2}).$$

In the odd case when $f\in S_{2k}(SL_2(\mathbb{Z}))$, the lift is of degree $n=2n'+1$ and of weight $2k+2n'$. By $2k-t(s,i)=t(1-s,2n+2-i)$, the standard L functions has functional equation of the form $s\mapsto 1-s$ and the gamma factor is the same. Hence the Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f and used as a pattern, extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of even weight ℓ, which equals in the lifted case to $\ell=2k+2n'$, where $k=(\ell-2n')/2=\ell/2-n'=\ell/2-n'$, when the Gamma factor of the standard zeta function with the symmetry $s\mapsto 1-s$ becomes (see p.55) $\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(s+\ell/2-n'+n'-i+(1/2))^2=$ $\prod_{i=1}^{n}\Gamma_{\mathbb{C}}(s+\ell/2-i+(1/2))^2=\prod_{i=0}^{n-1}\Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

Eisenstein series and the Rankin-Selberg method

The (Siegel-Hermite) Eisenstein series $E_{2\ell}^{(n)}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined by $E_{2\ell}^{(n)}(Z) = \sum_{g\in\Gamma_{K,\infty}^{(n)}\backslash\Gamma_K^{(n)}} (\det g)^{\ell} j(g,Z)^{-2\ell}$. The series converges absolutely for $\ell > n$. Define the normalized Eisenstein series $\mathcal{E}_{2\ell}^{(n)}(Z)$ by $\mathcal{E}_{2\ell}^{(n)}(Z) = 2^{-n}\prod_{i=1}^{n} L(i-2\ell,\theta^{i-1})\cdot E_{2\ell}^{(n)}(Z)$ If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)}\prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]}\det H.$$

Here, $\tilde{F}_p(H,X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in a Rankin-Selberg integral.

Also, we set , for $s \in \mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$,

$$E(Z,s,\ell,\psi) = \sum_{g\in C_\infty\backslash C} \psi(g)(\det g)^{\ell} j(g,Z)^{-2\ell}|(\det g) j(g,Z)|^{-s}.$$

A Rankin-Selberg method: the simplest case

Let us recall a Rankin-Selberg integral representation in the simplest elliptic modular case of GL_2.

Let $H = \{a = x+iy \mid y > 0\}$ be complex upper half plane on which the group $\mathrm{GL}_2^+(\mathbb{R})$ acts (of real 2×2 matrices $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix}$ of positive determinant). For a natural k the action of $\mathrm{GL}_2^+(\mathbb{R})$ on a function $f : H \to \mathbb{C}$ is defined by $(f|_k\gamma)(z)=$ $\det\gamma^{k/2} f(\gamma(z))(c_\gamma z + d_\gamma)^{-k}$

For a primitive cusp form $f = \sum_{n=1}^{\infty} a(n)e(nz)$ of conductor $C_f|N$, weight k and Dirichlet character ψ mod N (that is, a normalized new cusp Hecke eigenform $f \in \mathcal{S}_k^{new}(N,\psi) \subset \mathcal{S}_k(\Gamma_1(N))$), $a(1) = 1$, and another modular form $g = \sum_{n=1}^{\infty} b(n)e(nz) \in \mathcal{M}_\ell(N,\omega)$ of level N, weight ℓ and Dirichlet character ω mod N.

Classical convolution of Rankin-Selberg

For two Hecke eigenforms

$$f = \sum_{n=1}^{\infty} a(n)e(nz) \in \mathcal{S}_k(N,\psi) \quad g = \sum_{n=1}^{\infty} b(n)e(nz) \in \mathcal{M}_\ell(N,\omega)$$

their convolution is defined by $L(s,f,g) = \sum_{n=1}^{\infty} a(n)b(n)n^{-s}$. Moreover, one obtains an Euler product of degree 4 if

$$L(s,f) = \sum_{n=1}^{\infty} a(n)n^{-s} = \prod_{\text{primes } q} [(1-\alpha_q q^{-s})(1-\alpha'_q q^{-s})]^{-1}$$

$$L(s,g) = \sum_{n=1}^{\infty} b(n)n^{-s} = \prod_{\text{primes } q} [(1-\beta_q q^{-s})(1-\beta'_q q^{-s})]^{-1},$$

one has $\mathcal{D}(s,f,g) = L_N(2s+2-k-\ell,\omega\psi)L(s,f,g)$, where $\mathcal{D}(s,f,g)=$

$$\prod_{\text{primes } q} [(1-\alpha_q\beta_q q^{-s})(1-\alpha'_q\beta_q q^{-s})(1-\alpha_q\beta'_q q^{-s})(1-\alpha'_q\beta'_q q^{-s})]^{-1}$$

(Rankin's lemma).

Classical Rankin-Selberg integral representation

uses the Petersson product $\langle f,h\rangle_N$ on $\Gamma_0(N) = \{\begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})\}$

$$2(4\pi)^{-s}\Gamma(s)\mathcal{D}(s,f,g) = \langle f^\rho, gE(z,s-k+1)\rangle_N, \text{where}$$

$$E(z,s) = y^s {\sum}'_{(c,d)}(\psi\omega)(d)(Ncz+d)^{-(k-\ell)}|Ncz+d|^{-2s}.$$

This integral produces $\mathbb{C}$-analytic continuation from the right half plane $\{s\in\mathbb{C}|\mathrm{Re}(2s-k-\ell)>2\}$ where both the Eisestein series and $\mathcal{D}(s,f,g)$ converge, to the whole $\mathbb{C}$.

Also, it produces a $\mathbb{C}_p$-analytic continuation (or p-adic interpolatoon) of the critical values $\dfrac{\mathcal{D}(\ell+r,f,g(\chi))}{\pi^{1+\ell+2r}\langle f,f\rangle_N}$ (twisted by Dirichlet's characters χ mod p^v) which are algebraic numbers for all integers $r=0,\cdots,k-\ell-1$ assuming $k>\ell+2$, see [Hi85],

A Hermitian integral of Rankin-Selberg type

Theorem 4.1 (Shimura, Klosin), see [Bou16], p.13.
Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(C, \psi))$ of scalar weight ℓ, ψ mod $\mathfrak{c}$, such that $\forall \mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s + 3n/2, \mathbf{f}, \chi) =$$
$$\Lambda_\mathfrak{c}(s + 3n/2, \theta\psi\chi) \cdot C_0\langle \mathbf{f}, \theta_\mathcal{T}(\chi)\mathcal{E}(\bar{s} + n, \ell - \ell_\theta, \chi^\rho\psi)\rangle_{C''},$$

where $\mathcal{E}(Z, s, \ell - \ell_\theta, \psi)_{C''}$ is a normalized group theoretic Eisenstein series with components as above of level $\mathfrak{c}''$ divisible by $\mathfrak{c}$, and weight $\ell - \ell_\theta$. Here $\langle \cdot, \cdot \rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level $\mathfrak{c}''$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^\iota(s + h), \Gamma_n^\iota(s) = \pi^{\frac{n(n-1)}{2}} \prod_{j=0}^{n-1} \Gamma(s - j),$$

where $h = 0$ or 1, C_0 a subgroup index.

Euler factors of the standard zeta function, [Shi00], p. 171

The Euler factors $\mathcal{Z}_\mathfrak{q}(X)$ in the Hermitian modular case at the prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ are

(i) $\mathcal{Z}_\mathfrak{q}(X) = \prod_{i=1}^{n} \Big((1 - N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1 - N(\mathfrak{q})^n t_{\mathfrak{q},i}^{-1}X)\Big)^{-1}$,

if $\mathfrak{q}^\rho = \mathfrak{q}$ and $\mathfrak{q} \not| \mathfrak{c}$, (the inert case outside level $\mathfrak{c}$),

(ii) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{2n} \Big((1 - N(\mathfrak{q}_1)^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X_1)(1 - N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_2)\Big)^{-1}$,

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_1^\rho = \mathfrak{q}_2$ and $\mathfrak{q}_i \not| \mathfrak{c}$ for $i = 1, 2$ (the split case outside level) ,

(iii) $\mathcal{Z}_\mathfrak{q}(X) = \prod_{i=1}^{n} (1 - N(\mathfrak{q})^{n-1}t_{q,i}X)^{-1}$, if $\mathfrak{q}^\rho = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{c}$ (inert level divisors),

(iv) $\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{n} \Big((1 - N(\mathfrak{q}_1)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X_1)(1 - N(\mathfrak{q}_2)^{n-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X_2)\Big)^{-1}$,

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_i|\mathfrak{c}$ for $i = 1, 2$ (split level divisors).

where the $t_{?,i}$ above for $? = \mathfrak{q}, \mathfrak{q}_1\mathfrak{q}_2$, are the Satake parameters of the eigenform $\mathbf{f}$.

The standard motivic-normalized zeta $\mathcal{D}(s, \mathbf{f}, \chi)$

The standard zeta function of $\mathbf{f}$ is defined by means of the p-parameters as the following Euler product:

$$\mathcal{D}(s, \mathbf{f}, \chi) = \prod_p \prod_{i=1}^{2n} \left\{ \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)\alpha_{4n-i}(p)}{p^s}\right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. The p–parameters $\alpha_1(p), \ldots, \alpha_{4n}(p)$ of $\mathcal{D}(s, \mathbf{f}, \chi)$ for p not dividing the level C of the form $\mathbf{f}$ are related to the the $4n$ characteristic numbers

$$\alpha_1(p), \ \cdots, \ \alpha_{2n}(p), \ \alpha_{2n+1}(p), \ \cdots, \ \alpha_{4n}(p)$$

of the product of all $\mathfrak{q}$-factors $\mathcal{Z}_{\mathfrak{q}}(N\mathfrak{q}^{(n'+\frac{1}{2})}X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

There is a relation between the two normalizations $\mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f})$ explained below, see [Ha97] for general zeta functions $\mathcal{Z}(s, \mathbf{f})$ of type introduced in [Shi00], using reprsentation theory of unitary groups and Deligne's motivic L-functions.

