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Arithmeticity in the Theory of Automorphic Forms

Goro Shimura

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ABSTRACT. The main theme of the book is the arithmeticity of the critical values of certain zeta functions associated with algebraic groups. The author also included some basic material about arithmeticity of modular forms in general and a treatment of analytical properties of zeta functions and Eisenstein series on symplectic groups. The book can be viewed as a companion to the previous book, *Euler Products and Eisenstein Series*, by the same author (AMS, 1997).

For researchers and graduate students working in number theory and the theory of modular forms.

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TABLE OF CONTENTS

PREFACE

A preliminary idea of writing the present book was formed when I gave the Frank J. Hahn lectures at Yale University in March, 1992. The title of the lectures was "Differential operators, nearly holomorphic functions, and arithmetic." By "arithmetic" I meant the arithmeticity of the critical values of certain zeta functions, and I talked on the results I had on GL_2 and $GL_2 \times GL_2$. At that time the American Mathematical Society wrote me that they were interested in publishing my lectures in book form, but I thought that it would be desirable to discuss similar problems for symplectic groups of higher degree. Though I had satisfactory theories of differential operators and nearly holomorphic functions applicable to higher-dimensional cases, our knowledge of zeta functions on such groups was fragmentary and, at any rate, was not sufficient for discussing their critical values. Therefore I spent the next few years developing a reasonably complete theory, or rather, a theory adequate enough for stating general results of arithmeticity that cover the cases of all congruence subgroups of a symplectic group over an arbitrary totally real number field, including the case of half-integral weight.

On the other hand, I had been interested in arithmeticity problems on unitary groups for many years, and in fact had investigated some Eisenstein series on them. Therefore I thought that a book including the unitary case would be more appealing, and I took up that case as a principal topic of my NSF-CBMS lectures at the Texas Christian University in May, 1996. The expanded version of the lectures was eventually published by the AMS as "Euler products and Eisenstein series."

After this work, I felt that the time was ripe for bringing the original idea to fruition, which I am now attempting to do in this volume. To a large extent the present book may be viewed as a companion to the previous one just mentioned, and our arithmeticity concerns that of the Euler products and Eisenstein series discussed in it; I did not include the cases of GL_2 and $GL_2 \times GL_2$. Those cases are relatively well understood, and it is my wish to present something new. Though the arithmeticity in that sense is the main new feature, as will be explained in detail in the Introduction, I have also included some basic material concerning arithmeticity of modular forms in general, and also a treatment of analytic properties of zeta functions and Eisenstein series on symplectic groups which were not discussed in the previous book.

It is a pleasure for me to express my thanks to Haruzo Hida, who read the manuscript and contributed many useful suggestions.

Princeton
February, 2000

Goro Shimura

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NOTATION AND TERMINOLOGY

We denote by $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ the ring of rational integers, the fields of rational numbers, real numbers, and complex numbers, respectively. We put

$$\mathbf{T} = \{\, z \in \mathbf{C} \mid |z| = 1 \,\}.$$

We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ in $\mathbf{C}$, and for an algebraic number field K we denote by K_{ab} the maximum abelian extension of K. If p is a rational prime, $\mathbf{Z}_p$ and $\mathbf{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively.

For an associative ring R with identity element and an R-module M we denote by $R^\times$ the group of all its invertible elements and by M_n^m the R-module of all $m \times n$-matrices with entries in M; we put $M^m = M_1^m$ for simplicity. Sometimes an object with a superscript such as G^n in Section 23 is used with a different meaning, but the distinction will be clear from the context. For $x \in R_n^m$ and an ideal $\mathfrak{a}$ of R we write $x \prec \mathfrak{a}$ if all the entries of x belong to $\mathfrak{a}$. (There is a variation of this; see §1.8.)

The transpose, determinant, and trace of a matrix x are denoted by ${}^t x$, $\det(x)$, and $\mathrm{tr}(x)$. The zero element of R_n^m is denoted by 0_n^m or simply by 0, and the identity element of R_n^n by 1_n or simply by 1. The size of a zero matrix block written simply 0 should be determined by the size of adjacent nonzero matrix blocks. We put $GL_n(R) = (R_n^n)^\times$, and

$$SL_n(R) = \{\, \alpha \in GL_n(R) \mid \det(\alpha) = 1 \,\}$$

if R is commutative. If $x_1, \ldots, x_r$ are square matrices, $\mathrm{diag}[x_1, \ldots, x_r]$ denotes the matrix with $x_1, \ldots, x_r$ in the diagonal blocks and 0 in all other blocks. We shall be considering matrices x with entries in a ring with an anti-automorphism ρ (complex conjugation, for example), including the identity map. We then put $x^* = {}^t x^\rho$, and $\widehat{x} = (x^*)^{-1}$ if x is square and invertible.

For a complex number or more generally for a complex matrix α we denote by $\mathrm{Re}(\alpha)$, $\mathrm{Im}(\alpha)$, and $\overline{\alpha}$ the real part, the imaginary part, and the complex conjugate of α. For complex hermitian matrices x and y we write $x > y$ and $y < x$ if $x - y$ is positive definite, and $x \geq y$ and $y \leq x$ if $x - y$ is nonnegative. For $r \in \mathbf{R}$ we denote by $[r]$ the largest integer $\leq r$.

Given a set A, the identity map of A onto itself is denoted by id_A or 1_A. To indicate that a union $X = \bigcup_{i \in I} Y_i$ is disjoint, we write $X = \bigsqcup_{i \in I} Y_i$. We understand that $\prod_{i=\alpha}^{\beta} = 1$ and $\sum_{i=\alpha}^{\beta} = 0$ if $\alpha > \beta$. For a finite set X we denote by $\#X$ or $\#(X)$ the number of elements in X. If H is a subgroup of a group G, we put $[G : H] = \#(G/H)$. However we use also the symbol $[K : F]$ for the degree of an algebraic extension K of a field F. The distinction will be clear from the context. By a *Hecke character* χ of a number field K we mean a continuous $\mathbf{T}$-valued character of the idele group of K trivial on $K^\times$, and denote by χ^* the ideal character associated with χ. By a *CM-field* we mean a totally imaginary quadratic extension of a totally real algebraic number field.

INTRODUCTION

Our ultimate aim is to prove several theorems of arithmeticity on the values of an Euler product $\mathcal{Z}(s)$ and an Eisenstein series $E(z, s)$ at certain critical points s. We take these $\mathcal{Z}$ and E to be those of the types we treated in our previous book "Euler Products and Eisenstein Series," referred to as [S97] here. They are defined with respect to an algebraic group G, which is either symplectic or unitary. To illustrate the nature of our problems, let us take a CM-field K and put

$$G(\varphi) = G^\varphi = \{ \alpha \in GL_n(K) \mid \alpha\varphi \cdot {}^t\alpha^\rho = \varphi \}, \tag{0.1}$$

where ρ denotes complex conjugation and φ is an element of $GL_n(K)$ such that ${}^t\varphi^\rho = \varphi$. This group acts on a hermitian symmetric space which we write $\mathfrak{Z}^\varphi$. We shall often be interested in the special case where φ takes the form

$$\eta = \eta_q = \begin{bmatrix} 0 & 1_q \\ 1_q & 0 \end{bmatrix}. \tag{0.2}$$

In this case we write $\mathcal{H}_q$, or simply $\mathcal{H}$, instead of $\mathfrak{Z}^\varphi$ for the symmetric space.

Given a congruence subgroup Γ of G, a Hecke eigenform $\mathbf{f}$ of holomorphic type on $\mathfrak{Z}^\varphi$ with respect to Γ, and a Hecke character χ of K of algebraic type, but not necessarily of finite order, we can construct a "twisted Euler product" $\mathcal{Z}(s, \mathbf{f}, \chi)$, whose generic Euler p-factor for each rational prime p has degree $n[K : \mathbf{Q}]$. Then we shall eventually prove that

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) \in \pi^\varepsilon \mathfrak{q} \langle \mathbf{f}, \mathbf{f} \rangle \overline{\mathbf{Q}} \tag{0.3}$$

for σ_0 in a certain finite subset of $2^{-1}\mathbf{Z}$ and $\overline{\mathbf{Q}}$-rational $\mathbf{f}$. Here $\langle \mathbf{f}, \mathbf{f} \rangle$ is the inner product defined in a canonical way; ε is an integer determined by σ_0, the signature of φ, the weight of $\mathbf{f}$, and the archimedean factor of χ; $\mathfrak{q}$ is a certain "period symbol" determined by χ and φ. This is true for both isotropic and anisotropic φ, and even for a totally definite φ. In the simplest case in which $G = G^\eta$, we have $\mathfrak{q} = 1$.

Clearly such a result requires the definition of $\overline{\mathbf{Q}}$-rationality of automorphic forms. If G is of type G^η, then we can define the $\overline{\mathbf{Q}}$-rationality by the $\overline{\mathbf{Q}}$-rationality of the Fourier coefficients of a given automorphic form. If $[K : \mathbf{Q}] = 2$, for example, then $\mathcal{H}$ is a tube domain of the form $\mathcal{H} = \{ z \in \mathbf{C}_q^q \mid i(z^* - z) > 0 \}$, and a holomorphic automorphic form f has an expansion

$$f(z) = \textstyle\sum_h c(h) \exp\big(2\pi i \cdot \mathrm{tr}(hz)\big) \qquad (z \in \mathcal{H}) \tag{0.4}$$

with $c(h) \in \mathbf{C}$, where h runs over all nonnegative hermitian matrices belonging to a $\mathbf{Z}$-lattice in K_q^q. Then for a subfield M of $\mathbf{C}$ we say that f is *M-rational* if $c(h) \in M$ for every h. This definition may look simplistic, but actually it is intrinsically the right definition. To explain about this point, we first note that $\Gamma\backslash\mathfrak{Z}^\varphi$ has a structure of algebraic variety that has a natural model W defined over $\overline{\mathbf{Q}}$. We call then a Γ-automorphic function (that is, Γ-invariant meromorphic function on $\mathfrak{Z}^\varphi$ satisfying the cusp condition) *$\overline{\mathbf{Q}}$-rational* (or *arithmetic*) if it corresponds to a $\overline{\mathbf{Q}}$-rational function on W in the sense of algebraic geometry. Now there are two basic facts:

(0.5) *The value of a $\overline{\mathbf{Q}}$-rational automorphic function at any CM-point of $\mathfrak{Z}^\varphi$, if finite, is algebraic.*

(0.6) *If f and g are $\overline{\mathbf{Q}}$-rational automorphic forms of the same weight, then f/g is a $\overline{\mathbf{Q}}$-rational automorphic function.*

Here a CM-point on $\mathfrak{Z}^\varphi$ is defined to be the fixed point of a certain type of torus contained in G. If $G = G^\eta$ and $q = 1$, then $\mathcal{H}$ is the standard upper half plane, and any point of $\mathcal{H}$ belonging to an imaginary quadratic field is a CM-point and vice versa. In such a special case, (0.5) and (0.6) follow from the classical theory of complex multiplication of elliptic modular functions. In more general cases, (0.5) was established by the author in the framework of canonical models. As for (0.6), it makes sense if $G = G^\eta$, and we can indeed give a proof, if nontrivial, of (0.6) in such a case. For φ of a more general type, however, (0.6) is a meaningful statement only when we have defined the $\overline{\mathbf{Q}}$-rationality of automorphic forms. Thus it is one of our main tasks to define the notion so that (0.6) holds.

Turning our eyes to Eisenstein series, easily posable questions are as follows:

(i) *Assuming that $E(z, \sigma_0)$ is finite, is $E(z, \sigma_0)$ as a function of z holomorphic?*
(ii) *If that is so, is it $\overline{\mathbf{Q}}$-rational up to a well-defined constant?*

Here we take meromorphic continuation of $E(z, s)$ to the whole s-plane, as we proved in [S97], into account. Every researcher of automorphic forms should be able to accept such questions naturally, since the answers to them for $G = SL_2(\mathbf{Q})$ are well-known and fundamental. There is a marked difference between the $\overline{\mathbf{Q}}$-rationality here and the arithmeticity of $\mathcal{Z}(\sigma_0)$, since the latter concerns σ_0 in an interval which can be large, while $E(z, \sigma_0)$ can be holomorphic in z only at a single point σ_0. Now the interval, or rather the set of critical points belonging to the interval, is suggested by the functional equation for $\mathcal{Z}$, and we can find such a set even for $E(z, s)$ by means of its analytic properties. We cannot expect $E(z, \sigma_0)$ to be holomorphic in z for *every* critical point σ_0 in the set. We should also note a classical example in the elliptic modular case:

$$(0.7)\qquad (-4\pi^2)^{-1} \lim_{s\to+0} \sum_{0\neq(c,d)\in\mathbf{Z}^2} (cz+d)^{-2}|cz+d|^{-s} = (4\pi y)^{-1} - 12^{-1} + 2\sum_{n=1}^{\infty}\Bigl(\sum_{a|n} a\Bigr)e^{2\pi i n z}.$$

This is a nonholomorphic modular form of weight 2, and there are similar nonholomorphic forms of weight $(n+3)/2$ with respect to a congruence subgroup of $Sp(n, \mathbf{Z})$. Therefore our next questions are:

(iii) *What is the analytic nature of these $E(z, \sigma_0)$?*
(iv) *Can we still speak of the $\overline{\mathbf{Q}}$-rationality of such $E(z, \sigma_0)$?*

One of the main purposes of this book is to answer these questions, which are not only meaningful by themselves, but also closely connected with the arithmeticity of

$\mathcal{Z}(\sigma_0)$. In fact, the answers to (iii) and (iv) are indispensable for the proof of (0.3) as we shall explain later, but first let us describe our answers.

We first define the notion of *nearly holomorphic function* on any complex manifold with a fixed Kähler structure. Without going into details in the general case, let us just say that a function on such a manifold $\mathfrak{Z}$ is called nearly holomorphic if it is a polynomial of some functions $r_1, \ldots, r_m$ on $\mathfrak{Z}$, determined by the Kähler structure, over the ring of all holomorphic functions on $\mathfrak{Z}$. If $\mathfrak{Z}$ is the above $\mathcal{H}$ of tube type with a G-invariant Kähler structure, then the r_i are the entries of $(z^* - z)^{-1}$, where z is a variable matrix on $\mathcal{H}$. If $\mathfrak{Z}$ is a hermitian symmetric space, there is also a characterization of such functions in terms of the Lie algebra of the transformation group on $\mathfrak{Z}$.

Now we can naturally define nearly holomorphic automorphic forms by replacing holomorphy by near holomorphy in the definition of automorphic forms. If $G = G^\eta$, then such a form f on $\mathcal{H}$ has an expansion

$$f(z) = \textstyle\sum_h p_h\bigl([\pi i(z^* - z)]^{-1}\bigr) \exp\bigl(2\pi i \cdot \mathrm{tr}(hz)\bigr) \qquad (z \in \mathcal{H}), \tag{0.8}$$

where $\sum_h$ is the same as in (0.4) and $p_h(Y)$ is a polynomial function in the entries of Y whose degree is less than a constant depending on f. We say that f is *M-rational* if p_h has all its coefficients in a field M for every h. For example, the function of (0.7) is a $\mathbf{Q}$-rational nearly holomorphic modular form. We shall show that $E(z, \sigma_0)$ is indeed nearly holomorphic and $\overline{\mathbf{Q}}$-rational in this sense, up to a constant, which is a power of π if $G = G^\eta$. Moreover, here is a noteworthy consequence of our definition:

(0.9) *If f and g are $\overline{\mathbf{Q}}$-rational nearly holomorphic automorphic forms of the same weight, then the value of f/g at any CM-point of $\mathcal{H}$, if finite, is algebraic.*

It should be noted that this is anything but a direct consequence of (0.6). Also, for a general type of φ we cannot use (0.8). However, once we have the $\overline{\mathbf{Q}}$-rationality of holomorphic automorphic forms, we can at least define the $\overline{\mathbf{Q}}$-rationality of nearly holomorphic automorphic forms by property (0.9), though it is of course nontrivial to show that such a definition is indeed meaningful. So far we have taken G to be unitary, but the symplectic case can be handled too; in fact it is similar to and easier than G^η, though the case of half-integral weight requires special consideration.

Having thus presented our problems in rough forms, we can now set our program as follows:

(1) We first define the $\overline{\mathbf{Q}}$-rationality of automorphic forms so that (0.6) holds.

(2) We define nearly holomorhic automorphic forms and their $\overline{\mathbf{Q}}$-rationality so that (0.9) holds.

(3) We prove the near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z, \sigma_0)$ up to a power of π in the easiest cases, namely, when G is symplectic or of type G^η, and E is defined with respect to a parabolic subgroup whose unipotent radical is a commutative group of translations on $\mathcal{H}$. Let us call such an E a series of split type.

(4) We prove (0.3) by using the result of (3).

(5) Finally we prove the near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z, \sigma_0)$ up to a well-defined constant in the most general case.

Let us now briefly describe the technical aspect of how these can be achieved. One important point is that certain differential operators on $\mathcal{H}$ are essential to (2) and (3). In the above we tacitly assumed that our automorphic forms are scalar-valued, but in order to use differential operators effectively, it is necessary to consider vector-valued forms. If $[K : \mathbf{Q}] = 2$ and $G = G(\eta_q)$, such a form is defined relative to a representation $\{\rho, X\}$ of a group

$$\mathfrak{K} = \{ (a, b) \in GL_q(\mathbf{C}) \times GL_q(\mathbf{C}) \mid \det(a) = \det(b) \},$$

where X is a finite-dimensional complex vector space and ρ is a rational representation of $\mathfrak{K}$ into $GL(X)$. Put $T = \mathbf{C}_q^q$ and view it as a global holomorphic tangent space of $\mathcal{H}_q$; define a representation $\{\rho \otimes \tau, \mathrm{Hom}(T, X)\}$ of $\mathfrak{K}$ by $[(\rho\otimes\tau)(a, b)h](u) = \rho(a, b)h({}^t aub)$ for $(a, b) \in \mathfrak{K}$, $h \in \mathrm{Hom}(T, X)$, and $u \in T$. For a function $g : \mathcal{H} \to X$ we define $\mathrm{Hom}(T, X)$-valued function Dg and $D_\rho g$ on $\mathcal{H}$ by

$$(Dg)(u) = \textstyle\sum_{i,j=1}^q u_{ij}\partial g/\partial z_{ij} \qquad (u \in T),$$
$$(D_\rho g)(z) = \rho\bigl(\Xi(z)\bigr)^{-1} D\bigl[\rho\bigl(\Xi(z)\bigr)g(z)\bigr],$$

where $z = (z_{ij})_{i,j=1}^q \in \mathcal{H}$ and $\Xi : \mathcal{H} \to \mathfrak{K}$ is defined by $\Xi(z) = \bigl(i(\overline{z}-{}^t z),\, i(z^*-z)\bigr)$. These can also be defined on $\mathfrak{Z}^\varphi$ for φ of a general type. Then we can show that if g is an automorphic form of weight ρ, then $D_\rho g$ is a form of weight $\rho \otimes \tau$. If $q = 1$, then $\mathcal{H}$ is the standard upper half plane, $G^\eta \cap SL_2(K) = SL_2(\mathbf{Q})$, $\mathfrak{K} = \mathbf{C}^\times$, $\rho(a) = a^k$ for $a \in \mathbf{C}^\times$ with $k \in \mathbf{Z}$, and $\Xi(z) = (2y, 2y)$ where $y = \mathrm{Im}(z)$; we can easily identify Dg with $\partial g/\partial z$, so that $D_\rho g = y^{-k}(\partial/\partial z)(y^k g)$, and $(\rho \otimes \tau)(a) = a^{k+2}$. Thus D_ρ is the well-known operator that sends a form of weight k to a form of weight $k + 2$.

Now iteration of operators of this type, such as $D_{\rho\otimes\tau}D_\rho$, produces an automorphic form with values in a representation space of $\mathfrak{K}$ of a large dimension if $q > 1$, even if we start with $X = \mathbf{C}$. Decomposing the space into irreducible subspaces and looking particularly at the irreducible subspaces of dimension one, we can define a natural differential operator Δ that sends scalar-valued automorphic forms to scalar-valued forms of increased weight. The significance of these iterated operators and Δ are explained by the following fact, which is formulated only for Δ for simplicity:

(0.10) *If Δ is of total degree p in terms of $\partial/\partial z_{ij}$, then $\pi^{-p}\Delta$ preserves near holomorphy and $\overline{\mathbf{Q}}$-rationality.*

If $G = G^\eta$, this can be derived from our definition in terms of expression (0.8). Now property (0.9), if true, would imply that for a $\overline{\mathbf{Q}}$-rational holomorphic automorphic forms f and g such that Δf and g have the same weight, the value of $(\pi^{-p}\Delta f)/g$ at any CM-point, if finite, is algebraic. This is highly nontrivial, and in fact we first prove this special case of (0.9), and derive the general case from that result.

As for problem (3), we first investigate the Fourier expansion of $E(z, s)$ of split type. In fact, this was done in [S97], but here we examine the behavior of the Fourier coefficients at a critical value of s. Employing their explicit forms, we find that $E(z, \sigma)$ is holomorphic in z and $\overline{\mathbf{Q}}$-rational, or is of the type (0.7), if the weight of E and σ belong to certain special types. For a more general weight and a general σ_0, we prove that $cE(z, \sigma_0) = \Delta E'(z, \sigma)$ with a suitable Δ, a nonzero constant c, and a suitable E' belonging to those special types. Then (0.10) settles problem (3) for $E(z, \sigma_0)$.

To treat problems (4) and (5), let us now go back to the Euler product $\mathcal{Z}(s, \mathbf{f}, \chi)$ of (0.3) on G^φ; we refer the reader to [S97] for its precise definition. We consider G^ψ with $\psi = \mathrm{diag}[\varphi, \eta]$, where η is as in (0.2). Then $G^\varphi \times G^\eta$ can be embedded in G^ψ, and G^ψ has a parabolic subgroup whose reductive factor is $G^\varphi \times GL_q(K)$. Given a suitable congruence subgroup Γ' of G^ψ, we can define an Eisenstein series $E(z, s; \mathbf{f}, \chi)$ for $(z, s) \in \mathfrak{Z}^\psi \times \mathbf{C}$ with respect to that parabolic subgroup and the set of data $(\mathbf{f}, \chi, \Gamma')$. Now we easily see that $\mathrm{diag}[\psi, -\varphi]$ is equivalent to η_{n+q}, so that $G^\psi \times G^\varphi$ can be embedded into $G(\eta_{n+q})$, and $\mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$ can be embedded into $\mathcal{H}_{n+q}$. Pulling back an Eisenstein series on $\mathcal{H}_{n+q}$ of split type to $\mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$, we obtain a function $H(z, w; s)$ of $(z, w; s) \in \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi \times \mathbf{C}$, with which we proved in [S97] an equality that takes the form

$$(0.11) \qquad c(s)\mathcal{Z}(s, \mathbf{f}, \chi)E(z, s; \mathbf{f}, \chi) = \Lambda(s)\int_{\Gamma\backslash\mathfrak{Z}^\varphi} H(z, w; s)\mathbf{f}(w)\delta(w)^m \mathbf{d}w$$

in the simplest case, where Γ is a congruence subgroup of G^φ, c is an easy product of gamma functions, Λ is a product of some L-functions, $\mathbf{d}w$ is a G^φ-invariant measure on $\mathfrak{Z}^\varphi$, and $\delta(w)^m$ is a factor, similar to y^k in the one-dimensional case, that makes the integral meaningful. If $\psi = \varphi$, then (0.11) takes the form

$$(0.12) \qquad c'(s)\mathcal{Z}(s, \mathbf{f}, \chi)\mathbf{f}(z) = \Lambda'(s)\int_{\Gamma\backslash\mathfrak{Z}^\varphi} H'(z, w; s)\mathbf{f}(w)\delta(w)^m \mathbf{d}w.$$

We evaluate (0.11) and (0.12) at $s = \sigma_0$ for σ_0 belonging to a certain "critical set," and observe that $H(z, w; \sigma_0)$ is nearly holomorphic in $(z, w) \in \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$, and even $\overline{\mathbf{Q}}$-rational up to a power of π and a factor $\mathfrak{q}$ as in (0.3). Then we can show that

$$\Lambda(\sigma_0)H(z, w; \sigma_0) = \pi^\alpha \mathfrak{q} \sum_i g_i(z)\overline{h_i(w)}$$

with some $\alpha \in \mathbf{Z}$, and functions g_i on $\mathfrak{Z}^\psi$ and h_i on $\mathfrak{Z}^\varphi$, which are nearly holomorphic and $\overline{\mathbf{Q}}$-rational. The same is true for $\Lambda'H'$; both g_i and h_i are defined on $\mathfrak{Z}^\varphi$ then. This fact applied to (0.12) produces a proportionality relation

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) \in \pi^\beta \mathfrak{q} \langle \mathbf{p}', \mathbf{f} \rangle \overline{\mathbf{Q}}$$

with some $\beta \in \mathbf{Z}$ and a $\overline{\mathbf{Q}}$-rational nearly holomorhic $\mathbf{p}'$. Now we can show that $\mathcal{Z}(s, \mathbf{f}, \chi) \neq 0$ for $\mathrm{Re}(s) > 3q/2$ if $G = G(\eta_q)$ and for $\mathrm{Re}(s) > n$ if $G = G^\varphi$ with φ of a general type. There is one more crucial technical fact that we can replace $\mathbf{p}'$ by a $\overline{\mathbf{Q}}$-rational holomorphic cusp form $\mathbf{p}$ that belongs to the same Hecke eigenvalues as $\mathbf{f}$. Choosing σ_0 so that $\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) \neq 0$, we can show that $\langle \mathbf{p}, \mathbf{f} \rangle / \langle \mathbf{f}, \mathbf{f} \rangle \in \overline{\mathbf{Q}}$, and eventually (0.3) for σ_0 belonging to an appropriate set. Strictly speaking, (0.12) is true only under a consistency condition on $(\mathbf{f}, \chi)$, and the proof of (0.3) in the most general case is more complicated.

Next, we evaluate (0.11) at a critical σ_0 in a similar way, to find that

$$\mathcal{Z}(s_0, \mathbf{f}, \chi)E(z, \sigma_0; \mathbf{f}, \chi) = \pi^\gamma \mathfrak{q} \langle \mathbf{r}, \mathbf{f} \rangle g(z)$$

with some $\overline{\mathbf{Q}}$-rational nearly holomorphic function g on $\mathfrak{Z}^\psi$ and some $\mathbf{r}$ of the same type as the above $\mathbf{p}$. Dividing this equality by $\langle \mathbf{f}, \mathbf{f} \rangle$ and employing (0.3), we obtain the desired near holomorphy and $\overline{\mathbf{Q}}$-rationality of $E(z, \sigma_0; \mathbf{f}, \chi)$, which is the final main result of this book.

Since the title of each section can give a rough idea of its contents, we shall not describe them here for every section. However, there are some points which are not discussed in the above, and on which special attention may be paid. Let us note here some of the noteworthy aspects.

(A) As to the arithmeticity of automorphic forms, we stated only (0.6) as a basic requirement. However, there are other natural questions about arithmeticity whose answers become necessary in various applications. Let us mention here only a few facts we shall prove in this connection: (i) all automorphic forms are spanned by the $\overline{\mathbf{Q}}$-rational forms; (ii) the group action (defined relative to a fixed weight) preserves $\overline{\mathbf{Q}}$-rationality; (iii) in these statements $\overline{\mathbf{Q}}$ can be replaced by a smaller field such as $\mathbf{Q}$ or $\mathbf{Q}_{\mathrm{ab}}$ if the group and the weight are of special types.

(B) In Sections 19 through 25 we give a detailed treatment of $\mathcal{Z}(s, \mathbf{f}, \chi)$ and $E(z, s; \mathbf{f}, \chi)$ in the symplectic case, as well as in the case $G = G^{\eta}$. These cases were mentioned but not discussed in detail in the previous book [S97]. Also, in the symplectic case we can define $\mathcal{Z}$ and E even with respect to a half-integral weight, and we believe that the subject acquires the status of a complete theory only when that case is included. Therefore in this book we treat both integral and half-integral weights, and present the main results for both, though at a few points the details of the proof for a half-integral weight are referred to some papers of the author.

(C) We have spoken of a CM-point, which is naturally related to an abelian variety with complex multiplication. Thus it is necessary to view $\Gamma\backslash\mathfrak{Z}^{\varphi}$ as a space parametrizing a family of abelian varieties. This will be discussed in Sections 4 and 6. The topic was treated in [S98], but we prove here something which was not fully explained in that book. Namely, in Section 9, we prove the reciprocity-law for the value of an automorphic function at a CM-point, when $\Gamma\backslash\mathfrak{Z}^{\varphi}$ is associated with a PEL-type.

(D) In the elliptic modular case it is well-known that the space of all holomorphic modular forms is the direct sum of the space of cusp forms and the space of Eisenstein series. In Section 27 we prove several results of the same nature for symplectic and unitary groups. For example, we show that the orthogonal complement of the space of cusp forms in the space of all holomorphic automorphic forms is spanned by certain Eisenstein series, and the direct sum decomposition can be done $\overline{\mathbf{Q}}$-rationally. This will be proven for the weights with which the series are defined beyond the line of convergence.

(E) Though we are mainly interested in the higher-dimensional cases, in Section 18 we give an elementary theory of Eisenstein series in the Hilbert modular case, which leads to arithmeticity results on the critical values of an L-function of a CM-field. Also, in the Appendix we include some material of expository nature such as theta functions of a quadratic form and the estimate of the Fourier coefficients of a modular form. Many of them are well-known when the group is $SL_2(\mathbf{Q})$ or even $Sp(n, \mathbf{Q})$ for some statements, but the researchers have often had difficulties in finding references for the results on a more advanced level. Therefore we have expended conscious efforts in treating such standard topics in a rather general setting.

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CHAPTER I

AUTOMORPHIC FORMS AND FAMILIES OF ABELIAN VARIETIES

1. Algebraic preliminaries

1.1. The algebraic or Lie groups we treat in this book are symplectic and unitary, and the hermitian symmetric domains associated with them belong to Types A and C. Our methods are in fact applicable to groups and domains of other types, but it is naturally cumbersome to treat all cases. Therefore, in order to keep the book a reasonable length, we confine ourselves to those two types, though at some points we shall indicate that other cases can be handled in a similar way by citing relevant papers.

We take a basic field F of characteristic different from 2 and a couple (K, ρ) consisting of an F-algebra K of rank ≤ 2 and an F-linear automorphism ρ of K belonging to the following three types:

(I) $K = F$ and $\rho = \mathrm{id}_F$;
(II) K is a quadratic extension of F and ρ is the generator of $\mathrm{Gal}(K/F)$;
(III) $K = F \times F$ and $(x, y)^\rho = (y, x)$ for $(x, y) \in F \times F$.

In our later discussion, objects of type (III) will appear as the localizations of the global objects of type (II).

Given left K-modules V and W, we denote by $\mathrm{Hom}(W, V; K)$ the set of all K-linear maps of W into V. We then put $\mathrm{End}(V, K) = \mathrm{Hom}(V, V; K)$, $GL(V, K) = \mathrm{End}(V, K)^\times$, and $SL(V, K) = \{\, \alpha \in GL(V, K) \mid \det(\alpha) = 1 \,\}$. We drop the letter K if that is clear from the context. We let $\mathrm{Hom}(W, V)$ act on W on the right; namely we denote by $w\alpha$ the image of $w \in W$ under $\alpha \in \mathrm{Hom}(W, V)$.

Let V be a left K-module isomorphic to K_m^1. Given $\varepsilon = \pm 1$, by an *ε-hermitian form on V* we understand an F-linear map $\varphi : V \times V \to K$ such that

$$\varphi(x, y)^\rho = \varepsilon\varphi(y, x), \tag{1.1}$$

$$\varphi(ax, by) = a\varphi(x, y)b^\rho \quad \textit{for every} \quad a, b \in K. \tag{1.2}$$

Assuming φ to be nondegenerate, we define groups $GU(\varphi)$, $U(\varphi)$, and $SU(\varphi)$ by

$$GU(\varphi) = \{\, \alpha \in GL(V, K) \mid \varphi(x\alpha, y\alpha) = \nu(\alpha)\varphi(x, y) \quad \text{with} \quad \nu(\alpha) \in F^\times \,\}, \tag{1.3}$$

$$U(\varphi) = \{\, \alpha \in GU(\varphi) \mid \nu(\alpha) = 1 \,\}, \quad SU(\varphi) = U(\varphi) \cap SL(V, K). \tag{1.4}$$

We call φ *isotropic* if $\varphi(x, x) = 0$ for some $x \in V, \neq 0$; we call φ *anisotropic* if $\varphi(x, x) = 0$ only for $x = 0$.

Given (V, φ) and another structure (V', φ') of the same type, we denote by $(V, \varphi) \oplus (V', \varphi')$ the structure (W, ψ) given by $W = V \oplus V'$ and $\psi(x + x', y + y') =$

$\varphi(x, y) + \varphi'(x', y')$ for $x, y \in V$ and $x', y' \in V'$. We then view $U(\varphi) \times U(\varphi')$ as a subgroup of $U(\psi)$ in an obvious way.

1.2. We shall often express various objects by matrices. To simplify our notation, for a matrix x with entries in K we put

$$x^* = {}^t x^\rho, \qquad x^{-\rho} = (x^\rho)^{-1}, \qquad \widehat{x} = {}^t x^{-\rho}, \tag{1.5}$$

assuming x to be square and invertible if necessary. Now let $V = K_m^1$ and $\varphi = \varepsilon\varphi^* \in K_m^m$. Then we can define an ε-hermitian form φ_0 on V by $\varphi_0(x, y) = x\varphi y^*$ for $x, y \in V$. In this setting we shall always write simply φ for the form φ_0. Then we have

$$GU(\varphi) = \{\, \alpha \in GL_m(K) \,|\, \alpha\varphi\alpha^* = \nu(\alpha)\varphi \quad \text{with } \nu(\alpha) \in F^\times \,\}, \tag{1.6}$$

$$U(\varphi) = \{\, \alpha \in GL_m(K) \,|\, \alpha\varphi\alpha^* = \varphi \,\}, \quad SU(\varphi) = U(\varphi) \cap SL_m(K). \tag{1.7}$$

We shall often consider $U(\eta_n)$ with

$$\eta_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}. \tag{1.8}$$

Here we are taking $\varepsilon = -1$. In particular, if $K = F$, the group $U(\eta_n)$ is usually denoted by $Sp(n, F)$. More generally, for a commutative ring A with identity element we put

$$Sp(n, A) = \{\, \alpha \in GL_{2n}(A) \,|\, {}^t\alpha\eta_n\alpha = \eta_n \,\}, \tag{1.9}$$

$$Gp(n, A) = \{\, \alpha \in GL_{2n}(A) \mid {}^t\alpha\eta_n\alpha = \nu(\alpha)\eta_n \ \textit{with } \nu(\alpha) \in A^\times \,\}. \tag{1.10}$$

Notice that

$$\det(\alpha) = \nu(\alpha)^n \qquad (\alpha \in Gp(n, A)), \tag{1.11}$$

$$\det(\alpha)\det(\alpha)^\rho = \nu(\alpha)^m \qquad (\alpha \in GU(\varphi)). \tag{1.12}$$

The latter formula is obvious. To prove (1.11), let $\alpha \in Gp(n, A)$ and $\beta = \mathrm{diag}[1_n, \nu(\alpha)1_n]$. Then $\beta \in Gp(n, A)$ and $\nu(\beta) = \nu(\alpha)$; thus $\beta^{-1}\alpha \in Sp(n, A)$. It is well-known that $\det\big(Sp(n, A)\big) = 1$, and hence $\det(\alpha) = \det(\beta) = \nu(\alpha)^n$, which is (1.11). In particular $Gp(1, A) = GL_2(A)$ and $Sp(1, A) = SL_2(A)$.

Let $\xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_{2n}(K)$ with a, b, c, d of size n and let $s \in F^\times$. Then

$$\begin{aligned} \xi \in GU(\eta_n) \ \textit{and}\ \nu(\xi) = s &\iff a^*d - c^*b = s1_n,\ a^*c = c^*a,\ b^*d = d^*b, \\ &\iff ad^* - bc^* = s1_n,\ ab^* = ba^*,\ cd^* = dc^*. \end{aligned} \tag{1.13}$$

1.3. Lemma. (1) *Let A be a commutative ring with identity element. Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_{m+n}(A)$ and $x^{-1} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ with a, e of size m and d, h of size n. Then $\det(x)\det(e) = \det(d)$.*

(2) *If $\xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(\eta_n)$ with a, b, c, d of size n, then $\det(a)$, $\det(b)$, $\det(c)$, and $\det(d)$ all belong to F.*

(3) *Every element of $GU(\eta_n)$ is a diagonal matrix times an element of $SU(\eta_n)$.*

(4) *$GU(\varphi)/[F^\times U(\varphi)]$ is isomorphic to a subgroup of $F^\times/\{\, a^2 \,|\, a \in F^\times \,\}$. Consequently if λ is a homomorphism of $GU(\varphi)$ into a group whose kernel contains $F^\times U(\varphi)$, then $\lambda^2 = 1$.*

PROOF. For the proof of (1) and (2), see [S97, Lemmas 2.15 and 2.16]. To prove (3), let $\alpha \in GU(\eta_n)$, $p = \nu(\alpha)^{-1}$, and $\beta = \mathrm{diag}[1_n, p1_n]\alpha$. Then $\beta \in U(\eta_n)$.

This settles our problem if $K = F$, since $U(\eta_n) = SU(\eta_n)$; so assume $K \neq F$. Put $q = \det(\beta)$. Then $qq^\rho = 1$ by (1.12), and hence $q = r/r^\rho$ with some $r \in K^\times$. Put $\gamma = \mathrm{diag}[a^*, a^{-1}]$ with any diagonal matrix a such that $\det(a) = r$. Then $\gamma \in U(\eta_n)$ and $\det(\gamma\beta) = 1$. This proves (3) when $K \neq F$. Finally, consider the homomorphism $\nu : GU(\varphi) \to F^\times$. We easily see that $F^\times U(\varphi)$ is the inverse image of $\{\, a^2 \mid a \in F^\times \,\}$, and hence we obtain (4).

1.4. Let F be the field of quotients of a Dedekind domain $\mathfrak{g}$. By a $\mathfrak{g}$-*lattice* in a finite-dimensional vector space W over F we mean a finitely generated $\mathfrak{g}$-module in W that spans W over F. Every fractional ideal in F with respect to $\mathfrak{g}$ is a $\mathfrak{g}$-lattice in F and vice versa; we call it a $\mathfrak{g}$-ideal. A $\mathfrak{g}$-ideal is called *integral* if it is contained in $\mathfrak{g}$.

We now assume that F is an algebraic number field of finite degree. We denote by $\mathbf{a}$ and $\mathbf{h}$ the sets of archimedean primes and nonarchimedean primes of F; we put $\mathbf{v} = \mathbf{a} \cup \mathbf{h}$. Further we denote by $\mathfrak{g}$ the maximal order of F. For every $v \in \mathbf{v}$ we denote by F_v the v-completion of F. In particular, for $v \in \mathbf{h}$ and a $\mathfrak{g}$-ideal $\mathfrak{a}$ we denote by $\mathfrak{a}_v$ the v-closure of $\mathfrak{a}$ in F_v, which coincides with the $\mathfrak{g}_v$-linear span of $\mathfrak{a}$ in F_v. We denote by $N(\mathfrak{a})$ and $N(\mathfrak{a}_v)$ the norm of $\mathfrak{a}$ and $\mathfrak{a}_v$ as usual. They are positive rational numbers with the standard multiplicative property such that $N(\mathfrak{a}) = [\mathfrak{g} : \mathfrak{a}]$ if $\mathfrak{a}$ is integral and $N(\mathfrak{a}_v) = [\mathfrak{g}_v : \mathfrak{a}_v]$ if $\mathfrak{a}_v$ is integral.

Given a finite-dimensional vector space X over F and a $\mathfrak{g}$-lattice L in X, we put $X_v = X \otimes_F F_v$ for every $v \in \mathbf{v}$, and denote by L_v the $\mathfrak{g}_v$-linear span of L in X_v if $v \in \mathbf{h}$. Clearly L_v is a $\mathfrak{g}_v$-lattice in X_v, and is the closure of L in X_v. Notice also that every $\mathfrak{g}_v$-lattice in X_v is an open compact subgroup of X_v.

1.5. Lemma. *With F and X as above, let L be an arbitrarily fixed $\mathfrak{g}$-lattice in X. Then the following assertions hold:*

(1) *If M is a $\mathfrak{g}$-lattice in X, then $L_v = M_v$ for almost all v. Moreover, $L \subset M$ (resp. $L = M$) if $L_v \subset M_v$ (resp. $L_v = M_v$) for every $v \in \mathbf{h}$.*

(2) *Given a $\mathfrak{g}_v$-lattice N_v in X_v for each $v \in \mathbf{h}$ such that $N_v = L_v$ for almost all v, there exists a $\mathfrak{g}$-lattice M in X such that $M_v = N_v$ for every $v \in \mathbf{h}$.*

These assertions are well-known. For the proof, see [S97, Lemma 8.2].

1.6. Given an algebraic group G over F, we denote by $G_\mathbf{A}$ the adelization of G and by G_v for $v \in \mathbf{v}$ the localization of G at v. (The reader is referred to [S97, Section 8] for basic definitions and elementary facts on this topic.) We consider G a subgroup of $G_\mathbf{A}$ as usual. In particular, $F_\mathbf{A}$ and $F_\mathbf{A}^\times$ denote the adele ring and the idele group of F, respectively. The archimedean and nonarchimedean factors of $G_\mathbf{A}$ are denoted by $G_\mathbf{a}$ and $G_\mathbf{h}$. Namely $G_\mathbf{a} = \prod_{v\in\mathbf{a}} G_v$ and $G_\mathbf{h} = G_\mathbf{A} \cap \prod_{v\in\mathbf{h}} G_v$. For $x \in G_\mathbf{A}$ we denote by $x_\mathbf{a}$, $x_\mathbf{h}$, and x_v the projection of x to $G_\mathbf{a}$, $G_\mathbf{h}$, and G_v, respectively. If $G \subset GL(V)$ with a vector space V over F, then for $\alpha \in G_\mathbf{A}$ and a $\mathfrak{g}$-lattice L in V, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in V determined by $(L\alpha)_v = L_v\alpha_v$ for every $v \in \mathbf{h}$. The existence of such a lattice $L\alpha$ is guaranteed by Lemma 1.5 (2). In particular, for $x \in F_\mathbf{A}^\times$ we denote by $x\mathfrak{g}$ the fractional ideal such that $(x\mathfrak{g})_v = x_v\mathfrak{g}_v$. Also we put $|x|_\mathbf{A} = \prod_{v\in\mathbf{v}} |x_v|_v$, where $|\ \ |_v$ is the normalized valuation at v. To emphasize that this is defined on $F_\mathbf{A}^\times$, we shall also write $|x|_F$ for $|x|_\mathbf{A}$.

Given algebraic groups G and G' over F and an F-rational homomorphism f of G into G', we can extend f naturally to a homomorphism of $G_\mathbf{A}$ to $G'_\mathbf{A}$, which we shall denote by the same letter f. For example, we employ $\mathrm{Tr}_{F'/F}$ even for the map of $F'_{\mathbf{A}}$ into $F_{\mathbf{A}}$ derived from the map $\mathrm{Tr}_{F'/F} : F' \to F$ when F' is an algebraic extension of F.

Let W be a finite-dimensional vector space over F, and L a $\mathfrak{g}$-lattice in W. Taking an element a of $W_{\mathbf{h}}$, we define a function λ on $W_{\mathbf{h}}$ by $\lambda(x) = \prod_{v \in \mathbf{h}} \lambda_v(x_v)$ for $x \in W_{\mathbf{h}}$, where λ_v is the characteristic function of the coset $L_v + a_v$. We then denote by $\mathcal{S}(W_{\mathbf{h}})$ the vector space of all finite $\mathbf{C}$-linear combinations of such functions λ for all possible choices of (L, a). This is called *the Schwartz-Bruhat space of* $W_{\mathbf{h}}$. We view every $\ell \in \mathcal{S}(W_{\mathbf{h}})$ as a function on $W_{\mathbf{A}}$ by putting $\ell(x) = \ell(x_{\mathbf{h}})$ for $x \in W_{\mathbf{A}}$. In particular, $\ell(\xi)$ is meaningful for every $\xi \in W$. We can easily see that the restriction of the elements of $\mathcal{S}(W_{\mathbf{h}})$ to W gives an isomorphism of $\mathcal{S}(W_{\mathbf{h}})$ onto the set of all finite $\mathbf{C}$-linear combinations of functions, each of which is the characteristic function of a coset of W modulo a $\mathfrak{g}$-lattice in W. This is because $W_{\mathbf{A}} = W + Y$ with $Y = \{ y \in W_{\mathbf{A}} \mid y_{\mathbf{h}} \in \prod_{v \in \mathbf{h}} L_v \}$ for any fixed $\mathfrak{g}$-lattice L.

We now put

$$\mathbf{e}(z) = e^{2\pi i z} \qquad (z \in \mathbf{C}), \tag{1.14}$$

and define characters $\mathbf{e}_{\mathbf{A}} : F_{\mathbf{A}} \to \mathbf{T}$ and $\mathbf{e}_v : F_v \to \mathbf{T}$ for each $v \in \mathbf{v}$ as follows: if $v \in \mathbf{a}$, then $\mathbf{e}_v(x) = \mathbf{e}(x)$ for real v and $\mathbf{e}_v(x) = \mathbf{e}(x + \overline{x})$ for imaginary v; if $v \in \mathbf{h}$ and p is the rational prime divisible by v, then $\mathbf{e}_v(x) = \mathbf{e}_p\big(\mathrm{Tr}_{F_v/\mathbf{Q}_p}(x)\big)$, where $\mathbf{e}_p(z) = \mathbf{e}(-y)$ with $y \in \bigcup_{m=1}^{\infty} p^{-m}\mathbf{Z}$ such that $z - y \in \mathbf{Z}_p$. We then put $\mathbf{e}_{\mathbf{A}}(x) = \prod_{v \in \mathbf{v}} \mathbf{e}_v(x_v)$, $\mathbf{e}_{\mathbf{h}}(x) = \mathbf{e}_{\mathbf{A}}(x_{\mathbf{h}})$, and $\mathbf{e}_{\mathbf{a}}(x) = \mathbf{e}_{\mathbf{A}}(x_{\mathbf{a}})$ for $x \in F_{\mathbf{A}}$. We note here a basic property of $\mathbf{e}_v$:

$$\mathfrak{d}(F/\mathbf{Q})_v^{-1} = \{ x \in F_v \mid \mathbf{e}_v(xy) = 1 \ \textit{ for every } \ y \in \mathfrak{g}_v \} \qquad (v \in \mathbf{h}),$$

where $\mathfrak{d}(F/\mathbf{Q})$ denotes the different of F relative to $\mathbf{Q}$.

We insert here an easy fact as an application of Lemma 1.3 (4):

(*) *For every* $\alpha \in GU(\varphi)_{\mathbf{A}}$ *the map* $x \mapsto \alpha x \alpha^{-1}$ *of* $U(\varphi)_{\mathbf{A}}$ *onto itself leaves any fixed Haar measure of* $U(\varphi)_{\mathbf{A}}$ *invariant.*

Indeed, let μ be a Haar measure of $U(\varphi)_v$ for a fixed $v \in \mathbf{v}$. Then, for $\alpha \in GU(\varphi)_v$ we have $\mu(\alpha X \alpha^{-1}) = \lambda(\alpha)\mu(X)$ for every measurable set X in $U(\varphi)_v$ with a positive real number $\lambda(\alpha)$. Clearly λ is a homomorphism of $GU(\varphi)_v$ into $\mathbf{R}^{\times}$ and $\lambda(F_v^{\times}) = 1$. Also, $\lambda\big(U(\varphi)_v\big) = 1$ by [S97, Proposition 8.13 (1)]. Therefore, by Lemma 1.3 (4) we have $\lambda(\alpha) = 1$, since $\lambda(\alpha) > 0$.

1.7. Let A be a principal ideal domain and F the field of quotients of A. We call an element X of A_n^m *primitive* if $\mathrm{rank}(X) = \mathrm{Min}(m, n)$ and the elementary divisors of X are all equal to A. If $m = n$, clearly X is primitive if and only if $X \in GL_n(A)$. If $m < n$ (resp. $m > n$), then X is primitive if and only if X is the first m rows (resp. n columns) of an element of $GL_n(A)$ (resp. $GL_m(A)$). (For these and other properties of primitive matrices, see [S97, Lemmas 3.3 and 3.4].)

Given $x \in F_n^m$, we can find $c \in A_n^m$ and $d \in GL_m(F) \cap A_m^m$ such that $[c \ \ d]$ is primitive and $x = d^{-1}c$. We then call the last equality a *(left) reduced expression for* x, and define an integral ideal $\nu_0(x)$ by

$$\nu_0(x) = \det(d)A. \tag{1.15}$$

This is independent of the choice of c and d. We call $\nu_0(x)$ *the denominator ideal of* x. We easily see that $\nu_0(x + a) = \nu_0(x)$ if $a \prec A$ (see *Notation*). For these and other properties of the symbol ν_0 see [S97, Proposition 3.6, §3.7, and Lemma 3.8].

We now consider our setting to be that of §1.4, with an algebraic number field as F. Given $x \in (F_v)_n^m$ with $v \in \mathbf{h}$, we can naturally define $\nu_0(x)$ to be an integral $\mathfrak{g}_v$-ideal, taking $\mathfrak{g}_v$ to be A. Then we put

$$(1.16) \qquad \nu(x) = N(\nu_0(x)) = [\mathfrak{g}_v : \nu_0(x)].$$

If $x = d^{-1}c$ is a left reduced expression for x, then $\nu(x) = |\det(d)|_v^{-1}$. Moreover, if $uxv = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ with $u \in GL_m(\mathfrak{g}_v)$, $v \in GL_n(\mathfrak{g}_v)$, and $a = \mathrm{diag}[a_1, \ldots, a_r]$, then $\nu(x)^{-1}$ is the product of $|a_i|_v$ for all i such that $a_i \notin \mathfrak{g}_v$ (see [S97, Lemma 3.8 (2)]).

1.8. Take our setting to be the same as in Cases I and II in §1.1 with an algebraic number field as F. Namely, $K = F$ or K is a quadratic extension of F. We denote by $\mathfrak{r}$ the ring of algebraic integers in K and by $\mathbf{k}$ the set of all nonarchimedean primes of K. (Thus $\mathfrak{r} = \mathfrak{g}$ and $\mathbf{k} = \mathbf{h}$ if $K = F$.) Given $v \in \mathbf{k}$, an $\mathfrak{r}_v$-ideal $\mathfrak{a}$, and a matrix x with entries in K_v, we write $x \prec \mathfrak{a}$ if all the entries of x belong to $\mathfrak{a}$. Similarly, for a matrix y with entries in $K_{\mathbf{A}}$ and an $\mathfrak{r}$-ideal $\mathfrak{b}$, we write $y \prec \mathfrak{b}$ if all the entries of y_v belong to $\mathfrak{b}_v$ for every $v \in \mathbf{k}$.

Take two positive integers m and n. For $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_{m+n}(K)_{\mathbf{A}}$ with $a \in (K_{\mathbf{A}})_m^m$ and $d \in (K_{\mathbf{A}})_n^n$, write $a = a_x$, $b = b_x$, $c = c_x$, and $d = d_x$. With fixed $\mathfrak{r}$-ideals $\mathfrak{y}$ and $\mathfrak{z}$ such that $\mathfrak{y}\mathfrak{z} \subset \mathfrak{r}$, we put

$$(1.17) \qquad C[\mathfrak{y}, \mathfrak{z}] = \Big\{ x \in GL_{m+n}(K)_{\mathbf{A}} \;\Big|\; \det(x)_{\mathbf{h}} \in \textstyle\prod_{v \in \mathbf{h}} \mathfrak{r}_v^\times, \\ a_x \prec \mathfrak{r},\ b_x \prec \mathfrak{y},\ c_x \prec \mathfrak{z},\ d_x \prec \mathfrak{r} \Big\}.$$

We easily see that this is a subgroup of $GL_{m+n}(K)_{\mathbf{A}}$ (see [S97, §9.1]). We also note that if $x \in C[\mathfrak{y}, \mathfrak{z}]$ and $v|\mathfrak{y}\mathfrak{z}$, then $\big(\det(a_x)\det(d_x) - \det(x)\big)_v \in \mathfrak{y}_v\mathfrak{z}_v$, and hence we see that

(1.18) *The map* $x \mapsto \big((a_x, b_x)_v\big)_{v|\mathfrak{y}\mathfrak{z}}$ *defines a homomorphism of* $C[\mathfrak{y}, \mathfrak{z}]$ *into* $\prod_{v|\mathfrak{y}\mathfrak{z}} \big[GL_m(\mathfrak{r}_v/\mathfrak{y}_v\mathfrak{z}_v) \times GL_n(\mathfrak{r}_v/\mathfrak{y}_v\mathfrak{z}_v)\big]$.

1.9. Lemma. *Define a subgroup* $P^{(m,n)}$ *of* $GL_{m+n}(K)$ *by*

$$P^{(m,n)} = \{ x \in GL_{m+n}(K) \mid c_x = 0 \}.$$

Let C *denote the group* $C[\mathfrak{y}, \mathfrak{z}]$ *of (1.16). Then*

$$P_{\mathbf{A}}^{(m,n)}C = \{ x \in GL_{m+n}(K)_{\mathbf{A}} \mid (d_x)_v \in GL_n(K_v) \text{ and } \\ (d_x^{-1}c_x)_v \prec \mathfrak{z}_v \text{ for every } v|\mathfrak{y}\mathfrak{z} \}.$$

Moreover, assuming $m = n$, *let* G *denote* $GU(\eta_n)$, $U(\eta_n)$, *or* $SU(\eta_n)$ *with* η_n *of (1.8); put* $P = G \cap P^{(n,n)}$, *and* $D = G_{\mathbf{A}} \cap C$. *Then*

$$P_{\mathbf{A}}D = G_{\mathbf{A}} \cap P_{\mathbf{A}}^{(n,n)}C \\ = \{ x \in G_{\mathbf{A}} \mid (d_x)_v \in GL_n(K)_v \text{ and } (d_x^{-1}c_x)_v \prec \mathfrak{z}_v \text{ for every } v|\mathfrak{y}\mathfrak{z} \}.$$

In particular, $GL_{m+n}(K)_{\mathbf{A}} = P_{\mathbf{A}}^{(m,n)}C$ *and* $G_{\mathbf{A}} = P_{\mathbf{A}}D$ *if* $\mathfrak{y}\mathfrak{z} = \mathfrak{r}$.

PROOF. The assertions for $GL_{m+n}(K)$ and $U(\eta_n)$ are proved in [S97, Lemma 9.2]. Combining the result for $U(\eta_n)$ with [S97, Lemma 9.10 (2)], we obtain the assertion for $SU(\eta_n)$. As for $GU(\eta_n)$, let $x \in GU(\eta_n)_{\mathbf{A}}$, $p = \mathrm{diag}[1_n, \nu(\alpha)1_n]$, and $y = p^{-1}x$. Then $y \in U(\eta_n)$. Suppose $(d_x)_v \in GL_n(K)_v$ and $(d_x^{-1}c_x)_v \prec \mathfrak{z}_v$ for every $v|\mathfrak{y}\mathfrak{z}$. Then we easily see that y satisfies the same conditions, and so

$y \in \bigl(U(\eta_n)_{\mathbf{A}} \cap P_{\mathbf{A}}^{(n,n)}\bigr)\bigl(U(\eta_n)_{\mathbf{A}} \cap C\bigr)$. Since p is diagonal, we obtain the desired result for $GU(\eta_n)$.

1.10. The notation being as in §1.8, put $C[\mathfrak{z}] = C[\mathfrak{z}^{-1}, \mathfrak{z}]$. Then $GL_{m+n}(K)_{\mathbf{A}} = P_{\mathbf{A}}^{(m,n)}C[\mathfrak{z}]$ by the above lemma. Therefore every element x of $GL_{m+n}(K)_{\mathbf{A}}$ can be written $x = yz$ with $y \in P_{\mathbf{A}}^{(m,n)}$ and $z \in C[\mathfrak{z}]$. We then define an $\mathfrak{r}$-ideal $\mathrm{il}_{\mathfrak{z}}(x)$ by

$$\mathrm{il}_{\mathfrak{z}}(x) = \det(d_y)\mathfrak{r}, \tag{1.19}$$

where the right-hand side is an $\mathfrak{r}$-ideal defined in §1.6. We easily see that this is well-defined. (But it depends on m, n, and $\mathfrak{z}$.)

Next, given $x \in (K_{\mathbf{A}})_m^n$, we define an integral $\mathfrak{r}$-ideal $\nu_0(x)$ and a positive integer $\nu(x)$ by

$$\nu_0(x)_v = \nu_0(x_v) \quad \textit{for every} \quad v \in \mathbf{k}, \tag{1.20}$$

$$\nu(x) = N\bigl(\nu_0(x)\bigr). \tag{1.21}$$

Here $\nu_0(x_v)$ is defined by (1.15). In other words, take $g \in GL_n(K)_{\mathbf{A}}$ and $h \in (K_{\mathbf{A}})_m^n$ so that $x_v = g_v^{-1}h_v$ is a reduced expression for every $v \in \mathbf{k}$. Then $\nu_0(x) = \det(g)\mathfrak{r}$.

1.11. Lemma. *For $x \in GL_{m+n}(K)_{\mathbf{A}}$ the following assertions hold.*

(1) *If $\alpha = \mathrm{diag}[1_m, \kappa 1_n]$ with $\kappa \in K_{\mathbf{A}}^{\times}$, then $\mathrm{il}_{\kappa\mathfrak{z}}(\alpha x \alpha^{-1}) = \mathrm{il}_{\mathfrak{z}}(x)$.*

(2) *If $d_x \in GL_n(K)_{\mathbf{A}}$ and $\mathfrak{z} = \mu\mathfrak{r}$ with $\mu \in K_{\mathbf{A}}^{\times}$, then $\det(d_x)\mathrm{il}_{\mathfrak{z}}(x)^{-1} = \nu_0(\mu^{-1}d_x^{-1}c_x)$.*

(3) *If $x \in GL_{m+n}(K) \cap P_{\mathbf{A}}^{(m,n)}D[\mathfrak{y}, \mathfrak{z}]$ and $\mathfrak{y}\mathfrak{z} \neq \mathfrak{r}$, then $\det(d_x) \neq 0$ and $\det(d_x) \cdot \mathrm{il}_{\mathfrak{z}}(x)^{-1}$ is prime to $\mathfrak{y}\mathfrak{z}$.*

For the proof see [S97, Lemma 9.4].

1.12. By a *CM-field* we mean a totally imaginary quadratic extension of a totally real algebraic number field of finite degree. Given a CM-field K and an absolute equivalence class τ of representations of K by complex matrices, we call (K, τ) a *CM-type* if the direct sum of τ and its complex conjugate is equivalent to the regular representation of K over $\mathbf{Q}$. If (K, τ) is a CM-type and $[K : \mathbf{Q}] = 2n$, then τ is the class of $\mathrm{diag}[\tau_1, \ldots, \tau_n]$ with n isomorphic embeddings τ_i of K into $\mathbf{C}$ such that $\{\tau_1, \ldots, \tau_n, \tau_1\omega, \ldots, \tau_n\omega\}$ is exactly the set of all embeddings of K into $\mathbf{C}$, where ω denotes complex conjugation. In this setting we write $\tau = \{\tau_i\}_{i=1}^n$. If F is the totally real field over which K is quadratic and ρ is the generator of $\mathrm{Gal}(K/F)$, then we have $\tau_i\omega = \rho\tau_i$, because of the following easy fact:

(1.22) *If σ is an isomorphism of K onto a subfield of $\mathbf{C}$, then $x^{\rho\sigma}$ is the complex conjugate of x^{σ} for every $x \in K$.*

Therefore, if X is a matrix with entries in K, then $(X^*)^{\sigma} = {}^t\overline{(X^{\sigma})}$, where the bar is complex conjugation. Thus putting $Y^* = {}^t\overline{Y}$ for a complex matrix Y, we have $(X^*)^{\sigma} = (X^{\sigma})^*$.

Given (K, τ) as above, let K' be the field generated over $\mathbf{Q}$ by $\sum_{i=1}^n a^{\tau_i}$ for all $a \in K$. We call K' *the reflex field of* (K, τ). It can easily be shown that K' is a CM-field and contains $\prod_{i=1}^n a^{\tau_i}$ for all $a \in K$ (see [S98, pp.62-63, p.122, Lemma 18.2]).

2. Polarized abelian varieties

In this section we review some basic facts on polarized abelian varieties defined over a subfield of $\mathbf{C}$. The reader can find more detailed treatments, as well as further references, in [W58] and [S98].

2.1. By an *algebraic variety* we understand an affine or a projective variety which is absolutely irreducible, defined in an affine or a projective space with a fixed coordinate system. If V is an algebraic variety of dimension n, by a *divisor of* V we mean a finite $\mathbf{Z}$-linear combination of subvarieties of V of dimension $n-1$.

Suppose now V is an algebraic variety defined over a subfield of $\mathbf{C}$. Then by the same symbol V we mean the point set consisting of all the points with coordinates in $\mathbf{C}$ satisfying the defining equations for V. If V is nonsingular, then V has a natural structure of a complex manifold.

By an *abelian variety* we uderstand a projective algebraic variety A with a group structure such that the map $(x, y) \mapsto x+y$ of $A \times A$ into A and also the map $x \mapsto -x$ of A into A are both rational maps defined everywhere. We use the additive notation, since such a group is always commutative. Such an A must be nonsingular. We say that an abelian variety A is defined over a field k if the variety A and these maps are defined over k. Thus, we speak of an abelian variety A defined over a field k always in this sense.

Let A and B be two abelian varieties. By a *homomorphism* of A into B, or an *endomorphism* when $A = B$, we understand a rational map of A into B that is a group homomorphism. If such a map is birational, then it must be biregular, and we call it an *isomorphism*, or an *automorphism* when $A = B$. We denote by $\mathrm{Hom}(A, B)$ the set of all homomorphisms of A into B, defined over any extension of a given field of definition for A and B, and put $\mathrm{End}(A) = \mathrm{Hom}(A, A)$. We put also $\mathrm{Hom}_{\mathbf{Q}}(A, B) = \mathrm{Hom}(A, B) \otimes_{\mathbf{Z}} \mathbf{Q}$ and $\mathrm{End}_{\mathbf{Q}}(A) = \mathrm{End}(A) \otimes_{\mathbf{Z}} \mathbf{Q}$. An element of $\mathrm{Hom}(A, B)$ is called an *isogeny* if A and B have the same dimension and $\mathrm{Ker}(\lambda)$ is finite.

2.2. By a *lattice* in a finite-dimensional vector space W over $\mathbf{R}$ we understand a discrete subgroup of W that spans W over $\mathbf{R}$. If W is of dimension m, then a discrete subgroup D of W is a lattice in W if and only if D is isomorphic to $\mathbf{Z}^m$. (The distinction of a lattice in this sense from a $\mathfrak{g}$-lattice in the sense of §1.4 will be clear from the context, since the latter is never defined in a real vector space.)

Let D be a lattice in $\mathbf{C}^n$. Then D is isomorphic to $\mathbf{Z}^{2n}$. An $\mathbf{R}$-valued $\mathbf{R}$-bilinear form $E : \mathbf{C}^n \times \mathbf{C}^n \to \mathbf{R}$ is called a *Riemann form on* $\mathbf{C}^n$ *relative to* D, or simply a *Riemann form on* $\mathbf{C}^n/D$, if it satisfies the following conditions:

(2.1) $E(D, D) \subset \mathbf{Z}$.
(2.2) $E(x, y) = -E(y, x)$.
(2.3) *The form* $(x, y) \mapsto E(x, iy)$ *is symmeric and positive definite.*

Usually a Riemann form is defined with nonnegativity instead of positive definiteness in the last condition. In the present book, however, we always consider positive definite Riemann forms, and so we take the above (2.1-3) to be the conditions for a Riemann form.

Now, given an abelian variety defined over a subfield of $\mathbf{C}$, there is always a complex-analytic biregular map, which is also a group isomorphism, of A onto a complex torus. Conversely, it is a well-known fact that *a complex torus* $\mathbf{C}^n/D$ *is* *isomorphic to an abelian variety in that sense if and only if it has a Riemann form.* If ξ is such an isomorphism of $\mathbf{C}^n/D$ onto an abelian variety A, we can view ξ as a homomorphism of $\mathbf{C}^n$ onto A with kernel D. We then call $(\mathbf{C}^n/D, \xi)$ an *analytic coordinate system of* A.

2.3. Let A and $(\mathbf{C}^n/D, \xi)$ be as above. Given a Riemann form E on $\mathbf{C}^n/D$, put $H(u, v) = E(u, iv) + iE(u, v)$ for $u, v \in \mathbf{C}^n$. Then we can show that there is a nonzero holomorphic function f on $\mathbf{C}^n$ such that

$$f(u+\ell) = f(u)\psi(\ell)\exp\big(\pi \cdot H(\ell, u + (\ell/2))\big) \qquad \text{for every} \quad \ell \in D \tag{2.4}$$

with a map $\psi : D \to \mathbf{T}$. Then the zeros of f define a divisor on $\mathbf{C}^n/D$, which we write $\xi^{-1}(X)$ with a divisor X on A. In this situation we say that E *corresponds to* X or that X *determines* E with respect to ξ, since it can be shown that E is unique for X, though X is not uniquely determined by E.

Now we call a divisor on A *ample* if it determines a Riemann form on $\mathbf{C}^n/D$. By a *polarization of* A we mean a nonempty maximal set $\mathcal{C}$ of ample divisors of A with the following property: if $X, Y \in \mathcal{C}$, then there exist positive integers ℓ, m such that ℓX and mY determine the same Riemann form. By a *polarized abelian variety* we mean a structure $(A, \mathcal{C})$ formed by an abelian variety A and a polarization $\mathcal{C}$ of A. We shall often denote $(A, \mathcal{C})$ by a single letter $\mathcal{P}$. We call a member of $\mathcal{C}$ a *basic polar divisor* of $\mathcal{P}$ if the corresponding Riemann form E has the property that $E(D, D) = \mathbf{Z}$. Such a divisor X is characterized by the property that every member of $\mathcal{C}$ is algebraically equivalent to mX for some positive integer m, since two divisors on A determine the same Riemann form if and only if they are algebraically equivalent (cf. also [S98, pp.27-28]).

Let A' be another abelian variety of dimension n, and $(\mathbf{C}^n/D', \xi')$ an analytic coordinate system of A'. Every element λ of $\mathrm{Hom}(A', A)$ coresponds to a $\mathbf{C}$-linear endomorphism Λ of $\mathbf{C}^n$ such that $\Lambda D' \subset D$ by the relation $\lambda \circ \xi' = \xi \circ \Lambda$. Conversely every such Λ determines an element λ of $\mathrm{Hom}(A', A)$. Clearly λ is an isogeny if and only if $\det(\Lambda) \neq 0$. Similarly every element of $\mathrm{Hom}_{\mathbf{Q}}(A', A)$ coresponds to an element of $\mathrm{End}(\mathbf{C}^n, \mathbf{C})$ that sends $\mathbf{Q}D'$ into $\mathbf{Q}D$, where $\mathbf{Q}D$ denotes the $\mathbf{Q}$-linear span of D.

Let E be the Riemann form determined by a divisor X on A, and λ an isogeny of A' onto A. We can define a divisor X' on A' by $X' = \lambda^{-1}(X)$. Then X' is also ample, and determines the Riemann form E given by

$$E'(z, w) = E(\Lambda z, \Lambda w), \tag{2.5}$$

where Λ is determined by λ as above. This can be shown by considering $f \circ \Lambda$ with f satisfying (2.4).

Let $\mathcal{P} = (A, \mathcal{C})$ and $\mathcal{P}' = (A', \mathcal{C}')$ be two polarized abelian varieties of the same dimension. By an *isogeny* (resp. an *isomorphism*) of $\mathcal{P}'$ onto $\mathcal{P}$ we understand an isogeny (resp. an isomorphism) λ of A' onto A such that $\lambda^{-1}(Y) \in \mathcal{C}'$ for some $Y \in \mathcal{C}$. If that is so, then $\lambda^{-1}(X) \in \mathcal{C}'$ for every $X \in \mathcal{C}$.

2.4. Whenever we consider an algebraic variety V, by our convention of §2.1, V is defined in an affine or a projective space with a fixed coordinate system, so that we can speak of the coordinates of a point x on V. For $x \in V$ and a field of definition k for V we denote by $k(x)$ the field generated over k by the affine coordinates of x. (If V is a projective variety, the affine coordinates of x mean the quotients of the projective coordinates of x.)

We can also speak of *the smallest field of definition for V*. (See [W46, pp.71-72, Corollary 3 on page 71, in particular].) It is always finitely generated over the prime field. Also if V is defined over a field k and σ is an isomorphism of k onto a field k', Then V^σ is well-defined; for example, it is defined by the equations which are the images under σ of the k-rational defining equations for V. In particular, if V is a point, V^σ is the point whose coordinates are the images of the coordinates of V under σ. If f is a k-rational map of V into an algebraic variety V_1 rational over k, then f^σ is defined to be the rational map of V^σ into V_1^σ whose graph is the image of the graph of f under σ. If V is defined over k, we denote by $k(V)$ the field of all k-rational functions on V, that is, all k-rational maps of V into the one-dimensional affine line. If σ and k' are as above and $f \in k(V)$, then f^σ is a well-defined element of $k'(V^\sigma)$.

Let $\mathcal{P} = (A, \mathcal{C})$ be a polarized abelian variety. We say that $\mathcal{P}$ is *defined* (or *rational*) *over* k and that k *is a field of definition* (or *rationality*) *for* $\mathcal{P}$ if A is defined over k and $\mathcal{C}$ contains a k-rational divisor, say X. For σ as above, we can define $\mathcal{P}^\sigma$ by $\mathcal{P}^\sigma = (A^\sigma, \mathcal{C}^\sigma)$, where $\mathcal{C}^\sigma$ is the polarization of A^σ containing X^σ. (Here we need the fact that if X is ample, so is X^σ.)

