Constructions of p-adic L-functions and admissible measures for Hermitian modular forms

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Zeta values and Bernoulli Numbers

A key result in number theory is the expansion of the Riemann zeta-function $\zeta(s)$ into the Euler product:

$$\zeta(s) = \prod_p (1-p^{-s})^{-1} = \sum_{n=1}^{\infty} n^{-s} \qquad \text{(defined for } \mathrm{Re}(s) > 1\text{)}.$$

The set of arguments s for which $\zeta(s)$ is defined was extended by Riemann to all $s \in \mathbb{C}$, $s \neq 1$. The special values $\zeta(1-k)$ at negative integers are rational numbers: $\zeta(1-k) = -\frac{B_k}{k}$, satisfying certain Kummer congruences $\bmod\, p^m$, where B_k are Bernoulli numbers, defined by the

$$\sum_{n=0}^{\infty} \frac{B_n t^n}{n!} = \frac{te^t}{e^t-1}; B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = B_5 = \cdots = 0, B_4 = -\frac{1}{30},$$

$$B_6 = \frac{1}{42},\ B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = \frac{691}{2730},\ B_{14} = -\frac{7}{6}, \zeta(2k) = -\frac{(2\pi i)^{2k} B_{2k}}{2(2k)!},$$

Their denominators are small by the Sylvester-Lipschitz theorem

$$\forall c \in \mathbb{Z} \text{ implies } c^k(c^k-1)\frac{B_k}{k} \in \mathbb{Z} \quad \text{(see in [Mi-St])},$$

using the known formula for the sum of k-th powers via Bernoulli polynomials $B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i} = "(x+B)^k"$

$$S_k(N) = \sum_{n=1}^{N-1} n^k = \frac{1}{k+1}\left[B_{k+1}(N) - B_{k+1}\right],\ B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, \cdots$$

Kummer congruences and p-adic integration

Kubota and Leopoldt constructed [KuLe64] a p-adic interpolation of these special values, explained by Mazur via a p-adic measure μ_c on $\mathbb{Z}_p$ and Kummer congruences for the Bernoulli numbers, see [Ka78] (p is a prime number, $c > 1$ an integer prime to p). Writing the normalized values

$$\zeta_{(p)}^{(c)}(-k) = (1 - p^k)(1 - c^{k+1})\zeta(-k) = \int_{\mathbb{Z}_p^*} x^k d\mu_c(x)$$

produces the Kummer congruences in the form: for any polynomial $h(x) = \sum_{i=0}^n \alpha_i x^i$ over $\mathbb{Z}$,

$$\forall x \in \mathbb{Z}_p, \sum_{i=0}^n \alpha_i x^i \in p^m\mathbb{Z}_p \Longrightarrow \sum_{i=0}^n \alpha_i \zeta_{(p)}^{(c)}(-i) \in p^m\mathbb{Z}_p,$$

Indeed, integrating the above polynomial $h(x)$ over μ_c produces the congruences. The existence of μ_c is deduced from the above formula for the sum of k-th powers $S_k(p^r)$ for $r \to \infty$, restricted to numbers n, prime to p.
In order to define such a measure μ_c it suffices for any continuous function $\phi : \mathbb{Z}_p \to \mathbb{Z}_p$ to define its integral $\int_{\mathbb{Z}_p} \phi(x) d\mu_c$.
Approximating $\phi(x)$ by a polynomial (when the integral is already defined), pass to the limit (which is well defined due to Kummer congruences).

Kubota-Leopoldt p-adic zeta-function

The domain of definition of p-adic zeta functions is the p-adic analytic group $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ of all continuous p-adic characters of the profinite group $\mathbb{Z}_p^\times$, where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denotes the Tate field (completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$) (over complex numbers $\mathbb{C} = Hom_{cont}(\mathbb{R}_+^*, \mathbb{C}^*)$, y run the characters $t \mapsto t^s$.

Define $\zeta_p : \mathcal{Y}_p \to \mathbb{C}_p$ on the space as the p-adic Mellin transform

$$\zeta_p(y) = \frac{\int_{\mathbb{Z}_p^*} y(x) d\mu_c(x)}{1 - cy(c)} = \frac{\mathcal{L}_{\mu_c}(y)}{1 - cy(c)},$$

with a single simple pole at $y = y_p^{-1} \in \mathcal{Y}_p$, where $y_p(x) = x$ the inclusion character $\mathbb{Z}_p^* \hookrightarrow \mathbb{C}_p^*$ and $y(x) = \chi(x)x^{k-1}$ is a typical arithmetical character ($y = y_p^{-1}$ becomes $k = 0$, $s = 1 - k = 1$).

Explicitly: Mazur's measure is given by $\mu_c(a + p^v\mathbb{Z}_p) = \frac{1}{c}\left[\frac{ca}{p^v}\right] + \frac{1-c}{2c} = \frac{1}{c}B_1(\{\frac{ca}{p^v}\}) - B_1(\frac{a}{p^v})$, $B_1(x) = x - \frac{1}{2}$, ([LangMF], Ch.XIII), we see the zeta distribution $\mu_s|_{s=0}(a + (N)) = -B_1(\frac{a}{N})$.

Then the binomial formula

$\int_Z (1 + t)^z d\mu_c = \sum_{n=0}^\infty t^n \int_Z \binom{z}{n} d\mu_c$, gives the analyticity of $\zeta_p(y)$ on $t = y(1 + p) - 1$ in the unit disc $\{t \in \mathbb{C}_p \| \ |t|_p < 1\}$.

p-adic zeta functions of modular forms

From the p-adic zeta function of Kubota-Leopoldt, one extends p-adic zeta functions of various modular forms constructed, such as p-adic interpolation of the special values

$$L_\Delta(s,\chi)=\sum_{n=1}^{\infty}\chi(n)\tau(n)n^{-s},\ (s=1,2,\cdots,11)$$

for the Ramanujan function $\tau(n)$ defined by the expansion

$$q\prod_{m\geq 1}(1-q^m)^{24}=\sum_{n\geq 1}\tau(n)q^n=q-24q^2+252q^3-1472q^4+\cdots,$$

twisted by Dirichlet characters $\chi:(\mathbb{Z}/p^r\mathbb{Z})^*\to\mathbb{C}^*$; it was done in the elliptic and Hilbert modular cases by Yu.I.Manin and B.Mazur, via modular symbols and p-adic integration, see [Ma73], [Ma76]). In the Siegel modular case the p-adic standard zeta functions of Siegel modular forms were constructed in [Pa88], [Pa91] via Andrianov's identity (of Rankin-Selberg type).

PRESENT GOAL: To describe analytic p-adic continuation of the standard zeta function $L_F(s)$ of a Hermitian modular form $F=\sum_H A(H)q^H$ on the Hermitian upper half plane $\mathcal{H}_n$ of degree n, where $q^H=\exp(2\pi i\mathrm{Tr}(HZ))$, H runs through all semi-integral positive definite Hermitian matrices of degree n, i.e. $H\in\Lambda_n(\mathcal{O})$, in the integers $\mathcal{O}_K$ of an imaginary quadratic field $K=\mathbb{Q}(\sqrt{-D_K})$. Analytic p-adic continuation of their standard zeta functions is constructed via p-adic measures, bounded or growing.

Modular forms, zeta functions, L-functions

Eisenstein series $E_k = 1 + \frac{2}{\zeta(1-k)} \sum_{n=1}^{\infty} \sum_{d|n} d^{k-1} q^n \in \mathcal{M}_k$, a modular forms for even weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z})$, $q = e^{2\pi i z}$), and $E_2 \in \mathcal{QM}$ a quasimodular form. The ring of quasimodular forms, closed under differential operator $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, used in arithmetic, $\zeta(s)$ is the Riemann zeta function, $\zeta(-1) = -\frac{1}{12}$, $E_2 = 1 - 24\sum_{n=1}^{\infty}\sum_{d|n} dq^n$ is also a p-adic modular form (due to J.-P.Serre,[Se73], p.211)

Elliptic curves $E : y^2 = x^3 + ax + b$, $a, b \in \mathbb{Z}$, A.Wiles's modular forms $f_E = \sum_{n=1}^{\infty} a_n q^n$ with $a_p = p - CardE(\mathbb{F}_p)$ $(p \not| \, 4a^3 + 27b^2)$, and the L-function $L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$.