Lecture N°3. Distributions, measures, Kummer congruences.

Kubota-Leopoldt p-adic zeta function and Iwasawa algebra.

Zeta values and Bernoulli Numbers

A key result in number theory is the Euler product expansion of the Riemann zeta $\zeta(s)$:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \qquad \text{(defined for } \mathrm{Re}(s) > 1).$$

The set of arguments s for which $\zeta(s)$ is defined was extended by Riemann to all $s \in \mathbb{C}$, $s \neq 1$. The special values $\zeta(1-k)$ at negative integers are rational numbers: $\zeta(1-k) = -\frac{B_k}{k}$, satisfying certain Kummer congruences $\bmod p^m$, where B_k are Bernoulli numbers, defined by the

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{te^t}{e^t-1}; B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = B_5 = \cdots = 0, B_4 = -\frac{1}{30},$$

$$B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = \frac{691}{2730},\ B_{14} = -\frac{7}{6}, \zeta(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

The denominators of B_k are small (Sylvester-Lipschitz): $\forall c \in \mathbb{Z} \Longrightarrow c^k(c^k-1)\frac{B_k}{k} \in \mathbb{Z}$ (see in [Mi-St]), Bernoulli polynomials $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i} = "(x+B)^k"$

$$S_k(N) = \sum_{n=1}^{N-1} n^k = \frac{1}{k+1}\left[B_{k+1}(N) - B_{k+1}\right],$$

$B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$

Bernoulli numbers and Kummer congruences

Kubota and Leopoldt constructed [KuLe64] a p-adic interpolation of these special values, explained by Mazur via a p-adic measure μ_c on $\mathbb{Z}_p$ and Kummer congruences for the Bernoulli numbers, see [Ka78] (p is a prime number, $c > 1$ an integer prime to p). Writing the normalized values

$$\zeta_{(p)}^{(c)}(-k) = (1-p^k)(1-c^{k+1})\zeta(-k) = \int_{\mathbb{Z}_p^*} x^k d\mu_c(x)$$

produces the Kummer congruences in the form: for any polynomial $h(x) = \sum_{i=0}^{n} \alpha_i x^i$ over $\mathbb{Z}$,

$$\forall x \in \mathbb{Z}_p, \sum_{i=0}^{n} \alpha_i x^i \in p^m\mathbb{Z}_p \Longrightarrow \sum_{i=0}^{n} \alpha_i \zeta_{(p)}^{(c)}(-i) \in p^m\mathbb{Z}_p,$$

Indeed, integrating the above polynomial $h(x)$ over μ_c produces the congruences. The existence of μ_c is deduced from the above formula for the sum of k-th powers $S_k(p^r)$ for $r \to \infty$, restricted to numbers n, prime to p.
In order to define such a measure μ_c it suffices for any continuous function $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$ to define its integral $\int_{\mathbb{Z}_p} \phi(x) d\mu_c$.
Approximating $\phi(x)$ by a polynomial (when the integral is already defined), pass to the limit (which is well defined due to Kummer congruences).

Kubota-Leopoldt p-adic zeta-function

The domain of definition of p-adic zeta functions is the p-adic analytic group $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ of all continuous p-adic characters of the profinite group $\mathbb{Z}_p^\times$, where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$) (over complex numbers $\mathbb{C} = Hom_{cont}(\mathbb{R}_+^*, \mathbb{C}^*)$, y run the characters $t \mapsto t^s$.

Define $\zeta_p : \mathcal{Y}_p \to \mathbb{C}_p$ on the space as the p-adic Mellin transform

$$\zeta_p(y) = \frac{\int_{\mathbb{Z}_p^*} y(x) d\mu_c(x)}{1 - cy(c)} = \frac{\mathcal{L}_{\mu_c}(y)}{1 - cy(c)},$$

with a single simple pole at $y = y_p^{-1} \in \mathcal{Y}_p$, where $y_p(x) = x$ the inclusion character $\mathbb{Z}_p^* \hookrightarrow \mathbb{C}_p^*$ and $y(x) = \chi(x)x^{k-1}$ is a typical arithmetical character ($y = y_p^{-1}$ becomes $k = 0$, $s = 1 - k = 1$).

Explicitly: Mazur's measure is given by $\mu_c(a + p^v\mathbb{Z}_p) = \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v})$, $B_1(x) = x - \frac{1}{2}$, ([LangMF], Ch.XIII), we see the zeta distribution $\mu_s|_{s=0}(a + (N)) = -B_1(\frac{a}{N})$.

Then the binomial formula $\int_Z (1+t)^z d\mu_c = \sum_{n=0}^\infty t^n \int_Z \binom{z}{n} d\mu_c$, gives the analyticity of $\zeta_p(y)$ on $t = y(1+p) - 1$ in the unit disc $\{t \in \mathbb{C}_p \| \, |t|_p < 1\}$.

The abstract Kummer congruences, p-adic Mellin transform and the Iwasawa algebra

A useful criterion for the existence of a measure with given properties is:

Proposition (The abstract Kummer congruences, see [Ka78])

Let $\{f_i\}$ be a system of continuous functions $f_i \in \mathcal{C}(\mathcal{X}_p, O_p)$ in the ring $\mathcal{C}(\mathcal{X}_p, \mathcal{O}_p)$ of all continuous functions on the compact totally disconnected group $\mathcal{X}_p$ with values in the ring of integers O_p of $\mathbb{C}$ such that $\mathbb{C}_p$-linear span of $\{f_i\}$ is dense in $\mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$. Let also $\{a_i\}$ be any system of elements $a_i \in \mathcal{O}_p$. Then the existence of an $\mathcal{O}_p$-valued measure μ on $\mathcal{X}_p$ with the property

$$\int_{\mathcal{X}_p} f_i d\mu = a_i$$

is equivalent to the following congruences, for an arbitrary choice of elements $b_i \in \mathbb{C}_p$ almost all of which vanish

$$\sum_i b_i f_i(x) \in p^n\mathcal{O}_p \text{ for all } x \in \mathcal{X}_p \text{ implies } \sum_i b_i a_i \in p^n\mathcal{O}_p. \quad (8)$$

Remark. Since $\mathbb{C}_p$-measures are characterised as bounded $\mathbb{C}_p$-valued distributions, every $\mathbb{C}_p$-measures on Y becomes a $\mathcal{O}_p$-valued measure after multiplication by some non-zero constant.

Proof of proposition 8.

The necessity is obvious since

$$\sum_i b_i a_i = \int_{\mathcal{X}_p} (p^n \mathcal{O}_p - \text{valued function}) d\mu = $$
$$= p^n \int_{\mathcal{X}_p} (\mathcal{O}_p - \text{valued function}) d\mu \in p^n \mathcal{O}_p.$$

In order to prove the sufficiency we need to construct a measure μ from the numbers a_i. For a function $f \in \mathcal{C}(\mathcal{X}_p, \mathcal{O}_p)$ and a positive integer n there exist elements $b_i \in \mathbb{C}$ such that only a finite number of b_i does not vanish, and

$$f - \sum_i b_i f_i \in p^n \mathcal{C}(\mathcal{X}_p, \mathcal{O}_p),$$

according to the density of the $\mathbb{C}$-span of $\{f_i\}$ in $\mathcal{C}(\mathcal{X}_p, \mathbb{C})$. By the assumption (8) the value $\sum_i a_i b_i$ belongs to $\mathcal{O}_p$ and is well defined modulo p^n (i.e. does not depend on the choice of b_i). Following N.M. Katz ([Ka78]), we denote this value by "$\int_{\mathcal{X}_p} f d\mu \bmod p^n$ ". Then we have that the limit procedure

$$\int_{\mathcal{X}_p} f d\mu = \lim_{n\to\infty} \text{“}\int_{\mathcal{X}_p} f d\mu \bmod p^n\text{”} \in \varprojlim_n \mathcal{O}_p / p^n \mathcal{O}_p = \mathcal{O}_p,$$

gives the measure μ.