Suppose now V is an algebraic variety defined over a subfield of $\mathbf{C}$. Then, as we did in §2.1, we identify V with the point set consisting of all the points with coordinates in $\mathbf{C}$ satisfying the defining equations for V. We denote by $\mathbf{C}(V)$ the union of $k(V)$ for all the subfields k of $\mathbf{C}$. If $\sigma \in \mathrm{Aut}(\mathbf{C})$, then V^σ as a point set consists of the images under σ of all the points in V in that sense. Let k_0 be the smallest field of definition for V. Then, for $\sigma, \tau \in \mathrm{Aut}(\mathbf{C})$ we have $V^\sigma = V^\tau$ if and only if $\sigma = \tau$ on k_0. Also, if k is a subfield of $\mathbf{C}$ and $V^\sigma = V$ for every $\sigma \in \mathrm{Aut}(\mathbf{C}/k)$, then V is defined over k.

2.5. Let $(A, \mathcal{C})$ and $(\mathbf{C}^n/D, \xi)$ be as above; let E be the Riemann form on $\mathbf{C}^n/D$ determined by a divisor X in $\mathcal{C}$. Given $\lambda \in \mathrm{End}_{\mathbf{Q}}(A)$, take $\Lambda \in \mathrm{End}(\mathbf{C}^n, \mathbf{C})$ so that $\lambda \circ \xi = \xi \circ \Lambda$. Define an element Λ' of $\mathrm{End}(\mathbf{C}^n, \mathbf{C})$ by

$$E(\Lambda' x, y) = E(x, \Lambda y). \tag{2.6}$$

From (2.1) we see that $\Lambda'(\mathbf{Q}D) \subset \mathbf{Q}D$, and hence there is an element $\lambda' \in \mathrm{End}_{\mathbf{Q}}(A)$ such that $\lambda' \circ \xi = \xi \circ \Lambda'$. We can easily verify that the map $\lambda \mapsto \lambda'$ is a $\mathbf{Q}$-linear bijection of $\mathrm{End}_{\mathbf{Q}}(A)$ onto itself such that $(\lambda')' = \lambda$ and $(\lambda\mu)' = \mu'\lambda'$. We call this map *the involution of* $\mathrm{End}_{\mathbf{Q}}(A)$ *determined by* X, *or by* $\mathcal{C}$, since clearly it depends only on $\mathcal{C}$. Here are two easy facts:

(2.7) If R (resp. S) is an element of $\mathbf{Q}_{2n}^{2n}$ (resp. $\mathbf{C}_n^n$) that represents Λ with respect to a $\mathbf{Q}$-basis of $\mathbf{Q}D$ (resp. $\mathbf{C}$-basis of $\mathbf{C}^n$), then $R = T \cdot \mathrm{diag}[S, \overline{S}]T^{-1}$ for some $T \in GL_{2n}(\mathbf{C})$ independent of Λ.

(2.8) $\mathrm{tr}(\Lambda'\Lambda) > 0$ *if* $\Lambda \neq 0$.

Here $\mathrm{tr}(\Lambda)$ denotes the trace of Λ as an $\mathbf{R}$-linear endomorphism. Indeed, since a $\mathbf{Q}$-basis of $\mathbf{Q}D$ gives an $\mathbf{R}$-basis of $\mathbf{C}^n$, we easily obtain (2.7). As for the latter, we note that $E(\Lambda' x, iy) = E(x, i\Lambda y)$, which means that Λ' is the adjoint of Λ with respect to the positive definite form of (2.3), so that we obtain (2.8).

Suppose now A, X, and λ are rational over k; let σ be an isomorphism of k onto a field k'. Let $\mu \mapsto \mu''$ be the involution of $\mathrm{End}_{\mathbf{Q}}(A^\sigma)$ determined by X^σ. Then

(2.9) λ' *is rational over* k *and* $(\lambda^\sigma)'' = (\lambda')^\sigma$.

This is because λ' can be defined algebro-geometrically by means of the Picard variety of A without employing E, and we can let σ act on the defining formula for λ. For details, see [S98, §1.3, formula (5) on page 5 in particular, and the last paragraph of §3.3 on page 25].

2.6. Lemma. *Let $(\mathbf{C}^n/D, \xi)$ be an analytic coordinate system of an abelian variety A, and X a divisor on A corresponding to a Riemann form E on $\mathbf{C}^n/D$ via ξ. For $s = \xi(x)$ and $t = \xi(y)$ with $x, y \in N^{-1}D$, $0 < N \in \mathbf{Z}$, put*

$$\zeta_X(s, t) = \exp\bigl(2\pi i N \cdot E(x, y)\bigr). \tag{2.10}$$

Then $\zeta_X(s, t)$ is an N-th root of unity, and moreover, for every $\sigma \in \mathrm{Aut}(\mathbf{C})$,

$$\zeta_X(s, t)^\sigma = \zeta_{X^\sigma}(s^\sigma, t^\sigma). \tag{2.11}$$

Furthermore, if k is a field over which A, X, and all points on A of finite order are rational, then k contains the maximal abelian extension of $\mathbf{Q}$.

PROOF. That $\zeta_X(s, t)$ is an N-th root of unity can be seen from the fact that $E(\Lambda, \Lambda) \subset \mathbf{Z}$. Formula (7) in [S98, p.24] shows that $\zeta_X(s, t)$ is the number $e_{X,N}(s, t)$ defined in [S98, §1.4] in a purely algebraic fashion without E. This definition is due to Weil [W48]. Therefore its behavior under σ can easily be verified. Notice that if $E(\Lambda, \Lambda) = m\mathbf{Z}$ with $0 < m \in \mathbf{Z}$ and N is a multiple of m, then $\zeta_X(s, t)$ is a primitive (N/m)-th root of unity for suitable s and t. Therefore we obtain the last assertion from (2.11).

2.7. We now generalize the above notion of polarized abelian variety by considering a structure $\mathcal{P} = (A, \mathcal{C}, \iota; \{t_i\}_{i=1}^r)$ formed by a polarized abelian variety $(A, \mathcal{C})$ in the above sense, a ring-injection ι of a $\mathbf{Q}$-algebra W (with identity element) into $\mathrm{End}_{\mathbf{Q}}(A)$, and an ordered set of points $\{t_i\}_{i=1}^r$ of A of finite order. We always assume that $\iota(1)$ is the identity element of $\mathrm{End}(A)$. We say that $\mathcal{P}$ is *defined* (or *rational*) over a field k and that k *is a field of definition (or rationality) for* $\mathcal{P}$ if $(A, \mathcal{C})$, every element of $\iota(W) \cap \mathrm{End}(A)$, and every t_i are all rational over k. We always take such a k to be a subfield of $\mathbf{C}$. Given such a k and an isomorphism σ of k onto a field (contained in $\mathbf{C}$ for the moment), we put $\mathcal{P}^\sigma = (A^\sigma, \mathcal{C}^\sigma, \iota^\sigma; \{t_i^\sigma\}_{i=1}^r)$, where $\iota^\sigma(a) = \iota(a)^\sigma$. If $\mathcal{P}' = (A', \mathcal{C}', \iota'; \{t_i'\}_{i=1}^r)$ is another such structure with the same W, we understand by an *isomorphism* of $\mathcal{P}$ onto $\mathcal{P}'$ an isomorphism f of $(A, \mathcal{C})$ onto $(A', \mathcal{C}')$ such that $f \circ \iota(a) = \iota'(a) \circ f$ for every $a \in W$ and $f(t_i) = t_i'$ for every i. We call f an *automorphism* of $\mathcal{P}$ if $\mathcal{P} = \mathcal{P}'$, and denote by $\mathrm{Aut}(\mathcal{P})$ the group of all automorphisms of $\mathcal{P}$. Given an arbitrary $(A, \mathcal{C})$, we can construct $\mathcal{P}$ as above by taking $W = \mathbf{Q}$ and $\{t_i\}$ to be the set consisting of 0. We shall alwlays identify such a $\mathcal{P}$ with $(A, \mathcal{C})$.

2.8. Theorem. (1) *Given $\mathcal{P}$ as above, there exists a subfield k_1 of $\mathbf{C}$, called the field of moduli of $\mathcal{P}$*, which is uniquely characterized by the following properties: (i) *Every field of definition for $\mathcal{P}$ contains k_1*; (ii) *If $\mathcal{P}$ is defined over k and σ is an isomorphism of k onto a subfield of $\mathbf{C}$, then σ is the identity map on k_1 if and only if $\mathcal{P}^\sigma$ is isomorphic to $\mathcal{P}$.*

(2) *The field of moduli of $\mathcal{P}$ is algebraic over the field of moduli of $(A, \mathcal{C})$.*

(3) *If $\sum_{i=1}^r \mathbf{Z}t_i \supset \{\, t \in A \mid mt = 0 \,\}$ with an integer $m > 2$, then $\mathcal{P}$ has a model rational over its field of moduli.*

(4) *$\mathcal{P}$ has a model over a finite algebraic extension of the field of moduli of $(A, \mathcal{C})$.*

Assertion (1) is given in [S59] for $\mathcal{P}$ without $\{t_i\}$, and in [S65, §1.4] for $\mathcal{P}$ with $\{t_i\}$. Clearly the field of moduli of $\mathcal{P}$ is finitely generated over $\mathbf{Q}$. As for (2), see [S59, Proposition 8] and [S65, Proposition 1.11]; as for (3), see [S98, Proposition 21.1] and the remark after its proof, or [S65, Proposition 1.5]. Combining (2) and (3), we obtain (4).

2.9. The notation being as in §2.7, let ψ be an equivalence class of $\mathbf{Q}$-linear representations of W by complex matrices. We say that (A, ι) *is of type* (W, ψ), if we can find an analytic coordinate system $(\mathbf{C}^n/D, \xi)$ of A such that $\xi \circ \psi_0(a) = \iota(a) \circ \xi$ for every $a \in W$ with a representation ψ_0 belonging to the class of ψ. In particular, take W to be a CM-field K in the sense of §1.12. If A is of dimension n, then by (2.7) the direct sum of ψ and its complex conjugate is equivalent to a rational representation of K of degree $2n$, which must be a multiple of a regular representation of K. Thus $[K : \mathbf{Q}]$ divides $2n$. If $[K : \mathbf{Q}] = 2n$, then we see that (K, ψ) is a CM-type in the sense of §1.12. Conversely, given a CM-type (K, τ), we can always find (A, ι) of type (K, τ); for details, see [S98, §6].

As an easy generalization we consider a *CM-algebra* Y, by which we understand a finite direct sum $Y = K_1 \oplus \cdots \oplus K_t$ with CM-fields K_i. We take (A, ι) as above with Y as W. Suppose that $2\dim(A) = [Y : \mathbf{Q}]$; let e_i be the identity element of K_i and let $A_i = \iota(m_i e_i)A$ with a positive integer m_i such that $\iota(m_i e_i) \in \mathrm{End}(A)$. We easily see that A is isogenous to $A_1 \times \cdots \times A_t$ and ι embeds K_i into $\mathrm{End}_{\mathbf{Q}}(A_i)$. Denote this embedding by ι_i. Since $[K_i : \mathbf{Q}] \le 2\dim(A_i)$ as observed above, and $[Y : \mathbf{Q}] = 2\dim(A)$, we obtain $[K_i : \mathbf{Q}] = 2\dim(A_i)$. Thus (A_i, ι_i) determines a CM-type (K_i, Φ_i). Then we see that (A, ι) is of type (Y, Φ) with Φ defined by

$$\Phi\left(\textstyle\sum_{i=1}^{t} a_i e_i\right) = \mathrm{diag}[\Phi_1(a_1), \ldots, \Phi_t(a_t)] \qquad (a_i \in K_i). \tag{2.12}$$

3. Symmetric spaces and factors of automorphy

3.1. We first define our groups over $\mathbf{R}$, which will eventually be localizations of algebraic groups over a number field at archimedean primes. Using the notation of §1.2, we consider $Sp(n, \mathbf{R})$, $Gp(n, \mathbf{R})$ and the unitary groups of the following types:

$$U(\eta_n) = \{\, \alpha \in GL_{2n}(\mathbf{C}) \mid \alpha^* \eta_n \alpha = \eta_n \,\}, \tag{3.1}$$

$$GU(\eta_n) = \{\, \alpha \in GL_{2n}(\mathbf{C}) \mid \alpha^* \eta_n \alpha = \nu(\alpha)\eta_n \ \textit{with}\ \nu(\alpha) \in \mathbf{R}^\times \,\}, \tag{3.2}$$

$$U(m, n) = \{\, \alpha \in GL_r(\mathbf{C}) \mid \alpha^* I_{m,n} \alpha = I_{m,n} \,\}, \quad I_{m,n} = \mathrm{diag}[1_m, -1_n], \tag{3.3}$$

$$GU(m, n) = \{\, \alpha \in GL_r(\mathbf{C}) \mid \alpha^* I_{m,n} \alpha = \nu(\alpha) I_{m,n} \ \textit{with}\ \nu(\alpha) \in \mathbf{R}^\times \,\}. \tag{3.4}$$

Here η_n is the matrix of (1.8), m and n are nonnegative integers and $r = m + n > 0$; we put $Z^* = {}^t\overline{Z}$ for a complex matrix Z. We easily see that if α belongs to $Gp(n, A)$, $GU(\eta_n)$ or $GU(m, n)$, then ${}^t\alpha$ belongs to the same group and $\nu({}^t\alpha) = \nu(\alpha)$. Clearly $U(\eta_n)$ and $GU(\eta_n)$ are isomorphic to $U(n, n)$ and $GU(n, n)$ respectively. For some technical reasons, however, we consider various objects in the unitary case in two different ways, with respect to these two types, even though the objects defined for the former types can easily be transferred to those defined for the latter types. Thus our discussion will be made in three cases, referred to as Case SP, Case UT, and Case UB: these correspond to $Sp(n, \mathbf{R})$, $U(\eta_n)$, and $U(m, n)$.

We now define domains $\mathfrak{H}_n$, $\mathcal{H}_n$, and $\mathfrak{B}_{m,n}$ by

$$\mathfrak{H}_n = \{ z \in \mathbf{C}_n^n \mid {}^t z = z,\ \mathrm{Im}(z) > 0 \}, \tag{3.5}$$

$$\mathcal{H}_n = \{ z \in \mathbf{C}_n^n \mid i(z^* - z) > 0 \}, \tag{3.6}$$

$$\mathfrak{B}_{m,n} = \mathfrak{B}(m, n) = \{ z \in \mathbf{C}_n^m \mid 1_n - z^* z > 0 \} \qquad (mn > 0). \tag{3.7}$$

Here for a hermitian matrix ξ (in particular, for a real symmetric matrix ξ) we write $\xi > 0$ (resp. $\xi \geq 0$) to indicate that ξ is positive definite (resp. nonnegative). We then put

$$S^n = \{ h \in \mathbf{C}_n^n \mid h^* = h \}, \qquad S_n^+ = \{ h \in S^n \mid h > 0 \}, \tag{3.8}$$

$$B(z) = \begin{bmatrix} z^* & z \\ 1_n & 1_n \end{bmatrix}, \quad \xi(z) = i(\overline{z} - {}^t z), \quad \eta(z) = i(z^* - z) \qquad (z \in \mathbf{C}_n^n, \text{ Cases SP, UT}), \tag{3.9}$$

$$B(z) = \begin{bmatrix} 1_m & z \\ z^* & 1_n \end{bmatrix}, \quad \xi(z) = 1_m - \overline{z} \cdot {}^t z, \quad \eta(z) = 1_n - z^* z \qquad (z \in \mathbf{C}_n^m, \text{ Case UB}). \tag{3.10}$$

For the moment we assume $mn \neq 0$. We shall discuss the case $mn = 0$ later. By straightforward calculations we can verify that

$$i \cdot B(z)^* \eta_n B(z) = \mathrm{diag}[{}^t\xi(z),\ -\eta(z)] \qquad \text{(Cases SP, UT)}, \tag{3.11}$$

$$B(z)^* I_{m,n} B(z) = \mathrm{diag}[{}^t\xi(z),\ -\eta(z)] \qquad \text{(Case UB)}, \tag{3.12}$$

$$\det\left[B(z)\right] = \begin{cases} \det(z^* - z) & \text{(Cases SP, UT)}, \\ \det\left[\xi(z)\right] = \det\left[\eta(z)\right] & \text{(Case UB)}. \end{cases} \tag{3.13}$$

Now if $\eta(z) > 0$ in Case UB, then $\det\left[B(z)\right] \neq 0$, and hence the left-hand side of (3.12) has signature (m, n), so that $\xi(z) > 0$. Similarly $\eta(z) > 0$ if $\xi(z) > 0$.

3.2. Lemma. Case UT: *Let $\mathfrak{X}_u$ be the set of all matrices X in $\mathbf{C}_{2n}^{2n}$ such that $iX^*\eta_n X = \mathrm{diag}[v,\ -w]$ with $v,\ w \in S_n^+$. Then the map $(z, \lambda, \mu) \mapsto B(z)\mathrm{diag}[\overline{\lambda},\ \mu]$ gives a bijection of $\mathcal{H}_n \times GL_n(\mathbf{C}) \times GL_n(\mathbf{C})$ onto $\mathfrak{X}_u$.*

Case SP: *Let $\mathfrak{X}$ be the set of all X, belonging to the set $\mathfrak{X}_u$ defined above, of the form $X = [\overline{y}\ \ y]$ with $y \in \mathbf{C}_n^{2n}$. Then the map $(z, \mu) \mapsto B(z)\mathrm{diag}[\overline{\mu},\ \mu]$ gives a bijection of $\mathfrak{H}_n \times GL_n(\mathbf{C})$ onto $\mathfrak{X}$.*

Case UB: *Let $\mathfrak{X}$ be the set of all $X \in \mathbf{C}_{m+n}^{m+n}$ such that $X^* I_{m,n} X = \mathrm{diag}[v,\ -w]$ with $v \in S_m^+$ and $w \in S_n^+$. Then the map $(z, \lambda, \mu) \mapsto B(z)\mathrm{diag}[\overline{\lambda},\ \mu]$ gives a bijection of $\mathfrak{B}_{m,n} \times GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$ onto $\mathfrak{X}$.*

This lemma is completely elementary. The proof in Case SP was given in [S97, Lemma 7.10]. Cases UT nd UB can be proved similarly.

3.3. Write simply $\mathfrak{X}$ for the set $\mathfrak{X}_u$ defined in Case UT in the above lemma. Thus we have $\mathfrak{X}$ defined in each of the three cases SP, UT, and UB. If $\alpha \in Gp(n, \mathbf{R})$ (resp. $\alpha \in GU(\eta_n)$, $\alpha \in GU(m, n)$, $mn > 0$) with $\nu(\alpha) > 0$, then we easily see that $\alpha\mathfrak{X} \subset \mathfrak{X}$. Given such an α, we define the action of α on $\mathfrak{H}_n$ (resp. $\mathcal{H}_n$, $\mathfrak{B}_{m,n}$) and two factors of automorphy $\lambda(\alpha, z)$ and $\mu(\alpha, z)$ as follows. First we consider Case UB. Let $\alpha \in GU(m, n)$ and $z \in \mathfrak{B}_{m,n}$. By the above lemma $B(z) \in \mathfrak{X}$, and so $\alpha B(z) \in \mathfrak{X}$. By the same lemma we can put $\alpha B(z) = B(w)\mathrm{diag}[\overline{\lambda},\ \mu]$ with unique $w \in \mathfrak{B}_{m,n}$, $\lambda \in GL_m(\mathbf{C})$, and $\mu \in GL_n(\mathbf{C})$. Let us put $w = \alpha z = \alpha(z)$, $\lambda = \lambda(\alpha, z)$, and $\mu = \mu(\alpha, z)$. Then we have

$$\alpha B(z) = B(\alpha z)\mathrm{diag}\left[\overline{\lambda(\alpha, z)},\ \mu(\alpha, z)\right]. \tag{3.14}$$

We can do the same in the other two cases. In Case UT both $\lambda(\alpha, z)$ and $\mu(\alpha, z)$ belong to $GL_n(\mathbf{C})$; in Case SP we have $\lambda(\alpha, z) = \mu(\alpha, z) \in GL_n(\mathbf{C})$. To make our formulas short, we shall often put $\lambda(\alpha, z) = \lambda_\alpha(z)$ and $\mu(\alpha, z) = \mu_\alpha(z)$. For $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with d of size n let us put $a = a_\alpha$, $b = b_\alpha$, $c = c_\alpha$, and $d = d_\alpha$. Then from (3.14) we immediately see that $a_\alpha z + b_\alpha = \alpha(z)\mu_\alpha(z)$ and $\mu_\alpha(z) = c_\alpha z + d_\alpha$. Since μ is invertible, we obtain

$$\alpha z = (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1} \qquad \text{(all cases)}, \tag{3.15}$$

$$\lambda_\alpha(z) = \bar{c}_\alpha \cdot {}^t z + \bar{d}_\alpha, \quad \mu_\alpha(z) = c_\alpha z + d_\alpha \qquad \text{(Cases SP, UT)}, \tag{3.16}$$

$$\lambda_\alpha(z) = \bar{b}_\alpha \cdot {}^t z + \bar{a}_\alpha, \quad \mu_\alpha(z) = c_\alpha z + d_\alpha \qquad \text{(Case UB)}. \tag{3.17}$$

The first formula means that we can let α act on $\mathfrak{H}_n$, $\mathcal{H}_n$, or $\mathfrak{B}_{m,n}$ by defining αz by (3.15). Notice that in all three cases $\lambda_\alpha(z)$ and $\mu_\alpha(z)$ are holomorphic in z. Applying another element β with $\nu(\beta) > 0$ to (3.14), we see that $\beta(\alpha z) = (\beta\alpha)z$ and

$$\lambda(\beta\alpha, z) = \lambda(\beta, \alpha z)\lambda(\alpha, z), \quad \mu(\beta\alpha, z) = \mu(\beta, \alpha z)\mu(\alpha, z). \tag{3.18}$$

From (3.11), (3.12), and (3.14) we obtain

$$\lambda_\alpha(z)^*\xi(\alpha z)\lambda_\alpha(z) = \nu(\alpha)\xi(z), \quad \mu_\alpha(z)^*\eta(\alpha z)\mu_\alpha(z) = \nu(\alpha)\eta(z). \tag{3.19}$$

Define a scalar factor of automorphy $j_\alpha(z)$ and a real-valued function δ by

$$j_\alpha(z) = j(\alpha, z) = \det\left[\mu_\alpha(z)\right] \qquad \text{(all cases)}, \tag{3.20}$$

$$\delta(z) = \begin{cases} \det\left[2^{-1}\eta(z)\right] & \text{(Cases SP, UT)}, \\ \det\left[\eta(z)\right] & \text{(Case UB)}. \end{cases} \tag{3.21}$$

From (3.18) we obtain $j_{\beta\alpha}(z) = j_\beta(\alpha z) j_\alpha(z)$ and

$$\delta(\alpha z) = \nu(\alpha)^n |j_\alpha(z)|^{-2}\delta(z). \tag{3.22}$$

We note here an easy but essential relation:

$$\det\left[\lambda_\alpha(z)\right] = \det(\bar{\alpha})\nu(\alpha)^{-n} \cdot j_\alpha(z) \qquad \text{(Cases UT, UB)}, \tag{3.23}$$

which follows immediately from (3.13), (3.14), and (3.19).

So far we have assumed $mn > 0$ in Case UB. We now make the following convention: if $mn = 0$ in Case UB, then $\mathfrak{B}(m, n)$ consists of the single element 0, our group acts on it trivially, and $B(0) = 1_{m+n}$; $GL_0(\mathbf{C})$ denotes the group consisting only of the identity element 1, and $\det(1) = 1$; $\big(\lambda_\alpha(z), \mu_\alpha(z), j_\alpha(z)\big)$ and $\big(\xi(z), \eta(z), \delta(z)\big)$ are elements of $GL_m(\mathbf{C}) \times GL_n(\mathbf{C}) \times \mathbf{C}^\times$ determined by

$$\xi(z) = 1_m, \quad \lambda_\alpha(z) = \bar{\alpha}, \quad \textit{and} \quad j_\alpha(z) = 1 \qquad \textit{if} \quad n = 0, \tag{3.24a}$$

$$\eta(z) = 1_n, \quad \mu_\alpha(z) = \alpha, \quad \textit{and} \quad j_\alpha(z) = \det(\alpha) \qquad \textit{if} \quad m = 0, \tag{3.24b}$$

$$\delta(z) = 1 \qquad \textit{if} \quad mn = 0. \tag{3.24c}$$

Thus $\eta(z) = \mu_\alpha(z) = 1$ if $n = 0$, and $\xi(z) = \lambda_\alpha(z) = 1$ if $m = 0$.

Also we denote by $\mathbf{C}^0$ the 0-dimensional vector space $\{0\}$ and let $GL_0(\mathbf{C})$ act on $\mathbf{C}^0$ trivially. We identify $\mathbf{C}^m \times \mathbf{C}^n$ and $GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$ with $\mathbf{C}^{m+n}$ and $GL_{m+n}(\mathbf{C})$ in an obvious way, if $mn = 0$. Then we denote by $\mathrm{diag}[a, b]$ the element of $GL_{m+n}(\mathbf{C})$ identified with $(a, b) \in GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$. Then (3.14), (3.18), (3.19), (3.22), and (3.23) are valid.

3.4. Lemma. (1) *Let* $dz = (dz_{hk})$ *be a matrix of the same shape as* z *whose entries are 1-forms* dz_{hk} *on each space. Then for* α *in* $SP(n, \mathbf{R})$, $U(\eta_n)$, *or in* $U(q, r)$ *we have* $d(\alpha z) = {}^t\lambda_\alpha(z)^{-1} \cdot dz \cdot \mu_\alpha(z)^{-1}$.

(2) *The jacobian of the map* $z \mapsto \alpha z$ *for such an* α *is* $j_\alpha(z)^{-\kappa}$, *where* $\kappa = n+1$, $2n$, *and* $m+n$ *in Cases SP, UT, and UB, respectively.*

(3) *Define a differential form* $\mathbf{d}z$ *on our space by*

$$\mathbf{d}z = \begin{cases} \delta(z)^{-n-1} \prod_{h \le k} [(i/2)dz_{hk} \wedge d\bar{z}_{hk}] & \text{(Case SP)}, \\ \delta(z)^{-m-n} \prod_{h=1}^{m} \prod_{k=1}^{n} [(i/2)dz_{hk} \wedge d\bar{z}_{hk}] & \text{(Cases UT and UB)}, \end{cases}$$

where we put $m = n$ *in Case UT. Then* $\mathbf{d}z$ *is invariant under the map of (2), and therefore it defines an invariant measure on our space.*

PROOF. We first consider Case UB. For $z, w \in \mathfrak{B}_{m,n}$ we have

$$B(w)^* I_{m,n} B(z) = \begin{bmatrix} 1 - wz^* & z - w \\ w^* - z^* & w^*z - 1 \end{bmatrix}.$$

Changing w and z for αw and αz and employing (3.14), we find that

$$\alpha z - \alpha w = {}^t\lambda_\alpha(w)^{-1}(z - w)\mu_\alpha(z)^{-1}. \tag{3.25a}$$

Taking similarly $B(w)^*\eta_n B(z)$ in Cases SP and UT, we see that (3.25a) is true in all cases. From this we obtain (1). Computing the determinant of the linear map $x \mapsto {}^t\lambda_\alpha(z)^{-1} x \mu_\alpha(z)^{-1}$, we obtain (2), which together with (3.22) proves (3). Clearly (3.25a) combined with (3.23) implies

$$\det(\alpha z - \alpha w) = \det(\alpha) j_\alpha(z)^{-1} j_\alpha(w)^{-1} \det(z - w). \tag{3.25b}$$

3.5. We now take a totally real algebraic number field F of finite degree and denote by $\mathbf{a}$ the set of all archimedean primes of F. In Cases UT and UB we take a CM-type (K, τ) as in §1.12 with K containing F as its maximal real subfield, and denote by ρ the generator of $\mathrm{Gal}(K/F)$. We shall also employ the symbols $\mathfrak{g}$, $\mathfrak{r}$, $\mathbf{h}$, and $\mathbf{k}$ introduced in §1.4 and §1.8. Then τ can be written $\tau = \{\tau_v\}_{v \in \mathbf{a}}$ with an embedding $\tau_v : K \to \mathbf{C}$ which coincides with v on F. Hereafter we fix τ and for $a \in K$ denote by a_v the image of a under τ_v. Then we identify $\mathbf{a}$ with τ and view $\mathbf{a}$ also as the set of all archimedean primes of K. For $c \in F_{\mathbf{A}}^\times$ we write $c \gg 0$ if $c_v > 0$ for every $v \in \mathbf{a}$. To make our exposition uniform, we make a convention that $K = F$ and $\mathfrak{r} = \mathfrak{g}$ in Case SP; thus we use K and $\mathfrak{r}$ in all three cases.

Given a set X, we denote by $X^{\mathbf{a}}$ the product of $\mathbf{a}$ copies of X, that is, the set of all indexed elements $(x_v)_{v \in \mathbf{a}}$ with x_v in X. Then all the embeddings of F into $\mathbf{R}$ (resp. K into $\mathbf{C}$) given by the elements of $\mathbf{a}$ determine an isomorphism of $F \otimes_{\mathbf{Q}} \mathbf{R}$ onto $\mathbf{R}^{\mathbf{a}}$ (resp. $K \otimes_{\mathbf{Q}} \mathbf{R}$ onto $\mathbf{C}^{\mathbf{a}}$.). Similarly we obtain embeddings of F_n^m and K_n^m into $(\mathbf{R}_n^m)^{\mathbf{a}}$ and $(\mathbf{C}_n^m)^{\mathbf{a}}$. We view the former sets as subsets of the latter sets. Thus for $\alpha \in F_n^m$ and $v \in \mathbf{a}$ the v-component α_v of α considered in $(\mathbf{R}_n^m)^{\mathbf{a}}$ is the v-th conjugate of α. If $\alpha \in K_n^m$, then α_v is the image of α under τ_v.

We now define algebraic groups $\widetilde{G}$, G, and G_0 by

$$\widetilde{G} = Gp(n, F), \qquad G = Sp(n, F) \qquad \text{(Case SP)}, \tag{3.26}$$

$$\widetilde{G} = GU(\eta_n), \qquad G = U(\eta_n) \qquad \text{(Case UT)}, \tag{3.27}$$

$$\widetilde{G} = GU(\mathcal{T}), \qquad G = U(\mathcal{T}) \qquad \text{(Case UB)}, \tag{3.28}$$

$$G_0 = \left\{ \alpha \in \widetilde{G} \,\middle|\, \det(\alpha) = \nu(\alpha)^n,\ \nu(\alpha) \in \mathbf{Q} \right\} \quad \text{(Cases SP and UT)}. \tag{3.29}$$

Here we are using the notation of (1.6), (1.7), (1.8), (1.9), and (1.10) with the present (F, K, ρ); $\mathcal{T}$ is a fixed element of $GL_r(K)$ such that $\mathcal{T}^* = -\mathcal{T}$; for a matrix Z we put $Z^* = {}^tZ^\rho$ as we did in (1.5). Recall that in each case we have a homomorphism $\nu : \widetilde{G} \to F^\times$. Though Case UT is included in Case UB, we treat them in different ways. In Case SP we can ignore the condition $\det(\alpha) = \nu(\alpha)^n$ in the definition of G_0, since it is true for every $\alpha \in \widetilde{G}$ as noted in (1.11).

For each $v \in \mathbf{v}$ we have localizations $\widetilde{G}_v$ and G_v of $\widetilde{G}$ and G; recall that $\widetilde{G}_\mathbf{a} = \prod_{v\in\mathbf{a}} \widetilde{G}_v$ and $G_\mathbf{a} = \prod_{v\in\mathbf{a}} G_v$. In Case UB let (m_v, n_v) be the signature of $i\mathcal{T}_v$. We then put

$$\widetilde{G}_{\mathbf{A}+} = \left\{ \alpha \in G_\mathbf{A} \,\middle|\, \nu(\alpha)_\mathbf{a} \in F_{\mathbf{a}+}^\times \right\}, \qquad F_{\mathbf{a}+}^\times = \left\{ x \in F_\mathbf{a}^\times \,\middle|\, x \gg 0 \right\}, \tag{3.30}$$

$$\widetilde{G}_{\mathbf{a}+} = \widetilde{G}_\mathbf{a} \cap \widetilde{G}_{\mathbf{A}+}, \qquad \widetilde{G}_+ = \widetilde{G} \cap \widetilde{G}_{\mathbf{A}+}, \tag{3.31a}$$

$$(G_0)_{\mathbf{A}+} = (G_0)_\mathbf{A} \cap \widetilde{G}_{\mathbf{A}+}, \qquad G_{0+} = G_0 \cap \widetilde{G}_{\mathbf{A}+}. \tag{3.31b}$$

In particular, in Case UB we put

$$\widetilde{\mathfrak{G}} = \prod_{v\in\mathbf{a}} GU(m_v, n_v), \quad \mathfrak{G} = \prod_{v\in\mathbf{a}} U(m_v, n_v), \quad \widetilde{\mathfrak{G}}_+ = \left\{ \alpha \in \widetilde{\mathfrak{G}} \,\middle|\, \nu(\alpha) \gg 0 \right\}. \tag{3.32}$$

We define a space $\mathcal{H}$ by

$$\mathcal{H} = \begin{cases} (\mathfrak{H}_n)^\mathbf{a} & \text{(Case SP)}, \\ (\mathcal{H}_n)^\mathbf{a} & \text{(Case UT)}, \\ \prod_{v\in\mathbf{a}} \mathfrak{B}(m_v, n_v) & \text{(Case UB)}. \end{cases} \tag{3.33}$$

We now define the action of the elements of $\widetilde{G}_{\mathbf{A}+}$ on $\mathcal{H}$ as follows: In Cases SP and UT, given $\xi = (\xi_v)_{v\in\mathbf{v}} \in \widetilde{G}_{\mathbf{A}+}$ and $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$, we consider $\xi_\mathbf{a} = (\xi_v)_{v\in\mathbf{a}}$ and put $\xi(z) = \big(\xi_v(z_v)\big)_{v\in\mathbf{a}}$. In Case UB we take an element $Q_v \in GL_m(\overline{\mathbf{Q}})$ so that

$$i\mathcal{T}_v = Q_v I_{m_v, n_v} Q_v^*. \tag{3.34}$$

(The reason why we take $GL_m(\overline{\mathbf{Q}})$ instead of $GL_m(\mathbf{C})$ will be explained later.) Clearly the map

$$\alpha \mapsto (Q_v^{-1}\alpha_v Q_v)_{v\in\mathbf{a}} \tag{3.35}$$

sends $\widetilde{G}_\mathbf{A}$ into $\widetilde{\mathfrak{G}}$. We then let $\widetilde{G}_{\mathbf{A}+}$ act on $\mathcal{H}$ through the map $\widetilde{G}_{\mathbf{A}+} \to \widetilde{\mathfrak{G}}_+$. For $\alpha \in \widetilde{G}_{\mathbf{A}+}$ and $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$ we put

$$\lambda_v(\alpha, z) = \lambda(\alpha_v, z_v), \quad \mu_v(\alpha, z) = \mu(\alpha_v, z_v) \quad \text{(Cases SP, UT)}, \tag{3.36}$$

$$\lambda_v(\alpha, z) = \lambda\big(Q_v^{-1}\alpha_v Q_v, z_v\big), \quad \mu_v(\alpha, z) = \mu\big(Q_v^{-1}\alpha_v Q_v, z_v\big) \quad \text{(Case UB)}. \tag{3.37}$$

Then, in Case UB, from (3.14) we obtain

$$\alpha_v Q_v B(z_v) = Q_v B\big((\alpha z)_v\big) \cdot \operatorname{diag}\big[\overline{\lambda_v(\alpha, z)}, \mu_v(\alpha, z)\big]. \tag{3.38}$$

3.6. It is sometimes necessary to send $\mathfrak{H}_n$ and $\mathcal{H}_n$ onto bounded domains. In Case UT the domain $\mathcal{H}_n$ can be sent onto $\mathfrak{B}(n, n)$ as will be shown below. To deal with problems of this type and to define the domain in Case SP, we put

(3.39) $$\mathfrak{B}_n = \{ z \in \mathfrak{B}(n, n) \mid {}^t z = z \},$$

(3.40) $$\mathfrak{E} = \{ \alpha \in GL_{2n}(\mathbf{C}) \mid \alpha^* \eta_n \alpha = -iI_{n,n} \},$$

(3.41) $$G' = Sp(n, \mathbf{C}) \cap U(n, n), \qquad \mathfrak{E}' = Sp(n, \mathbf{C}) \cap \mathfrak{E}.$$

Clearly $\mathfrak{E} = U(\eta_n)\alpha = \alpha U(n, n)$ for every $\alpha \in \mathfrak{E}$ and $\mathfrak{E}' = Sp(n, \mathbf{R})\beta = \beta G'$ for every $\beta \in \mathfrak{E}'$. As an exemplary element of $\mathfrak{E}'$ we note

(3.42) $$\beta_0 = \begin{bmatrix} \varepsilon 1_n & \varepsilon 1_n \\ -\bar{\varepsilon} 1_n & \bar{\varepsilon} 1_n \end{bmatrix}$$

with a complex number ε such that $\varepsilon^2 = i/2$.

Now, given $\alpha \in \mathfrak{E}$ and $z \in \mathfrak{B}_{n,n}$, we see that $\alpha B(z)$ belongs to the set $\mathfrak{X}_u$ of Lemma 3.2 in Case UT, and so, by that lemma, $\alpha B(z) = \begin{bmatrix} w^* & w \\ 1 & 1 \end{bmatrix} \operatorname{diag}[\lambda, \mu]$ with $\lambda, \mu \in GL_n(\mathbf{C})$ and $w \in \mathcal{H}_n$. Then we put $w = \alpha z, \lambda_\alpha(z) = i\bar{\lambda}$, and $\mu_\alpha(z) = \mu$. Thus

(3.43) $$\alpha \begin{bmatrix} 1_n & z \\ z^* & 1_n \end{bmatrix} = \begin{bmatrix} (\alpha z)^* & \alpha z \\ 1_n & 1_n \end{bmatrix} \begin{bmatrix} i \cdot \overline{\lambda_\alpha(z)} & 0 \\ 0 & \mu_\alpha(z) \end{bmatrix} \qquad (z \in \mathfrak{B}_{n,n}).$$

In this way $z \mapsto \alpha z$ sends $\mathfrak{B}_{n,n}$ into $\mathcal{H}_n$. A simple calculation shows that $\alpha z = (a_\alpha z + b_\alpha)(c_\alpha z + d_\alpha)^{-1}$.

Since $\varepsilon = i\bar{\varepsilon}$ and $\mathfrak{E}' = Sp(n, \mathbf{R})\beta_0$ with β_0 of (3.42), we easily see that every element of $\mathfrak{E}'$ is of the form $[i\bar{p} \;\; p]$ with $p \in \mathbf{C}_n^{2n}$. Therefore if $\alpha \in \mathfrak{E}'$ and ${}^t z = z$, then we see that $\alpha B(z) = [i\bar{y} \;\; y]$ with $y \in \mathbf{C}_n^{2n}$. From this and Lemma 3.2 (Case SP) we see that $\lambda_\alpha(z) = \mu_\alpha(z)$ and $\alpha z \in \mathfrak{H}_n$. Thus α sends $\mathfrak{B}_n$ into $\mathfrak{H}_n$.

Considering similarly $\alpha^{-1}B(w)$ with $w \in \mathcal{H}_n$, we find that every α in $\mathfrak{E}$ gives a bijection of $\mathfrak{B}_{n,n}$ onto $\mathcal{H}_n$, and also a bijection of $\mathfrak{B}_n$ onto $\mathfrak{H}_n$ if $\alpha \in \mathfrak{E}'$. The action is associative in the sense that $(\alpha\gamma)z = \alpha(\gamma z)$ for $\gamma \in U(n, n)$ and $(\beta\alpha)z = \beta(\alpha z)$ for $\beta \in U(\eta_n)$. Furthermorer, (3.18) holds for $(\beta, \alpha) \in \mathfrak{E} \times U(n, n)$ and also for $(\beta, \alpha) \in U(\eta_n) \times \mathfrak{E}$; (3.19) is valid for $\alpha \in \mathfrak{E}$ and $z \in \mathfrak{B}_{n,n}$ if we understand that $\xi(\alpha z) = {}^t\eta(\alpha z) = i(\overline{\alpha z} - {}^t(\alpha z))$, as defined in (3.9). Finally we can show that (3.25a) is true for $z, w \in \mathfrak{B}_{n,n}$ and $\alpha \in \mathfrak{E}'$, and hence

(3.44) $$d(\alpha z) = {}^t\lambda_\alpha(z)^{-1} \cdot dz \cdot \mu_\alpha(z)^{-1} \qquad (\alpha \in \mathfrak{E}, \; z \in \mathfrak{B}_{n,n}).$$

These are essentially special cases of [S97, Lemma A2.3] and the formulas stated in its proof.

4. Families of polarized abelian varieties

4.1. Let us now consider a structure $\mathcal{P} = (A, \mathcal{C}, \iota; \{t_i\}_{i=1}^s)$ of §2.7 under the following conditions:

(4.1) *ι is a ring-injection of a field K into* $\operatorname{End}_{\mathbf{Q}}(A)$.

(4.2) *$\iota(K)$ is stable under the involution $\alpha \mapsto \alpha'$ of* $\operatorname{End}_{\mathbf{Q}}(A)$ *determined by $\mathcal{C}$ as in §2.5.*

Here K is as in §3.5; namely $K = F$ in Case SP, and K is a CM-field in Cases UT and UB. Given $\mathcal{P}$ satisfying these conditions, let d be the dimension of A. Let $(\mathbf{C}^d/\Lambda, \xi)$ be an analytic coordinate system for A in the sense of §2.2. Then we find a ring-injection $\Psi : K \to \mathbf{C}_d^d$ given by $\iota(a)\xi(u) = \xi(\Psi(a)u)$ for $a \in K$ and $u \in \mathbf{C}^d$. Let $\mathbf{Q}\Lambda$ denote the $\mathbf{Q}$-linear span of Λ in $\mathbf{C}^d$. Then $\mathbf{Q}\Lambda$ is a $2d$-dimensional vector space over $\mathbf{Q}$, and is stable under $\Psi(K)$. Thus it has a structure of a vector space over K, so that $\mathbf{Q}\Lambda$ is isomorphic to K_r^1 with the integer r such that $2d = r[K:\mathbf{Q}]$. Since $K_{\mathbf{a}} = K \otimes_{\mathbf{Q}} \mathbf{R}$, the isomorphism of K_r^1 onto $\mathbf{Q}\Lambda$ can be extended to an $\mathbf{R}$-linear isomorphism q of $(K_{\mathbf{a}})_r^1$ onto $\mathbf{C}^d$ such that $q(ax) = \Psi(a)q(x)$ for $a \in K$ and $x \in K_r^1$. Let L be the inverse image of Λ. Then we obtain a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & (K_{\mathbf{a}})_r^1 & \longrightarrow & (K_{\mathbf{a}})_r^1/L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle q} & & \downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \mathbf{C}^d & \xrightarrow{\xi} & A & \longrightarrow & 0 \end{array} \tag{4.3}$$

Let $E_X(x, y)$ be the Riemann form determined by a divisor X in $\mathcal{C}$, and ρ the involution of K such that $\iota(a^\rho) = \iota(a)'$, where α' is as in (4.2). Then, by (2.6),

$$E_X\big(\Psi(a)u,\, v\big) = E_X\big(u,\, \Psi(a^\rho)v\big) \quad \textit{for every} \quad a \in W. \tag{4.4}$$

Let $\mathrm{tr}\big(\Psi(a)\big)$ denote the trace of $\Psi(a)$ as an $\mathbf{R}$-linear map. Clearly $\mathrm{tr}\big(\Psi(a)\big) = r \cdot \mathrm{Tr}_{K/\mathbf{Q}}(a)$, and so by (2.8), $\mathrm{Tr}_{K/\mathbf{Q}}(aa^\rho) > 0$ for $a \neq 0$. Then it is an easy exercise to show that ρ is the identity map if $K = F$, and ρ is the Galois involution of K/F if $K \neq F$ (see [S98, p.37, Lemma 2]).

4.2. Put $f(x, y) = E_X\big(q(x), q(y)\big)$. Then $(x, y) \mapsto f(x, y)$ is a $\mathbf{Q}$-valued alternating form on $K_r^1 \times K_r^1$ such that $f(ax, y) = f(x, a^\rho y)$. For fixed x, y we consider a $\mathbf{Q}$-linear map $a \mapsto f(ax, y)$ of K into $\mathbf{Q}$, and find an element $g(x, y) \in K$ such that $f(ax, y) = \mathrm{Tr}_{K/\mathbf{Q}}\big(a \cdot g(x, y)\big)$. Then we easily see that $(x, y) \mapsto g(x, y)$ is a skew-hermitian form. Putting $g(x, y) = x\mathcal{T}y^*$ with an element $\mathcal{T}$ of $GL_r(K)$, where $y^* = {}^t y^\rho$ as we defined in (1.5), we thus obtain

$$\mathcal{T}^* = -\mathcal{T}, \tag{4.5}$$

$$E_X\big(q(x), q(y)\big) = \mathrm{Tr}_{K/\mathbf{Q}}\big(x\mathcal{T}y^*\big) \quad \textit{for every} \quad x, y \in K_r^1. \tag{4.6}$$

With a $\mathbf{Z}$-lattice L in K_r^1 and $\{u_i\}_{i=1}^s \subset K_r^1$ we consider a set of data

$$\Omega = \big\{\, K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^s \,\big\}. \tag{4.7}$$

We call such an Ω a *PEL-type*. We note here an easy fact:

(4.8) *The direct sum of Ψ and its complex conjugate is equivalent to a $\mathbf{Q}$-rational representation of K.*

This follows from (2.7). Given such an Ω, we say that $\mathcal{P} = (A, \mathcal{C}, \iota; \{t_i\}_{i=1}^s)$ *is of type* Ω (with respect to ξ and q) if there is an $\mathbf{R}$-linear isomorphism $q : (K_{\mathbf{a}})_r^1 \to \mathbf{C}^d$ with which we have (4.3) and such that $q(ax) = \Psi(a)q(x)$ and $\iota(a) \circ \xi = \xi \circ \Psi(a)$ for every $a \in K$, $q(u_i) = t_i$ for every i, and (4.6) holds with some $X \in \mathcal{C}$.

4.3. Let us now classify all structures of type Ω. We first treat Case SP, in which $K = F$. Then ${}^t\mathcal{T} = -\mathcal{T}$. Changing the coordinate system in F_r^1, we may assume that $\mathcal{T} = \eta_n$ with $n = r/2$. This means that r must be even; then $d = n[F:\mathbf{Q}]$. Also any $\mathbf{Q}$-rational representation of F must be equivalent to a multiple of the regular representation of F over $\mathbf{Q}$, and hence (4.8) shows that Ψ must be equivalent to the direct sum of n copies of the regular representation of F over $\mathbf{Q}$. Namely we can decompose $\mathbf{C}^d$ into the direct sum $\bigoplus_{v \in \mathbf{a}} V_v$ so that each V_v is isomorphic to $\mathbf{C}^n$ and $\Psi(a)$ acts on V_v as a scalar a_v for each $a \in F$.

Next take K to be a CM-field, and $\tau = \{\tau_v\}_{v \in \mathbf{a}}$ to be a CM-type as in §3.5. Let m_v resp. n_v be the multiplicity of τ_v resp. $\rho\tau_v$ in Ψ. Then (4.8) shows that

$m_v + n_v$ must be the same for all $v \in \mathbf{a}$. Then $m_v + n_v = r$ since $2d = r[K:\mathbf{Q}]$. This time we can decompose $\mathbf{C}^d$ into the direct sum $\bigoplus_{v\in\mathbf{a}} V_v$ so that each V_v is isomorphic to $\mathbf{C}^r$ and $\Psi(a)$ acts on V_v as $\mathrm{diag}[\overline{a}_v 1_{m_v},\, a_v 1_{n_v}]$ for each $a \in K$.

Putting $l = n$ if $K = F$ and $l = r$ if $K \neq F$, we can thus put

$$\mathbf{C}^d = (\mathbf{C}^l)^{\mathbf{a}}, \qquad \Psi(a) = \mathrm{diag}\big[\Psi_v(a)\big]_{v\in\mathbf{a}}, \tag{4.9}$$

$$\Psi_v(a) = \begin{cases} a_v 1_n & (a \in F = K), \\ \mathrm{diag}[\overline{a}_v 1_{m_v},\, a_v 1_{n_v}] & (a \in K \neq F). \end{cases} \tag{4.10}$$

Then (4.3) can be written

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & (K_{\mathbf{a}})^1_r & \longrightarrow & (K_{\mathbf{a}})^1_r/L & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow{\scriptstyle q} & & \big\downarrow & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & (\mathbf{C}^l)^{\mathbf{a}} & \xrightarrow{\ \xi\ } & A & \longrightarrow & 0 \end{array} \tag{4.11}$$

with a map q such that

$$q(ax) = \Psi(a)q(x) \qquad (a \in K_{\mathbf{a}},\ x \in (K_{\mathbf{a}})^1_r), \tag{4.12}$$

$$E_X\big(q(x),\, q(y)\big) = \mathrm{Tr}_{K_{\mathbf{a}}/\mathbf{R}}\big(x\mathcal{T}y^*\big) \quad \textit{for every} \quad x,\, y \in (K_{\mathbf{a}})^1_r, \tag{4.13}$$

$$q(u_i) = t_i \quad \textit{for every} \quad i. \tag{4.14}$$

Conditions (2.1) and (2.3) can be written in the forms

$$\mathrm{Tr}_{K/\mathbf{Q}}\big(x\mathcal{T}y^*\big) \in \mathbf{Z} \quad \textit{for every} \quad x, y \in L, \tag{4.15}$$

(4.16) $E_X\big(q(x),\, i\cdot q(y)\big)$ *is symmetric in* $(x,\, y)$ *and positive definite.*

Condition (4.15) concerns only L, and so we consider Ω with L satisfying (4.15). We shall later prove the following compatibility condition on Ψ and $\mathcal{T}$:

(4.17) *If a structure of type Ω exists with $K \neq F$, then the hermitian form $i\mathcal{T}_v$ has signature $(m_v,\, n_v)$ for every $v \in \mathbf{a}$.*

Once this is established, we take Q_v as in (3.34), consider $\widetilde{G}$, G, and other symbols as in §3.5 with the present $\mathcal{T}$ and $(m_v,\, n_v)$. We call this setting Case UB in accordance with what we said in Section 3.

Now, still with a CM-field as K, suppose that $\mathcal{T} = \eta_n$. Then (4.17) implies that $m_v = n_v$ for every $v \in \mathbf{a}$. We call this setting Case UT. In this case we put $n = r/2$; then $m_v = n_v = n$.

4.4. We are going to determine all $\mathcal{P}$ of type Ω of (4.7), which amounts to the classification of all possible maps q. Let $\{\, e_k \,\}_{k=1}^{r}$ be the standard basis of K^1_r. Given any map $q : (K_{\mathbf{a}})^1_r \to (\mathbf{C}^l)^{\mathbf{a}}$ satisfying (4.12) (but not necessarily injective), we observe that q is determined by the vectors $q(e_k)$, and define a matrix $X_v(q) \in \mathbf{C}^r_r$ for each $v \in \mathbf{a}$ by

$$X_v(q) = \begin{bmatrix} x_v^1 & \cdots & x_v^{2n} \\ \overline{x}_v^1 & \cdots & \overline{x}_v^{2n} \end{bmatrix}, \quad q(e_k)_v = x_v^k \qquad \text{(Case SP)}, \tag{4.18}$$

$$X_v(q) = \begin{bmatrix} x_v^1 & \cdots & x_v^r \\ \overline{y}_v^1 & \cdots & \overline{y}_v^r \end{bmatrix}, \quad q(e_k)_v = \begin{bmatrix} x_v^k \\ y_v^k \end{bmatrix} \qquad \text{(Cases UT and UB)}. \tag{4.19}$$

Here $x_v^k \in \mathbf{C}^n$ in Case SP and $x_v^k \in \mathbf{C}^{m_v}$, $y_v^k \in \mathbf{C}^{n_v}$ in Cases UT and UB. We easily see that given any $\omega \in (\mathbf{C}_{2n}^n)^{\mathbf{a}}$, (resp. $Y \in (\mathbf{C}_r^r)^{\mathbf{a}}$) we can find q in Case SP (resp. Cases UT and UB) satisfying (4.12) such that $X_v(q) = \begin{bmatrix} \omega_v \\ \overline{\omega}_v \end{bmatrix}$ (resp. $X_v(q) = Y_v$) for every $v \in \mathbf{a}$. For the moment we disregard (4.13), (4.15), and (4.16).

4.5. Lemma. (1) *For q as above and $\beta \in (K_{\mathbf{a}})_r^r$ put $q^\beta(x) = q(x\beta)$ for $x \in (K_{\mathbf{a}})_r^1$. Then q^β satisfies (4.12) and $X_v(q^\beta) = X_v(q)\beta_v^*$.*

(2) *The map q is injective (and hence surjective) if and only if* $\det X_v(q) \neq 0$ *for every $v \in \mathbf{a}$.*

PROOF. Assertion (1) can easily be verified. Suppose $q\left(\sum_{i=1}^r b_i e_i\right) = 0$ for some $b = (b_i) \in (K_{\mathbf{a}})_r^1$. Let β be the element of $(K_{\mathbf{a}})_r^r$ whose rows are all equal to b. Then $q^\beta(e_k) = q(e_k\beta) = 0$ for every k, and hence $0 = X_v(q^\beta) = X_v(q)\beta_v^*$. If $\det X_v(q) \neq 0$ for every $v \in \mathbf{a}$, then $\beta_v = 0$ for every $v \in \mathbf{a}$, that is, q is injective. To prove the converse, we take q_0 so that $X_v(q_0) = \begin{bmatrix} 1_n & i1_n \\ 1_n & -i1_n \end{bmatrix}$ in Case SP and $X_v(q_0) = 1_r$ in Cases UT and UB for every $v \in \mathbf{a}$. This is possible by virtue of the observation at the end of §4.4. Then q_0 is injective since $X_v(q_0)$ is invertible. Given an injective q, $q_0^{-1}q$ is a $K_{\mathbf{a}}$-automorphism of $(K_{\mathbf{a}})_r^1$, and so we have $(q_0^{-1}q)(x) = x\beta$ with $\beta \in GL_r(K_{\mathbf{a}})$. Then $q = q_0^\beta$, and hence $X_v(q) = X_v(q_0)\beta_v^*$, which is invertible. This completes the proof.

4.6. Given an injective q, we see that the map $x \mapsto q^{-1}\big(i \cdot q(x)\big)$ is a $K_{\mathbf{a}}$-automorphism of $(K_{\mathbf{a}})_r^1$, and hence we have $i \cdot q(x) = q(xC)$ with $C \in GL_r(K_{\mathbf{a}})$. Then $i \cdot q(e_k) = q^C(e_k)$, and so from (4.18), (4.19), and (1) of Lemma 4.5 we obtain

$$i \cdot I_v X_v(q) = X_v(q)C_v^* \quad \textit{with} \quad I_v = \operatorname{diag}[1_{m_v}, -1_{n_v}], \tag{4.20}$$

where $m_v = n_v = n$ in Cases SP and UT. We now take (4.13) and (4.16) into account. We have

$$\begin{aligned} E_X\big(q(x), i \cdot q(y)\big) &= E_X\big(q(x), q(yC)\big) = \operatorname{Tr}_{K_{\mathbf{a}}/\mathbf{R}}\big(x\mathcal{T}(yC)^*\big) \\ &= [K : F] \textstyle\sum_{v \in \mathbf{a}} \operatorname{Re}\big(x_v \mathcal{T}_v C_v^* \ y_v^*\big). \end{aligned}$$

This must be symmetric in (x, y) and positive definite, which is so if and only if $\mathcal{T}_v C_v^*$ is hermitian and positive definite for every $v \in \mathbf{a}$. Fixing our attention to one v, dropping the subscript v, and putting simply $X = X_v(q)$, from (4.20) we obtain $\mathcal{T}C^* = i\mathcal{T}X^{-1}IX$. This is hermitian if and only if $X\mathcal{T}^{-1}X^*$ commutes with I, that is, if and only if $X\mathcal{T}^{-1}X^*$ is of the form $\operatorname{diag}[\beta, \gamma]$ with $\beta \in GL_m(\mathbf{C})$ and $\gamma \in GL_n(\mathbf{C})$. Then

$$X(\mathcal{T}C^*)^{-1}X^* = -iIX\mathcal{T}^{-1}X^* = \operatorname{diag}[-i\beta, i\gamma],$$

so that both $-i\beta$ and $i\gamma$ are hermitian and positive definite. Thus $iX\mathcal{T}^{-1}X^* = \operatorname{diag}[i\beta, i\gamma]$. This proves (4.17). Applying Lemma 3.2 to X^* (resp. $Q^{-1}X^*$), and reinstating the subscript v, we thus obtain

$$X_v(q) = \begin{cases} \operatorname{diag}[\xi_v, \zeta_v]B(z_v)^* & \text{(Cases SP and UT)}, \\ \operatorname{diag}[\xi_v, \zeta_v]B(z_v)Q_v^* & \text{(Case UB)}, \end{cases} \tag{4.21}$$

with $\xi_v \in GL_{m_v}(\mathbf{C})$, $\zeta_v \in GL_{n_v}(\mathbf{C})$, and $(z_v) \in \mathcal{H}$; $\zeta_v = \overline{\xi}_v$ in Case SP. Conversely, given $X_v(q)$ in this fashion, we see that q is injective, and reversing our reasoning, we find that $\mathcal{T}_v C_v^*$ is hermitian and positive definite, and hence (4.16) holds.

4.7. Given $z = (z_v) \in \mathcal{H}$, define $p : (K_{\mathbf{a}})_r^1 \times \mathcal{H} \to (\mathbf{C}^l)^{\mathbf{a}}$ and $p_z : (K_{\mathbf{a}})_r^1 \to (\mathbf{C}^l)^{\mathbf{a}}$ by specifying $X_v(p_z)$ as follows:

$$X_v(p_z) = \begin{cases} B(z_v)^* & \text{(Cases SP and UT)}, \\ B(z_v)Q_v^* & \text{(Case UB)}, \end{cases} \tag{4.22}$$

$$p(x, z) = p_z(x) \qquad \big(x \in (K_{\mathbf{a}})_r^1,\ z \in \mathcal{H}\big). \tag{4.23}$$

It can easily be seen that $p_z(x)$ is holomorphic in z. In particular, in Cases SP and UT we have

$$p_z(x) = \begin{cases} \big([z_v \quad 1_n] \cdot {}^t x_v\big)_{v \in \mathbf{a}} & \big(x \in (F_{\mathbf{a}})_{2n}^1,\ z \in \mathfrak{H}^{\mathbf{a}}\big), \\ \big([z_v \quad 1_n]x_v^*,\ [{}^t z_v \quad 1_n] \cdot {}^t x_v\big)_{v \in \mathbf{a}} & \big(x \in (K_{\mathbf{a}})_{2n}^1,\ z \in \mathcal{H}^{\mathbf{a}}\big). \end{cases} \tag{4.24}$$

As observed at the end of §4.6, from p_z we obtain a structure $\mathcal{P}_z$ of type Ω for which (4.11) holds with p_z as q. To be more precise, by Lemma 4.5(2), $p_z(L)$ is a lattice in $(\mathbf{C}^l)^{\mathbf{a}}$ so that $(\mathbf{C}^l)^{\mathbf{a}}/p_z(L)$ is a complex torus. Define E_z by

$$E_z\big(p_z(x), p_z(y)\big) = \mathrm{Tr}_{K_{\mathbf{a}}/\mathbf{R}}(x\mathcal{T}y^*). \tag{4.25}$$

Since (4.15) and (4.16) are satisfied, E_z is a Riemann form, so that $(\mathbf{C}^l)^{\mathbf{a}}/p_z(L)$ has a structure of an abelian variety; call it A_z and denote by $\mathcal{C}_z$ the polarization of A_z given by E_z. For $a \in K$ denote by $\iota_z(a)$ the element of $\mathrm{End}_{\mathbf{Q}}(A_z)$ represented by $\Psi(a)$ of (4.9) and (4.10), and by $t_i(z)$ the point of A_z represented by $p_z(u_i)$. Thus we obtain a nonempty family of polarized abelian varieties

$$\mathcal{F}(\Omega) = \big\{\, \mathcal{P}_z \,\big|\, z \in \mathcal{H} \,\big\}, \qquad \mathcal{P}_z = (A_z, \mathcal{C}_z, \iota_z;\ \{t_i(z)\}_{i=1}^s) \tag{4.26}$$

under the condition, which we hereafter assume, that

(4.27) *In Case UB the hermitian form $i\mathcal{T}_v$ has signature (m_v, n_v) for every $v \in \mathbf{a}$.*

4.8. Theorem. (1) *$\mathcal{P}_z$ is of type (4.7) for every $z \in \mathcal{H}$.*

(2) *A structure of type (4.7) is isomorphic to $\mathcal{P}_z$ for some $z \in \mathcal{H}$.*

(3) *$\mathcal{P}_z$ and $\mathcal{P}_w$ are isomorphic if and only if $w = \gamma z$ for some $\gamma \in \Gamma$, where*

$$\Gamma = \big\{\, \alpha \in G \,\big|\, L\alpha = L \text{ and } u_i\alpha - u_i \in L \text{ for every } i \,\big\}. \tag{4.28}$$

PROOF. Assertion (1) is obvious. Given $\mathcal{P}$ of type Ω, take q as in (4.11); then we obtain ξ_v, η_v, ζ_v, and $z \in \mathcal{H}$ as in (4.21). Define $S \in GL_l(\mathbf{C})^{\mathbf{a}}$ by $S = \mathrm{diag}[\kappa_v]_{v \in \mathbf{a}}$ with $\kappa_v = \mathrm{diag}[\xi_v]$ in Case SP and $\kappa_v = \mathrm{diag}[\xi_v, \overline{\zeta}_v]$ in Cases UT and UB. Clearly $S\Psi(a) = \Psi(a)S$ for every $a \in K$. Now (4.21) and (4.22) show that $q(e_k) = Sp_z(e_k)$ for every k, that is, $q = S \circ p_z$. We easily see that S gives an isomorphism of $\mathcal{P}_z$ onto $\mathcal{P}$, which proves (2). Before proving (3) we make some preliminary observations.

If $\sum_{v \in \mathbf{a}} m_v n_v = 0$ in Case UB, then $\mathcal{H}$ consists of a single point. Thus, under (4.27) there is exactly one isomorphism-class of structures $\mathcal{P}$ of type Ω. In fact, it can be shown that this $\mathcal{P}$ is isogenous to the product of r copies of an abelian variety belonging to a CM-type; see [S98, Theorem 24.15].

4.9. For $\alpha \in \widetilde{G}_+$ and $z \in \mathcal{H}$ define $M(\alpha, z) \in GL_l(\mathbf{C})^{\mathbf{a}}$ by

$$M(\alpha, z) = \begin{cases} \mathrm{diag}\big[\mu_v(\alpha, z)\big]_{v \in \mathbf{a}} & \text{(Case SP)}, \\ \mathrm{diag}\big[\lambda_v(\alpha, z),\ \mu_v(\alpha, z)\big]_{v \in \mathbf{a}} & \text{(Cases UT and UB)}. \end{cases} \tag{4.29}$$

Formula (3.14) or (3.38), combined with (4.22) and Lemma 4.5(1), gives

$$X_v(p_z^\alpha) = X_v(p_z)\alpha_v^* = \operatorname{diag}\left[{}^t\lambda_v(\alpha, z),\ \mu_v(\alpha, z)^*\right]X_v(p_{\alpha z}). \tag{4.30}$$

(See §3.3 for the meaning of the symbols when $m_v n_v = 0$.) This means that $p_z^\alpha(e_k) = {}^tM(\alpha, z)p_{\alpha z}(e_k)$ for every k. Since ${}^tM(\alpha, z)$ commutes with $\Psi(a)$ for every $a \in K$, we have $p_z(x\alpha) = p_z^\alpha(x) = {}^tM(\alpha, z)p_{\alpha z}(x)$, that is,

$$p(x\alpha, z) = {}^tM(\alpha, z)p(x, \alpha z) \qquad \left(x \in (K_{\mathbf{a}})_r^1,\ \alpha \in \widetilde{G}_+,\ z \in \mathcal{H}\right). \tag{4.31}$$

If α belongs to Γ of (4.28), then we easily see that ${}^tM(\alpha, z)$ sends $p_{\alpha z}(L)$ onto $p_z(L)$, and also sends E_z back to $E_{\alpha z}$ by virtue of (4.25). By (2.5) this means that it defines an isomorphism of $\mathcal{P}_{\alpha z}$ onto $\mathcal{P}_z$, which proves the "if"-part of Theorem 4.8 (3). Conversely, suppose that there is an isomorphism of $\mathcal{P}_w$ to $\mathcal{P}_z$; represent it by $S \in GL_l(\mathbf{C})^{\mathbf{a}}$. Then $p_z^{-1} \circ S \circ p_w$ defines an element α of $GL_r(K_{\mathbf{a}})$, that is, $Sp_w(x) = p_z(x\alpha)$. Since S sends $\left(p_w(L), \mathcal{C}_w, p_w(u_i)\right)$ to $\left(p_z(L), \mathcal{C}_z, p_z(u_i)\right)$, we see that $\alpha \in \Gamma$ by virtue of (2.5). This completes the proof of Theorem 4.8.

4.10. Given a $\mathfrak{g}$-lattice L in K_r^1 and an integral $\mathfrak{g}$-ideal $\mathfrak{c}$, we put

$$\Gamma(L, \mathfrak{c}) = \left\{\alpha \in \widetilde{G} \,\middle|\, L\alpha = L \ \text{and}\ L(1-\alpha) \subset \mathfrak{c}L\right\}. \tag{4.32}$$

We call a subgroup Γ of $\widetilde{G}$ (resp. G) a *congruence subgroup* of $\widetilde{G}$ (resp. G) if Γ contains $\Gamma(L, \mathfrak{c})$ (resp. $\Gamma(L, \mathfrak{c}) \cap G$) as a subgroup of finite index for some L and $\mathfrak{c}$. Clearly the group of (4.28) is a congruence subgroup of G. We note here two easy facts:

(4.33) $\det(\alpha)$ *and* $\nu(\alpha)$ *for* α *in such a* Γ *are units;* $\det(\alpha)$ *is a root of unity if* $\alpha \in \Gamma \cap G$.

(4.34) $\det(\alpha) = 1$ *if* $\alpha \in G \cap \Gamma(L, m\mathfrak{g})$ *with* $2 < m \in \mathbf{Z}$.

The first part of (4.33) is obvious. If $\alpha \in \Gamma \cap G$, then $\det(\alpha)$ is a unit in K, and $\det(\alpha)\det(\alpha)^\rho = 1$ by (1.12), and so $|\det(\alpha)|_v = 1$ for every $v \in \mathbf{a}$ by (1.22). Therefore $\det(\alpha)$ is a root of unity. If $\alpha \in \Gamma(L, m\mathfrak{g})$, then $\det(\alpha) - 1$ is divisible by m, and hence we obtain (4.34).

It is well-known that for a congruence subgroup Γ of G the quotient space $\Gamma\backslash\mathcal{H}$ has a compactification, called *the Satake compactification,* which has a structure of complex analytic space, and as such, is isomorphic to a normal projective variety V^*; moreover, $\Gamma\backslash\mathcal{H}$ is mapped onto a Zariski open subset V of V^*. Let φ denote the Γ-invariant map $\mathcal{H} \to V$ that gives this isomorphism. We then call (V, φ) a *model* of $\Gamma\backslash\mathcal{H}$.

4.11. We now introduce the notion of *CM-points* on $\mathcal{H}$. We take a CM-algebra $Y = K_1 \oplus \cdots \oplus K_t$ with CM-fields K_i as in §2.9. We assume that $K \subset K_i$ for every i and $r = [Y : K]$. We denote by F_i the maximal real subfield of K_i, and by ρ the automorphism of Y that coincides with the Galois involution of K_i/F_i for every i. Let us now consider a K-linear ring-injection $h : Y \to K_r^r$ satisfying

$$h(a^\rho) = \mathcal{T}h(a)^*\mathcal{T}^{-1} \qquad (a \in Y). \tag{4.35}$$

Put $Y^u = \left\{a \in Y \,\middle|\, aa^\rho = 1\right\}$. Then clearly $h(Y^u) \subset G$. Since Y^u is contained in a compact subgroup of $(Y \otimes_{\mathbf{Q}} \mathbf{R})^\times$, we easily see that the projection of $h(Y^u)$ to $G_{\mathbf{a}}$ is contained in a compact subgroup of $G_{\mathbf{a}}$, and hence $h(Y^u)$ has a common fixed point in $\mathcal{H}$. Moreover, $h(Y^u)$ has only one common fixed point as will be shown in

Lemma 4.12 below. We call a point on $\mathcal{H}$ which is obtained as such a fixed point a *CM-point on $\mathcal{H}$ with rèspect to G*. If w is a CM-point and $\beta \in \widetilde{G}_+$, then $\beta(w)$ is also a CM-point, since h' defined by $h'(a) = \beta h(a)\beta^{-1}$ satisfies (4.35) and $\beta(w)$ is fixed by $h'(Y^u)$.

Let us now show that in Case SP an injection h of type (4.35) always exists for any given Y. Take an element ζ of $Y^\times$ such that $\zeta^\rho = -\zeta$. Then $(x, y) \mapsto \mathrm{Tr}_{Y/F}(\zeta x y^\rho)$ is an F-valued nondegenerate alternating form on $Y \times Y$. Therefore we can find an F-linear bijection $r : Y \to F_{2n}^1$ such that $\mathrm{Tr}_{Y/F}(\zeta x y^\rho) = r(x)\eta_n \cdot {}^t r(y)$. Since multiplication by $a \in Y$ is an F-linear endomorphism of Y, we can define an F-linear map $h : Y \to F_{2n}^{2n}$ by the relation $r(ax) = r(x)h(a)$ for every $a, x \in Y$. Then we can easily verify (4.35).