Zeta-functions or L-functions

They are attached to various mathematical objects as certain Euler products.

- L-functions link such objects to each other (a general form of functoriality);
- Special L-values answer fundamental questions about these objects in the form of a number (complex or p-adic).

Computing these numbers use integration theory of Dirichlet-Hecke characters along p-adic and complex valued measures.
This approach originates in the Dirichlet class number formula using the L-values in order to compute class numbers of algebraic number fields through Dirichlet's L-series $L(s,\chi)$: for an imaginary quadratic field K of discriminant $-D < -4$, $\chi_D(n) = \left(\frac{-D}{n}\right)$

$$h_D = \frac{\sqrt{D}L(1,\chi_D)}{2\pi} = L(0,\chi) = -\frac{1}{D}\sum_{a=1}^{D-1}\chi_D(a)a.$$

(Example: $\mathrm{disc}(\mathbb{Q}(\sqrt{-5}))) = -20$, $h_{20} = 2$; in PARI/GP $\chi_{20}(n) =$ kronecker(-20,n), `gp > -sum(x=1,19,x*kronecker(-20,x))/20`
`% 29 = 2`
Another famous example: the Millenium BSD Conjecture gives the rank of an elliptic curve E as the order of $L(E,s)$ at s=1 (i.e. the residue of its logarithmic derivative, see [MaPa], Ch.6).

A short story of critical values, see [YS]

Euler discovered $\zeta(2) = \frac{\pi^2}{6}$, and $\frac{2\zeta(2n)}{(2\pi i)^{2n}} = -\frac{B_{2n}}{(2n)!} \in \mathbb{Q}, (n \geq 1)$.

These are examples of critical values (in the sense of Deligne): for a more general zeta function $\mathcal{D}(s)$ the critical values are defined using its gamma factor $\Gamma_{\mathcal{D}}(s)$ such that the product $\Gamma_{\mathcal{D}}(s)\mathcal{D}(s)$ satisfies a standard functional equation under the symmetry $s \mapsto v - s$. Then $\mathcal{D}(n)$, $n \in \mathbb{Z}$ is a critical value of $\mathcal{D}(s)$ if both $\Gamma_{\mathcal{D}}(n)$ and $\Gamma_{\mathcal{D}}(v-n)$ are finite.

Hurwitz [Hur1899] showed a striking analogy to Euler's theorem:

$$\frac{\sum'_{\alpha \in \mathbb{Z}[i]} \alpha^{-4m}}{\Omega^{4m}} = \frac{H_m}{(4m)!} \in \mathbb{Q}, \Omega = 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = 2.6220575542\cdots$$

for $1 \leq m \in \mathbb{Z}$, where $\alpha = a + ib$, $a, b \in \mathbb{Z}$ are non-zero Gaussian integers and H_m are Hurwitz numbers (recursively computed, [Sl]): $H_1, H_2, \cdots = \frac{1}{10}, \frac{3}{10}, \frac{567}{130}, \frac{43659}{170}, \frac{392931}{10}, \cdots$. Recall the formula: Let $\wp$ be the Weierstrass $\wp$-function satisfying $\wp'^2 = 4\wp^3 - 4\wp$. Then $\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2^{4n} H_n z^{4n-2}}{4n(4n-2)!}$. A rapid computation of these values: take the Fourier expansion of the Eisenstein series at $z = i$, $q = e^{-2\pi}$:

$$G_{4m}(z) = \sum_{a,b}{}'(az+b)^{-4m} = 2\zeta(4m) + \frac{2(2\pi)^{4m}}{(4m-1)!} \sum_{d \geq 1} \frac{d^{4m-1} q^d}{(1-q^d)},$$

$\frac{G_{4m}(i)}{\Omega^{4m}} = \frac{H_m}{(4m)!}$, π, Ω – periods of $\zeta(s)$ and of $E : y^2 = 4x^3 - 4x$.

Analytic p-adic theory: zeta values vs. coefficients

It was much developed in the 60th in [Iw], [Se73] and [Wa].

Modular methods are applicable to the p-adic analytic continuation of $\zeta(s)$ itself through the normalized Eisenstein series:

$$\frac{(k-1)!}{2(2\pi i)^k}G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty}\sum_{d|n} d^{k-1}q^n = -\frac{B_k}{2k} + \sum_{d\geq 1}\frac{d^{k-1}q^d}{1-q^d},$$

modular forms of even weight $k \geq 4$ for $\mathrm{SL}_2(\mathbb{Z})$ as follows:

J.-P.Serre noticed [Se73], p.206, that the constant term

$$\frac{\zeta(1-k)}{2}(1-p^{k-1}) \text{ expresses by } \sigma^*_{k-1}(n) = \sum_{d|n} d^{k-1}\ (p \nmid d, n \geq 1),$$

the higher coefficients of the normalized Eisenstein series $\bmod p^r$. In this way $\zeta_p^*(1-k)$ can be continually extended to $s \in \mathbb{Z}_p$ with a single simple pole at $s=1$ starting from $s = 1-k$ (see [Se73]).

The Hurwitz numbers naturally appear as the critical values of the Hecke L-function of ideal character $L(s,\psi) = \sum_{\mathfrak{a}} \psi(\mathfrak{a})N\mathfrak{a}^{-s}$,

$\psi((\alpha)) = \alpha^m, \alpha \equiv 1 \bmod (2+2i)$, also defined for any imaginary quadratic field K, and $g_\psi = \sum_{\mathfrak{a}} \psi(\mathfrak{a})q^{N\mathfrak{a}}$ is a modular form of weight $m+1$. Its p-adic analytic continuation over m and s was constructed by Yu.I.Manin and M.M.Vishik (1974, [Ma-Vi]).

Complex and p-adic analytic continuation

A classical example of analytic continuation is given by the Riemenn zeta function with

$$\zeta(s) = \frac{(2\pi)^{s/2}}{2\Gamma(s/2)} \int_0^\infty (\theta(iy) - 1) y^{(s/2)-1} dy \quad (\mathrm{Re}(s) > 1),$$

through the theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{2\pi i n^2 z}$ which is a modular form of weight $1/2$ on the complex upper half plane $\mathcal{H}$.

For a Dirichlet L-function $L(s,\chi)$, an integral representation uses
I) theta function with Dirichlet character χ mod N

$$\theta(z,\chi) = \sum_{n\in\mathbb{Z}} \chi(n) n^\nu e^{2\pi i n^2 z}, \quad \chi(-1) = (-1)^\nu, \nu = 0, 1, \text{ or}$$

II) meromorphic zeta distributions

$\mu_s(a+(N)) := \sum_{\substack{n\geq 1 \\ n\equiv a \bmod N}} n^{-s} = N^{-s} \sum_{n\geq 1} (n + (\frac{a}{N}))^{-s}$: the integral

$L(s,\chi) = \int_X \chi(x) d\mu_s(x) = \sum_{a \bmod N} \chi(a)\mu_s(a+(N)) =: \mu_s(\chi)$ over

$X = \hat{\mathbb{Z}}$ or $\mathbb{Z}_p$ is a finite sum of partial series, $= -N^{k-1}\dfrac{B_k(\frac{a}{N})}{k}$.

Methods of constructing p-adic L-functions

Our long term purposes are to define and to use the p-adic L-functions in a way similar to complex L-functions via the following methods:
(1) Tate, Godement-Jacquet;
(2) the method of Rankin-Selberg;
(3) the method of Euler subgroups of Piatetski-Shapiro and the doubling method of Rallis-Böcherer (integral representations on a subgroup of $G \times G$);
(4) Shimura's method (the convolution integral with theta series);
(5) Shahidi's method.
There exist already advances for (1) to (4), and we also tried to develop (5), see [GMPS14].
We used the Eisenstein series and a p-adic integral of Shahidi's type for the reciprocal of a product of certain L-functions.