Mazur's measure over $\mathcal{X}_S = \mathbb{Z}_S$

Let $c > 1$ be a positive integer coprime to $M_0 = \prod_{q\in S} q$ with S being a fixed set of primes containing p. Using the criterion of the proposition 8 we show that the $\mathbb{Q}$ -valued distribution defined by the formula

$$E_k^c(f) = E_k(f) - c^k E_k(f_c), \qquad f_c(x) = f(cx), \tag{9}$$

turns out to be a measure where $E_k(f)$ are defined in [LangMF], $f \in \mathrm{Step}(\mathcal{X}, \mathbb{Q}_p)$ and the field $\mathbb{Q}$ is viewed as a subfield of $\mathbb{C}_p$. Define the generalized Bernoulli polynomials $B_{k,f}^{(M)}(X)$ as

$$\sum_{k=0}^{\infty} B_{k,f}^{(M)}(X) \frac{t^k}{k!} = \sum_{a=0}^{M-1} f(a) \frac{te^{(a+X)t}}{e^{Mt} - 1}, \tag{10}$$

and the generalized sums of powers

$$S_{k,f}(M) = \sum_{a=0}^{M-1} f(a) a^k.$$

Then the definition (10) formally implies that

$$\frac{1}{k}[B_{k,f}^{(M)}(M) - B_{k,f}^{(M)}(0)] = S_{k-1,f}(M), \tag{11}$$

and also we see that

$$B_{k,f}^{(M)}(X) = \sum_{i=0}^{k} \binom{k}{i} B_{i,f} X^{k-i} = B_{k,f} + kB_{k-1,f}X + \cdots + B_{0,f}X^k. \tag{12}$$

The last identity can be rewritten symbolically as

$$B_{k,f}(X) = (B_f + X)^k.$$

The equality (11) enables us to calculate the (generalized) sums of powers in terms of the (generalized) Bernoulli numbers. In particular this equality implies that the Bernoulli numbers $B_{k,f}$ can be obtained by the following p-adic limit procedure (see [LangMF]):

$$B_{k,f} = \lim_{n\to\infty} \frac{1}{Mp^n} S_{k,f}(Mp^n) \quad \text{(a } p\text{-adic limit)}, \tag{13}$$

where f is a $\mathbb{C}_p$-valued function on $\mathcal{X}_p = \mathbb{Z}_S$. Indeed, if we replace M in (11) by Mp^n with growing n and let D be the common denominator of all coefficients of the polynomial $B_{k,f}^{(M)}(X)$. Then we have from (12) that

$$\frac{1}{k}\left[B_{k,f}^{(Mp^n)}(M) - B_{k,f}^{(M)}(0)\right] \equiv B_{k-1,f}(Mp^n) \pmod{\frac{1}{kD}p^2 n}. \tag{14}$$

The proof of (13) is accomplished by division of (14) by Mp^n and by
73 application of the formula (11).

Now we can directly show that the distribution E_k^c defined by (9) are in fact bounded measures. If we use (8) and take the functions $\{f_i\}$ to be all of the functions in $\mathrm{Step}(\mathcal{X}_p, O_p)$. Let $\{b_i\}$ be a system of elements $b_i \in \mathbb{C}_p$ such that for all $x \in \mathcal{X}_p$ the congruence

$$\sum_i b_i f_i(x) \equiv 0 \pmod{p^n} \tag{15}$$

holds. Set $f = \sum_i b_i f_i$ and assume (without loss of generality) that the number n is large enough so that for all i with $b_i \neq 0$ the congruence

$$B_{k,f_i} \equiv \frac{1}{Mp^n} S_{k,f_i}(Mp^n) \pmod{p^n} \tag{16}$$

is valid in accordance with (13). Then we see that

$$B_{k,f} \equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}, \tag{17}$$

hence we get by definition (9):

$$\begin{aligned} E_k^c(f) &= B_{k,f} - c^k B_{k,f_c} \qquad\qquad (18) \\ &\equiv (Mp^n)^{-1} \sum_i \sum_{a=0}^{Mp^n-1} b_i \left[f_i(a)a^k - f_i(ac)(ac)^k\right] \pmod{p^n}. \end{aligned}$$

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Let $a_c \in \{0, 1, \cdots, Mp^n - 1\}$, such that $a_c \equiv ac \pmod{Mp^n}$, then the map $a \longmapsto a_c$ is well defined and acts as a permutation of the set $\{0, 1, \cdots, Mp^n - 1\}$, hence (18) is equivalent to the congruence

$$E_k^c(f) = B_{k,f} - c^k B_{k,f_c} \equiv \sum_i \frac{a_c^k - (ac)^k}{Mp^n} \sum_{a=0}^{Mp^n-1} b_i f_i(a) a^k \pmod{p^n}. \tag{19}$$

Now the assumption (14) formally inplies that $E_k^c(f) \equiv 0 \pmod{p^n}$, completing the proof of the abstact congruences and the construction of measures E_k^c.

Remark

The formula (18) *also implies that for all* $f \in \mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$ *the following holds*

$$E_k^c(f) = kE_1^c(x_p^{k-1} f) \tag{20}$$

where $x_p : \mathcal{X}_p \longrightarrow \mathbb{C}_p \in \mathcal{C}(\mathcal{X}_p, \mathbb{C}_p)$ *is the composition of the projection* $\mathcal{X}_p \longrightarrow \mathbb{Z}_p$ *and the embedding* $\mathbb{Z}_p \hookrightarrow \mathbb{C}_p$.

Indeed if we put $a_c = ac + Mp^n t$ for some $t \in \mathbb{Z}$ then we see that

$$a_c^k - (ac)^k = (ac + Mp^n t)^k - (ac)^k \equiv kMp^n t(ac)^{k-1} \pmod{(Mp^n)^2},$$

and we get that in (19):

$$\frac{a_c^k - (ac)^k}{Mp^n} \equiv k(ac)^{k-1} \frac{a_c - ac}{Mp^n} \pmod{Mp^n}.$$

The last congruence is equivalent to saying that the abstract Kummer congruences (8) are satisfied by all functions of the type $x_p^{k-1} f_i$ for the measure E_1^c with $f_i \in \mathrm{Step}(\mathcal{X}_p, \mathbb{C}_p)$ establishing the identity (20).

The domain of definition of the non-Archimedean zeta functions

In the classical case the set on which zeta functions are defined is the set of complex numbers $\mathbb{C}$ which may be viewed equally as the set of all continuous characters (more precisely, quasicharacters) via the following isomorphism:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\sim} & \mathrm{Hom}_{\mathrm{cont}}(\mathbb{R}_+^\times, \mathbb{C}^\times) \\ s & \longmapsto & (x \longmapsto x^s) \end{array} \tag{21}$$

The construction which associates to a function $h(x)$ on $\mathbb{R}_+^\times$ (with certain growth conditions as $x \to \infty$ and $x \to 0$) the following integral

$$L_h(s) = \int_{\mathbb{R}_+^\times} h(x) x^s \frac{dx}{x}$$

(which converges probably not for all values of s) is called the *Mellin transform*.

The case of the Riemann zeta function

For example, if $\zeta(s) = \sum_{n\geq 1} n^{-s}$ then the function $\zeta(s)\Gamma(s)$ is the Mellin transform of the function $h(x) = 1/(1-e^{-x})$:

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{1-e^{-x}} x^s \frac{dx}{x}, \tag{22}$$

so that the integral and the series are absolutely convergent for $\mathrm{Re}(s) > 1$. For an arbitrary function of type $f(z) = \sum_{n=1}^{\infty} a(n) e^{2i\pi nz}$ with $z = x + iy \in \mathbb{H}$ in the upper half plane $\mathbb{H}$ and with the growth condition $a(n) = O(n^c)$ $(c > 0)$ on its Fourier coefficients, we see that the zeta function $L(s,f) = \sum_{n=1}^{\infty} a(n) n^{-s}$, essentially coincides with the Mellin transform of $f(z)$, that is, for $\mathrm{Re}(s) > 1 + c$

$$\frac{\Gamma(s)}{(2\pi)^s} L(s,f) = \int_0^\infty f(iy) y^s \frac{dy}{y} \text{ using } \Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y} \tag{23}$$

p-adic Mellin transform

In the theory of the non-Archimedean integration one considers the group $\mathbb{Z}_S^\times$ (the group of units of the S-adic completion of the ring of integers $\mathbb{Z}$) instead of the group $\mathbb{R}_+^\times$, and the Tate field $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ (the completion of an algebraic closure of $\mathbb{Q}_p$) instead of the complex field $\mathbb{C}$. The domain of definition of the p-adic zeta functions is the p-adic analytic group

$$\mathcal{Y}_S = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_S^\times, \mathbb{C}_p^\times) = \mathcal{Y}(\mathbb{Z}_S^\times), \tag{24}$$

where:

$$\mathbb{Z}_S^\times \cong \oplus_{q\in S} \mathbb{Z}_q^\times,$$

and the symbol

$$\mathcal{Y}(G) = \mathrm{Hom}_{\mathrm{cont}}(G, \mathbb{C}_p^\times) \tag{25}$$

denotes the functor of all p-adic characters of a topological group G.

The analytic structure of $\mathcal{Y}_S$

Let us consider in detail the structure of the topological group $\mathcal{Y}_S$. Define

$$U_p = \{x \in \mathbb{Z}_p^\times \quad | \quad x \equiv 1 \pmod{p^\nu}\},$$

where $\nu = 1$ or $\nu = 2$ according as $p > 2$ or $p = 2$. Then we have the natural decomposition

$$\mathcal{Y}_S = \mathcal{Y}\left((\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times\right) \times \mathcal{Y}(U_p). \tag{26}$$

The analytic structure on $\mathcal{Y}(U_p)$ is defined by the following isomorphism (which is equivalent to a special choice of a local parameter):

$$\varphi : \mathcal{Y}(U_p) \xrightarrow{\sim} T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\},$$

where $\varphi(x) = x(1+p^\nu)$, $1+p^\nu$ being a topoplogical generator of the multiplicative group $U_p \cong \mathbb{Z}_p$. An arbitrary character $\chi \in \mathcal{Y}_S$ can be uniquely represented in the form $\chi = \chi_0\chi_1$ where χ_0 is trivial on the component U_p, and χ_1 is trivial on the other component

$$(\mathbb{Z}/p^\nu\mathbb{Z})^\times \times \prod_{q \neq p} \mathbb{Z}_q^\times.$$

The character χ_0 is called the *tame component*, and χ_1 the *wild component* of the character χ. We denote by the symbol $\chi_{(t)}$ the (wild) character which is uniquely determined by the condition

$$\chi_{(t)}(1+p^\nu) = t$$

with $t \in \mathbb{C}_p$, $|t|_p < 1$.