In the unitary case the matter is not so simple. However, we can find at least one (Y, h) as follows. We first consider Case UB. Changing the coordinate system of K_r^1, we may assume that $\mathcal{T}$ is diagonal. Take Y to be the direct sum of r copies of K and define h by $h(a) = \mathrm{diag}[a_1, \dots, a_r]$ for $a = (a_i)_{i=1}^r \in Y$ with $a_i \in K$. Clearly (4.35) is satisfied. In Case UT we can find $\sigma \in GL_{2n}(K)$ such that $\sigma\eta_n\sigma^*$ is diagonal, and so the argument in Case UB produces a map in Case UT. Thus $\mathcal{H}$ has at least one CM-point with respect to G. Also we can always find CM-points on $\mathcal{H}$ associated with infinitely many different CM-fields Y, as shown in [S64, Proposition 4.10] and [S66b, pp.379-381].

Now let w denote the CM-point obtained as the common fixed point of $h(Y^u)$ as above in all three cases. Putting $X_v = X_v(p_w)$, from (4.30) and (4.31) we obtain

$$h(\alpha)_v X_v^* = X_v^* \mathrm{diag}\left[\overline{\lambda_v\big(h(\alpha), w\big)},\ \mu_v\big(h(\alpha), w\big)\right] \qquad (v \in \mathbf{a}), \tag{4.36a}$$

$$p_w\big(xh(\alpha)\big) = {}^t M\big(h(\alpha), w\big) p_w(x) \qquad (x \in (K_\mathbf{a})_r^1,\ \alpha \in Y^u). \tag{4.36b}$$

Since Y^u spans Y over $\mathbf{Q}$ as will be shown in Lemma 4.12 below, we can extend these equalities $\mathbf{Q}$-linearly to Y and define $\psi_v : Y \to \mathbf{C}_{m_v}^{m_v}$, $\varphi_v : Y \to \mathbf{C}_{n_v}^{n_v}$, and $\Phi : Y \to \mathrm{End}\big((\mathbf{C}^n)^\mathbf{a}\big)$ so that

$$\psi_v(\alpha) = \lambda_v\big(h(\alpha), w\big) \ \ \textit{and}\ \ \varphi_v(\alpha) = \mu_v\big(h(\alpha), w\big) \quad \textit{for} \quad \alpha \in Y^u, \tag{4.37}$$

$$h(a)_v X_v^* = X_v^* \mathrm{diag}\big[\overline{\psi_v(a)},\ \varphi_v(a)\big] \qquad (a \in Y,\ v \in \mathbf{a}), \tag{4.38}$$

$$p_w\big(xh(a)\big) = {}^t\Phi(a) p_w(x) \qquad (x \in (K_\mathbf{a})_r^1,\ a \in Y), \tag{4.39}$$

$$\Phi(a) = \mathrm{diag}\,[\Phi_v(a)]_{v\in\mathbf{a}}, \quad \Phi_v(a) = \begin{cases} \varphi_v(a) & (K = F), \\ \mathrm{diag}\big[\psi_v(a),\ \varphi_v(a)\big] & (K \neq F). \end{cases} \tag{4.40}$$

(In Case SP we have $\psi_v = \varphi_v$ and $m_v = n_v = n$.) Notice that the last equality follows from (4.29). From (4.38) and (4.40) we see that $\Phi_v(a) = \Psi_v(a)$ for every $a \in K$. Therefore the restriction of Φ to K is Ψ. Clearly ${}^t\Phi(\alpha)$ for each $\alpha \in Y$ defines an element of $\mathrm{End}_\mathbf{Q}(A_w)$; denote it by $\iota'(\alpha)$. Then ι' is a ring-injection of Y into $\mathrm{End}_\mathbf{Q}(A_w)$ that coincides with ι_w on K. Thus we find that $\mathcal{P}_w$, together with ι', defines a structure considered in §2.9 with Y and ${}^t\Phi$ as W and Φ there, and obtain a CM-type (K_i, Φ_i) for each i such that Φ is equivalent to the direct sum of $\Phi_1, \dots, \Phi_t$ in the sense of (2.12). Moreover, from (4.25), (4.35), and (4.39) we see that $E_w\big({}^t\Phi(\alpha)u,\ v\big) = E_w\big(u,\ {}^t\Phi(\alpha^\rho)v\big)$. Thus the automorphism ρ of Y corresponds to the involution of $\mathrm{End}_\mathbf{Q}(A_w)$ determined by $\mathcal{C}_w$ in the sense of §2.5.

4.12. Lemma. *Let (Y, h) be as in §4.11. Then Y is spanned by Y^u over $\mathbf{Q}$, and there exists an element β of Y^u such that $Y = \mathbf{Q}[\beta]$. Moreover, $h(\beta)$ for any such β has only one fixed point in $\mathcal{H}$.*

PROOF. Clearly $Y \otimes_{\mathbf{Q}} \mathbf{R}$ as an $\mathbf{R}$-algebra is isomorphic to $\mathbf{C}^d$. For $y \in Y \otimes_{\mathbf{Q}} \mathbf{R}$ denote by y_i the i-th coordinate of y viewed as an element of $\mathbf{C}^d$. Then $(y_i)^\rho = (y^\rho)_i$. Since Y is dense in $\mathbf{C}^d$, we can find an element x of $Y^\times$ such that $x_i/x_j \notin \mathbf{R}$ for $i \neq j$, and $x_i^\rho/x_j \notin \mathbf{R}$ for every (i, j). Put $\beta = x^\rho/x$. Then $\beta \in Y^u$, and $\beta_1, \ldots, \beta_d, \beta_1^\rho, \ldots, \beta_d^\rho$ are all different. Thus $Y = \mathbf{Q}[\beta]$, and hence Y is spanned by Y^u over $\mathbf{Q}$, since the powers of β belong to Y^u. To prove the uniqueness of the fixed point of $h(\beta)$, we first consider Case SP. Employing the symbols of §3.6, observe that

$$\{\alpha \in G' \mid \alpha(0) = 0\} = \{\operatorname{diag}[u, \overline{u}] \mid u \in U(n)\}, \tag{4.41}$$

where $U(n) = U(n, 0)$ in the sense of (3.3). Let w be a fixed point of $h(Y^u)$ as in §4.11 in Case SP. Then for each $v \in \mathbf{a}$ we can find $\xi_v \in \mathfrak{E}'$ such that $\xi_v(0) = w_v$ and

$$\xi_v^{-1} h(\alpha)_v \xi_v = \operatorname{diag}[\sigma_v(\alpha), \overline{\sigma}_v(\alpha)] \qquad (\alpha \in Y^u) \tag{4.42}$$

with a map $\sigma_v : Y^u \to U(n)$. Suppose w' is another fixed point of $h(Y^u)$ on $\mathcal{H}$; put $z'_v = \xi_v^{-1}(w'_v)$. Then $z'_v = \sigma_v(\alpha) z'_v \cdot {}^t\sigma_v(\alpha)$ for every $\alpha \in Y^u$. Let $c_{v1}, \ldots, c_{vn}$ be the characteristic roots of $\sigma_v(\alpha)$. Diagonalizing $\sigma_v(\alpha)$, we easily see that z' must be 0 if $c_{vj}c_{vk} \neq 1$ for every (j, k). Now take α to be the above β. From (4.42) we see that $c_{v1}, \ldots, c_{vn}, \overline{c}_{v1}, \ldots, \overline{c}_{vn}$ are the characteristic roots of $h(\beta)_v$, so that $c_{vj} \neq \overline{c}_{vk}$ for every (j, k). Thus $c_{vj}c_{vk} \neq 1$ for every (j, k) and every $v \in \mathbf{a}$. Then $w'_v = w_v$, which proves the desired fact. Case UT can be handled by the same technique. In Case UB we take an element β of $G_{\mathbf{a}}$ so that $\beta(0) = w$. Then taking $\mathfrak{B}(m_v, n_v)$ itself in place of $\mathfrak{B}_{n,n}$, we can prove the uniqueness of the fixed point in the same manner.

4.13. Lemma. *If w is a CM-point on $\mathcal{H}$, then the entries of w_v, $X_v(p_w)$ and $\psi_v(a)$, $\varphi_v(a)$ of (4.38) are algebraic for every $v \in \mathbf{a}$ and every $a \in Y$.*

PROOF. The point w can be obtained as the unique fixed point of $h(\alpha)$ with some $\alpha \in Y^u$ as above. Since we took Q_v of (3.34) to be algebraic, the action of $h(\alpha)$ on $\mathcal{H}$ is $\overline{\mathbf{Q}}$-rational. (Notice that $\mathcal{H}$ has an obvious $\overline{\mathbf{Q}}$-rational structure.) Therefore the fixed point w_v of $Q_v^{-1}h(\alpha)_v Q_v$ has algebraic entries. Then from (4.22) and (4.38) we immediately see the algebraicity of the other quantities in question.

If $\widetilde{G} = Gp(1, F) = GL_2(F)$, then $\mathcal{H} = \mathfrak{H}_1^{\mathbf{a}}$. This is the so-called Hilbert modular case. In this case the CM-points on $\mathcal{H}$ can be described in a clear-cut way as follows:

4.14. Proposition. *Let $\widetilde{G} = Gp(1, F) = GL_2(F)$. Let $(K, \{\tau_v\}_{v\in\mathbf{a}})$ be a CM-type with K containing F as the maximal real subfield and an injection $\tau_v : K \to \mathbf{C}$ which extends $v : F \to \mathbf{R}$. Take an element w_0 of K so that $\operatorname{Im}(\tau_v(w_0)) > 0$ for every $v \in \mathbf{a}$, and put $w = (w_v)_{v\in\mathbf{a}}$ with $w_v = \tau_v(w_0)$. Then w is a CM-point on $\mathfrak{H}_1^{\mathbf{a}}$ with respect to $GL_2(F)$, τ_v coincides with φ_v of (4.38), and (A_w, ι') is of type $(K, \{\tau_v\}_{v\in\mathbf{a}})$. Conversely every CM-point on $\mathfrak{H}_1^{\mathbf{a}}$ can be obtained in such a manner.*

PROOF. Given such a CM-type and w_0, we can define an F-linear ring-injection $h : K \to F_2^2$ by $h(\alpha)\begin{bmatrix} w_0 \\ 1 \end{bmatrix} = \begin{bmatrix} w_0 \\ 1 \end{bmatrix}\alpha$ for $\alpha \in K$. (Notice that (4.35) with $\mathcal{T} = \eta_1$ is satisfied by every F-linear ring-injection h of K into F_2^2.) Taking the image under τ_v, we obtain (4.38) for every $\alpha \in K$ with τ_v in place of φ_v, which shows that w is a fixed point of $h(K^\times)$ and (A_w, ι') is of type $(K, \{\tau_v\}_{v\in\mathbf{a}})$. Thus w is a CM-point. Conversely, consider a CM-point w obtained from an F-linear ring-injection $h : K \to F_2^2$. Then we obtain maps $\varphi_v : K \to \mathbf{C}$ satisfying (4.38). Take $\alpha \in K$ so that $h(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $c \neq 0$. Then (4.38) shows that $c_v w_v + d_v = \varphi_v(\alpha)$. Put $w_0 = c^{-1}(\alpha - d)$. Then $w_v = \varphi_v(w_0)$. This proves the converse part.

4.15. Remark. It should be noted that our formulation is essentially the same as in [S98] and some earlier papers of the author, [S63], [S65], and [S79], for example, but some symbols are defined differently. In particular, in those articles we set $\Psi_v(a) = \mathrm{diag}[a_v 1_{m_v}, \bar{a}_v 1_{n_v}]$ for $a \in K \neq F$, and formulas (3.34), (3.35), (3.37), and (3.38) were given in accordance with this change; see [S98, §§23 and 24]. Also, the families associated with more general PEL-types are treated in [S63].

5. Definition of automorphic forms

5.1. Our setting is the same as in §3.5; thus $K = F$ and $\mathfrak{r} = \mathfrak{g}$ in Case SP and $K \neq F$ in Cases UT and UB. We now define a symbol $\mathbf{b}$ as follows. In Case SP we put $\mathbf{b} = \mathbf{a}$. In Cases UT and UB we denote by $\mathbf{b}$ the set of all isomorphic embeddings of K into $\mathbf{C}$, and view $\mathbf{a}$ as a subset of $\mathbf{b}$. In all cases, for each $v \in \mathbf{b}$ and $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, or more generally for $\sigma \in \mathrm{Aut}(\mathbf{C})$, we denote by $v\sigma$ the element of $\mathbf{b}$ that is the composed map of v and σ. Then $\mathbf{b} = \mathbf{a}\rho \cup \mathbf{a}$ in Cases UT and UB, where ρ is complex conjugation.

In (3.36) and (3.37) we defined factors of automorphy λ_v and μ_v for $v \in \mathbf{a}$. We now put, for $\alpha \in \widetilde{G}_{\mathbf{A}+}$ and $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$,

$$M_\alpha(z) = M(\alpha, z) = \big(\mu_v(\alpha, z)\big)_{v\in\mathbf{b}}, \tag{5.1}$$

$$\mu_{v\rho}(\alpha, z) = \lambda_v(\alpha, z), \qquad n_{v\rho} = m_v \qquad (v \in \mathbf{a},\ K \neq F), \tag{5.2}$$

$$j_\alpha(z) = j(\alpha, z) = \big(j_v(\alpha, z)\big)_{v\in\mathbf{b}}, \quad j_v(\alpha, z) = \det\big[\mu_v(\alpha, z)\big] \quad (v \in \mathbf{b}). \tag{5.3}$$

The symbol $M_\alpha(z)$ is the same as $M(\alpha, z)$ of (4.29); in Case SP we shall write also $\mu_\alpha(z)$ for $M_\alpha(z)$. In Cases UT and UB, we hereafter use $(\mu_v)_{v\in\mathbf{b}}$ and $(n_v)_{v\in\mathbf{b}}$ instead of $(\lambda_v, \mu_v)_{v\in\mathbf{a}}$ and $(m_v, n_v)_{v\in\mathbf{a}}$. According to the convention of §3.3 in Case UB, for $v \in \mathbf{a}$, the pair $(j_{v\rho}, j_v)$ is either $\big(\det(\overline{\alpha}), 1\big)$ or $\big(1, \det(\alpha)\big)$ according as $n_v = 0$ or $n_{v\rho} = 0$.

To simplify our notation, for $x,\ y \in \mathbf{C}^{\mathbf{b}}$ and $\kappa \in \mathbf{C}$ we put

$$x^y = \textstyle\prod_{v\in\mathbf{b}} x_v^{y_v}, \tag{5.4a}$$

$$x^{\kappa\mathbf{a}} = \textstyle\prod_{v\in\mathbf{a}} x_v^{\kappa}, \qquad x^{\kappa\mathbf{b}} = \prod_{v\in\mathbf{b}} x_v^{\kappa}. \tag{5.4b}$$

The factors $x_v^{y_v}$ and x_v^κ must be understood according to the context. If $0 < x_v \in \mathbf{R}$, we always put $x_v^{y_v} = \exp\big(y_v \log x_v\big)$ with real $\log x_v$. If we identify $\kappa\mathbf{a}$ (resp. $\kappa\mathbf{b}$) with the element of $\mathbf{Z}^{\mathbf{a}}$ (resp. $\mathbf{Z}^{\mathbf{b}}$) whose components are all equal to κ, then (5.4b) is a special case of (5.4a). We shall later speak of $x^{\kappa\mathbf{a}+\lambda}$ with $\lambda \in \mathbf{C}^{\mathbf{a}}$.

To define automorphic forms on $\mathcal{H}$, we naturally assume that $\dim(\mathcal{H}) > 0$. Then we take a rational representation

$$\omega : \prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}) \to GL(X) \tag{5.5}$$

with a finite-dimensional complex vector space X. We understand that $n_v = n$ in Cases SP and UT, and $GL_{n_v}(\mathbf{C}) = 1$ if $n_v = 0$ in Case UB (see §3.3). Notice that $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}) = \prod_{v\in\mathbf{a}} \left[GL_{m_v}(\mathbf{C}) \times GL_{n_v}(\mathbf{C})\right]$ in Cases UT and UB. In all cases $M_\alpha(z)$ is an element of $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})$. We shall often write an element of $\alpha \in \prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})$ in the form $\alpha = (a,\, b)$ according to the following convention: $a = b \in GL_n(\mathbf{C})^{\mathbf{a}}$ in Case SP; $a = (\alpha_{v\rho})_{v\in\mathbf{a}} \in \prod_{v\in\mathbf{a}} GL_{m_v}(\mathbf{C})$ and $b = (\alpha_v)_{v\in\mathbf{a}} \in \prod_{v\in\mathbf{a}} GL_{n_v}(\mathbf{C})$ in Cases UT and UB.

Given a map $f : \mathcal{H} \to X$ and $\alpha \in \widetilde{G}_{\mathbf{A}+}$, we define $f\|_\omega\alpha : \mathcal{H} \to X$ and $f|_\omega\alpha : \mathcal{H} \to X$ by

$$(f\|_\omega\alpha)(z) = \omega\bigl(M_\alpha(z)\bigr)^{-1} f(\alpha z) \qquad (z \in \mathcal{H}), \tag{5.6a}$$

$$f|_\omega\alpha = f\|_\omega\bigl(\nu(\alpha)_{\mathbf{a}}^{-1/2}\alpha\bigr), \tag{5.6b}$$

where $\nu(\alpha)_{\mathbf{a}} = (\nu(\alpha)_v)_{v\in\mathbf{a}}$. We easily see that $(f\|_\omega\alpha)\|_\omega\beta = f\|_\omega(\alpha\beta)$ and $(f|_\omega\alpha)|_\omega\beta = f|_\omega(\alpha\beta)$.

Given $k = (k_v)_{v\in\mathbf{b}} \in \mathbf{Z}^{\mathbf{b}}$, we can define a representation $\omega : \prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}) \to GL(\mathbf{C}) = \mathbf{C}^\times$ by $\omega(x) = \det(x)^k$ using the notation of (5.4a). Then we write $f\|_k\alpha$ for $f\|_\omega\alpha$; thus

$$(f\|_k\alpha)(z) = j_\alpha(z)^{-k} f(\alpha z) \qquad (z \in \mathcal{H}). \tag{5.7}$$

We shall often write $f\|\alpha$ for $f\|_\omega\alpha$ or $f\|_k\alpha$ when ω or k is clear from the context. We use $f|\alpha$ in a similar sense.

5.2. Given a congruence subgroup Γ of $\widetilde{G}$ or G contained in $\widetilde{G}_+$ and ω as in (5.5), we denote by $\mathcal{M}_\omega(\Gamma)$ the set of all functions $f : \mathcal{H} \to X$ satisfying the following conditions:

(5.8) *f is holomorphic;*

(5.9) *$f|_\omega\gamma = f$ for every $\gamma \in \Gamma$;*

(5.10) *f is holomorphic at every cusp.*

If $\omega(x) = \det(x)^k$ as above, we write $\mathcal{M}_k(\Gamma)$ for $\mathcal{M}_\omega(\Gamma)$. In particular $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma)$ and $\mathcal{M}_{\kappa\mathbf{b}}(\Gamma)$ are meaningful for $\kappa \in \mathbf{Z}$. Condition (5.10) is necessary or meaningful only in the following exceptional cases: (Cases SP and UT) $F = \mathbf{Q}$ *and* $n = 1$; (Case UB) $F = \mathbf{Q}$, $r = 2$, *and* $x^*Tx = 0$ *for some* $x \in K^2$, $\neq 0$. In these cases Γ is commensurable with a "conjugate" of $SL_2(\mathbf{Z})$. The precise meaning of (5.10) will be explained in §5.6 below. In our later treatment, we will have to prove that certain functions are elements of $\mathcal{M}_\omega(\Gamma)$. In order to do so, we have to verify (5.10) in those exceptional cases. However, since the verification is always easy and we are mainly interested in the higher-dimensional case, we will not give the proof of the fact on each occasion, leaving the task to the reader.

If $\Gamma \subset G$, we can take $f\|_\omega\gamma$ instead of $f|_\omega\gamma$ in (5.9), but there is a natural example of Γ not contained in G, for which (5.9) is the right condition. For example, we can take $\Gamma = GL_2(\mathfrak{g}) \cap \widetilde{G}_+$ in the Hilbert modular case. In the present book, however, we consider almost exclusively Γ contained in G and also $f\|_\omega\gamma$ instead of $f|_\omega\gamma$. The only exception is Theorem 10.4, in which a group Γ not contained in G and the symbol $f|_\omega\gamma$ appear.

An element of $\mathcal{M}_\omega(\Gamma)$ is called a *(holomorphic) automorphic form of weight* ω (or, *of weight* k, if $\omega(x) = \det(x)^k$) *with respect to* Γ. An automorphic form is also called a *modular form* usually in Case SP.

5.3. Since it is often convenient not to specify Γ, we denote by $\mathcal{M}_\omega$ (resp. $\mathcal{M}_k$) the union of $\mathcal{M}_\omega(\Gamma)$ (resp. $\mathcal{M}_k(\Gamma)$) for all congruence subgroups Γ of G, and put

$$\mathcal{A}_\omega = \bigcup_e \{\, g^{-1}f \mid f \in \mathcal{M}_{\tau_e},\, 0 \neq g \in \mathcal{M}_e \,\}, \tag{5.11}$$

$$\mathcal{A}_\omega(\Gamma) = \{\, h \in \mathcal{A}_\omega \mid h\|_\omega \gamma = h \text{ for every } \gamma \in \Gamma \,\} \qquad (\Gamma \subset G), \tag{5.12}$$

where e runs over $\mathbf{Z}^\mathbf{b}$, and τ_e denotes the representation defined by $\tau_e(x) = \det(x)^e\omega(x)$. If $\omega(x) = \det(x)^k$, we denote these by $\mathcal{A}_k$ and $\mathcal{A}_k(\Gamma)$.

5.4. If $\alpha \in G_\mathbf{A}$ in Cases UT and UB, from (3.23) we obtain $j_{v\rho}(\alpha, z) = \det(\alpha)_v^{-1} j_v(\alpha, z)$ for every $v \in \mathbf{a}$. Therefore, for $k \in \mathbf{Z}^\mathbf{b}$ we have

$$j_\alpha(z)^k = \prod_{v \in \mathbf{a}} \det(\alpha)_v^{-k_{v\rho}} j_v(\alpha, z)^{k_{v\rho}+k_v} \quad \textit{if} \quad \alpha \in G_\mathbf{A}. \tag{5.13}$$

This means that $j_\alpha(z)^k = \det(\alpha)^p j_\alpha(z)^q$ with $p, q \in \mathbf{Z}^\mathbf{a}$, and so $f\|_k\alpha$ and $\mathcal{M}_k(\Gamma)$ can be defined in terms of p and q instead of k. (That is what we did in [S97, §10.4].) Also, by (4.34), $\det(\alpha) = 1$ if α belongs to a sufficiently small congruence subgroup. Therefore $\mathcal{M}_k = \mathcal{M}_l$ if $k_{v\rho} + k_v = l_{v\rho} + l_v$ for every $v \in \mathbf{a}$ such that the v-factor of $\mathcal{H}$ is nontrivial; in particular $\mathcal{M}_{\kappa\mathbf{b}} = \mathcal{M}_{2\kappa\mathbf{a}}$. In that sense our definition of $\mathcal{M}_k$ with k in $\mathbf{Z}^\mathbf{b}$ may look awkward, but we shall later see that this is a natural definition from an arithmetical viewpoint. However, since $\mathcal{M}_k = \mathcal{M}_q$ with some $q \in \mathbf{Z}^\mathbf{a}$, we can restrict e to $\mathbf{Z}^\mathbf{a}$ in (5.11).

5.5. An element of $\mathcal{A}_0(\Gamma)$ is a Γ-invariant meromorphic function on $\mathcal{H}$, and it is known that conversely every Γ-invariant meromorphic function on $\mathcal{H}$ belongs to $\mathcal{A}_0(\Gamma)$ if we exclude the exceptional cases mentioned in §5.2. An element of $\mathcal{A}_0(\Gamma)$ is called an *automorphic function with respect to* Γ or a *Γ-automorphic function* on $\mathcal{H}$.

Now, let (V, φ) be a model of $\Gamma\backslash\mathcal{H}$ in the sense of §4.10, and let $\mathbf{C}(V)$ be the field of all functions on V in the sense of algebraic geometry, as defined in §2.4. Then $\mathcal{A}_0(\Gamma)$ consists of the functions $g \circ \varphi$ for all $g \in \mathbf{C}(V)$. In this sense $\mathcal{A}_0(\Gamma)$ can be identified with $\mathbf{C}(V)$ if we identify $\Gamma\backslash\mathcal{H}$ with V.

5.6. Hereafter until the end of Section 8 we confine ourselves to Cases SP and UT; we shall return to Case UB in Sections 9 and 11. In Case UT we identify $(\mathbf{C}^n)^\mathbf{b}$ with $(\mathbf{C}^n \times \mathbf{C}^n)^\mathbf{a}$ through the map $(x_v)_{v\in\mathbf{b}} \mapsto (x_{v\rho}, x_v)_{v\in\mathbf{a}}$, where $x_v \in \mathbf{C}^n$. Notice that (5.5) becomes $\omega : GL_n(\mathbf{C})^\mathbf{b} \to GL(X)$ in Cases SP and UT. We now put

$$\mathbf{e}(c) = \exp(2\pi i c) \qquad (c \in \mathbf{C}), \tag{5.14}$$

$$\mathbf{e}_\mathbf{a}(x) = \exp\bigl(2\pi i \textstyle\sum_{v\in\mathbf{a}} x_v\bigr) \qquad (x \in \mathbf{C}^\mathbf{a}), \tag{5.15}$$

$$\mathbf{e}_\mathbf{a}^n(X) = \mathbf{e}_\mathbf{a}\bigl(\mathrm{tr}(X)\bigr) \qquad \bigl(X \in (\mathbf{C}_n^n)^\mathbf{a}\bigr), \tag{5.16}$$

$$S = S^n = \{\, \sigma \in K_n^n \mid \sigma^* = \sigma \,\}. \tag{5.17}$$

Observe that $\begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \in G$ for every $\sigma \in S$ and $\mathrm{diag}[a, \widehat{a}] \in G$ for every $a \in GL_n(K)$, and for a function $f : \mathcal{H} \to X$ we have

$$\left(f\|_\omega \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix}\right)(z) = f(z+\sigma), \tag{5.18}$$

$$(f\|_\omega \mathrm{diag}[a,\, \widehat{a}]) = \omega({}^t a,\, a^*) f(aza^*), \tag{5.19}$$

where we view $({}^t a,\, a^*)$ as an element of $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})$ according to our convention of §5.1. Given Γ as above, we can find a $\mathbf{Z}$-lattice M in S and a subgroup U of $GL_n(\mathfrak{r})$ of finite index such that $\begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \in \Gamma$ for every $\sigma \in M$ and $\mathrm{diag}[a,\, \widehat{a}] \in \Gamma$ for every $a \in U$. Thus, if $f \in \mathcal{M}_\omega(\Gamma)$, then

$$f(z+\sigma) = f(z) \quad \textit{for every} \quad \sigma \in M, \tag{5.20}$$

$$f(aza^*) = \omega({}^t a,\, a^*)^{-1} f(z) \quad \textit{for every} \quad a \in U. \tag{5.21}$$

Notice that $\mathrm{tr}(\sigma\sigma') \in F$ for $\sigma,\, \sigma' \in S$, and $(\sigma,\, \sigma') \mapsto \mathrm{Tr}_{F/\mathbf{Q}}\big(\mathrm{tr}(\sigma\sigma')\big)$ defines a nondegenerate pairing $S \times S \to \mathbf{Q}$. Let $L = \big\{\, h \in S \,\big|\, \mathrm{Tr}_{F/\mathbf{Q}}\big(\mathrm{tr}(hM)\big) \subset \mathbf{Z} \,\big\}$. Then (5.20) guarantees an expansion of the form

$$f(z) = \sum_{h\in L} c(h)\mathbf{e}_\mathbf{a}^n(hz) \tag{5.22a}$$

with $c(h) \in X$. (For the proof, see [S97, Lemma A1.4].) We shall often put

$$f(z) = \sum_{h\in S} c(h)\mathbf{e}_\mathbf{a}^n(hz) \tag{5.22b}$$

by defining $c(h)$ to be 0 for $h \in S,\, \notin L$. Usually we call the right-hand side of (5.22a) or (5.22b) *the Fourier expansion of* f, and call the $c(h)$ *the Fourier coefficients of* f. If $n = 1$ and $F = \mathbf{Q}$, we take $\omega(x) = \det(x)^k$ with $k \in \mathbf{Z}$. Then (5.10) means:

(5.23) *For every* $\alpha \in SL_2(\mathbf{Q})$ *we have*

$$(f\|_k\, \alpha)(z) = \textstyle\sum_{m=0}^{\infty} c_{\alpha m}\mathbf{e}(mz/N_\alpha)$$

with $c_{\alpha m} \in \mathbf{C}$ *and a positive integer* N_α.

In this situation we say that f *satisfies the cusp condition.* Now if $n > 1$ or $F \neq \mathbf{Q}$, no condition of this nature is necessary, because of the following fact:

5.7. Proposition. *Suppose* $n > 1$ *or* $F \neq \mathbf{Q}$; *let* f *be a holomorphic function on* $\mathcal{H}$ *of the form (5.22a) satisfying (5.21) with a subgroup* U *of* $GL_n(\mathfrak{r})$ *of finite index. Then* $c(h) \neq 0$ *only if* h_v *is nonnegative for every* $v \in \mathbf{a}$.

PROOF. First we observe that $f(x+iy) = \sum_{h\in L} c(h)\mathbf{e}_\mathbf{a}^n(ihy)\mathbf{e}_\mathbf{a}^n(hx)$, and hence

$$\mathbf{e}_\mathbf{a}^n(ihy)c(h) = A\int_{S_\mathbf{a}/M} f(x+iy)\mathbf{e}_\mathbf{a}^n(-hx)dx, \tag{5.24}$$

where $A = \mathrm{vol}(S_\mathbf{a}/M)^{-1}$. Taking $X = \mathbf{C}^t$ with some t, put $\|w\| = \big(\sum_{k=1}^t |w_k|^2\big)^{1/2}$ for $w \in X$, and put also $\|\alpha\| = \mathrm{Max}_{\|w\|=1}\|\alpha w\|$ for $\alpha \in \mathrm{End}(X,\, \mathbf{C})$. Taking $y_v = (2\pi)^{-1}1_n$ for every $v \in \mathbf{a}$ in (5.24), we obtain $\|c(h)\| \leq B\exp\big(\sum_{v\in\mathbf{a}}\mathrm{tr}(h)_v\big)$ with a constant B independent of h. Now from (5.21) we obtain $c(h) = \omega({}^t a,\, a^*)^{-1}c(a^*ha)$ for every $a \in U$, and hence

$$\|c(h)\| \leq B\|\omega({}^t a,\, a^*)^{-1}\|\exp\big(\textstyle\sum_{v\in\mathbf{a}}\mathrm{tr}(a^*ha)_v\big) \quad \text{for every } a \in U. \tag{*}$$

Now suppose $n > 1$; let h be an element of L such that h_u is not nonnegative for some $u \in \mathbf{a}$. We consider Case UT; Case SP can be handled with obvious modifications. We can find $x \in (\mathbf{C}^n)^\mathbf{a}$ such that

$$(^{**})\qquad (x_1x_1^\rho + x_2x_2^\rho)_u(x^*hx)_u < -\textstyle\sum_{u\neq v\in\mathbf{a}} \left|(x_1x_1^\rho + x_2x_2^\rho)_v(x^*hx)_v\right|,$$

where x_1 and x_2 are the first two components of x. (They are elements of $\mathbf{C}^\mathbf{a}$.) Since K is dense in $\mathbf{C}^\mathbf{a}$, we can take such an x in K^n. Multiplying it by a positive integer, we may assume that $x \in \mathfrak{r}^n$. Let $y = [-x_2\ x_1\ 0\ \cdots\ 0]$ and $b = xy$; here $y \in (\mathbf{C}_n^1)^\mathbf{a}$ and $b \in (\mathbf{C}_n^n)^\mathbf{a}$. Then $b \in \mathfrak{r}_n^n$ and $b^2 = 0$ since $yx = 0$. We have

$$(^{***})\qquad \mathrm{tr}(b^*hb) = \mathrm{tr}(y^*x^*hxy) = yy^*x^*hx = (x_1x_1^\rho + x_2x_2^\rho)(x^*hx).$$

Put $a = (1+b)^m$ with $0 < m \in \mathbf{Z}$. Since $1 + b \in SL_n(\mathfrak{r})$, we see that $a \in U$ if m is a multiple of some positive integer N. We have

$$\mathrm{tr}(a^*ha) = \mathrm{tr}\big((1+mb)^*h(1+mb)\big) = \mathrm{tr}(h) + m\cdot\mathrm{tr}\big(h(b+b^*)\big) + m^2\cdot\mathrm{tr}(b^*hb).$$

Thus $\sum_{v\in\mathbf{a}}\mathrm{tr}(a^*ha)_v = p + mq + m^2r$ with $p,\, q,\, r \in \mathbf{R}$, and $r = \sum_{v\in\mathbf{a}}(x_1x_1^\rho + x_2x_2^\rho)_v(x^*hx)_v < 0$ by (***) and (**). Now by (*),

$$\|c(h)\| \le B\|\omega(1+{}^tb,\, 1+b^*)^{-1}\|^m\exp(p + mq + m^2r)$$

for $0 < m \in N\mathbf{Z}$. Making m large, we find that $c(h) = 0$ as expected.

Next suppose $n = 1$ and $F \neq \mathbf{Q}$. In this case we have $S = F$ and we can take U to be a subgroup of $\mathfrak{g}^\times$ of finite index. Let h be an element of F such that $h_u < 0$ for some $u \in \mathbf{a}$. Since $F \neq \mathbf{Q}$, we can find an element $a \in U$ such that $|a_u| > 1$ and $|a_v| < 1$ for $u \neq v \in \mathbf{a}$. Now $\mathrm{tr}\big((a^m)^*ha^m\big) = ha^{2m}$ in this case, and therefore we obtain $\|c(h)\| = 0$ by taking a^m in place of a in (*) and making m large.

Another type of proof of Proposition 5.7 for the elements of $\mathcal{M}_k$, $k \in \mathbf{Z}^\mathbf{a}$, is given in [S97, Proposition A4.2]. See also [S78b, Proposition 3.1] and [S97, Proposition A4.5] for some results of the same nature in different settings.

5.8. Let $f \in \mathcal{M}_\omega(\Gamma)$ and $\alpha \in \widetilde{G}_+$. Then we easily see that $f\|_\omega\,\alpha \in \mathcal{M}_\omega(\alpha^{-1}\Gamma\alpha)$. Thus $\mathcal{M}_\omega$ is stable under the map $f \mapsto f\|_\omega\,\alpha$ for every $\alpha \in \widetilde{G}_+$. Now for $f \in \mathcal{M}_\omega(\Gamma)$ and $\alpha \in \widetilde{G}_+$ we have an expansion

$$(f\|_\omega\,\alpha)(z) = \sum_{h\in S} c_\alpha(h)\mathbf{e}_\mathbf{a}^n(hz). \tag{5.25}$$

By Proposition 5.7, $c_\alpha(h) \neq 0$ only if h_v is nonnegative for every $v \in \mathbf{a}$. We call f a *cusp form* if $c_\alpha(h) = 0$ for every $\alpha \in G$ and for every h such that $\det(h) = 0$, and denote by $\mathcal{S}_\omega(\Gamma)$ (resp. $\mathcal{S}_\omega$) the set of all cusp forms contained in $\mathcal{M}_\omega(\Gamma)$ (resp. $\mathcal{M}_\omega$). In view of Lemma 1.3 (3), if f is a cusp form, then $c_\alpha(h) = 0$ for every $\alpha \in \widetilde{G}_+$ and for every h such that $\det(h) = 0$. If $f \in \mathcal{S}_\omega(\Gamma)$ and $\alpha \in \widetilde{G}_+$, then $f\|_\omega\,\alpha \in \mathcal{S}_\omega(\alpha^{-1}\Gamma\alpha)$.

To consider the arithmeticity of modular forms, let us hereafter assume that $(X,\,\omega)$ has a $\mathbf{Q}$-structure in the sense that $X = X_0 \otimes_\mathbf{Q} \mathbf{C}$ with a fixed vector space X_0 over $\mathbf{Q}$, and ω is the natural extension of a rational representation $\omega_0 : GL_n(\mathbf{Q})^\mathbf{b} \to GL(X_0)$. (Often $X = \mathbf{C}^m$, $X_0 = \mathbf{Q}^m$, and ω_0 is a representation $GL_n(\mathbf{Q})^\mathbf{b} \to GL_m(\mathbf{Q})$.) Then, given a subfield D of $\mathbf{C}$, we say that f of (5.23) is *D-rational* if $c(h) \in X_0 \otimes_\mathbf{Q} D$ for all $h \in S$, and denote by $\mathcal{M}_\omega(D)$ the set of all D-rational elements of $\mathcal{M}_\omega$. Then we put

$$\mathcal{A}_\omega(D) = \bigcup_e \{\, p^{-1}q \mid q \in \mathcal{M}_{\tau_e}(D),\ 0 \neq p \in \mathcal{M}_e(D)\,\}, \tag{5.26a}$$

$$\mathcal{M}_\omega(\Gamma,\, D) = \mathcal{M}_\omega(\Gamma) \cap \mathcal{M}_\omega(D), \quad \mathcal{A}_\omega(\Gamma,\, D) = \mathcal{A}_\omega(\Gamma) \cap \mathcal{A}_\omega(D), \tag{5.26b}$$

$$\mathcal{S}_\omega(\Gamma,\, D) = \mathcal{S}_\omega(\Gamma) \cap \mathcal{M}_\omega(D), \quad \mathcal{S}_\omega(D) = \mathcal{S}_\omega \cap \mathcal{M}_\omega(D), \tag{5.26c}$$

where $\tau_e(x) = \det(x)^e\omega(x)$, $e \in \mathbf{Z}^{\mathbf{b}}$. We use the subscript k instead of ω (that is, we write $\mathcal{A}_k(D)$ and $\mathcal{S}_k$ for $\mathcal{A}_\omega(D)$ and $\mathcal{S}_\omega$, for example) if $\omega(x) = \det(x)^k$.

Clearly $\mathcal{A}_0(D)$ (resp. $\mathcal{A}_0(\Gamma, D)$) is a subfield of $\mathcal{A}_0$ (resp. $\mathcal{A}_0(\Gamma)$). Also $E\mathcal{A}_0(\Gamma, D) \subset \mathcal{A}_0(\Gamma, E)$ (resp. $E\mathcal{A}_0(D) \subset \mathcal{A}_0(E)$) if E is an extension of D. The equalities $E\mathcal{A}_0(\Gamma, D) = \mathcal{A}_0(\Gamma, E)$ and $E\mathcal{A}_0(D) = \mathcal{A}_0(E)$ are true in some cases, but they are not necessarily true in general. It should also be noted that $\mathcal{M}_k$ can be $\{0\}$ even if $k_v > 0$ for every $v \in \mathbf{a}$; $\mathcal{S}_k$ can be $\{0\}$ even if $\mathcal{M}_k \neq \{0\}$; see Proposition 6.16 below.

5.9. For $h \in S$ we write $0 \leq h$ or $h \geq 0$ if h_v is nonnegative for every $v \in \mathbf{a}$. By Proposition 5.7 the expansion of f in (5.22a) can be written

$$f(z) = \sum_{0 \leq h \in L} c(h)\mathbf{e}_{\mathbf{a}}^n(hz). \tag{5.27}$$

Let $\mathfrak{T}(L)$ denote the set of all formal series of the form (5.27) with $c(h) \in \mathbf{C}$, and $\mathfrak{T}$ the union of $\mathfrak{T}(L)$ for all $\mathbf{Z}$-lattices L in S. Then $\mathfrak{T}$ has a natural ring-structure. Put $[x] = \mathrm{Tr}_{F/\mathbf{Q}}(\mathrm{tr}(x))$ for $x \in S$ and let m be the dimension of S over $\mathbf{Q}$. Then we can find a $\mathbf{Q}$-basis $\{s_1, \ldots, s_m\}$ of S such that $[s_iL] \subset \mathbf{Z}$ and $s_{iv} > 0$ for every i and every $v \in \mathbf{a}$. Clearly $[s_ih] \geq 0$ if $0 \leq h \in L$. Taking m independent indeterminates $\xi_1, \ldots, \xi_m$, for $f \in \mathfrak{T}(L)$ as in (5.27) we put

$$\psi(f) = \sum_{0 \leq h \in L} c(h) \prod_{i=1}^{m} \xi_i^{[s_ih]}. \tag{5.28}$$

We easily see that ψ defines a ring-injection of $\mathfrak{T}(L)$ into the ring $\mathbf{C}[[\xi_1, \ldots, \xi_m]]$ of all formal power series in $\xi_1, \ldots, \xi_m$ with coefficients in $\mathbf{C}$. Therefore $\mathfrak{T}(L)$ is an integral domain, and the same is true for $\mathfrak{T}$. Given $\sigma \in \mathrm{Aut}(\mathbf{C})$, we obtain automorphisms of $\mathbf{C}[[\xi_1, \ldots, \xi_m]]$ and $\mathfrak{T}$ by applying σ to the coefficients; denote by f^σ the image of f under these automorphisms. This means that for $f \in \mathfrak{T}(L)$ as in (5.27) we have

$$f^\sigma = \sum_{0 \leq h \in L} c(h)^\sigma \mathbf{e}_{\mathbf{a}}^n(hz), \tag{5.29}$$

and clearly $\psi(f)^\sigma = \psi(f^\sigma)$. We can in fact define f^σ formally in the same manner even when $c(h) \in X$, since σ acts naturally on X. This action of σ can be extended to $\mathcal{A}_\omega$. Indeed, for $f = p^{-1}q \in \mathcal{A}_\omega$ with $0 \neq p \in \mathcal{M}_e$ and $q \in \mathcal{M}_{\tau_e}$ as in (5.26a) we put $f^\sigma = (p^\sigma)^{-1}q^\sigma$. This is a vector whose components belong to the field of quotients of $\mathfrak{T}$. Clearly this is well-defined. For the moment, f^σ is merely defined formally, and, in general, not defined as a function on $\mathcal{H}$. However, we shall later show that it is always meaningful as a function on $\mathcal{H}$.

Let us now prove

$$\mathcal{A}_\omega(D) \cap \mathcal{M}_\omega = \mathcal{M}_\omega(D). \tag{5.30}$$

If $f \in \mathcal{A}_\omega(D) \cap \mathcal{M}_\omega$, then $f = p^{-1}q$ with $0 \neq p \in \mathcal{M}_e(D)$ and $q \in \mathcal{M}_{\tau_e}(D)$. Then for $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$ we have $pf^\sigma = (pf)^\sigma = q^\sigma = q = pf$. Since p is not a zero-divisor, we obtain $f^\sigma = f$, that is, $f \in \mathcal{M}_\omega(D)$.

There are several natural, but highly nontrivial, questions concerning $\mathcal{M}_\omega$:

(Q1) *If $f \in \mathcal{M}_\omega$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, does the formal series for f^σ define an automorphic form? If so, what is its weight?*

(Q2) *Can we find an algebraic number field D such that $\mathcal{M}_\omega = \mathcal{M}_\omega(D) \otimes_D \mathbf{C}$? When can we take $D = \mathbf{Q}$?*

(Q3) *Given* $\alpha \in \widetilde{G}_+$ *and* $f \in \mathcal{M}_\omega$, *how is* $(f\|_\omega \alpha)^\sigma$ *related to* f^σ ? *Can we find* $\beta \in \widetilde{G}_+$ *and a weight* ψ *such that* $(f\|_\omega \alpha)^\sigma = f^\sigma\|_\psi \beta$?

We can ask similar questions on $\mathcal{S}_\omega$ and $\mathcal{A}_\omega$. We shall answer these questions in Sections 9 and 10. In connection with (Q2) we note here an easy lemma.

5.10. Lemma. (1) *If* D *is a subfield of* $\mathbf{C}$ *and* $f_1, \ldots, f_m$ *are elements of* $\mathcal{M}_\omega(D)$ *linearly independent over* D, *then they are linearly independent over* $\mathbf{C}$.
(2) *If* $\mathcal{M}_\omega(\Gamma, D)$ *spans* $\mathcal{M}_\omega(\Gamma)$ *over* $\mathbf{C}$, *then* $\mathcal{M}_\omega(\Gamma) = \mathcal{M}_\omega(\Gamma, D) \otimes_D \mathbf{C}$.

PROOF. Let $f_i(z) = \sum_h c_i(h)\mathbf{e}_\mathbf{a}^n(hz)$. Put $W_h = \{ x \in \mathbf{C}^m \mid \sum_{i=1}^m x_i c_i(h) = 0 \}$ for $h \in S$ and $Y = \bigcap_{h \in S} W_h$. Then each W_h, as well as Y, is a vector subspace of $\mathbf{C}^m$ defined over D. Since Y has no D-rational point other than 0, we have $Y = \{0\}$, which proves (1). Then (2) follows immediately from (1).

5.11. There is one phenomenon peculiar to Case UT. First put ${}^t z = ({}^t z_v)_{v \in \mathbf{a}}$ for $z = (z_v)_{v \in \mathbf{a}} \in \mathcal{H}$. Let $\alpha \in \widetilde{G}_+$. Then $\alpha^\rho \in \widetilde{G}_+$ and from (3.14), (3.15), and (3.16) we can easily derive that

$$\alpha^\rho({}^t z) = {}^t(\alpha z), \qquad \lambda(\alpha, z) = \mu(\alpha^\rho, {}^t z). \tag{5.31}$$

Given $\{\omega, X\}$ and $f : \mathcal{H} \to X$, define $f' : \mathcal{H} \to X$ by $f'(z) = f({}^t z)$. Then we easily see that

$$(f\|_\omega \alpha)' = f'\|_{\omega^\rho} \alpha^\rho, \tag{5.32}$$

where ω^ρ is defined by $\omega^\rho(a, b) = \omega(b, a)$. Therefore $f' \in \mathcal{M}_{\omega^\rho}$ if $f \in \mathcal{M}_\omega$.

Define also $\widetilde{f} = f^\sim : \mathcal{H} \to X$ by $\widetilde{f}(z) = \overline{f(-z^*)}$ and put $\alpha' = \varepsilon\alpha\varepsilon$ for $\alpha \in \widetilde{G}_+$ with $\varepsilon = \mathrm{diag}[1_r, -1_r]$. Then we can easily verify that

$$\alpha(-z^*) = -(\alpha' z)^*, \qquad \lambda(\alpha, -z^*) = \overline{\mu(\alpha', z)}, \tag{5.33}$$

$$(f\|_\omega \alpha)^\sim = \widetilde{f}\|_{\omega^\rho} \alpha', \tag{5.34}$$

provided ω is $\mathbf{R}$-rational, in which case $\widetilde{f} \in \mathcal{M}_{\omega^\rho}$ if $f \in \mathcal{M}_\omega$.

5.12. The measure $\mathbf{d}z$ of Lemma 3.4 in Case SP can be written

$$\mathbf{d}(x + iy) = \det(y)^{-n-1} dx\, dy \qquad \text{(Case SP)} \tag{5.35}$$

with $dx = \prod_{h \le k} dx_{hk}$ and $dy = \prod_{h \le k} dy_{hk}$ for real symmetric matrices x and y. In Case UT, for each fixed $v \in \mathbf{a}$ we have $\mathbf{C}_n^n = S_v \oplus iS_v$ with S of (5.17). Now S_v is the vector space of all hermitian matrices of size n. We can identify S_v with $\mathbf{R}^{n^2}$ through the map $x \mapsto \big(x_{hh}, \mathrm{Re}(x_{hk}), \mathrm{Im}(x_{hk})\ (h < k)\big)$, and define a measure dx on S_v by pulling back the standard measure on $\mathbf{R}^{n^2}$. Writing $z = (z_{hk}) \in \mathbf{C}_n^n$ in the form $z = x + iy$ with $x, y \in S_v$, we have a measure $dxdy$ on $\mathbf{C}_n^n$ with the measures dx and dy on S_v given as above. It should be noted that *this does not coincide* with the standard measure $\prod_{h,k}[(i/2)dz_{hk} \wedge d\overline{z}_{hk}]$ on $\mathbf{C}_n^n$. In fact, we easily see that

$$\prod_{h,k} \big[(i/2)dz_{hk} \wedge d\overline{z}_{hk}\big] = 2^{n(n-1)} dxdy. \tag{5.36}$$

Thus the measure $\mathbf{d}z$ of Lemma 3.4 in Case UT can be given by

$$\mathbf{d}(x + iy) = 2^{n(n-1)} \det(y)^{-2n} dxdy \qquad (x^* = x,\ y^* = y;\ \text{Case UT}). \tag{5.37}$$

6. Parametrization by theta functions

6.1. In this section we introduce certain theta functions, by which we parametrize our abelian varieties in Cases SP and UT. If the basic field is $\mathbf{Q}$, they are the classical θ and its modification φ given by

$$\theta(u,\, z;\, r,\, s) = \sum_{g - r \in \mathbf{Z}^n} \mathbf{e}\big(2^{-1} \cdot {}^t g z g + {}^t g(u + s)\big), \tag{6.1}$$

$$\varphi(u,\, z;\, r,\, s) = \mathbf{e}\big(2^{-1} \cdot {}^t u (z - \bar{z})^{-1} u\big)\theta(u,\, z;\, r,\, s). \tag{6.2}$$

Here $u \in \mathbf{C}^n$, $z \in \mathfrak{H}_n$, r, $s \in \mathbf{R}^n$, and $\mathbf{e}(c) = \exp(2\pi i c)$ as we set in (5.14). We are going to consider the pullbacks of these (with larger n) to $(\mathbf{C}^n)^{\mathbf{b}} \times \mathcal{H}$, which may be viewed also as generalizations of (6.1) and (6.2).

We need some new symbols. First, we denote by $\mathbf{Q}_{\mathrm{ab}}$ the maximal abelian extension of $\mathbf{Q}$ in $\mathbf{C}$. Next, given a finite-dimensional vector space W over $\mathbf{Q}$, we denote by $\mathcal{S}(W_{\mathbf{h}})$ the Schwartz-Bruhat space of $W_{\mathbf{h}}$ (see §1.6). Also, for a square matrix x of size $2n$ we denote by a_x, b_x, c_x, and d_x the $n \times n$-blocks of x in the sense of §1.8.

Let $[F : \mathbf{Q}] = e$. In this section we often identify $\mathbf{a}$ with $\{1, \ldots, e\}$ so that $X^{\mathbf{a}}$ may be written X^e for various symbols X. For example, $(\mathbf{C}^n)^{\mathbf{a}}$ can be identified with $\mathbf{C}^{en}$, and $\mathrm{diag}[a_v]_{v \in \mathbf{a}}$ with $\mathrm{diag}[a_1, \ldots, a_e]$.

6.2. Let $\{\beta_1, \ldots, \beta_e\}$ be a $\mathbf{Q}$-basis of F, and $\{\gamma_1, \ldots, \gamma_e\}$ be another $\mathbf{Q}$-basis of F determined by the condition $\mathrm{Tr}_{F/\mathbf{Q}}(\beta_i \gamma_j) = \delta_{ij}$. Define a $\mathbf{Q}$-linear map $g : \mathbf{Q}^1_{2en} \to F^1_{2n}$ by

$$g(x_1, \ldots, x_e, y_1, \ldots, y_e) = \left(\sum_{i=1}^{e} \beta_i x_i,\, \sum_{i=1}^{e} \gamma_i y_i\right) \qquad (x_i,\, y_i \in \mathbf{Q}^1_n). \tag{6.3}$$

A simple calculation shows that

$$\mathrm{Tr}_{F/\mathbf{Q}}\big(g(u)\eta_n \cdot {}^t g(u')\big) = u\eta_{en} \cdot {}^t u' \tag{6.4}$$

for u, $u' \in \mathbf{Q}^1_{2en}$. Denoting the j-th conjugate of β_i by β_{ij}, put

$$B = \begin{bmatrix} \beta_{11} 1_n & \cdots & \beta_{1e} 1_n \\ \cdots & \cdots & \cdots \\ \beta_{e1} 1_n & \cdots & \beta_{ee} 1_n \end{bmatrix}, \tag{6.5}$$

$$\psi(a) = \mathrm{diag}[a_v]_{v \in \mathbf{a}} = \mathrm{diag}[a_1, \ldots, a_e] \qquad \big((a_v)_{v \in \mathbf{a}} \in (\mathbf{C}^n_n)^{\mathbf{a}} = (\mathbf{C}^n_n)^e\big), \tag{6.6}$$

$$\omega(\alpha) = \begin{bmatrix} B & 0 \\ 0 & {}^t B^{-1} \end{bmatrix} \begin{bmatrix} \psi(a_\alpha) & \psi(b_\alpha) \\ \psi(c_\alpha) & \psi(d_\alpha) \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & {}^t B \end{bmatrix} \qquad \big(\alpha \in F^{2n}_{2n}\big). \tag{6.7}$$

Then we can easily verify that $\omega(\alpha) \in \mathbf{Q}^{2en}_{2en}$ and

$$g\big(u\omega(\alpha)\big) = g(u)\alpha \qquad (u \in \mathbf{Q}^1_{2en},\ \alpha \in F^{2n}_{2n}). \tag{6.8}$$

Let us now define a subgroup $G_0 = G_0(F, \eta_n)$ of $Gp(n, F)$ by

$$G_0 = G_0(F,\, \eta_n) = \big\{\, \alpha \in Gp(n,\, F) \,\big|\, \nu(\alpha) \in \mathbf{Q} \,\big\}. \tag{6.9}$$

This is the same as the group of (3.29). In view of (6.4) we can show that ω defines an injective homomorphism of $G_0(F, \eta_n)$ into $Gp(en, \mathbf{Q})$, and also an injection of $Sp(n, F)$ into $Sp(en, \mathbf{Q})$. Define also an embedding $\varepsilon : \mathfrak{H}^{\mathbf{a}}_n \to \mathfrak{H}_{en}$ by

$$\varepsilon(z) = B \cdot \mathrm{diag}[z_v]_{v \in \mathbf{a}} \cdot {}^t B \qquad \big(z = (z_v)_{v \in \mathbf{a}} \in \mathfrak{H}^{\mathbf{a}}_n\big). \tag{6.10}$$

Then we can easily verify that

$$\omega(\alpha)\varepsilon(z) = \varepsilon(\alpha z) \qquad (\alpha \in G_0(F, \eta_n),\ \nu(\alpha) > 0), \tag{6.11a}$$

$$\mu\big(\omega(\alpha), \varepsilon(z)\big) = {}^tB^{-1}\mathrm{diag}[\mu_v(\alpha, z)]_{v\in\mathbf{a}} \cdot {}^tB. \tag{6.11b}$$

6.3. Let us describe the above embedding ε in terms of the families of abelian varieties established in §4.7. We consider two families

$$\mathcal{F}(\Omega_0) = \{\, \mathcal{P}_z \mid z \in \mathfrak{H}_{en} \,\}, \qquad \mathcal{P}_z = \big(A_z, \mathcal{C}_z, \iota_z;\ \{t_i(z)\}_{i=1}^s\big),$$
$$\mathcal{F}(\Omega_F) = \{\, \mathcal{P}_z \mid z \in \mathfrak{H}_n^{\mathbf{a}} \,\}, \qquad \mathcal{P}_z = \big(A_z, \mathcal{C}_z, \iota_z;\ \{t_i(z)\}_{i=1}^s\big),$$
$$\Omega_0 = \{\, \mathbf{Q}, \text{id.}, L_0, \eta_{en}, \{u_i\}_{i=1}^s \,\},$$
$$\Omega_F = \{\, F, \Psi, L_F, \eta_n, \{g(u_i)\}_{i=1}^s \,\}.$$

Here L_0 resp. L_F is a $\mathbf{Z}$-lattice in $\mathbf{Q}^1_{2en}$ resp. F^1_n. By (4.15) we have to assume that $\mathrm{Tr}_{F/\mathbf{Q}}(x\eta_n \cdot {}^ty) \in \mathbf{Z}$ for every $x, y \in L_F$. We take $L_0 = g^{-1}(L_F)$. By (6.4), $u\eta_{en} \cdot {}^tu' \in \mathbf{Z}$ for every $u, u' \in L_0$. Thus Ω_0 is meaningful for this L_0. Now recall that for each $z \in \mathfrak{H}_n^{\mathbf{a}}$ the variety A_z can be given by $(\mathbf{C}^n)^{\mathbf{a}}/p_z(L_F)$ with the map p_z of (4.24). Disregarding the endomorphism algebra $\iota_z(F)$, we observe that $\mathcal{P}_z$ is of type Ω_0, and so isomorphic to $\mathcal{P}_w$ with some $w \in \mathfrak{H}_{en}$. We obtain such a w by checking the period matrix $X(p_z \circ g)$ of (4.18). In fact, a simple calculation shows that the upper half of $X(p_z \circ g)$ is of the form

$$\begin{bmatrix} \beta_{11}z_1 & \cdots & \beta_{e1}z_1 & \gamma_{11}1_n & \cdots & \gamma_{e1}1_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \beta_{1e}z_e & \cdots & \beta_{ee}z_e & \gamma_{1e}1_n & \cdots & \gamma_{ee}1_n \end{bmatrix}.$$

This equals $B^{-1}\,[\,\varepsilon(z) \quad 1_{en}\,]$. Since $p_z\big(g(x)\big)$ equals this matrix times ${}^tx$ for $x \in \mathbf{Q}^1_{2en}$, we obtain

$$B \cdot p_z\big(g(x)\big) = p_{\varepsilon(z)}(x) \qquad (x \in \mathbf{Q}^1_{2en}). \tag{6.12}$$

Therefore the map $u \mapsto Bu$ for $u \in (\mathbf{C}^n)^{\mathbf{a}}$ gives an isomorphism of $(\mathbf{C}^n)^{\mathbf{a}}/p_z(L_F)$ onto $\mathbf{C}^{en}/p_{\varepsilon(z)}(L_0)$, or rather, an isomorphism of $\mathcal{P}_z$ (minus ι_z) to $\mathcal{P}_{\varepsilon(z)}$ because of (6.4).

6.4. We now define theta functions θ_F and φ_F by

$$\theta_F(u, z; r, s) = \sum_{h-r\in M} \mathbf{e_a}\big(2^{-1} \cdot {}^thzh + {}^th(u+s)\big), \tag{6.13}$$

$$\varphi_F(u, z; r, s) = \mathbf{e_a}\big(2^{-1} \cdot {}^tu(z-\overline{z})^{-1}u\big)\theta_F(u, z; r, s). \tag{6.14}$$

Here $u \in (\mathbf{C}^n)^{\mathbf{a}}$, $z \in \mathfrak{H}_n^{\mathbf{a}}$, $r, s \in (\mathbf{R}^n)^{\mathbf{a}}$, and $\mathbf{e_a}(x) = \exp\big(\sum_{v\in\mathbf{a}} x_v\big)$ for $x \in \mathbf{C}^{\mathbf{a}}$ as we set in (5.15); M will be specified after (6.15). If $F = \mathbf{Q}$ and $M = \mathbf{Z}^n$, then these coincide with the functions of (6.1) and (6.2). By an easy calculation we can verify that these are the pullbacks of (6.1) and (6.2) in the sense that

$$\varphi_F\big(u, z; g_1(r), g_2(s)\big) = \varphi\big(Bu, \varepsilon(z); r, s\big) \tag{6.15}$$
$$\big(u \in (\mathbf{C}^n)^{\mathbf{a}},\ z \in \mathfrak{H}_n^{\mathbf{a}},\ r, s \in \mathbf{Q}^{2en}\big),$$

where $g_1(q) = \sum_{i=1}^e \beta_i q_i$ and $g_2(q) = \sum_{i=1}^e \gamma_i q_i$ for ${}^tq = [{}^tq_1 \ \cdots \ {}^tq_e]$ with $q_i \in \mathbf{Q}^n$, and the same type of equality holds with θ in place of φ. Now $M = g_1(\mathbf{Z}^{en})$. It is not difficult to show that the right-hand side of (6.13) is locally uniformly convergent on $(\mathbf{C}^n)^{\mathbf{a}} \times (\mathfrak{H}_n)^{\mathbf{a}}$, and so defines a holomorphic function in (u, z).

For our later purposes it is convenient to consider functions of the following types:

$$\theta_F(u, z; \lambda) = \sum_{h \in F^n} \lambda(h)\mathbf{e}_{\mathbf{a}}\big(2^{-1} \cdot {}^t hzh + {}^t hu\big), \tag{6.16}$$

$$\varphi_F(u, z; \lambda) = \mathbf{e}_{\mathbf{a}}\big(2^{-1} \cdot {}^t u(z - \overline{z})^{-1}u\big)\theta_F(u, z; \lambda). \tag{6.17}$$

Here $\lambda \in \mathcal{S}(F_{\mathbf{h}}^n)$. Clearly (6.13) (resp. (6.14)) is a special case of (6.16) (resp. (6.17)). As we said in §1.6, we view λ as a function on $F_{\mathbf{A}}^n$, so that $\lambda(h)$ for $h \in F^n$ is meaningful.

6.5. Let K be a totally imaginary quadratic extension of F as in §3.5. Take $\zeta \in K$ so that $\zeta^\rho = -\zeta$ and $K = F(\zeta)$. Define an F-linear map $h : F_{4n}^1 \longrightarrow K_{2n}^1$ by

$$h(x_1, x_2, y_1, y_2) = (x_1 - \zeta x_2, 2^{-1}y_1 + (2\zeta)^{-1}y_2) \qquad (x_i, y_i \in F_n^1). \tag{6.18}$$

Then we have $\mathrm{Tr}_{K/F}\big(h(u)\eta_n h(u')^*\big) = u\eta_{2n} \cdot {}^t u'$ for $u, u' \in F_{4n}^1$.

We can now define embeddings $\tau : K_{2n}^{2n} \longrightarrow F_{4n}^{4n}$ and $\psi : \mathcal{H}_n^{\mathbf{a}} \longrightarrow \mathfrak{H}_{2n}^{\mathbf{a}}$ as follows:

$$A = \begin{bmatrix} 1_n & 1_n \\ \zeta 1_n & -\zeta 1_n \end{bmatrix}, \qquad \sigma(a) = \mathrm{diag}[a^\rho, a] \qquad (a \in K_n^n), \tag{6.19}$$

$$\tau(\alpha) = \begin{bmatrix} A & 0 \\ 0 & \widehat{A} \end{bmatrix} \begin{bmatrix} \sigma(a_\alpha) & \sigma(b_\alpha) \\ \sigma(c_\alpha) & \sigma(d_\alpha) \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & A^* \end{bmatrix} \qquad \big(\alpha \in K_{2n}^{2n}\big), \tag{6.20}$$

$$\psi(w) = (A_v \cdot \mathrm{diag}[{}^t w_v, w_v]A_v^*)_{v \in \mathbf{a}} \qquad \big(w = (w_v)_{v \in \mathbf{a}} \in \mathcal{H}_n^{\mathbf{a}}\big). \tag{6.21}$$

Then we can easily verify that τ defines an injective homomorphism of $GU(\eta_n)$ into $Gp(n, F)$ and

$$h\big(u\tau(\alpha)\big) = h(u)\alpha \qquad (u \in F_{4n}^1, \ \alpha \in F_{4n}^{4n}), \tag{6.22a}$$

$$\tau(\alpha)\psi(w) = \psi(\alpha w) \qquad \big(\alpha \in GU(\eta_n), \ 0 \ll \nu(\alpha) \in F\big), \tag{6.22b}$$

$$\mu_v\big(\tau(\alpha), \psi(w)\big) = \widehat{A}_v \mathrm{diag}[\lambda_v(\alpha, w), \mu_v(\alpha, w)]A_v^*. \tag{6.22c}$$

Taking a $\mathbf{Z}$-lattice L_K in K_{2n}^1 such that $\mathrm{Tr}_{K/\mathbf{Q}}(x\eta_n y^*) \in \mathbf{Z}$ for every $x, y \in L_K$, we consider a family of polarized abelian varieties in Case UT defined in §4.7:

$$\mathcal{F}(\Omega_K) = \{\, \mathcal{P}_w \mid w \in \mathcal{H}_n^{\mathbf{a}} \,\}, \qquad \mathcal{P}_w = \big(A_w, \mathcal{C}_w, \iota_w; \{t_i(w)\}_{i=1}^s\big), \tag{6.23a}$$

$$\Omega_K = \{\, K, \Psi, L_K, \eta_n, \{h(u_i)\}_{i=1}^s \,\}. \tag{6.23b}$$

Take the map p_z of (4.24) for $z \in \mathfrak{H}_{2n}^{\mathbf{a}}$ and similarly p_w for $w \in \mathcal{H}_n^{\mathbf{a}}$. Then a simple calculation shows that the upper half of $X_v(p_w \circ h)$ is of the form

$$\begin{bmatrix} w & \zeta w & 2^{-1}1_n & -(2\zeta)^{-1}1_n \\ {}^t w & -\zeta \cdot {}^t w & 2^{-1}1_n & (2\zeta)^{-1}1_n \end{bmatrix}_v. \tag{6.24a}$$

This equals $\overline{A}_v^{-1}\,[\,\psi(w)_v \quad 1_{2n}\,]$. We also see that

$$\overline{A} \cdot p_w\big(h(a)\big) = p_{\psi(w)}(a) \qquad (a \in F_{4n}^1), \tag{6.24b}$$

where $\overline{A} = \mathrm{diag}\,\big[\overline{A}_v\big]_{v \in \mathbf{a}}$. Define a $\mathbf{Z}$-lattice L_F in F_{4n}^1 by $L_F = h^{-1}(L_K)$. Then the map $u \mapsto \overline{A}u$ for $u \in (\mathbf{C}^{2n})^{\mathbf{a}}$ gives an isomorphism of $(\mathbf{C}^{2n})^{\mathbf{a}}/p_w(L_K)$ onto $(\mathbf{C}^{2n})^{\mathbf{a}}/p_{\psi(w)}(L_F)$, or rather, an isomorphism of $\mathcal{P}_w$ to $\mathcal{P}_{\psi(w)}$ if we restrict ι_w to F.

6.6. In Case UT we identify $(\mathbf{C}^n)^{\mathbf{b}}$ with $(\mathbf{C}^n)^{\mathbf{a}} \times (\mathbf{C}^n)^{\mathbf{a}}$ via the map $(x_v)_{v \in \mathbf{b}} \mapsto (x_{v\rho})_{v \in \mathbf{a}} \times (x_v)_{v \in \mathbf{a}}$. We now define theta functions θ_K and φ_K by

$$\theta_K(u,\, w;\, \ell) = \sum_{g\in K^n} \ell(g)\mathbf{e}_{\mathbf{a}}\big({}^tgw\bar{g} + {}^tgx + g^*y\big), \tag{6.25}$$

$$\varphi_K(u,\, w;\, \ell) = \mathbf{e}_{\mathbf{a}}\big({}^ty(w-w^*)^{-1}x\big)\theta_K(u;\, w;\, \ell), \tag{6.26}$$

$$\left(u = \begin{bmatrix} x \\ y \end{bmatrix} \in (\mathbf{C}^n)^{\mathbf{b}},\ w \in \mathcal{H}_n^{\mathbf{a}},\ \ell \in \mathcal{S}(K_{\mathbf{h}}^n)\right).$$

By an easy calculation we can verify that

$$\varphi_K\big(u,\, w;\, \ell\big) = \varphi_F\left(\overline{A}u,\, \psi(w);\, \ell\circ p\right) \qquad (\ell \in \mathcal{S}(K_{\mathbf{h}}^n)), \tag{6.27}$$

where $p : F^{2n} \to K^n$ is defined by $p\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = a - \zeta b$ for $a,\, b \in F^{2n}$, and the same type of equality holds with θ in place of φ.

6.7. In Case SP we have $G = Sp(n,\, F)$. We define, in this case, open subgroups C^0 and C^θ of $G_{\mathbf{A}}$ and arithmetic subgroups Γ^0 and Γ^θ of G by

$$C^0 = \left\{\, \xi \in G_{\mathbf{A}} \,\middle|\, a_\xi \prec \mathfrak{g},\, b_\xi \prec \mathfrak{d}^{-1},\, c_\xi \prec \mathfrak{d},\, d_\xi \prec \mathfrak{g} \,\right\}, \tag{6.28}$$

$$C^\theta = \big\{\, \xi \in C^0 \mid (a_\xi \cdot {}^tb_\xi)_{ii} \prec 2\mathfrak{d}^{-1} \ \ \textit{and} \ \ (c_\xi \cdot {}^td_\xi)_{ii} \prec 2\mathfrak{d} \ \ \textit{for } 1 \le i \le n\big\}, \tag{6.29}$$

$$\Gamma^0 = G \cap C^0, \qquad \Gamma^\theta = G \cap C^\theta, \tag{6.30}$$

where $\mathfrak{d}$ is the different of F relative to $\mathbf{Q}$. Notice that $C^0 = G_{\mathbf{A}} \cap C[\mathfrak{d}^{-1},\, \mathfrak{d}]$ with $C[\ ,\ ]$ defined by (1.17) with $m = n$, and so C^0 is a subgroup of $G_{\mathbf{A}}$. That C^θ is indeed a subgroup will be shown in §A2.3. We are going to define a factor of automorphy of weight 1/2 in Case SP. To make our formulas short, for $\alpha \in \widetilde{G}_+$ and $\kappa \in \mathbf{Z}$ we put

$$j_\alpha(z)^{\kappa\mathbf{a}} = \textstyle\prod_{v\in\mathbf{a}} j_v(\alpha,\, z)^\kappa \qquad \text{(Cases SP and UT)} \tag{6.31}$$

with j_v of (5.3), in accordance with (5.4b). It should be noted that we use $\mathbf{a}$, not $\mathbf{b}$; naturally we write this $j_\alpha(z)^{\mathbf{a}}$ if $\kappa = 1$.