Hermitian modular group $\Gamma_{n,K}$ and the standard zeta function $\mathcal{Z}(s; \mathbf{f})$ (definitions)

Let $\theta = \theta_K$ be the quadratic character attached to $K, n' = \left[\frac{n}{2}\right]$.

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) | M\eta_n M^* = \eta_n \right\}, \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$$

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(via Hecke's eigenvalues: $\mathbf{f} | T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}, \mathfrak{a} \subset \mathcal{O}_K$)

$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$(an Euler product over primes $\mathfrak{q} \subset \mathcal{O}_K$,

with $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$, the Satake parameters $t_{i,\mathfrak{q}}, i = 1, \cdots, n$),

$\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ (Motivically normalized standard zeta function

with a functional equation $s \mapsto \ell - s; \mathrm{rk} = 4n$)

Main result: p-adic interpolation of all critical values $\mathcal{D}(s, \mathbf{f}, \chi)$, $n \le s \le \ell - n, \chi \bmod p^r$.

The idea of motivic normalization: Ikeda's lifting [Ike08]

The Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f and used as a pattern, extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of even weight ℓ, which equals in the lifted case to $\ell = 2k + 2n'$, where $k = (\ell - 2n')/2$ $= \ell/2 - n' = \ell/2 - n'$, when the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1 - s$ becomes (see p.43)
$\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s + \ell/2 - n' + n' - i + (1/2))^2 =$
$\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s + \ell/2 - i + (1/2))^2 = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s + \ell/2 - i - (1/2))^2.$
This Gamma factor suggests the following motivic normalization $\mathcal{D}(s) = \mathcal{Z}(s - (\ell/2) + (1/2))$ for which $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{Z}}(s - (\ell/2) + (1/2))^2$, and the L-function becomes $\mathcal{D}(s) = \mathcal{Z}(s - (\ell/2) + (1/2))$ with symmetry $s \mapsto 2(\ell/2) - 1 + 1 - s = \ell - s$ of motivic weight $\ell - 1$ and

$$\Gamma_{\mathcal{D}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s - i)^2,$$

with the slopes $2 \cdot 0, 2 \cdot 1, \ldots 2 \cdot (n-1)$, $2 \cdot (\ell - n), \cdots, 2 \cdot (\ell - 1)$, so that Deligne's critical values are at $s = n, \ldots, s = \ell - n$.

General zeta functions: critical values and coefficients

More general zeta functions are Euler products of degree d

$$\mathcal{D}(s,\chi) = \sum_{n=1}^{\infty} \chi(n) a_n n^{-s} = \prod_p \frac{1}{\mathcal{D}_p(\chi(p)p^{-s})},\ \Lambda_{\mathcal{D}}(s,\chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s,\chi),$$

where $\deg \mathcal{D}_p(X) = d$ for all but finitely many p, and $\mathcal{D}_p(0) = 1$.

In many cases algebraicity of the zeta values was proven as

$$\frac{\mathcal{D}^*(s_0,\chi)}{\Omega^{\pm}_{\mathcal{D}}} \in \mathbb{Q}(\{\chi(n), a_n\}_n),\ \text{where } \mathcal{D}^*(s,\chi) \text{ is normalized by } \Gamma_{\mathcal{D}},$$

at critical points $s_0 \in \mathbb{Z}_{crit}$ as linear combinations of coefficients a_n dividing out periods $\Omega^{\pm}_{\mathcal{D}}$, where $\mathcal{D}^*(s_0,\chi) = \Lambda_{\mathcal{D}}(s_0,\chi)$ if $h^{\ell,\ell} = 0$.

In p-adic analysis, the Tate field is used $\mathbb{C}_p = \hat{\bar{\mathbb{Q}}}_p$, the completion of an algebraic closure $\bar{\mathbb{Q}}_p$, in place of $\mathbb{C}$. Let us fix embeddings $\begin{cases} i_p : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p \\ i_\infty : \bar{\mathbb{Q}} \hookrightarrow \mathbb{C}, \end{cases}$ and try to continue analytically these zeta values to $s \in \mathbb{Z}_p$, $\chi \bmod p^r$.

Main result stated with Hodge/Newton polygons of $\mathcal{D}(s)$

The Hodge polygon $P_H(t) : [0, d] \to \mathbb{R}$ of the function $\mathcal{D}(s)$ and the Newton polygon $P_{N,p}(t) : [0, d] \to \mathbb{R}$ at p are piecewise linear:

The Hodge polygon of pure weight w has the slopes j of $length_j = h^{j,w-j}$ given by Serre's Gamma factors of the functional equation of the form $s \mapsto w + 1 - s$, relating $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ and $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$, where ρ is the complex conjugation of a_n, and $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$ equals to the product $\Gamma_{\mathcal{D}}(s) = \prod_{j \le \frac{w}{2}} \Gamma_{j,w-j}(s)$, where

$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h_+^{j,j}} \Gamma_{\mathbb{R}}(s-j+1)^{h_-^{j,j}}, & \text{if } 2j = w, \end{cases} \text{ where}$$

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right), \Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s),$$
$$h^{j,j} = h_+^{j,j} + h_-^{j,j}, \sum_j h^{j,w-j} = d.$$

The Newton polygon at p is the convex hull of points $(i, \mathrm{ord}_p(a_i))$ $(i = 0, \dots, d)$; its slopes λ are the p-adic valuations $\mathrm{ord}_p(\alpha_i)$ of the inverse roots α_i of $\mathcal{D}_p(X) \in \bar{\mathbb{Q}}[X] \subset \mathbb{C}_p[X]$: $length_\lambda = \sharp\{i \mid \mathrm{ord}_p(\alpha_i) = \lambda\}$.

p-adic analytic interpolation of $\mathcal{D}(s, \mathbf{f}, \chi)$

The result expresses the zeta values as integrals with respect to p-adic Mazur-type measures. These measures are constructed from the Fourier coefficients of Hermitian modular forms, and from eigenvalues of Hecke operators on the unitary group.

Pre-ordinary case: $P_H(t) = P_{N,p}(t)$ **at** $t = \frac{d}{2}$ The integrality of measures is proven representing $\mathcal{D}^*(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$ as a Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce certain bounded measures $\mu_{\mathcal{D}}$ from integral representations and Petersson product, [CourPa]. For the case of p inert in K, see [Bou16].

Admissible case: $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2}) > 0$ The zeta distributions are unbounded, but their sequence produce h-admissible (growing) measures of Amice-Vélu-type, allowing to integrate any continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) = \mathcal{Y}_p$. A general result is used on the existence of h-admissible (growing) measures from binomial congruences for the coefficients of Hermitian modular forms. Their p-adic Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y(x) d\mu_{\mathcal{D}}(x)$, $\mathcal{L}_{\mathcal{D}} : \mathcal{Y}_p \to \mathbb{C}_p$ give p-adic analytic interpolation of growth $\log_p^h(\cdot)$ of the L-values: the values $\mathcal{L}_{\mathcal{D}}(\chi x_p^m)$ are integrals given by $i_p\left(\frac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$.

A Hermitian modular form of weight ℓ with character σ

is a holomorphic function F on $\mathcal{H}_n$ ($n \geq 2$) such that $F(g\langle Z\rangle) = \sigma(g)F(Z)j(g,Z)^\ell$ for any $g \in \Gamma_{n,K}$. Here σ be a character of $\Gamma_K^{(n)}$, trivial on $\left\{\begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix}\right\}$, and for $Z \in \mathcal{H}_n$, put $g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}$, $j(g,Z) = \det(CZ+D)$.

Fourier expansions: a semi-integral Hermitian matrix is a Hermitian matrix $H \in (\sqrt{-D_K})^{-1}M_n(\mathcal{O})$ whose diagonal entries are integral. Denote the set of semi-integral Hermitian matrices by $\Lambda_n(\mathcal{O})$, the subset of its positive definite elements is $\Lambda_n(\mathcal{O})^+$.

A Hermitian modular form F is called a cusp form if it has a Fourier expansion of the form $F(Z) = \sum_{H \in \Lambda_n(\mathcal{O})^+} A(H)q^H$. Denote the space of cusp forms of weight ℓ with character σ by $\mathcal{S}_\ell(\Gamma_{n,K},\sigma)$.