In some cases it is convenient to use another local coordinate s which is analogous to the classical argument s of the Dirichlet series:

$$\begin{array}{ccc} \mathcal{O}_p & \longrightarrow & \mathcal{Y}_S \\ s & \longmapsto & \chi^{(s)}, \end{array}$$

where $\chi^{(s)}$ is given by $\chi^{(s)}((1+p^\nu)^\alpha) = (1+p^\nu)^{\alpha s} = \exp(\alpha s \log(1+p^\nu))$. The character $\chi^{(s)}$ is defined only for such s for which the series exp is p-adically convergent (i.e. for $|s|_p < p^{\nu - 1/(p-1)}$). In this domain of values of the argument we have that $t = (1+p^\nu)^s - 1$. But, for example, for $(1+t)^{p^n} = 1$ there is certainly no such value of s (because $t \neq 1$), so that the s-coordonate parametrizes a smaller neighborhood of the trivial character than the t-coordinate (which parametrizes all wild characters) (see [Ma73], [Ma76]).

Recall that an analytic function $F : T \longrightarrow \mathbb{C}_p$ ($T = \{z \in \mathbb{C}_p^\times \quad | \quad |z-1|_p < 1\}$), is defined as the sum of a series of the type $\sum_{i\geq 0} a_i(t-1)^i$ ($a_i \in \_Cp$), which is assumed to be absolutely convergent for all $t \in T$. The notion of an analytic function is then obviously extended to the whole group $\mathcal{Y}_S$ by shifts. The function

$$F(t) = \sum_{i=0}^{\infty} a_i(t-1)^i$$

is bounded on T iff all its coefficients a_i are universally bounded. This last fact can be easily deduced for example from the basic properties of the Newton polygon of the series $F(t)$ (see [Ko80], [Am-V]). If we apply to these series the Weierstrass preparation theorem (see [Ko80], [Ma73]), we see that in this case the function F has only a finite number of zeroes on T (if it is not identically zero).

p-adic analytic functions on $\mathcal{Y}_S$

Consider the torsion subgroup $\mathcal{Y}_S^{\mathrm{tors}} \subset \mathcal{Y}_S$. This subgroup is discrete in $\mathcal{Y}_S$ and its elements $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ can be obviously identified with primitive Dirichlet characters $\chi \mod M$ such that the support $S(\chi) = S(M)$ of the conductor of χ is containded in S. This identification is provided by a fixed embedding denoted

$$i_p : \overline{\mathbb{Q}}^\times \hookrightarrow \mathbb{C}_p^\times$$

if we note that each character $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ can be factored through some finite factor group $(\mathbb{Z}/M\mathbb{Z})^\times$:

$$\chi : \mathbb{Z}_S^\times \to (\mathbb{Z}/M\mathbb{Z})^\times \to \overline{\mathbb{Q}}^\times \overset{i_p}{\hookrightarrow} \mathbb{C}_p^\times,$$

and the smallest number M with the above condition is the conductor of $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$.

The symbol x_p will denote the composition of the natural projection $\mathbb{Z}_S^\times \to \mathbb{Z}_p^\times$ and of the natural embedding $\mathbb{Z}_p^\times \to \mathbb{C}_p^\times$, so that $x_p \in \mathcal{Y}_S$ and all integers k can be considered as the characters $x_p^k : y \longmapsto y^k$.

Let us consider a bounded $\mathbb{C}_p$-analytic function F on $\mathcal{Y}_S$. The above statement about zeroes of bounded $\mathbb{C}_p$-analytic functions implies now that the function F is uniquely determined by its values $F(\chi_0\chi)$, where χ_0 is a fixed character and χ runs through all elements $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ with possible exclusion of a finite number of characters in each analyticity component of the decomposition (26). This condition is satisfied, for example, by the set of characters $\chi \in \mathcal{Y}_S^{\mathrm{tors}}$ with the S-complete conductor (i.e. such that $S(\chi) = S$), and even for a smaller set of characters, for example for the set obtained by imposing the additional assumption that the character χ^2 is not trivial (see [Ma73]). Let μ be a (bounded) $\mathbb{C}_p$-valued measure on $\mathbb{Z}_S^\times$. Then the *non-Archimedean Mellin transform* of the measure μ is defined by

$$L_\mu(x) = \mu(x) = \int_{\mathbb{Z}_S^\times} x \mathrm{d}\mu, \qquad (x \in \mathcal{Y}_S), \tag{27}$$

which represents a bounded $\mathbb{C}_p$-analytic function

$$L_\mu : \mathcal{Y}_S \longrightarrow \mathbb{C}_p. \tag{28}$$

Indeed, the boundedness of the function L_μ is obvious since all characters $x \in \mathcal{Y}_S$ take values in O_p and μ also is bounded. The analyticity of this function expresses a general property of the integral (27), namely that it depends analytically on the parameter $x \in \mathcal{Y}_S$. However, we give below a pure algebraic proof of this fact which is based on a description of the Iwasawa algebra. This description will also imply that every bounded $\mathbb{C}_p$-analytic function on $\mathcal{Y}_S$ is the Mellin transform of a certain measure μ.

The Iwasawa algebra

Let $\mathcal{O}$ be a closed subring in $\mathcal{O}_p = \{z \in \mathbb{C}_p \quad | \quad |z|_p \leq 1\}$,

$$G = \varprojlim_i G_i, \qquad (i \in I),$$

a profinite group. Then the canonical homomorphism $G_i \overset{\pi_{ij}}{\longleftarrow} G_j$ induces a homomorphism of the corresponding group rings

$$\mathcal{O}[G_i] \longleftarrow \mathcal{O}[G_j].$$

Then the *completed group ring* $\mathcal{O}[[G]]$ is defined as the projective limit

$$\mathcal{O}[[G]] = \varprojlim_i \mathcal{O}[[G_i]], \qquad (i \in I).$$

Let us consider also the set $\mathrm{Dist}(G, \mathcal{O})$ of all $\mathcal{O}$-valued distributions on G which itself is an $\mathcal{O}$-module and a ring with respect to multiplication given by the *convolution of distributions*, which is defined in terms of families of functions

$$\mu_1^{(i)}, \mu_2^{(i)} : G_i \longrightarrow \mathcal{O},$$

(see the previous section) as follows:

We noticed above that the theorem 9 would imply a description of $\mathbb{C}_p$-analytic bounded functions on $\mathcal{Y}_S$ in terms of measures. Indeed, these functions are defined on analyticity components of the decomposition (26) as certain power series with p-adically bounded coefficients, that is, power series, whose coefficients belong to $\mathcal{O}_p$ after multiplication by some constant from $\mathbb{C}_p{}^{\times}$. Formulas for coefficients of these series can be also deduced from the proof of the theorem. However, we give a more direct computation of these coefficients in terms of the corresponding measures. Let us consider the component aU_p of the set $\mathbb{Z}_S^{\times}$ where

$$a \in (\mathbb{Z}/p^{\nu}\mathbb{Z})^{\times} \times \prod_{q \neq} \mathbb{Z}_q^{\times},$$

and let $\mu_a(x) = \mu(ax)$ be the corresponding measure on U_p defined by restriction of μ to the subset $aU_p \subset \mathbb{Z}_S^{\times}$.

Consider the isomorphism $U_p \cong \mathbb{Z}_p$ given by:

$$y = \gamma^x \quad (x \in \mathbb{Z}_p, y \in U_p),$$

with some choice of the generator γ of U_p (for example, we can take $\gamma = 1 + p^{\nu}$). Let μ'_a be the corresponding measure on $\mathbb{Z}_p$. Then this measure is uniquely determined by values of the integrals

$$\int_{\mathbb{Z}_p} \binom{x}{i} d\mu'_a(x) = a_i, \qquad (32)$$

with the interpolation polynomials $\binom{x}{i}$, since the $\mathbb{C}_p$-span of the family

$$\left\{ \binom{x}{i} \right\} \qquad (i \in \mathbb{Z}, i \geq 0)$$

is dense in $\mathcal{C}(\mathbb{Z}_p, \mathcal{O}_p)$ according to Mahler's interpolation theorem for continuous functions on $\mathbb{Z}_p$). Indeed, from the basic properties of the interpolation polynomials it follows that

$$\sum_i b_i \binom{x}{i} \equiv 0 \pmod{p^n} \quad (\text{for all } x \in \mathbb{Z}_p) \Longrightarrow b_i \equiv 0 \pmod{p^n}.$$

We can now apply the abstract Kummer congruences (see proposition 8), which imply that for arbitrary choice of numbers $a_i \in \mathcal{O}_p$ there exists a measure with the property (32).