6.8. Theorem (Case SP). *There is a holomorphic function $h_\alpha(z)$, written also $h(\alpha,\, z)$, on $\mathfrak{H}_n^{\mathbf{a}}$ defined for each $\alpha \in \Gamma^\theta$ with the following properties:*

(1) $h_\alpha(z)^2 = \zeta_\alpha j_\alpha(z)^{\mathbf{a}}$ *with a root of unity* ζ_α.
(2) $h_{\alpha\beta}(z) = h_\alpha(\beta z)h_\beta(z)$ *for every α and β in Γ^θ.*
(3) $h_\gamma(z) = 1$ *if* $c_\gamma = 0$.
(4) *Given $\lambda \in \mathcal{S}(F_{\mathbf{h}}^n)$, there is an open subgroup D_λ of C^θ such that*

$$\varphi_F\big({}^t\mu_\alpha(z)^{-1}u,\, \alpha z;\, \lambda\big) = h_\alpha(z)\varphi_F(u,\, z;\, \lambda') \tag{6.32}$$

with $\lambda'(x) = \lambda(dx)$ if $\alpha \in G \cap \big(D_\lambda \mathrm{diag}[{}^td^{-1},\, d]\big)$ with $d \in \prod_{v\in\mathbf{h}} GL_n(\mathfrak{g}_v)$, where μ_α is defined by (5.1). In particular, we can take $D_\lambda = C^\theta$ if λ is the characteristic function of $\mathfrak{g}^n$.

(5) *Let ε be the Hecke character of F corresponding to the quadratic extension $F\big(\sqrt{-1}\big)/F$ and let $\mathfrak{d}$ be the different of F relative to $\mathbf{Q}$. If $\gamma \in \Gamma^0$, $b_\gamma \prec 2\mathfrak{d}^{-1}$, and $c_\gamma \prec 2\mathfrak{d}$, then $\det(d_\gamma)$ is prime to 2, and*

$$h_\gamma(z)^2 = \textstyle\prod_{v|2} \varepsilon_v\big(\det(d_\gamma)\big) j_\gamma(z)^{\mathbf{a}}.$$

This theorem, as well as the following one, will be proven in §A2.9. The function h_γ may be called the *factor of automorphy of weight* $\mathbf{a}/2$ (or simply, of weight 1/2).

For the moment, it is defined only for $\gamma \in \Gamma^\theta$. We cannot define such for every $\gamma \in Sp(n, F)$ consistently so that property (2) above holds in general. Howevere, we can define h_γ for γ in a certain set larger than Γ^θ, as will be shown in Theorem A2.4 in the Appendix.

6.9. Theorem (Case SP). (1) *Given $\alpha \in \widetilde{G}_+$, let $r(z)$ be a holomorphic function on $\mathfrak{H}_n^{\mathbf{a}}$ such that $r(z)^2 = \zeta \cdot j_\alpha(z)^{\mathbf{a}}$ with a constant $\zeta \in \mathbf{C}^\times$. Then there is a congruence subgroup Δ of Γ^θ depending on (α, r) such that $\alpha\Delta\alpha^{-1} \subset \Gamma^\theta$ and*

$$h(\alpha\gamma\alpha^{-1}, \alpha z)r(z) = r(\gamma z)h_\gamma(z) \quad \textit{for every} \quad \gamma \in \Delta.$$

(2) *Given α and $r(z)$ as in (1), suppose $\alpha \in G$. Then for every $\lambda \in \mathcal{S}(F_{\mathbf{h}}^n)$ we have*

$$\varphi_F\big({}^t\mu_\alpha(z)^{-1}u,\ \alpha z;\ \lambda\big) = r(z)\varphi_F(u,\ z;\ \lambda')$$

with an element $\lambda' \in \mathcal{S}(F_{\mathbf{h}}^n)$, which is determined by α, r, and λ. In particular, if $\alpha = \eta_n$ and $r(z) = \prod_{v\in\mathbf{a}} \det(-iz_v)^{1/2}$, then λ' is given by

$$\lambda'(x) = |D_F|^{-n/2} \int_{\mathbf{F}_{\mathbf{h}}^n} \lambda(y)\mathbf{e}_{\mathbf{h}}({}^t xy)dy. \tag{6.33}$$

Here the branch of r is chosen so that $r(z) > 0$ if $\mathrm{Re}(z) = 0$, D_F is the discriminant of F, $\mathbf{e}_{\mathbf{h}}$ is the character of $F_{\mathbf{h}}$ defined in §1.6, and dy is the Haar measure such that the measure of $\prod_{v\in\mathbf{h}} \mathfrak{g}_v^n$ is 1.

Put $\Gamma_\lambda = G \cap D_\lambda$ with D_λ of Theorem 6.8 (4). Taking $d = 1_n$ in Theorem 6.8 (4), we obtain

$$\varphi_F\big({}^t\mu_\gamma(z)^{-1}u,\ \gamma z;\ \lambda\big) = h_\gamma(z)\varphi_F(u,\ z;\ \lambda) \quad \textit{for every} \quad \gamma \in \Gamma_\lambda. \tag{6.34}$$

Now let $s \in F$, $\gg 0$; take $\alpha = \mathrm{diag}[s1_n,\ 1_n]$ and $r(z) = 1$ in Theorem 6.9 (1); put $\beta = \alpha\gamma\alpha^{-1}$. Then $\mu_\beta(sz) = \mu_\gamma(z)$ and $h_\beta(sz) = h_\gamma(z)$ for γ in a suitable Δ. Therefore from (6.34) with β in place of γ, we obtain

$$\varphi_F\big({}^t\mu_\gamma(z)^{-1}u,\ s \cdot (\gamma z);\ \lambda\big) = h_\gamma(z)\varphi_F(u,\ sz;\ \lambda) \quad \textit{for every} \quad \gamma \in \Gamma \tag{6.35}$$

with a congruence subgroup Γ of G depending on λ and s.

6.10. Let us now define, in Case SP, modular forms of half-integral weight. First, by an *integral weight* we mean an element of $\mathbf{Z}^{\mathbf{a}}$. By a *half-integral weight* we mean an element $k = (k_v)_{v\in\mathbf{a}}$ of $2^{-1}\mathbf{Z}^{\mathbf{a}}$ such that $k_v = m_v + (1/2)$ with $m_v \in \mathbf{Z}$ for every $v \in \mathbf{a}$. Given such a k, $\alpha \in \Gamma^\theta$, and a $\mathbf{C}$-valued function f on $\mathfrak{H}_n^{\mathbf{a}}$, we define a $\mathbf{C}$-valued function $f\|_k\alpha$ on $\mathfrak{H}_n^{\mathbf{a}}$ by

$$(f\|_k\alpha)(z) = h_\alpha(z)^{-1}j_\alpha(z)^{-m}f(\alpha z). \tag{6.36}$$

For a congruence subgroup Γ of G contained in Γ^θ we denote by $\mathcal{M}_k(\Gamma)$ the set of all holomorphic functions f on $\mathfrak{H}_n^{\mathbf{a}}$ which satisfy $f\|_k\gamma = f$ for every $\gamma \in \Gamma$ (and also the cusp condition if $n = 1$ and $F = \mathbf{Q}$). We then denote by $\mathcal{M}_k$ the union of $\mathcal{M}_k(\Gamma)$ for all such Γ, and define $\mathcal{A}_k$ and $\mathcal{A}_k(\Gamma)$ in the same manner as in §5.3. If $f \in \mathcal{M}_k$, by Theorem 6.8 (3) we have $f(z + \sigma) = f(z)$ for every σ in a suitable lattice in S and $f(ax \cdot {}^ta) = \det(a)^{-m}f(z)$ for every a in a suitable subgroup of $GL_n(\mathfrak{g})$ of finite index. Thus Proposition 5.7 is applicable to f, and hence f has an expansion of type (5.27). Now, given a subfield D of $\mathbf{C}$, we can define $\mathcal{M}_k(D)$, $\mathcal{A}_k(D)$, $\mathcal{M}_k(\Gamma,\ D)$, and $\mathcal{A}_k(\Gamma,\ D)$ in the same manner as in §5.8. Notice that (5.30) is true for the present $\mathcal{M}_k(D)$ too.

Suppose $k \in \mathbf{Z}^{\mathbf{a}}$ and $k_v = \kappa$ for every $v \in \mathbf{a}$ with $\kappa \in 2^{-1}\mathbf{Z}$ in Case SP and $\kappa \in \mathbf{Z}$ in Case UT. (This means that $j_\alpha(z)^k = j_\alpha(z)^{\kappa\mathbf{a}}$ if $\kappa \in \mathbf{Z}$; also $k_v = 0$ for $v \in \mathbf{b}, \notin \mathbf{a}$ in Case UT.) We then employ $\kappa\mathbf{a}$ instead of the subscript k (that is, we write $f\|_{\kappa\mathbf{a}}\alpha$ and $\mathcal{M}_{\kappa\mathbf{a}}(D)$ for $f\|_k\alpha$ and $\mathcal{M}_k(D)$, for example).

In Case UT we can formulate similar results in a more clear-cut way. Indeed we have:

6.11. Theorem (Case UT). *Every element α of G gives a $\mathbf{C}$-linear automorphism of $\mathcal{S}(K_\mathbf{h}^n)$, written $\ell \mapsto \ell^\alpha$ for $\ell \in \mathcal{S}(K_\mathbf{h}^n)$, with the following properties:*

(1) *$\varphi_K\big({}^tM_\alpha(z)^{-1}u,\, \alpha z;\, \ell\big) = j_\alpha(z)^{\mathbf{a}}\varphi_K(u,\, z;\, \ell^\alpha)$, where M_α is defined by (5.1) and $j_\alpha(z)^{\mathbf{a}}$ by (6.31).*

(2) *$\ell^{\alpha\beta} = (\ell^\alpha)^\beta$.*

(3) *For every $\ell \in \mathcal{S}(K_\mathbf{h}^n)$ there exists a congruence subgroup Γ_ℓ of G such that $\ell^\gamma = \ell$ for every $\gamma \in \Gamma_\ell$.*

(4) *In particular, we have*

$$(\ell^\eta)(x) = (-i)^{n[F:\mathbf{Q}]}|D_K|^{-n/2}\int_{\mathbf{K}_\mathbf{h}^n} \ell(y)\mathbf{e}_\mathbf{h}\big(\mathrm{Tr}_{K/F}(y^*x)\big)dy,$$

where D_K is the discriminant of K, $\mathbf{e}_\mathbf{h}$ is the character of $F_\mathbf{h}$ defined in §1.6, and dy is the Haar measure such that the measure of $\prod_{v\in\mathbf{h}} \mathfrak{r}_v^n$ is 1.

PROOF. This is essentially a special case of [S97, Theorem A7.4]. Indeed, define $f(z;\, u,\, u';\, \ell)$ of [S97, (A7.3.2)] with $q = 1$ and $H = Q = A = 1$. Then we can easily verify that

$$\varphi_K\left(\begin{bmatrix} x \\ y \end{bmatrix},\, z;\, \ell\right) = f(2z;\, y,\, x;\, \ell). \tag{6.37}$$

In [S97, Theorem A7.4] we proved the results of type (1, 2, 3) of our theorem for the function f in a stronger form for the elements of the group $G_1 = G \cap SL_{2n}(K)$. Given $\alpha \in G_1$, put $\beta = \xi^{-1}\alpha^{-1}\xi$ with $\xi = \mathrm{diag}[1_n,\, 2\cdot 1_n]$; write ℓ^α for the symbol ${}^\beta\ell$ defined in that theorem. Then we obtain the present theorem at least for the elements of G_1. Now G is generated by G_1 and the elements of the form $\mathrm{diag}[a,\, \widehat{a}]$ with $a \in GL_n(K)$. Therefore we can easily extend the results to G. As for (4), if $\alpha = \eta$, then $\beta = \tau\eta$ with $\tau = \mathrm{diag}[-2,\, -2^{-1}]$, so that $\ell^\eta = {}^\beta\ell = {}^\tau({}^\eta\ell)$. Combining (5) and (6) of [S97, Theorem A7.4], we obtain ℓ^η as stated in (4).

Notice that in Case UT the facts corresponding to Theorem 6.9 (1) and (6.35) (with $j_\gamma^{\mathbf{a}}$ in place of h_γ) are trivial. Also, we shall later give Theorem A5.4 which essentially includes Theorem 6.11, as well as [S97, Theorem A7.4].

6.12. Theorem. *Let $\mathcal{F}(\Omega_K)$ be defined in Cases SP and UT as in §§6.3 and 6.5. Then there exist a finite set Λ of $\mathbf{Z}$-valued elements of $\mathcal{S}(K_\mathbf{h}^n)$, a positive integer p, and a congruence subgroup Γ_0 of G with the following properties, in which it is understood that $K = F$ in Case SP:*

(1) *The quotient $\theta_K(u,\, p^{-1}z;\, \lambda)/\theta_K(u,\, p^{-1}z;\, \lambda')$ is invariant under $u \mapsto u + \ell$ for every $\lambda,\, \lambda' \in \Lambda$ and every $\ell \in p_z\big(L_K\big)$.*

(2) *For every $(u_0,\, z_0) \in (\mathbf{C}^n)^{\mathbf{b}} \times \mathcal{H}$ there exists an element λ of Λ such that $\theta_K(u_0,\, p^{-1}z_0;\, \lambda) \neq 0$. Let $\Theta_K(u;\, z)$ denote the point in the complex projective space $P^m(\mathbf{C})$ whose homogeneous coordinates are $\big(\theta_K(u,\, p^{-1}z;\, \lambda)\big)_{\lambda\in\Lambda}$, where $m = \#(\Lambda) - 1$.*

(3) *For each fixed* $z \in \mathcal{H}$ *the map* $u \mapsto \Theta_K(u, z)$ *defines a biregular projective embedding of* $(\mathbf{C}^n)^{\mathbf{b}}/p_z(L_K)$ *onto an abelian variety.*

(4) *The polarization on the image abelian variety determined by its hyperplane sections corresponds to the Riemann form of (4.13).*

(5) $\Theta_K({}^tM_\gamma(z)u, z) = \Theta_K(u, \gamma z)$ *for every* $\gamma \in \Gamma_0$.

PROOF. We first consider $\mathbf{C}^n/p_z(L)$ for $z \in \mathfrak{H}_n$ and a $\mathbf{Z}$-lattice L in $\mathbf{Q}^1_{2n}$ such that $x\eta_n \cdot {}^ty \in \mathbf{Z}$ for every $x, y \in L$. As is well-known, we can find an element α of $\mathbf{Z}^{2n}_{2n}$ such that $\mathbf{Z}^1_{2n}\alpha = L$ and $\alpha\eta_n \cdot {}^t\alpha = \begin{bmatrix} 0 & -\delta \\ \delta & 0 \end{bmatrix}$ with a diagonal matrix δ. Put $\tau_\delta = \mathrm{diag}[1_n, \delta]$. Then $\alpha^{-1}\tau_\delta$ gives an isomorphism of (η_n, L) onto $(\eta_n, \mathbf{Z}^1_{2n}\tau_\delta)$. Thus we may assume, without losing generality, that $L = \mathbf{Z}^1_{2n}\tau_\delta$. Then $p_z(L) = \{ za + \delta b \mid a, b \in \mathbf{Z}^n \}$. Take an integer p so that every entry of $p\delta$ is divisible by an integer greater than 2, and put $f_r(u) = \varphi(pu, pz; r, 0)$ for $r \in (p\delta)^{-1}\mathbf{Z}^n$. Let $\mathfrak{T}$ be the vector space of all the holomorphic functions f on $\mathbf{C}^n$ such that

$$f(u+\ell) = \mathbf{e}\big((p/2)\cdot {}^ta\delta b\big)\mathbf{e}\big(p\ell^*(z-\overline{z})^{-1}(u+(\ell/2))f(u)\big) \qquad \textit{for every} \quad \ell = za + \delta b \textit{ with } a, b \in \mathbf{Z}^n.$$

(This is essentially the same as (2.4).) Let R be a complete set of representatives for $(p\delta)^{-1}\mathbf{Z}^n/\mathbf{Z}^n$. Then $\{ f_r \mid r \in R \}$ is a $\mathbf{C}$-basis of $\mathfrak{T}$, and the map $u \mapsto \big(f_r(u)\big)_{r\in R}$ defines a biregular projective embedding of $\mathbf{C}^n/p_z(L)$. These are well-known classical results. (For detailed treatments, see [W58] and [S98, Section 27]. Observe that $u \mapsto pu$ gives an isomorphism of $\mathbf{C}^n/p_z(\mathbf{Z}^1_{2n}\tau_\delta)$ onto $\mathbf{C}^n/p_{pz}(\mathbf{Z}^1_{2n}\tau_{p\delta})$; apply the standard facts as stated in [S98, §27.12] to the last complex torus.) Since the Riemann form corresponds to the zero divisor of a nonzero f, it corresponds to the hyperplane sections of the image variety. Now we obtain the desired Λ as follows. In Case SP, we may assume that $g^{-1}(L_F) = \mathbf{Z}^1_{2en}\tau_\delta$ with δ of size en. Then we consider $\varphi_F(pu, pz; r, 0)$ with $r \in g_1\big((p\delta)^{-1}\mathbf{Z}^{en}\big)/g_1(\mathbf{Z}^{en})$. We can express this as $\varphi_F(u, p^{-1}z; \lambda_r)$ with some $\lambda_r \in \mathcal{S}(F^n_{\mathbf{h}})$ that is $\mathbf{Z}$-valued. We then take $\Lambda = \{ \lambda_r \}$ with all such r's. Since $u \mapsto Bu$ for $u \in (\mathbf{C}^n)^{\mathbf{a}}$ gives an isomorphism of $(\mathbf{C}^n)^{\mathbf{a}}/p_z(L)$ onto $\mathbf{C}^{en}/p_{\varepsilon(z)}(L_0)$, from (6.15) and the above classical results we obtain the first four assertions in Case SP. The existence of Γ_0 satisfying (5) follows from (6.35). Case UT can be handled in a similar manner by means of (6.27) and what we said at the end of §6.5.

6.13. With a fixed choice of Λ let A_w denote the image abelian variety of the map of Theorem 6.12(3). Hereafter we understand that the symbol A_w in $\mathcal{P}_w$ belonging to our families $\mathcal{F}(\Omega_F)$ and $\mathcal{F}(\Omega_K)$ is this projective variety. From Theorem 6.12 (5) we see that

$$\text{(6.38)} \qquad \begin{array}{ccc} (\mathbf{C}^n)^{\mathbf{b}}/p_{\gamma z}(L_K) & \xrightarrow{\Theta_{\gamma z}} & A_{\gamma z} \\ {\scriptstyle {}^tM_\gamma(z)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ (\mathbf{C}^n)^{\mathbf{b}}/p_z(L_K) & \xrightarrow{\Theta_z} & A_z \end{array}$$

is a commutative diagram, where $\Theta_z(u) = \Theta_K(u, z)$, and $K = F$ in Case SP.

6.14. Proposition. Case SP: *Let* $\ell \in \mathcal{S}(F^n_{\mathbf{h}})$, $b, c \in F^n$, *and* $r \in F, \gg 0$. *Define* $\ell' \in \mathcal{S}(F^n_{\mathbf{h}})$ *by* $\ell'(h) = \ell(h-b)\mathbf{e}_{\mathbf{a}}\big({}^tc(h-b)\big)$. *Then*

$$\mathbf{e_a}\big((r/2)\cdot {}^t bzb\big)\theta_F\big(rzb+c,\, rz;\, \ell\big)=\theta_F(0,\, rz;\, \ell'). \tag{6.39}$$

Moreover, $\theta_F(0, rz; \ell)$ as a function of z belongs to $\mathcal{M}_{\mathbf{a}/2}$; it belongs to $\mathcal{M}_{\mathbf{a}/2}(D)$ if ℓ is D-valued for any subfield D of $\mathbf{C}$.

Case UT: *Let $\ell \in \mathcal{S}(K^n_{\mathbf{h}})$, $b, c \in K^n$, and $r \in F, \gg 0$. Define $\ell' \in \mathcal{S}(K^n_{\mathbf{h}})$ by $\ell'(h)=\ell(h-b)\mathbf{e}\big(\mathrm{Tr}_{K/\mathbf{Q}}(c^*(h-b))\big)$. Then*

$$\mathbf{e_a}(r\cdot {}^t bw\bar{b})\theta_K(rw\bar{b}+\bar{c},\, r\cdot {}^t wb+c;\, rw;\, \ell)=\theta_K(0,\, rw;\, \ell'). \tag{6.40}$$

Moreover, $\theta_K(0, rz; \ell)$ as a function of z belongs to $\mathcal{M}_{\mathbf{a}}$; it belongs to $\mathcal{M}_{\mathbf{a}}(D)$ if ℓ is D-valued for any subfield D of $\mathbf{C}$.

PROOF. Since $\theta_F(0; z; \ell)=\varphi_F(0; z; \ell)$, we see from (6.35) that $\theta_F(0, rz; \ell)\in \mathcal{M}_{\mathbf{a}/2}$ for any ℓ. The D-rationality for D-valued ℓ is obvious. Equality (6.39) can be verified by a direct calculation. Case UT is similar; we need Theorem 6.11 (1, 3) instead of (6.35).

6.15. Proposition (Case SP). *Let $f_i \in \mathcal{A}_{k_i}$ with an integral or a half-integral weight k_i for $1\le i\le m$. Then $f_1\cdots f_m \in \mathcal{A}_k$ with $k=\sum_{i=1}^m k_i$.*

PROOF. From Theorem 6.8 (5) we see that $h_\gamma(z)^2 = j_\gamma(z)^{\mathbf{a}}$ if $\gamma\in\Gamma^0$, $b_\gamma \prec 2\mathfrak{d}^{-1}$, $c_\gamma \prec 2\mathfrak{d}$, and $\det(d_\gamma)-1\in 4\mathfrak{g}$. Our assertion can easily be derived from this fact.

6.16. Proposition. *For an integral or a half-integral weight k let $0\ne f(z)=\sum_{h\in S} c(h)\mathbf{e}^n_{\mathbf{a}}(hz)\in\mathcal{M}_k$ and let $r=\mathrm{Max}\{\mathrm{rank}(h)\mid c(h)\ne 0\}$. Then*

$$r=n \iff \begin{cases} k_v \ge n/2 & \text{for every } v\in\mathbf{a} \quad \text{(Case SP)},\\ k_{v\rho}+k_v\ge n & \text{for every } v\in\mathbf{a} \quad \text{(Case UT)};\end{cases} \tag{6.41a}$$

$$r<n \iff \begin{cases} k_v = r/2 < n/2 & \text{for every } v\in\mathbf{a} \quad \text{(Case SP)},\\ k_{v\rho}+k_v = r< n & \text{for every } v\in\mathbf{a} \quad \text{(Case UT)}.\end{cases} \tag{6.41b}$$

For the proof, see [S94b, Theorem 5.6 and Corollary 5.7]. Clearly it follows that $\mathcal{M}_0=\mathbf{C}$ in both cases. If $0\ne f\in\mathcal{S}_k$, then $r=n$, and hence

$$\mathcal{S}_k\ne\{0\} \Longrightarrow \begin{cases} k_v \ge n/2 & \text{for every } v\in\mathbf{a} \quad \text{(Case SP)},\\ k_{v\rho}+k_v\ge n & \text{for every } v\in\mathbf{a} \quad \text{(Case UT)}.\end{cases} \tag{6.42}$$

6.17. Lemma. *Let $0<\kappa\in 2^{-1}\mathbf{Z}$ in Case SP and let $0<\kappa\in\mathbf{Z}$ in Case UT. Then, given $z_0\in\mathcal{H}$, there exists an element $f\in\mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q})$ such that $f(z_0)\ne 0$.*

PROOF. By Theorem 6.12 (2) there is a $\mathbf{Z}$-valued λ such that $\theta_K(0, p^{-1}z_0; \lambda)\ne 0$. Put $g(z)=\theta_K(0, p^{-1}z; \lambda)$. By Proposition 6.14, $g\in\mathcal{M}_{\mathbf{a}/2}(\mathbf{Q})$ in Case SP and $g\in\mathcal{M}_{\mathbf{a}}(\mathbf{Q})$ in Case UT. Thus a suitable power of g gives the desired f.

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CHAPTER II

ARITHMETICITY OF AUTOMORPHIC FORMS

7. The field $\mathcal{A}_0(\mathbf{Q}_{ab})$

The principal result of this section is Theorem 7.10. We start with some auxiliarly lemmas.

7.1. Lemma. *Let h and k be subfields of $\mathbf{C}$ with countably many elements. Then the following assertions hold:*

(i) *If k is stable under $\mathrm{Aut}(\mathbf{C}/h)$, then the composite field hk is a finite or an infinite Galois extension of h.*

(ii) *If every element of $\mathrm{Aut}(\mathbf{C}/h)$ gives the identity map on k, then $k \subset h$.*

(iii) *If Y is an algebro-geometric object such as a variety, a divisor, or a rational map, and $Y^\sigma = Y$ for every $\sigma \in \mathrm{Aut}(\mathbf{C}/h)$, then Y is rational over h.*

The proof, being completely elementary, is left to the reader. In the following sections we shall often make use of these principles, though we shall not explicitly mention them in each instance.

7.2. Lemma. *Let Φ and Ψ be extensions of a field k which are linearly disjoint over k; let Φ' be a subfield of Φ containing k such that $\Psi\Phi = \Psi\Phi'$. Then $\Phi = \Phi'$.*

PROOF. Let $f \in \Phi$. Then $f \in \Psi\Phi = \Psi\Phi'$, so that $f = \sum_\kappa a_\kappa p_\kappa / \sum_\lambda b_\lambda q_\lambda$ with $a_\kappa,\, b_\lambda \in \Psi$ and $p_\kappa,\, q_\lambda \in \Phi'$. Take a finite set $\{c_\mu\}$ of elements of Ψ linearly independent over k so that $\sum_\kappa k a_\kappa + \sum_\lambda k b_\lambda = \sum_\mu k c_\mu$. Expressing a_κ and b_λ as k-linear combinations of c_μ, we can put $f = \sum_\mu c_\mu g_\mu / \sum_\mu c_\mu h_\mu$ with $g_\mu, h_\mu \in \Phi'$. Then $\sum_\mu c_\mu (f h_\mu - g_\mu) = 0$. Since $f h_\mu - g_\mu \in \Phi$, the linear disjointness shows that $f h_\mu = g_\mu$ for every μ. Thus $f = g_\mu / h_\mu$ for some μ, and hence $f \in \Phi'$ as expected.

7.3. Lemma. *Let $\{\, f_\nu \mid \nu \in N \,\}$ be a set of meromorphic functions in a connected open subset D of $\mathbf{C}^d$, indexed by an at most countable set N. Let k be a subfield of $\mathbf{C}$ with only countably many elements. Then there exists a point z_0 of D such that the specialization $\{\, f_\nu \,\}_{\nu \in N} \mapsto \{\, f_\nu(z_0) \,\}_{\nu \in N}$ defines an isomorphism of the field $k(f_\nu \mid \nu \in N)$ onto $k\big(f_\nu(z_0) \mid \nu \in N\big)$ over k.*

PROOF. We may assume that $N = \{\, 1, 2, 3, \dots \}$ (finite or not). By induction we can find a subset $M = \{\, \nu_1, \nu_2, \dots \}$ of N such that: (i) $\nu_1 < \nu_2 < \cdots$; (ii) $f_{\nu_1}, f_{\nu_2}, \dots$ are algebraically independent over k; and (iii) $f_1, \dots, f_n$ are algebraic over $k(f_\nu \mid \nu \in M,\, \nu \le n)$. Let S_m be the set of all polynomials $P(X_1, \dots, X_m) \neq 0$ in m indeterminates with coefficients in k, and W_ν the set of the points of D where f_ν is not holomorphic. For each $P \in S_m$ put

$$E_P = \left\{\, z \in D - \textstyle\bigcup_{i=1}^m W_{\nu_i} \;\middle|\; P\big(f_{\nu_1}(z), \dots, f_{\nu_m}(z)\big) = 0 \,\right\}.$$

The closure of E_P in D has no interior point of D. Now observe that S_m has only countably many elements. Recall a well-known fact that if D is covered by countably many closed subsets, then at least one of them has an interior point. Therefore we find a point z_0 of D not belonging to the countable union $\left[\bigcup_{\nu\in N} W_\nu\right] \cup \left[\bigcup_{m=1}^{\infty}\bigcup_{P\in S_m} E_P\right]$. Then our construction shows that $k(f_1, \ldots, f_n)$ has the same transcendence degree as $k\bigl(f_1(z_0), \ldots, f_n(z_0)\bigr)$ over k for every n. Therefore the specialization $f_\nu \mapsto f_\nu(z_0)$ defines an isomorphism of these fields as expected.

We say that z_0 is *generic* for $\{\, f_\nu \,|\, \nu \in N \,\}$ over k if z_0 has the property as in the above lemma. If z_0 is such a point, $\{\, g_\lambda \,|\, \lambda \in L \,\}$ is another countable set of meromorphic functions on D, and each g_λ is algebraic over the field generated by the f_ν over k, then clearly z_0 is generic for $\{\, f_\nu \,|\, \nu \in N \,\} \cup \{\, g_\lambda \,|\, \lambda \in L \,\}$ over k.

Hereafter until the end of Section 8 we treat only Cases SP and UT. We first consider the fields $\mathcal{A}_0(k)$ and $\mathcal{A}_0(\Gamma, k)$ defined in §5.8.

7.4. Lemma. (1) *For every subfield k of $\mathbf{C}$, the fields $\mathcal{A}_0(k)$ and $\mathbf{C}$ are linearly disjoint over k.*

(2) *Let Γ be a congruence subgroup of G and let (V, φ) be a model of $\Gamma\backslash\mathcal{H}$ in the sense of §4.10. Let k be a subfield of $\mathbf{C}$ such that V is defined over k and $k(V)\circ\varphi \subset \mathcal{A}_0(\Gamma, k)$. Then $k(V)\circ\varphi = \mathcal{A}_0(\Gamma, k)$, where $k(V)$ is defined as in §2.4.*

PROOF. To prove (1), let $f_1, \ldots, f_m$ be elements of $\mathcal{A}_0(k)$ linearly independent over k; put $f_i = p_i/q_i$ with $p_i,\, q_i \in \mathcal{M}_{\nu_i}(k)$, $\nu_i \in \mathbf{Z}^{\mathbf{b}}$, and $r_i = q_1\cdots q_m f_i$. Then the r_i are automorphic forms of the same weight linearly independent over k, and hence, linearly independent over $\mathbf{C}$ by virtue of Lemma 5.10 (1). Therefore we obtain (1). As for (2), we have $\mathbf{C}k(V)\circ\varphi \subset \mathbf{C}\mathcal{A}_0(\Gamma, k) \subset \mathcal{A}_0(\Gamma) = \mathbf{C}(V)\circ\varphi = \mathbf{C}k(V)\circ\varphi$, and hence $\mathbf{C}k(V)\circ\varphi = \mathbf{C}\mathcal{A}_0(\Gamma, k)$. Therefore, by (1) and Lemma 7.2 we obtain the desired equality of (2).

7.5. Lemma. *Let $P = \{\, \xi \in \widetilde{G} \,|\, c_\xi = 0 \,\}$, where c_ξ is the c-block of ξ (see Lemma 1.9). Put $G_1 = G \cap SL_{2n}(K)$ and denote by $\mathcal{G}$ any of the groups $\widetilde{G}$, $\widetilde{G}_+$, G, and G_1; put $\mathcal{P} = P \cap \mathcal{G}$. Then $\mathcal{G}$ is dense in $\mathcal{G}_{\mathbf{a}}$, and $\mathcal{G}$ (resp. $\mathcal{G}_{\mathbf{a}}$) is generated by η_n and $\mathcal{P}$ (resp. $\mathcal{P}_{\mathbf{a}}$), where we understand that $(\widetilde{G}_+)_{\mathbf{a}} = \widetilde{G}_{\mathbf{a}+}$ and $\mathcal{P}_{\mathbf{a}} = P_{\mathbf{a}} \cap \mathcal{G}_{\mathbf{a}}$.*

PROOF. Put $\mathcal{B} = \{\, \xi \in \mathcal{G} \,|\, \det(c_\xi) \neq 0 \,\}$ and $\mathcal{B}_{\mathbf{a}} = \{\, \xi \in \mathcal{G}_{\mathbf{a}} \,|\, \det(c_\xi) \in K_{\mathbf{a}}^{\times} \,\}$. Then $\mathcal{B} = \mathcal{P}\eta\mathcal{P}$ and $\mathcal{B}_{\mathbf{a}} = \mathcal{P}_{\mathbf{a}}\eta\mathcal{P}_{\mathbf{a}}$. Indeed, we can easily verify that $\mathcal{P}\eta\mathcal{P} \subset \mathcal{B}$ and $\mathcal{P}_{\mathbf{a}}\eta\mathcal{P}_{\mathbf{a}} \subset \mathcal{B}_{\mathbf{a}}$. That $\mathcal{B} \subset \mathcal{P}\eta\mathcal{P}$ and $\mathcal{B}_{\mathbf{a}} \subset \mathcal{P}_{\mathbf{a}}\eta\mathcal{P}_{\mathbf{a}}$ can be seen from an equality

$$(*) \qquad \xi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} s\cdot 1_n & ac^{-1} \\ 0 & 1_n \end{bmatrix} \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix} \begin{bmatrix} c & d \\ 0 & \widehat{c} \end{bmatrix},$$

where $\xi \in GU(\eta_n)$, $s = \nu(\xi)$, and c is invertible. (Notice that by Lemma 1.3 (2), if $\nu(\xi) = \det(\xi) = 1$, then $\det(c) \in F$, so that the last matrix of (*) has determinant 1.) Now $\mathcal{B}_{\mathbf{a}}$ is open and dense in $\mathcal{G}_{\mathbf{a}}$. Clearly $\mathcal{P}$ is dense in $\mathcal{P}_{\mathbf{a}}$, and so $\mathcal{B}$ is dense in $\mathcal{B}_{\mathbf{a}}$. Therefore $\mathcal{G}$ is dense in $\mathcal{G}_{\mathbf{a}}$. Now given $\alpha \in \mathcal{G}$, the set $\{\xi \in \mathcal{G}_{\mathbf{a}} \,|\, \det(c_\xi c_{\alpha\xi}) \in K_{\mathbf{a}}^{\times}\}$ is open in $\mathcal{G}_{\mathbf{a}}$. Therefore we can find $\xi \in \mathcal{G}$ such that $\det(c_\xi c_{\alpha\xi}) \neq 0$. Then both ξ and $\alpha\xi$ belong to $\mathcal{P}\eta\mathcal{P}$, and so $\alpha = \alpha\xi\xi^{-1} \in \mathcal{P}\eta_n\mathcal{P}\eta_n\mathcal{P}$, which gives the desired fact for $\mathcal{G}$. The assertion for $\mathcal{G}_{\mathbf{a}}$ can be proved in the same manner.

7.6. We now take our setting to be that of Section 6. We fix a positive integer $p \geq 3$, a subset Λ of $\mathcal{S}(K_{\mathbf{h}}^n)$, and a congruence subgroup Γ_0 of G as in Theorem 6.12; we then consider the map Θ_K defined with these p and Λ as in (3) of that theorem. Let $P^*(\mathbf{C})$ be the complex projective space in which the map Θ_K takes its values. Then A_w of §6.13 is a subvariety of $P^*(\mathbf{C})$. We retain the convention that $K = F$ and $\mathfrak{r} = \mathfrak{g}$ in Case SP (see §3.5). The symbol p_z of (4.23) is a map of $(K_{\mathbf{a}})^1_{2n}$ into $(\mathbf{C}^n)^{\mathbf{b}}$. Recall the formula $p_z(x\alpha) = {}^t\mu_\alpha(z)p_{\alpha z}(x)$ stated in (4.31). Substituting $p_{\gamma z}(a)$ with $a \in K^1_{2n}$ for u in Theorem 6.12 (5), we obtain

$$(7.1) \qquad \Theta_K\big(p_z(a\gamma),\, z\big) = \Theta_K\big(p_{\gamma z}(a),\, \gamma z\big) \quad \textit{for every} \quad a \in K^1_{2n} \quad \textit{and} \quad \gamma \in \Gamma_0.$$

Define $t : K^1_{2n} \times \mathcal{H} \to P^*(\mathbf{C})$ and $t_z : K^1_{2n} \to P^*(\mathbf{C})$ for $z \in \mathcal{H}$ by

$$(7.2) \qquad t(a,\, z) = t_z(a) = \Theta_K\big(p_z(a),\, z\big) = \big(\theta_K(p_z(a),\, p^{-1}z;\, \lambda)\big)_{\lambda\in\Lambda}.$$

Then from (7.1) we obtain

$$(7.3) \qquad t(a\gamma,\, z) = t(a,\, \gamma z) \quad \textit{for every} \quad a \in K^1_{2n} \quad \textit{and} \quad \gamma \in \Gamma_0.$$

7.7. We fix an $\mathfrak{r}$-lattice L in K^1_{2n}, and for each positive integer N we fix a subset $\{u_i\}_{i=1}^s$ of $N^{-1}L$ so that $N^{-1}L = L + \sum_{i=1}^s \mathbf{Z}u_i$. We then consider the families of §§6.3 and 6.5 in the following form:

$$(7.4a) \qquad \mathcal{F}(\Omega^N) = \big\{\,\mathcal{P}^N_z \,\big|\, z \in \mathcal{H}\,\big\}, \quad \mathcal{P}^N(z) = \mathcal{P}^N_z = \big(A_z,\, \mathcal{C}_z,\, \iota_z;\, \{t_i(z)\}_{i=1}^s\big),$$

$$(7.4b) \qquad \Omega^N = \big\{\,K,\, \Psi,\, L,\, \eta_n,\, \{u_i\}_{i=1}^s\,\big\}.$$

We write Ω^N and $\mathcal{P}^N_z$ instead of Ω and $\mathcal{P}_z$ in order to emphasize N. We simply write L instead of L_K in Theorem 6.12. Naturally $t_i(z) = t_z(u_i)$, and so

$$(7.4c) \qquad \big\{\,t \in A_z \,\big|\, Nt = 0\,\big\} = \textstyle\sum_{i=1}^s \mathbf{Z}t_i(z).$$

For $c \in K$ we have $p_z(ca) = \Psi(c)p_z(a)$ and $\iota_z(c)$ is represented by $\Psi(c)$, and so we obtain

$$(7.5) \qquad \iota_z(c)t_z(a) = t_z(ca) \quad \text{for every } c \in \mathfrak{r} \text{ and } a \in K^1_{2n}.$$

Define a congruence subgroup Γ^N of G by

$$(7.6) \qquad \Gamma^N = \big\{\,\gamma \in G \,\big|\, L\gamma = L,\ L(\gamma - 1) \subset L\,\big\}.$$

By Theorem 4.8, $\mathcal{P}^N_z$ and $\mathcal{P}^N_w$ are isomorphic if and only if $z = \gamma w$ for some $\gamma \in \Gamma^N$, since Γ of (4.28) coincides with Γ^N for the present family.

Let us now fix a model $(V_N,\, \varphi_N)$ of $\Gamma_N\backslash\mathcal{H}$. If N divides M, then there exists a rational map $p^M_N : V_M \to V_N$ such that $\varphi_N = p^M_N \circ \varphi_M$. We can find a subfield k_0 of $\mathbf{C}$ with countably many elements over which the varieties V_N and the maps p^M_N are defined for all M and N. Put

$$(7.7) \qquad \mathfrak{F} = \bigcup_{N=1}^{\infty} \mathfrak{F}_N, \qquad \mathfrak{F}_N = \big\{\,f \circ \varphi_N \,\big|\, f \in k_0(V_N)\,\big\}.$$

Then $\mathcal{A}_0(\Gamma_N) = \mathbf{C}\mathfrak{F}_N$ and $\mathcal{A}_0 = \mathbf{C}\mathfrak{F}$. Since $\mathfrak{F}_N$ is finitely generated over k_0, we see that $\mathfrak{F}$ is countable. Now let $\mathfrak{K}$ be the field generated over $\mathbf{Q}$ by the functions on $\mathcal{H}$ of the form

$$(7.8) \qquad \theta_K\big(p_z(a),\, p^{-1}z;\, \ell\big)/\theta_K\big(p_z(a),\, p^{-1}z;\, \ell'\big)$$

for all $\ell,\, \ell' \in \Lambda$ and all $a \in K^1_{2n}$, where Λ is the set of Theorem 6.12. By Proposition 6.14 each such quotient belongs to $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$; thus $\mathfrak{K} \subset \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. For each $w \in \mathcal{H}$ let $\mathfrak{K}[w]$ denote the field generated over $\mathbf{Q}$ by the values $f(w)$ for every $f \in \mathfrak{K}$ finite at w. Since any affine coordinate of $t_z(a)$ is of the form (7.8), $t_w(a)$ is rational over $\mathfrak{K}[w]$ for every $a \in K^1_{2n}$, and hence A_w is rational over $\mathfrak{K}[w]$. Also, from (7.5) we see that $\iota_w(c)$ is rational over $\mathfrak{K}[w]$. Since the polarization is determined by hyperplane sections, $\mathcal{P}_w^N$ is rational over $\mathfrak{K}[w]$ for every N. Now by Lemma 2.6, $\mathbf{Q}_{\mathrm{ab}}$ is contained in the field generated over $\mathbf{Q}$ by the affine coordinates of the points $t_z(a)$, and hence $\mathbf{Q}_{\mathrm{ab}} \subset \mathfrak{K}[w]$. Taking w to be generic for the elements of $\mathfrak{K}$ over $\mathbf{Q}$, we see that $\mathbf{Q}_{\mathrm{ab}} \subset \mathfrak{K}$.

7.8. Lemma. *Given $w, w' \in \mathcal{H}$, suppose that there exist an isomorhism σ of $\mathfrak{K}[w]$ onto $\mathfrak{K}[w']$ and an $\mathfrak{r}$-linear automorphism λ of K_{2n}^1/L such that $t(a, w)^\sigma = t(\lambda(a), w')$ for every a and $\lambda(u_i) = u_i$ for every i. Then $(\mathcal{P}_w^N)^\sigma = \mathcal{P}_{w'}^N$, where $(\mathcal{P}_w^N)^\sigma$ is defined as in §2.7.*

PROOF. Since the points $t_w(a)$ are dense in A_w, we have $(A_w)^\sigma = A_{w'}$. Clearly σ sends the hyperplane sections of A_w to those of $A_{w'}$; also $t_w(u_i)^\sigma = t_{w'}(u_i)$. Now, from (7.5) we obtain $\iota_w(c)^\sigma t_w(a)^\sigma = t_w(ca)^\sigma = t_{w'}(c\lambda(a)) = \iota_{w'}(c)t_{w'}(\lambda(a)) = \iota_{w'}(c)t_w(a)^\sigma$ for every $c \in \mathfrak{r}$ and $a \in K_{2n}^1$; thus $\iota_w(c)^\sigma = \iota_{w'}(c)$. This completes the proof.

7.9. Since $\mathfrak{K} \subset \mathcal{A}_0 = \mathbf{C}\mathfrak{F}$, we can find a countable subfield k of $\mathbf{C}$ containing k_0 such that $\mathfrak{K} \subset k\mathfrak{F}$. Replacing k by its algebraic closure in $\mathbf{C}$ if necessary, we assume that k is algebraically closed. Now let z_0 be a generic point for the elements of $\mathfrak{F}$ over k. Take and fix a positive integer N. Since $\mathcal{P}_{z_0}^N$ is defined over $\mathfrak{K}[z_0]$, we can find elements $g_1, \dots, g_m$ of $\mathfrak{K}$ such that $\mathcal{P}_{z_0}^N$ is defined over $\mathbf{Q}(g_1(z_0), \dots, g_m(z_0))$. Since the coordinates of $t_{z_0}(a)$ are algebraic over this field, $\mathfrak{K}[z_0]$ is algebraic over $\mathbf{Q}(g_1(z_0), \dots, g_m(z_0))$. Therefore $\mathfrak{K}$ is algebraic over $\mathbf{Q}(g_1, \dots, g_m)$.

Let V be the affine locus of the point $(g_1(z_0), \dots, g_m(z_0))$ over k. Take a multiple M of N so that the g_i belong to $k\mathfrak{F}_M$. Since z_0 is generic for $\mathfrak{F}$ over k, $\varphi_M(z_0)$ is a generic point of V_M over k, and we can define a k-rational map $q : V_M \to V$ by $q(\varphi_M(z_0)) = g(z_0)$, where we write $g = (g_1, , \dots, g_m)$.

Let us now prove that

$$k(\varphi_N(z_0)) \subset k(g(z_0)). \tag{7.9}$$

Let $\sigma \in \mathrm{Aut}(\mathbf{C}/k(g(z_0)))$. Since V_M is defined over k, the point $\varphi_M(z_0)^\sigma$ belongs to V_M, and so $\varphi_M(z_0)^\sigma = \varphi_M(z_1)$ with $z_1 \in \mathcal{H}$. This means that z_1 is generic for $\mathfrak{F}_M$ over k, and for $\mathfrak{F}$ over k as well, since $\mathfrak{F}$ is algebraic over $\mathfrak{F}_M$. Thus $f(z_0) \mapsto f(z_1)$ for $f \in k\mathfrak{F}$ is an isomorphism; denote it by τ. Then $g(z_0)^\tau = q(\varphi_M(z_0))^\tau = q(\varphi_M(z_1)) = q(\varphi_M(z_0))^\sigma = g(z_0)^\sigma = g(z_0)$. Since $\mathcal{P}_{z_0}^N$ is defined over $k(g(z_0))$, we have $(\mathcal{P}_{z_0}^N)^\tau = \mathcal{P}_{z_0}^N$. On the other hand, $t_{z_0}(a)^\tau = t_{z_1}(a)$ for every $a \in K_{2n}^1$, so that $(\mathcal{P}_{z_0}^N)^\tau = \mathcal{P}_{z_1}^N$ by Lemma 7.8. Thus $\mathcal{P}_{z_0}^N = \mathcal{P}_{z_1}^N$, and hence $\varphi_N(z_0) = \varphi_N(z_1)$. Since $\varphi_N(z_1) = p_N^M(\varphi_M(z_1)) = p_N^M(\varphi_M(z_0))^\sigma = \varphi_N(z_0)^\sigma$, we obtain $\varphi_N(z_0) = \varphi_N(z_0)^\sigma$, which proves (7.9).

7.10. Theorem. (1) *$\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ is generated over $\mathbf{Q}$ by all the quotients of the form $\theta_K(0, rz; \lambda)/\theta_K(0, rz; \lambda')$ with $\mathbf{Q}_{\mathrm{ab}}$-valued λ, λ' in $\mathcal{S}(K_{\mathbf{h}}^n)$ and $0 \ll r \in F$.*

(2) $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}}) = \mathfrak{K}$.

(3) $\mathcal{A}_0(\Phi) = \Phi\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ *for every subfield Φ of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$; in particular, $\mathcal{A}_0 = \mathbf{C}\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$.*

(4) $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ *is stable under the map $f \mapsto f \circ \alpha$ for every $\alpha \in \widetilde{G}_+$.*

PROOF. From (7.9) we obtain $\mathfrak{F}^N \subset k\mathfrak{K}$, and consequently $\mathcal{A}_0 = \mathbf{C}\mathfrak{F} = \mathbf{C}\mathfrak{K}$. Since $\mathbf{Q}_{\mathrm{ab}} \subset \mathfrak{K} \subset \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$, we obtain $\mathcal{A}_0 = \mathbf{C}\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. Now, given a subfield Φ of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$, we have $\Phi\mathfrak{K} \subset \Phi\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}}) \subset \mathcal{A}_0(\Phi)$ and $\mathbf{C}\mathcal{A}_0(\Phi) \subset \mathcal{A}_0 = \mathbf{C}\mathfrak{K} =$

$\mathbf{C}\Phi\mathfrak{K}$, so that $\mathbf{C}\mathcal{A}_0(\Phi) = \mathbf{C}\Phi\mathfrak{K}$. By Lemma 7.4 (1), $\mathcal{A}_0(\Phi)$ and $\mathbf{C}$ are linearly disjoint over Φ. Therefore, by Lemma 7.2, $\Phi\mathfrak{K} = \mathcal{A}_0(\Phi)$. Taking $\Phi = \mathbf{Q}_{\mathrm{ab}}$, we obtain (2), and hence (3) as well. As for (1), Proposition 6.14 shows that any quotient of the form (7.8) is of the form described in (1). Now let f and g denote the numerator and the denominator of the quotient of (1). In Case SP, by Propositions 6.14 and 6.15, both fg and g^2 belong to $\mathcal{M}_1(\mathbf{Q}_{\mathrm{ab}})$, and hence $f/g \in \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. Clearly the same conclusion is true in Case UT. This proves (1). As for (4), by Lemma 7.5 it is sufficient to prove the cases $\alpha = \eta_n$ and $\alpha \in P \cap \widetilde{G}_+$. Since $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}}) = \mathfrak{K}$, our task is to show that α sends a quotient of the form (7.8) into $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. If $\alpha = \eta_n$, we have $r \cdot \eta(z) = \eta(r^{-1}z)$. By Theorem 6.9 (2) or Theorem 6.11 (1) we have

$$\begin{aligned}\theta_K(0,\, r\cdot\eta(z);\, \lambda_1)/\theta_K(0,\, r\cdot\eta(z);\, l_2) &= \theta_K(0,\, \eta(r^{-1}z);\, \lambda_1)/\theta_K(0,\, \eta(r^{-1}z);\, \lambda_2)\\ &= \theta_K(0,\, r^{-1}z;\, \lambda_1')/\theta_K(0,\, r^{-1}z;\, \lambda_2'),\end{aligned}$$

where λ_i' can be obtained from λ_i by (6.33) or Theorem 6.11 (4). We easily see that λ_i' is $\mathbf{Q}_{\mathrm{ab}}$-valued if λ_i is so. Thus η sends $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ into itself. The action of $P \cap \widetilde{G}_+$ will be discussed in the proof of the following theorem.

7.11. Theorem. *Let $\kappa \in \mathbf{Z}$ in Case UT and $\kappa \in 2^{-1}\mathbf{Z}$ in Case SP. Given $\alpha \in \widetilde{G}_+$, let $q(z) = \left(j_\alpha(z)^{\mathbf{a}}\right)^\kappa$, where we take any branch of the square root of $j_\alpha(z)^{2\kappa\mathbf{a}}$ to be $q(z)$ if $\kappa \notin \mathbf{Z}$. Let Φ be a subfield of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$. If $\kappa \neq 0$ in Case UT, then assume either $\det(\alpha) = \nu(\alpha)^n$ or Φ contains the reflex field K' of §1.12 defined for (K, τ) of §3.5. Then $q(z)^{-1}f(\alpha z) \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$ for every $f \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$. Moreover, in Case UT, $f\|_{\kappa\mathbf{b}}\,\alpha \in \mathcal{M}_{\kappa\mathbf{b}}(\Phi)$ for every $f \in \mathcal{M}_{\kappa\mathbf{b}}(\Phi)$, every $\alpha \in \widetilde{G}_+$, and every subfield Φ of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$.*

PROOF. We first assume $K' \subset \Phi$ in Case UT. By Lemma 7.5 it is sufficient to prove the cases $\alpha = \eta_n$ and $\alpha \in P \cap \widetilde{G}_+$. (Notice that $\eta_n^{-1} = -\eta_n \in \eta_n(P \cap \widetilde{G}_+)$.) Let $\alpha = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in P \cap \widetilde{G}_+$, $s = \nu(\alpha)$, and $f(z) = \sum_h c(h)\mathbf{e}_{\mathbf{a}}^n(hz) \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$. Then $a^*d = s1_n$, $q(z) = \prod_{v\in\mathbf{a}} \det(d)_v^\kappa \in K'$, and

$$q(z)^{-1}f(\alpha z) = q(z)^{-1}\sum_h c(h)\mathbf{e}_{\mathbf{a}}^n(s^{-1}hba^*)\mathbf{e}_{\mathbf{a}}^n(s^{-1}a^*haz).$$

Since $ba^* \in S$ by (1.13), we see that $\mathrm{tr}(s^{-1}hba^*) \in F$, and hence $\mathbf{e}_{\mathbf{a}}^n(s^{-1}hba^*)$ is a root of unity; thus $q(z)^{-1}f(\alpha z) \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$. In Case SP the same reasoning is valid for any Φ containing $\mathbf{Q}_{\mathrm{ab}}$.

Returning to the proof of Theorem 7.10 (4), we apply the same technique to $f = \theta_K(0, rz; \lambda)$ with a $\mathbf{Q}_{\mathrm{ab}}$-valued λ. Then for $\alpha \in P \cap \widetilde{G}_+$ the above reasoning shows that $f \circ \alpha$ belongs to $\mathcal{M}_{\mathbf{a}/2}(\mathbf{Q}_{\mathrm{ab}})$ or $\mathcal{M}_{\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$, and so $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ is stable under $P \cap \widetilde{G}_+$. This completes the proof of Theorem 7.10.

Next we consider the action of η on $\mathcal{M}_{\kappa\mathbf{a}}(\Phi)$. Given $f \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$, put $m = 2\kappa/[K : F]$ and $t = fg^{-m}$ with a nonzero function g of the form $g(z) = \theta_K(0, z; \lambda)$ with a $\mathbf{Q}_{\mathrm{ab}}$-valued λ. Then $g^m \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ and $t \in \mathcal{A}_0(\Phi)$. By Theorem 7.10 (3), (4), $\mathcal{A}_0(\Phi)$ is stable under $\widetilde{G}_+$. In Case SP we have $q(z) = \zeta \cdot h_\eta(z)^m$ with a root of unity ζ, and $q(z)^{-1}f(\eta z) = \zeta^{-1}(g\|\eta)^m(t \circ \eta)$, which belongs to $\mathcal{A}_{\kappa\mathbf{a}}(\Phi)$, since $t \circ \eta \in \mathcal{A}_0(\Phi)$ and $(g\|\eta)^m \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$. The last fact follows from the behavior of θ_K under η as discussed at the end of the proof of Theorem 7.10 (cf. Proposition 6.15). By (5.30), $q(z)^{-1}f(\eta z) \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$. Next, take $\alpha \in \widetilde{G}_+$ such that $\det(\alpha) = \nu(\alpha)^n$ in Case UT. Given such an α, put $\tau = \mathrm{diag}[\nu(\alpha)1_n,\, 1_n]$

and $\beta = \tau^{-1}\alpha$. Then $\beta \in G_1$. By Lemma 7.5, G_1 is generated by η and $G_1 \cap P$. If $\gamma \in G_1 \cap P$ or $\gamma = \tau$, then $\det(d_\gamma) \in F$, so that $j_\gamma(z)^{\mathbf{a}} \in \mathbf{Q}$. Therefore our reasoning is valid for the group of elements α satisfying $\det(\alpha) = \nu(\alpha)^n$ without the condition $K' \subset \Phi$. Finally take $\mathcal{M}_{\kappa\mathbf{b}}$ in Case UT. Since $j_\gamma^{\mathbf{b}} \in \mathbf{Q}$ for $\gamma \in P \cap \widetilde{G}_+$, we do not have to assume that $K' \subset \Phi$. Thus we obtain the last assertion.

7.12. Lemma. *If $(\mathcal{P}_w^N)^\sigma$ is isomorphic to $\mathcal{P}_z^N$ for $\sigma \in \mathrm{Aut}(\mathbf{C})$ and some $z, w \in \mathcal{H}$, then $\mathbf{e}(1/N)^\sigma = \mathbf{e}(1/N)$.*

PROOF. Let s be the positive integer such that $\{\mathrm{Tr}_{K/\mathbf{Q}}(x\eta_n y^*) \mid x, y \in L\} = s\mathbf{Z}$. Then there exists a divisor X_z on A_z that determines the Riemann form E_z of (4.25) with $\mathcal{T} = s^{-1}\eta_n$. Then X_z is a basic polar divisor of $\mathcal{P}_z$, and by (2.10) we have

$$\zeta_{X_z}\big(t_z(a), t_z(b)\big) = \mathbf{e}\big(m \cdot E_z(p_z(a), p_z(b))\big) = \mathbf{e}\big((m/s)\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*)\big) \tag{7.10}$$

for every $a, b \in m^{-1}L$, $0 < m \in \mathbf{Z}$. Let ε be an isomorphism of $(\mathcal{P}_w^N)^\sigma$ onto $\mathcal{P}_z^N$. Then $\varepsilon t_w(a)^\sigma = t_z(a)$ for every $a \in N^{-1}L$. Let $Y = \varepsilon(X_w^\sigma)$. Since X_w is a basic polar divisor of $\mathcal{P}_w$, $(X_w)^\sigma$ is a basic polar divisor of $(\mathcal{P}_w)^\sigma$. (This follows from the characterization of a basic polar divisor in terms of algebraic equivalence mentioned in §2.3.) Therefore Y is a basic polar divisor of $\mathcal{P}_z$, so that Y determines E_z. Therefore for $a, b \in N^{-1}L$ we have

$$\begin{aligned}\mathbf{e}\big((N/s)\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*)\big) &= \zeta_Y\big(t_z(a), t_z(b)\big) = \zeta_{X_w^\sigma}\big(t_w(a)^\sigma, t_w(b)^\sigma\big)\\ &= \zeta_{X_w}\big(t_w(a), t_w(b)\big)^\sigma = \mathbf{e}\big((N/s)\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*)\big)^\sigma\end{aligned}$$

by (2.11). We can take a and b in $N^{-1}L$ so that $\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*) = sN^{-2}$. Then we obtain $\mathbf{e}(1/N)^\sigma = \mathbf{e}(1/N)$ as expected.

8. Action of certain elements of $\widetilde{G}_{\mathbf{A}}$ on $\mathfrak{K}$

8.1. It is well-known that $\mathbf{Q}^\times\mathbf{Q}_{\mathbf{a}+}^\times$ is closed in $\mathbf{Q}_{\mathbf{A}}^\times$, and there is a canonical homomorphism of $\mathbf{Q}_{\mathbf{A}}^\times$ onto $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ with kernel $\mathbf{Q}^\times\mathbf{Q}_{\mathbf{a}+}^\times$. For $t \in \mathbf{Q}_{\mathbf{A}}^\times$ we denote by $[t, \mathbf{Q}]$ the element of $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ that is the image of t under that homomorphism. To simplify our notation, let us now put

$$\mathbf{Z}_{\mathbf{h}}^\times = \prod_p \mathbf{Z}_p^\times, \tag{8.1}$$

where the product is taken over all rational primes p. Observing that $\mathbf{Q}_{\mathbf{A}}^\times/\mathbf{Q}^\times\mathbf{Q}_{\mathbf{a}+}^\times$ is isomorphic to $\mathbf{Z}_{\mathbf{h}}^\times$, we see that the map $t \to [t, \mathbf{Q}]$ gives an isomorphism of $\mathbf{Z}_{\mathbf{h}}^\times$ onto $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$. Now $\mathbf{Q}/\mathbf{Z}$ is canonically isomorphic to $\mathbf{Q}_{\mathbf{A}}/(\mathbf{Q}_{\mathbf{a}}\prod_p \mathbf{Z}_p)$, and we can let $\mathbf{Z}_{\mathbf{h}}^\times$ act on the last group by multiplication. For $t \in \mathbf{Z}_{\mathbf{h}}^\times$ and $x \in \mathbf{Q}/\mathbf{Z}$ denote by tx the image of x under t. Clearly $x \to \mathbf{e}_{\mathbf{h}}(x) = \prod_p \mathbf{e}_p(x)$ for $x \in \mathbf{Q}$ gives an isomorphism of $\mathbf{Q}/\mathbf{Z}$ onto the group of all roots of unity. (See §1.6 for the definition of $\mathbf{e}_p$. Notice also that $\mathbf{e}_{\mathbf{h}}(x) = \mathbf{e}(-x)$ for every $x \in \mathbf{Q}$.) Then we can easily show that

$$\mathbf{e}_{\mathbf{h}}(x)^{[t, \mathbf{Q}]} = \mathbf{e}_{\mathbf{h}}(t^{-1}x) \qquad (t \in \mathbf{Z}_{\mathbf{h}}^\times,\ x \in \mathbf{Q}/\mathbf{Z}). \tag{8.2}$$

In particular, given $t \in \mathbf{Z}_{\mathbf{h}}^\times$ and a positive integer N, take a positive integer r so that $rt_p - 1 \in N\mathbf{Z}_p$ for every prime p. Then (8.2) means $\mathbf{e}(1/N)^{[t, \mathbf{Q}]} = \mathbf{e}(r/N)$, which is the classical reciprocity-law in cyclotomic fields.

8.2. Our setting is the same as in Sections 6 and 7; see also §5.6. We retain the convention that $K=F$ and $\mathfrak{r}=\mathfrak{g}$ in Case SP. We use the symbols $\widetilde{G}$, G, and G_0 of (3.26), (3.27), and (3.29) in both cases. Put $G_1=G\cap SL_{2n}(K)$ as in Lemma 7.5. Then $G_0\cap G=G_1$; $G=G_1=Sp(n, F)$ in Case SP. Define $\iota : \mathbf{Q}_{\mathbf{A}}^{\times} \to (G_0)_{\mathbf{A}}$ by

$$\iota(s) = \mathrm{diag}[1_n,\, s^{-1}1_n] \qquad (s \in \mathbf{Q}_{\mathbf{A}}^{\times}). \tag{8.3}$$

We now take the $\mathfrak{r}$-lattice L of (7.4b) to be $\mathfrak{r}_{2n}^1$ (Actually our treatment is applicable to a more general type of L, $L = \mathfrak{r}_{2n}^1\xi$, for example, with any diagonal ξ in $G_{\mathbf{h}}$ such that $\mathrm{Tr}_{K/\mathbf{Q}}(x\eta_n y^*) \in \mathbf{Z}$ for every $x,\, y \in L$.) We then put

$$U^1 = \{\, x \in (G_0)_{\mathbf{A}+} \,|\, Lx = L \,\}, \tag{8.4}$$

$$U^N=\{\, x \in U^1 \,|\, L_v(x_v - 1) \subset NL_v \ \text{ for every } \ v \in \mathbf{h} \,\} \quad (0<N\in\mathbf{Z}), \tag{8.5}$$

$$T^N = \iota(\mathbf{Z}_{\mathbf{h}}^{\times})U^N. \tag{8.6}$$

Clearly U^1 is a subgroup of $(G_0)_{\mathbf{A}+}$ and U^N is a normal aubgroup of U^1; since $\iota(\mathbf{Z}_{\mathbf{h}}^{\times}) \subset U^1$, T^N is a subgroup of U^1 and $T^N = U^N\iota(\mathbf{Z}_{\mathbf{h}}^{\times})$. We also employ Γ^N of (7.6).

8.3. Lemma. (1) $(G_0)_{\mathbf{A}+} = G_{0+}T^N = T^N G_{0+}$ *for every* N.

(2) $\widetilde{G} \cap T^N\widetilde{G}_{\mathbf{a}+} = \Gamma^N \cap T^N = U^N \cap G_1$ *for every* N.

(3) $\Gamma^N \subset U^N \cap G_1$ *for* $N > 2$ *in Case UT and for every* N *in Case SP.*

(4) *Given* $x \in T^N\Gamma^N$ *and a multiple* M *of* N, *there exists an element* γ *of* Γ^N *and an element* y *of* U^M *such that* $x = \iota(r)y\gamma$, *where* $r = \nu(x)^{-1}$.

PROOF. Let $x \in (G_0)_{\mathbf{A}+}$. Then $0 \ll \nu(x) \in \mathbf{Q}_{\mathbf{A}}^{\times}$, and hence $\nu(x) = abc$ with $0 < a \in \mathbf{Q}^{\times}$, $b \in \mathbf{Z}_{\mathbf{h}}^{\times}$, and $0 < c \in \mathbf{Q}_{\mathbf{a}}^{\times}$. Put $y = \iota(a)x\iota(bc)$. Then $\nu(y) = 1$ and $y \in (G_0)_{\mathbf{A}}$, so that $y \in (G_1)_{\mathbf{A}}$. By strong approximation in G_1 we have $(G_1)_{\mathbf{A}} \subset G_1U^N$ for every N, and hence $x = \iota(a)^{-1}y\iota(bc)^{-1} \in G_{0+}G_1U^N\iota(b)^{-1} \subset G_{0+}T^N$, from which we can easily derive (1). To prove (2), let $\gamma \in \widetilde{G} \cap T^N\widetilde{G}_{\mathbf{a}+}$. Then $\gamma \in \iota(s)x\widetilde{G}_{\mathbf{a}+}$ with $s \in \mathbf{Z}_{\mathbf{h}}^{\times}$ and $x \in U^N$. We see that $\nu(\gamma)$ is a totally positive unit belonging to $\mathbf{Q}$, and so $\nu(\gamma) = 1$. Similarly $\det(\gamma) = 1$. Thus $s = \nu(x)_{\mathbf{h}}$. Since $x \in U^N$, we have $s - 1 \prec N\mathbf{Z}$, so that $\gamma - 1 \prec N\mathfrak{r}$. From these facts we easily obtain (2). By (4.34), $\Gamma^N \subset G_1$ for $N > 2$ in Case UT and for every N, in Case SP, which implies (3). Clearly it is sufficient to prove (4) for $x \in T^N$. Given $x \in T^N$, put $r = \nu(x)^{-1}$ and $z = \iota(r)^{-1}x$. Then $\nu(z) = 1$; also $z \in T^MG_0$ by (1), and hence $z = \iota(s)w\gamma$ with $s \in \mathbf{Z}_{\mathbf{h}}^{\times}$, $w \in U^M$, and $\gamma \in G_0$. Then $\gamma \in G_0 \cap T^N \subset \Gamma^N$ by (2), and hence $\nu(\gamma) = 1$. Thus $s = \nu(w)$, and $s - 1 \prec M\mathbf{Z}$ since $w \in U^M$. Put $y = \iota(s)w$; then $y \in U^M$ and $x = \iota(r)y\gamma$. This proves (4).

8.4. We are going to define the action of a certain subgroup of $\widetilde{G}_{\mathbf{A}}$ on $\mathfrak{K}$. First, in view of Theorem 7.10 (4), the map $f \to f \circ \alpha$ is clearly an automorphism of $\mathfrak{K}$ for every $\alpha \in \widetilde{G}_+$.

Observe that K_{2n}^1/L is canonically isomorphic to $(K_{2n}^1)_{\mathbf{A}}/\big[(K_{2n}^1)_{\mathbf{a}}\prod_{v\in\mathbf{h}} L_v\big]$. Therefore we can define $t(a,\, z)$ for $a \in (K_{2n}^1)_{\mathbf{A}}$ by putting $t(a,\, z) = t(b,\, z)$ with $b \in K_{2n}^1$ such that $a_v - b \in L_v$ for every $v \in \mathbf{h}$. In particular, if $a \in K_{2n}^1$ and $x \in T^1$, then $ax \pmod L$ and $t(ax,\, w)$ are meaningful, and they depend only on a modulo L. Notice that $t(ax,\, z) = t(a,\, z)$ for every $a \in K_{2n}^1$ if and only if $x \in \widetilde{G}_{\mathbf{a}+}$.

We now consider the action of $\mathrm{Aut}(\mathbf{C})$ on the field of quotients of the formal series defined in §5.9. In particular we let $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ act on $\mathfrak{K} = \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. For the moment f^{σ} for $f \in \mathfrak{K}$ and $\sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ is just the quotient of two formal series, but the following lemma will show that $f^\sigma \in \mathfrak{K}$. For a fixed $a \in K_{2n}^1$ the quotients of the projective coordinates of $t(a, z)$ as functions of z belong to $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$, and so we can let $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}})$ act on $t(a, z)$.

8.5. Lemma. (1) *We have*

$$(8.7)\qquad t(a, z)^{[s,\,\mathbf{Q}]} = t\big(a\iota(s),\, z\big) \quad \textit{for every} \quad s \in \mathbf{Z}_{\mathbf{h}}^\times \quad \textit{and} \quad \textit{every} \quad a \in K_{2n}^1.$$

Consequently $\mathfrak{K}$ *is stable under* $\mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$.

(2) *Every element of* $\mathfrak{K}$ *is of the form* q/r *with* $q \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ *and* $0 \neq r \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q})$, *for some positive integer* κ, *such that* $q^\sigma \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ *for every* $\sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$.

(3) *For every subfield* D *of* $\mathbf{Q}_{\mathrm{ab}}$ *we have*

$$\mathcal{A}_0(D) = \big\{\, f \in \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}}) \,\big|\, f^\sigma = f \quad \textit{for every} \quad \sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/D) \,\big\}.$$

PROOF. We prove (1) only in Case UT. Case SP is similar and simpler. Take Λ as in Theorem 6.12 and §7.6. Let $a = ({}^t b,\, {}^t c) \in K_{2n}^1$ with $b, c \in N^{-1}\mathfrak{r}_1^n$, $0 < N \in \mathbf{Z}$. By (4.24), $p_z(a) = (z\bar{b} + \bar{c},\, {}^t z b + c)$, and hence, by Proposition 6.14, for $\ell_1, \ell_2 \in \Lambda$ we have

$$\theta_K\big(p_z(a),\, p^{-1}z;\, \ell_1\big)/\theta_K\big(p_z(a),\, p^{-1}z;\, \ell_2\big) = \theta_K\big(0,\, p^{-1}z;\, \ell_1'\big)/\theta_K\big(0,\, p^{-1}z;\, \ell_2'\big),$$

where $\ell_i'(h) = \ell_i(h - pb)\mathbf{e}\big(\mathrm{Tr}_{K/\mathbf{Q}}\big(c^*(h - pb)\big)\big)$ for $h \in K^n$. Take a multiple M of N so that $\ell(h) = 0$ for every $\ell \in \Lambda$ if $Mh \notin \mathfrak{r}^n$. Given $s \in \mathbf{Z}_{\mathbf{h}}^\times$, take $c_1 \in K^n$ so that $(c_1 - s^{-1}c)_v \in M\mathfrak{r}_v^n$ for every $v \in \mathbf{h}$. Since ℓ_i is $\mathbf{Z}$-valued, by (8.2), $\sigma = [s,\, \mathbf{Q}]$ sends the last quotient to $\theta_K\big(0;\, p^{-1}z;\, \lambda_1\big)/\theta_K\big(0;\, p^{-1}z;\, \lambda_2\big)$ with $\lambda_i(h) = \ell_i(h - pb)\mathbf{e}\big(\mathrm{Tr}_{K/\mathbf{Q}}\big(c_1^*(h - pb)\big)\big)$. This means that $t_z(a)^\sigma = t_z\big(({}^t b,\, {}^t c_1)\big)$, which proves (1).