The standard zeta function of a Hermitian modular form

Fix an integral ideal $\mathfrak{c}$ of $\mathcal{O}_K$. Denote by $C \subset \Gamma_{n,K}$ the congruence subgroup of level $\mathfrak{c}$; the group is essentially a principal congruence subgroup; it is an analogue of the group $\Gamma_0(N)$ in the elliptic modular case. Write $T(\mathfrak{a})$ for the Hecke operator associated to it as it is defined in [Shi00], page 162, using the action of double cosets $C\xi C$ with $\xi = \mathrm{diag}(\hat{D}, D)$, $(\det(D)) = (\alpha)$, $\hat{D} = (D^*)^{-1}$.

Consider a non-zero Hermitian modular form $\mathbf{f} \in \mathcal{M}_k(C, \psi)$ and assume $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbb{C}$ for all integral ideals $\mathfrak{a} \in \mathcal{O}$. Then

$$\mathcal{Z}(s, \mathbf{f}) = \left(\prod_{i=1}^{2n} L_{\mathfrak{c}}(2s - i + 1, \theta^{i-1})\right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

the sum is over all integral ideals of $\mathcal{O}_K$.

This series has an Euler product representation $\mathcal{Z}(s, \mathbf{f}) = \prod_{\mathfrak{q}} (\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1}$, where the product is over all prime ideals of $\mathcal{O}_K$, $\mathcal{Z}_{\mathfrak{q}}(X)$ is the numerator of the series $\sum_{r \geq 0} \lambda(\mathfrak{q}^r) X^r \in \mathbb{C}(X)$, computed by Shimura as follows.

Euler factors of the standard zeta function, [Shi00], p. 171

The Euler factors $\mathcal{Z}_{\mathfrak{q}}(X)$ in the Hermitian modular case at the prime ideal $\mathfrak{q}$ of $\mathcal{O}_K$ are

(i) $$\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n} \Big((1 - N(\mathfrak{q})^{n-1} t_{\mathfrak{q},i} X)(1 - N(\mathfrak{q})^{n} t_{\mathfrak{q},i}^{-1} X) \Big)^{-1},$$

if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q} \nmid \mathfrak{c}$, (the inert case outside level $\mathfrak{c}$),

(ii) $$\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{2n} \Big((1 - N(\mathfrak{q}_1)^{2n} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2) \Big)^{-1},$$

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_1^{\rho} = \mathfrak{q}_2$ and $\mathfrak{q}_i \nmid \mathfrak{c}$ for $i = 1, 2$ (the split case outside level) ,

(iii) $$\mathcal{Z}_{\mathfrak{q}}(X) = \prod_{i=1}^{n} \Big(1 - N(\mathfrak{q})^{n-1} t_{q,i} X\Big)^{-1},$$ if $\mathfrak{q}^{\rho} = \mathfrak{q}$ and $\mathfrak{q}|\mathfrak{c}$ (inert level divisors),

(iv) $$\mathcal{Z}_{\mathfrak{q}_1}(X_1)\mathcal{Z}_{\mathfrak{q}_2}(X_2) = \prod_{i=1}^{n} \Big((1 - N(\mathfrak{q}_1)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1} X_1)(1 - N(\mathfrak{q}_2)^{n-1} t_{\mathfrak{q}_1\mathfrak{q}_2,i} X_2) \Big)^{-1},$$

if $\mathfrak{q}_1 \neq \mathfrak{q}_2, \mathfrak{q}_i|\mathfrak{c}$ for $i = 1, 2$ (split level divisors).

where the $t_{?,i}$ above for $? = \mathfrak{q}, \mathfrak{q}_1\mathfrak{q}_2$, are the Satake parameters of the eigenform $\mathbf{f}$.

Notice the important dychotomy for the L-factors

in the Siegel modular case (that is, of symplectic type) vs. the Hermite modular case (of unitary type). In these cases the correspopnding complex component of the Langlands L-group is either $GSpinO(2n+1)(\mathbb{C})$, with the Euler factors of degree $2n+1$ (the standard representation of $GO(2n+1)$, resp. of degree 2^n (the spinor representation of the L-group) (the symplectic case), or, in the Hermite case, the complex component of the L-group is $GL_{2n}(\mathbb{C}) \times GL_{2n}(\mathbb{C})$, with the Euler factors of degree $4n$ (the standard representation of the L-group), see also 16.16, p.133, in particular, formula (16.16.2) at p.134 of [Shi97a] or [Shi97b] for a concise exposition.

The standard motivic-normalized zeta $\mathcal{D}(s, \mathbf{f}, \chi)$

The standard zeta function of $\mathbf{f}$ is defined by means of the p-parameters as the following Euler product:

$$\mathcal{D}(s, \mathbf{f}, \chi) = \prod_p \prod_{i=1}^{2n} \left\{ \left(1 - \frac{\chi(p)\alpha_i(p)}{p^s}\right) \left(1 - \frac{\chi(p)\alpha_{4n-i}(p)}{p^s}\right) \right\}^{-1},$$

where χ is an arbitrary Dirichlet character. The p–parameters $\alpha_1(p), \ldots, \alpha_{4n}(p)$ of $\mathcal{D}(s, \mathbf{f}, \chi)$ for p not dividing the level C of the form $\mathbf{f}$ are related to the the $4n$ characteristic numbers

$$\alpha_1(p),\ \cdots,\ \alpha_{2n}(p),\ \alpha_{2n+1}(p),\ \cdots,\ \alpha_{4n}(p)$$

of the product of all $\mathfrak{q}$-factors $\mathcal{Z}_\mathfrak{q}(N\mathfrak{q}^{(n'+\frac{1}{2})}X)^{-1}$ for all $\mathfrak{q}|p$, which is a polynomial of degree $4n$ of the variable $X = p^{-s}$ (for almost all p) with coefficients in a number field $T = T(\mathbf{f})$.

There is a relation between the two normalizations $\mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f})$ explained below, see [Ha97] for general zeta functions $\mathcal{Z}(s, \mathbf{f})$ of type introduced in [Shi00], using reprsentation theory of unitary groups and Deligne's motivic L-functions.

Description of the Main theorem

Let $\Omega_{\mathbf{f}}$ be a period attached to an Hermitian cusp eigenform $\mathbf{f}$, $\mathcal{D}(s,\mathbf{f}) = \mathcal{Z}(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$ the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left(\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i}\right) p^{-n(n+1)}, \quad h = \mathrm{ord}_p(\alpha_{\mathbf{f},p}),$$

The number $\alpha_{\mathbf{f}}$ turns out to be an eigenvalue of Atkin's type operator $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$ on some $\mathbf{f_0}$, and $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$.

Definition. Let M be a $\mathcal{O}$-module of finite rank where $\mathcal{O} \subset \mathbb{C}_p$. For $h \geq 1$, consider the following $\mathbb{C}_p$-vector spaces of functions on $\mathbb{Z}_p^*$: $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$. Then

- a continuous homomorphism $\mu : \mathcal{C} \to M$ is called a (bounded) measure M-valued measure on $\mathbb{Z}_p^*$.
- $\mu : \mathcal{C}^h \to M$ is called an h admissible measure M-valued measure on $\mathbb{Z}_p^*$ measure if the following growth condition is satisfied

$$\left|\int_{a+(p^v)} (x-a)^j d\mu\right|_p \leq p^{-v(h-j)}$$

for $j = 0, 1, ..., h-1$, and et $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ be the space of definition of p-adic Mellin transform

Theorem ([Am-V], [MTT]) For an h-admissible measure μ, the Mellin transform $\mathcal{L}_\mu : \mathcal{Y}_p \to \mathbb{C}_p$ exists and has growth $o(\log^h)$ (with infinitely many zeros).

Main Theorem.