Coefficients of power series and the Iwasawa isomorphism

We state that the Mellin transform L_{μ_a} of the measure μ_a is given by the power series $F_a(t)$ with coefficients as in (32), that is

$$\int_{U_p} \chi_{(t)}(y) \mathrm{d}\mu(ay) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu_a'(x) \right) (t-1)^i \qquad (33)$$

for all wild characters of the form $\chi_{(t)}$, $\chi_{(t)}(\gamma) = t$, $|t-1|_p < 1$. It suffices to show that (33) is valid for all characters of the type $y \longmapsto y^m$, where m is a positive integer. In order to do this we use the binomial expansion

$$\gamma^{mx} = (1 + (\gamma^m - 1))^x = \sum_{i=0}^{\infty} \binom{x}{i} (\gamma^m - 1)^i,$$

which implies that

$$\int_{U_p} y^m \mathrm{d}\mu(ay) = \int_{\mathbb{Z}_p} \gamma^{mx} \mathrm{d}\mu_a'(x) = \sum_{i=0}^{\infty} \left(\int_{\mathbb{Z}_p} \binom{x}{i} \mathrm{d}\mu_a'(x) \right) (\gamma^m - 1)^i,$$

establishing (33).

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6.2.7 Zeta Functions, L-Functions and Motives

(cf. [Man68], [Del79]). As we have seen with the example of the Dedekind zeta function $\zeta_K(s)$, the zeta function $\zeta(X, s)$ of an arithmetic scheme X can often be expressed in terms of L–functions of certain Galois representations. This link seems to be universal in the following sense.

Let $X \to \operatorname{Spec} \mathcal{O}_K$ be an arithmetic scheme over the maximal order $\mathcal{O}_K$ of a number field K such that the generic fiber $X_K = X \otimes_{\mathcal{O}_K} K$ is a smooth projective variety of dimension d, and let

$$\zeta(X, s) = \prod_{\mathfrak{p}} \zeta(X(\mathfrak{p}), s)$$

be its zeta function, where $X(\mathfrak{p}) = X \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{p})$ is the reduction of X modulo a maximal ideal $\mathfrak{p} \subset \mathcal{O}_K$. The shape of the function $\zeta(X(\mathfrak{p}), s)$ is described by the Weil conjecture (W4). If we assume that all $X(\mathfrak{p})$ are smooth projective varieties over $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_q$ then we obtain the following expressions for $\zeta(X, s)$:

$$\zeta(X,s) = \prod_{i=0}^{2d} L_i(X,s)^{(-1)^{i+1}}, \tag{6.2.56}$$

where

$$L_i(X,s) = \prod_{\mathfrak{p}} P_{i,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

and $P_{i,\mathfrak{p}}(X,t) \in \overline{\mathbb{Q}}[t]$ denote polynomials from the decomposition of the zeta function

$$\zeta(X(\mathfrak{p}),s) = \prod_{i=0}^{2d} P_{i,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{(-1)^{i+1}}.$$

In order to prove the conjecture (W4) ("the Riemann Hypothesis over a finite field"), Deligne identified the functions $L_i(X,s)$ with the L–functions of certain rational l–adic Galois representations

$$\rho_{X,i} : G_K \to \mathrm{Aut}\ H^i_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l); \quad L_i(X,s) = L(\rho_{X,i}, s)$$

defined by a natural action of the Galois group G_K on the l–adic cohomology groups $H^*_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l)$ using the transfer of structure

$$\begin{array}{ccc} X_{\overline{K}} & = X_K \otimes \overline{K} & \\ \downarrow & & \\ \mathrm{Spec}\ \overline{K} & \xrightarrow{\sigma} \mathrm{Spec}\ \overline{K} & (\sigma \in \mathrm{Aut}\ \overline{K}). \end{array}$$

If X_K is an algebraic curve then there are G_K–module isomorphisms

$$H^1_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) \cong V_l(J) = T_l(X) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$$

(the Tate module of the Jacobian of X),

$$H^0_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) = \mathbb{Q}_l, \quad H^2_{\acute{e}t}(X_{\overline{K}}, \mathbb{Q}_l) \cong V_l(\mu)$$

($V_l(\mu) = T_l(\mu) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$ the Tate module of l–power roots of unity). This implies the following explicit expressions for the L–functions

$$L_0(X,s) = \zeta_K(s), \qquad L_2(X,s) = \zeta_K(s-1),$$

and the zeta function

$$L_1(X,s) = L(X,s) = \prod_{\mathfrak{p}} P_{1,\mathfrak{p}}(X, \mathrm{N}\mathfrak{p}^{-s})^{-1},$$

(where $\deg P_{1,\mathfrak{p}}(X,t) = 2g$, g is the genus of the curve X_K) is often called the L–function of the curve X.

For topological varieties cohomology classes can be represented using cycles (by Poincaré duality), or using cells if the variety is a CW–complex. Grothendieck has conjectured that an analogue of CW–decomposition must exist for algebraic varieties over K. In view of this decomposition the factorization of the zeta function (6.2.56) should correspond to the decomposition of the variety into "generalized cells", which are no longer algebraic varieties but *motives*, elements of a certain larger category $\mathcal{M}_K$. This category is constructed in several steps, starting from the category $\mathcal{V}_K$ of smooth projective varieties over K.

Step 1). One constructs first an additive category $\mathcal{M}'_K$ in which $\mathrm{Hom}(M, N)$ are $\mathbb{Q}$–linear vector spaces, and one constructs a contravariant functor H^* from $\mathcal{V}_K$ to $\mathcal{M}'_K$, which is bijective on objects (i.e. with objects $H^*(X)$ one for each $X \in \mathcal{O}b(\mathcal{V}_K)$). This category is endowed with the following additional structures:

a) a tensor product $\otimes$ satisfying the standard commutativity, associativity and distributivity constraints;
b) the functor H^* takes disjoint unions of varieties into direct sums and products into tensor products (by means of a natural transformation compatible with the commutativity and associativity).

In this definition the group $\mathrm{Hom}(H^*(X), H^*(Y))$ is defined as a certain group of classes of correspondences between X and Y. For a smooth projective variety X over K denote by $Z^i(X)$ the vector space over $\mathbb{Q}$ whose basis is the set of all irreducible closed subschemes of codimension i, and denote by $Z^i_R(X)$ its quotient space modulo cohomological equivalence of cycles. Then in Grothendieck's definition, for fields K of characteristic zero one puts

$$\mathrm{Hom}(H^*(Y), H^*(X)) = Z_R^{\dim(Y)}(X \times Y).$$

Step 2. The category $\mathcal{M}_{\mathrm{eff},K}$ of false effective motives. This is obtained from $\mathcal{M}'_K$ by formally adjoining the images of all projections (i.e. of idempotent morphisms). In this category every projection arises from a direct sum decomposition. Categories with a tensor product and with the latter property are called *caroubien* or *pseudo–Abelian* categories; $\mathcal{M}_{\mathrm{eff},K}$ is the pseudo–Abelian envelope of $\mathcal{M}'_K$, cf. [Del79].

Step 3. The category $\overset{\circ}{\mathcal{M}}_K$ of false motives. Next we adjoin to $\mathcal{M}_{\mathrm{eff},K}$ all powers of the Tate object $\mathbb{Q}(1) = \underline{\mathrm{Hom}}(L, \mathbb{Q})$, where $L = \mathbb{Q}(-1) = H^2(\mathbb{P}^1)$ is the Lefschetz object and $\underline{\mathrm{Hom}}$ denotes the internal Hom in $\mathcal{M}_{\mathrm{eff},K}$. As a result we get the category $\overset{\circ}{\mathcal{M}}_K$ of "false motives". The category $\overset{\circ}{\mathcal{M}}_K$ can be obtained by a universal construction which converts the functor $M \to M \otimes \mathbb{Q}(-1) = M(-1)$ into an invertible functor. Each object of $\overset{\circ}{\mathcal{M}}_K$ has the form $M(n)$ with some M from $\mathcal{M}_{\mathrm{eff},K}$.
Note that for $X \in \mathcal{O}b(\mathcal{V}_K)$ the objects $H^i(X)$ are defined as the images of appropriate projections and

$$H^*(X) = \bigoplus_{i=0}^{2d} H^i(X).$$

The category $\overset{\circ}{\mathcal{M}}_K$ is a $\mathbb{Q}$–linear rigid Abelian category with the commutativity rule

$$\Psi^{r,s} : H^r(X) \otimes H^s(Y) \cong H^s(Y) \otimes H^r(X), u \otimes v \mapsto (-1)^{rs} v \otimes u,$$

which implies that the rank $\mathrm{rk}(H(X)) = \sum(-1)^r \dim H^r(X)$ could be negative (in fact it coincides with the *Euler characteristic* of X).

Step 4. The category $\mathcal{M}_K$ of true motives is obtained from $\overset{\circ}{\mathcal{M}}_K$ by a modification of the above commutativity constraint, in which the sign $(-1)^{rs}$ is dropped. This is a $\mathbb{Q}$–linear *Tannakian category*, formed by direct sums of factors of the type $M \subset H^r(X)(m)$, see [Del79].

Tannakian categories are characterized by the property that every such category (endowed with a fiber functor) can be realized as the category of finite dimensional representations of some (pro–) algebraic group.
In particular, the thus obtained category of motives can be regarded as the category of finite dimensional representations of a certain (pro–) algebraic group (the so-called *motivic Galois group*).
Each standard cohomology theory $\mathcal{H}$ on $\mathcal{V}_K$ (a functor from $\mathcal{V}_K$ to an Abelian category with the Künneth formula and with some standard functoriality properties) can be extended to the category $\mathcal{M}_K$. This extension thus defines the $\mathcal{H}$–realizations of motives.