Next, from this argument we can easily derive that every element of $\mathfrak{K}$ is of the form g_1/g with $g, g_1 \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ for some κ such that $g_1^\sigma, g^\sigma \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ for every $\sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$. (In Case SP, multiplying both denominator and numerator of (7.8) by the denominator, and employing Proposition 6.15, we can avoid half-integral weights.) Also $g \in \mathcal{M}_{\kappa\mathbf{a}}(\Phi)$ with a finite extension Φ of $\mathbf{Q}$ contained in $\mathbf{Q}_{\mathrm{ab}}$. Then taking r to be the product of g^σ for all $\sigma \in \mathrm{Gal}(\Phi/\mathbf{Q})$ and putting $q = g_1 r/g$, we obtain (2). Assertion (3) follows immediately from (2).

We now consider the following condition on Γ^N :

$$(8.8)\qquad t(a\gamma,\, z) = t(a,\, \gamma z) \quad \textit{for every} \quad a \in K_{2n}^1 \quad \textit{and} \quad \gamma \in \Gamma^N.$$

Let Γ_0 be the group as in Theorem 6.12 which we fixed in §7.6. Then, by (7.3), condition (8.8) is satified if $\Gamma^N \subset \Gamma_0$, which is so for any multiple N of a suitably chosen positive integer.

8.6. Lemma. *Under condition (8.8), given* $x \in T^N$, *there is a unique automorphism* $\tau(x)$ *of* $\mathfrak{K}$ *such that* $t(a, z)^{\tau(x)} = t(ax, z)$ *for every* $a \in K_{2n}^1$. *Moreover,* $\tau(x) = [\nu(x)^{-1},\, \mathbf{Q}]$ *on* $\mathbf{Q}_{\mathrm{ab}}$, $\tau(xy) = \tau(x)\tau(y)$, *and* $f^{\tau(x)} = f \circ x$ *for* $f \in \mathfrak{K}$ *if* $x \in \Gamma^N$.

PROOF. The uniqueness is obvious, since $\mathfrak{K}$ is generated by the affine coordinates of $t(a, z)$. Also the statement is true if $x \in \Gamma^N$, since (8.8) shows that the desired automorhism can be given by $f \to f \circ x$, which is the last assertion. Take a multiple M of N. Let Φ_M denote the field generated over $\mathbf{Q}$ by the affine coordinates of $t(a, w)$ for all $a \in M^{-1}L$. Now, given $x \in T^N$, by Lemma 8.3 (4) we can put $x = \iota(r)y\gamma$ with $r = \nu(x)^{-1}$, $y \in U^M$, and $\gamma \in \Gamma^N$. Then for $a \in M^{-1}L$ we have

$$t(ax, z) = t\big(a\iota(r)y\gamma, z\big) = t\big(a\iota(r)y, \gamma z\big) = t\big(a\iota(r), \gamma z\big) = \big(t(a, z)^{[r, \mathbf{Q}]}\big) \circ \gamma$$

by (8.7) and (8.8). This means that there is an automorphism σ of Φ_M that sends $t(a, z)$ to $t(ax, z)$ for every $a \in M^{-1}L$; moreover, $f^\sigma = (f^{[r, \mathbf{Q}]}) \circ \gamma$. Thus $\sigma = [r, \mathbf{Q}]$ on $\mathbf{Q}_{\mathrm{ab}} \cap \Phi_M$. Taking all multiples M of N, we obtain the desired automorphism $\tau(x)$. The equality $\tau(xy) = \tau(x)\tau(y)$ is obvious from the definition.

8.7. Lemma. *Under (8.8) let $\alpha, \beta \in \widetilde{G}_+$ and $x, y \in T^N$; suppose $y\beta \in \alpha x \widetilde{G}_{\mathbf{a}+}$. Then $(f \circ \alpha)^{\tau(x)} = f^{\tau(y)} \circ \beta$ for every $f \in \mathfrak{K}$.*

PROOF. Put $w = x^{-1}\alpha^{-1}y\beta$. Then $w \in \widetilde{G}_{\mathbf{a}+} \cap (G_0)_{\mathbf{A}} \subset T^N$ and $\tau(x) = \tau(xw)$. Therefore, changing x for xw, we may assume that $\alpha x = y\beta$. Changing also α and β for $c\alpha$ and $c\beta$ with a suitable $c \in \mathbf{Q}$, we may assume that $\alpha^{-1}, \beta^{-1} \prec \mathfrak{r}$. Take a generic point z_0 for $\mathfrak{F}$ over k as in §7.9. Since the map $f \mapsto f(z_0)$ is an isomorphism, we can define an automorphism σ of $\mathfrak{K}[z_0]$ by $f(z_0)^\sigma = f^{\tau(x)}(z_0)$ for $f \in \mathfrak{K}$. Then our assertion is equivalent to the equality $t(a, \alpha z_0)^\sigma = t(ay, \beta z_0)$. To simplify our notation, put $\alpha = \alpha_1$, $\beta = \alpha_2$, and $z_\nu = \alpha_\nu z_0$. Observe that $\mathfrak{K}[z_\nu] = \mathfrak{K}[z_0]$ and $A(z_0)^\sigma = A(z_0)$ since $t(a, z_0)^\sigma = t(ax, z_0)$ by our definition. Now we consider the following commutative diagram:

$$\begin{array}{ccccccc}
K^1_{2n} & \xrightarrow{p_{z_0}} & (\mathbf{C}^n)^{\mathbf{b}} & \longrightarrow & (\mathbf{C}^n)^{\mathbf{b}}/p_{z_0}(L) & \xrightarrow{\Theta_{z_0}} & A(z_0) \\
{\scriptstyle \alpha_\nu^{-1}}\downarrow & & \downarrow{\scriptstyle \Lambda_\nu} & & \downarrow & & \downarrow{\scriptstyle \lambda_\nu} \\
K^1_{2n} & \xrightarrow{p_{z_\nu}} & (\mathbf{C}^n)^{\mathbf{b}} & \longrightarrow & (\mathbf{C}^n)^{\mathbf{b}}/p_{z_\nu}(L) & \xrightarrow{\Theta_{z_\nu}} & A(z_\nu)
\end{array}$$

Here $\Lambda_\nu = {}^tM(\alpha_\nu, z_0)^{-1}$ with the symbol M of (4.29). From (4.31) we obtain $\Lambda_\nu p(a, z_0) = p(a\alpha_\nu^{-1}, z_\nu)$, which gives the leftmost part of the diagram; $\Theta_z(u) = \Theta_K(u, z)$ with Θ_K of Theorem 6.12; λ_ν is an isogeny determined by Λ_ν with the property that $\lambda_\nu t(a, z_0) = t(a\alpha_\nu^{-1}, z_\nu)$ and $\mathrm{Ker}(\lambda_\nu) = t(L\alpha_\nu, z_0)$. Notice that $\lambda_\nu t(a, z_0) = t(a\alpha_\nu^{-1}, z_\nu)$ holds even for $a \in (K^1_{2n})_{\mathbf{A}}$. Since $t(a, z_0)^\sigma = t(ax, z_0)$, we have $\mathrm{Ker}(\lambda_1)^\sigma = t(L\alpha_1 x, z_0) = t(Ly\alpha_2, z_0) = t(L\alpha_2, z_0) = \mathrm{Ker}(\lambda_2)$, and hence there exists an isomorphism ε of $A(z_2)$ onto $A(z_1)^\sigma$ such that $\lambda_1^\sigma = \varepsilon\lambda_2$. Observe that $A(z_\nu)$ and λ_ν are rational over $\mathfrak{K}[z_0]$. Now for every $a \in K^1_{2n}$, we have $t(a\alpha_1^{-1}, z_1)^\sigma = \lambda_1^\sigma t(a, z_0)^\sigma = \lambda_1^\sigma t(ax, z_0) = \varepsilon\lambda_2 t(ax, z_0) = \varepsilon t(ax\alpha_2^{-1}, z_2) = \varepsilon t(a\alpha_1^{-1}y, z_2)$, which shows that

$$(*) \qquad t(a, z_1)^\sigma = \varepsilon t(ay, z_2).$$

Put $\mathcal{P}(z) = (A_z, \mathcal{C}_z, \iota_z)$. Then ε gives an isomorphism of $\mathcal{P}(z_2)$ onto $\mathcal{P}(z_1)^\sigma$, which will be proven at the end of the proof. Taking (y^{-1}, z_2) in place of (x, z_0), we can define an automorphism σ' of $\mathfrak{K}[z_2] = \mathfrak{K}[z_0]$ such that $t(a, z_2)^{\sigma'} = t(ay^{-1}, z_2)$; then $\mathcal{P}(z_2)^{\sigma'} = \mathcal{P}(z_2)$ by Lemma 7.8. Put $\varepsilon' = \varepsilon^{\sigma'}$. Then

$$(**) \qquad t(a, z_1)^{\sigma\sigma'} = \big(\varepsilon t(ay, z_2)\big)^{\sigma'} = \varepsilon' t(a, z_2).$$

Now let N' be a multiple of N; taking N' to be N of §7.9, consider $g = (g_1, \ldots, g_m)$, M, V, and q as in §7.9. We can find a finitely generated extension k' of $\mathbf{Q}$ contained in k such that V_M, $V_{N'}$, $p^M_{N'}$, V, and q are all k'-rational and $g_i \in k'(V_M)$. Since $\mathfrak{K} = \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$, by Lemma 7.4 (1), $\mathfrak{K}$ and $k'\mathbf{Q}_{\mathrm{ab}}$ are linearly disjoint over $\mathbf{Q}_{\mathrm{ab}}$, and hence $\mathfrak{K}[z_0]$ and $k'\mathbf{Q}_{\mathrm{ab}}$ are linearly disjoint over $\mathbf{Q}_{\mathrm{ab}}$.

Therefore the automorphism $\sigma\sigma'$ of $\mathfrak{K}[z_0]$ over $\mathbf{Q}_{\mathrm{ab}}$ can be extended to an automorphism τ of $k'\mathfrak{K}[z_0]$ over $k'\mathbf{Q}_{\mathrm{ab}}$. Clearly z_ν is generic for $k'\mathfrak{F}$ over k'. Now $\varphi_M(z_1)^\tau = \varphi_M(w)$ with $w \in \mathcal{H}$ and $g(z_1)^\tau = q(\varphi_M(z_1))^\tau = q(\varphi_M(w)) = g(w)$. From this we can derive that $(\mathcal{P}_{z_1}^{N'})^\tau = \mathcal{P}_w^{N'}$. Indeed, w is generic for g over k' and so generic for $\mathfrak{K}$ over k', since $\mathfrak{K}$ is algebraic over $\mathbf{Q}(g)$. Thus $f(z_1) \mapsto f(w)$ for $f \in \mathfrak{K}$ is an isomorphism, which coincides with τ on $\mathbf{Q}(g(z_1))$, and which sends $\mathcal{P}_{z_1}^{N'}$ to $\mathcal{P}_w^{N'}$ by Lemma 7.8. Since $\mathcal{P}_{z_1}^{N'}$ is defined over $\mathbf{Q}(g(z_1))$, this shows that $(\mathcal{P}_{z_1}^{N'})^\tau = \mathcal{P}_w^{N'}$, so that $t(a, z_1)^\tau = t(a, w)$ for every $a \in (N')^{-1}L$. Since $\mathcal{P}(z_2)^{\sigma'} = \mathcal{P}(z_2)$, (**) together with (7.5a) shows that ε' is an isomorphism of $\mathcal{P}_{z_2}^{N'}$ to $(\mathcal{P}_{z_1}^{N'})^{\sigma\sigma'}$. Thus $z_2 = \gamma w$ with some $\gamma \in \Gamma^{N'}$, and for $a \in (N')^{-1}L$ we have $t(a, z_2) = t(a, \gamma w) = t(a\gamma, w) = t(a, w) = t(a, z_1)^\tau$. This shows that $t(a, z_1)^{\sigma\sigma'} = t(a, z_2)$ for every $a \in K_{2n}^1$, since our reasoning is applicable to every multiple N' of N. Thus $t(a, \alpha z_0)^{\sigma\sigma'} = t(a, z_1)^{\sigma\sigma'} = t(a, z_2) = t(ay, z_2)^{\sigma'}$. Applying σ'^{-1} to this, we obtain the expected equality.

It remains to prove that ε is an isomorphism of $\mathcal{P}(z_2)$ onto $\mathcal{P}(z_1)^\sigma$. We first observe that $\nu(yx^{-1}) = \nu(\alpha_1\alpha_2^{-1}) \in F \cap F_{\mathbf{a}+}\mathbf{Z}_{\mathbf{h}}^\times$, so that $\nu(x) = \nu(y)$. Applying σ to (7.5a) and employing (*), we obtain $\iota_{z_1}(c)^\sigma \varepsilon t(ay, z_2) = \iota_{z_1}(c)^\sigma t(a, z_1)^\sigma = t(ca, z_1)^\sigma = \varepsilon t(cay, z_2) = \varepsilon\iota_{z_2}(c)t(ay, z_2)$, so that $\iota_{z_1}(c)^\sigma\varepsilon = \varepsilon\iota_{z_2}(c)$. Next, let X_ν denote the divisor X_z for $z = z_\nu$ given in the proof of Lemma 7.12. By (7.10) we have $\zeta_{X_\nu}(t(a, z_\nu), t(b, z_\nu)) = \mathbf{e}((m/s)\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*))$ for every $a, b \in m^{-1}L$, $0 < m \in \mathbf{Z}$. Let $Y = \varepsilon^{-1}(X_1^\sigma)$ and let E' be the Riemann form determined by Y. Then by (*), (2.10), and (2.11) we have, for such a and b,

$$\begin{aligned}\mathbf{e}\{mE'(p_{z_2}(ay), p_{z_2}(by))\} &= \zeta_Y(t(ay, z_2), t(by, z_2)) = \zeta_{X_1^\sigma}(\varepsilon t(ay, z_2), \varepsilon t(by, z_2)) \\ &= \zeta_{X_1}(t(a, z_1), t(b, z_1))^\sigma = \mathbf{e}((m/s)\mathrm{Tr}_{K/\mathbf{Q}}(a\eta_n b^*))^\sigma.\end{aligned}$$

In view of (8.2) and (7.10), the last quantity equals $\mathbf{e}\{mE_{z_2}(p_{z_2}(ay), p_{z_2}(by))\}$, since $\sigma = [\nu(x)^{-1}, \mathbf{Q}]$ on $\mathbf{Q}_{\mathrm{ab}}$ and $\nu(x) = \nu(y)$. Thus $E' = E_{z_2}$, which means that ε sends $\mathcal{C}_{z_2}$ to $(\mathcal{C}_{z_1})^\sigma$. This completes the proof.

8.8. Put

$$\mathcal{G} = (G_0)_{\mathbf{A}}\widetilde{G}\widetilde{G}_{\mathbf{a}+}, \qquad \mathcal{G}_+ = \mathcal{G} \cap \widetilde{G}_{\mathbf{A}+}. \tag{8.9}$$

Since $(G_0)_{\mathbf{A}}$ and $\widetilde{G}_{\mathbf{a}+}$ are normal subgroups of $\widetilde{G}_{\mathbf{A}}$, $\mathcal{G}$ and $\mathcal{G}_+$ are subgroups of $\widetilde{G}_{\mathbf{A}}$; moreover,

$$\mathcal{G}_+ = (G_0)_{\mathbf{A}+}\widetilde{G}_+\widetilde{G}_{\mathbf{a}+} = T^N\widetilde{G}_+\widetilde{G}_{\mathbf{a}+} = \widetilde{G}_+T^N\widetilde{G}_{\mathbf{a}+} \quad \textit{for every} \quad N. \tag{8.10}$$

Indeed, let $\alpha \in (G_0)_{\mathbf{A}}$ and $\beta \in \widetilde{G}$; suppose $\nu(\alpha\beta) \gg 0$. Since $\nu(\alpha) \in \mathbf{Q}_{\mathbf{A}}^\times$, we can find an element γ of G_0 such that that $\nu(\alpha\gamma) \gg 0$. Then $\alpha\beta = \alpha\gamma\gamma^{-1}\beta$, $\alpha\gamma \in (G_0)_{\mathbf{A}+}$, and $\gamma^{-1}\beta \in \widetilde{G}_+$. The equalities of (8.10) follow from this fact and Lemma 8.3 (1).

Now given $x \in \mathcal{G}_+$, take $c \in \mathbf{Q}_{\mathbf{A}}^\times$ so that $\nu(x) \in cF^\times F_{\mathbf{a}+}^\times$. We then define $\sigma(x)$ to be $[c^{-1}, \mathbf{Q}]$ $(\in \mathrm{Aut}(\mathbf{Q}_{\mathrm{ab}}))$. Then we easily see that $\sigma(x)$ is well-defined independently of the choice of c, and thus we obtain a homomorphism

$$\sigma : \mathcal{G}_+ \longrightarrow \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q}). \tag{8.11}$$

8.9. Lemma. (1) $\alpha U^N\alpha^{-1} = U^N$ *for every* N *if* $\alpha \in \widetilde{G}_{\mathbf{A}}$ *and* $L\alpha = L$.
(2) $\Gamma^N U^N$ *and* $\Gamma^N T^N$ *are subgroups of* $\widetilde{G}_{\mathbf{A}}$.

(3) *$\Gamma^N U^N$ is a normal subgroup of $\Gamma^1 T^1$.*
(4) *$\Gamma^N T^N/\Gamma^N U^N$ is isomorphic to T^N/U^N.*

PROOF. Since $(G_0)_{\mathbf{A}}$ is normal in $\widetilde{G}_{\mathbf{A}}$, we easily obtain (1). In particular $\gamma U^N\gamma^{-1} = U^N$ for every $\gamma \in \Gamma^N$, so that $\Gamma^N U^N$ is a subgroup of $\widetilde{G}_{\mathbf{A}}$. Let $x \in T^1$ and $\gamma \in \Gamma^N$. Then $x^{-1}\gamma x\gamma^{-1} - 1 \prec N\mathfrak{r}$, so that $x^{-1}\gamma x\gamma^{-1} \in U^N$, and hence $\gamma x\gamma^{-1} \in T^N$ if $x \in T^N$. Thus $\Gamma^N T^N$ is a subgroup of $\widetilde{G}_{\mathbf{A}}$. Clearly $\alpha(\Gamma^N U^N)\alpha^{-1} = \Gamma^N U^N$ for $\alpha \in \Gamma^1$. Now let $x \in T^1$. Then $xU^Nx^{-1} = U^N$. For $\gamma \in \Gamma^N$, we have $x^{-1}\gamma x \in U^N\gamma \subset \Gamma^N U^N$. Thus $x\Gamma^N x^{-1} \subset \Gamma^N U^N$. This proves (3). From Lemma 8.3 (2) we obtain $\Gamma^N U^N \cap T^N = U^N$, from which (4) follows.

8.10. Theorem. *There exists a homomorphism $\tau : \mathcal{G}_+ \longrightarrow \mathrm{Aut}(\mathfrak{K})$ with the following properties:*
(1) *$\tau(\xi) = \sigma(\xi)$ on $\mathbf{Q}_{\mathrm{ab}}$.*
(2) *$f^{\tau(\alpha)} = f \circ \alpha$ for every $f \in \mathfrak{K}$ and every $\alpha \in \widetilde{G}_+$.*
(3) *$f^{\tau(\xi)} = f^{[s,\,\mathbf{Q}]}$ if $\xi = \iota(s)$ with $s \in \mathbf{Z}_{\mathbf{h}}^{\times}$.*
(4) *$\mathrm{Ker}(\tau) = K^{\times}\widetilde{G}_{\mathbf{a}+}$.*
(5) *For any fixed $f \in \mathfrak{K}$ the set $\{\,\xi \in \mathcal{G}_+ \mid f^{\tau(\xi)} = f\,\}$ contains U^N for some N.*
(6) *Let $k_N = \mathbf{Q}(\mathbf{e}(1/N))$. Then, for every N we have*

$$\mathcal{A}_0(\Gamma^N,\, k_N) = \{\, f \in \mathfrak{K} \mid f^{\tau(x)} = f \quad \text{for every } \ x \in \Gamma^N U^N \,\}, \tag{8.12}$$

$$\mathcal{A}_0(\Gamma^N,\, \mathbf{Q}) = \{\, f \in \mathfrak{K} \mid f^{\tau(x)} = f \quad \text{for every } \ x \in \Gamma^N T^N \,\}. \tag{8.13}$$

Here we remind the reader that $\mathfrak{K} = \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$.

PROOF. Fix any N as in (8.8). By (8.10), given $\xi \in \mathcal{G}_+$, we can find $x \in T^N$ and $\alpha \in \widetilde{G}_+$ such that $\xi \in x\alpha\widetilde{G}_{\mathbf{a}+}$ with $x \in T^N$ and $\alpha \in \widetilde{G}_+$. We then define an automorphism $\tau(\xi)$ of $\mathfrak{K}$ by $f^{\tau(\xi)} = f^{\tau(x)} \circ \alpha$. To show that this is well-defined, let $x\alpha \in x_1\alpha_1\widetilde{G}_{\mathbf{a}+}$ with $x_1 \in T^M$ and $\alpha_1 \in \widetilde{G}_+$ with a multiple M of N. Put $\gamma = \alpha_1\alpha^{-1}$. By Lemma 8.3 (2), $\gamma \in \Gamma^N \cap T^N$, and $f^{\tau(x)} \circ \alpha = f^{\tau(x_1\gamma)} \circ \alpha = f^{\tau(x_1)} \circ \gamma\alpha = f^{\tau(x_1)} \circ \alpha_1$, since $g^{\tau(\gamma)} = g \circ \gamma$ as shown in Lemma 8.6. Thus $\tau(\xi)$ is well-defined independently of $(N,\, x,\, \alpha)$. Next let $\xi' \in x'\alpha'\widetilde{G}_{\mathbf{a}+}$ with $x' \in T^N$ and $\alpha' \in \widetilde{G}_+$. By (8.10) we have $\alpha x' \in y\beta\widetilde{G}_{\mathbf{a}+}$ with $y \in T^N$ and $\beta \in \widetilde{G}_+$. Then $\xi\xi' \in xy\beta\alpha'\widetilde{G}_{\mathbf{a}+}$, and employing Lemma 8.7, we can easily verify that $\tau(\xi)\tau(\xi') = \tau(\xi\xi')$. Now $\tau(\xi) = \tau(x)$ on $\mathbf{Q}_{\mathrm{ab}}$, and $\tau(x) = [\nu(x)^{-1},\, \mathbf{Q}]$ on $\mathbf{Q}_{\mathrm{ab}}$ by Lemma 8.6. Since $\nu(\xi) \in \nu(x)F^{\times}F_{\mathbf{a}+}^{\times}$, this proves (1). Property (3) is clear from our definition and Lemmas 8.5, 8.6. To prove (4), let $\xi \in \alpha\widetilde{G}_{\mathbf{a}+}$ with $\alpha \in K^{\times}$. Then $t(a,\, z)^{\tau(\xi)} = t(a,\, \alpha z) = t(a,\, z)$, so that $\tau(\xi) = \mathrm{id}$. Suppose conversely $\tau(\xi) = \mathrm{id}$. on $\mathfrak{K}$ and $\xi \in x\alpha\widetilde{G}_{\mathbf{a}+}$ with $x \in T^N$ and $\alpha \in \widetilde{G}_+$. Then

$$t(a,\, z) = t(a,\, z)^{\tau(\xi)} = t(ax,\, \alpha z) \quad \text{for every } \ a \in K_{2n}^1. \tag{8.14}$$

Since $\sigma(\xi) = 1$, we have $[\nu(x)^{-1}, \mathbf{Q}] = \sigma(x) = 1$. Now $\nu(x) \in \nu(T^N) = \mathbf{Z}_{\mathbf{h}}^{\times}\mathbf{Q}_{\mathbf{a}+}^{\times}$, so that $\nu(x)_{\mathbf{h}} = 1$. Then we easily see that $x \in U^N$. Fix a generic point $z_0 \in \mathcal{H}$ as before. Then (8.14) shows that $t(a,\, z_0) = t(ax,\, \alpha z_0)$, and hence $\mathcal{P}^N(z_0) = \mathcal{P}^N(\alpha z_0)$ by Lemma 7.8, since $x - 1 \prec N\mathfrak{r}$. Thus $\alpha z_0 = \gamma z_0$ with $\gamma \in \Gamma^N$. Taking z_0 generic even for the action of $\widetilde{G}_+$, we find that $\alpha \in K^{\times}\gamma$. Now $t(a,\, z) = t(ax,\, \gamma z) = t(ax\gamma,\, z)$ for every $a \in K_{2n}^1$, and hence $x\gamma \in \widetilde{G}_{\mathbf{a}+}$, so that $\xi = x\alpha \in K^{\times}\widetilde{G}_{\mathbf{a}+}$. This proves (4). To prove (5), we observe that $t(a,\, z)^{\tau(x)} = t(ax,\, z) =$

$t(a, z)$ if $x \in U^N$ and $a \in N^{-1}L$. Since $\mathfrak{K}$ is generated by the coordinates of such points $t(a, z)$, we easily obtain (5).

To prove (6) (in which we do not assume (8.8)), let us write simply f^x for $f^{\tau(x)}$. Let f be an element of the right-hand side of (8.12). Then $f \circ \gamma = f$ for every $\gamma \in \Gamma^N$. Moreover, if $s \in \mathbf{Z}_\mathbf{h}^\times$ and $s - 1 \prec N\mathbf{Z}$, then $\iota(s) \in U^N$, and so $f^{\iota(s)} = f$. Thus, by Lemma 8.5 (3), $f \in \mathcal{A}_0(\Gamma^N, k_N)$. Similarly if f belongs to the right-hand side of (8.13), then the same type of argument shows that $f \in \mathcal{A}_0(\Gamma^N, \mathbf{Q})$. Conversely, let $f \in \mathcal{A}_0(\Gamma^N, k_N)$. By (5) there exists a multiple M of N such that $f^\xi = f$ for every $\xi \in U^M$. Given $x \in U^N$, by Lemma 8.3 (4) we can put $x = \iota(r)y\gamma$ with $r = \nu(x)^{-1}$, $y \in U^M$, and $\gamma \in \Gamma^N$. Since $f \in \mathcal{A}_0(\Gamma^N, k_N)$, we have $f^{\iota(r)} = f = f \circ \gamma$, and hence $f^x = f^{\iota(r)y\gamma} = f^y \circ \gamma = f$, which proves (8.12). Next, let $f \in \mathcal{A}_0(\Gamma^N, \mathbf{Q})$ and $x \in T^N$. Then $x = \iota(p)x'$ with $p \in \mathbf{Z}_\mathbf{h}^\times$ and $x' \in U^N$. Since $f^{\iota(p)} = f$ and $f^{x'} = f$ by (8.12), we obtain $f^x = f$. This proves (8.13).

8.11. Theorem. (1) *$\mathfrak{K}$ is algebraic over $\mathcal{A}_0(\Gamma^1, \mathbf{Q})$.*

(2) *$\mathcal{A}_0(\Gamma^N, \mathbf{Q})$ and $\mathbf{C}$ are linearly disjoint over $\mathbf{Q}$. Moreover, $\mathcal{A}_0(\Gamma^N, k) = k\mathcal{A}_0(\Gamma^N, \mathbf{Q})$ for every subfield k of $\mathbf{C}$; in particular $\mathcal{A}_0(\Gamma^N) = \mathbf{C}\mathcal{A}_0(\Gamma^N, \mathbf{Q})$.*

(3) *$\mathcal{A}_0(\mathbf{Q})$ and $\mathbf{C}$ are linearly disjoint over $\mathbf{Q}$. Moreover $\mathcal{A}_0(k) = k\mathcal{A}_0(\mathbf{Q})$ for every subfield k of $\mathbf{C}$; in particular $\mathcal{A}_0 = \mathbf{C}\mathcal{A}_0(\mathbf{Q})$.*

(4) *Given $f \in \mathcal{A}_0$, there exists a finitely generated extension k of $\mathbf{Q}$ such that $f \in \mathcal{A}_0(k)$.*

(5) *Given $f \in \mathcal{A}_0$ and $\sigma \in \mathrm{Aut}(\mathbf{C})$, the element f^σ formally defined in §5.9 is indeed an element of $\mathcal{A}_0$.*

PROOF. The linear disjointness in (2) and (3) follows immediately from Lemma 7.4 (1). Now observe that T^N/U^N is isomorphic to $(\mathbf{Z}/N\mathbf{Z})^\times$ (resp. to $\mathrm{Gal}(k_N/\mathbf{Q})$) via the map $x \mapsto \nu(x)$ (resp. via $x \mapsto \sigma(x)$) for $x \in T^N$. From Lemma 8.9 (3) and (8.12) we see that $\mathcal{A}_0(\Gamma^N, k_N)$ is stable under $\tau(T^N)$. Thus we obtain a homomorphism of T^N/U^N into the group of automorphisms of $\mathcal{A}_0(\Gamma^N, k_N)$. This is injective, since T^N/U^N is isomorphic to $\mathrm{Gal}(k_N/\mathbf{Q})$. By (8.13) the fixed subfield of T^N/U^N is $\mathcal{A}_0(\Gamma^N, \mathbf{Q})$, so that $\mathcal{A}_0(\Gamma^N, k_N)$ is a Galois extension of $\mathcal{A}_0(\Gamma^N, \mathbf{Q})$ whose Galois group is isomorphic to T^N/U^N. By the linear disjointness we have $[k_N\mathcal{A}_0(\Gamma^N, \mathbf{Q}) : \mathcal{A}_0(\Gamma^N, \mathbf{Q})] = [k_N : \mathbf{Q}]$, and hence we obtain

$$\mathcal{A}_0(\Gamma^N, k_N) = k_N\mathcal{A}_0(\Gamma^N, \mathbf{Q}). \tag{8.15}$$

Now $\mathcal{A}_0 = \mathbf{C}\mathfrak{K}$ by Theorem 7.10 (3) and every quotient of (7.8) belongs to $\mathcal{A}_0(\Gamma^N, k_N)$ for some N. Therefore we can conclude that $\mathcal{A}_0 = \mathbf{C}\mathcal{A}_0(\mathbf{Q})$. Then (4) and (5) follow immediately from this fact. Now, for any subfield k of $\mathbf{C}$, we have $\mathbf{C}\mathcal{A}_0(k) = \mathcal{A}_0 = \mathbf{C}k\mathcal{A}_0(\mathbf{Q})$, and $k\mathcal{A}_0(\mathbf{Q}) \subset \mathcal{A}_0(k)$. Since $\mathbf{C}$ and $\mathcal{A}_0(k)$ are linearly disjoint over k, we obtain $\mathcal{A}_0(k) = k\mathcal{A}_0(\mathbf{Q})$ by Lemma 7.2. This completes the proof of (3).

To prove the main part of (2), given N, take a finitely generated extension k of $\mathbf{Q}$ so that V_N is defined over k. Then $k(V_N) \circ \varphi_N = k(f_1, \dots, f_m)$ with suitable $f_i \in \mathcal{A}_0(\Gamma^N, \mathbf{C})$. By (4) we can find a finitely generated extension k' of k so that $f_i \in \mathcal{A}_0(\Gamma^N, k')$ for every i. Then $k'(V_N) \circ \varphi_N = k'(f_1, \dots, f_m) \subset \mathcal{A}_0(\Gamma^N, k')$. By Lemma 7.4 (2) we have $k'(f_1, \dots, f_m) = \mathcal{A}_0(\Gamma^N, k')$. Since $\mathcal{A}_0(k') = k'\mathcal{A}_0(\mathbf{Q})$, we can find $g_1, \dots, g_r \in \mathcal{A}_0(\mathbf{Q})$ so that $f_1, \dots, f_m \in k'(g_1, \dots, g_r)$. Take a multiple M of N so that $g_1, \dots, g_r \in \mathcal{A}_0(\Gamma^M, \mathbf{Q})$. Replacing k' by $k'k_M$, we may assume that $k_M \subset k'$. From (8.12) and Lemma 8.9 (3) we see that $\mathcal{A}_0(\Gamma^M, k_M)$

is stable under Γ^N. Since $\mathcal{A}_0(\Gamma^N, k_M)$ consists of the Γ^N-invariant elements of $\mathcal{A}_0(\Gamma^M, k_M)$, we see that $\mathcal{A}_0(\Gamma^M, k_M)$ is a Galois extension of $\mathcal{A}_0(\Gamma^N, k_M)$. By Lemma 7.4 (1), $\mathcal{A}_0(k_M)$ and k' are linearly disjoint over k_M, so that $k'\mathcal{A}_0(\Gamma^M, k_M)$ is a Galois extension of $k'\mathcal{A}_0(\Gamma^N, k_M)$ with the same Galois group. Now the elements of $\mathcal{A}_0(\Gamma^N, k')$ are contained in $k'\mathcal{A}_0(\Gamma^M, k_M)$ and Γ^N-invariant, so that $\mathcal{A}_0(\Gamma^N, k') = k'\mathcal{A}_0(\Gamma^N, k_M)$. Now we can let $\Gamma^N U^N/\Gamma^M U^M$ act on $\mathcal{A}_0(\Gamma^M, k_M)$, and observe that $\Gamma^N U^N/\Gamma^M U^M$ can be mapped onto the Galois group of $\mathcal{A}_0(\Gamma^M, k_M)$ over $\mathcal{A}_0(\Gamma^N, k_N)$. (Taking $N = 1$ here, we obtain (1).) We easily see that $\tau(\alpha x)$, with $\alpha \in \Gamma^N$ and $x \in U^N$, gives the identity map on $k_M\mathcal{A}_0(\Gamma^N, k_N)$ if and only if $\nu(x) - 1 \prec M\mathbf{Z}$. Given such an x, by Lemma 8.3 (4) we have $x = \iota(r)y\gamma$ with $r = \nu(x)^{-1}$, $y \in U^M$, and $\gamma \in \Gamma^N$. Then $[r, \mathbf{Q}] =$ id. on k_M, so that for $f \in \mathcal{A}_0(\Gamma^N, k_M)$ and $\alpha \in \Gamma^N$ we have $f^{\tau(\alpha x)} = f^{\tau(y\gamma)} = f$. By Galois theory this shows that $\mathcal{A}_0(\Gamma^N, k_M) = k_M\mathcal{A}_0(\Gamma^N, k_N)$. This combined with (8.15) shows that $\mathcal{A}_0(\Gamma^N, k_M) = k_M\mathcal{A}_0(\Gamma^N, \mathbf{Q})$. Thus $\mathcal{A}_0(\Gamma^N, k') = k'\mathcal{A}_0(\Gamma^N, k_M) = k'\mathcal{A}_0(\Gamma^N, \mathbf{Q})$. Consequently $\mathcal{A}_0(\Gamma^N) = \mathbf{C}(V_N)\circ\varphi_N = \mathbf{C}k'(V_N)\circ\varphi_N = \mathbf{C}\mathcal{A}_0(\Gamma^N, k') = \mathbf{C}\mathcal{A}_0(\Gamma^N, \mathbf{Q})$. This proves the last statement of (2). Then for any subfield k_0 of $\mathbf{C}$ we have $\mathbf{C}\mathcal{A}_0(\Gamma^N, k_0) \subset \mathcal{A}_0(\Gamma^N, \mathbf{C}) = \mathbf{C}\mathcal{A}_0(\Gamma^N, \mathbf{Q}) = \mathbf{C}k_0\mathcal{A}_0(\Gamma^N, \mathbf{Q}) \subset \mathbf{C}\mathcal{A}_0(\Gamma^N, k_0)$, so that $\mathbf{C}k_0 \cdot \mathcal{A}_0(\Gamma^N, \mathbf{Q}) = \mathbf{C}\mathcal{A}_0(\Gamma^N, k_0)$. Since $\mathcal{A}_0(\Gamma^N, k_0)$ and $\mathbf{C}$ are linearly disjoint over k_0 by Lemma 7.4 (1), we obtain $k_0\mathcal{A}_0(\Gamma^N, \mathbf{Q}) = \mathcal{A}_0(\Gamma^N, k_0)$ by Lemma 7.2. This completes the proof.

8.12. Theorem. *For a point w on $\mathcal{H}$ and a positive integer N, let $\mathfrak{K}_N[w]$ denote the field consisting of $f(w)$ for all $f \in \mathfrak{K}_N$, where $\mathfrak{K}_N = \mathcal{A}_0(\Gamma^N, k_N)$. Then $\mathfrak{K}_N[w]$ is the field of moduli of $\mathcal{P}_w^N$.*

PROOF. We shall prove this in the proof of Theorem 9.3, invoking some results of [S66a]. Here we prove the case where w is generic for $\mathfrak{K}$ over $\mathbf{Q}$, using the results obtained in this section so far. In §7.9 we observed that $\mathcal{P}_w^N$ is rational over $\mathfrak{K}[w]$. Therefore we can find a multiple M of N such that $\mathcal{P}_w^N$ is rational over $\mathfrak{K}_M[w]$. Since $f \mapsto f(w)$ is an isomorphism of $\mathfrak{K}$ onto $\mathfrak{K}[w]$, we can let $\mathcal{G}_+$ act on $\mathfrak{K}[w]$ by putting $f(w)^x = f^{\tau(x)}(w)$ for $x \in \mathcal{G}_+$ and $f \in \mathfrak{K}$. As shown in the proof of Theorem 8.11, $\Gamma^N U^N/\Gamma^M U^M$ is mapped onto $\mathrm{Gal}(\mathfrak{K}_M[w]/\mathfrak{K}_N[w])$. Let Φ be the field of moduli of $\mathcal{P}_w^N$. To show that $\Phi \subset \mathfrak{K}_N[w]$, take $x \in U^N\Gamma^N$. By Lemma 8.3 (4), $x = \iota(r)y\gamma$ with $r = \nu(x)^{-1}$, $y \in U^M$, and $\gamma \in \Gamma^N$. Since $t(a, w)^{\iota(r)} = t(a\iota(r), w)$ and $r - 1 \prec N\mathbf{Z}$, we obtain $(\mathcal{P}_w^N)^{\iota(r)} = \mathcal{P}_w^N$ by Lemma 7.8. Since $\mathcal{P}_w^N$ is rational over the field $\mathfrak{K}_M[w]$ on which y gives the identity map, we have $(\mathcal{P}_w^N)^y = \mathcal{P}_w^N$. Thus $(\mathcal{P}_w^N)^x = (\mathcal{P}_w^N)^\gamma = \mathcal{P}_{\gamma w}^N$, as $f(w)^\gamma = f(\gamma w)$ for every $f \in \mathfrak{K}$. Since $\gamma \in \Gamma^N$, $\mathcal{P}_{\gamma w}^N$ is isomorphic to $\mathcal{P}_w^N$. Therefore the property of Φ stated in Theorem 2.8 (1), (ii) implies that x gives the identity map on Φ, so that $\Phi \subset \mathfrak{K}_N[w]$.

To show that $\mathfrak{K}_N[w] \subset \Phi$, we consider V_N, φ_N, and p_N^M as in §7.9. We take a finitely generated extension k of k_M over which V_M, V_N, and p_N^M are rational. Since $k(V_M)\circ\varphi_M \subset \mathcal{A}_0(\Gamma^M) \subset \mathbf{C}\mathcal{A}_0(\Gamma^M, \mathbf{Q})$, changing k suitably, we may assume that $k(V_M) \circ \varphi_M \subset \mathcal{A}_0(\Gamma^M, k)$. Then $k(V_M) \circ \varphi_M = \mathcal{A}_0(\Gamma^M, k)$ by Lemma 7.4 (2). Since $\varphi_N = p_N^M \circ \varphi_M$, we obtain $k(V_N) \circ \varphi_N = \mathcal{A}_0(\Gamma^N, k)$. Now $k_N \subset \Phi$ by Lemma 7.12. To prove our theorem, we may assume that w is generic for $k\mathfrak{K}$ over k. Then $\varphi_N(w)$ (resp. $\varphi_M(w)$) is a generic point of V_N (resp. V_M) over k. Let σ be an isomorphism of $\mathfrak{K}_N[w]$ onto a subfield of $\mathbf{C}$ over Φ. Since k and $\mathfrak{K}_N[w]$ are linearly disjoint over k_N, we can extend σ to an isomorphism of $k\mathfrak{K}_N[w]$ onto a subfield of $\mathbf{C}$ over $k\Phi$. Extend this further to $k\mathfrak{K}_M[w]$. Then $\varphi_M(w)^\sigma$ is a point of V_M, so that $\varphi_M(w)^\sigma = \varphi_M(w')$ with some $w' \in \mathcal{H}$. Since p_N^M is k-rational, we have $\varphi_N(w)^\sigma = \varphi_N(w')$, and w' is generic for $k\mathfrak{K}_M$ over k. Since $\mathcal{P}_w^N$ is rational over $k\mathfrak{K}_M[w] = k(\varphi_M(w))$, we have $(\mathcal{P}_w^N)^\sigma = \mathcal{P}_{w'}^N$. Since $\sigma = \mathrm{id.}$ on Φ, $(\mathcal{P}_w^N)^\sigma$ must be isomorphic to $\mathcal{P}_w^N$, so that $w' \in \Gamma^N w$. Therefore $\varphi_N(w)^\sigma = \varphi_N(w') = \varphi_N(w)$, and hence $\sigma = \mathrm{id.}$ on $\mathfrak{K}_N[w]$. Thus $\mathfrak{K}_N[w] \subset \Phi$, which completes the proof.

9. The reciprocity-law at CM-points and rationality of automorphic forms

The first main purpose of this section is to study the behavior of the values of the elements of $\mathfrak{K}$ at a CM-point under certain automorphisms, which may be viewed as an explicit reciprocity law of a certain abelian extension. The next task is to extend the results of the previous two sections to automorphic forms of nontrivial weight. First we quote a theorem without proof:

9.1. Theorem. *Given a PEL-type $\Omega = \{K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^s\}$ as in (4.7) (in all three cases), consider the family $\mathcal{F}(\Omega) = \{\mathcal{P}_z \mid z \in \mathcal{H}\}$ as in (4.26) and define Γ by (4.28). Then there exist an algebraic number field k_Ω of finite degree and a model (V, φ) of $\Gamma\backslash\mathcal{H}$ with the following properties:*

(1) *Let K_Ψ be the field generated over $\mathbf{Q}$ by the numbers $\mathrm{tr}(\Psi(c))$ for all $c \in K$. Then $K_\Psi \subset k_\Omega$.*

(2) *If $\alpha \in \mathrm{Aut}(\mathbf{C})$ and $\mathcal{P}$ is a structure of type Ω, then $\mathcal{P}^\alpha$ is of type Ω if and only if $\alpha = \mathrm{id.}$ on k_Ω, where $\mathcal{P}^\alpha$ is defined as in §2.7.*

(3) *V is defined over k_Ω.*

(4) *$k_\Omega(\varphi(w))$ is the field of moduli of $\mathcal{P}_w$ for every $w \in \mathcal{H}$.*

This is a simplified form of [S66a, Theorems 5.1 and 6.2], which are applicable to a PEL-type of a more general type. We call (V, φ) a *canonical model* for Ω. We have $K_\Psi = \mathbf{Q}$ in Cases SP and UT, and so (1) is trivial in those cases; we will show in Theorem 9.3 below that $k_\Omega = k_N$ if Ω is Ω^N of (7.4b). In Case UB we can show that $K_\Psi = \mathbf{Q}$ if $m_v = n_v$ for every $v \in \mathbf{a}$ and K_Ψ is a CM-field otherwise; here m_v, n_v are as in (4.10).

It may be added that no complete proof of the existence of (V, φ) with property (4) was given, even when $\Gamma = Sp(n, \mathbf{Z})$, in any paper published before 1966.

9.2. Lemma. *Let σ be an isomorphism of k_Ω onto a subfield k' of $\overline{\mathbf{Q}}$ over K_Ψ. Then there exists a PEL-type Ω' with the following properties:*

(1) *$\Omega' = \{K, \Psi, L', \lambda\mathcal{T}, \{u_i'\}_{i=1}^s\}$ with the same K, Ψ, $\mathcal{T}$ as Ω, a totally positive element λ of F, and some L', $\{u_i'\}_{i=1}^s$; we can take $\lambda = 1$ in Cases SP and UT.*

(2) *$k' = k_{\Omega'}$.*

(3) *If $\alpha \in \mathrm{Aut}(\mathbf{C})$, $\alpha = \sigma$ on k_Ω, and $\mathcal{P}$ is of type Ω, then $\mathcal{P}^\alpha$ is of type Ω'.*

(4) *Let (V', φ') be a canonical model for Ω' and let $\mathcal{F}(\Omega') = \{\mathcal{P}_z' \mid z \in \mathcal{H}\}$. Then there exists a k'-rational biregular map f of V^σ onto V' with the property that $f(\varphi(z)^\alpha) = \varphi'(w)$ whenever α is as in (3) and $(\mathcal{P}_z)^\alpha$ is isomorphic to $\mathcal{P}_w'$.*

PROOF. Pick any $\mathcal{P}$ of type Ω; then $\mathcal{P}^\alpha$ is of type Ω' for some PEL-type Ω'. This Ω' does not depend on the choice of $\mathcal{P}$ by virtue of [S66a, Proposition 4.1], as explained in [S66a, p.323, lines 1∼4]. From Theorem 9.1 (2) we see that $k' = k_{\Omega'}$. For $K \neq F$ assertion (1) was proved in [S64, Proposition 5.2]. Recall that, by virtue of (4.17), Ψ is determined by the signature of $\mathcal{T}_v$ for $v \in \mathbf{a}$, and vice versa. If $K = F$, we can always put $\mathcal{T} = \eta_n$ as explained in §4.3, and hence $\mathcal{T}' = \eta_n$ too.

In the proof of the following theorem we shall determine Ω' in Cases Sp and UT without invoking [S64]. In any case, the space $\mathcal{H}$ is common to Ω and Ω'.

Now, to any structure $\mathcal{P}$ of type Ω we assign a point $\mathfrak{v}(\mathcal{P})$ of V as follows: by Theorem 4.8 (2), $\mathcal{P}$ is isomorphic to $\mathcal{P}_z$ for some $z \in \mathcal{H}$; we then put $\mathfrak{v}(\mathcal{P}) = \varphi(z)$. This symbol $\mathfrak{v}$ has the following properties:

(i) $k_\Omega(\mathfrak{v}(\mathcal{P}))$ *is the field of moduli of* $\mathcal{P}$.
(ii) $\mathfrak{v}(\mathcal{P}_1) = \mathfrak{v}(\mathcal{P}_2)$ *if and only if* $\mathcal{P}_1$ *is isomorphic to* $\mathcal{P}_2$.
(iii) *If* $\mathfrak{p}$ *is a* $\mathbf{C}$*-valued discrete place of a field of rationality for* $\mathcal{P}$ *of type* Ω *such that* $\mathfrak{p}(a) = a$ *for every* $a \in k_\Omega$ *and* $\mathcal{P}_0$ *is the reduction of* $\mathcal{P}$ *modulo* $\mathfrak{p}$, *then* $\mathcal{P}_0$ *is of type* Ω *and* $\mathfrak{p}(\mathfrak{v}(\mathcal{P})) = \mathfrak{v}(\mathcal{P}_0)$.

Properties (i) and (ii) follow immediately from our definition; (iii) was given in [S66a, Theorem 6.2]. For each $\mathcal{Q}$ of type Ω' we define $\mathfrak{v}'(\mathcal{Q}) \in V'$ in the same manner. Now take any α as in (3). Then $\mathcal{Q}^{\alpha^{-1}}$ is of type Ω, and so $\mathfrak{v}(\mathcal{Q}^{\alpha^{-1}})$ is meaningful. We then define a point $\mathfrak{v}^\sigma(\mathcal{Q})$ of V^σ by $\mathfrak{v}^\sigma(\mathcal{Q}) = \mathfrak{v}(\mathcal{Q}^{\alpha^{-1}})^\alpha$. In view of (iii) above, $\mathfrak{v}^\sigma(\mathcal{Q})$ is well-defined independently of the choice of α. Now by [S66a, Theorem 6.7], $(V, \mathfrak{v})$ can be characterized by properties (ii) and (iii). Since we easily see that $(V^\sigma, \mathfrak{v}^\sigma)$ has these properties for Ω', that theorem guaratees a k'-rational biregular map f of V^σ onto V' such that $f(\mathfrak{v}^\sigma(\mathcal{Q})) = \mathfrak{v}'(\mathcal{Q})$ for every $\mathcal{Q}$ of type Ω'. If $(\mathcal{P}_z)^\alpha$ is isomorphic to $\mathcal{P}'_w$, then $\mathfrak{v}^\sigma((\mathcal{P}_z)^\alpha) = \mathfrak{v}(\mathcal{P}_z)^\alpha = \varphi(z)^\alpha$, and so $f(\varphi(z)^\alpha) = \mathfrak{v}'((\mathcal{P}_z)^\alpha) = \mathfrak{v}'(\mathcal{P}'_w) = \varphi'(w)$, which is (4). This completes the proof.

9.3. Theorem. *Suppose that* Ω *is* Ω^N *of (7.4b) with* $L = \mathfrak{r}^1_{2n}$ *in Cases SP and UT; let* (V, φ) *and* k_Ω *be as in Theorem 9.1 and let* $k_N = \mathbf{Q}(\mathbf{e}(1/N))$. *Then the following assertions hold:*

(1) $k_\Omega = k_N$.
(2) $k_\Omega(V) \circ \varphi = \mathcal{A}_0(\Gamma^N, k_N)$.
(3) *There exists a model* (W, ψ) *of* $\Gamma^N \backslash \mathcal{H}$ *such that* W *is defined over* $\mathbf{Q}$ *and* $\mathbf{Q}(W) \circ \psi = \mathcal{A}_0(\Gamma^N, \mathbf{Q})$.

PROOF. Once (1) and (2) are established, Theorem 9.1 (4) implies Theorem 8.12 (for an arbitrary w). Now, given $\Omega = \Omega^N$ as in (7.4b) with $L = \mathfrak{r}^1_{2n}$, for each $s \in \mathbf{Z}_\mathbf{h}^\times$ we put $\Omega_s = \{K, \Psi, L, \eta_n, \{u_i\iota(s)\}\}$, where $u_i\iota(s)$ is an element of $N^{-1}L$ determined modulo L as explained in §8.4; let $\mathcal{F}(\Omega_s) = \{\mathcal{P}_{z,s} \mid z \in \mathcal{H}\}$. Take a point z_0 of $\mathcal{H}$ generic for $\mathfrak{K}$ over $\mathbf{Q}_{\mathrm{ab}}$; fixing s, define an automorphism ξ of $\mathfrak{K}[z_0]$ by $f(z_0)^\xi = f^{\tau(\iota(s))}(z_0)$ for $f \in \mathfrak{K}$. By Lemma 8.6 this means $t(a, z_0)^\xi = t(a\iota(s), z_0)$ for every $a \in K^1_{2n}$. By Lemma 7.8, or rather by its proof, $(\mathcal{P}_{z_0})^\xi = \mathcal{P}_{z_0,s}$. Let (V_s, φ_s) be a canonical model for Ω_s; put $k_s = k_{\Omega_s}$ and $\mathfrak{K}_N = \mathcal{A}_0(\Gamma^N, k_N)$. By Theorem 8.12, $\mathfrak{K}_N[z_0]$ is the field of moduli of $\mathcal{P}_{z_0,s}$, which equals $k_s(\varphi_s(z_0))$. Since k_N is algebraically closed in $\mathfrak{K}_N$, we see that $k_s = k_N$, and hence $k_N(\varphi_s(z_0)) = \mathfrak{K}_N[z_0]$. From this we can conclude that $k_N = k_\Omega$ and $k_N(V_s) \circ \varphi_s = \mathfrak{K}_N$, which gives (1) and(2) if we take $s = 1$.

Let σ be the restriction of ξ to k_N. Taking Ω' of Lemma 9.2 to be Ω_s, we obtain a k_N-rational biregular map of V^σ to V_s. This means that we may assume that $V^\sigma = V_s$, and

(9.1) $\varphi(z)^\alpha = \varphi_s(w)$ if $\alpha \in \mathrm{Aut}(\mathbf{C})$, $\alpha = \sigma$ on k_N, and $(\mathcal{P}_z)^\alpha$ is isomorphic to $\mathcal{P}_{w,s}$. In particular, $\varphi(z_0)^\xi = \varphi_s(z_0)$.

Clearly Ω_s depends only on $\{\, s_p \pmod{N\mathbf{Z}_p} \,\}_p$. Writing Ω_σ, φ_σ, and $\mathcal{P}_{z,\sigma}$ for Ω_s, φ_s, and $\mathcal{P}_{z,s}$, we thus have a canonical model $(V^\sigma, \varphi_\sigma)$ for Ω_σ for each $\sigma \in \mathrm{Gal}(k_N/\mathbf{Q})$. Since $(V^\sigma, \varphi_\sigma)$ is a model of $\Gamma^N\backslash\mathcal{H}$, there exists a biregular map $g_\sigma : V \to V^\sigma$ such that $\varphi_\sigma = g_\sigma \circ \varphi$. Then $k_N\big(\varphi(z_0)\big) = \mathfrak{K}_N[z_0] = k_N\big(\varphi_\sigma(z_0)\big)$. Since $\varphi(z_0)$ and $\varphi_\sigma(z_0)$ are generic on V and V^σ over k_N, there exists a k_N-rational map of V to V^σ that sends $\varphi(z_0)$ to $\varphi_\sigma(z_0)$. Clearly this map must coincide with g_σ; thus g_σ is k_N-rational. Now for each $\sigma \in \mathrm{Gal}(k_N/\mathbf{Q})$ and $f \in \mathfrak{K}_N$ we have a well-defined element f^σ of $\mathfrak{K}_N$, which is the same as $f^{\tau(\iota(s))}$ if $\sigma = [s, \mathbf{Q}]$ on k_N. Define an automorphism α of $\mathfrak{K}[z_0]$ by $f(z_0)^\alpha = f^\sigma(z_0)$ for $f \in \mathfrak{K}$. Given $\tau \in \mathrm{Gal}(k_N/\mathbf{Q})$, define similarly β by $f(z_0)^\beta = f^\tau(z_0)$. Then $(\mathcal{P}_{z_0})^\alpha = \mathcal{P}_{z_0,\sigma}$ as observed at the beginning, and similarly $(\mathcal{P}_{z_0})^\beta = \mathcal{P}_{z_0,\tau}$ and $(\mathcal{P}_{z_0})^{\alpha\beta} = \mathcal{P}_{z_0,\sigma\tau}$. Therefore $\varphi(z_0)^\alpha = \varphi_\sigma(z_0)$, $\varphi(z_0)^\beta = \varphi_\tau(z_0)$, and $\varphi(z_0)^{\alpha\beta} = \varphi_{\sigma\tau}(z_0)$. Thus $\big((g_\sigma)^\tau \circ g_\tau\big)\big(\varphi(z_0)\big) = (g_\sigma)^\tau\big(\varphi_\tau(z_0)\big) = (g_\sigma)^\tau\big(\varphi(z_0)^\beta\big) = g_\sigma\big(\varphi(z_0)\big)^\beta = \varphi_\sigma(z_0)^\beta = \varphi(z_0)^{\alpha\beta} = \varphi_{\sigma\tau}(z_0) = g_{\sigma\tau}\big(\varphi(z_0)\big)$, and so $(g_\sigma)^\tau \circ g_\tau = g_{\sigma\tau}$, since $\varphi(z_0)$ is generic on V over k_N. Applying a well-known criterion of [W56, Theorem 3] to $\{\, V^\sigma, g_\sigma \,\}$, we find a $\mathbf{Q}$-rational variety W and a k_N-rational biregular map h of V onto W such that $h = h^\sigma \circ g_\sigma$ for every $\sigma \in \mathrm{Gal}(k_N/\mathbf{Q})$. Put $\psi = h \circ \varphi$. Clearly (W, ψ) is a model of $\Gamma^N\backslash\mathcal{H}$. Now $h\big(\varphi(z_0)\big)^\alpha = h^\sigma\big(\varphi_\sigma(z_0)\big) = (h^\sigma \circ g_\sigma)\big(\varphi(z_0)\big) = h\big(\varphi(z_0)\big)$. Thus $h\big(\varphi(z_0)\big)$ is rational over the subfield of $\mathfrak{K}_N[z_0]$ fixed by the automorphisms α. This subfield corresponds to the subfield of $\mathfrak{K}_N = \mathcal{A}_0(\Gamma^N, k_N)$ fixed by $\mathrm{Gal}(k_N/\mathbf{Q})$, which is $\mathcal{A}_0(\Gamma^N, \mathbf{Q})$. Since $k_N\big(h(\varphi(z_0))\big) = \mathfrak{K}_N[z_0]$, we see that $\mathbf{Q}\big(h(\varphi(z_0))\big)$ corresponds to $\mathcal{A}_0(\Gamma^N, \mathbf{Q})$. This means that $\mathbf{Q}(W) \circ \psi = \mathcal{A}_0(\Gamma^N, \mathbf{Q})$, which completes the proof.

9.4. Given an algebraic number field M of finite degree contained in $\mathbf{C}$, we denote its maximal abelian extension contained in $\mathbf{C}$ by M_{ab}. By class field theory there exists a canonical homomorphism of $M_\mathbf{A}^\times$ onto $\mathrm{Gal}(M_{\mathrm{ab}}/M)$ whose kernel is the closure of the product of $M^\times$ and the identity component of $M_\mathbf{a}^\times$. We denote by $[a, M]$ the element of $\mathrm{Gal}(M_{\mathrm{ab}}/M)$ which is the image of $a \in M_\mathbf{A}^\times$. (This includes $[t, \mathbf{Q}]$ of §8.1 as a special case.)

Let us now briefly recall the notion of a reflex of a CM-type. (For details, see [S98, §§8.3 and 18.5].) Given a CM-type (K, Φ) in the sense of §2.9, take a Galois extension L of $\mathbf{Q}$ containing K and take elements φ_ν of G so that $\Phi = \{\, \varphi_1, \ldots, \varphi_n \,\}$. (Thus $n = [K : \mathbf{Q}]$.) Let K^* be the field generated over $\mathbf{Q}$ by the elements $\sum_{\nu=1}^n x^{\varphi_\nu}$ for all $x \in K$. Then it can be shown that K^* is a CM-field. Put $G = \mathrm{Gal}(L/\mathbf{Q})$, $H = \mathrm{Gal}(L/K)$, and $H^* = \mathrm{Gal}(L/K^*)$. Then we have

$$(9.2) \qquad \bigsqcup_{\nu=1}^n \varphi_\nu^{-1} H = \bigsqcup_{\mu=1}^m H^* \tau_\mu$$

with $\tau_\mu \in G$, where $m = [K^* : \mathbf{Q}]/2$. We can show that (K^*, Φ^*) with $\Phi^* = \{\, \tau_\mu \,\}$ is a CM-type, which we call *the reflex of* (K, Φ). The field K^* is called *the reflex field of* (K, Φ). We can also define a map $g : (K^*)^\times \to K^\times$ by

$$(9.3) \qquad g(a) = \prod_{\mu=1}^m a^{\tau_\mu} \qquad (a \in (K^*)^\times).$$

The map g can be extended naturally to a homomorphism $(K^*)_\mathbf{A}^\times \to K_\mathbf{A}^\times$. We shall denote this map g also by $\det \Phi^*$. Since $\{\, \varphi_\nu, \varphi_\nu\rho \,\}_{\nu=1}^n$ gives the set of all embeddings of K into $\mathbf{C}$, we obtain:

$$(9.4) \qquad g(a)g(a)^\rho = N_{K^*/\mathbf{Q}}(a) \qquad \big(a \in (K^*)_\mathbf{A}^\times\big).$$

We now consider a CM-algebra $Y = K_1 \oplus \cdots \oplus K_t$, such that $[Y : K] = 2n$, a K-linear ring-injection $h : Y \to K_{2n}^{2n}$ such that

$$h(a^\rho)\eta_n = \eta_n h(a)^*, \tag{9.5}$$

and the fixed point w of $h(Y^u)$ as in §4.11. Define CM-types (K_i, Φ_i) as noted at the end of that subsection. Let (K_i^*, Φ_i^*) be the reflex of (K_i, Φ_i), and Y^* the composite field of the K_i^*. (Note: Y is a K-algebra, but Y^* is a subfield of $\mathbf{C}$.) We then define a map $g : (Y^*)_{\mathbf{A}}^\times \to Y_{\mathbf{A}}^\times$ by

$$g(x) = \Big(\det \Phi_i^*\big(N_{Y^*/K_i^*}(x)\big)\Big)_{i=1}^t . \tag{9.6}$$

9.5. Lemma. *The notation being as above, for every $x \in (Y^*)_{\mathbf{A}}^\times$ we have $\nu\big(h(g(x))\big) = g(x)g(x)^\rho = N_{Y^*/\mathbf{Q}}(x)$ and $\det\big[h(g(x))\big] = N_{Y^*/\mathbf{Q}}(x)^n$. Consequently $h(g(x)) \in (G_0)_{\mathbf{A}+}$.*

PROOF. That $g(x)g(x)^\rho = N_{Y^*/\mathbf{Q}}(x)$ follows immediately from (9.4) and (9.6). Also from this and (9.5) we see that $h(g(x)) \in \widetilde{G}_{\mathbf{A}+}$ and $\nu\big(h(g(x))\big) = g(x)g(x)^\rho$. This proves our lemma in Case SP. To compute $\det\big[h(g(x))\big]$ in Case UT, take the Galois closure over $\mathbf{Q}$ of the composite of the K_i and put $G = \mathrm{Gal}(L/\mathbf{Q})$, $H_i = \mathrm{Gal}(L/K_i)$, $H_i^* = \mathrm{Gal}(L/K_i^*)$, $H^* = \mathrm{Gal}(L/Y^*)$, and $J = \mathrm{Gal}(L/K)$. Let R be the group-ring of G over $\mathbf{Z}$; for $x = \sum_{\gamma\in G} c_\gamma\gamma \in R$ with $c_\gamma \in \mathbf{Z}$ put $x' = \sum_{\gamma\in G} c_\gamma\gamma^{-1}$; for a subgroup H of G put $[H] = \sum_{\alpha\in H}\alpha$. Take $\varphi_{i\nu}$ and $\tau_{i\mu}$ in G so that $\Phi_i = \{\varphi_{i\nu}\}_\nu$ and $\Phi_i^* = \{\tau_{i\mu}\}_\mu$. Then $\big(\sum_\nu [H_i]\varphi_{i\nu}\big)' = \sum_\mu [H_i^*]\tau_{i\mu}$ by (9.2). Since Ψ is equivalent to n times a regular representation of K over $\mathbf{Q}$, we have $n[G] = \sum_{i,\nu}[J]\varphi_{i\nu}$. Given $x \in (Y^*)_{\mathbf{A}}^\times$, put $x_i = N_{Y^*/K_i^*}(x)$ and $y_i = \det\Phi_i^*(x_i)$. Then $g(x) = (y_i)_{i=1}^t$ and $\det\big[h(g(x))\big] = \prod_{i=1}^t N_{K_i/K}(y_i)$, since h is equivalent to a regular representation of Y over K. Now $N_{K_i/K}(y_i) = \prod_{\gamma\in H_i\backslash J} y_i^\gamma = \prod_\mu\prod_{\gamma\in H_i\backslash J} x_i^{\tau_{i\mu}\gamma}$, and $\sum_\mu [H_i^*]\tau_{i\mu}\sum_{\gamma\in H_i\backslash J}\gamma = \sum_\nu \varphi_{i\nu}^{-1}[H_i]\sum_{\gamma\in H_i\backslash J}\gamma = \sum_\nu\varphi_{i\nu}^{-1}[J]$. Since $H^* \subset H_i^*$, we can write $\sum_\nu\varphi_{i\nu}^{-1}[J] = \sum_\lambda [H^*]\alpha_{i\lambda}$ with some $\alpha_{i\lambda} \in G$. Then $N_{K_i/K}(y_i) = \prod_\lambda x^{\alpha_{i\lambda}}$, so that $\det\big[h(g(x))\big] = \prod_{i=1}^t\prod_\lambda x^{\alpha_{i\lambda}}$. Since $\sum_{i,\nu}\varphi_{i\nu}^{-1}[J] = n[G]$, the $\alpha_{i\lambda}$ give n times $H^*\backslash G$. Therefeore the last double product is $N_{Y^*/\mathbf{Q}}(x)^n$, which completes the proof.

Now our main theorem on the reciprocity-law can be stated as follows:

9.6. Theorem. *Let Y, h, w, and Y^* be as in §9.4 in Case SP or UT. Then for every $f \in \mathfrak{K}$ defined at w, the value $f(w)$ belongs to Y_{ab}^*. Moreover, if $b \in (Y^*)_{\mathbf{A}}^\times$, then $f^{\tau(r)}$ with $r = h\big(g(b)^{-1}\big)$ is finite at w and $f(w)^{[b,\,Y^*]} = f^{\tau(r)}(w)$. (Notice that $\tau(r)$ is meaningful, since $r \in (G_0)_{\mathbf{A}+}$.)*

PROOF. Fixing N, we use the same symbols as in the proof of Theorem 9.3. Given $b \in (Y^*)_{\mathbf{A}}^\times$ and r as above, by Lemma 8.3 (1) we can put $r = y\iota(s)\alpha$ with $y \in U^N$, $s \in \mathbf{Z}_{\mathbf{h}}^\times$, and $\alpha \in G_{0+}$. By Lemma 9.5, $N_{Y^*/\mathbf{Q}}(b) = \nu(r)^{-1} = s\cdot\nu(y\alpha)^{-1}$. Take $\varepsilon \in \mathrm{Aut}(\mathbf{C}/Y^*)$ so that $\varepsilon = [b,\, Y^*]$ on Y_{ab}^*; let σ be the restriction of ε to k_N. Then $\sigma = [s,\, \mathbf{Q}]$ on k_N, and we have a canonical model (V^σ, φ_s) for $\Omega_\sigma = \Omega_s$. Now let $f = f_1\circ\varphi$ with $f_1 \in k_N(V)$. Then $f \in \mathfrak{K}_N$ and f_1^σ is meaningful as an element of $k_N(V^\sigma)$. Since $\varphi(z_0)$ is a generic point of V over k_N, we have $f_1^\sigma\big(\varphi(z_0)^\xi\big) = f_1\big(\varphi(z_0)\big)^\xi = f(z_0)^\xi$. The last quantity is $f^{\tau(\iota(s))}(z_0)$ by the definition of ξ. Since $\varphi(z_0)^\xi = \varphi_s(z_0)$ as noted in (9.1), we obtain $f_1^\sigma\big(\varphi_s(z_0)\big) = f^{\tau(\iota(s))}(z_0)$. Since z_0 is generic for $\mathfrak{K}$ over k_N, we have thus

$$(f_1\circ\varphi)^{\tau(\iota(s))} = f_1^\sigma\circ\varphi_s \quad \textit{for every} \quad f_1 \in k_N(V). \tag{9.7}$$

Next we consider the member $\mathcal{P}_w = \bigl(A_w, \mathcal{C}_w, \iota_w, \{t_i\}\bigr)$ of our family as in Theorem 9.3 for the CM-point w in question. Put $\mathcal{Q} = \bigl(A_w, \mathcal{C}_w, \iota'\bigr)$ with $\iota' : Y \to \mathrm{End}_{\mathbf{Q}}(A_w)$ as in §4.11. Take $e_0 \in K_{2n}^1$ so that $K_{2n}^1 = e_0h(Y)$ and define $q : Y_{\mathbf{a}} \to (\mathbf{C}^n)^{\mathbf{b}}$ by $q(a) = p_w\bigl(e_0h(a)\bigr)$ for $a \in Y_{\mathbf{a}}$. Then (4.39) shows that $q(ca) = {}^t\Phi(c)q(a)$ for $c \in Y$. Take a $\mathbf{Z}$-lattice $\mathfrak{a}$ in Y so that $L = e_0h(\mathfrak{a})$. Then $q(\mathfrak{a}) = p_w(L)$, and t_w of (7.2) gives an isomorphism of $(\mathbf{C}^n)^{\mathbf{b}}/q(\mathfrak{a})$ onto A_w; $t_i = t_w\bigl(q(v_i)\bigr)$ with an element $v_i \in Y$ such that $e_0h(v_i) = u_i$. Take $\zeta \in Y$ so that $\mathrm{Tr}_{K/\mathbf{Q}}\bigl(e_0h(a)\eta_n e_0^*\bigr) = \mathrm{Tr}_{Y/\mathbf{Q}}(\zeta a)$ for every $a \in Y$. Then from (4.25) and (9.5) we obtain

$$E_w\bigl(q(a),\, q(b)\bigr) = \mathrm{Tr}_{K/\mathbf{Q}}\bigl(e_0h(a)\eta_n h(b)^* e_0^*\bigr) = \mathrm{Tr}_{K/\mathbf{Q}}\bigl(e_0h(ab^\rho)\eta_n e_0^*\bigr) = \mathrm{Tr}_{Y/\mathbf{Q}}(\zeta ab^\rho).$$

Thus $\mathcal{Q}$ is of type $\{\, Y,\, {}^t\Phi,\, \mathfrak{a},\, \zeta \,\}$ with respect to t_w in the sense of [S98, §§18.4, 18.7]. We now apply the main theorem of complex multiplication of abelian varieties, as stated in [S98, Theorem 18.8], to $\mathcal{Q}^\varepsilon$. Then $\mathcal{Q}^\varepsilon$ is of type $\{\, Y,\, {}^t\Phi,\, g(b)^{-1}\mathfrak{a},\, \mu\zeta \,\}$ with respect to an isomorphism ξ' of $(\mathbf{C}^n)^{\mathbf{b}}/q\bigl(g(b)^{-1}\mathfrak{a}\bigr)$ to $(A_w)^\varepsilon$, where μ is the positive rational number such that $\mu\mathbf{Z} = N_{Y^*/\mathbf{Q}}(b)\mathbf{Z}$; besides, $t_w\bigl(q(a)\bigr)^\varepsilon = \xi'\bigl(q(g(b)^{-1}a)\bigr)$ for every $a \in Y/\mathfrak{a}$. (The present q is not exactly the same as the map q of [S98, §18.4]. However, the only property of q we need in the proof of [S98, Theorems 18.6 and 18.8] is that $q(cx) = \Phi(c)q(x)$ for $c,\, x \in Y$. The present ${}^t\Phi$ corresponds to Φ there. Thus there is no problem with the present q.) Now for $a \in Y/\mathfrak{a}$ we have

$$q\bigl(ag(b)^{-1}\bigr) = p_w\bigl(e_0h\bigl(ag(b)^{-1}\bigr)\bigr) = p_w\bigl(e_0h(a)r\bigr) = p_w\bigl(e_0h(a)y\iota(s)\alpha\bigr),$$

so that $q\bigl(\mathfrak{a}g(b)^{-1}\bigr) = p_w\bigl(L\alpha\bigr)$ and $q\bigl(v_ig(b)^{-1}\bigr) = p_w\bigl(u_i\iota(s)\alpha\bigr)$. Thus $(\mathcal{P}_w)^\varepsilon$ is of type $\Omega' = \{\, K,\, \Psi,\, L\alpha,\, \mu\eta_n, \{u_i\iota(s)\alpha\} \,\}$ with respect to p_w and ξ'. In other words, $(\mathcal{P}_w)^\varepsilon$ is isomorphic to the member $\mathcal{P}'_w$ of the family $\mathcal{F}(\Omega')$ at w. Put $w' = \alpha(w)$ and $\Lambda = {}^tM(\alpha, w)$. By (4.31) we have $p_w(x\alpha) = \Lambda p_{w'}(x)$ for $x \in (K_{\mathbf{a}})_{2n}^1$. Since $N_{Y^*/\mathbf{Q}}(b) = s \cdot \nu(y\alpha)^{-1}$, we have $\mu = \nu(\alpha)^{-1}$. Therefore we easily see that Λ gives an isomorphism of $\mathcal{P}_{w',s}$ onto $\mathcal{P}'_w$. Thus $(\mathcal{P}_w)^\varepsilon$ is isomorphic to $\mathcal{P}_{w',s}$, so that $\varphi(w)^\varepsilon = \varphi_s(w')$ by (9.1). Let $f \in \mathfrak{K}_N$; take $f_1 \in k_N(V)$ so that $f = f_1 \circ \varphi$. Then $f(w)^\varepsilon = f_1\bigl(\varphi(w)\bigr)^\varepsilon = f_1^\sigma\bigl(\varphi(w)^\varepsilon\bigr) = f_1^\sigma\bigl(\varphi_s(w')\bigr) = (f_1^\sigma \circ \varphi_s)(w') = f^{\tau(\iota(s))}(\alpha w)$ by (9.7). (Here notice that if f is finite at w, then f_1 is finite at $\varphi(w)$, and $f^{\tau(\iota(s))}$ is finite at αw.) Since $f^{\tau(y)} = f$ by (8.12), we have $f^{\tau(r)}(w) = f^{\tau(\iota(s)\alpha)}(w) = f^{\tau(\iota(s))}(\alpha w) = f(w)^\varepsilon$. Since $\tau(r)$ depends only on b, we see that $f(w)^\varepsilon$ depends only on the restriction of ε to Y^*_{ab}. Therefore $f(w) \in Y^*_{\mathrm{ab}}$, and we obtain the desired equality of our theorem for $f \in \mathfrak{K}_N$. Since N is arbitrary, this completes the proof.