Let $\mathbf{f}$ be a Hermitian cusp eigenform of degree $n \geq 2$ and of weight $\ell > 4n+2$. There exist distributions $\mu_{\mathcal{D},s}$ for $s = n, \cdots, \ell - n$ with the properties:

i) for all pairs (s, χ) such that $s \in \mathbb{Z}$ with $n \leq s \leq \ell - n$,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s,\chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \overline{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion i_p), with elementary factors $A_p(s,\chi) = \prod_{\mathfrak{q}|p} A_{\mathfrak{q}}(s,\chi)$ including a finite Euler product, gaussian sums, the conductor of χ; the integral is a finite sum.

(ii) if $\mathrm{ord}_p\left((\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i})p^{-n(n+1)}\right) = 0$ then the above distributions $\mu_{\mathcal{D},s}$ are bounded measures, we set $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$ and the integral is defined for all continuous characters $y \in \mathrm{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$.

Their Mellin transforms $\mathcal{L}_{\mu_{\mathcal{D}}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \to \mathbb{C}_p$, give bounded p-adic analytic interpolation of the above L-values to on the $\mathbb{C}_p$-analytic group $\mathcal{Y}_p$; and these distributions are related by:

$$\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} d\mu_{\mathcal{D}}^*,\ X = \mathbb{Z}_p^*, \text{ where } s^* = \ell - n,\ s_* = n.$$

(iii) in the admissible case assume that

$0 < h \le \dfrac{s^* - s_* + 1}{2} = \dfrac{\ell + 1 - 2n}{2}$, where

$h = \mathrm{ord}_p\left((\prod_{\mathfrak{q}|p}\prod_{i=1}^{n} t_{\mathfrak{q},i})p^{-n(n+1)}\right) > 0$, Then there exist h–admissible measures $\mu_{\mathcal{D}}$ whose integrals $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$ are given by

$i_p\left(A_p(s,\chi)\dfrac{\mathcal{D}^*(s,\mathbf{f},\overline{\chi})}{\Omega_{\mathbf{f}}}\right) \in \mathbb{C}_p$ with $A_p(s,\chi)$ as in (i); their Mellin transforms $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$, belong to the type $o(\log x_p^h)$.

(iv) the functions $\mathcal{L}_{\mathcal{D}}$ are determined by (i)-(iii).

Remarks.

(a) Interpretation of s^*: the smallest of the "big slopes" of P_H

(b) Interpretation of $s_* - 1$: the biggest of the "small slopes" of P_H.

Eisenstein series and congruences (KEY POINT!)

The (Siegel-Hermite)Eisenstein series $E^{(n)}_{2\ell}(Z)$ of weight 2ℓ, character $\det^{-\ell}$, is defined by $E^{(n)}_{2\ell}(Z) = \sum_{g \in \Gamma^{(n)}_{K,\infty} \backslash \Gamma^{(n)}_K} (\det g)^{\ell} j(g, Z)^{-2\ell}$. The series converges absolutely for $\ell > n$. Define the normalized Eisenstein series $\mathcal{E}^{(n)}_{2\ell}(Z)$ by $\mathcal{E}^{(n)}_{2\ell}(Z) = 2^{-n} \prod_{i=1}^{n} L(i - 2\ell, \theta^{i-1}) \cdot E^{(n)}_{2\ell}(Z)$ If $H \in \Lambda_n(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}^{(n)}_{2\ell}(Z)$ is polynomial over $\mathbb{Z}$ in $\{p^{\ell-(n/2)}\}_p$, and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here, $\tilde{F}_p(H, X)$ is a certain Laurent polynomial in the variables $\{X_p = p^{-s}, X_p^{-1}\}_p$ over $\mathbb{Z}$. This polynomial is a key point in proving congruences for the modular forms in a Rankin-Selberg integral.

Also, we set , for $s \in \mathbb{C}$ and a Hecke ideal character ψ mod $\mathfrak{c}$,

$$E(Z, s, \ell, \psi) = \sum_{g \in C_\infty \backslash C} \psi(g)(\det g)^{\ell} j(g, Z)^{-2\ell} |(\det g) j(g, Z)|^{-s}.$$

An integral representation of Rankin-Selberg type

The integral representation of Rankin-Selberg type in the Hermitian modular case:

Theorem 4.1 (Shimura, Klosin), see [Bou16], p.13.

Let $0 \neq \mathbf{f} \in \mathcal{M}_\ell(C,\psi))$ of scalar weight ℓ, ψ mod $\mathfrak{c}$, such that $\forall \mathfrak{a}, \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$, and assume that $2\ell \geq n$, then there exists $\mathcal{T} \in S_+ \cap \mathrm{GL}_n(K)$ and $\mathcal{R} \in \mathrm{GL}_n(K)$ such that

$$\Gamma((s))\psi(\det(\mathcal{T}))\mathcal{Z}(s+3n/2,\mathbf{f},\chi) =$$
$$\Lambda_\mathfrak{c}(s+3n/2,\theta\psi\chi)\cdot C_0\langle \mathbf{f}, \theta_\mathcal{T}(\chi)\mathcal{E}(\bar{s}+n,\ell-\ell_\theta,\chi^\rho\psi)\rangle_{C''},$$

where $\mathcal{E}(Z,s,\ell-\ell_\theta,\psi)_{C''}$ is a normalized group theoretic Eisenstein series with components as above of level $\mathfrak{c}''$ divisible by $\mathfrak{c}$, and weight $\ell-\ell_\theta$. Here $\langle\cdot,\cdot\rangle_{C''}$ is the normalized Petersson inner product associated to the congruence subgroup C'' of level $\mathfrak{c}''$.

$$\Gamma((s)) = (4\pi)^{-n(s+h)}\Gamma_n^\iota(s+h), \Gamma_n^\iota(s) = \pi^{\frac{n(n-1)}{2}}\prod_{j=0}^{n-1}\Gamma(s-j),$$

where $h=0$ or 1, C_0 a subgroup index.

The Hodge polygon of the Hermitian zeta function

Starting from the Gamma factors of the standard Hermitian L-function $\mathcal{D}(s, \mathbf{f}, \chi)$ let us describe the Hodge polygon for $F = \mathbb{Q}$. The explicit form of the Gamma factors of the standard Hermitian L-function $\mathcal{Z}(s, \mathbf{f})$ was studied in (cf. [Shi00], p.179, [Ha97], [Ha14], [Kl], [Bou16], [Ge16]), and that of $\mathcal{D}(s, \mathbf{f}, \chi)$ follows with the Gamma factor

$$\Gamma_{\mathcal{D}}(s) = L_\infty(s, \mathbf{f}, \chi) = \prod_{j=0}^{n-1} \Gamma_{\mathbb{C}}(s-j)^2,$$

with the symmetry $s \mapsto \ell - s$.

These factors suggest the following form of the Hodge polygon of $\mathcal{D}(s, \mathbf{f}, \chi)$ of rank $d = 4n$ as that of the Hodge numbers $h^{j, w-j}$ below (in the increasing order of slopes j, with weight $w = \ell - 1$):

$$\begin{aligned} &2 \cdot (0, \ell - 1), \dots, 2 \cdot (n-1, \ell - n), \\ &2 \cdot (\ell - n, n-1), \dots, 2 \cdot (\ell - 1, 0), \end{aligned}$$

following Serre's recipe [Se70], p.11.

Geometric study in the p-ordinary case

This case corresponds to the coincidence of the Hodge polygon and the Newton polygon, it was considered in [EHLS] using methods of algebraic geometry and the theory of algebraic modular forms, These methods use infinite dimensional towers of spaces over $\bar{\mathbb{Q}}$ containing automorphic forms of all levels of type Np^r, and their specializations at CM-points on Shimra varieties.

On the other hand, the case p inert in K was studied in [Bou16], based on methods in [CourPa].

The present method treats all p unramified in K and coprime to the level $\mathfrak{c}$ of $\mathbf{f}$; it is based on a modular construction of admissible measures as sequences of zeta distributions via an integral representation of Rankin-Selberg type. This method allows to reduce consideration to congruences between Hermitian modular forms of fixed level $\mathfrak{c}p$.