In order to construct L–functions of motives one uses the following realizations:

a) *The Betti realization H_B*: for a field K embedded in $\mathbb{C}$ and $X \in \mathcal{O}b(\mathcal{V}_K)$ the singular cohomology groups (vector spaces over $\mathbb{Q}$) are defined

$$\mathcal{H} : X \mapsto H^*(X(\mathbb{C}), \mathbb{Q}) = H_B(X).$$

One has a Hodge decomposition of the complex vector spaces

$$H_B(M) \otimes \mathbb{C} = \oplus H_B^{p,q}(M) \quad (h^{p,q} = \dim_{\mathbb{C}} H_B^{p,q}(M)),$$

so that $\overline{H_B^{p,q}(M)} = H_B^{q,p}(M)$. If $K \subset \mathbb{R}$ then the complex conjugation on $X(\mathbb{C})$ defines a canonical involution F_∞ on $H_B(M)$, which may be viewed as the Frobenius element at infinity.

b) *The l–adic realizations H_l*: if Char $K \neq l$, $X \in \mathcal{O}b(\mathcal{V}_K)$ then the l–adic cohomology groups are defined as certain vector spaces over $\mathbb{Q}_l$

$$\mathcal{H} : X \mapsto H^*_{ét}(X_K, \mathbb{Q}_l) = H_l(X).$$

There is a natural action of the Galois group G_K on $H_l(X)$ by way of which one assigns an l–adic representation to a motive $M \in \mathcal{M}_K$

$$\rho_{M,l} : G_K \longrightarrow \text{Aut } H_l(M).$$

A non–trivial fact is that these representations are E–rational for some E, $E \subset \mathbb{C}$ in the sense of §6.2.1.

Using the general construction of 6.2.1 one defines the L–functions

$$L(M,s) = \prod_v L_v(M,s) \quad (v \text{ finite}),$$

where $L_v(M,s)^{-1} = L_{\mathfrak{p}_v}(M, \mathrm{N}\mathfrak{p}_v^{-s})^{-1}$ are certain polynomials in the variable $t = \mathrm{N}\mathfrak{p}_v^{-s}$ with coefficients in E.

For Archimedean places v one chooses a complex embedding $\tau_v : K \to \mathbb{C}$ defining v. Then the factors $L_v(M,s)$ are constructed using the Hodge decomposition $H_B(M) \otimes \mathbb{C} = \oplus H_B^{p,q}(M)$ and the action of the involution F_∞ (see the table in 5.3. of [Del79]).

According to a general conjecture the product

$$\Lambda(M,s) = \prod_v L_v(M,s) \quad (v \in \Sigma_K).$$

admits an analytic (meromorphic) continuation to the entire complex plane and satisfies a certain (conjectural) functional equation of the form

$$\Lambda(M,s) = \varepsilon(M,s)\Lambda(M^\vee, 1-s),$$

where $M^\vee$ is the motive dual to M (its realizations are duals of those of M), and $\varepsilon(M,s)$ is a certain function of s which is a product of an exponential function and a constant.

One has the following equation

$$\Lambda(M(n),s) = \Lambda(M, s+n).$$

A motive M is called pure of weight w if $h^{p,q} = 0$ for $p+q \neq w$. In this case we put $\mathrm{Re}(M) = -\frac{w}{2}$. The Weil conjecture W4) (see section 6.1.3) implies that for a sufficiently large finite set S of places of K the corresponding Dirichlet series (and the Euler product)

$$L_S(M,s) = \prod_{v \notin S} L_v(M,s)$$

converges absolutely for $\mathrm{Re}(M) + \mathrm{Re}(s) > 1$.

For points s on the boundary of absolute convergence (i.e. for $\mathrm{Re}(M) + \mathrm{Re}(s) = 1$ there is the following general conjecture (generalizing the theorem of Hadamard and de la Vallée–Poussin):

a) the function $L_S(M, s)$ does not vanish for $\mathrm{Re}(M) + \mathrm{Re}(s) = 1$;
b) the function $L_S(M, s)$ is entire apart from the case when M has even weight $-2n$ and contains as a summand the motive $\mathbb{Q}(n)$; in the last case there is a pole at $s = 1 - n$.

For example, for the motive $\mathbb{Q}(-1)$ one has

$$H_B(\mathbb{Q}(-1)) = H^2(\mathbb{P}^1(\mathbb{C}), \mathbb{Q}), \quad H_l(\mathbb{Q}(-1)) \cong V_l(\mu) = T_l(\mu) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l,$$

$w = 2$, $n = -1$ and the L -function

$$L(\mathbb{Q}(-1), s) = \zeta_K(s - 1)$$

has a simple pole at $s = 2$.

There are some very general conjectures on the existence of a correspondence between motives and compatible systems of l–adic representations. Nowadays these conjectures essentially determine key directions in arithmetical research ([CR01], [Tay02], [BoCa79], [Bor79], [Ta79]). We mention only a remarkable fact that in view of the proof of the theorem of G. Faltings (see §5.5) an Abelian variety is uniquely determined upto isogeny by the corresponding l–adic Galois representation on its Tate module.

This important result is cruicial also in Wiles' marvelous proof: in order to show that every semistable elliptic curve E over $\mathbb{Q}$ admits a modular parametrisation (see §7.2), it is enough (due to Faltings) to check that for some prime p the L–function of the Galois representation $\rho_{p,E}$ coinsides with the Mellin transform of a modular form of weight two (Wiles has used $p = 3$ and $p = 5$). In other words, the generating series of such a representation, defined starting from the traces of Frobenius elements, is a modular form of weight two which is proved by counting all possible deformations of the Galois representation in question taken modulo p.

Lecture N°4. p-adic L-functions on classical groups.

Ordinary case. Admissible measures, special values. ("Fonctions L p-adiques sur les groupes classiques : cas ordinaire, mesures admissibles, valeurs spéciales").

Admissible measures: Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then
- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, \cdots, h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform.

Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Complex and p-adic L-functions on classical groups.

- Automorphic forms and their weights. Complex analytic weight space. Motivic weights, introduction [EHLS].
- Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$
- Main Theorem in the Hermitian case
- Appendix 1. Zeta Functions, L-Functions and Motives. [MaPa]
- Appendix 2. Automorphic L-functions (A.Borel, [Bor79])

Modular forms as a tool in arithmetic

We view modular forms as:

1) q-power series

$$f = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$$ and as

2) holomorphic functions on the upper half plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}\ z > 0\}$

where $q = \exp(2\pi i z)$, $z \in \mathbb{H}$, and define L-function

$$L(f, s, \chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s}$$

for a Dirichlet character $\chi : (\mathbb{Z}/N\mathbb{Z})^* \to \mathbb{C}^*$ (its Mellin transfor

A famous example: the Ramanujan function $\tau(n)$

The function Δ (of the variable z) is defined by the formal expansion
$\Delta = \sum_{n=1}^{\infty} \tau(n) q^n$
$= q \prod_{m=1}^{\infty} (1 - q^m)^{24}$
$= q - 24q^2 + 252q^3 + \cdots$
is a cusp form of weight $k = 12$ for the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$).

$\tau(1) = 1, \tau(2) = -24,$
$\tau(3) = 252, \tau(4) = -1472$
$\tau(m)\tau(n) = \tau(mn)$
for $(n, m) = 1$,
$|\tau(p)| \le 2p^{11/2}$
for all primes p .

Analytic p-adic theory: zeta values vs. coefficients

It was much developed in the 60th in [Iw], [Se73] and [Wa].

Modular methods are applicable to the p-adic analytic continuation of $\zeta(s)$ itself through the normalized Eisenstein series:

$$\frac{(k-1)!}{2(2\pi i)^k} G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n = -\frac{B_k}{2k} + \sum_{d \ge 1} \frac{d^{k-1} q^d}{1 - q^d},$$

modular forms of even weight $k \ge 4$ for $\mathrm{SL}_2(\mathbb{Z})$ as follows:

J.-P.Serre noticed [Se73], p.206, that the constant term

$$\frac{\zeta(1-k)}{2}(1 - p^{k-1}) \text{ expresses by } \sigma^*_{k-1}(n) = \sum_{d|n} d^{k-1} \ (p \nmid d, n \ge 1),$$

the higher coefficients of the normalized Eisenstein series mod p^r.
In this way $\zeta^*_p(1-k)$ can be continually extended to $s \in \mathbb{Z}_p$ with a single simple pole at $s = 1$ starting from $s = 1 - k$ (see [Se73]).

The Hurwitz numbers naturally appear as the critical values of the Hecke L-function of ideal character $L(s, \psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) N\mathfrak{a}^{-s}$,

$\psi((\alpha)) = \alpha^m, \alpha \equiv 1 \bmod (2 + 2i)$, also defined for any imaginary quadratic field K, and $g_\psi = \sum_{\mathfrak{a}} \psi(\mathfrak{a}) q^{N\mathfrak{a}}$ is a modular form of weight $m + 1$. Its p-adic analytic continuation over m and s was constructed by Yu.I.Manin and M.M.Vishik (1974, [Ma-Vi]).

Recall: Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s; \mathbf{f})$

Let $\theta = \theta_K$ be the quadratic character attached to $K, n' = \left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\}, \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$$

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(via Hecke's eigenvalues: $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$$ (an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ (Motivically normalized standard zeta function

with a functional equation $s \mapsto \ell - s; \mathrm{rk} = 4n$)

Main result: p-adic interpolation of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$, $n \le s \le \ell - n, \chi \bmod p^r$.