In the elliptic modular case we have $G_0 = \widetilde{G} = GL_2(\mathbf{Q})$ and $\mathcal{G}+ = \widetilde{G}_{\mathbf{A}+}$, and we take an imaginary quadratic field as Y. Then Y^* is the isomorphic image of Y in $\mathbf{C}$. Thus the above theorem specialized to that case is exactly the principal result of the classical theory of complex multipication given in [S71, Theorem 6.31].

In this book we consider only canonical models associated with PEL-types introduced in Section 4. Actually we can define canonical models of arithmetic quotients of hermitian symmetric spaces that are not necessarily associated with PEL-types; we can even prove a reciprocity-law similar to that of Theorem 9.6 in such cases. For details the reader is referred to [S67], [S70], and [S98, Section 26].

9.7. Let the notation be as in Theorem 9.1, and let $\kappa \in \mathbf{Z}$. For every nonzero element h of $\mathcal{A}_{\kappa\mathbf{a}}(\Gamma)$ we can speak of its divisor on V, which is a divisor on the variety V in the sense of algebraic geometry. We denote it by $\mathrm{div}(h)$. Given a subfield k of $\mathbf{C}$ containing k_Ω and a k-rational divisor X on V, we define the standard symbols $\mathcal{L}(X)$ and $\mathcal{L}(X, k)$ of linear systems as follows:

$$\mathcal{L}(X) = \{ f \in \mathbf{C}(V) \,|\, \operatorname{div}(f) \succ -X \}, \tag{9.8a}$$

$$\mathcal{L}(X, k) = \mathcal{L}(X) \cap k(V). \tag{9.8b}$$

It is well-known that $\mathcal{L}(X) = \mathcal{L}(X, k) \otimes_k \mathbf{C}$. Also we easily see that $\mathcal{L}(\operatorname{div}(h))$ is $\mathbf{C}$-linearly isomorphic to $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma)$ via the map $f \mapsto fh$. (Here we have to exclude the one-dimensional case that requires the cusp condition. Therefore, strictly speaking, we have to modify the proofs of the following theorems in that case, a task we leave to the reader.)

Hereafter we consider only Cases SP and UT, and denote by (V_N, φ_N) a model of $\Gamma^N\backslash\mathcal{H}$ such that $\mathbf{Q}(V_N) \circ \varphi_N = \mathcal{A}_0(\Gamma^N, \mathbf{Q})$ established in Theorem 9.3 above. Then for any subfield k of $\mathbf{C}$ we have $k(V) \circ \varphi_N = k\mathcal{A}_0(\Gamma^N, \mathbf{Q}) \subset \mathcal{A}_0(\Gamma^N, k)$. By Lemma 7.4 we have $k(V) \circ \varphi_N = \mathcal{A}_0(\Gamma^N, k)$. Therefore we can take $\mathcal{A}_0(\Gamma^N)$ and $\mathcal{A}_0(\Gamma^N, k)$ in place of $\mathbf{C}(V)$ and $k(V)$ in (9.8a, b).

9.8. Proposition. (1) *If* $0 \neq h \in \mathcal{A}_{\kappa\mathbf{a}}(\Gamma^N, D)$ *with* $\kappa \in \mathbf{Z}$ *and a subfield* D *of* $\mathbf{C}$, *then* $\operatorname{div}(h)$ *considered on* V_N *is* D*-rational.*

(2) *There exist a positive integer* λ *and a nonzero element* $g \in \mathcal{A}_\lambda(\Gamma^1, \mathbf{Q})$ *such that* $\operatorname{div}(g)$ *considered on* V_N *for every* N *is* $\mathbf{Q}$*-rational.*

PROOF. We prove this only in Case UT; Case SP can be treated in a similar and much simpler way. To prove (2), let m be the complex dimension of $\mathcal{H}$. Take m algebraically independent functions in $\mathcal{A}_0(\Gamma^1, \mathbf{Q})$. Multiplying these by the product of the denominators, we obtain algebraically independent functions $f_1/f_0, \ldots, f_m/f_0$ in $\mathcal{A}_0(\Gamma^1, \mathbf{Q})$ with $f_\nu \in \mathcal{M}_{\mu\mathbf{a}}(\Gamma^M, \mathbf{Q})$ with some M and $\mu > 0$. Let z^v_{ab} be the (a, b)-entry of the matrix z_v which is the v-th component of the variable $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$. Let $h_p = f_p/f_0$ and

$$r = (2\pi i)^{-m}\partial(h_1, \ldots, h_m)/\partial(z_1, \ldots, z_m),$$

where $z_1, \ldots, z_m$ are an arbitrarily fixed arrangement of the variables z^v_{ab} for all $v \in \mathbf{a}$ and $1 \le a, b \le n$. In view of Lemma 3.4 (2), we see that $r \in \mathcal{A}_{2n\mathbf{a}}(\Gamma^1)$. Now for a function of the form $f(z) = \sum_h c(h)\mathbf{e}^n_{\mathbf{a}}(hz)$ as in (5.22a) we have

$$\partial f/\partial z^v_{ab} = 2\pi i \textstyle\sum_h c(h) h^v_{ba}\mathbf{e}^n_{\mathbf{a}}(hz),$$

where h^v_{ba} is the image of h_{ba} under v. We have also

$$(2\pi i)^{-1} f_0^2 \partial h_p/\partial z^v_{ab} = (2\pi i)^{-1}\big(f_0 \partial f_p/\partial z^v_{ab} - f_p \partial f_0/\partial z^v_{ab}\big). \tag{9.9}$$

Therefore we easily see that $f_0^{2m} r \in \mathcal{M}_{l\mathbf{a}}(\Gamma^M, \Phi)$, where Φ is the Galois closure of K over $\mathbf{Q}$ and $l = 2m\mu + 2n$. Moreover, $\operatorname{div}(r)$, considered on V_N for any fixed N, is the same as the divisor of $dh_1 \wedge \cdots \wedge dh_m$ on V_N, which is $\mathbf{Q}$-rational. Now we can view $f_0^{2m} r$ as the determinant of a matrix of size $n^2\#\mathbf{a}$ whose (v, a, b)-th column is $(2\pi i)^{-1}\big[f_0^2 \partial h_p/\partial z^v_{ab}\big]_{p=1}^m$. Call this column vector $(f; v, a, b)$. Let $\sigma \in \operatorname{Gal}(\Phi/\mathbf{Q})$. Applying σ to (9.9), we observe that

$$(f; v, a, b)^\sigma = \begin{cases} (f; v', a, b) & \text{if } v\sigma = v' \text{ on } K, \\ (f; v', b, a) & \text{if } v\sigma = v'\rho \text{ on } K, \end{cases}$$

where ρ is complex conjugation. Therefore we have $r^\sigma = \pm r$, and hence $\operatorname{div}(r^\sigma) = \operatorname{div}(dh_1 \wedge \cdots \wedge dh_m) = \operatorname{div}(r)$. Then we obtain the desired function g of (2) by

$g = \prod_\sigma r^\sigma$, where σ runs over $\mathrm{Gal}(\Phi/\mathbf{Q})$. Let $0 \neq h \in \mathcal{A}_{\kappa\mathbf{a}}(\Gamma^N, D)$. Take g as above. Then $h^\lambda/g^\kappa \in \mathcal{A}_0(\Gamma_N, D)$ and $\mathrm{div}(h^\lambda/g^\kappa) = \lambda \cdot \mathrm{div}(h) - \kappa \cdot \mathrm{div}(g)$. Since this divisor and $\mathrm{div}(g)$ are D-rational, $\mathrm{div}(h)$ must be D-rational. This proves (1).

9.9. Theorem. *Let $0 < \kappa \in \mathbf{Z}$ and $0 < N \in \mathbf{Z}$; let D be an arbitrary sufield of $\mathbf{C}$. Then the following assertions hold:*

(1) *$\mathcal{M}_{\kappa\mathbf{a}}(\Gamma^N) = \mathcal{M}_{\kappa\mathbf{a}}(\Gamma^N, D) \otimes_D \mathbf{C}$ provided $\mathcal{A}_{\kappa\mathbf{a}}(\Gamma^N, D) \neq \{0\}$.*

(2) *$\mathcal{M}_{\kappa\mathbf{a}} = \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{C}$.*

(3) *Given $f \in \mathcal{A}_{\kappa\mathbf{a}}$, there exists a finitely generated extension E of $\mathbf{Q}$ such that $f \in \mathcal{A}_{\kappa\mathbf{a}}(E)$.*

(4) *Let $f \in \mathcal{M}_{\kappa\mathbf{a}}$ (resp. $f \in \mathcal{A}_{\kappa\mathbf{a}}$) and $\sigma \in \mathrm{Aut}(\mathbf{C})$. Then f^σ, defined as a formal element in §5.9, is indeed an element of $\mathcal{M}_{\kappa\mathbf{a}}$ (resp. $\mathcal{A}_{\kappa\mathbf{a}}$).*

PROOF. Let $0 \neq h \in \mathcal{A}_{\kappa\mathbf{a}}(\Gamma^N, D)$. By Proposition 9.8 (1), $\mathrm{div}(h)$ is D-rational. Then $f \mapsto hf$ is an isomorphism of $\mathcal{L}(\mathrm{div}(h))$ onto $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma^N)$, and this maps $\mathcal{L}(\mathrm{div}(h), D)$ onto $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma^N, D)$. Since $\mathcal{L}(\mathrm{div}(h)) = \mathcal{L}(\mathrm{div}(h), D) \otimes_D \mathbf{C}$, we obtain (1). Now by Lemma 6.17 there exists a nonzero element p in $\mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q})$. Thus $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma^N, \mathbf{Q}) \neq \{0\}$ for sufficiently large N. Therefore (2) follows from (1). From (2) we see that every element of $\mathcal{M}_{\kappa\mathbf{a}}$ is a finite $\mathbf{C}$-linear combination of elements of $\mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q})$. Clearly this implies (4) for $f \in \mathcal{M}_{\kappa\mathbf{a}}$. Given $f \in \mathcal{A}_{\kappa\mathbf{a}}$, we see that $p^{-1}f \in \mathcal{A}_0$, so that by Theorem 8.11 (4) and (5), $p^{-1}f \in \mathcal{A}_0(E)$ with a finitely generated extension E of $\mathbf{Q}$, and $(p^{-1}f)^\sigma \in \mathcal{A}_0$. Then we obtain (3) and (4) for $f \in \mathcal{A}_{\kappa\mathbf{a}}$.

9.10. In (5.1) we defined the symbol $M_\alpha(z) = \big(\mu_v(\alpha, z)\big)_{v\in\mathbf{b}}$. Let ω be a representation of $GL_n(\mathbf{C})^{\mathbf{b}}$ given in the form $\omega(x) = \bigotimes_{v\in\mathbf{b}} \omega_v(x_v)$ with $\mathbf{Q}$-rational representations ω_v of $GL_n(\mathbf{C})$. We recall that

$$(f\|_\omega\alpha)(z) = \omega(M_\alpha(z))^{-1} f(\alpha z) \tag{9.10}$$

in both Cases SP and UT.

Now, for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, we define a representation ω^σ of $GL_n(\mathbf{C})^{\mathbf{b}}$ by

$$\omega^\sigma(x) = \bigotimes_{v\in\mathbf{b}} \omega_v(x_{v\sigma}). \tag{9.11}$$

Thus both ω and ω^σ have the same representation space.

9.11. Proposition. *For each fixed $v \in \mathbf{b}$ define a representation $\tau_v : GL_n(\mathbf{C})^{\mathbf{b}} \to GL_n(\mathbf{C})$ by*

$$\tau_v(x) = \det(x)^{\mathbf{b}} x_v \quad \text{for} \quad x \in GL_n(\mathbf{C})^{\mathbf{b}}, \tag{9.12}$$

where $a^{\mathbf{b}} = \prod_{v\in\mathbf{b}} a_v$. (Thus $(\tau_v)^\sigma = \tau_{v\sigma}$.) Given $z_0 \in \mathcal{H}$, there exist a set of $\mathbf{C}^n_n$-valued functions $\{R_v\}_{v\in\mathbf{b}}$ and a congruence subgroup Γ of G with the following properties:

(1) *The columns of R_v belong to $\mathcal{M}_{\tau_v}(\Gamma, K^v)$, where K^v is the image of K under v.*

(2) *$R_v^\sigma = R_{v\sigma}$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.*

(3) *$\det\big(R_v(z_0)\big) \neq 0$ for every $v \in \mathbf{b}$.*

(4) *The columns of $R_v\|_{\tau_v}\alpha$ belong to $\mathcal{M}_{\tau_v}(K^v\mathbf{Q}_{\mathrm{ab}})$ for every $\alpha \in \widetilde{G}_+$.*

Moreover, these assertions are true with the following modifications: Replace $\mathcal{M}_{\tau_v}$ by $\mathcal{A}_{\sigma_v}$, where $\sigma_v(x) = x_v$; (3) should then read: R_v is holomorphic at z_0 and $\det\big(R_v(z_0)\big) \neq 0$ for every $v \in \mathbf{b}$.

PROOF. We first prove (1, 2, 3) for $\mathcal{M}_{\tau_v}$ in Case SP with $F = \mathbf{Q}$. Put

$$\psi(u,\, z;\, \lambda) = (2\pi i)^{-1}\left[(\partial/\partial u_j)\varphi(u,\, z;\, \lambda)\right]_{j=1}^{n}, \tag{9.13}$$

where φ is φ_F of (6.17) with $F = \mathbf{Q}$. We view this as a column vector. Take $\Lambda \subset \mathcal{S}(\mathbf{Q}_{\mathbf{h}}^n)$ as in Theorem 6.12 and take Γ_λ as in (6.34). Differentiating (6.34), we obtain

$$\psi(u,\, \gamma z;\, \lambda) = h_\gamma(z)\mu_\gamma(z)\psi\big({}^t\mu_\gamma(z)u,\, z;\, \lambda\big) \quad \textit{for every} \quad \gamma \in \Gamma_\lambda. \tag{9.14}$$

Put $\Gamma_0 = \bigcap_{\lambda\in\Lambda}\Gamma_\lambda$. Denoting the elements of Λ by $\lambda_1, \ldots, \lambda_m$, put

$$\varphi_k(u,\, z) = \varphi(u,\, z;\, \lambda_k), \quad \psi_k(u,\, z) = \psi(u,\, z;\, \lambda_k) \qquad (1 \le k \le m).$$

Since $u \mapsto \big(\varphi_k(u,\, z)\big)_{k=1}^m$ is the biregular embedding of Theorem 6.12 (3), we have

$$\operatorname{rank}\begin{bmatrix}\varphi_1(u,\, z) & \cdots & \varphi_m(u,\, z)\\ \psi_1(u,\, z) & \cdots & \psi_m(u,\, z)\end{bmatrix} = n+1$$

for every $(u,\, z) \in \mathbf{C}^n \times \mathfrak{H}$. Therefore, given $z_0 \in \mathfrak{H}_n$, changing the order of λ_k, we may assume that $\det\big[\psi_1(0,\, z_0) \ \cdots\ \psi_n(0,\, z_0)\big] \neq 0$. Also we can find an index j such that $\varphi_j(0,\, z_0) \neq 0$. Define a $\mathbf{C}_n^n$-valued function R by

$$R(z) = \varphi_j(0,\, z)\big[\psi_1(0,\, z) \ \cdots\ \psi_n(0,\, z)\big]. \tag{9.15}$$

Then (6.34) together with (9.14) and Theorem 6.8 (5) shows that $R\big(\gamma(z)\big) = j_\gamma(z)\mu_\gamma(z)R(z)$ for every γ in a subgroup Γ of Γ_0 of finite index. We easily see that $\psi_k(0,\, z)$ has Fourier coefficients in $\mathbf{Q}$. This proves (1, 2, 3) in the case $F = \mathbf{Q}$.

Next we consider Case SP with $F \neq \mathbf{Q}$. We take the embedding $\varepsilon : \mathfrak{H}_n^{\mathbf{a}} \to \mathfrak{H}_{en}$ of (6.10), and given $z_0 \in \mathfrak{H}_n^{\mathbf{a}}$, choose the above R on $\mathfrak{H}_{en}$ (that is, with en instead of n) so that $\det\big[R\big(\varepsilon(z_0)\big)\big] \neq 0$. With B as in (6.5) put

$$\begin{bmatrix}R_1(z)\\ \vdots\\ R_e(z)\end{bmatrix} = {}^tBR\big(\varepsilon(z)\big)Q \qquad (z \in \mathfrak{H}_n^{\mathbf{a}}) \tag{9.16}$$

with $Q \in \mathbf{Q}_n^{en}$. Here each R_j is an $n \times n$-matrix. We can choose Q so that $\det\big[R_j(z_0)\big] \neq 0$ for every j. (To see this, we first note an easy fact: Given a nonzero polynomial $p(x_1, \ldots, x_m)$, with complex coefficients, there exist rational numbers $q_1, \ldots, q_m$ such that $p(q_1, \ldots, q_m) \neq 0$. Then take a variable $en \times n$-matrix X and put ${}^tBR(\varepsilon(z_0))X = \begin{bmatrix}Y_1\\ \vdots\\ Y_e\end{bmatrix}$, where each Y_j is square and of size n, and apply the above fact to $p(X) = \prod_{j=1}^e \det(Y_j)$.) Writing $\{R_v\}_{v\in\mathbf{a}}$ for $\{R_j\}$ and employing (6.11a, b), we can easily verify that the R_v satisfy (1, 2, 3) with a suitable choice of $\Gamma \subset Sp(n,\, F)$.

Finally take Case UT. Using the symbols of §6.5, denote by T_v the above function R_v with $2n$ in place of n such that $\det\big[T_v\big(\psi(z_0)\big)\big] \neq 0$ for a given $z_0 \in \mathcal{H}_n^{\mathbf{a}}$. Then we define $n \times n$-matrices S_v and R_v by

$$\begin{bmatrix}S_v(w)\\ R_v(w)\end{bmatrix} = A_v^*T_v\big(\psi(w)\big)Y \qquad (w \in \mathcal{H}_n^{\mathbf{a}},\ v \in \mathbf{a}) \tag{9.17}$$

with $Y \in \mathbf{Q}_n^{2n}$. By virtue of (6.22b, c) we easily find a congruence subgroup Γ of G such that

$$S_v(\alpha w) = j_\alpha(w)^{\mathbf{b}}\lambda_v(\alpha,\, w)S_v(w) \quad \textit{and} \quad R_v(\alpha w) = j_\alpha(w)^{\mathbf{b}}\mu_v(\alpha,\, w)R_v(w)$$

for every $\alpha \in \Gamma$. Also, with a suitable choice of Y, we have $\det\left[S_v(z_0)R_v(z_0)\right] \neq 0$. Putting $R_{v\rho} = S_v$, we can easily verify that the set $\{R_v\}_{v\in\mathbf{b}}$ satisfies (1, 2, 3).

As for (4), since every element of $\widetilde{G}_+$ is a product of an element of G and an element of the form $\mathrm{diag}[1_n,\, c1_n]$ with $c \in F,\gg 0$, it is sufficient to treat the case where $\alpha \in G$. Then by means of (6.11a, b), (6.22b, c), (9.16), and (9.17), we can reduce our problem to the case $G = Sp(n,\, \mathbf{Q})$. In this case we prove (4) in a stronger form:

(9.18) *If $X \in \mathcal{M}_\tau(\mathbf{Q}_{\mathrm{ab}})$, then $X\|_\tau\alpha \in \mathcal{M}_\tau(\mathbf{Q}_{\mathrm{ab}})$ for every $\alpha \in Sp(n,\, \mathbf{Q})$.*

This is clear if $\alpha \in P \cap G$ with P of Lemma 7.5. Therefore, by that lemma, it is sufficient to prove (9.18) when $\alpha = \eta$. First we take X to be R of (9.15). Let $r(z) = \det(-iz)^{1/2}$. By Theorem 6.9 (2) we have

$$\varphi(u,\, \eta z;\, \lambda_k) = r(z)\varphi({}^t\mu_\eta(z)u,\, z;\, \lambda'_k)$$

with λ'_k given by (6.33) with λ_k as λ and $F = \mathbf{Q}$. Clearly λ'_k is $\mathbf{Q}_{\mathrm{ab}}$-valued. Then differentiation shows that $\psi(u,\, \eta z;\, \lambda_k) = r(z)\mu_\eta(z)\psi({}^t\mu_\eta(z)u,\, z;\, \lambda'_k)$. Therefore we obtain

$$(R\|\eta)(z) = (-i)^n\varphi(0,\, z;\lambda'_j)\big[\psi(0,\, z;\lambda'_1) \quad \cdots \quad \psi(0,\, z;\lambda'_n\big].$$

This is clearly $\mathbf{Q}_{\mathrm{ab}}$-rational. Now given X as in (9.18), put $Z = R^{-1}X$. Then the entries of Z belong to $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$, and so by Theorem 7.10 (4) the entries of $Z \circ \eta$ belong to $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. Thus $X\|\eta = (R\|\eta)(Z \circ \eta) \in \mathcal{M}_\tau(\mathbf{Q}_{\mathrm{ab}})$, which proves (9.18). This completes the proof of (4).

As for the assertions for $\mathcal{A}_{\sigma_v}$ instead of $\mathcal{M}_{\tau_v}$, the desired functions can be obtained by taking $\varphi_j(0,\, z)^{-1}$ in place of $\varphi_j(0,\, z)$ in (9.15). Then the above proof is applicable to this case too, with no other changes.

9.12. Lemma. *Given $\gamma = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in SL_2(\mathbf{Z})$ with $s > 0$, put $\beta = \begin{bmatrix} p1_n & q1_n \\ r1_n & s1_n \end{bmatrix}$. Then for some positive integer M we can choose the function R of Proposition 9.11 with $R_v \in \mathcal{A}_{\sigma_v}$ so that it has the following property: If $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, $t \in \mathbf{Z}_{\mathbf{h}}^\times$, $\tau = [t,\, \mathbf{Q}]$ on $\mathbf{Q}_{\mathrm{ab}}$, and $\gamma - \mathrm{diag}[t^{-1},\, t] \prec M\mathbf{Z}$, then $(R_v\|_{\sigma_v}\eta)^\tau = R_{v\tau}\|_{\sigma_{v\tau}}\beta\eta$.*

PROOF. We first consider Case SP with $F = \mathbf{Q}$. Given $\lambda \in \mathcal{S}(\mathbf{Q}_{\mathbf{h}}^n)$ and $t \in \mathbf{Z}_{\mathbf{h}}^\times$, define $\lambda_t,\, \lambda' \in \mathcal{S}(\mathbf{Q}_{\mathbf{h}}^n)$ by $\lambda_t(x) = \lambda(tx)$ for $x \in \mathbf{Q}_{\mathbf{h}}^n$ and

$$\lambda'(x) = \int_{\mathbf{Q}_{\mathbf{h}}^n} \lambda(y)\mathbf{e}_{\mathbf{h}}({}^t xy)dy \tag{9.19}$$

as in (6.33). Also, for $\tau \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ define λ^τ by $\lambda^\tau(x) = \lambda(x)^\tau$. Put $\varphi(z,\, \lambda) = \varphi(0,\, z;\, \lambda)$ and $\psi(z,\, \lambda) = \psi(0,\, z;\, \lambda)$. Then our construction shows that each column of R is of the form $f(z) = \varphi(z;\, \lambda)^{-1}\psi(z,\, \ell)$ with some $\mathbf{Q}$-valued elements λ, ℓ of $\mathcal{S}(\mathbf{Q}_{\mathbf{h}}^n)$. By Theorem 6.9 (2) we have $f\|\eta = \varphi(z;\, \lambda')^{-1}\psi(z,\, \ell')$, and hence $(f\|\eta)^\tau = \varphi\big(z;\, (\lambda')^\tau\big)^{-1}\psi\big(z,\, (\ell')^\tau\big)$. Take a positive integer M so that

$$\{\, \xi \in Sp(n,\, \mathbf{Q})_{\mathbf{A}} \,|\, \xi - 1_{2n} \prec M\mathbf{Z} \,\} \subset D_\lambda \cap D_\ell$$

for D_λ and D_ℓ of Theorem 6.8 (4). Then, by (6.32), $f\|\beta = \varphi(z;\, \lambda_t)^{-1}\psi(z,\, \ell_t)$. Applying τ to (9.19), we find that $(\lambda')^\tau = (\lambda_t)'$ and $(\ell')^\tau = (\ell_t)'$, and hence $(f\|\eta)^\tau = \varphi\big(z;\, (\lambda_t)'\big)^{-1}\psi\big(z,\, (\ell_t)'\big) = f\|\beta\eta$, which proves the case $G = Sp(n,\, \mathbf{Q})$.

To prove the case of $G = Sp(n,\, F)$, we employ the symbols of §6.2. Let β be the element of G defined in $Sp(n,\, F)$ as in our lemma, and let $\beta' = \begin{bmatrix} p1_{en} & q1_{en} \\ r1_{en} & s1_{en} \end{bmatrix} \in$

$Sp(en, \mathbf{Q})$. With B as in (6.5), put $A = B \cdot {}^tB$ and $\alpha = \mathrm{diag}[A^{-1}, A]$. Let M' be the integer with which R on $Sp(en, \mathbf{Q})$ has the desired property. We take it so that $R\|\Gamma^{M'} = R$. Now take a positive multiple M of M' so that $M\alpha \prec M'\mathbf{Z}$. With ω as in (6.7), observe that $\omega(\eta_n) = \eta_{en}\alpha$ and $\omega(\beta) = \xi\beta'$ with

$$\xi = \begin{bmatrix} 1 + qr(1 - A) & pq(A - 1) \\ sr(A^{-1} - 1) & 1 + qr(1 - A^{-1}) \end{bmatrix}.$$

Therefore $\xi - 1 \prec M'\mathbf{Z}$ if $q, r \in M\mathbf{Z}$. Combining our result in the case $F = \mathbf{Q}$ with (9.16), we obtain

$$(*) \qquad \bigl((R_v\|\eta)^\tau\bigr)_{v\in\mathbf{a}} = {}^tB^\tau(R\|\eta_{en}\alpha)^\tau\bigl(\varepsilon(z)\bigr)Q = {}^tB^\tau(R\|\beta'\eta_{en}\alpha)\bigl(\varepsilon(z)\bigr)Q,$$

since $(g\|\alpha)^\tau = g^\tau\|\alpha$ holds for $g \in \mathcal{A}_\sigma$. On the other hand

$$(R_v\|\beta\eta)_{v\in\mathbf{a}} = {}^tB(R\|\xi\beta'\eta_{en}\alpha)\bigl(\varepsilon(z)\bigr)Q = {}^tB(R\|\beta'\eta_{en}\alpha)\bigl(\varepsilon(z)\bigr)Q$$

since $R\|\xi = R$. Comparing this with (*), we obtain the desired equality for R_v on $Sp(n, F)$. Case UT can be handled in a similar manner by means of (9.17).

9.13. Theorem. (1) *Let $f \in \mathcal{M}_\omega$ (resp. $f \in \mathcal{A}_\omega$) with ω as in §9.10 and let $\sigma \in \mathrm{Aut}(\mathbf{C})$. Then f^σ, defined as a formal element in §5.9, is indeed an element of $\mathcal{M}_{\omega^\sigma}$ (resp. $\mathcal{A}_{\omega^\sigma}$). Thus $(\mathcal{A}_\omega)^\sigma = \mathcal{A}_{\omega^\sigma}$ and $(\mathcal{M}_\omega)^\sigma = \mathcal{M}_{\omega^\sigma}$.*

(2) *Given $f \in \mathcal{A}_\omega$, there exists a finitely generated extension k of $\mathbf{Q}$ such that $f \in \mathcal{A}_\omega(k)$.*

(3) *$\mathcal{M}_\omega(D)$ is stable under $f \mapsto f\|_\omega\,\alpha$ for every $\alpha \in \widetilde{G}_+$ and every subfield D of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$ and the Galois closure of K over $\mathbf{Q}$.*

PROOF. We may assume that ω is irreducible. Then there is an integer e such that $\omega(cy) = c^e\omega(y)$ for $c \in \mathbf{C}^\times$. Take Γ and $R = (R_v)_{v\in\mathbf{b}}$ as in Proposition 9.11. Then we see that $\omega(R) \circ \alpha = j_\alpha^{e\mathbf{b}}\omega(\mu_\alpha R)$ for every $\alpha \in \Gamma$. Take a positive integer m so that $m > e$ and $\det(y)^m\omega_v(y)^{-1}$ is a polynomial in y for every $v \in \mathbf{b}$; put $s(x) = \det(x)^{-m\mathbf{b}}\omega(x)$ for $x \in GL_n(\mathbf{C})^{\mathbf{b}}$. Then $s(R) \circ \alpha = j_\alpha^{-\kappa\mathbf{b}}\omega(\mu_\alpha)s(R)$ with $\kappa = m(1 + n|\mathbf{b}|) - e$. Given $f \in \mathcal{A}_\omega$, put $g = s(R)^{-1}f$, and observe that the components of g belong to $\mathcal{A}_{\kappa\mathbf{b}}$ and that $s(R)^\sigma = s(R^\sigma) = s^\sigma(R)$. Identify the representation space of ω with $\mathbf{C}^t$ with some t so that $\omega\bigl(GL_n(\mathbf{Q})^{\mathbf{b}}\bigr)$ acts on $\mathbf{Q}^t$. Then $g \in \bigl(\mathcal{A}_{\kappa\mathbf{b}}\bigr)^t$. By Theorem 9.9 (4), $g^\sigma \in \bigl(\mathcal{A}_{\kappa\mathbf{b}}\bigr)^t$, so that f^σ, being equal to $s(R^\sigma)g^\sigma$, must be defined as an element of $\mathcal{A}_{\omega^\sigma}$. Suppose $f \in \mathcal{M}_\omega$. Then $g \in \bigl(\mathcal{M}_{\kappa\mathbf{b}}\bigr)^t$, since $s(R)^{-1}$ is holomorphic everywhere. Now for any point z_0, take R so that $\det\bigl[\prod_{v\in\mathbf{b}} R_v(z_0)\bigr] \neq 0$. Now $g^\sigma \in \bigl(\mathcal{M}_{\kappa\mathbf{b}}\bigr)^t$ by Theorem 9.9 (4), so that $f^\sigma = s(R)^\sigma g^\sigma$ is holomorphic at z_0. Thus f^σ is holomorphic everywhere, and so $f^\sigma \in \mathcal{M}_{\omega^\sigma}$. This proves (1). Now $s(R)$ is rational over the Galois closure of K. Since $f = s(R)g$, we obtain (2) by applying Theorem 9.9 (3) to g. To prove (3), we take R_v of Proposition 9.11 with $R_v \in \mathcal{A}_{\sigma_v}$. Given $f \in \mathcal{M}_\omega(D)$, put $g = \omega(R)^{-1}f$. Then g has components in $\mathcal{A}_0(D)$. Now $R_v\|_{\sigma_v}\alpha \in \mathcal{A}_{\sigma_v}(K^v\mathbf{Q}_{\mathrm{ab}})$ as stated in Proposition 9.11, and $g \circ \alpha$ has components in $\mathcal{A}_0(D)$ by Theorem 7.10, and hence $f\|_\omega\,\alpha = \bigl(\omega(R)\|_\omega\,\alpha\bigr)(g \circ \alpha) \in \mathcal{A}_\omega(D)$. This together with (5.30) proves (3).

10. Automorphisms of the spaces of automorphic forms

10.1. We are going to prove a theorem analogous to Theorem 8.10, taking automorphic forms instead of automorphic functions. Let ω and ψ be two $\mathbf{Q}$-rational representations of $GL_n(\mathbf{C})^{\mathbf{b}}$. For $f \in \mathcal{A}_\omega$ and $g \in \mathcal{A}_\psi$ we denote by $f \otimes g$ the element of $\mathcal{A}_{\omega\otimes\psi}$ defined in a natural way.

We now define a subgroup $\mathfrak{G}$ of $\mathcal{G}_+ \times \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by

$$\mathfrak{G} = \left\{ (\xi,\, \sigma) \in \mathcal{G}_+ \times \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \,\middle|\, \sigma(\xi) = \sigma \quad on \quad \mathbf{Q}_{\mathrm{ab}} \right\}, \tag{10.1}$$

where $\sigma(\xi)$ is defined in the paragraph preceding (8.11).

10.2. Theorem. *Each element $(\xi,\, \sigma)$ of $\mathfrak{G}$ gives a $\mathbf{Q}$-linear bijection of $\mathcal{A}_\omega\left(\overline{\mathbf{Q}}\right)$ onto $\mathcal{A}_{\omega^\sigma}\left(\overline{\mathbf{Q}}\right)$, written $f \mapsto f^{(\xi,\sigma)}$, with the following properties:*

(1) $\mathcal{M}_\omega\left(\overline{\mathbf{Q}}\right)^{(\xi,\sigma)} = \mathcal{M}_{\omega^\sigma}\left(\overline{\mathbf{Q}}\right)$.

(2) *If $f \in \mathcal{A}_0\left(\mathbf{Q}_{\mathrm{ab}}\right)$, then $f^{(\xi,\sigma)} = f^{\tau(\xi)}$ with τ of Theorem 8.10.*

(3) $(af)^{(\xi,\sigma)} = a^\sigma f^{(\xi,\sigma)}$ *for every $a \in \overline{\mathbf{Q}}$.*

(4) $\left(f^{(\xi,\sigma)}\right)^{(\zeta,\tau)} = f^{(\xi\zeta,\sigma\tau)}$.

(5) $f^{(\alpha,1)} = f\|_\omega\alpha$ *if $\alpha \in \widetilde{G}_+$ and $f \in \mathcal{A}_\omega$.*

(6) $(f \otimes g)^{(\xi,\sigma)} = f^{(\xi,\sigma)} \otimes g^{(\xi,\sigma)}$.

(7) *$f^{(\xi,\sigma)}$ coincides with f^σ of §5.9 and Theorem 9.13 (1) if $\xi = \iota(s)$ with $s \in \mathbf{Z}_{\mathbf{h}}^\times$.*

(8) *If $f \in \mathcal{A}_\omega(\Gamma^N,\, \overline{\mathbf{Q}})$ and $(\xi,\, \sigma) \in \mathfrak{G}$ with $\xi \in T^N$, then $f^{(\xi,\sigma)} = f^\sigma$.*

PROOF. We first consider the action of $\mathfrak{G}$ on $\mathcal{A}_0\left(\overline{\mathbf{Q}}\right)$. By Lemma 7.4 (1) and Theorem 7.10 (3), $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ and $\overline{\mathbf{Q}}$ are linearly disjoint over $\mathbf{Q}_{\mathrm{ab}}$, and $\mathcal{A}_0\left(\overline{\mathbf{Q}}\right) = \overline{\mathbf{Q}}\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. Therefore, given $(\xi,\, \sigma) \in \mathfrak{G}$, we can find an automorphism of $\mathcal{A}_0\left(\overline{\mathbf{Q}}\right)$ that coincides with σ on $\overline{\mathbf{Q}}$ and with $\tau(\xi)$ on $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$. Writing this action by putting $(\xi,\, \sigma)$ on the upper right, we can easily verify all the assertions restricted to $\mathcal{A}_0\left(\overline{\mathbf{Q}}\right)$. (As for (7) and (8), we can derive them from Theorem 8.10 (3), Theorem 8.11 (2), and (8.13).)

Next, to treat $\mathcal{A}_\omega$ with ω of a general type, let $R = (R_v)_{v\in\mathbf{b}}$ denote the set of functions obtained in Proposition 9.11 with $R_v \in \mathcal{A}_{\sigma_v}$; take a positive integer M so that the columns of R_v and $R_v\|\eta$ belong to $\mathcal{A}_{\sigma_v}(\Gamma^M)$ for every $v \in \mathbf{b}$ and have the property stated in Lemma 9.12. By Proposition 9.11 (2), the columns of $\omega(R)^\sigma$ belong to $\mathcal{A}_{\omega^\sigma}(\Gamma^M)$ for every $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Given $f \in \mathcal{A}_\omega\left(\overline{\mathbf{Q}}\right)$ and $(\xi,\, \sigma) \in \mathfrak{G}$, by (8.10) we have $\xi \in x\alpha\widetilde{G}_{\mathbf{a}+}$ with $x \in T^M$ and $\alpha \in \widetilde{G}_+$. Observe that the components of $\omega(R)^{-1}f$ belong to $\mathcal{A}_0\left(\overline{\mathbf{Q}}\right)$, and so $\left(\omega(R)^{-1}f\right)^{(\xi,\,\sigma)}$ is meaningful. Then we define $f^{(\xi,\,\sigma)}$ by

$$f^{(\xi,\,\sigma)} = \left(\omega^\sigma(R)\|_{\omega^\sigma}\alpha\right)\left(\omega(R)^{-1}f\right)^{(\xi,\,\sigma)}. \tag{10.2}$$

By Proposition 9.11 (4) the columns of $\omega^\sigma(R)\|_{\omega^\sigma}\alpha$ belong to $\mathcal{M}_{\omega^\sigma}\left(\overline{\mathbf{Q}}\right)$, and hence $f^{(\xi,\,\sigma)}$ is indeed an element of $\mathcal{A}_{\omega^\sigma}\left(\overline{\mathbf{Q}}\right)$. Also we can easily verify that this does not depend on the choice of x and α. Then clearly (5) and (6) hold. If $\xi = \iota(s)$ with $s \in \mathbf{Z}_{\mathbf{h}}^\times$, we can take $x = \iota(s)$ and $\alpha = 1$; then we obtain (7). If ω is the trivial representation, it is consistent with the above action on $\mathcal{A}_0$. Also, if $f \in \mathcal{A}_\omega(\Gamma^N,\, \overline{\mathbf{Q}})$ with a multiple N of M and $x \in T^N$, then

$$\left(\omega(R)^{-1}f\right)^{(\xi,\,\sigma)} = \left(\omega(R)^{-1}f\right)^{(x,\,\sigma)(\alpha,\,1)} = \left(\omega(R)^{-1}f\right)^\sigma \circ \alpha$$

since (4), (5), and (8) are true for $\mathcal{A}_0(\overline{\mathbf{Q}})$, and hence we easily see that

$$f^{(\xi,\,\sigma)} = f^\sigma\|_{\omega^\sigma}\,\alpha. \tag{10.3}$$

This combined with Theorem 9.13 (1) proves (1). We also note that

$$\omega(R)^{(x\alpha,\,\sigma)} = \omega^{\sigma}(R)\|_{\omega^{\sigma}}\alpha \quad \textit{if} \quad x \in T^{M} \quad \textit{and} \quad \alpha \in \widetilde{G}_{+}, \tag{10.4a}$$

which is a special case of (10.2). Also from this and (10.2) with R_v as f, we obtain

$$\omega(R)^{(\xi,\,\sigma)} = \omega\big((R'_v)_{v\in\mathbf{b}}\big) \quad \text{with} \quad R'_v = R_v^{(\xi,\,\sigma)}. \tag{10.4b}$$

Let us now prove (8) assuming (4). Let $f \in \mathcal{A}_{\omega}(\Gamma^{N},\,\overline{\mathbf{Q}})$ and $(\xi,\,\sigma) \in \mathfrak{G}$ with $\xi \in T^{N}$. Take a common multiple N' of M and N; let $\xi \in \beta x\widetilde{G}_{\mathbf{a}+}$ with $\beta \in \widetilde{G}_{+}$ and $x \in T^{N'}$. Then $\beta \in \Gamma^{N}$ by Lemma 8.3 (2), and so $f^{(\xi,\,\sigma)} = \big(f^{(\beta,\,1)}\big)^{(x,\,\sigma)} = f^{(x,\,\sigma)}$. Taking $(x,\,\sigma)$ and 1 as $(\xi,\,\sigma)$ and α in (10.3), we obtain $f^{(x,\,\sigma)} = f^{\sigma}$. This proves (8).

To prove the associativity of (4), we first observe that it follows easily from our definition (10.2) if both ξ and ζ are contained in T^{M}, or $\zeta \in \widetilde{G}_{+}$ and $\tau = 1$. Now assume that

$$\big(g^{(\alpha,1)}\big)^{(\zeta,\tau)} = g^{(\alpha\zeta,\tau)} \quad \textit{for every} \quad g \in \mathcal{A}_{\omega}\left(\overline{\mathbf{Q}}\right),\ \alpha \in \widetilde{G}_{+},\ (\zeta,\,\tau) \in \mathfrak{G}. \tag{*}$$

Given $(\xi,\,\sigma)$ and $(\zeta,\,\tau)$ in $\mathfrak{G}$, let $\xi \in x\alpha\widetilde{G}_{\mathbf{a}+}$ as above and $\alpha\zeta \in y\beta\widetilde{G}_{\mathbf{a}+}$ with $y \in T^{M}$ and $\beta \in \widetilde{G}_{+}$. Then, assuming (*), we have

$$\begin{aligned}\big(g^{(\xi,\sigma)}\big)^{(\zeta,\tau)} &= \big((g^{(x,\,\sigma)})^{(\alpha,\,1)}\big)^{(\zeta,\tau)} = (g^{(x,\,\sigma)})^{(\alpha\zeta,\tau)} = (g^{(x,\,\sigma)})^{(y\beta,\tau)}\\ &= \big((g^{(x,\,\sigma)})^{(y,\,\tau)}\big)^{(\beta,\,1)} = \big((g^{(xy,\,\sigma\tau)}\big)^{(\beta,\,1)} = g^{(xy\beta,\,\sigma\tau)} = g^{(\xi\zeta,\,\sigma\tau)},\end{aligned}$$

which is the desired equality of (4). Thus our task is to prove (*). Observe that if (*) is true for fixed $(\alpha,\,g)$ and an arbitrary ζ, then it is true for $(\alpha^{-1},\,g)$ and an arbitrary ζ; if it is true for some fixed α, $(\zeta,\,\tau)$, and for the columns of $\omega(R)$, then it is true for all $g \in \mathcal{A}_{\omega}\left(\overline{\mathbf{Q}}\right)$ and the same α, $(\zeta,\,\tau)$, since the problem can be reduced to $\omega(R)^{-1}g$, whose components belong to $\mathcal{A}_{0}\left(\overline{\mathbf{Q}}\right)$. Therefore it is sufficient to prove that

$$\big(\omega(R)^{(\alpha,\,1)}\big)^{(\zeta,\,\tau)} = \omega(R)^{(\alpha\zeta,\,\tau)}$$

for α belonging to a set of generators, say B, of $\widetilde{G}_{+}$. Given $\alpha \in B$ and $(\zeta,\,\tau) \in \mathfrak{G}$, let $\zeta \in y\beta\widetilde{G}_{\mathbf{a}+}$ with $y \in T^{M}$ and $\beta \in \widetilde{G}_{+}$. For simplicity put $S = \omega(R)$. Suppose $\big(S^{(\alpha,\,1)}\big)^{(y,\tau)} = S^{(\alpha y,\tau)}$. Then

$$(S^{(\alpha,\,1)})^{(\zeta,\,\tau)} = \big((S^{(\alpha,\,1)})^{(y,\,\tau)}\big)^{(\beta,\,1)} = (S^{(\alpha y,\,\tau)})^{(\beta,\,1)} = S^{(\alpha y\beta,\,\tau)} = S^{(\alpha\zeta,\,\tau)}.$$

Thus it is sufficient to prove

$$\big(S^{(\alpha,\,1)}\big)^{(y,\tau)} = S^{(\alpha y,\tau)} \quad \textit{for every} \quad \alpha \in B \quad \textit{and} \quad y \in T^{M}. \tag{**}$$

By Lemma 7.5 we can take $B = (P \cap \widetilde{G}_{+}) \cup \{\eta\}$. We first consider the case $\alpha = \eta$. Let $y \in T^{M}$. Then $y \in \iota(t)U^{M}$ with $t \in \mathbf{Z}_{\mathbf{h}}^{\times}$. Take $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in SL_2(\mathbf{Z})$ so that $s > 0$ and $\begin{bmatrix} p & q \\ r & s \end{bmatrix} - \begin{bmatrix} t^{-1} & 0 \\ 0 & t \end{bmatrix} \prec M\mathbf{Z}$; put $\beta = \begin{bmatrix} p1_n & q1_n \\ r1_n & s1_n \end{bmatrix}$ and $w = \eta y\eta^{-1}\beta^{-1}$. Then $w \in T^{M}$, and $S^{(\eta y,\,\tau)} = S^{(w\beta\eta,\,\tau)} = S^{\tau}\|_{\omega^{\tau}}\beta\eta$ by (10.4a). On the other hand, taking $S\|\eta$ as f in (10.2), we have $(S\|\eta)^{(y,\,\tau)} = S^{\tau}\cdot\big[S^{-1}(S\|\eta)\big]^{(y,\,\tau)}$. Since $S^{-1}(S\|\eta)$ has components in $\mathcal{A}_{0}(\Gamma^{M})$, assertion (8) for $\mathcal{A}_{0}$ shows that $\big[S^{-1}(S\|\eta)\big]^{(y,\,\tau)} = (S^{-1})^{\tau}(S\|\eta)^{\tau}$, so that $(S\|\eta)^{(y,\,\tau)} = (S\|\eta)^{\tau}$. Thus (**) with $\alpha = \eta$ can be written

$$(***) \qquad (S\|_\omega\eta)^\tau = S^\tau\|_{\omega^\tau}\beta\eta.$$

Since $S = \omega(R)$, the desired equality (***) follows immediately from Lemma 9.12.

Next, let $\alpha = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in P \cap \widetilde{G}_+$. Since P is generated by $P \cap \mathfrak{r}_{2n}^{2n}$, we may assume that $\alpha \prec \mathfrak{r}$. Let $N = pM \cdot N_{K/\mathbf{Q}}(\det(\alpha))$ with a positive integer p which will be determined afterward. Given (y, τ) as in (**), let $y \in w\beta\widetilde{G}_{\mathbf{a}+}$ with $w \in T^N$ and $\beta \in \widetilde{G}_+$. Then $w \in \iota(s)U^N$with $s \in \mathbf{Z}_{\mathbf{h}}^\times$. Take a positive integer r so that $sr - 1 \prec N\mathbf{Z}$; put $\gamma = \begin{bmatrix} a & rb \\ 0 & d \end{bmatrix}$ and $x = \alpha w \gamma^{-1}$. Then we can easily verify that $x \in T^M$. Therefore $S^{(\alpha y, \tau)} = S^{(x\gamma\beta, \tau)} = S^\tau\|\gamma\beta$ by (10.4a). On the other hand $(S\|\alpha)^{(y, \tau)} = (S\|\alpha)^{(w\beta, \tau)} = (S\|\alpha)^\tau\|\beta$ by (10.3), since $S\|\alpha \in \mathcal{A}_\omega(\Gamma^N, \overline{\mathbf{Q}})$. We can put $S = q^{-1}S_1$ with $q \in \mathcal{M}_k(\overline{\mathbf{Q}})$, $k \in \mathbf{Z}^{\mathbf{b}}$, and a matrix S_1 whose columns belong to $\mathcal{M}_\psi(\overline{\mathbf{Q}})$, $\psi(x) = \det(x)^k\omega(x)$. Put $q(z) = \sum_h a(h)\mathbf{e}_{\mathbf{a}}^n(hz)$ with $a(h) \in \overline{\mathbf{Q}}$ and $S_1(z) = \sum_h c(h)\mathbf{e}_{\mathbf{a}}^n(hz)$ with $\overline{\mathbf{Q}}$-rational matrices $c(h)$. We choose p so that $a(h) \neq 0$ or $c(h) \neq 0$ only if $h \prec p^{-1}\mathfrak{r}$. Then $Nd^{-1}hb \prec \mathfrak{r}$ for every such h, and hence

$$(S_1\|\alpha)^\tau(z) = \omega^\tau(d)^{-1}\textstyle\sum_h c(h)^\tau\mathbf{e}_{\mathbf{a}}^n(rd^{-1}hb)\mathbf{e}_{\mathbf{a}}^n(d^{-1}haz) = (S_1^\tau\|\gamma)(z).$$

Similarly $(q\|\alpha)^\tau = q^\tau\|\gamma$, and hence $(S\|\alpha)^\tau = S^\tau\|\gamma$. Therefore $(S\|\alpha)^{(y, \tau)} = (S\|\alpha)^\tau\|\beta = S^\tau\|\gamma\beta = S^{(\alpha y, \tau)}$, which is (**) for the present α. This completes the proof.

10.3. Lemma. *Let $c_1, \ldots, c_m$ be m elements of $\mathbf{C}$ linearly independent over a subfield D of $\overline{\mathbf{Q}}$. Then there exists a set $\{\sigma\}$ of m automorphisms of $\mathbf{C}$ over D such that* $\det(c_\nu^\sigma)_{\sigma,\nu} \neq 0$.

PROOF. For $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$ let $H_\sigma = \{ x \in \mathbf{C}^m \mid \sum_{\nu=1}^m c_\nu^\sigma x_\nu = 0 \}$ and let J be the intersection of H_σ for all such σ. Since J is a vector subspace of $\mathbf{C}^m$ stable under $\mathrm{Aut}(\mathbf{C}/D)$, it is defined over D. Then the linear independence of the c_ν shows that $J = \{0\}$. Therefore we can find a set $\{\sigma\}$ of m elements of $\mathrm{Aut}(\mathbf{C}/D)$ such that $\bigcap_{\sigma\in\{\sigma\}} H_\sigma = \{0\}$. Then $\det(c_\nu^\sigma)_{\sigma,\nu} \neq 0$ as desired.

10.4. Theorem. *Suppose that $\omega(c1_n)$ is a scalar matrix for every $c \in F_{\mathbf{a}}^\times$ (which is the case if ω is irreducible). Let W be a subgroup of $\widetilde{G}_{\mathbf{A}+}$ containing an open subgroup of $(G_1)_{\mathbf{A}}$. Suppose that $W \cap \widetilde{G}_{\mathbf{h}}$ is contained in an open compact subgroup of $\widetilde{G}_{\mathbf{h}}$ and $xWx^{-1} = W$ for every $x \in \iota(Z_{\mathbf{h}}^\times)$; let $\Gamma = \widetilde{G} \cap W$. Further, given a subfield D of $\mathbf{C}$ and a character $\chi : \Gamma \to \mathbf{T}$ of finite order such that $\Gamma^N \cap \Gamma \subset \mathrm{Ker}(\chi)$ for some N, put*

$$(10.5) \qquad \mathcal{M}_\omega(\Gamma, \chi) = \{ f \in \mathcal{M}_\omega \mid f|_\omega\gamma = \chi(\gamma)f \ \ \textit{for every} \ \ \gamma \in \Gamma \},$$

$$(10.6) \qquad \mathcal{M}_\omega(\Gamma, D, \chi) = \mathcal{M}_\omega(D) \cap \mathcal{M}_\omega(\Gamma, \chi).$$

Then the following assertions hold:

(1) *Γ is contained in a congruence subgroup of $\widetilde{G}$, and contains Γ^N of (7.6) for some N.*

(2) *$\mathcal{M}_\omega(\Gamma, \chi)^\tau = \mathcal{M}_{\omega^\tau}(\Gamma, \chi_\tau)$ for every $\tau \in \mathrm{Aut}(\mathbf{C})$, where χ_τ is a character of Γ of finite order determined by χ, ω, and τ. If $\Gamma \subset G$ and χ is trivial, then χ_τ is trivial.*

(3) $\mathcal{M}_\omega(\Gamma, \chi) = \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}}, \chi) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.

(4) *Let Φ be the Galois closure of K over $\mathbf{Q}$ in $\overline{\mathbf{Q}}$. Then $\mathcal{M}_\omega(\Gamma) = \mathcal{M}_\omega(\Gamma, \Phi)\otimes_\Phi \mathbf{C}$ provided $\Gamma \subset G$.*

(5) *Given $k = (k_v)_{v\in\mathbf{b}} \in \mathbf{Z}^{\mathbf{b}}$, put $k^\sigma = (k_v^\sigma)_{v\in\mathbf{b}}$ with $k_v^\sigma = k_{v\sigma^{-1}}$ for every $\sigma \in \mathrm{Gal}(\Phi/\mathbf{Q})$; let Φ_k be the subfield of Φ determined by*

$$\mathrm{Gal}(\Phi/\Phi_k) = \left\{ \sigma \in \mathrm{Gal}(\Phi/\mathbf{Q}) \mid k^\sigma = k \right\}. \tag{10.7}$$

Then $\mathcal{M}_k(\Gamma) = \mathcal{M}_k(\Gamma, \Phi_k) \otimes_{\Phi_k} \mathbf{C}$ provided $\Gamma \subset G$.

PROOF. Clearly Γ is contained in a congruence subgroup of $\widetilde{G}$. Take $N > 2$ so that $U^N \cap (G_1)_\mathbf{A} \subset W$. Then by Lemma 8.3 (3), $\Gamma^N \subset G_1 \cap W \subset \Gamma$, which proves (1). Changing N suitably, we may assume that χ is trivial on Γ^N. Let $\tau \in \mathrm{Aut}(\mathbf{C})$ and $f \in \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}}, \chi)$. Put $x = \iota(s)$ with an element s of $\mathbf{Z}_\mathbf{h}^\times$ such that $[s, \mathbf{Q}] = \tau$ on $\mathbf{Q}_{\mathrm{ab}}$. Given $\gamma \in \Gamma$, we can find $\alpha \in \Gamma$ such that $\alpha \in \gamma G_1$ and $x\gamma x^{-1} \in \alpha(W \cap U^N)$. Indeed, since $\gamma^{-1}x\gamma x^{-1} \in (G_1)_\mathbf{A}$, by strong approximation we have $\gamma^{-1}x\gamma x^{-1} \in \varepsilon(W \cap U^N)$ with $\varepsilon \in G_1$. Then $\gamma\varepsilon$ gives the desired α. Now we can define a function χ_s on Γ by $\chi_s(\gamma) = \chi(\alpha)$ with such an α. Indeed, if β is another element of Γ such that $\beta \in \gamma G_1$ and $x\gamma x^{-1} \in \beta(W \cap U^N)$, then $\alpha^{-1}\beta \in U^N \cap G_1 = \Gamma^N$, so that $\chi(\alpha) = \chi(\beta)$. Thus χ_s is well-defined. Moreover we can easily show that χ_s is a character of Γ, since N can be changed for any larger integer. Put $x\gamma = \alpha y$. Then $y \in U^N x$. Since $f \in \mathcal{M}_\omega(\Gamma^N, \overline{\mathbf{Q}})$, by Theorem 10.2 (8) we have $f^{(x,\tau)} = f^{(y,\tau)} = f^\tau$, so that $f^\tau\|\gamma = f^{(x\gamma,\tau)} = f^{(\alpha y,\tau)} = (f\|\alpha)^{(y,\tau)}$. By our assumption on ω, we have $\omega(c1_n) = c^m = \prod_{v\in\mathbf{a}} c_v^{m_v}$ for every $c \in F_\mathbf{a}^\times$ with some $m \in \mathbf{Z}^\mathbf{a}$. Thus $f\|\alpha = \nu(\alpha)^{-m/2}f|\alpha = \nu(\alpha)^{-m/2}\chi(\alpha)f$, and hence $f^\tau|\gamma = \chi_\tau(\gamma)f^\tau$ with

$$\chi_\tau(\gamma) = \chi_s(\gamma)^\tau \textstyle\prod_{v\in\mathbf{a}} \nu(\gamma)_{v\tau}^{m_v/2}\left(\nu(\gamma)_v^{-m_v/2}\right)^\tau. \tag{10.8}$$

Clearly χ_τ is a character of Γ of finite order; if $\Gamma \subset G$ and χ is trivial, then clearly χ_τ is trivial. This proves that the left-hand side of the following equality is contained in the right-hand side.

$$\mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}}, \chi)^\tau = \mathcal{M}_{\omega^\tau}(\Gamma, \overline{\mathbf{Q}}, \chi_\tau). \tag{10.9}$$

Since we easily see that $(\chi_\tau)_{\tau^{-1}} = \chi$, applying τ^{-1} to the right-hand side, we obtain the opposite inclusion, which proves (10.9).

Taking Γ^M with an arbitrary M and a trivial character as Γ and χ, we see that $\mathcal{M}_\omega(\overline{\mathbf{Q}})^\tau = \mathcal{M}_{\omega^\tau}(\overline{\mathbf{Q}})$. Now we employ the notation of the proof of Theorem 9.13. Given $f \in \mathcal{M}_\omega$, put $g = s(R)^{-1}f$ as in that proof. Then $g \in \left(\mathcal{M}_{\kappa\mathbf{b}}\right)^t$. By Theorem 9.9 (2) we can put $g = \sum_{\nu=1}^m c_\nu g_\nu$ with $c_\nu \in \mathbf{C}$ and $g_\nu \in \mathcal{M}_{\kappa\mathbf{b}}(\overline{\mathbf{Q}})^t$. Changing $\{ c_\nu \}$ for a $\overline{\mathbf{Q}}$-basis of $\sum_{\nu=1}^m \overline{\mathbf{Q}}c_\nu$, we may assume that the c_ν are linearly independent over $\overline{\mathbf{Q}}$. Put $f_\nu = s(R)g_\nu$. Then $f_\nu \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$ and $f^\sigma = \sum_{\nu=1}^m c_\nu^\sigma f_\nu$ for every $\sigma \in \mathrm{Aut}\left(\mathbf{C}/\overline{\mathbf{Q}}\right)$. Now $f^\sigma \in \mathcal{M}_\omega$, and hence from Lemma 10.3 we see that $f_\nu \in \mathcal{M}_\omega$. Since $f_\nu \in \mathcal{A}_\omega\left(\overline{\mathbf{Q}}\right)$, we obtain $f_\nu \in \mathcal{M}_\omega(\overline{\mathbf{Q}})$ by (5.30). This shows that $\mathcal{M}_\omega = \mathcal{M}_\omega\left(\overline{\mathbf{Q}}\right) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$. Suppose $f \in \mathcal{M}_\omega(\Gamma, \chi)$. Then for every $\gamma \in \Gamma$ we have $\chi(\gamma)f = f|\gamma = \sum_{\nu=1}^m c_\nu f_\nu|\gamma$. Since $f_\nu|\gamma \in \mathcal{M}_\omega(\overline{\mathbf{Q}})$ and the c_ν are linearly independent over $\overline{\mathbf{Q}}$, we obtain $f_\nu|\gamma = \chi(\gamma)f_\nu$ for every ν and every $\gamma \in \Gamma$. This proves (3). Combining this with (10.9), we obtain (2).

To prove (4), suppose $\Gamma \subset G$; let $f \in \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}})$. By Theorem 9.13 (2) we can find a finite extension Ξ of Φ contained in $\overline{\mathbf{Q}}$ such that $f \in \mathcal{A}_\omega(\Xi)$. Clearly we may assume that Ξ is a Galois extension of Φ. Then for every $\sigma \in \mathrm{Gal}(\Xi/\Phi)$ we have, by (2), $f^\sigma \in \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}}) \cap \mathcal{A}_\omega(\Xi) = \mathcal{M}_\omega(\Gamma, \Xi)$. Therefore $\sum_\sigma (af)^\sigma \in \mathcal{M}_\omega(\Gamma, \Phi)$ for every $a \in \Xi$, where the sum is taken over all $\sigma \in \mathrm{Gal}(\Xi/\Phi)$. Then we see that $f \in \mathcal{M}_\omega(\Gamma, \Phi) \otimes_\Phi \Xi$. Combining this with (3), we obtain (4). To prove (5), take $\omega(x) = \det(x)^k$ with k as in (5). Since $\omega^\sigma = \omega$ if $\sigma = \mathrm{id.}$ on Φ_k, we have $\mathcal{M}_\omega(\Gamma, \Phi)^\tau = \mathcal{M}_\omega(\Gamma, \Phi)$ for every $\tau \in \mathrm{Aut}(\mathbf{C}/\Phi_k)$. Then $\sum_\sigma (af)^\sigma \in \mathcal{M}_\omega(\Gamma, \Phi_k)$ for every $f \in \mathcal{M}_\omega(\Gamma, \Phi)$ and every $a \in \Phi$, where the sum is taken over all $\sigma \in \mathrm{Gal}(\Phi/\Phi_k)$. This proves that $\mathcal{M}_\omega(\Gamma, \Phi) = \mathcal{M}_\omega(\Gamma, \Phi_k) \otimes_{\Phi_k} \Phi$, which combined with (4) proves (5).

It may be added that $\chi_s(\gamma)$ in (10.8) is often $\chi(\gamma)$. Indeed, suppose $\chi(\gamma)$ depends only on a_γ and d_γ modulo some $\mathfrak{r}$-ideal. Then taking a suitable N in the above proof, we see that $a_\alpha - a_\gamma \prec N\mathfrak{r}$ and $d_\alpha - d_\gamma \prec N\mathfrak{r}$, and therefore $\chi_s(\gamma) = \chi(\gamma)$. Thus $\chi_\tau(\gamma) = \chi(\gamma)^\tau$ for such a χ if $\nu(\gamma) = 1$. For example, take $W = C[\mathfrak{y}, \mathfrak{z}] \cap (G_0)_{\mathbf{A}+}$ with $C[\mathfrak{y}, \mathfrak{z}]$ of (1.17). Here $\mathfrak{y}$ and $\mathfrak{z}$ are $\mathfrak{r}$-ideals such that $\mathfrak{y}\mathfrak{z} \subset \mathfrak{r}$; we naturally take $m = n$ in (1.17). Taking a character φ of $(\mathfrak{r}/\mathfrak{y}\mathfrak{z})^\times$, put $\chi(\gamma) = \varphi\big(\det(d_\gamma)\big)$ for $\gamma \in \Gamma$. Then we have $\chi_\tau = \chi^\tau$.

10.5. Lemma. *Given $\tau \in \mathrm{Aut}(\mathbf{C})$, a positive integer N, and $\alpha \in \widetilde{G}_+$, there exist two elements β and γ of $\widetilde{G}_+$ such that $f^\tau \|_{\omega^\tau} \alpha = (f\|_\omega \beta)^\tau$ and $(f\|_\omega \alpha)^\tau = f^\tau \|_{\omega^\tau} \gamma$ for every $f \in \mathcal{M}_\omega(\Gamma^N)$ and every $\mathbf{Q}$-rational representation ω. Moreover, if $\alpha \in G$, then β and γ can be taken from G.*

PROOF. Decomposing ω into irreducible representations, we may assume that ω satisfies the condition of Theorem 10.4. Also, by Theorem 10.4 (4), we may assume that $f \in \mathcal{M}_\omega(\Gamma^N, \overline{\mathbf{Q}})$. Then we may take $\tau \in \mathrm{Aut}(\overline{\mathbf{Q}})$. Take a multiple M of $2N$ so that $\Gamma^M \subset \alpha^{-1}\Gamma^N\alpha$. We have $f^\tau \in \mathcal{M}_{\omega^\tau}(\Gamma^N)$ by Theorem 10.4 (2), and hence $f^\tau\|\alpha \in \mathcal{M}_{\omega^\tau}(\Gamma^M)$. Let $x = \iota(r)$ with an element r of $\mathbf{Z}_{\mathbf{h}}^\times$ such that $\tau = [r, \mathbf{Q}]$ on $\mathbf{Q}_{\mathrm{ab}}$. By (8.10) we can put $x\alpha = \beta y \widetilde{G}_{\mathbf{a}+}$ with $\beta \in \widetilde{G}_+$ and $y \in T^M$. Then $f\|\beta = f^{(x\alpha y^{-1}, 1)} = \big(f^{(x\alpha, \tau)}\big)^{(y, \tau)^{-1}} = (f^\tau\|\alpha)^{\tau^{-1}}$ by Theorem 10.2 (8). Thus $f^\tau\|\alpha = (f\|\beta)^\tau$ as desired. Similarly we can put $\alpha x \in z\gamma\widetilde{G}_{\mathbf{a}+}$ with $z \in T^M$ and $\gamma \in \widetilde{G}_+$. By Theorem 10.2 (8), $(f\|\alpha)^{(x,\tau)} = (f\|\alpha)^\tau$ and $f^{(z,\tau)} = f^\tau$, so that $(f\|\alpha)^\tau = f^{(\alpha x, \tau)} = f^{(z\gamma, \tau)} = f^\tau\|\gamma$ as expected. The last assertion is clear from our choice of β and γ.

Returning to questions (Q1), (Q2), and (Q3) of §5.9, Theorems 9.13 and 10.4 answer (Q1) and (Q2); the above lemma answers (Q3).

10.6. To treat forms of half-integral weight, we naturally confine our discussion to Case SP. Thus $G = Sp(n, F)$. By a *quasi-representation* of $GL_n(\mathbf{C})^{\mathbf{a}}$ we understand a symbol ψ given by

$$\psi(x) = \det(x)^{\mathbf{a}/2}\omega(x) \tag{10.10}$$

with a representation ω of $GL_n(\mathbf{C})^{\mathbf{a}}$ as in §9.10. Given γ in the group Γ^θ of (6.30) and a function f on $\mathfrak{H}_n^{\mathbf{a}}$ with values in the representation space of ω, we put

$$(f\|_\psi \gamma)(z) = h_\gamma(z)^{-1}(f\|_\omega \gamma)(z) \tag{10.11}$$

with h_γ of Theorem 6.8. For a congruence subgroup Γ of G contained in Γ^θ we define $\mathcal{M}_\psi(\Gamma)$ by conditions (5.8), (5.9), and (5.10), taking $f\|_\psi \gamma$ instead of $f|_\omega \gamma$ in (5.9); we then denote by $\mathcal{M}_\psi$ the union of $\mathcal{M}_\psi(\Gamma)$ for all such Γ's. If $f \in \mathcal{M}_\psi$, we have (5.20) and (5.21) with suitable M and U, in view of Theorem 6.8 (3).