Proof of the Main Theorem (ii): Kummer congruences

Let us se the notation $\mathcal{D}_p^{alg}(m, \mathbf{f}, \chi) = A_p(s, \chi)\dfrac{\mathcal{D}^*(m, \mathbf{f}, \chi)}{\Omega_{\mathbf{f}}}$

The integrality of measures is proven representing $\mathcal{D}_p^{alg}(m, \chi)$ as Rankin-Selberg type integral at critical points $s = m$. Coefficients of modular forms in this integral satisfy Kummer-type congruences and produce bounded measures $\mu_{\mathcal{D}}$ whose construction reduces to congruences of Kummer type between the Fourier coefficients of modular forms, see also [Bou16]. Suppose that we are given infinitely many "critical pairs" (s_j, χ_j) at which one has an integral representation $\mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi_j) = A_p(s, \chi)\dfrac{\langle \mathbf{f}, h_j \rangle}{\Omega_{\mathbf{f}}}$ with all $h_j = \sum_{\mathcal{T}} b_{j,\mathcal{T}} q^{\mathcal{T}} \in \mathcal{M}$ in a certain finite-dimensional space $\mathcal{M}$ containing $\mathbf{f}$ and defined over $\bar{\mathbb{Q}}$. We prove the following Kummer-type congruences:

$$\forall x \in \mathbb{Z}_p^*, \sum_j \beta_j \chi_j x^{k_j} \equiv 0 \mod p^N \Longrightarrow \sum_j \beta_j \mathcal{D}_p^{alg}(s_j, \mathbf{f}, \chi) \equiv 0 \mod p^N$$

$$\beta_j \in \bar{\mathbb{Q}}, k_j = s^* - s_j, \text{ where } s^* = \ell - n \text{ in our case.}$$

Computing the Petersson products of a given modular form $\mathbf{f}(Z) = \sum_H a_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ by another modular form $h(Z) = \sum_H b_H q^H \in \mathcal{M}_*(\bar{\mathbb{Q}})$ uses a linear form $\ell_{\mathbf{f}} : h \mapsto \dfrac{\langle \mathbf{f}, h \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}$ defined over a subfield $k \subset \bar{\mathbb{Q}}$.

Admissible Hermitian case

Let $\mathbf{f} \in \mathcal{S}_k(C;\psi)$ be a Hecke eigenform for the congruence subgroup C of level $\mathfrak{c}$. Let $\mathfrak{p}$ be a prime of K prime to $\mathfrak{c}$, which is inert over F. Then we say that $\mathbf{f}$ is pre-ordinary at $\mathfrak{p}$ if there exists an eigenform $0 \neq \mathbf{f}_0 \in \mathcal{M}_{\{p\}} \subset \mathcal{S}_k(Cp,\psi)$ with Satake parameters $t_{\mathfrak{p},i}$ such that

$$\left\| \left(\prod_{i=1}^{n} t_{\mathfrak{p},i} \right) N(\mathfrak{p})^{-\frac{n(n+1)}{2}} \right\|_p = 1,$$

where $\|\|_p$ the normalized absolute value at p.

The admissible case corresponds to

$$\left\| \left(\prod_{\mathfrak{q}|p} \prod_{i=1}^{n} t_{\mathfrak{q},i} \right) p^{-n(n+1)} \right\|_p = p^{-h} \text{ for a positive } h > 0.$$

An interpretation of h as the difference $h = P_{N,p}(d/2) - P_H(d/2)$ comes from the above explicit relations.

Existence of h-admissible measures

of Amice-Vélu-type gives an unbounded p-adic analytic interpolation of the L-values of growth $\log_p^h(\cdot)$, using the Mellin transform of the constructed measures. This condition says that the product $\prod_{i=1}^n t_{\mathfrak{p},i}$ is nonzero and divisible by a certain power of p in $\mathcal{O}$:

$$\mathrm{ord}_p\left(\prod_{\mathfrak{q}|p}\left(\prod_{i=1}^{n} t_{\mathfrak{q},i}\right)p^{-n(n+1)}\right) = h.$$

We use an easy condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathcal{O}_K \otimes \mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa] and check congruences of the type

$$U^{\varkappa v}\Big(\sum_{j'=0}^{j}\binom{j}{j'}(-a_p^0)^{j-j'}\Phi_{j'}(a+(p^v)\Big) \in Cp^{vj}\mathcal{O}[[q]]$$

for all $j = 0, 1, \ldots, \varkappa h - 1$. Here $s = j' + s_*$, $\Phi_{j'}(a + (p^v))$ a certain convolution, i.e.

$$\Phi_{j'}(\chi) = \theta(\chi) \cdot \mathcal{E}(s, \chi)$$

of a Hermitian theta series $\theta(\chi)$ and an Eisenstein series $\mathbf{E}(s, \chi)$ with any Dirichlet character χ mod p^r. We use a general sufficient condition of admissibility of a sequence of modular distributions Φ_j on $X = \mathbb{Z}_p$ with values in $\mathcal{O}[[q]]$ as in Theorem 4.8 of [CourPa].

Using algebraic and p-adic modular forms

There are several methods to compute various L-values starting from the constant term of the Eisenstein series in [Se73],

$$G_k(z) = \frac{\zeta(1-k)}{2} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \frac{\Gamma(k)}{(2\pi i)^k} \sum_{(c,d)}{}' (cz+d)^{-k} \ (\text{for} k \geq 4),$$

and using Petersson products of nearly-holomorphic Siegel modular forms and arithmetical automorphic forms as in [Shi00]:
the Rankin-Selberg method,
the doubling method (pull-back method).
A known example is the standard zeta function $D(s, f, \chi)$ of a Siegel cusp eigenform $f \in \mathcal{S}_n^k(\Gamma)$ of genus n (with local factors of degree $2n+1$) and χ a Dirichlet character.
Theorem (the case of even genus n ([Pa91], [CourPa]), via the Rankin-Selberg method) gives a p-adic interpolation of the normailzed critical values $D^*(s, f, \chi)$ using Andrianov-Kalinin integral representation of these values $1+n-k \leq s \leq k-n$ through the Petersson product $\langle f, \theta_{T_0} \delta^r E \rangle$ where δ^r is a certain composition of Maass-Shimura differential operators, θ_{T_0} a theta-series of weight $n/2$, attached to a fixed $n \times n$ matrix T_0.

Proof of the Main Theorem (iii): (admissible case)

Using a Rankin-Selberg integral representation for $\mathcal{D}^{alg}(s, \mathbf{f}, \chi)$ and an eigenfunction $\mathbf{f}_0$ of Atkin's operator $U(p)$ of eigenvalue $\alpha_{\mathbf{f}}$ on $\mathbf{f}_0$ the Rankin-Selberg integral of $\mathcal{F}_{s,\chi} := \theta(\chi) \cdot \mathcal{E}(s, \chi)$ gives

$$\mathcal{D}^{alg}(s, \mathbf{f}, \chi) = \frac{\langle \mathbf{f}_0, \theta(\chi) \cdot \mathcal{E}(s, \chi) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} \text{ (the Petersson product on } G = GU(\eta_n))$$

$$= \alpha_{\mathbf{f}}^{-v} \frac{\langle \mathbf{f}_0, U(p^v)(\theta(\chi) \cdot \mathcal{E}(s, \chi)) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle} = \alpha_{\mathbf{f}}^{-v} \frac{\langle f_0, U(p^v)(\mathcal{F}_{s,\chi}) \rangle}{\langle \mathbf{f}, \mathbf{f} \rangle}.$$

Modication in the admissible case: instead of Kummer congruences,

to estimate p-adically the integrals of test functions: $M = p^v$:

$$\int_{a+(M)} (x-a)^j d\mathcal{D}^{alg} := \sum_{j'=0}^{j} \binom{j}{j'} (-a)^{j-j'} \int_{a+(M)} x^{j'} d\mathcal{D}^{alg}, \text{ using}$$

the orthogonality of characters and the sequence of zeta distributions

$$\int_{a+(M)} x^j d\mathcal{D}^{alg} = \frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times} \sum_{\chi \bmod M} \chi^{-1}(a) \int_X \chi(x) x^j d\mathcal{D}^{alg},$$

$\int_X \chi d\mathcal{D}^{alg}_{s_-+j} = \mathcal{D}^{alg}(s^* - j, f, \chi) =: \int_X \chi(x) x^j d\mathcal{D}^{alg}$.