Automorphic forms, p-adic theory of weights.

p-adic analytic weight space. Motivic and arithmetical weights, introduction to [EHLS]

[Lan13] , Arithmetic compactifications of PEL-type shimura varieties, London Mathematical Society Monographs, vol. 36, Princeton University Press, 2013.

For the purposes of subsequently defining p-adic modular forms for unitary groups we assume that the PEL data considered also satisfy:

- B has no type D factor;
- $\langle \cdot, \cdot \rangle$: $L \otimes \mathbb{Z}_p \times L \otimes \mathbb{Z}_p \to \mathbb{Z}_p(1)$ is a perfect pairing;
- p $\not|$ Disc$(\mathcal{O}_B)$, where Disc$(\mathcal{O}_B)$ is the discriminant of $(\mathcal{O}_B)$ over $\mathbb{Z}$ defined in [Lan13, Def. 1.1.1.6]; this condition implies that $(\mathcal{O}_B) \otimes (\mathcal{O}_B)$ is a maximal $(\mathcal{O}_B)$-order in B and that $\mathcal{O}_B \otimes \mathbb{Z}_p$ is a product of matrix algebras.

Associate a group scheme $G = G_P$ over $\mathbb{Z}$ with such a PEL datum P: for any $\mathbb{Z}$-algebra R

$$G(R) = \{(g, \nu) \in \mathrm{GL}_{\mathcal{O}_B \otimes R}(L \otimes R) \times R^\times : \langle gx, gy \rangle = \nu \langle x, y \rangle \forall x, y \in L \otimes R\}.$$

Then $G_{/\mathbb{Q}}$ is a reductive group, and by our hypotheses with respect to p, $G_{/\mathbb{Q}}$ is smooth and $G(\mathbb{Z}_p)$ is a hyperspecial maximal compact of $G(\mathbb{Q}_p)$.

Definitions of $\mathcal{X}_p$ $\mathcal{Y}_p$ through B, T.

Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$

The Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0, d] \to \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j\le\frac{w}{2}} \Gamma_{j,w-j}(s)$, where

$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h^{j,j}_+}\Gamma_{\mathbb{R}}(s-j+1)^{h^{j,j}_-}, & \text{if } 2j = w, \end{cases} \text{ where}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s),$$
$$h^{j,j} = h^{j,j}_+ + h^{j,j}_-, \sum_j h^{j,w-j} = d.$$

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \dots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$.

The Hodge polygon of the Hermitian zeta function

Starting from the Gamma factors of the standard Hermitian L-function $\mathcal{D}(s, \mathbf{f}, \chi)$ let us describe the Hodge polygon for $F = \mathbb{Q}$. The explicit form of the Gamma factors of the standard Hermitian L-function $\mathcal{Z}(s, \mathbf{f})$ was studied in (cf. [Shi00], p.179, [Ha97], [Ha14], [Kl], [Bou16], [Ge16]), and that of $\mathcal{D}(s, \mathbf{f}, \chi)$ follows with the Gamma factor

$$\Gamma_{\mathcal{D}}(s) = L_\infty(s, \mathbf{f}, \chi) = \prod_{j=0}^{n-1} \Gamma_{\mathbb{C}}(s-j)^2,$$

with the symmetry $s \mapsto \ell - s$.

These factors suggest the following form of the Hodge polygon of $\mathcal{D}(s, \mathbf{f}, \chi)$ of rank $d = 4n$ as that of the Hodge numbers $h^{j,w-j}$ below (in the increasing order of slopes j, with weight $w = \ell - 1$):

$$2 \cdot (0, \ell - 1), \dots, 2 \cdot (n-1, \ell - n),$$
$$2 \cdot (\ell - n, n-1), \dots, 2 \cdot (\ell - 1, 0),$$

following Serre's recipe [Se70], p.11.

Main Theorem (the Hermitian case)

Let $\Omega_{\mathbf{f}} = \langle \mathbf{f}, \mathbf{f} \rangle$ be the period attached to a Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f}_0$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n + 2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s, \chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s, \chi)$ including a finite Euler product, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}) p^{-n(n+1)} \right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by:
$\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^* - s} d\mu_{\mathcal{D}}^*$, $X = \mathbb{Z}_p^*$, where $s^* = \ell - n$, $s_* = n$.

(iii) in the admissible case assume that
$0 < h \leq \dfrac{s^* - s_* + 1}{2} = \dfrac{\ell + 1 - 2n}{2}$, where
$h = \mathrm{ord}_p \left((\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i}) p^{-n(n+1)} \right) > 0$, Then there exist h–admissible measures $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by
$i_p \left(A_p(s, \chi) \dfrac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$ with $A_p(s, \chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.

(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.

(a) Interpretation of s^*: the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

Eisenstein series and congruences (KEY POINT!)

The (Siegel-Hermite)Eisenstein series $E_{2\ell}^{(n)}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined by $E_{2\ell}^{(n)}(Z) = \sum_{g\in\Gamma_{K,\infty}^{(n)}\backslash\Gamma_K^{(n)}} (\det g)^{\ell} j(g,Z)^{-2\ell}$. The series converges absolutely for $\ell > n$. Define the normalized Eisenstein series $\mathcal{E}_{2\ell}^{(n)}(Z)$ by $\mathcal{E}_{2\ell}^{(n)}(Z) = 2^{-n}\prod_{i=1}^{n} L(i-2\ell,\theta^{i-1})\cdot E_{2\ell}^{(n)}(Z)$ If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2\ell}^{(n)}(Z)$ is polynomial over $\mathbb{Z}$ in $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]}\det H.$$

Here, $\tilde{F}_p(H,X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in a Rankin-Selberg integral.

Also, we set , for $s\in\mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$,

$$E(Z,s,\ell,\psi) = \sum_{g\in C_\infty\backslash C} \psi(g)(\det g)^{\ell} j(g,Z)^{-2\ell}|(\det g)j(g,Z)|^{-s}.$$

Recall :Hermitian Rankin-Selberg type integral

Theorem 4.1 (Shimura, Klosin), see [Bou16], p.13.
Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(C,\psi))$ of scalar weight ℓ, ψ mod $\mathfrak{c}$, such that $\forall\mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s+3n/2,\mathbf{f},\chi) =$$
$$\Lambda_{\mathfrak{c}}(s+3n/2,\theta\psi\chi)\cdot C_0\langle \mathbf{f}, \theta_{\mathcal{T}}(\chi)\mathcal{E}(\bar{s}+n,\ell-\ell_\theta,\chi^\rho\psi)\rangle_{C''},$$

where $\mathcal{E}(Z,s,\ell-\ell_\theta,\psi)_{C''}$ is a normalized group theoretic Eisenstein series with components as above of level $\mathfrak{c}''$ divisible by $\mathfrak{c}$, and weight $\ell-\ell_\theta$. Here $\langle\cdot,\cdot\rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level $\mathfrak{c}''$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^{\iota}(s+h), \Gamma_n^{\iota}(s) = \pi^{\frac{n(n-1)}{2}}\prod_{j=0}^{n-1}\Gamma(s-j),$$

where $h = 0$ or 1, C_0 a subgroup index.

Proof of the Main Theorem (ii): Kummer congruences

Let us se the notation $\mathcal{D}_p^{alg}(m,\mathbf{f},\chi)=A_p(s,\chi)\dfrac{\mathcal{D}^*(m,\mathbf{f},\chi)}{\Omega_{\mathbf{f}}}$

The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m,\chi)$ as Rankin-Selberg type integral at critical points $s=m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_{\mathcal{D}}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j,χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j,\mathbf{f},\chi_j)=A_p(s,\chi)\dfrac{\langle \mathbf{f},h_j\rangle}{\Omega_{\mathbf{f}}}$ with all $h_j=\sum_{\mathcal{T}} b_{j,\mathcal{T}}q^{\mathcal{T}}\in\mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing $\mathbf{f}$ and defined over $\bar{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x\in\mathbb{Z}_p^*,\ \sum_j \beta_j\chi_j x^{k_j}\equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \mathcal{D}_p^{alg}(s_j,\mathbf{f},\chi)\equiv 0 \mod p^N$$

$\beta_j\in\bar{\mathbb{Q}},\ k_j=s^*-s_j,$ where $s^*=\ell-n$ in our case.

Computing the Petersson products of a given modular form $\mathbf{f}(Z)=\sum_H a_H q^H\in\mathcal{M}_*(\bar{\mathbb{Q}})$ by another modular form $h(Z)=\sum_H b_H q^H\in\mathcal{M}_*(\bar{\mathbb{Q}})$ uses a linear form $\ell_{\mathbf{f}}: h\mapsto \dfrac{\langle \mathbf{f},h\rangle}{\langle \mathbf{f},\mathbf{f}\rangle}$ defined over a subfield $k\subset\bar{\mathbb{Q}}$.

Admissible Hermitian case

Let $\mathbf{f}\in\mathcal{S}_k(C;\psi)$ be a Hecke eigenform for the congruence subgroup C of level $\mathfrak{c}$. Let $\mathfrak{p}$ be a prime of K prime to $\mathfrak{c}$, which is inert over F. Then we say that $\mathbf{f}$ is pre-ordinary at $\mathfrak{p}$ if there exists an eigenform $0\neq\mathbf{f}_0\in\mathcal{M}_{\{p\}}\subset\mathcal{S}_k(Cp,\psi)$ with Satake parameters $t_{\mathfrak{p},i}$ such that

$$\left\|\left(\prod_{i=1}^n t_{\mathfrak{p},i}\right)N(\mathfrak{p})^{-\frac{n(n+1)}{2}}\right\|_p=1,$$

where $\|\|_p$ the normalized absolute value at p.