Therefore we have an expansion of type (5.22a, b) for f. Also, Proposition 5.7 is valid for the present f, since what is needed in the proof is (5.20) and (5.21). Now for $\sigma \in \mathrm{Aut}(\mathbf{C})$ we can define f^σ as a formal series by (5.29). In the following theorem we shall prove that f^σ defines an element of $\mathcal{M}_{\psi^\sigma}$ with a certain quasi-representation ψ^σ. Given $\alpha \in \widetilde{G}_+$, let $p(z)$ be any branch of the square root of $j_\alpha(z)^{\mathbf{a}}$. Then by Theorem 6.9 (1) we can show that $p(z)^{-1}f\|_\omega \alpha \in \mathcal{M}_\psi$, and so we have an expansion

$$p(z)^{-1}(f\|_\omega \alpha)(z) = \textstyle\sum_{h\in S} c_{\alpha,\,p}(h)\mathbf{e}_{\mathbf{a}}^n(hz). \tag{10.12}$$

We call f a *cusp form* if $c_{\alpha,\,p}(h) = 0$ for every $(\alpha,\,p)$ and for every h such that $\det(h) = 0$, and denote by $\mathcal{S}_\psi(\Gamma)$ (resp. $\mathcal{S}_\psi$) the set of all cusp forms contained in $\mathcal{M}_\psi(\Gamma)$ (resp. $\mathcal{M}_\psi$). We can restrict α to $Sp(n,\,F)$ by virtue of Lemma 1.3 (3).

Further, given a subfield D of $\mathbf{C}$ and a character $\chi : \Gamma \to \mathbf{T}$ of finite order, we define $\mathcal{M}_\psi(D)$, $\mathcal{S}_\psi(D)$, $\mathcal{A}_\psi(D)$, $\mathcal{M}_\psi(\Gamma,\,D)$, $\mathcal{S}_\psi(\Gamma,\,D)$, and $\mathcal{A}_\psi(\Gamma,\,D)$ in the same manner as in §5.8, and put

$$\mathcal{M}_\psi(\Gamma,\,\chi) = \left\{\, f \in \mathcal{M}_\psi \;\middle|\; f\|_\psi \gamma = \chi(\gamma)f \ \text{ for every } \ \gamma \in \Gamma \,\right\},$$
$$\mathcal{M}_\psi(\Gamma,\,D,\,\chi) = \mathcal{M}_\psi(D) \cap \mathcal{M}_\psi(\Gamma,\,\chi).$$

We put also $\mathcal{A}_\psi(\Gamma) = \mathcal{A}_\psi(\Gamma,\,\mathbf{C})$ and $\mathcal{A}_\psi = \mathcal{A}_\psi(\mathbf{C})$.

Let k be a half-integral weight and let $m = (m_v)_{v\in\mathbf{a}}$ with $m_v = k_v - 1/2$ as in §6.10. If $\omega(x) = \det(x)^m$, then $f\|_\psi \gamma$ coincides with $f\|_k\gamma$ of (6.36), and hence the symbol $\mathcal{M}_\psi$ and $\mathcal{A}_\psi$ coincide with $\mathcal{M}_k$ and $\mathcal{A}_k$ of §6.10. We write $\mathcal{S}_k$ for $\mathcal{S}_\psi$ in this case.

10.7. Theorem (Case SP). *Let Γ and χ be as in Theorem 10.4, and ψ be as in (10.10); let Φ be the Galois closure of F over $\mathbf{Q}$ in $\overline{\mathbf{Q}}$. Suppose that $\Gamma \subset \Gamma^\theta$ and that $b_\gamma \prec 2\mathfrak{d}^{-1}$ and $c_\gamma \prec 2\mathfrak{d}$ for every $\gamma \in \Gamma$. Then the following assertions hold:*

(1) *Given $f \in \mathcal{A}_\psi$, there exists a finitely generated extension D of $\mathbf{Q}$ such that $f \in \mathcal{A}_\psi(D)$.*

(2) *$(\mathcal{M}_\psi)^\tau = \mathcal{M}_{\psi^\tau}$, $(\mathcal{A}_\psi)^\tau = \mathcal{A}_{\psi^\tau}$, and $\mathcal{M}_\psi(\Gamma,\,\chi)^\tau = \mathcal{M}_{\psi^\tau}(\Gamma,\,\chi_\tau)$ for every $\tau \in \mathrm{Aut}(\mathbf{C})$, where ψ^τ is defined by $\psi^\tau(x) = \det(x)^{\mathbf{a}/2}\omega^\tau(x)$, and χ_τ is a character of Γ of finite order determined by χ, ω, and τ as in Theorem 10.4 (2).*

(3) *$\mathcal{M}_\psi(\Gamma,\,\chi) = \mathcal{M}_\psi\big(\Gamma,\,\overline{\mathbf{Q}},\,\chi\big) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.*

(4) *$\mathcal{M}_\psi(\Gamma) = \mathcal{M}_\psi(\Gamma,\,\Phi) \otimes_\Phi \mathbf{C}$.*

(5) *Given a half-integral weight $k = (k_v)_{v\in\mathbf{a}} \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$, put $k^\sigma = (k_v^\sigma)_{v\in\mathbf{a}}$ with $k_v^\sigma = k_{v\sigma^{-1}}$ for every $\sigma \in \mathrm{Gal}(\Phi/\mathbf{Q})$; let Φ_k be the subfield of Φ such that*

$$\mathrm{Gal}(\Phi/\Phi_k) = \left\{\, \sigma \in \mathrm{Gal}(\Phi/\mathbf{Q}) \;\middle|\; k^\sigma = k \,\right\}.$$

Then $\mathcal{M}_k(\Gamma) = \mathcal{M}_k\big(\Gamma,\,\Phi_k\big) \otimes_{\Phi_k} \mathbf{C}$.

(6) *Let D be a subfield of $\mathbf{C}$ containing $\mathbf{Q}_{\mathrm{ab}}$ and Φ. Given $\alpha \in \widetilde{G}_+$, let $p(z)$ be any branch of the square root of $j_\alpha(z)^{\mathbf{a}}$. For $f \in \mathcal{M}_\psi(D)$ put $g(z) = p(z)^{-1}\cdot(f\|_\omega \alpha)$. Then $g \in \mathcal{M}_\psi(D)$.*

(7) *Let $\alpha \in \widetilde{G}_+$ and $\tau \in \mathrm{Aut}(\mathbf{C})$; let $p(z)$ be as in (6). Then there exists an element β of $\widetilde{G}_+$ and a branch $q(z)$ of the square root of $j_\beta(z)^{\mathbf{a}}$ such that $\big(q(z)^{-1}(f\|_\omega \beta)\big)^\tau = p(z)^{-1}\cdot(f^\tau\|_{\omega^\tau} \alpha)$ for every $f \in \mathcal{M}_\psi(\Gamma)$. Moreover, if $\alpha \in G$, then β can be taken from G.*

PROOF. Put $\theta(z) = \sum_{a\in\mathfrak{g}^n} \mathbf{e_a}({}^t aza/2)$ for $z \in \mathfrak{H}_n^{\mathbf{a}}$ and $\zeta(x) = \det(x)^{\mathbf{a}}\omega(x)$. Define a character φ of Γ^θ by $h_\gamma(z)^2 = \varphi(\gamma)j_\gamma^{\mathbf{a}}$ for $\gamma \in \Gamma^\theta$. By Theorem 6.8 (5), $\varphi(\gamma) = \prod_{v|2}\varepsilon_v\big(\det(d_\gamma)\big)$ if $\gamma \in \Gamma$ with ε defined there. Let $\tau \in \mathrm{Aut}(\mathbf{C})$. If $f \in \mathcal{M}_\psi(\Gamma, \chi)$, then we easily see that $\theta f \in \mathcal{M}_\zeta(\Gamma, \varphi\chi)$, so that $(\theta f)^\tau \in \mathcal{M}_{\zeta^\tau}(\Gamma, (\varphi\chi)_\tau)$ by Theorem 10.4 (2); observe that $(\varphi\chi)_\tau = \varphi\chi_\tau$. (Notice that we get the same χ_τ for both ζ and ω, since $\Gamma \subset G$.) We define a vector-valued meromorphic function f' on $\mathfrak{H}_n^{\mathbf{a}}$ by $f' = \theta^{-1}(\theta f)^\tau$. Then $f'\|_{\psi^\tau}\gamma = \chi_\tau(\gamma)f'$ for every $\gamma \in \Gamma$. Now $f \otimes f \in \mathcal{M}_\rho$ with $\rho(x) = \det(x)^{\mathbf{a}}(\omega\otimes\omega)(x)$, and $f'\otimes f' = \theta^{-2}(\theta f \otimes \theta f)^\tau = (f\otimes f)^\tau$. Therefore $f'\otimes f'$ is holomorphic everywhere, and hence the square of any component of f' is holomorphic everywhere. Thus f' has a Fourier expansion, which, multiplied by θ, equals $(\theta f)^\tau = \theta f^\tau$. Since θ is not a zero-divisor in the ring of formal series of §5.9, we see that f^τ gives the Fourier expansion of f'. This proves that $f^\tau \in \mathcal{M}_{\psi^\tau}(\Gamma, \chi_\tau)$. Considering the action of τ^{-1} in the same way, we obtain the last equality of (2), which clearly implies the first two equalities. Next, by Theorem 10.4 (3), $\theta f = \sum_{\nu=1}^m c_\nu g_\nu$ with $c_\nu \in \mathbf{C}$ and $g_\nu \in \mathcal{M}_\zeta(\Gamma, \overline{\mathbf{Q}}, \varphi\chi)$. Changing $\{c_\nu\}$ for a $\overline{\mathbf{Q}}$-basis of $\sum_{\nu=1}^m \overline{\mathbf{Q}}c_\nu$, we may assume that the c_ν are linearly independent over $\overline{\mathbf{Q}}$. Now $f^\sigma = \sum_{\nu=1}^m c_\nu^\sigma\theta^{-1}g_\nu$ for every $\sigma \in \mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$. Since f^σ is holomorphic everywhere, from Lemma 10.3 we see that $\theta^{-1}g_\nu$ is holomorphic everywhere. Thus $\theta^{-1}g_\nu \in \mathcal{M}_\psi(\Gamma, \overline{\mathbf{Q}}, \chi)$. This proves (3). Assertions (4) and (5) can be proved by the same technique as in the proof of Theorem 10.4 (4) and (5).

Next, let the notation be as in (6). Since $\theta f \in \mathcal{M}_\zeta(D)$, we have $(\theta f)\|_\zeta\,\alpha \in \mathcal{M}_\zeta(D)$ by Theorem 9.13 (3). Now $(\theta f)\|_\zeta\,\alpha = p^{-1}(\theta\circ\alpha)g$. By Theorem 7.11, $p^{-1}(\theta\circ\alpha) \in \mathcal{M}_{\mathbf{a}/2}(\mathbf{Q}_{\mathrm{ab}})$, and hence $g \in \mathcal{A}_\psi(D)$. Clearly g is holomorphic, so that $g \in \mathcal{M}_\psi(D)$. This proves (6). To prove (7), take $N > 2$ so that $\theta^2 \in \mathcal{M}_{\mathbf{a}}(\Gamma^N)$, $\Gamma^N \subset \Gamma$, and $\theta f \in \mathcal{M}_\zeta(\Gamma^N)$ for every $f \in \mathcal{M}_\psi(\Gamma)$. Fix such an f; given $\alpha \in \widetilde{G}_+$, take β as in Lemma 10.5; take any branch $q(z)$ of the square root of $j_\beta(z)^{\mathbf{a}}$. We have $\theta^2\|\alpha = (\theta^2\|\beta)^\tau$ and $(\theta f)^\tau\|_{\zeta^\tau}\,\alpha = \big((\theta f)\|_\zeta\beta\big)^\tau$, and so we see that $p^{-1}\big(f^\tau\|_{\omega^\tau}\,\alpha\big) = \pm\big(q^{-1}(f\|_\omega\,\beta)\big)^\tau$. Now let f and g be two elements of $\mathcal{M}_\psi(\Gamma)$ linearly independent over $\mathbf{Q}$. Then $p^{-1}\big((af^\tau + bg^\tau)\|_{\omega^\tau}\,\alpha\big) = \varepsilon_{a,b}\big(q^{-1}((af+bg)\|_\omega\,\beta)\big)^\tau$ with $\varepsilon_{a,b} = \pm1$ for every $a, b \in \mathbf{Q}$. By an elementary argument we easily see that $\varepsilon_{a,b}$ is a constant. Changing q accordingly, we obtain (7). Finally, to prove (1), let $f = g^{-1}h$ with $g \in \mathcal{M}_e$, $e \in \mathbf{Z}^{\mathbf{a}}$ and $h \in \mathcal{M}_{\tau_e}$, where $\tau_e(x) = \det(x)^e\psi(x)$. Taking τ_e to be ψ in (4), we see that (1) is true for h; it is also true for g by Theorem 9.13 (2). Therefore we obtain (1) for f.

10.8. Theorem. *Let ω be as in §9.10 in Cases SP and UT; let ψ be as in (10.10) in Case SP; let Φ be the Galois closure of K over $\mathbf{Q}$ in $\mathbf{C}$ in both cases. Then the following assertions hold:*

(1) $(\mathcal{S}_\omega)^\tau = \mathcal{S}_{\omega^\tau}$ *and* $(\mathcal{S}_\psi)^\tau = \mathcal{S}_{\psi^\tau}$ *for every* $\tau \in \mathrm{Aut}(\mathbf{C})$.

(2) $\mathcal{S}_\omega(\Gamma) = \mathcal{S}_\omega(\Gamma, \Phi)\otimes_\Phi \mathbf{C}$ *and* $\mathcal{S}_\psi(\Gamma) = \mathcal{S}_\psi(\Gamma, \Phi)\otimes_\Phi \mathbf{C}$, *where Γ is a group as in Theorem 10.4 (4) or Theorem 10.7 (4).*

PROOF. Let $f \in \mathcal{S}_\omega$. Given $\alpha \in \widetilde{G}_+$ and $\tau \in \mathrm{Aut}(\mathbf{C})$, take β as in Lemma 10.5. Applying τ to the Fourier expansion of $f\|\beta$, we find that f^τ is a cusp form. From this we obtain $(\mathcal{S}_\omega)^\tau = \mathcal{S}_{\omega^\tau}$. The same type of reasoning applies to $\mathcal{S}_\psi$, if we take β as in Theorem 10.7 (7). Thus we obtain (1). To prove (2), take $f \in \mathcal{S}_\psi(\Gamma)$. By Theorem 10.7 (4), $f = \sum_{\nu=1}^m c_\nu g_\nu$ with $c_\nu \in \mathbf{C}$ and $g_\nu \in \mathcal{M}_\psi(\Gamma, \overline{\mathbf{Q}})$.

We may assume that the c_ν are linearly independent over $\mathbf{C}$. Take a set of m automorphisms $\{\sigma\}$ of $\mathbf{C}$ over $\overline{\mathbf{Q}}$ as in Lemma 10.3. Then $\sum_{\nu=1}^{m} c_\nu^\sigma g_\nu = f^\sigma \in \mathcal{S}_\psi(\Gamma)$ by (1), and so $g_\nu \in \mathcal{S}_\psi(\Gamma)$. This proves that $\mathcal{S}_\psi(\Gamma) = \mathcal{S}_\psi(\Gamma, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$. Then we can prove that $\mathcal{S}_\psi(\Gamma, \overline{\mathbf{Q}}) = \mathcal{S}_\psi(\Gamma, \Phi) \otimes_\Phi \overline{\mathbf{Q}}$ by a Galois-theoretical argument as in the proof of Theorem 10.4 (4). The same technique applies to $\mathcal{S}_\omega$. Thus we obtain (2).

10.9. Theorem. *The symbols Y, Y^*, w, b, and r being the same as in Theorem 9.6, let σ be an element of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *such that $\sigma = [b, Y^*]$ on Y^*_{ab}. Then the following assertions hold:*

(1) *If an element f of $\mathcal{A}_0(\overline{\mathbf{Q}})$ is finite at w, then $f^{(r,\sigma)}$ is finite at w and $f^{(r,\sigma)}(w) = f(w)^\sigma$. (This is a generalization of Theorem 9.6.)*

(2) *If an element f of $\mathcal{A}_\omega(\overline{\mathbf{Q}})$ is finite at w, then $f^{(r,\sigma)}$ is finite at w.*

(3) *Let Q be a square matrix whose columns belong to $\mathcal{A}_\omega(\overline{\mathbf{Q}})$. If Q is finite and invertible at w, then $Q^{(r,\sigma)}$ is finite and invertible at w.*

PROOF. To prove (1), let $f \in \mathcal{A}_0(\Gamma^N, \overline{\mathbf{Q}})$, and put $f = g \circ \varphi_N$ with $g \in \overline{\mathbf{Q}}(V_N)$ with (V_N, φ_N) of §9.7. Let z_0 be a point of $\mathcal{H}$ generic for $\mathcal{A}_0(\overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$. Then $\varphi_N(z_0)$ is a generic point of V_N over $\overline{\mathbf{Q}}$, and so if f is finite at w, then g is defined at $\varphi_N(w)$, so that $g = p(\varphi_N(z_0))/q(\varphi_N(z_0))$ with polynomials p and q such that $q(\varphi_N(w)) \neq 0$. Writing p and q as $\overline{\mathbf{Q}}$-linear combinations of $\mathbf{Q}$-rational polynomials, we find that $f = \sum_i a_i t_i / (\sum_j b_j s_j)$ with a_i, $b_j \in \overline{\mathbf{Q}}$ and t_i, $s_j \in \mathcal{A}_0(\mathbf{Q})$ finite at w such that $\sum_j b_j s_j(w) \neq 0$. Then $f^{(r,\sigma)} = \sum_i a_i^\sigma t_i^{\tau(r)} / (\sum_j b_j^\sigma s_j^{\tau(r)})$. By Theorem 9.6, $t_i^{\tau(r)}$ and $s_j^{\tau(r)}$ are finite at w; moreover $\sum_i a_i^\sigma t_i^{\tau(r)}(w) = \left(\sum_i a_i t_i(w)\right)^\sigma$ and $\sum_j b_j^\sigma s_j^{\tau(r)}(w) = \left(\sum_j b_j s_j(w)\right)^\sigma \neq 0$. Therefore we obtain (1).

Let us now prove

(10.13) *If $f \in \mathcal{A}_{\kappa\mathbf{a}}(\overline{\mathbf{Q}})$ with $\kappa \in \mathbf{Z}$ and f is finite at w, then $f^{(r,\sigma)}$ is finite at w; moreover, if $f(w) \neq 0$, then $f^{(r,\sigma)}(w) \neq 0$.*

This follows from (1) if $\kappa = 0$. Put $g(z) = \theta_K(0, rz; \lambda)$ with $0 < r \in \mathbf{Q}$ and a $\mathbf{Q}$-valued λ. By Proposition 6.14, $g \in \mathcal{M}_{\mathbf{a}/2}(\mathbf{Q})$ in Case SP and $g \in \mathcal{M}_{\mathbf{a}}(\mathbf{Q})$ in Case UT. Choosing suitable r and λ, we may assume that $g(w) \neq 0$, by virtue of Theorem 6.12 (2). Assuming $\kappa > 0$, put $h = g^{2\kappa}$ in Case SP and $h = g^\kappa$ in Case UT. If f belongs to $\mathcal{A}_{\kappa\mathbf{a}}(\overline{\mathbf{Q}})$ and is finite at w, then f/h belongs to $\mathcal{A}_0(\overline{\mathbf{Q}})$ and is finite at w, and hence $f^{(r,\sigma)}/h^{(r,\sigma)}$ is finite at w by (1); therefore $f^{(r,\sigma)}$ is finite at w. Suppose $f(w) \neq 0$; put $p = h^{(r,\sigma)^{-1}}/f$. Then p belongs to $\mathcal{A}_0(\overline{\mathbf{Q}})$ and is finite at w. We have $p^{(r,\sigma)}(w) f^{(r,\sigma)}(w) = h(w) \neq 0$. By (1), $p^{(r,\sigma)}$ is finite at w, and hence $f^{(r,\sigma)}(w) \neq 0$. This proves (10.13) for $\kappa > 0$. Next let $f \in \mathcal{A}_{-\kappa\mathbf{a}}(\overline{\mathbf{Q}})$, $\kappa > 0$; suppose f is finite at w. If $f(w) \neq 0$, then applying our result to f^{-1}, we obtain the desired fact. If $f(w) = 0$, then apply the last result to $f + h^{-1}$. In this way we obtain (10.13) for every $\kappa \in \mathbf{Z}$.

To prove (2) and (3), take $R_v \in \mathcal{M}_{\sigma_v}(\overline{\mathbf{Q}})$ as in Proposition 9.11 so that R_v is finite at w and $\det\left(R_v(w)\right) \neq 0$ for every $v \in \mathbf{b}$; put $q = \prod_{v \in \mathbf{b}} \det(R_v)$. Then $q \in \mathcal{A}_{\mathbf{b}}(\overline{\mathbf{Q}})$. Given $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$ finite at w, put $g = \omega(R)^{-1} f$. Then g has components in $\mathcal{A}_0(\overline{\mathbf{Q}})$ and is finite at w. We have $f^{(r,\sigma)} = \omega(R)^{(r,\sigma)} g^{(r,\sigma)}$, and $g^{(r,\sigma)}$ is finite at w by (1). Since $q(w) \neq 0$, we have $q^{(r,\sigma)}(w) \neq 0$ by (10.13). Therefore $\det(R_v)^{(r,\sigma)}(w) \neq 0$, so that by (10.4b), $\omega(R)^{(r,\sigma)}$ is finite at w. Thus $f^{(r,\sigma)}$ is finite at w, which proves (2).

Let Q be as in (3); suppose Q is finite and invertible at w. Observe that the columns of ${}^tQ^{-1}$ belongs to $\mathcal{A}_\rho(\overline{\mathbf{Q}})$ with $\rho = {}^t\omega^{-1}$. By (2) both $Q^{(r,\sigma)}$ and $({}^tQ^{-1})^{(r,\sigma)}$ are finite at w. Therefore $Q^{(r,\sigma)}$ is invertible at w. This completes the proof.

Some results, similar to, but somewhat different from, Theorem 10.9, for $G = SP(n, \mathbf{Q})$ were given in [S78b, Section 2, Theorem 3.12, and Proposition 3.13].

We insert here an easy lemma which is necessary for our later purposes.

10.10. Lemma. *Let Φ be as in Theorem 10.4 (4), and K' be the reflex field defined for (K, τ) of §3.5 as in §1.12. Let ω and ψ be as in §9.10 and (10.10). Given $\alpha \in \tilde{G}_+$, take $p(z)$ as in Theorem 10.7 (6) and put $\bigl(f\|_\psi(\alpha, p)\bigr)(z) = p(z)^{-1}(f\|_\omega \alpha)(z)$; let $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q}_{\mathrm{ab}}\Phi)$. Then*

$$(f\|_\omega \alpha)^\sigma = f^\sigma\|_\omega \alpha \quad \textit{for every} \quad f \in \mathcal{M}_\omega,$$
$$\bigl(f\|_\psi (\alpha, p)\bigr)^\sigma = f^\sigma\|_\psi (\alpha, p) \quad \textit{for every} \quad f \in \mathcal{M}_\psi.$$

Moreover, if $\omega(x) = \det(x)^{\kappa\mathbf{a}}$ with $\kappa \in \mathbf{Z}$, then these hold for every $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q}_{\mathrm{ab}})$ in Case SP and $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q}_{\mathrm{ab}}K')$ in Case UT.

PROOF. Let $D = \mathbf{Q}_{\mathrm{ab}}$ in Case SP and $D = \mathbf{Q}_{\mathrm{ab}}K'$ in Case UT if $\omega(x) = \det(x)^{\kappa\mathbf{a}}$; let $D = \mathbf{Q}_{\mathrm{ab}}\Phi$ in the general case. The desired facts for f in $\mathcal{M}_\omega(D)$ or $\mathcal{M}_\psi(D)$ follow from Theorem 7.11, Theorem 9.13 (3), and Theorem 10.7 (6). Then, for an arbitrary f in $\mathcal{M}_\omega$ or $\mathcal{M}_\psi$, we obtain our assertion from Theorem 10.4 (5) and Theorem 10.7 (5).

11. Arithmeticity at CM-points

11.1. Let us now consider, in all three cases SP, UT, and UB, the family $\mathcal{F}(\Omega) = \{\mathcal{P}_z \,|\, z \in \mathcal{H}\}$, $\mathcal{P}_z = (A_z, \mathcal{C}_z, \iota_z;\ \{t_i(z)\}_{i=1}^s)$, of (4.26) with a PEL-type $\Omega = \{K, \Psi, L, \mathcal{T}, \{u_i\}_{i=1}^s\}$ of (4.7). With Γ as in (4.28), let (V, φ) be a model of $\Gamma\backslash\mathcal{H}$ as in Theorem 9.1. As explained in §5.4, we can identify $\mathcal{A}_0(\Gamma)$ with the function field of V. Let $\mathcal{A}_0(\Gamma, \overline{\mathbf{Q}})$ denote the set of all functions of the form $f \circ \varphi$ with a $\overline{\mathbf{Q}}$-rational function f on V, which is meaningful since V is defined over $\overline{\mathbf{Q}}$. Then let $\mathcal{A}_0(\overline{\mathbf{Q}})$ denote the union of the fields $\mathcal{A}_0(\Gamma, \overline{\mathbf{Q}})$ for all such Γ. Clearly $\mathcal{A}_0(\Gamma) = \mathbf{C}\mathcal{A}_0(\Gamma, \overline{\mathbf{Q}})$ and $\mathcal{A}_0 = \mathbf{C}\mathcal{A}_0(\overline{\mathbf{Q}})$. For an arbitrary congruence subgroup Γ' of G, we put $\mathcal{A}_0(\Gamma', \overline{\mathbf{Q}}) = \mathcal{A}_0(\overline{\mathbf{Q}}) \cap \mathcal{A}_0(\Gamma')$. For each $w \in \mathcal{H}$, let $\mathfrak{F}_w$ denote the field generated over $\mathbf{Q}$ by $f(w)$ for all f in $\mathcal{A}_0(\overline{\mathbf{Q}})$ finite at w. From Theorem 9.1 (4) and Theorem 2.8 (2) we see that $\mathfrak{F}_w$ contains the field of moduli of $\mathcal{P}_w$, which is algebraic over the field of moduli of $(A_w, \mathcal{C}_w)$. By Theorem 2.8 (3) we can find a model of $\mathcal{P}_w$ rational over $\mathfrak{F}_w$. In this section, whenever we speak of $\mathcal{P}_w$, we take it to be rational over $\mathfrak{F}_w$.

In Cases SP and UT we already defined $\mathcal{A}_0(\overline{\mathbf{Q}})$ in §5.8 in terms of the Fourier coefficients of automorphic forms. That this coincides with $\mathcal{A}_0(\overline{\mathbf{Q}})$ defined above can be seen from Theorem 9.3, since $\mathcal{A}_0(\overline{\mathbf{Q}}) = \bigcup_{N=1}^\infty \mathcal{A}_0(\Gamma^N, \overline{\mathbf{Q}})$ and $\mathcal{A}_0(\Gamma^N, \overline{\mathbf{Q}}) = \overline{\mathbf{Q}}\mathcal{A}_0(\Gamma^N, \mathbf{Q})$ by Theorem 8.11 (2). We can also speak of the $\overline{\mathbf{Q}}$-rationality of automorphic forms in Cases SP and UT. The main purpose of this section is to define such in Case UB, or rather in all three cases, in terms of a certain property at each CM-point.

Let us hereafter denote by $\mathcal{H}_{\mathrm{CM}}$ the set of all CM-points of $\mathcal{H}$. The field of moduli of $(A_w, \mathcal{C}_w)$ for $w \in \mathcal{H}_{\mathrm{CM}}$ is contained in $\overline{\mathbf{Q}}$ as shown in [S98, Proposition

26 on p.96 or Corollary 18.9]. Therefore if $w \in \mathcal{H}_{\mathrm{CM}}$, then $k_\Omega(\varphi(w)) \subset \overline{\mathbf{Q}}$; hence if an element f of $\mathcal{A}_0(\overline{\mathbf{Q}})$ is finite at w, then $f(w) \in \overline{\mathbf{Q}}$, so that $\mathfrak{F}_w \subset \overline{\mathbf{Q}}$. (In Cases SP and UT, Theorem 9.6 gives a more precise result, but we do not need it in this section.) Thus $\mathcal{P}_w$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{H}_{\mathrm{CM}}$.

11.2. Proposition. (1) *If $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$ and $\alpha \in \widetilde{G}_+$, then $f \circ \alpha \in \mathcal{A}_0(\overline{\mathbf{Q}})$.*

(2) *Let $\mathcal{W}$ be a subset of $\mathcal{H}_{\mathrm{CM}}$ dense in $\mathcal{H}$. If $f \in \mathcal{A}_0$ and $f(w) \in \overline{\mathbf{Q}}$ for every $w \in \mathcal{W}$ where f is finite, then $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$.*

PROOF. Given $f \in \mathcal{A}_0$, we can find Ω so that $f \in \mathcal{A}_0(\Gamma)$ for Γ of (4.28). Then $f = g \circ \varphi$ with $g \in \mathbf{C}(V)$. Let $\mathcal{W}'$ be the subset of $\mathcal{W}$ consisting of the points where f is defined. Take a field of rationality k for g containing $\overline{\mathbf{Q}}$ and take also an isomorphism σ of k onto a subfield of $\mathbf{C}$ over $\overline{\mathbf{Q}}$. Suppose $f(w) \in \overline{\mathbf{Q}}$ for every $w \in \mathcal{W}'$. Now $\varphi(w)$ is $\overline{\mathbf{Q}}$-rational, and hence $g^\sigma(\varphi(w)) = g(\varphi(w))^\sigma = f(w)^\sigma = f(w) = g(\varphi(w))$. Since $\varphi(\mathcal{W}')$ is dense in V, we obtain $g^\sigma = g$, that is, g is $\overline{\mathbf{Q}}$-rational, and so $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$. This proves (2). Clearly (1) follows from (2), since an element of $\widetilde{G}_+$ maps $\mathcal{H}_{\mathrm{CM}}$ into $\mathcal{H}_{\mathrm{CM}}$ as noted in §4.11.

It should of course be remarked that $\mathcal{W}$ as in (2) exists. Indeed, we have at least one CM-point w_0 in $\mathcal{H}$, as remarked in §4.11. Now $G_{\mathbf{a}}$ acts on $\mathcal{H}$ transitively, and the projection of G to $G_{\mathbf{a}}$ is dense in $G_{\mathbf{a}}$. (The last fact in Cases SP and UT is proved in Lemma 7.5. In Case UB, it can be proved by means of the Cayley parametrization; see [S98, Proposition 23.5].) Then we can take the set of points $\alpha(w_0)$ for all $\alpha \in G$ as $\mathcal{W}$.

11.3. Given a CM-field K, we denote by J_K the set of all embeddings of K into $\mathbf{C}$, and by id_K the identity embedding of K into $\mathbf{C}$. We then denote by I_K the free $\mathbf{Z}$-module generated by the elements of J_K. If (K, Φ) is a CM-type, then we naturally view Φ as a subset of J_K, and denote by the same letter Φ the element of I_K that is the sum of the members of Φ. To make our exposition easier, we assume that every CM-field in this section is a subfield of $\mathbf{C}$. We always denote complex conjugation by ρ.

Let us now recall some basic properties of the *period symbol* associated with a CM-field K, which is a bilinear map $p_K : J_K \times J_K \to \mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ with certain properties (see [S98, Theorem 32.5]). Here we note a fundamental property:

Given a CM-type (K, Φ), take (A, ι) of type (K, Φ) in the sense of §2.9, rational over $\overline{\mathbf{Q}}$. Let $n = \dim(A)$ and $\Phi = \sum_{\nu=1}^n \varphi_\nu$ with $\varphi_\nu \in J_K$. Then for each ν there exists a holomorphic 1-form ω on A such that $\delta\iota(a)\omega = a^{\varphi_\nu}\omega$ if $a \in K$ and $\iota(a) \in \mathrm{End}(A)$, where $\delta\iota(a)$ denotes the action of $\iota(a)$ on the space of 1-forms. Clearly $\mathbf{C}\omega$ depends only on φ_ν. In this setting we have (see [S98, Theorem 32.2])

(11.1) *ω is $\overline{\mathbf{Q}}$-rational if and only if the value of the integral $\int_c \omega$ belongs to the coset $\pi p_K(\varphi_\nu, \Phi)\overline{\mathbf{Q}}$ for every 1-cycle c on A with coefficients in $\mathbf{Z}$.*

Take a CM-algebra $Y = K_1 \oplus \cdots \oplus K_t$ with CM-fields K_i. We denote by J_Y the set of all nontrivial ring-homomorphisms of Y into $\mathbf{C}$, and identify J_Y with the union $\bigcup_{i=1}^t J_{K_i}$ in an obvious way. We also denote by I_Y the free $\mathbf{Z}$-module generated by the elements of J_Y, and identify I_Y with the direct sum $I_{K_1} \oplus \cdots \oplus I_{K_t}$. Given $\alpha = (\alpha_i)_{i=1}^t$ and $\beta = (\beta_i)_{i=1}^t$ in I_Y with $\alpha_i, \beta_i \in I_{K_i}$, we define an element

$p_Y(\alpha, \beta)$ of $\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$ by

$$p_Y(\alpha, \beta) = \prod_{i=1}^{t} p_{K_i}(\alpha_i, \beta_i). \tag{11.2}$$

In the following treatment we denote any nonzero complex number belonging to the coset $p_Y(\alpha, \beta)$ by the same symbol $p_Y(\alpha, \beta)$. The same convention applies also to the symbols $\mathfrak{p}_v(w)$, $\mathfrak{P}_\omega(w)$, and $\mathfrak{P}_k(w)$ which will be defined below.

11.4. Returning to the family $\mathcal{F}(\Omega)$ of §11.1, we take a CM-algebra Y, a map $h : Y \to K_r^r$, the fixed point w of $h(Y^u)$, and the injection $\iota' : Y \to \mathrm{End}_{\mathbf{Q}}(A_w)$ as in §4.11. We have seen there that (A_w, ι') is of type (Y, Φ) with some Φ whose restriction to K is equivalent to Ψ. We take here a $\overline{\mathbf{Q}}$-rational model of $\mathcal{P}_w$ as explained in §11.1. Let Φ_v, ψ_v, and φ_v be as in (4.38) and (4.40) for the present Y. For $\alpha \in Y^u$ we have $\psi_v(\alpha) = \lambda_v(h(\alpha), w)$ and $\varphi_v(\alpha) = \mu_v(h(\alpha), w)$. Moreover, $\mathrm{diag}[{}^t\psi_v(\alpha), {}^t\varphi_v(\alpha)]$, being the v-component of ${}^tM(h(\alpha), w)$, represents $\iota_w(\alpha)$ on the v-th factor of $(\mathbf{C}^l)^{\mathbf{a}}$ (see §4.3). By Lemma 4.13, $\psi_v(a)$ and $\varphi_v(a)$ have algebraic entries for every $a \in Y$. Take $B_v \in GL_{m_v}(\overline{\mathbf{Q}})$ and $C_v \in GL_{n_v}(\overline{\mathbf{Q}})$ so that

$$B_v\psi_v(a)B_v^{-1} = \mathrm{diag}[\psi_{v1}(a), \dots, \psi_{vm_v}(a)], \tag{11.3a}$$

$$C_v\varphi_v(a)C_v^{-1} = \mathrm{diag}[\varphi_{v1}(a), \dots, \varphi_{vn_v}(a)] \tag{11.3b}$$

for every $a \in Y$ with $\psi_{vi}, \varphi_{vj} \in J_Y$. In Case SP we have $\psi_v = \varphi_v$ and $m_v = n_v = n$, and so we take $\psi_{vi} = \varphi_{vi}$ and $B_v = C_v$.

We now define an element $\mathfrak{p}(w) = (\mathfrak{p}_v(w))_{v\in\mathbf{b}}$ of $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})$ in Cases UT and UB by

$$\mathfrak{p}_{v\rho}(w) = B_v^{-1}\mathrm{diag}[p_Y(\psi_{v1}, \Phi), \dots, p_Y(\psi_{vm_v}, \Phi)]B_v, \tag{11.4a}$$

$$\mathfrak{p}_v(w) = C_v^{-1}\mathrm{diag}[p_Y(\varphi_{v1}, \Phi), \dots, p_Y(\varphi_{vn_v}, \Phi)]C_v, \tag{11.4b}$$

where $v \in \mathbf{a}$. In Case SP we define $\mathfrak{p}_v(w)$ for each $v \in \mathbf{a}$ by (11.4b), and so $\mathfrak{p}(w)$ is an element of $GL_n(\mathbf{C})^{\mathbf{a}}$. Here recall the convention made in §§3.5 and 5.1 that $\mathbf{b} = \mathbf{a}\rho \cup \mathbf{a} = J_K$ in Cases UT and UB, and $\mathbf{a}$ is identified with a fixed CM-type of K; also $n_{v\rho} = m_v$ for $v \in \mathbf{a}$. As we said in §3.3, we put $GL_0(\mathbf{C}) = \{1\}$. Thus we define $\mathfrak{p}_v(w)$ to be the element 1 of $GL_0(\mathbf{C})$ if $n_v = 0$. If we replace C_v by another matrix C'_v in (11.3a), then $C'_vC_v^{-1}$ is diagonal, and hence $\mathfrak{p}_v(w)$ does not change. Thus $\mathfrak{p}(w)$ is independent of the choice of B_v and C_v. Also, if we view $p_Y(\ ,\ )$ as a coset of $\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$, then we view $\mathfrak{p}(w)$ as a coset of $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})/\prod_{v\in\mathbf{b}} GL_{n_v}(\overline{\mathbf{Q}})$.

It may happen that the same CM-point can be obtained from different (Y, h). However, we have:

11.5. Proposition. (1) *The coset $\mathfrak{p}_v(w)GL_{n_v}(\overline{\mathbf{Q}})$ is determined by w independently of the choice of (Y, h).*

(2) *$\mathfrak{p}(\beta w)\mathcal{G} = M_\beta(w)\mathfrak{p}(w)\mathcal{G}$ for every $\beta \in \widetilde{G}_+$, where $\mathcal{G} = \prod_{v\in\mathbf{b}} GL_{n_v}(\overline{\mathbf{Q}})$.*

PROOF. Here we prove only (2). The proof of (1) will be completed in §§11.9 and 11.10. Given Y, h, and w as above, put $h'(a) = \beta h(a)\beta^{-1}$. Then βw is the fixed point of $h'(Y^u)$ as observed in §4.11. Put $\xi_v = \lambda_v(\beta, w)$. Then we have $\lambda_v(h'(\alpha), \beta w)\xi_v = \xi_v\lambda_v(h(\alpha), w)$ for every $\alpha \in Y^u$. Therefore, if we define ψ'_v and φ'_v by (4.37) with h' in place of h, then $B_v\xi_v^{-1}\psi'_v(a)\xi_vB_v^{-1} = \mathrm{diag}[\psi_{v1}(a), \dots, \psi_{vm_v}(a)]$ for every $a \in Y$. Thus $\mathfrak{p}_{v\rho}(\beta w) = \xi_v\mathfrak{p}_{v\rho}(w)\xi_v^{-1}$. A similar formula holds for $\mathfrak{p}_v(w)$ and $\mathfrak{p}_v(\beta w)$. This proves (2).

11.6. Putting $2d = r[K : \mathbf{Q}]$, we are going to define an embedding of $\mathcal{H}$ into $\mathfrak{H}_d$. In Cases SP and UT this was done in §§6.2 and 6.5. Here we treat all cases uniformly. (Recall that $r = 2n$ in Cases SP and UT; also the present d is the same as that of (4.9).) Since $(x, y) \mapsto \mathrm{Tr}_{K/\mathbf{Q}}(x\mathcal{T}y^*)$ with $\mathcal{T}$ of (4.13) is a nondegenerate alternating form, we can find a $\mathbf{Q}$-linear map $g : K_r^1 \to \mathbf{Q}_{2d}^1$ so that

$$\mathrm{Tr}_{K/\mathbf{Q}}(x\mathcal{T}y^*) = g(x)\eta_d \cdot {}^t g(y) \qquad (x, y \in K_r^1). \tag{11.5}$$

Let $\{ e_k \}_{k=1}^{2d}$ be the standard basis of $\mathbf{Q}_{2d}^1$. Given $z \in \mathcal{H}$, we consider the map $p_z : (K_{\mathbf{a}})_r^1 \to \mathbf{C}^d$ of (4.23) and put $x_k = p_z\big(g^{-1}(e_k)\big)$. Now we can take $(\mathbf{Q},\, p_z \circ g^{-1},\, \eta_d)$ as $(W, q, \mathcal{T})$ in §§4.3 and 4.4, and we can write (4.18) and (4.21) in the form

$$X\big(p_z \circ g^{-1}\big) = \begin{bmatrix} x_1 & \cdots & x_{2d} \\ \overline{x}_1 & \cdots & \overline{x}_{2d} \end{bmatrix} = \begin{bmatrix} \kappa & 0 \\ 0 & \overline{\kappa} \end{bmatrix} \begin{bmatrix} Z & 1_d \\ \overline{Z} & 1_d \end{bmatrix} \tag{11.6}$$

with $\kappa \in GL_d(\mathbf{C})$ and $Z \in \mathfrak{H}_d$. Put $\kappa = \kappa(z)$ and $Z = \varepsilon(z)$. Then

$$\varepsilon(z) = \kappa(z)^{-1} [x_1 \quad \cdots \quad x_d], \quad \kappa(z) = [x_{d+1} \quad \cdots \quad x_{2d}]. \tag{11.7}$$

Since p_z is holomorphic in z as noted in §4.7, we see that both $\kappa(z)$ and $\varepsilon(z)$ are holomorphic in z. Thus we have a holomorphic embedding

$$\varepsilon : \mathcal{H} \to \mathfrak{H}_d. \tag{11.8}$$

Now $(p_z \circ g^{-1})\big(\sum_{k=1}^m c_k e_k\big) = \kappa(z)[\varepsilon(z) \quad 1_d]c$ for every $c \in \mathbf{Q}^{2d}$, that is,

$$p_z(x) = \kappa(z)[\varepsilon(z) \quad 1_d] \cdot {}^t g(x) \qquad (x \in K_r^1). \tag{11.9}$$

For $\alpha \in K_r^r$ define $\widetilde{\alpha} \in \mathbf{Q}_{2d}^{2d}$ by $g(x\alpha) = g(x)\widetilde{\alpha}$. From (11.5) we see that $\widetilde{\alpha} \in Sp(d, \mathbf{Q})$ if $\alpha \in U(\mathcal{T}) = G$. Also, from (4.31) and (11.9) we obtain

$$\begin{aligned} &{}^t M(\alpha, z)\kappa(\alpha z)[\varepsilon(\alpha z) \quad 1] \cdot {}^t g(x) = {}^t M(\alpha, z) p_{\alpha z}(x) = p_z(x\alpha) \\ &= \kappa(z)[\varepsilon(z) \quad 1] \cdot {}^t\widetilde{\alpha} \cdot {}^t g(x) = \kappa(z) \cdot {}^t\mu\big(\widetilde{\alpha}, \varepsilon(z)\big)[\widetilde{\alpha}(\varepsilon(z)) \quad 1] \cdot {}^t g(x). \end{aligned} \tag{11.10}$$

This proves that $\varepsilon(\alpha z) = \widetilde{\alpha}\big(\varepsilon(z)\big)$ and

$$\mu\big(\widetilde{\alpha}, \varepsilon(z)\big) = {}^t\kappa(\alpha z) M(\alpha, z) \cdot {}^t\kappa(z)^{-1} \qquad (\alpha \in U(\mathcal{T}),\, z \in \mathcal{H}). \tag{11.11}$$

To avoid confusion, we hereafter denote by $\mathfrak{A}_0(\overline{\mathbf{Q}})$ (resp. $\mathfrak{A}_s(\overline{\mathbf{Q}})$) the field $\mathcal{A}_0(\overline{\mathbf{Q}})$ (resp. the vector space $\mathcal{A}_s(\overline{\mathbf{Q}})$) defined on $\mathfrak{H}_d$ with respect to $Sp(d, \mathbf{Q})$, where s is a $\mathbf{Q}$-rational representation of $GL_d(\mathbf{C})$. Then we note a simple fact:

(11.12) *If $g \in \mathfrak{A}_0(\overline{\mathbf{Q}})$ and $g \circ \varepsilon$ is finite, then $g \circ \varepsilon \in \mathcal{A}_0(\overline{\mathbf{Q}})$.*

To prove this, take a CM-point w on $\mathcal{H}$ fixed by $h(Y^u)$ with $h : Y \to K_r^r$ satisfying (4.35). Define $\widetilde{h} : Y \to \mathbf{Q}_{2d}^{2d}$ by $\widetilde{h}(a) = \widetilde{h(a)}$. Then (4.35) together with (11.5) shows that $\widetilde{h}(a^\rho) = \eta_d \cdot {}^t\widetilde{h}(a)\eta_d^{-1}$, and $\varepsilon(w)$ is the fixed point of $\widetilde{h}(Y^u)$. Thus $\varepsilon(w)$ is a CM-point on $\mathfrak{H}_d$. Therefore (11.12) follows from Proposition 11.2 (2).

We note here a basic fact on 1-forms on an abelian variety:

11.7. Lemma. *Let A be an abelian variety defined over a subfield k of $\mathbf{C}$ which is algebraic over the field of moduli of $(A, \mathcal{C})$ with a polarization $\mathcal{C}$ of A. Suppose that A is isomorphic to $\mathbf{C}^d/\big([Z_0 \quad 1_d]\mathbf{Z}^{2d}\big)$ with $Z_0 \in \mathfrak{H}_d$. Then there exists a $(d \times d)$-matrix P whose columns belong to $\mathfrak{A}_\sigma(\overline{\mathbf{Q}})$ and such that P is finite and invertible at Z_0, where σ is the identity map of $GL_d(\mathbf{C})$ onto itself. Moreover, with any such P define 1-forms $\xi_1, \ldots, \xi_n$ on $\mathbf{C}^d/\big([Z_0 \quad 1_d]\mathbf{Z}^{2d}\big)$ by*

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} = \pi \cdot {}^tP(Z_0) \begin{bmatrix} du_1 \\ \vdots \\ du_d \end{bmatrix},$$

where $u_1, \dots, u_d$ are the coordinate functions on $\mathbf{C}^d$. Then $\xi_1, \dots, \xi_d$, viewed as 1-forms on A, form a basis of holomorphic 1-forms rational over the algebraic closure $\overline{k}$ of k.

This is a simplified version of [S98, Theorem 30.3]. The existence of P is a special case of Proposition 9.11. Notice that $\overline{k}$ does not depend on the choice of $\mathcal{C}$.

11.8. We now restrict our discussion to Case UB. Take σ and P as in Lemma 11.7 with any fixed point Z_0 on $\mathfrak{H}_d$ in the setting of §11.6. By Lemma 4.12, we can take $b \in K$ so that $K = \mathbf{Q}(b)$ and $bb^\rho = 1$; put $\alpha = b1_r$ and

$$U(z) = V\big(\varepsilon(z)\big), \quad V(Z) = P(Z)^{-1}(P\|_\sigma\widetilde{\alpha})(Z) \qquad (z \in \mathcal{H},\ Z \in \mathfrak{H}_d).$$

By Proposition 9.11 (4) the entries of V belong to $\mathfrak{A}_0(\overline{\mathbf{Q}})$, and hence the entries of U belong to $\mathcal{A}_0(\overline{\mathbf{Q}})$ by (11.12). From (3.17), (3.37), (4.9), (4.10), and (4.29) we see that $M(\alpha, z) = \Psi(b)$, which combined with (11.11) shows that $\mu\big(\widetilde{\alpha}, \varepsilon(z)\big) = {}^t\kappa(z)\Psi(b) \cdot {}^t\kappa(z)^{-1}$, and hence $U(z)^{-1} = X(z)^{-1}\Psi(b)X(z)$ with $X(z) = {}^t\kappa(z)^{-1}P\big(\varepsilon(z)\big)$. Since $K = \mathbf{Q}[b]$, we see that $a \mapsto X(z)^{-1}\Psi(a)X(z)$ is a ring-injection of K into $\mathcal{A}_0(\overline{\mathbf{Q}})_d^d$. Therefore we can find an element W of $GL_d\big(\mathcal{A}_0(\overline{\mathbf{Q}})\big)$ such that $X(z)^{-1}\Psi(a)X(z) = W^{-1}\Psi(a)W$ for every $a \in K$. In view of (4.10) we have

$$(11.13) \qquad {}^t\kappa(z)^{-1}P\big(\varepsilon(z)\big) = X(z) = \mathrm{diag}[S_v,\ R_v]_{v\in\mathbf{a}} \cdot W(z)$$

with square matrices S_v and R_v of size m_v and n_v, whose entries are meromorphic functions on $\mathcal{H}$. Employing (11.11) and (4.29), we easily see that

$$(11.14) \qquad S_v\big(\gamma(z)\big) = \lambda_v(\gamma, z)S_v(z), \quad R_v\big(\gamma(z)\big) = \mu_v(\gamma, z)R_v(z) \qquad (v \in \mathbf{a}),$$

if γ belongs to a sufficiently small congruence subgroup Γ of G.

11.9. We took $\mathcal{F}(\Omega)$ in §11.1 with a PEL-type $\Omega = \{K, \Psi, L, \mathcal{T}\}$. We now assume that $L = g^{-1}\big(\mathbf{Z}_{2d}^1\big)$ with $g : K_r^1 \to \mathbf{Q}_{2d}^1$ as in §11.6. Let $w \in \mathcal{H}_{\mathrm{CM}}$. From Lemma 4.13 we see that $p_w(x)$ has algebraic components for every $x \in K_r^1$, and hence (11.6) shows that the entries of $\varepsilon(w)$ and $\kappa(w)_v$ for every $v \in \mathbf{a}$ are all algebraic.

Let $\mathcal{W}_0$ be the set of all CM-points on $\mathcal{H}$ where both $P \circ \varepsilon$ and W are finite and invertible. Then, by (11.13), R_v and S_v are finite and invertible at every point of $\mathcal{W}_0$.

To prove (1) of Proposition 11.5 in Case UB, we first assume that $w \in \mathcal{W}_0$. Take (A_w, ι') of type (Y, Φ) we considered in §11.3. Recall that A_w is isomorphic to $\mathbf{C}^d/p_w(L)$. Also, we see that $u \mapsto \kappa(w)^{-1}u$ for $u \in \mathbf{C}^d$ gives an isomorphism of $\mathbf{C}^d/p_w(L)$ onto $\mathbf{C}^d/[\varepsilon(w) \quad 1_d]\mathbf{Z}^{2d}$. By Lemma 11.7 we obtain $\overline{\mathbf{Q}}$-rational 1-forms ξ_k on A_w by putting

$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_d \end{bmatrix} = \pi \cdot {}^tP(\varepsilon(w))\kappa(w)^{-1} \begin{bmatrix} du_1 \\ \vdots \\ du_d \end{bmatrix},$$

where $u_1, \dots, u_d$ are the standard coordinate functions on $\mathbf{C}^d$.

We have a decomposition $\mathbf{C}^d = \bigoplus_{v\in\mathbf{a}} V_v$ with V_v, isomorphic to $\mathbf{C}^r$, on which Ψ_v of (24.1b) acts (see §4.3). Let $x_1^v, \dots, x_{m_v}^v, y_1^v, \dots, y_{n_v}^v$ be the coordinate functions on V_v. Then these for all v are renamings of the u_k. Since $W(w)$ is $\overline{\mathbf{Q}}$-rational, we find that the components of

$$(11.15) \qquad \pi \cdot {}^tS_v(w)\begin{bmatrix} dx_1^v \\ \vdots \\ dx_{m_v}^v \end{bmatrix} \quad \text{and} \quad \pi \cdot {}^tR_v(w)\begin{bmatrix} dy_1^v \\ \vdots \\ dy_{n_v}^v \end{bmatrix}$$

correspond to $\overline{\mathbf{Q}}$-rational 1-forms on A_w. Take B_v as in (11.3a) and put

$$\begin{bmatrix} du_1^v \\ \vdots \\ du_{m_v}^v \end{bmatrix} = {}^tB_v^{-1}\begin{bmatrix} dx_1^v \\ \vdots \\ dx_{m_v}^v \end{bmatrix}.$$

Then we find that $\delta\iota(a)du_i^v = \psi_{vi}(a)du_i^v$. Now the periods of dx_i^v are entries of $p_w(L)$, which are algebraic, and hence so are the periods of du_i^v. Therefore by (11.1) we see that $\pi p_Y(\psi_{vi}, \Phi)du_i^v$, viewed as a 1-form on A_w, must be $\overline{\mathbf{Q}}$-rational. Comparing this result with (11.15), we find the first of the following two inclusions:

$$\mathrm{diag}\big[p_Y(\psi_{v1}, \Phi), \ldots, p_Y(\psi_{vm_v}, \Phi)\big] \in B_vS_v(w)GL_{m_v}(\overline{\mathbf{Q}}),$$
$$\mathrm{diag}\big[p_Y(\varphi_{v1}, \Phi), \ldots, p_Y(\varphi_{vn_v}, \Phi)\big] \in C_vR_v(w)GL_{n_v}(\overline{\mathbf{Q}}).$$

The latter can be shown similarly. These can be written

$$(11.16) \qquad \mathfrak{p}_{v\rho}(w) \in S_v(w)GL_{m_v}(\overline{\mathbf{Q}}) \quad \text{and} \quad \mathfrak{p}_v(w) \in R_v(w)GL_{n_v}(\overline{\mathbf{Q}})$$

for every $v \in \mathbf{a}$. Now $S_v(w)$ and $R_v(w)$ depend only on w (and P), and are independent of the choice of (Y, h). Therefore we obtain (1) of Proposition 11.5 in Case UB at least for $w \in \mathcal{W}_0$.

Now take any CM-point w. Then we can find $\beta \in \widetilde{G}_+$ so that $\beta^{-1}w \in \mathcal{W}_0$. Put $w_1 = \beta^{-1}w$. By (2) of Proposition 11.5, $\mathfrak{p}(w)\mathcal{G} = M_\beta(w_1)\mathfrak{p}(w_1)\mathcal{G}$, where $\mathfrak{p}(w)$ and $\mathfrak{p}(w_1)$ are defined with a fixed (Y, h) and (Y, h') as in the proof there. Now $\mathfrak{p}(w_1)$ is independent of the choice of (Y, h), and therefore the same is true for $\mathfrak{p}(w)$. This completes the proof of Proposition 11.5 in Case UB.

11.10. Let us now prove (1) of Proposition 11.5 in Cases SP and UT. In Case SP, (6.12) shows that (11.9) is true for $z \in \mathfrak{H}_n^{\mathbf{a}}$ with $\kappa(z) = B^{-1}$, and (6.11b) is exactly (11.11). Now we consider R_v of Proposition 9.11 belonging to $\mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$. Let $\mathfrak{R}(z) = \mathrm{diag}\big[R_v(z)\big]_{v\in\mathbf{a}}$ and $W(z) = \mathfrak{R}(z)^{-1} \cdot {}^tBP\big(\varepsilon(z)\big)$, where P is the function on $\mathfrak{H}_d$ as in Lemma 11.7. Then W has entries in $\mathcal{A}_0$. Checking the Fourier coefficients, we see that the entries of W are $\overline{\mathbf{Q}}$-rational. Since ${}^tBP\big(\varepsilon(z)\big) = \mathrm{diag}\big[R_v(z)\big]_{v\in\mathbf{a}}W(z)$, (11.13) is true with the present symbols, if we ignore S_v. Therefore we can repeat our argument of §11.9 in Case SP to find that (11.16) is true in the present setting, and we obtain the desired fact.

In Case UT, combining (6.24b) with (6.12), we obtain an element E of $GL_d(\overline{\mathbf{Q}})$ and an embedding ε of $\mathcal{H} = \mathcal{H}_n^{\mathbf{a}}$ into $\mathfrak{H}_d$ such that $p_z(x) = E\big[\varepsilon(z) \;\; 1_d\big] \cdot {}^tg(x)$ with a map $g : K_{2n}^1 \to \mathbf{Q}_{2d}^1$ satisfying (11.5) with $\mathcal{T} = \eta_n$. (Using the symbols of §§6.2 and 6.5, $g^{-1} \circ h^{-1}$ gives the present g.) Therefore, considering the functions R_v of Proposition 9.11, we can handle Case UT in the same manner as in Case SP.

As a by-product of this reasoning we obtain:

11.11. Proposition. *Let ω be a $\overline{\mathbf{Q}}$-rational representation of $GL_n(\mathbf{C})^{\mathbf{b}}$ in Cases SP and UT. Let $\mathcal{W}$ be a subset of $\mathcal{H}_{\mathrm{CM}}$ dense in $\mathcal{H}$. Then an element f*

of $\mathcal{A}_\omega$ *belongs to* $\mathcal{A}_\omega(\overline{\mathbf{Q}})$ *if and only if* $\omega(\mathfrak{p}(w)^{-1})f(w)$ *is* $\overline{\mathbf{Q}}$*-rational for every* $w \in \mathcal{W}$ *where* f *is finite.*

PROOF. Take $R = (R_v)_{v\in\mathbf{b}}$ with $R_v \in \mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$ as in Proposition 9.11. Let $\mathcal{W}'$ be the subset of $\mathcal{W}$ consisting of the points w where R_v is finite and invertible for every $v \in \mathbf{b}$. Then $\mathcal{W}'$ is dense in $\mathcal{H}$. Put $g = \omega(R)^{-1}f$. Then g has components in $\mathcal{A}_0$. Suppose $\omega(\mathfrak{p}(w)^{-1})f(w)$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{W}$ where f is finite. Then $g(w) = \omega(R(w)^{-1}\mathfrak{p}(w))\omega(\mathfrak{p}(w)^{-1})f(w)$. The reasoning in §11.10 shows that (11.16) is true in Cases SP and UT with the present R_v and $S_v = R_{v\rho}$, that is, $R(w)^{-1}\mathfrak{p}(w)$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{W}'$, and hence $g(w)$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{W}'$ where g is finite. By Proposition 11.2 (2), g has components in $\mathcal{A}_0(\overline{\mathbf{Q}})$. Since $f = \omega(R)g$, we see that $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$. This proves the 'if'-part. Conversely, if $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$, then $g \in \mathcal{A}_0(\overline{\mathbf{Q}})$. For every $w \in \mathcal{W}$ we can choose R so that R_v is finite and invertible at w for every $v \in \mathbf{b}$. If f is finite at w, then so is g, and $g(w)$ is $\overline{\mathbf{Q}}$-rational. We have $\omega(\mathfrak{p}(w)^{-1})f(w) = \omega(\mathfrak{p}(w)^{-1}R(w))g(w)$, which is $\overline{\mathbf{Q}}$-rational. This completes the proof.

11.12. Take a $\overline{\mathbf{Q}}$-rational representation $\omega : \prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}) \to GL(V)$ with a finite-dimensional complex vector space V with a $\overline{\mathbf{Q}}$-rational structure. Then for each CM-point w we put

$$\mathfrak{P}_\omega(w) = \omega(\mathfrak{p}(w)), \tag{11.17}$$

$$\mathfrak{P}_k(w) = \det(\mathfrak{p}(w))^k \qquad (k \in \mathbf{Z}^{\mathbf{b}}), \tag{11.17a}$$

and denote by $\mathcal{A}_\omega(\overline{\mathbf{Q}})$ the set of all $f \in \mathcal{A}_\omega$ such that $\mathfrak{P}_\omega(w)^{-1}f(w)$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{H}_{\mathrm{CM}}$ where f is finite. We call such an f $\overline{\mathbf{Q}}$*-rational,* and put $\mathcal{M}_\omega(\overline{\mathbf{Q}}) = \mathcal{M}_\omega \cap \mathcal{A}_\omega(\overline{\mathbf{Q}})$. Proposition 11.11 shows that this is consistent with what we already have in Cases SP and UT. Thus the point of our definition in this subsection is mainly in Case UB.

11.13. Proposition (Case UB). (1) *If* $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$ *and* $\beta \in \widetilde{G}_+$, *then both* $f|_\omega\beta$ *and* $f\|_\omega\beta$ *belong to* $\mathcal{A}_\omega(\overline{\mathbf{Q}})$.

(2) *Let* $\mathcal{W}$ *be a subset of* $\mathcal{H}_{\mathrm{CM}}$ *dense in* $\mathcal{H}$. *Then an element* f *of* $\mathcal{A}_\omega$ *belongs to* $\mathcal{A}_\omega(\overline{\mathbf{Q}})$ *if* $\mathfrak{P}_\omega(w)^{-1}f(w)$ *is* $\overline{\mathbf{Q}}$*-rational for every* $w \in \mathcal{W}$ *where* f *is finite.*

PROOF. Assertion (1) follows immediately from our definition of $\mathcal{A}_\omega(\overline{\mathbf{Q}})$ and Proposition 11.5 (2). Assertion (2) will be proved in the proof of the following proposition.

11.14. Proposition (Case UB). *For each fixed* $v \in \mathbf{b}$ *define two* $GL_{n_v}(\mathbf{C})$*-valued representations* σ_v *and* τ_v *of* $\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C})$ *by*

$$\sigma_v(x) = x_v \quad \text{and} \quad \tau_v(x) = \det(x)^{\mathbf{b}}x_v \quad \text{for} \quad x \in \textstyle\prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}), \tag{11.18}$$

where $a^{\mathbf{b}} = \prod_{v\in\mathbf{b}} a_v$. *Given* $z_0 \in \mathcal{H}$, *there exist two sets of matrix-valued functions* $\{R_v\}_{v\in\mathbf{b}}$ *and* $\{Q_v\}_{v\in\mathbf{b}}$ *on* $\mathcal{H}$ *with the following properties:*

(1) R_v *and* Q_v *are square matrices of size* n_v, *and the columns of* R_v *(resp.* Q_v*) belong to* $\mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$ *(resp.* $\mathcal{M}_{\tau_v}(\overline{\mathbf{Q}})$*).*

(2) R_v *is finite at* z_0, *and* $\det(R_v(z_0))\det(Q_v(z_0)) \neq 0$ *for every* $v \in \mathbf{b}$.

If $n_v = 0$, we have $GL_{n_v}(\mathbf{C}) = \{1\}$ by our stipulation. Therefore, if $n_v = 0$, we simply take R_v and Q_v to be the constant on $\mathcal{H}$ whose values are 1 in this trivial group. Notice also that Proposition 9.11 gives similar results in Cases SP and UT in stronger forms.

PROOF. Let the notation be as in §§11.8 and 11.9. Put $R_{v\rho} = S_v$ for each $v \in \mathbf{a}$. Then (11.14) shows that $R_v \in \mathcal{A}_{\sigma_v}$ for every $v \in \mathbf{b}$. Moreover, by (11.16), $\mathfrak{p}_v(w)^{-1}R_v(w)$ is $\overline{\mathbf{Q}}$-rational for every $w \in \mathcal{W}_0$. Let w_1 be a CM-point where R_v (for a fixed v) is finite. Take $\alpha \in G$ so that $w_1 = \alpha w_0$ with $w_0 \in \mathcal{W}_0$. Put $R'_v = R_v\|_{\sigma_v}\alpha$. Then $R_v^{-1}R'_v$ has entries in $\mathcal{A}_0$. Take a CM-point w such that both w and αw belong to $\mathcal{W}_0$. Then $(R_v^{-1}R'_v)(w) \in GL_{n_v}(\overline{\mathbf{Q}})\mathfrak{p}_v(w)^{-1}\mu_v(\alpha, w)^{-1}R_v(\alpha w) = GL_{n_v}(\overline{\mathbf{Q}})\mathfrak{p}_v(\alpha w)^{-1}R_v(\alpha w) = GL_{n_v}(\overline{\mathbf{Q}})$. By Proposition 11.2 (2), $R_v^{-1}R'_v$ has entries in $\mathcal{A}_0(\overline{\mathbf{Q}})$. Therefore $\mathfrak{p}_v(w_1)^{-1}R_v(w_1) \in GL_{n_v}(\overline{\mathbf{Q}})\mathfrak{p}_v(w_0)^{-1}\mu_v(\alpha, w_0)^{-1}R_v(\alpha w_0) \subset GL_{n_v}(\overline{\mathbf{Q}})(R_v^{-1}R'_v)(w_0) \subset GL_{n_v}(\overline{\mathbf{Q}})$. This shows that the columns of R_v belong to $\mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$. Now given $z_0 \in \mathcal{H}$, we can find $\beta \in G$ such that R_v is finite and invertible at βz_0 for every $v \in \mathbf{b}$. Changing R_v for $R_v\|\beta$, we obtain the desired elements of $\mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$ in view of Proposition 11.13 (1).

Let us now prove (2) of Proposition 11.13. Given $\mathcal{W}$ and f in that assertion, take $R = (R_v)_{v\in\mathbf{b}}$ as in the present proposition and put $g = \omega(R)^{-1}f$. Then we can show that g has entries in $\mathcal{A}_0(\overline{\mathbf{Q}})$ by the same technique as in the proof of Proposition 11.11. Now let w be a CM-point where f is finite. Choose R so that R_v is finite and invertible at w for every $v \in \mathbf{b}$. Then g is finite at w and $g(w)$ is algebraic. Since $f(w) = \omega\big(R(w)\big)g(w)$, we see that $\omega\big(\mathfrak{p}(w)\big)^{-1}f(w)$ is algebraic, so that $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$ as desired.

To prove the existence of $(Q_v)_{v\in\mathbf{b}}$, define $\tau : GL_d(\mathbf{C}) \to GL_d(\mathbf{C})$ by $\tau(x) = \det(x)x$. By Proposition 9.11, given any $z_0 \in \mathcal{H}$, there exists a $d\times d$-matrix E whose columns belong to $\mathcal{M}_\tau(\mathbf{Q})$ and such that $\det E\big(\varepsilon(z_0)\big) \neq 0$. Put $S(z) = \det\big(\kappa(z)\big)^{-1}\cdot {}^t\kappa(z)^{-1}E\big(\varepsilon(z)\big)$ for $z \in \mathcal{H}$. From (11.11) and (4.29) we easily see that $S\big(\gamma(z)\big) = j_\gamma(z)^{\mathbf{b}}M(\gamma, z)S(z)$ for every γ in a congruence subgroup Γ of G. Let S_v be the submatrix of S composed of the n_v rows of S corresponding to the component $\mu_v(\gamma, z)$ of $M(\gamma, z)$. Then $S_v\big(\gamma(z)\big) = j_\gamma(z)^{\mathbf{b}}\mu_v(\gamma, z)S_v(z)$ for every $\gamma \in \Gamma$. Since $\det E\big(\varepsilon(z_0)\big) \neq 0$, we can find suitable n_v columns of S_v which make nonzero determinant at z_0. Call Q_v the $(n_v\times n_v)$-matrix composed of those n_v columns, which clearly belong to $\mathcal{A}_{\tau_v}$. Now by (11.13) and (11.16), for every $w \in \mathcal{W}_0$ we have

$$P\big(\varepsilon(w)\big) \in {}^t\kappa(w)\mathrm{diag}[\mathfrak{p}_v(w)]_{v\in\mathbf{b}}GL_d(\overline{\mathbf{Q}}).$$

Since $P \in \mathcal{A}_\sigma(\overline{\mathbf{Q}})$, this together with Proposition 11.11 implies that

$$\mathfrak{p}\big(\varepsilon(w)\big) \in {}^t\kappa(w)\mathrm{diag}[\mathfrak{p}_v(w)]_{v\in\mathbf{b}}GL_d(\overline{\mathbf{Q}}).$$

Put $\mathfrak{q}(w) = \prod_{v\in\mathbf{b}}\det\big(\mathfrak{p}_v(w)\big)$. Then $\det\big[\mathfrak{p}\big(\varepsilon(w)\big)\big] \in \mathfrak{q}(w)\overline{\mathbf{Q}}$, and hence

$$\mathfrak{q}(w)^{-1}\mathrm{diag}[\mathfrak{p}_v(w)]_{v\in\mathbf{b}}^{-1}S(w) \in GL_d(\overline{\mathbf{Q}})\tau\big(\mathfrak{p}\big(\varepsilon(w)\big)\big)^{-1}E\big(\varepsilon(w)\big) = GL_d(\overline{\mathbf{Q}}).$$

Therefore $\tau_v\big(\mathfrak{p}(w)\big)^{-1}Q_v(w)$ is algebraic for every $w \in \mathcal{W}_0$. By Proposition 11.13 (2), the columns of Q_v belong to $\mathcal{M}_{\tau_v}(\overline{\mathbf{Q}})$. This completes the proof.

11.15. Proposition. *For every ω as in §11.12 and every congruence subgroup Γ of G we have $\mathcal{M}_\omega(\Gamma) = \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}}) \otimes \mathbf{C}$.*

PROOF. In Cases SP and UT we already proved a stronger result in Theorem 10.4. Thus our question here is in Case UB. We first prove that if $f_1, \dots, f_t$ are elements of $\mathcal{M}_\omega(\overline{\mathbf{Q}})$ linearly independent over $\overline{\mathbf{Q}}$, then they are linearly independent over $\mathbf{C}$. Indeed, for each $w \in \mathcal{H}_{\mathrm{CM}}$ put $X_w = \{\, x \in \mathbf{C}^t \mid \sum_{i=1}^t x_i \mathfrak{P}_\omega(w)^{-1} f_i(w) = 0 \}$; let Y be the intersection of X_w for all such w. Clearly we can omit $\mathfrak{P}_\omega(w)^{-1}$ in the definition of X_w. Since $\mathfrak{P}_\omega(w)^{-1} f_i(w)$ is algebraic, each X_w, as well as Y, is a vector subspace of $\mathbf{C}^t$ defined over $\overline{\mathbf{Q}}$. If Y contains a nonzero element c, then $\sum_{i=1}^t c_i f_i(w) = 0$ for all CM-points w so that $\sum_{i=1}^t c_i f_i = 0$. Since the f_i are linearly independent over $\overline{\mathbf{Q}}$, Y has no such c in $\overline{\mathbf{Q}}^t$, so that $Y = \{0\}$, which shows that the f_i are linearly independent over $\mathbf{C}$.