Congruences between the coefficients of the Hermitian modular forms

In order to integrate any locally-analytic function on X, it suffices to check the following binomial congruences for the coefficients of the Hermitian modular form $\mathcal{F}_{s^*-j,\chi} = \sum_\xi v(\xi, s^*-j, \chi)q^\xi$: for $v \gg 0$, and a constant C

$$\frac{1}{\sharp(\mathcal{O}/M\mathcal{O})^\times} \sum_{j'=0}^{j} \binom{j}{j'} (-a)^{j-j'} \sum_{\chi \bmod M} \chi^{-1}(a) v(p^v\xi, s^*-j', \chi) q^\xi$$

$$\in Cp^{vj}\mathcal{O}[[q]] \quad \text{(This is a quasimodular form if } j' \neq s^*\text{)}$$

The resulting measure $\mu_{\mathcal{D}}$ allows to integrate all continuous characters in $\mathcal{Y}_p = \mathrm{Hom}_{cont}(X, \mathbb{C}_p^*)$, including Hecke characters, as they are always locally analytic.

Its p-adic Mellin transform $\mathcal{L}_{\mu_{\mathcal{D}}}$ is an analytic function on $\mathcal{Y}_p$ of the logarithmic growth $\mathcal{O}(\log^h)$, $h = \mathrm{ord}_p(\alpha)$.

Proof of the main congruences

Thus the Petersson product in $\ell_{\mathbf{f}}$ can be expressed through the Fourier coeffcients of h in the case when there is a finite basis of the dual space consisting of certain Fourier coeffcients: $\ell_{\mathcal{T}_i} : h \mapsto b_{\mathcal{T}_i} (i = 1, \ldots, n)$. It follows that $\ell_{\mathbf{f}}(h) = \sum_i \gamma_i b_{\mathcal{T}_i}$, where $\gamma_i \in k$.

Using the expression for $\ell_f(h_j) = \sum_i \gamma_{i,j} b_{j,\mathcal{T}_i}$, the above congruences reduce to

$$\sum_{i,j} \gamma_{i,j} \beta_j b_{j,\mathcal{T}_i} \equiv 0 \mod p^N.$$

The last congruence is done by an elementary check on the Fourier coefficients $b_{j,\mathcal{T}_i}$.

The abstract Kummer congruences are checked for a family of test elements.

In the admissible case it suffices to check binomial congruences for the Fourier coefficients as above in place of Kummer congruences.

Appendix A. Rewriting the local factor at p with character θ

Notice that if θ is the quadratic character attached to $K/\mathbb{Q}$ then

$$(1-\alpha_p X)(1-\alpha_p\theta(p)X) = \begin{cases} (1-\alpha_p X)^2 & \text{if } \theta(p)=1, p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2, N(\mathfrak{q}_i)=p, \\ (1-\alpha_p^2 X^2), & \text{if } \theta(p)=-1, p\mathfrak{r}=\mathfrak{q}, N(\mathfrak{q})=p^2, \\ (1-\alpha_p X) & \text{if } \theta(p)=0, p\mathfrak{r}=\mathfrak{q}^2, N(\mathfrak{q})=p. \end{cases}$$

Thus, if $X=p^{-s}$, $X^2=p^{-2s}$, $N(\mathfrak{q})=p$, $\mathcal{Z}_{\mathfrak{q}}(X)^{-1}$

$$= \begin{cases} \prod_{i=1}^{2n}(1-N(\mathfrak{q}_1)^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}X)(1-N(\mathfrak{q}_2)^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}X), & \text{if } \theta(p)=1, \\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X^2)(1-N(\mathfrak{q})^{n}t_{\mathfrak{q},i}^{-1}X^2), & \text{if } \theta(p)=-1, \\ \prod_{i=1}^{n}(1-N(\mathfrak{q})^{n-1}t_{\mathfrak{q},i}X)(1-N(\mathfrak{q})^{n}t_{\mathfrak{q},i}^{-1}X), & \text{if } \theta(p)=0. \end{cases}$$

$$= \begin{cases} \prod_{i=1}^{n}(1-\gamma_{p,i}X)^2\prod_{i=1}^{n}(1-\delta_{p,i}X)^2 & \text{if } \theta(p)=1, \text{ i.e. } p\mathfrak{r}=\mathfrak{q}_1\mathfrak{q}_2, \\ \prod_{i=1}^{n}(1-\alpha_{p,i}^2X^2)\prod_{i=1}^{n}(1-\beta_{p,i}^2X^2), & \text{if } \theta(p)=-1, \text{ i.e. } p\mathfrak{r}=\mathfrak{q}, \\ \prod_{i=1}^{n}(1-\alpha'_{p,i}X)\prod_{i=1}^{n}(1-\beta'_{p,i}X) & \text{if } \theta(p)=0, \text{ i.e. } p\mathfrak{r}=\mathfrak{q}^2, \end{cases}$$

where $\alpha'_{p,i}=p^{n-1}t_{\mathfrak{q},i}$, $\beta'_{p,i}p^n t_{\mathfrak{q},i}^{-1}$, $\gamma_{p,i}=p^{2n}t_{\mathfrak{q}_1\mathfrak{q}_2,i}^{-1}$, $p^{-1}t_{\mathfrak{q}_1\mathfrak{q}_2,i}$. It follows that $\prod_{\mathfrak{q}|p}\mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-n-(1/2)}X)=X^{4n}+\cdots$

Appendix A (continued).Relations between $\alpha_i(p)$ and $t_{i,\mathfrak{q}}$

were studied and explained by M.Harris [Ha97] for general Hermitian zeta functions $\mathcal{Z}(s, \mathbf{f})$ of type introduced in [Shi00], using reprsentation theory of unitary groups and Deligne's approach to L-functions, see [De79], in terms of a n-dimensional Galois representations $\rho_\lambda : \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}(M_{\mathbf{f},\lambda}) \cong \mathrm{GL}_n(E_\lambda)$ over a completion E_λ of a number field E containing K and the Hecke eigenvalues of a vector-valued Hermitian modular form $\mathbf{f}$:

$$\mathcal{Z}(s - n' - \frac{1}{2}, \mathbf{f}) = \mathcal{D}(s, \mathbf{f}) = L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$$

for an algebraic Hecke ideal character ψ as above of the infinity type m_ψ, see [GH16], p.20. Here the symbol $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ denotes the Rankin-Selberg type convolution (it corresponds to tensor product of Galois representations). Notice that $L(s, M_{\mathbf{f},\lambda})$ is of degree $2n$, and $L(s, M_{\mathbf{f},\lambda} \boxtimes M(\psi))$ is of degree $4n$ because $L(s, \psi) = L(s, R(\psi))$ is of degree 2.
Moreover, M.Harris suggested a general description of $\mathcal{D}(s)$ with given Gamma factors and analytic properties as some $\mathcal{D}(s, \mathbf{f})$ some under natural conditions on Gamma factors, giving higher versions of Shimura-Taniyama-Weil conjecture (i.e. higher Wiles' modularity theorem). This can be stated also over a totally real field F (instead of $\mathbb{Q}$), and its quadratic totally imaginary extension K, see [GH16], [Pa94].

Appendix B. Shimura's Theorem: algebraicity of critical values in Cases Sp and UT, p.234 of [Shi00]

Let $\mathbf{f} \in \mathcal{V}(\bar{\mathbb{Q}})$ be a non zero arithmetical automorphic form of type Sp or UT. Let χ be a Hecke character of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbb{Z}^{\mathbf{a}}$, and let $\sigma_0 \in 2^{-1}\mathbb{Z}$. Assume, in the notations of Chapter 7 of [Shi00] on the weights k_v, μ_v, ℓ_v, that

Case Sp $\quad 2n + 1 - k_v + \mu_v \le 2\sigma_0 \le k_v - \mu_v,$
where $\mu_v = 0$ if $[k_v] - l_v \in 2\mathbb{Z}$
and $\mu_v = 1$ if $[k_v] - l_v \notin 2\mathbb{Z}$; $\sigma_0 - k_v + \mu_v$
for every $v \in \mathbf{a}$ if $\sigma_0 > n$ and
$\sigma_0 - 1 - k_v + \mu_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$ if $\sigma_0 \le n$.