The admissible case corresponds to

$$\left\|\left(\prod_{\mathfrak{q}|p}\prod_{i=1}^n t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right\|_p=p^{-h}\text{ for a positive } h>0.$$

An interpretation of h as the difference $h=P_{N,p}(d/2)-P_H(d/2)$ comes from the above explicit relations.

Existence of h-admissible measures

of Amice-Vélu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p\left(\prod_{\mathfrak{q}|p}\left(\prod_{i=1}^n t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right)=h.$$

We use an easy condition of admissibility of a sequence of modular distributions Φ_j on $X=\mathcal{O}_K\otimes\mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa] and check congruences of the type

$$U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}(a+(p^v)\Big)\in Cp^{vj}\mathcal{O}[[q]]$$

for all $j=0,1,\dots,\varkappa h-1$. Here $s=j'+s_*$, $\Phi_{j'}(a+(p^v))$ a certain convolution, i.e.

$$\Phi_{j'}(\chi)=\theta(\chi)\cdot\mathcal{E}(s,\chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathcal{E}(s,\chi)$ with any Dirichlet character χ mod p^r. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X=\mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

Proof of the Main Theorem (iii): (admissible case)

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s,\mathbf{f},\chi)$ and an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on $\mathbf{f}_0$ the Rankin-Selberg integral of $\mathcal{F}_{s,\chi}:=\theta(\chi)\cdot\mathcal{E}(s,\chi)$ gives

$$\mathcal{D}^{alg}(s,\mathbf{f},\chi)=\frac{\langle\mathbf{f}_0,\theta(\chi)\cdot\mathcal{E}(s,\chi)\rangle}{\langle\mathbf{f},\mathbf{f}\rangle}\ \text{(the Petersson product on } G=GU(\eta_n))$$

$$=\alpha_{\mathbf{f}}^{-v}\frac{\langle\mathbf{f}_0,U(p^v)(\theta(\chi)\cdot\mathcal{E}(s,\chi))\rangle}{\langle\mathbf{f},\mathbf{f}\rangle}=\alpha_{\mathbf{f}}^{-v}\frac{\langle\mathbf{f}_0,U(p^v)(\mathcal{F}_{s,\chi})\rangle}{\langle\mathbf{f},\mathbf{f}\rangle}.$$

Modication in the admissible case: instead of Kummer congruences, to estimate p-adically the integrals of test functions: $M=p^v$:

$$\int_{a+(M)}(x-a)^j\,d\mathcal{D}^{alg}:=\sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\int_{a+(M)}x^{j'}\,d\mathcal{D}^{alg},\ \text{using}$$

the orthogonality of characters and the sequence of zeta distributions

$$\int_{a+(M)}x^j\,d\mathcal{D}^{alg}=\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times}\sum_{\chi \bmod M}\chi^{-1}(a)\int_X\chi(x)x^j\,d\mathcal{D}^{alg},$$

$\int_X\chi\,d\mathcal{D}^{alg}_{s_-+j}=\mathcal{D}^{alg}(s^*-j,f,\chi)=:\int_X\chi(x)x^j\,d\mathcal{D}^{alg}$.

Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j,\chi} = \sum_\xi v(\xi, s^*-j,\chi)q^\xi$: for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times}\sum_{j'=0}^{j}\binom{j}{j'}(-a)^{j-j'}\sum_{\chi \bmod M}\chi^{-1}(a)v(p^v\xi, s^*-j',\chi)q^\xi$$

$$\in Cp^{vj}\mathcal{O}[[q]] \quad \text{(This is a quasimodular form if } j' \neq s^*\text{)}$$

The resulting measure $\mu_{\mathcal{D}}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X, \mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.
Its p-adic Mellin transform $\mathcal{L}_{\mu_{\mathcal{D}}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \mathrm{ord}_p(\alpha)$.

Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients:
$\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i}(i = 1, \ldots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$.
Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j}\gamma_{i,j}\beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.
The abstract Kummer congruences are checked for a family of test elements.
In the admissible case it suffices to check binomial congruences for the Fourier coefficients as above in place of Kummer congruences.

Appendix A. Rewriting the local factor at p with character θ

Notice that if θ is the quadratic character attached to $K/\mathbb{Q}$ then

$$(1-\alpha_p X)(1-\alpha_p\theta(p)X) = \begin{cases} (1-\alpha_p X)^2 & \text{if } \theta(p)=1, p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2, N(\mathfrak{q}_i)=p,\\ (1-\alpha_p^2X^2), & \text{if } \theta(p)=-1, p\mathfrak{r}=\mathfrak{q}, N(\mathfrak{q})=p^2,\\ (1-\alpha_p X) & \text{if } \theta(p)=0, p\mathfrak{r}=\mathfrak{q}^2, N(\mathfrak{q})=p.\end{cases}$$

Thus, if $X=p^{-s}$, $X^2=p^{-2s}$, $N(\mathfrak{q})=p$, $\mathcal{Z}_{\mathfrak{q}}(X)^{-1}$

$$= \begin{cases} \prod_{i=1}^{2n}(1-N(\mathfrak{q}_1)^{2n}t^{-1}_{\mathfrak{q}_1\mathfrak{q}_2,i}X)(1-N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X), & \text{if } \theta(p)=1,\\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X^2)(1-N(\mathfrak{q})^{n}t^{-1}_{\mathfrak{q},i}X^2), & \text{if } \theta(p)=-1,\\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1-N(\mathfrak{q})^{n}t^{-1}_{\mathfrak{q},i}X), & \text{if } \theta(p)=0.\end{cases}$$

$$= \begin{cases} \prod_{i=1}^{n}(1-\gamma_{p,i}X)^2\prod_{i=1}^{n}(1-\delta_{p,i}X)^2 & \text{if } \theta(p)=1, \text{ i.e. } p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2,\\ \prod_{i=1}^{n}(1-\alpha^2_{p,i}X^2)\prod_{i=1}^{n}(1-\beta^2_{p,i}X^2), & \text{if } \theta(p)=-1, \text{ i.e. } p\mathfrak{r}=\mathfrak{q},\\ \prod_{i=1}^{n}(1-\alpha'_{p,i}X)\prod_{i=1}^{n}(1-\beta'_{p,i}X) & \text{if } \theta(p)=0, \text{ i.e. } p\mathfrak{r}=\mathfrak{q}^2,\end{cases}$$

where $\alpha'_{p,i}=p^{n-1}t_{\mathfrak{q},i}$, $\beta'_{p,i}p^n t^{-1}_{\mathfrak{q},i}$, $\gamma_{p,i}=p^{2n}t^{-1}_{\mathfrak{q}_1\mathfrak{q}_2,i}$, $p^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}$. It follows that $\prod_{\mathfrak{q}|p}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-n-(1/2)}X)=X^{4n}+\cdots$

Appendix B. Shimura's Theorem: algebraicity of critical values in Cases Sp and UT, p.234 of [Shi00]

Let $\mathbf{f}\in\mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x)=x^{\ell}_{\mathbf{a}}|x_{\mathbf{a}}|^{-\ell}$ with $\ell\in\mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0\in 2^{-1}\mathbb{Z}$. Assume, in the notations of Chapter 7 of [Shi00] on the weights k_v,μ_v,ℓ_v, that

Case Sp $\quad 2n+1-k_v+\mu_v\le 2\sigma_0\le k_v-\mu_v$,
where $\mu_v=0$ if $[k_v]-l_v\in 2\mathbb{Z}$
and $\mu_v=1$ if $[k_v]-l_v\notin 2\mathbb{Z}$; $\sigma_0-k_v+\mu_v$
for every $v\in\mathbf{a}$ if $\sigma_0>n$ and
$\sigma_0-1-k_v+\mu_v\in 2\mathbb{Z}$ for every $v\in\mathbf{a}$ if $\sigma_0\le n$.

Case UT $\quad 4n-(2k_{v\rho}+\ell_v)\le 2\sigma_0\le m_v-|k_v-k_{v\rho}-\ell_v|$
and $2\sigma_0-\ell_v\in 2\mathbb{Z}$ for every $v\in\mathbf{a}$.

Appendix B. Shimura's Theorem (continued)

Further exclude the following cases

(A) Case Sp $\sigma_0 = n+1, F = \mathbb{Q}$ and $\chi^2 = 1$;
(B) Case Sp $\sigma_0 = n+(3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$
(C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$;
(D) Case Sp $0 < \sigma_0 \leq n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is $\mathfrak{g}$;
(E) Case UT $2\sigma_0 = 2n+1, F = \mathbb{Q}, \chi_1 = \theta,$ and $k_v - k_{v\rho} = \ell_v$;
(F) Case UT $0 \leq 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is $\mathfrak{r}$

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi)/\langle \mathbf{f}, \mathbf{f}\rangle \in \pi^{n|m|+d\varepsilon}\bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp}, k \in \mathbb{Z}^{\mathbf{a}}, \text{ and } \sigma_0 > n_0), \\ n\sigma_0 - n^2, & \text{Case Sp}, k \notin \mathbb{Z}^{\mathbf{a}}, \text{or} \sigma_0 \leq n_0), \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon} \in \mathbb{Z}$ in all cases; if $k \notin \mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0 + k_v \in \mathbb{Z}$, so that $n|m| + d\varepsilon \in \mathbb{Z}$.

Thanks for your attention!

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