Case UT $\quad 4n - (2k_{v\rho} + \ell_v) \le 2\sigma_0 \le m_v - |k_v - k_{v\rho} - \ell_v|$
and $2\sigma_0 - \ell_v \in 2\mathbb{Z}$ for every $v \in \mathbf{a}$.

Appendix B. Shimura's Theorem (continued)

Further exclude the following cases

(A) Case Sp $\sigma_0 = n+1, F = \mathbb{Q}$ and $\chi^2 = 1$;

(B) Case Sp $\sigma_0 = n + (3/2), F = \mathbb{Q}; \chi^2 = 1$ and $[k] - \ell \in 2\mathbb{Z}$

(C) Case Sp $\sigma_0 = 0, \mathfrak{c} = \mathfrak{g}$ and $\chi = 1$;

(D) Case Sp $0 < \sigma_0 \le n, \mathfrak{c} = \mathfrak{g}, \chi^2 = 1$ and the conductor of χ is $\mathfrak{g}$;

(E) Case UT $2\sigma_0 = 2n+1, F = \mathbb{Q}, \chi_1 = \theta,$ and $k_v - k_{v\rho} = \ell_v$;

(F) Case UT $0 \le 2\sigma_0 < 2n, \mathfrak{c} = \mathfrak{g}, \chi_1 = \theta^{2\sigma_0}$ and the conductor of χ is $\mathfrak{r}$

Then

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi)/\langle \mathbf{f}, \mathbf{f} \rangle \in \pi^{n|m|+d\varepsilon}\bar{\mathbb{Q}},$$

where $d = [F : \mathbb{Q}]$, $|m| = \sum_{v \in \mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n, & \text{Case Sp}, k \in \mathbb{Z}^{\mathbf{a}}, \text{ and } \sigma_0 > n_0), \\ n\sigma_0 - n^2, & \text{Case Sp}, k \notin \mathbb{Z}^{\mathbf{a}}, \text{or} \sigma_0 \le n_0), \\ 2n\sigma_0 - 2n^2 + n & \text{Case UT} \end{cases}$$

Notice that $\pi^{n|m|+d\varepsilon} \in \mathbb{Z}$ in all cases; if $k \notin \mathbb{Z}^{\mathbf{a}}$, the above parity condition on σ_0 shows that $\sigma_0 + k_v \in \mathbb{Z}$, so that $n|m| + d\varepsilon \in \mathbb{Z}$.

Appendix C. Examples of Hermitian cusp forms

The Hermitian Ikeda lift, [Ike08]. Assume $n = 2n'$ even.

Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in \mathcal{S}_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \theta(p)p^{2k-2s})^{-1} \prod_{p | D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \theta(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \theta(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X)$$

For $p|D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$A(H) = |\gamma(H)|^k \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_{2n}.$$

Appendix C (continued).The first theorem (even case)

Theorem 5.1 (Case E) of [Ike08] Assume that $n = 2n'$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

In the case when n is odd, consider a similar lifting for a normalized Hecke eigenform $n = 2n' + 1$ is odd. Let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N$ $\in \mathcal{S}_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p, define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha^{-1}X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), H \in \Lambda_n(\mathcal{O})^+$$

$$F(H) = \sum_{H \in \Lambda_n(O)^+} A(H)q^H, Z \in \mathcal{H}_n.$$

Appendix C (continued). The second theorem (odd case)

Theorem 5.2 (Case O) of [Ike08]. Assume that $n = 2n' + 1$ is odd. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in \mathcal{S}_{2k+2n'}(\Gamma_K^{(n)}, \det^{-k-n'})$.

The lift $Lift^{(n)}(f)$ of f is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem 13.6).

Theorem 18.1 of [Ike08]. Let n, n', and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(n)}(f) \neq 0$. Let $L(s, Lift^{(n)}(f), st)$ be the L-function of $Lift^{(n)}(\mathbf{f})$ associated to $st : {}^L\mathcal{G} \to \mathrm{GL}_{4n}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(n)}(f), st)$ is equal to

$$\prod_{i=1}^{n} L(s+k+n'-i+\frac{1}{2}, f)L(s+k+n'-i+\frac{1}{2}, f, \theta).$$

Moreover, the $4n$ charcteristic roots of $L(s, Lift^{(n)}(f), st)$ given as follows: for $i = 1, \cdots, n$

$$\alpha_p p^{-k-n'+i-\frac{1}{2}}, \alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p p^{-k-n'+i-\frac{1}{2}}, \theta(p)\alpha_p^{-1} p^{-k-n'+i-\frac{1}{2}}$$

Functional equation of the lift (thanks to Sho Takemori!)

There are two cases [Ike08]: the even case (E) and the odd case (O):

$$\begin{cases} f \in S_{2k+1}(\Gamma_0(D),\theta), F = Lift^{(n)}(f) & (E) \\ \text{(the lift is of even degree } n = 2n' \text{ and of weight } 2k+2n') & \\ f \in S_{2k}(\mathrm{SL}(\mathbb{Z})), F = Lift^{(n)}(f) & (O) \\ \text{(the lift is of odd degree } n = 2n'+1 \text{ and of weight } 2k+2n'). & \end{cases}$$

Then, up to bad Euler factors, the standard L-function of $F = Lift^{(n)}(f)$ is given by

$$\prod_{i=1}^{n} L(s+k+n'-i+\tfrac{1}{2}, f)L(s+k+n'-i+\tfrac{1}{2}, f,\theta)$$

$$= \begin{cases} \prod_{i=1}^{2n'} L(s+k+n'-i+\frac{1}{2}, f)L(s+k+n'-i+\frac{1}{2}, f,\theta) & (E) \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s,2n'+1-i), f) & \\ L(t(s,i), f,\theta)L(t(s,2n'+1-i), f,\theta) & \\ \prod_{i=1}^{2n'+1} L(s+k+n'-i+\frac{1}{2}, f) & \\ \times L(s+k+n'-i+\frac{1}{2}, f,\theta) & (O) \\ = L(s+k-\frac{1}{2}, f)L(s+k-\frac{1}{2}, f,\theta) & \\ \prod_{i=1}^{n'} L(t(s,i), f)L(t(s,2n'+2-i), f) & \\ L(t(s,i), f,\theta)L(t(s,2n'+2-i), f,\theta) & \end{cases}$$

where $t(s,i) = s+k+n'-i+\frac{1}{2}$.

The Gamma factor $\Gamma_{\mathcal{Z}}(s)$ of Ikeda's lift

In the even case since $(2k+1) - t(s,i) = t(1-s, 2n'+1-i)$, using the Hecke functional equation in the symmetric terms of the product, gives the functional equation of the standard L function of the form $s \mapsto 1-s$, and the gamma factor is given by

$$\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+k+n'-i+1/2)^2 = \Gamma_{\mathfrak{D}}(s+n'+\frac{1}{2}).$$

In the odd case when $f \in S_{2k}(SL_2(\mathbb{Z}))$, the lift is of degree $n = 2n'+1$ and of weight $2k+2n'$. By $2k - t(s,i) = t(1-s, 2n+2-i)$, the standard L functions has functional equation of the form $s \mapsto 1-s$ and the gamma factor is the same. Hence the Gamma factor of Ikeda's lifting, denoted by $\mathbf{f}$, of an elliptic modular form f and used as a pattern, extends to a general (not necessarily lifted) Hermitian modular form $\mathbf{f}$ of even weight ℓ, which equals in the lifted case to $\ell = 2k+2n'$, where $k = (\ell - 2n')/2 = \ell/2 - n' {=} \ell/2 - n'$, when the Gamma factor of the standard zeta function with the symmetry $s \mapsto 1-s$ becomes (see p.43) $\prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-n'+n'-i+(1/2))^2 = \prod_{i=1}^{n} \Gamma_{\mathbb{C}}(s+\ell/2-i+(1/2))^2 = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$.

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