Our next task is to show that $\mathcal{M}_\omega$ can be spanned by $\mathcal{M}_\omega(\overline{\mathbf{Q}})$ over $\mathbf{C}$. Denote $\mathcal{A}_\omega$ and $\mathcal{M}_\omega$ by $\mathcal{A}_{\kappa\mathbf{b}}$ and $\mathcal{M}_{\kappa\mathbf{b}}$ if $\omega(x) = \det(x)^{\kappa\mathbf{b}}$ with an integer κ (see (5.4b)). In this special case our assertion is

$$\mathcal{M}_{\kappa\mathbf{b}} = \mathcal{M}_{\kappa\mathbf{b}}(\overline{\mathbf{Q}}) \otimes \mathbf{C}. \tag{11.19}$$

We shall prove this after the proof of Proposition 14.8. Assuming this result, we are going to prove the desired result for $\mathcal{M}_\omega$. We may assume that ω is irreducible. Then there is an integer e such that $\omega(cy) = c^e\omega(y)$ for $c \in \mathbf{C}^\times$. Take $Q = (Q_v)_{v\in\mathbf{b}}$ as in Proposition 11.14 with any $z_0 \in \mathcal{H}$. Then $\omega(Q) \in \mathcal{M}_{\omega'}(\overline{\mathbf{Q}})$ with $\omega'(x) = \det(x)^{e\mathbf{b}}\omega(x)$. Take a positive integer p so that $p > e$ and $\det(x)^{p\mathbf{b}}\omega(x)^{-1}$ is a polynomial in x; put $\zeta(x) = \det(x)^{-p\mathbf{b}}\omega(x)$. Then $\zeta(Q) \in \mathcal{A}_\xi(\overline{\mathbf{Q}})$ with $\xi(x) = \det(x)^{-\kappa\mathbf{b}}\omega(x)$, where $\kappa = p(1 + r|\mathbf{a}|) - e$. Given $f \in \mathcal{M}_\omega$, put $g = \zeta(Q)^{-1}f$; then g has components in $\mathcal{A}_{\kappa\mathbf{b}}$. Since ζ^{-1} is a polynomial function, g is holomorphic, so that g has components in $\mathcal{M}_{\kappa\mathbf{b}}$, or rather, $g \in (\mathcal{M}_{\kappa\mathbf{b}})^t$, where t is the dimension of the representation space of ω. Take a $\overline{\mathbf{Q}}$-basis $\{c_\nu\}$ of $\mathbf{C}$. By (11.18) we can express g as a finite sum $g = \sum_\nu c_\nu g_\nu$ with $g_\nu \in (\mathcal{M}_{\kappa\mathbf{b}}(\overline{\mathbf{Q}}))^t$. Then $f = \sum_\nu c_\nu f_\nu$ with $f_\nu = \zeta(Q)g_\nu$. Clearly $f_\nu \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$. Suppose $f = \sum_\nu c_\nu f'_\nu$ with $f'_\nu \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$. Let w be a CM-point where f_ν and f'_ν are finite. Then $0 = \sum_\nu c_\nu \mathfrak{P}_\omega(w)^{-1}\bigl(f_\nu(w) - f'_\nu(w)\bigr)$, so that $f_\nu(w) = f'_\nu(w)$. Since all such points w form a dense subset of $\mathcal{H}$, we obtain $f_\nu = f'_\nu$. This means that the f_ν are unique for f, once $\{c_\nu\}$ is chosen. Now, given $z_0 \in \mathcal{H}$, take Q so that $\det Q_v(z_0) \neq 0$ for every $v \in \mathbf{b}$. Then $\zeta(Q)$ is finite at z_0, so that f_ν is finite at z_0. Thus f_ν is finite everywhere, and hence $f_\nu \in \mathcal{M}_\omega(\overline{\mathbf{Q}})$. Now suppose $f \in \mathcal{M}_\omega(\Gamma)$. Then $f = f\|\gamma = \sum_\nu c_\nu f_\nu\|\gamma$ for every $\gamma \in \Gamma$. Since $f_\nu\|\gamma \in \mathcal{M}_\omega(\overline{\mathbf{Q}})$ by Proposition 11.13 (1), the uniqueness of f_ν just proved shows that $f_\nu\|\gamma = f_\nu$, and hence $f_\nu \in \mathcal{M}_\omega(\Gamma, \overline{\mathbf{Q}})$. This proves our proposition.

11.16. Before proceeding further, let us introduce some symbols. Let L be a CM-field containing K. For $\alpha \in J_K$ and $\beta \in J_L$ we denote by $\mathrm{Inf}_{L/K}(\alpha)$ the sum of all the elements of J_L which coincide with α on K, and by $\mathrm{Res}_{L/K}(\beta)$ the restriction of β to K. We then extend these to additive maps

$$\mathrm{Inf}_{L/K} : I_K \to I_L, \qquad \mathrm{Res}_{L/K} : I_L \to I_K. \tag{11.20}$$

We recall here three basic properties of the period symbol p_K (see [S98, Theorem 32.5]):

$$p_K\bigl(\xi, \mathrm{Res}_{L/K}(\zeta)\bigr) = p_L\bigl(\mathrm{Inf}_{L/K}(\xi), \zeta\bigr) \quad \text{if } \xi \in I_K \text{ and } \zeta \in I_L, \tag{11.21}$$

$$p_K\bigl(\mathrm{Res}_{L/K}(\zeta), \xi\bigr) = p_L\bigl(\zeta, \mathrm{Inf}_{L/K}(\xi)\bigr) \quad \text{if } \xi \in I_K \text{ and } \zeta \in I_L, \tag{11.22}$$

$$p_K(\xi\rho, \eta) = p_K(\xi, \eta\rho) = p_K(\xi, \eta)^{-1} \quad \text{for every } \xi, \eta \in I_K. \tag{11.23}$$

Here $p_K(\cdots)$ and $p_L(\cdots)$ are elements of $\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$; also in the following proposition we view $\det\big(\mathfrak{p}_v(w)\big)$ for each $v \in \mathbf{b}$ as an element of $\mathbf{C}^\times/\overline{\mathbf{Q}}^\times$.

11.17. Theorem (Case UB). *Let $\tau = \{\tau_v\}_{v\in\mathbf{a}}$ be a CM-type as in §3.5, and Ψ the representation of K as in (4.9) and (4.10); identify Ψ with the element $\sum_{v\in\mathbf{a}}(m_v\tau_v\rho + n_v\tau_v)$ of I_K. Then the following assertions hold:*

(1) $\det\big(\mathfrak{p}_v(w)\big)/\det\big(\mathfrak{p}_{v\rho}(w)\big) = p_K(\tau_v,\,\Psi)$ *for every $w \in \mathcal{H}_{\mathrm{CM}}$. In particular, if $m_v = n_v$ for every $v \in \mathbf{a}$, then $\det\big(\mathfrak{p}_v(w)\big) = \det\big(\mathfrak{p}_{v\rho}(w)\big)$ for every $v \in \mathbf{a}$.*

(2) *Let $\omega(x) = \zeta(x)\prod_{v\in\mathbf{a}}\det(x_v)^{a_v}\det(x_{v\rho})^{b_v}$ with $a,\, b \in \mathbf{Z}^{\mathbf{a}}$ such that $a_v + b_v = 0$ whenever $m_v n_v \neq 0$. Then $\mathcal{A}_\omega = \mathcal{A}_\zeta$ and $\mathcal{A}_\omega(\overline{\mathbf{Q}}) = \mathfrak{q}\cdot\mathcal{A}_\zeta(\overline{\mathbf{Q}})$, where*

$$\mathfrak{q} = p_K\left(\sum_{v\in\mathbf{a}} a_v\tau_v - \sum_{v\in\mathbf{a},\,n_v=0}(a_v+b_v)\tau_v,\ \Psi\right).$$

In particular, $\mathcal{A}_\omega(\overline{\mathbf{Q}}) = \mathcal{A}_\zeta(\overline{\mathbf{Q}})$ if $m_v = n_v$ for every $v \in \mathbf{a}$.

Remark. In view of (11.23), we can replace Ψ in these assertions by $\sum_{v\in\mathbf{a}}(n_v - m_v)\tau_v$. Since $\mathfrak{q}$ can often be transcendental, the last assertion shows that the $\overline{\mathbf{Q}}$-rationality of automorphic forms in Case UB is far more complex than that in Cases SP and UT.

PROOF. If $(Y,\,\Phi)$ is as in §11.4, then we have CM-types $(K_i,\,\Phi_i)$ such that $Y = \bigoplus_{i=1}^t K_i$ and $\Phi = \sum_{i=1}^t \Phi_i$ as in §4.11. Using the symbols of (11.3a, b), for each fixed $v \in \mathbf{a}$ we observe that the $\psi_{vi}\rho$ and the φ_{vj} are exactly the elements of I_Y whose restrictions to K coincide with τ_v. Thus $\sum_{i=1}^{m_v}\psi_{vi}\rho + \sum_{j=1}^{n_v}\varphi_{vj} = \sum_{i=1}^t \mathrm{Inf}_{K_i/K}(\tau_v)$. Also $\Psi = \sum_{i=1}^t \mathrm{Res}_{K_i/K}(\Phi_i)$. Therefore from (11.4a, b), (11.23), (11.2), and (11.21) we obtain

$$\begin{aligned}\det\big(\mathfrak{p}_v(w)\big)/\det\big(\mathfrak{p}_{v\rho}(w)\big) &= \textstyle\prod_{i=1}^t p_{K_i}\big(\mathrm{Inf}_{K_i/K}(\tau_v),\,\Phi_i\big)\\ &= \textstyle\prod_{i=1}^t p_K\big(\tau_v,\,\mathrm{Res}_{K_i/K}(\Phi_i)\big) = p_K(\tau_v,\,\Psi).\end{aligned}$$

This proves the first half of (1). Suppose $m_v = n_v$ for every $v \in \mathbf{a}$; then $\Psi = \sum_{i=1}^t n_v(\tau_v\rho + \tau_v)$, and hence $p_K(\tau_v,\,\Psi) = 1$ for every $v \in \mathbf{a}$ by (11.23). This proves the latter half of (1). Let the notation be as in (2). From (3.23) and (4.34) we see that $\mathcal{A}_\omega = \mathcal{A}_\zeta$. (We already made the same type of observation in §5.4.) Now $\omega(x) = \zeta(x)\prod_{v\in\mathbf{a}}\big[\det(x_v)/\det(x_{v\rho})\big]^{a_v}\det(x_{v\rho})^{a_v+b_v}$, and $a_v + b_v \neq 0$ only if $m_v n_v = 0$. If $n_v = 0$, then $\det\big(\mathfrak{p}_v(w)\big) = 1$, so that $\det\big(\mathfrak{p}_{v\rho}(w)\big) = p_K(-\tau_v,\,\Psi)$ by (1); if $m_v = 0$, then $n_v \neq 0$ and $\det\big(\mathfrak{p}_{v\rho}(w)\big) = 1$. Therefore by (1), $\mathfrak{P}_\omega(w) = \mathfrak{q}\mathfrak{P}_\zeta(w)$ with $\mathfrak{q}$ given as above, and $\mathfrak{q} = 1$ if $m_v = n_v$ for every $v \in \mathbf{a}$. Therefore we obtain (2) from our definion of $\mathcal{A}_\omega(\overline{\mathbf{Q}})$.

11.18. Proposition. *With $\widetilde{G} = Gp(1,\,F) = GL_2(F)$ let w be a CM-point on $\mathcal{H} = \mathfrak{H}_1^{\mathbf{a}}$ obtained from $\big(K,\,\{\tau_v\}_{v\in\mathbf{a}}\big)$ and $w_0 \in K$ as in Proposition 4.14. Then for $k \in \mathbf{Z}^{\mathbf{a}}$ we have $\mathfrak{P}_k(w) = p_K\big(\sum_{v\in\mathbf{a}}k_v\tau_v,\,\sum_{v\in\mathbf{a}}\tau_v\big)$.*

PROOF. In Proposition 4.14 we have seen that $\tau_v = \varphi_v$ and $\Phi = \sum_{v\in\mathbf{a}}\tau_v$, so that $\mathfrak{p}_v(w) = p_K(\tau_v,\,\Phi)$. Therefore $\mathfrak{P}_k(w) = \mathfrak{p}(w)^k = p_K\big(\sum_{v\in\mathbf{a}}k_v\tau_v,\,\Phi\big)$.

11.19. Proposition. *The coset $p_Y(\alpha,\,\beta)$ of (11.2) can be represented by a real number.*

PROOF. It is sufficient to consider the case where Y is a CM-field K. Let K, Φ, w_0, and w be as in Proposition 11.18; fix $u \in \mathbf{a}$ and put $\Phi' = \Phi - \tau_u + \tau_u\rho$.

Then Φ' is a CM-type, and $p_K(\alpha, \tau_u)^2 = p_K(\alpha, \tau_u - \tau_u\rho) = p_K(\alpha, \Phi)p_K(\alpha, \Phi')^{-1}$. Therefore, to prove our proposition, it is sufficient to show that $p_K(\tau_u, \Phi)$ can be represented by a real number for every CM-type Φ and every $u \in \mathbf{a}$. Now we can choose w_0 so that $\tau_v(w_0)$ is pure imaginary for every $v \in \mathbf{a}$. Take $f = R_u \in \mathcal{A}_{\sigma_u}(F^u)$ as in Proposition 9.11 with $n = 1$ in Case SP so that $f(w)$ is finite and nonzero. By Proposition 11.18, $f(w)$ represents $p_K(\tau_u, \Phi)$. Since f is F^u-rational and the components of w are pure imaginary, we see that $f(w) \in \mathbf{R}$. This proves our proposition.

11.20. Remark. (A) In the elliptic modular case, if $f \in \mathcal{A}_0(D)$ with a subfield D of $\overline{\mathbf{Q}}$, then $(2\pi i)^{-1}df/dz$ belongs to $\mathcal{A}_2(D)$. Therefore, by the above lemma the value $(df/dz)(w)$ for a CM-point w belongs to $\pi p_K(\tau, \tau)^2\overline{\mathbf{Q}}$, where $K = \mathbf{Q}(w)$ and τ is the identity injection of K into $\mathbf{C}$. We can naturally ask the nature of the derivatives of an element of $\mathcal{A}_0(\overline{\mathbf{Q}})$ in the general case. As one can easily imagine, this problem in the higher-dimensional case is highly nontrivial, and especially so when $\Gamma\backslash\mathcal{H}$ is compact. For example, suppose $F \neq \mathbf{Q}$ and $\sum_{v\in\mathbf{a}} m_v n_v = 1$ in Case UB. Then $\dim(\mathcal{H}) = 1$, and so, given a nonconstant element g of $\mathcal{A}_0(\overline{\mathbf{Q}})$, we have a well-defined derivation d/dg of $\mathcal{A}_0(\overline{\mathbf{Q}})$. If z is the variable on $\mathcal{H}$, then for every $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$ we have $(df/dz)/(dg/dz) = df/dg \in \mathcal{A}_0(\overline{\mathbf{Q}})$. Therefore the value of $(df/dz)/(dg/dz)$ at a CM-point w is algebraic. But what exactly is the nature of the coset $(df/dz)(w)\overline{\mathbf{Q}}$? Or more precisely, can we express it by means of the symbol $\mathfrak{p}(w)$? In fact, $(df/dz)(w) \in \pi\mathfrak{p}_v(w)\overline{\mathbf{Q}}$, where v is the element of $\mathbf{a}$ for which $m_v n_v = 1$. In the next chapter we shall present a systematic treatment of differential operators acting on automorphic forms, especially such problems of arithmeticity in view. For example, the last fact on $(df/dz)(w)$ is an easy special case of Proposition 14.5 below.

(B) There is a somewhat more intrinsic way of defining the symbol $\mathfrak{p}(w)$ than what was done in §11.4. In the setting of §4.11, we put $Y_\mathbf{R} = Y \otimes_\mathbf{Q} \mathbf{R}$ and $Y_\mathbf{R}^u = \{x \in Y_\mathbf{R} \mid xx^\rho = 1\}$, where ρ is extended $\mathbf{R}$-linearly to $Y_\mathbf{R}$. Clearly $Y_\mathbf{R}^u$ can be identified with $\{(x_\sigma)_{\sigma\in J_Y} \in \mathbf{C}^{J_Y} \mid x_\sigma x_{\sigma\rho} = 1\}$. Define an element $\mathfrak{q}_w$ of $Y_\mathbf{R}^u$ by

$$\mathfrak{q}_w = \bigl(p_Y(\sigma, \Phi)\bigr)_{\sigma\in J_Y}. \tag{11.24}$$

We extend ψ_v and φ_v of (4.37) to $\mathbf{R}$-linear maps of $Y_\mathbf{R}$ into $\mathbf{C}_{m_v}^{m_v}$ and $\mathbf{C}_{n_v}^{n_v}$; we then define $\lambda_v\bigl(h(\alpha), w\bigr)$ and $\mu_v\bigl(h(\alpha), w\bigr)$ for $\alpha \in Y_\mathbf{R}^u$ by (4.37), so that $M\bigl(h(\alpha), w\bigr) = \bigl(\lambda_v\bigl(h(\alpha), w\bigr), \mu_v\bigl(h(\alpha), w\bigr)\bigr)_{v\in\mathbf{a}}$ is meaningful for $\alpha \in Y_\mathbf{R}^u$. Then (11.3a, b) and (11.4a, b) show that $\lambda_v\bigl(h(\mathfrak{q}_w), w\bigr) = \psi_v(\mathfrak{q}_w) = \mathfrak{p}_{v\rho}(w)$ and $\mu_v\bigl(h(\mathfrak{q}_w), w\bigr) = \varphi_v(\mathfrak{q}_w) = \mathfrak{p}_v(w)$. Therefore we can define $\mathfrak{p}(w)$ by

$$\mathfrak{p}(w) = M\bigl(h(\mathfrak{q}_w), w\bigr). \tag{11.25}$$

(C) All arithmetic quotients treated in this book are associated with PEL-types. We can also construct canonical models of certain types of arithmetic quotients not associated with PEL-types, as we already noted it after the proof of Theorem 9.6, and then can define arithmeticity of automorphic forms in such cases. In each case we can speak of a CM-point, say w, obtained from a CM-algebra Y; then we have an element Φ of I_Y and define the symbol $\mathfrak{p}(w)$ in terms of $p_Y(\alpha, \Phi)$ with $\alpha \in J_Y$ by natural analogues of (11.24) and (11.25). These are given in [S80, §4 resp. §5] when the group is a quaternion unitary group resp. an orthogonal group. One noteworthy aspect is that Φ is not necessarily a collection of CM-types for such groups.

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CHAPTER III

ARITHMETIC OF DIFFERENTIAL OPERATORS AND NEARLY HOLOMORPHIC FUNCTIONS

12. Differential operators on symmetric spaces

12.1. In this section we deal with two types of irreducible hermitian symmetric spaces H of noncompact type, called Types A and C, which we already discussed in Section 3. To treat the space given in various different forms uniformly, we use symbols different from those of Section 3. For each type of H we have a simple Lie group G, its maximal compact subgroup K such that G/K can be identified with H, and a complex vector space T which can be identified with the holomorphic (or nonholomorphic) tangent space of H, on which the complexification K^c of K acts; G is unitary for Type A and symplectic for Type C. The space H can be presented as a "ball," or a "tube" under a certain condition. We refer to Types AB and CB if H is a ball, and to Types AT and CT if it is a tube. If it is unnecessary to specify the distinction between a ball and a tube, we simply speak of Types A and C. In fact, we can treat all types of classical hermitian symmetric spaces of noncompact type by the same methods, but for simplicity we restrict our discussion to those two types; we refer the reader to [S94b] for a detailed treatment in the general case.

Now G, K^c, H, and T for each type can be given explicitly in the table below. In each case we view H as a subset of T; we also give positive definite hermitian matrices $\xi(z)$ and $\eta(z)$ defined for $z \in H$, which are closely connected with the Kähler structure of H and the canonical factor of automorphy for the elements of G. We do not need K, nor any conceptual or geometric meaning of these objects, for the moment. Strictly speaking, K^c is merely isomorphic, and not exactly equal, to the complexification of K. Here, however, we define K^c formally as follows. Its connection with the maximal compact subgroup will be discussed in §A8.2.

Type AB: $\quad G = SU(m,\, n) = \{\, \alpha \in SL_{m+n}(\mathbf{C}) \,|\, \alpha^* I_{m,n}\alpha = I_{m,n} \,\}$,

$$I_{m,n} = \mathrm{diag}[1_m,\, -1_n], \quad K^c = \{\, (a,\, b) \in GL_m(\mathbf{C}) \times GL_n(\mathbf{C}) \,|\, \det(a) = \det(b) \,\},$$
$$T = \mathbf{C}_n^m, \qquad H = \{\, z \in T \,|\, zz^* < 1_m \,\},$$
$$\xi(z) = 1_m - \overline{z} \cdot {}^t z, \qquad \eta(z) = 1_n - z^* z.$$

Type CB: $\quad G = Sp(n,\, \mathbf{C}) \cap SU(n,\, n), \quad K^c = GL_n(\mathbf{C})$,

$$T = \{\, z \in \mathbf{C}_n^n \,|\, {}^t z = z \,\}, \quad H = \{\, z \in T \,|\, z^* z < 1_n \,\},$$
$$\xi(z) = \eta(z) = 1_n - z^* z.$$

The space H of Type CB is equivalent to a tube; however, H of Type AB is equivalent to a tube if and only if $m = n$. We present here the explicit forms of the objects associated to H of tube form; we also present a real vector subspace U of T such that $T=U\otimes_{\mathbf{R}}\mathbf{C}$ and that $H=U+iP$ with a domain of positivity P in U.

Type AT: $$G=SU(\eta_n)=\left\{\alpha\in SL_{2n}(\mathbf{C})\,\middle|\,\alpha^*\eta_n\alpha=\eta_n\right\},\quad \eta_n=\begin{bmatrix}0 & -1_n\\ 1_n & 0\end{bmatrix},$$
$$K^c=\left\{(a,\,b)\in GL_n(\mathbf{C})\times GL_n(\mathbf{C})\,\middle|\,\det(a)=\det(b)\right\},$$
$$T=\mathbf{C}_n^n,\quad U=\left\{x\in\mathbf{C}_n^n\,\middle|\,x^*=x\right\},\quad H=\left\{z\in T\,\middle|\,i(z^*-z)>0\right\},$$
$${}^t\xi(z)=\eta(z)=i(z^*-z).$$

Type CT: $$G=Sp(n,\,\mathbf{R}),\qquad K^c=GL_n(\mathbf{C}),$$
$$T=\left\{z\in\mathbf{C}_n^n\,\middle|\,{}^tz=z\right\},\quad U=\left\{x\in\mathbf{R}_n^n\,\middle|\,{}^tx=x\right\},$$
$$\xi(z)=\eta(z)=i(\overline{z}-z).$$

The space H of Type CT, AT, or AB is exactly $\mathfrak{H}_n$, $\mathcal{H}_n$, or $\mathfrak{B}_{m,n}$ in Case SP, UT, or UB of §3.1; H of Type CB is $\mathfrak{B}_n$ of (3.39). The functions ξ and η are the same as those of (3.9) and (3.10). The space H of Type CB or CT is contained in H of Type AB (with $m=n$) or AT; ξ and η for Type C are just restrictions of the corresponding functions for Type A.

In Section 3 we defined the action of G on H and also two factors of automorphy $\lambda_\alpha(z)$ and $\mu_\alpha(z)$; see (3.15), (3.16), and (3.17); $\lambda_\alpha(z)=\mu_\alpha(z)$ for Type C. We recall here a few basic formulas:

(12.1a) $$\lambda_\alpha(z)^*\xi(\alpha z)\lambda_\alpha(z)=\xi(z),\qquad \mu_\alpha(z)^*\eta(\alpha z)\mu_\alpha(z)=\eta(z),$$

(12.1b) $$d(\alpha z)={}^t\lambda_\alpha(z)^{-1}\cdot dz\cdot\mu_\alpha(z)^{-1}\qquad(\alpha\in G).$$

These were given in (3.19) and Lemma 3.4 (1). We also need a scalar factor of automorphy $j_\alpha(z)$ and a scalar-valued function δ defined in (3.20) and (3.21). Since our group G of Type AB or AT is $SU(m,\,n)$ or $SU(\eta_n)$, the factors $\det(\alpha)$ and $\nu(\alpha)$ in the formulas in Section 3 are all equal to 1. In particular we note that

(12.2) $$\det\left(\lambda_\alpha(z)\right)=\det\left(\mu_\alpha(z)\right)=j_\alpha(z).$$

We have $K^c=GL_n(\mathbf{C})$ for Type C. We view it as a subgroup of $GL_n(\mathbf{C})\times GL_n(\mathbf{C})$ by the embedding $a\mapsto(a,\,a)$. We make a convention that m means n for Types AT, CT, and CB. Then, in all cases K^c is a subgroup of $GL_m(\mathbf{C})\times GL_n(\mathbf{C})$. Hereafter we shall write an element of K^c in the form $(a,\,b)\in GL_m(\mathbf{C})\times GL_n(\mathbf{C})$ with the understanding that $a=b$ for Type C. For example, $\left(\xi(z),\,\eta(z)\right)\in K^c$ for all types. (Notice that $\det\xi(z)=\det\eta(z)$ in all cases; see (3.13) for Type AB.) We let K^c act on T by $(a,\,b)u=au\cdot{}^tb$ for $(a,\,b)\in K^c$ and $u\in T$. For T of Type C, this is the action of $GL_n(\mathbf{C})$ on the space of complex symmetric matrices defined by $u\mapsto au\cdot{}^ta$ for $a\in GL_n(\mathbf{C})$.

12.2. Throughout the rest of this section we denote by G an algebraic group of the following types:

$$G=SU(T)\qquad\text{(Type AB)},$$
$$G=SU(\eta_n)\qquad\text{(Type AT)},$$
$$G=Sp(n,\,F)\qquad\text{(Type CT)},$$
$$G=SU(\eta_n)\cap Sp(n,\,K)\qquad\text{(Type CB)}.$$

Here F is a totally real algebraic number field and K is a totally imaginary quadratic extension of F; $\mathcal{T}$ is an antihermitian element of $GL_r(K)$. Thus our group of Types AB, AT, and CT is exactly the group G_1 of §8.2, which is a subgroup of the group G defined in §3.5 in Cases UB, UT, and SP. In this sense our notation is not consistent, but since in this section we exclusively consider the elements of G_1, we simply denote it by G; we do not consider larger groups like $Gp(n, F)$ and $GU(\eta_n)$ in this and the next sections. Now we have

$$G_{\mathbf{a}} = \prod_{v \in \mathbf{a}} G_v, \tag{12.3}$$

and we see that each G_v is either compact or a group belonging to the four types of §1.1; G_v is not compact for G of Types AT, CT and CB.

For each $v \in \mathbf{a}$ such that G_v is not compact, we take the objects H, T, K^c, etc. associated with G_v, and denote them by H_v, T_v, K_v^c, etc. If G_v is compact, that is, if $G_v = SU(m, n)$ with $mn = 0$, then we put $K_v^c = SL_{m+n}(\mathbf{C})$, $T_v = \{0\}$, and let H_v denote the space consisting of a single element 0 (see §3.3). We then put

$$\mathcal{H} = \prod_{v \in \mathbf{a}} H_v, \qquad \mathfrak{K}_0 = \prod_{v \in \mathbf{a}} K_v^c, \tag{12.4a}$$

$$\alpha z = \alpha(z) = (\alpha_v z_v)_{v \in \mathbf{a}}, \quad \Xi(z) = \bigl(\xi_v(z_v), \eta_v(z_v)\bigr)_{v \in \mathbf{a}} \ (\in \mathfrak{K}_0), \tag{12.4b}$$

$$M_\alpha(z) = \bigl(\lambda(\alpha_v, z_v), \mu(\alpha_v, z_v)\bigr)_{v \in \mathbf{a}} \ (\in \mathfrak{K}_0) \tag{12.4c}$$

for $z = (z_v)_{v \in \mathbf{a}} \in \mathcal{H}$ and $\alpha = (\alpha_v)_{v \in \mathbf{a}} \in G_{\mathbf{a}}$, where ξ_v and η_v denote the functions ξ and η defined on H_v. If G_v is compact, we understand that the v-component of Ξ is 1, and the v-component of M_α is $\overline{\alpha}$ or α according to the convention of (32.4a, b). We also put

$$\mathbf{a}' = \bigl\{ v \in \mathbf{a} \bigm| G_v \text{ is not compact} \bigr\}. \tag{12.5}$$

In this section, by a *representation* $\{\sigma, W\}$ of a topological group $\mathcal{G}$, we mean a pair formed by a complex vector space W of finite dimension and a continuous homomorphism σ of $\mathcal{G}$ into $GL(W)$. We now take a representation $\{\rho, X\}$ of $\mathfrak{K}_0$ such that ρ is complex analytic. Given $f \in C^\infty(\mathcal{H}, X)$ and $\alpha \in G_{\mathbf{a}}$, we define $f\|_\rho \alpha \in C^\infty(\mathcal{H}, X)$ by

$$(f\|_\rho \alpha)(z) = \rho\bigl(M_\alpha(z)\bigr)^{-1} f(\alpha z) \qquad (z \in \mathcal{H}). \tag{12.6}$$

In the previous sections we considered automorphic forms with respect to a representation $\{\omega, V\}$ of $\prod_{v \in \mathbf{b}} GL_{n_v}(\mathbf{C})$. For Type C we have $\mathfrak{K}_0 = \prod_{v \in \mathbf{b}} GL_{n_v}(\mathbf{C})$, but for Type A, $\mathfrak{K}_0$ is smaller than $\prod_{v \in \mathbf{b}} GL_{n_v}(\mathbf{C})$. Also, in this section we are taking the group SU instead of GU or U. For these reasons we use the letter ρ instead of ω. As we said in §12.1, we can develop the whole theory for an arbitray hermitian symmetric space, and our exposition, being almost axiomatic, can easily be generalized, as shown in [S94b]. In order to deal with such a general case, it is easier to consider semisimple groups, which is the reason why we consider SU here. However, we shall later reinstate ω, and take ρ to be the restriction of ω to $\mathfrak{K}_0$. Anyway (12.6) is consistent with (5.6a).

12.3. Given a positive integer p and finite-dimensional complex vector spaces X and Y, we denote by $Ml_p(Y, X)$ the vector space of all $\mathbf{C}$-multilinear maps of $Y \times \cdots \times Y$ (p copies) into X, and by $S_p(Y, X)$ the vector space of all homogeneous polynomial maps of Y into X of degree p. Thus $S_1(Y, X) = Ml_1(Y, X)$, and it is the vector space of all $\mathbf{C}$-linear maps of Y into X. We put $S_0(Y, X) = Ml_0(Y, X) =$

X and $S_p(Y) = S_p(Y, \mathbf{C})$. We call an element g of $Ml_p(Y, X)$ *symmetric* if $g(y_{\pi(1)}, \ldots, y_{\pi(p)}) = g(y_1, \ldots, y_p)$ for every permutation π of $\{1, \ldots, p\}$.

We are going to define differential operators on $\mathcal{H}$ with respect to each component z_v of the variable point $z = (z_v)_{v\in\mathbf{a}}$ on $\mathcal{H}$ such that G_v is not compact. Given such a v, $0 \le p \in \mathbf{Z}$, and a representation $\{\rho, X\}$ of $\mathfrak{K}_0$ as above, we define representations $\{\rho\otimes\tau_v^p, Ml_p(T_v, X)\}$ and $\{\rho\otimes\sigma_v^p, Ml_p(T_v, X)\}$ of $\mathfrak{K}_0$ by

$$[(\rho\otimes\tau_v^p)(a, b)h](u_1, \ldots, u_p) = \rho(a, b)h({}^ta_vu_1b_v, \ldots, {}^ta_vu_pb_v), \tag{12.7a}$$

$$[(\rho\otimes\sigma_v^p)(a, b)h](u_1, \ldots, u_p) = \rho(a, b)h(a_v^{-1}u_1{}^tb_v^{-1}, \ldots, a_v^{-1}u_p{}^tb_v^{-1}), \tag{12.7b}$$

for $(a, b) \in \mathfrak{K}_0$, $h \in Ml_p(T_v, X)$, and $u_i \in T_v$. We use the same symbols $\rho\otimes\tau_v^p$ and $\rho\otimes\sigma_v^p$ for their restrictions to $S_p(T_v, X)$, and write them simply τ_v^p and σ_v^p if $X = \mathbf{C}$ and ρ is trivial. Here we use (a, b) to denote an element of $\mathfrak{K}_0$ with $a = (a_v)_{v\in\mathbf{a}} \in \prod_{v\in\mathbf{a}} GL_{m_v}(\mathbf{C})$ and $b = (b_v)_{v\in\mathbf{a}} \in \prod_{v\in\mathbf{a}} GL_{n_v}(\mathbf{C})$, ignoring a_v or b_v according as $m_v = 0$ or $n_v = 0$.

12.4. Lemma. (1) *$S_p(Y)$ is spanned by g^p for all $g \in S_1(Y)$.*

(2) *Given $h \in S_p(Y, X)$, there is a unique symmetric element of $Ml_p(Y, X)$, which we write h_*, such that $h(y) = h_*(y, \ldots, y)$.*

PROOF. We prove (1) by induction on $\dim(Y)$. Let $S_p'(Y)$ be the subspace of $S_p(Y)$ spanned by g^p for all $g \in S_1(Y)$. Clearly $S_p'(Y) = S_p(Y)$ if $\dim(Y) = 1$. Now we consider $\mathbf{C}\oplus Y$, and take a variable $(t, y) \in \mathbf{C}\oplus Y$. For $c \in \mathbf{C}$ and $f \in S_1(Y)$ we have $\bigl(ct + f(y)\bigr)^p = \sum_{m=0}^p \binom{p}{m} c^mt^mf(y)^{p-m}$. From this we see that the functions of the form $t^mf(y)^{p-m}$ belong to $S_p'(\mathbf{C}\oplus Y)$. By induction, $S_{p-m}'(Y) = S_{p-m}(Y)$. Since $S_p(\mathbf{C}\oplus Y)$ is spanned by the functions $t^mg(y)$ with $g \in S_{p-m}(Y)$, we obtain (1) for $\mathbf{C}\oplus Y$. Clearly it is sufficient to prove (2) when $X = \mathbf{C}$. To prove the existence, by (1), we may assume that $h(y) = g(y)^p$ with $g \in S_1(Y)$. Then put $h_*(x_1, \ldots, x_p) = g(x_1)\cdots g(x_p)$. Clearly h_* is symmetric and $h_*(y, , \ldots, y) = h(y)$. We prove the uniqueness of h_* by induction on p. Suppose $p > 1$ and f is a symmetric element of $Ml_p(Y, \mathbf{C})$ such that $f(x, \ldots, x) = 0$. Then

$$0 = f(x+y, \ldots, x+y) = \sum_{m=0}^p \binom{p}{m} f(\overbrace{x, \ldots, x}^{m}, \overbrace{y, \ldots, y}^{p-m}).$$

Viewing this as a polynomial function of y, we find that each term on the right-hand side must vanish; in particular, $f(x, y, \ldots, y) = 0$. Fixing x and applying our induction to $f(x, y, \ldots, y)$, we obtain $f(x, x_1, \ldots, x_{p-1}) = 0$ for arbitrary x_i, which completes the proof, since the case $p \le 1$ is obvious.

12.5. We view T_v as its own dual over $\mathbf{C}$ by the pairing $(u, v) \mapsto \mathrm{tr}({}^tuv)$. Then, for $g \in S_p(T_v)$ and $h \in S_p(T_v, X)$ we put

$$[g, h] = \sum g_*(a_{\nu_1}, \ldots, a_{\nu_p})h_*(b_{\nu_1}, \ldots, b_{\nu_p}), \tag{12.8}$$

where $\{a_\nu\}_{\nu\in N}$ and $\{b_\nu\}_{\nu\in N}$ are dual bases of T_v, and $(\nu_1, \ldots, \nu_p)$ runs over N^p. Then $[g, h]$ is an element of X determined independently of the choice of dual bases. From (12.7a, b) we easily obtain

$$\bigl[\sigma_v^p(\alpha)g, (\rho\otimes\tau_v^p)(\alpha)h\bigr] = \bigl[\tau_v^p(\alpha)g, (\rho\otimes\sigma_v^p)(\alpha)h\bigr] = \rho(\alpha)[g, h] \quad (\alpha \in \mathfrak{K}_0). \tag{12.9}$$

In particular, taking $X = \mathbf{C}$, we can view $S_p(T_v)$ as its own dual by the pairing $(g, h) \mapsto [g, h]$, which is indeed nondegenerate because of a simple formula

(12.10) $\qquad [g, h] = g(x) \quad \textit{if} \quad h(u) = \mathrm{tr}({}^t xu)^p \quad \textit{with a fixed } x \in T_v.$

To prove this, we observe that $h_*(u_1, \ldots, u_p) = \prod_{i=1}^{p} \mathrm{tr}({}^t xu_i)$; therefore $[g, h] = \sum g_*(a_\nu, a_\mu, \cdots)\mathrm{tr}({}^t xb_\nu)\mathrm{tr}({}^t xb_\mu) \cdots$. Since $x = \sum_\nu \mathrm{tr}({}^t xb_\nu)a_\nu$, we obtain (12.10).

12.6. Let us now recall some elementary facts on the polynomial representations of $GL_n(\mathbf{C})$. Every polynomial representation of $GL_n(\mathbf{C})$ can be decomposed into a direct sum of irreducible representations. The equivalence classes of irreducible polynomial representations of $GL_n(\mathbf{C})$ are in one-to-one correspondence with the ordered sets of integers $\{r_1, \ldots, r_n\}$ such that $r_1 \geq \cdots \geq r_n \geq 0$. If $\{\sigma, W\}$ is such an irreducible representation corresponding to $\{r_1, \ldots, r_n\}$, then

$$\mathrm{tr}\{\sigma(\mathrm{diag}[a_1, \ldots, a_n])\} = a_1^{r_1} \cdots a_n^{r_n} + \textstyle\sum_{s<r} \lambda_s a_1^{s_1} \cdots a_n^{s_n}$$

for $a_i \in \mathbf{C}^\times$ with $\lambda_s \in \mathbf{Z}$, where $<$ in the last sum is the lexicographic ordering. We call then σ an *irreducible representation of highest weight* $\{r_1, \ldots, r_n\}$. A nonzero vector q of W is called an *eigenvector of highest weight* if $\sigma(\mathrm{diag}[a_1, \ldots, a_n])q = a_1^{r_1} \cdots a_n^{r_n} q$.

For $x \in \mathbf{C}_n^m$ and $1 \leq i \leq \mathrm{Min}(m, n)$ we denote by $\det_i(x)$ the determinant of the uper left i^2 entries of x. Let R_n denote the subgroup of $GL_n(\mathbf{C})$ consisting of all the upper triangular matrices. Given an irreducible representation $\{\sigma, W\}$ of $GL_n(\mathbf{C})$, there is a unique common eigenvector p of $\sigma(R_n)$ in W. If $\{r_1, \ldots, r_n\}$ is the highest weight of σ, then

(12.11) $\qquad \sigma(a)p = \prod_{i=1}^{n} \det_i(a)^{e_i} p \quad \textit{for every} \quad a \in R_n,$

where $e_i = r_i - r_{i+1}$ for $i < n$ and $e_n = r_n$.

We are interested in the representations $\{S_r(T), \tau^r\}$ of a group K^c defined in §12.3. We drop the subscript v. Thus we are looking at the objects as follows:

(Type A) $T = \mathbf{C}_n^m$, $[\tau^r(a, b)h](u) = h({}^t aub)$ for $(a, b) \in \mathcal{K} = GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$ and $h \in S_r(T)$;

(Type C) $T = \{x \in \mathbf{C}_n^n, \mid {}^t x = x\}$, $[\tau^r(a)h](u) = h({}^t aua)$ for $a \in \mathcal{K} = GL_n(\mathbf{C})$ and $h \in S_r(T)$.

We have $\mathcal{K} = K^c$ for Type C, but $\mathcal{K}$ is larger than K^c for Type A. However, clearly we can extend the representation τ^r to $\mathcal{K}$ as above, and the $\mathcal{K}$-irreducibility of a subspace of $S_r(T)$ is the same as that of K^c. Therefore we consider $\mathcal{K}$ instead of K^c. Also, for Type A we recall a well-known fact that every irreducible representation of $\mathcal{K}$ is of the form $\rho \otimes \sigma$ with irreducible representations ρ of $GL_m(\mathbf{C})$ and σ of $GL_n(\mathbf{C})$.

12.7. Theorem. Type A: *Let ρ and σ be irreducible representations of $GL_m(\mathbf{C})$ and $GL_n(\mathbf{C})$, respectively. Then $\rho \otimes \sigma$ occurs in τ^r if and only if ρ and σ have the highest weights*

$$\{r_1, \ldots, r_n, 0, \ldots, 0\} \quad \textit{and} \quad \{r_1, \ldots, r_n\} \quad \textit{when} \quad m \geq n,$$
$$\{r_1, \ldots, r_m\} \quad \textit{and} \quad \{r_1, \ldots, r_m, 0, \ldots, 0\} \quad \textit{when} \quad n \geq m,$$

with the r_i such that $r_1 + \cdots + r_\mu = r$ and $r_\mu \geq 0$, where $\mu = \mathrm{Min}(m, n)$. Such a $\rho \otimes \sigma$ has multiplicity one in τ^r, and the corresponding irreducible subspace of $S_r(T)$ contains a polynomial p defined by

$$p(x) = \textstyle\prod_{i=1}^{\mu} \det_i(x)^{e_i} \qquad (x \in T = \mathbf{C}_n^m)$$

as an eigenvector of highest weight with respect to both ρ *and* σ*, where* $e_i = r_i - r_{i+1}$ *for* $i < \mu$ *and* $e_\mu = r_\mu$.

Type C: *An irreducible representation* σ *of* $GL_n(\mathbf{C})$ *of highest weight* $\{r_1, \dots, r_n\}$ *occurs in* τ^r *if and only if all* r_i *are even,* $r_n \geq 0$*, and* $r_1 + \cdots + r_n = 2r$*. Such a* σ *has multiplicity one in* τ^r*, and the corresponding irreducible subspace of* $S_r(T)$ *contains a polynomial* p *defined by*

$$p(x) = \textstyle\prod_{i=1}^{n} \det_i(x)^{e_i} \qquad (x \in T)$$

as an eigenvector of highest weight, where $e_i = (r_i - r_{i+1})/2$ *for* $i < n$ *and* $e_n = r_n/2$.

The decomposition of $S_r(T)$ into irreducible subspaces with highest weights as described above is due to L.-K. Hua; the highest weight vector was determined by K. D. Johnson. The proof, as well as references to these and other related investigations, can be found in [S84b].

12.8. Lemma. (1) *Let* Z *and* W *be different irreducible subspaces of* $S_r(T)$*. Then* $[f, g] = 0$ *for every* $f \in Z$ *and every* $g \in W$.

(2) *The form* $(f, g) \mapsto [f, g]$ *is nondegenerate on any* K^c*-stable subspace of* $S_r(T)$.

PROOF. We prove this for Type A; Type C can be handled in a similar and simpler way. Take Z to be the space described in Theorem 12.7 and take $p \in Z$ as in that theorem. Define similarly a function q in W. By (12.9) we have $[\tau^r({}^ta, {}^tb)f, g] = [f, \tau^r(a, b)g]$ for $f, g \in S_r(T)$. Taking a and b to be diagonal, we see that $[p, q] = 0$. Since p and q are eigenvectors of $R_m \times R_n$, from (12.9) we obtain $[\tau^r({}^tR_m, {}^tR_n)p, q] = 0$, and consequently $[\tau^r({}^tR_mR_m, {}^tR_nR_n)p, q] = 0$. Now ${}^tR_nR_n = \{\, x \in GL_n(\mathbf{C}) \mid \det_i(x) \neq 0 \text{ for every } i \leq n \,\}$, and hence ${}^tR_mR_m \times {}^tR_nR_n$ is dense in $GL_m(\mathbf{C}) \times GL_n(\mathbf{C})$. Therefore we see that $[Z, q] = 0$, which combined with (12.9) proves (1). Since the form $(f, g) \mapsto [f, g]$ is nondegenerate on $S_r(T)$ as we noted in §12.5, (2) follows immediately from (1).

12.9. Given C^∞ manifolds V and W, we denote by $C^\infty(V, W)$ the set of all C^∞ maps of V into W. For the most part we take $\mathcal{H}$ as V. In fact, we will have to consider the derivatives of C^∞ functions which are defined only on an open dense subset of $\mathcal{H}$, such as meromorphic functions on $\mathcal{H}$. For simplicity, however, we state our definitions and formulas only for C^∞ functions on $\mathcal{H}$, but we will apply them to functions of more general types, without any additional comment, as the validity of such applications is obvious in each case.

Since T_v has a natural $\mathbf{R}$-structure, we can speak of an $\mathbf{R}$-rational basis of T_v over $\mathbf{C}$. Take any such basis $\{\varepsilon_\nu\}_{\nu \in N}$, and for $u \in T_v$ define $u_\nu \in \mathbf{C}$ by $u = \sum_{\nu \in N} u_\nu \varepsilon_\nu$. We also put $z_v = \sum_{\nu \in N} z_{v\nu}\varepsilon_\nu$ with $z_{v\nu} \in \mathbf{C}$ for the variable z_v on H_v. Then, for $f \in C^\infty(\mathcal{H}, X)$ we define $D_vf, \overline{D}_vf, C_vf, E_vf \in C^\infty\big(\mathcal{H}, S_1(T_v, X)\big)$ by

$$(D_vf)(u) = \sum_{\nu \in N} u_\nu \partial f/\partial z_{v\nu}, \quad (\overline{D}_vf)(u) = \sum_{\nu \in N} u_\nu \partial f/\partial \overline{z}_{v\nu}, \tag{12.12a}$$

$$(C_vf)(u) = (D_vf)({}^t\xi_v u \eta_v), \quad (E_vf)(u) = (\overline{D}_vf)(\xi_v u \cdot {}^t\eta_v) \tag{12.12b}$$

for $u \in T_v$. These are independent of the choice of $\{\varepsilon_\nu\}_{\nu \in N}$. The last two formulas can be written $C_vf = \tau_v^1(\Xi)D_vf$ and $E_vf = \sigma_v^1(\Xi^{-1})\overline{D}_vf$ with Ξ of (12.4b), and the first two are equivalent to the expression

$$df = \sum_{v\in\mathbf{a}}(D_v f)(dz_v) + \sum_{v\in\mathbf{a}}(\overline{D}_v f)(d\bar{z}_v). \tag{12.13}$$

Notice that $E_v f = 0$ if and only if f is holomorphic in z_v. Substituting $f \circ \alpha$ for f in (12.13) and employing (12.1b), we find that

$$\big((D_v f)\circ\alpha\big)(u) = D_v(f\circ\alpha)\big({}^t\lambda(\alpha_v,\, z_v)u\mu(\alpha_v,\, z_v)\big), \tag{12.14a}$$

$$\big((\overline{D}_v f)\circ\alpha\big)(u) = \overline{D}_v(f\circ\alpha)\big(\lambda(\alpha_v,\, z_v)^* u\overline{\mu(\alpha_v\, z_v)}\big) \tag{12.14b}$$

for $u \in T_v$ and $\alpha \in G_{\mathbf{a}}$.

We also define $D_v^e f$, $\overline{D}_v^e f$, $C_v^e f$, and $E_v^e f$ for $0 \le e \in \mathbf{Z}$ by

$$D_v^e f = D_v D_v^{e-1} f, \quad \overline{D}_v^e f = \overline{D}_v \overline{D}_v^{e-1} f, \quad D_v^0 f = \overline{D}_v^0 f = f, \tag{12.15a}$$

$$C_v^e f = C_v C_v^{e-1} f, \quad E_v^e f = E_v E_v^{e-1} f, \quad C_v^0 f = E_v^0 f = f. \tag{12.15b}$$

These have values in $Ml_e(T_v,\, X)$ in the sense that

$$(A^e f)(u_1,\, \dots,\, u_e) = A\big\{(A^{e-1}f)(u_1,\, \dots,\, u_{e-1})\big\}(u_e) \qquad (u_i \in T_v), \tag{12.15c}$$

where A is D_v, $\overline{D}_v$, C_v or E_v. Clearly $D_v^e f$ and $\overline{D}_v^e f$ have symmetric elements of $Ml_e(T_v,\, X)$ as their values; the same is true for $C_v^e f$ and $E_v^e f$ as will be shown in §13.9. Therefore we can view them as elements of $C^\infty(\mathcal{H},\, S_e(T_v,\, X))$. For example, we have

$$(D_v^p f)(u) = p!h(u) \quad \textit{if} \quad f(z) = h(z_v) \quad \textit{with} \quad h \in S_p(T_v). \tag{12.16}$$

Indeed, $h(z_v) = h_*(z_v,\, \dots,\, z_v)$ with h_* as in Lemma 12.4 (2), and hence $(D_v f)(u) = ph_*(u,\, z_v,\, \dots,\, z_v)$, and similarly $(D_v^2 f)(u,\, v) = p(p-1)h_*(u,\, v,\, z_v,\, \dots,\, z_v)$. Repeating this procedure, we obtain (12.16).

We now define $D_{\rho,v}^e f \in C^\infty(\mathcal{H},\, S_e(T_v,\, X))$ by

$$D_{\rho,v}^e f = (\rho\otimes\tau_v^e)(\Xi)^{-1} C_v^e[\rho(\Xi)f]. \tag{12.17}$$

In particular, since $C_v f = \tau_v^1(\Xi)D_v f$, writing $D_{\rho,v} = D_{\rho,v}^1$, we have

$$(D_{\rho,v}f)(u) = \rho(\Xi)^{-1}D_v[\rho(\Xi)f](u) \qquad (u \in T_v). \tag{12.18}$$

12.10. Proposition. (1) $D_{\rho,v}^{e+1} = D_{\rho\otimes\tau_v^e,\, v}D_{\rho,v}^e = D_{\rho\otimes\tau_v,\, v}^e D_{\rho,v}$.
(2) $D_{\rho,v}^e(f\|_\rho\,\alpha) = (D_{\rho,v}^e f)\|_{\rho\otimes\tau_v^e}\,\alpha, \quad E_v^e(f\|_\rho\,\alpha) = (E_v^e f)\|_{\rho\otimes\sigma_v^e}\,\alpha. \qquad (\alpha \in G_{\mathbf{a}})$.

PROOF. Identifying $\rho\otimes\tau_v^e\otimes\tau_v$ with $\rho\otimes\tau_v^{e+1}$, by (12.17) and (12.18) we have

$$\begin{aligned} D_{\rho\otimes\tau_v^e,\, v}\big(D_{\rho,v}^e f\big) &= (\rho\otimes\tau_v^{e+1})(\Xi)^{-1}C_v\big[(\rho\otimes\tau_v^e)(\Xi)(D_{\rho,v}^e f)\big] \\ &= (\rho\otimes\tau_v^{e+1})(\Xi)^{-1}C_vC_v^e\big[\rho(\Xi)f\big] = D_{\rho,v}^{e+1}f, \end{aligned}$$

which proves the first part of (1). The second part can be proved similarly. Now for $\alpha \in G_{\mathbf{a}}$ and $u \in T_v$ we have, employing (12.1a) and (12.14a),

$$\begin{aligned} &D_{\rho,\, v}(f\|_\rho\,\alpha)(u) = \rho(\Xi)^{-1}D_v\big[\rho(\Xi M_\alpha^{-1})(f\circ\alpha)\big](u) \\ &= \rho(\Xi)^{-1}D_v\big[\rho(M_\alpha^*)(\{\rho(\Xi)f\}\circ\alpha)\big](u) = \rho(\Xi^{-1}M_\alpha^*)D_v\big[(\rho(\Xi)f)\circ\alpha\big](u) \\ &= \rho\big(M_\alpha^{-1}(\Xi\circ\alpha)^{-1}\big)\big\{D_v\big(\rho(\Xi)f\big)\circ\alpha\big\}\big({}^t\lambda(\alpha_v,\, z_v)^{-1}u\mu(\alpha_v,\, z_v)^{-1}\big) \\ &= \rho(M_\alpha^{-1})\big\{(D_{\rho,\, v}f)\circ\alpha\big\}\big({}^t\lambda(\alpha_v,\, z_v)^{-1}u\mu(\alpha_v,\, z_v)^{-1}\big). \end{aligned}$$

Observing that the last quantity can be written $\big[(D_{\rho,\, v}f)\|_{\rho\otimes\tau_v}\,\alpha\big](u)$, we obtain the first formula of (2) for $e = 1$. Then the general case can be proved by induction on e, by virtue of (1). As for E_v, we have similarly, by (12.14b) and (12.1a),

$$\begin{aligned}E_v(f\|_\rho\,\alpha)(u) &= E_v\{\rho(M_\alpha^{-1})(f\circ\alpha)\}(u) = \rho(M_\alpha)^{-1}\overline{D}_v(f\circ\alpha)(\xi_v u\cdot{}^t\eta_v)\\ &= \rho(M_\alpha)^{-1}\big((\overline{D}_v f)\circ\alpha\big)\big({}^t\overline{\lambda(\alpha_v,\,z_v)}^{-1}\xi_v u\cdot{}^t\eta_v\overline{\mu(\alpha_v,\,z_v)}^{-1}\big)\\ &= \rho(M_\alpha)^{-1}\big((\overline{D}_v f)\circ\alpha\big)\big((\xi_v\circ\alpha)\lambda(\alpha_v,\,z_v)u\cdot{}^t\mu(\alpha_v,\,z_v)({}^t\eta_v\circ\alpha)\big)\\ &= \big\{(\rho\otimes\sigma_v)(M_\alpha^{-1})\big((E_v f)\circ\alpha\big)\big\}(u) = \big((E_v f)\|_{\rho\times\sigma_v}\,\alpha\big)(u).\end{aligned}$$

This proves the second formula of (2) for $e = 1$. The general case can be proved by induction on e.

12.11. The representation τ_v^p or σ_v^p of $\mathfrak{K}_0$ on $S_p(T_v)$ is essentially that of K_v^c. By Theorem 12.7 it is the direct sum of irreducible representations, and each irreducible constituent has multiplicity one. (Since the τ_v^p-irreducibility is the same as the σ_v^p-irreducibility, we shall simply speak of an irreducible subspace of $S_p(T_v)$.) Thus, for each $\mathfrak{K}_0$-stable subspace Z of $S_p(T_v)$, we can define the projection map φ_Z of $S_p(T_v)$ onto Z. Now we can identify $S_p(T_v,\,X)$ with $S_p(T_v)\otimes X$ by the rule

(12.19) $$(h\otimes x)(u) = h(u)x \quad \textit{for } h\in S_p(T_v),\ x\in X,\ \textit{ and } u\in T_v.$$

(This justifies the notation $\rho\otimes\tau_v^p$ and $\rho\otimes\sigma_v^p$.) Using the same symbol φ_Z for the map $h\otimes x\mapsto(\varphi_Z h)\otimes x$ of $S_p(T_v)\otimes X$ to $Z\otimes X$, we define $D_\rho^Z f,\ E^Z f\in C^\infty(\mathcal{H},\,Z\otimes X)$ by

(12.20) $$D_\rho^Z f = \varphi_Z D_{\rho,v}^p f, \qquad E^Z f = \varphi_Z E_v^p f.$$

Let τ_Z and σ_Z denote the restrictions of τ_v^p and σ_v^p to Z. Then $\rho\otimes\tau_Z$ and $\rho\otimes\sigma_Z$ are the restrictions of $\rho\otimes\tau_v^p$ and $\rho\otimes\sigma_v^p$ to $Z\otimes X$. Then $(\varphi_Z f)\|_{\rho\otimes\tau_Z}\,\alpha = \varphi_Z(f\|_{\rho\otimes\tau_v^e}\,\alpha)$ and $(\varphi_Z f)\|_{\rho\otimes\sigma_Z}\,\alpha = \varphi_Z(f\|_{\rho\otimes\sigma_v^e}\,\alpha)$. Therefore from Proposition 12.10 (2) we obtain, for every $\alpha\in G_{\mathbf{a}}$,

(12.21) $$D_\rho^Z(f\|_\rho\,\alpha) = (D_\rho^Z f)\|_{\rho\otimes\tau_Z}\,\alpha, \quad E^Z(f\|_\rho\,\alpha) = (E^Z f)\|_{\rho\otimes\sigma_Z}\,\alpha.$$

By Lemma 12.8, Z is its own dual with respect to the bilinear form of (12.8). Therefore we can identify $Z\otimes X$ with $S_1(Z,\,X)$ by the rule

(12.22) $$(\omega\otimes x)(\zeta) = [\zeta,\,\omega]x \quad \textit{for } \omega,\ \zeta\in Z \textit{ and } x\in X.$$

Then φ_Z as a map $S_p(T_v,\,X)\to S_1(Z,\,X)$ can be given by $(\varphi_Z g)(\zeta) = [\zeta,\,g]$ for $g\in S_p(T_v,\,X)$ and $\zeta\in Z$, and hence $D_\rho^Z f$ and $E^Z f$ as $S_1(Z,\,X)$-valued functions can be given by

(12.23) $$(D_\rho^Z f)(\zeta) = [\zeta,\,D_{\rho,v}^p f], \qquad (E^Z f)(\zeta) = [\zeta,\,E_v^p f] \qquad (\zeta\in Z).$$

The symbols $\rho\otimes\tau_Z$ and $\rho\otimes\sigma_Z$ as representations on the space $S_1(Z,\,X)$ can be given by

(12.24a) $$\big[(\rho\otimes\tau_Z)(a,\,b)h\big](\zeta) = \rho(a,\,b)h\big(\tau_Z({}^t a_v,\,{}^t b_v)\zeta\big),$$

(12.24b) $$\big[(\rho\otimes\sigma_Z)(a,\,b)h\big](\zeta) = \rho(a,\,b)h\big(\sigma_Z({}^t a_v,\,{}^t b_v)\zeta\big)$$

for $h\in S_1(Z,\,X)$, $\zeta\in Z$, and $(a,\,b)\in\mathfrak{K}_0$ as in (12.7a, b). Take, for example, $Z = \mathbf{C}\psi\subset S_{en}(T)$ with $\psi(u) = \det(u)^e$, $0\le e\in\mathbf{Z}$, assuming that $m = n$ for Type AB. Then, from (12.21) and (12.24a, b) we obtain

(12.24c) $$D_\rho^Z(f\|_\rho\alpha)(\psi) = \big[(D_\rho^Z f)(\psi)\big]\|_{\rho'}\,\alpha, \quad E^Z(f\|_\rho\alpha)(\psi) = \big[(E^Z f)(\psi)\big]\|_{\rho''}\,\alpha,$$

where $\rho'(a,\,b) = \det(a)^e\det(b)^e\rho(x,\,y)$ and $\rho''(a,\,b) = \det(a)^{-e}\det(b)^{-e}\rho(x,\,y)$.

12.12. Fixing $v \in \mathbf{a}$ and taking $\mathbf{R}$-rational dual bases $\{\varepsilon_\nu\}_{\nu\in N}$ and $\{\varepsilon'_\nu\}_{\nu\in N}$ of T_v over $\mathbf{C}$, put $z_v = \sum_{\nu\in N} z_{v\nu}\varepsilon_\nu$ as in §12.9, and

$$\mathcal{D}_v = \sum_{\nu\in N} \varepsilon'_\nu \partial/\partial z_{v\nu}, \qquad \overline{\mathcal{D}}_v = \sum_{\nu\in N} \varepsilon'_\nu \partial/\partial \overline{z}_{v\nu}. \tag{12.25}$$

These are independent of the choice of bases. For $g \in S_p(T_v)$ we define $g(\mathcal{D}_v)$ by

$$g(\mathcal{D}_v) = g_*(\mathcal{D}_v, \dots, \mathcal{D}_v) = \sum g_*(\varepsilon'_{\nu_1}, \dots, \varepsilon'_{\nu_p})\partial^p/\partial z_{v\nu_1} \cdots \partial z_{v\nu_p}, \tag{12.26}$$

where $(\nu_1, \dots, \nu_p)$ runs over N^p, and define $g(\overline{\mathcal{D}}_v)$ similarly. Now we have

$$g(\mathcal{D}_v)f = [g, D_v^p f] \quad \textit{for every} \quad f \in C^\infty(\mathcal{H}, X), \tag{12.27}$$

$$g(\mathcal{D}_v)h = p!\,[g, h] \quad \textit{for every} \quad h \in S_p(T_v) \tag{12.28}$$

with $[g, h]$ of (12.8). In (12.27) we view $D_v^p f$ as $S_p(T_v, X)$-valued; in (12.28) we view h as a function on $\mathcal{H}$ by putting $h(z) = h(z_v)$. To prove these, suppress the subscript v. Since $(Df)(\varepsilon_\nu) = \partial f/\partial z_\nu$, we have $(D^p f)(\varepsilon_{\nu_1}, \dots, \varepsilon_{\nu_p}) = \partial^p f/\partial z_{\nu_1} \cdots \partial z_{\nu_p}$, which combined with (12.8) and (12.26) proves (12.27). Then (12.28) follows from (12.16) and (12.27).

For example, since $\left[D_v^p \exp(\mathrm{tr}({}^t x z_v))\right](u) = \mathrm{tr}({}^t x u)^p \exp\left(\mathrm{tr}({}^t x z_v)\right)$ for $u, x \in T_v$, from (12.10) and (12.27) we obtain

$$g(\mathcal{D}_v)\exp\left(\mathrm{tr}({}^t x z_v)\right) = g(x)\exp\left(\mathrm{tr}({}^t x z_v)\right). \tag{12.29}$$

In particular, for an element g of $S_{nk}(T_v)$ given by $g(u) = \det(u)^k$, we denote the operator $g(\mathcal{D}_v)$ by $\det(\mathcal{D}_v)^k$.

In the following theorem, we drop the subscript v for simplicity.

12.13. Theorem. *Let Z be the irreducible subspace of $S_r(T)$ described in Theorem 12.7 and r_i be the integers in that theorem; let $\zeta \in Z$, $s \in \mathbf{C}$, and*

$$L = \begin{cases} \{ (c, d) \in \mathbf{C}_m^n \times \mathbf{C}_n^n \mid \mathrm{rank}[c \ \ d] = n \} & \text{(Type A)}, \\ \{ (c, d) \in \mathbf{C}_n^n \times \mathbf{C}_n^n \mid \mathrm{rank}[c \ \ d] = n,\ c \cdot {}^t d = d \cdot {}^t c \} & \text{(Type C)}. \end{cases}$$

Then, for any fixed $(c, d) \in L$ and any fixed branch of $\det(cz+d)^s$ in an open subset of T on which $\det(cz+d)^s$ is meaningful, we have

$$\zeta(\mathcal{D})\det(cz+d)^s = \psi_Z(s)\det(cz+d)^s \zeta\left({}^t c \cdot {}^t(cz+d)^{-1}\right)$$

$$\textit{with} \quad \psi_Z(s) = \begin{cases} \displaystyle\prod_{h=1}^{\mu}\prod_{i=1}^{r_h}(s-i+h), \quad \mu = \mathrm{Min}(m, n) & \text{(Type A)}, \\ \displaystyle\prod_{h=1}^{n}\prod_{i=1}^{r_h/2}\left(s-i+\frac{h+1}{2}\right) & \text{(Type C)}. \end{cases}$$

For the proof, see [S84b, Theorem 4.3].

12.14. The notation being as in §12.11, we define a contraction operator θ : $Z \otimes Z \otimes X \to X$ by $\theta(\zeta \otimes \omega \otimes x) = [\zeta, \omega]x$. This as a map $S_1\big(Z, S_1(Z, X)\big) \to X$ can be given by

$$\theta h = \textstyle\sum_\mu h(\zeta_\mu, \omega_\mu) \quad \textit{for} \quad h \in S_1\big(Z, S_1(Z, X)\big) \tag{12.30a}$$

with bases $\{\zeta_\mu\}$ and $\{\omega_\mu\}$ of Z such that $[\zeta_\mu, \omega_\nu] = \delta_{\mu\nu}$. From (12.9) we obtain

$$\theta \circ (\rho \otimes \tau_Z \otimes \sigma_Z)(a, b) = \theta \circ (\rho \otimes \sigma_Z \otimes \tau_Z)(a, b) = \rho(a, b) \circ \theta \quad \textit{for every} \quad (a, b) \in \mathfrak{K}_0. \tag{12.30b}$$

If $g \in C^\infty(\mathcal{H}, S_1(Z, X))$, then $D^Z_{\rho\otimes\sigma_Z} g$ and $E^Z g$ have values in $S_1(Z, S_1(Z, X))$, so that $\theta D^Z_{\rho\otimes\sigma_Z} g$ and $\theta E^Z g$ are meaningful as X-valued functions. In particular, for $f \in C^\infty(\mathcal{H}, X)$ the symbols $\theta D^Z_{\rho\otimes\sigma_Z} E^Z f$ and $\theta E^Z D^Z_\rho f$ are elements of $C^\infty(\mathcal{H}, X)$. We then put

$$(12.31) \qquad L^Z_\rho f = (-1)^p \theta D^Z_{\rho\otimes\sigma_Z} E^Z f, \qquad M^Z_\rho f = (-1)^p \theta E^Z D^Z_\rho f.$$

Then, by (12.21) and (12.30b), for every $\alpha \in G_{\mathbf{a}}$ we have

$$(12.32) \qquad L^Z_\rho(f\|_\rho\,\alpha) = (L^Z_\rho f)\|_\rho\,\alpha, \qquad M^Z_\rho(f\|_\rho\,\alpha) = (M^Z_\rho f)\|_\rho\,\alpha.$$

If $Z = S_1(T_v)$ and this is identified with T_v, then the map θ can be viewed as a map $S_1(T_v, S_1(T_v, X)) \to X$, and is given by

$$(12.33) \qquad \theta h = \textstyle\sum_{\nu\in N} h(a_\nu, b_\nu) \quad \text{for} \quad h \in S_1(T_v, S_1(T_v, X))$$

with dual bases $\{a_\nu\}$ and $\{b_\nu\}$ of T_v.

Given a congruence subgroup Γ of G, we denote by $C_\rho(\Gamma)$ the set of all $f \in C^\infty(\mathcal{H}, X)$ such that $f\|_\rho\gamma = f$ for every $\gamma \in \Gamma$, and by C_ρ the union of $C_\rho(\Gamma)$ for all such Γ. Now X has an inner product $\langle x, y\rangle_X$ which is $\mathbf{C}$-linear in y and $\mathbf{C}$-antilinear in x and which satisfies

$$(12.34) \qquad \langle\, x,\, \rho(a, b)y\,\rangle_X = \langle\, \rho(a^*, b^*)x,\, y\,\rangle_X$$

for $(a, b) \in \mathfrak{K}_0$, where $(a^*, b^*) = (a^*_v, b^*_v)_{v\in\mathbf{a}}$. This will be proven after (12.35b). Then, for $f, g \in C_\rho$ we define their inner product $\langle f, g\rangle$ by

$$(12.35\text{a}) \qquad \langle\, f, g\,\rangle = \mu(\Phi)^{-1}\int_\Phi \langle\, f(z),\, \rho(\Xi(z))g(z)\,\rangle_X\, d\mu(z)$$

whenever the integral is convergent, where $d\mu(z)$ is a fixed $G_{\mathbf{a}}$-invariant measure on $\mathcal{H}$, $\mu(\Phi) = \int_\Phi d\mu(z)$, and $\Phi = \Gamma\backslash\mathcal{H}$ with Γ such that $f, g \in C_\rho(\Gamma)$. The inner product is independent of the choice of Γ, and

$$(12.35\text{b}) \qquad \langle\, f\|_\rho\alpha,\, g\|_\rho\alpha\,\rangle = \langle\, f, g\,\rangle \quad \text{for every} \quad \alpha \in G.$$

Let us now prove the existence of $\langle\ ,\ \rangle_X$ satisfying (12.34). In view of the structure of $\mathfrak{K}_0$, it is sufficient to prove that if (Y, σ) is an irreducible polynomial representation of $GL_n(\mathbf{C})$, then Y has an inner product $\langle\ ,\ \rangle_Y$ such that $\langle x, \sigma(a)y\rangle_Y = \langle\sigma(a^*)x, y\rangle_Y$ for $a \in GL_n(\mathbf{C})$. Now (Y, σ) is a direct summand of (W, ω), where $W = \mathbf{C}^n\otimes\cdots\otimes\mathbf{C}^n$ and $\omega(x) = x\otimes\cdots\otimes x$ with $\mathbf{C}^n$ and x repeated m times for some m. Clearly W has an inner product with the required property with respect to ω, and we only have to restrict it to Y.

12.15. Theorem. *Let Z be an irreducible subspace of $S_p(T_v)$. Then $Z\otimes X$ has an inner product satisfying (12.34) with $(Z\otimes X, \rho\otimes\tau_Z)$ and $(Z\otimes X, \rho\otimes\sigma_Z)$ in place of (X, ρ) and with the property that for $f, f' \in C_\rho$, $g \in C_{\rho\otimes\tau_Z}$, and $h \in C_{\rho\otimes\sigma_Z}$ we have*

$$\langle\, D^Z_\rho f, g\,\rangle = (-1)^p\langle\, f, \theta E^Z g\,\rangle, \qquad \langle\, E^Z f, h\,\rangle = (-1)^p\langle\, f, \theta D^Z_{\rho\otimes\sigma_Z} h\,\rangle,$$
$$\langle\, L^Z_\rho f, f'\,\rangle = \langle\, f, L^Z_\rho f'\,\rangle, \qquad \langle\, M^Z_\rho f, f'\,\rangle = \langle\, f, M^Z_\rho f'\,\rangle,$$
$$\langle\, L^Z_\rho f, f\,\rangle \geq 0, \qquad \langle\, M^Z_\rho f, f\,\rangle \geq 0$$

under suitable convergence conditions (see below).

The theory of differential operators in this section was developed in [S80], [S81b], [S84a], [S84b], and [S86]. In particular, the last theorem was obtained in [S84a, Theorem 11.5, Corollary 11.8] and [S84b, p.486]. An equivalent result formulated on G_v was given in [S90, Theorem 4.3, Corollary 4.4]. See also [S90, Lemma 4.2] for the definition of the inner product on $Z \otimes X$. The operators on G_v correspond to D_ρ^Z and E^Z, but the choice of Ξ in [S90, (7.7)] is the same as that of (12.4b) only for Types AB and CB; for Types AT and CT the latter is twice the former, and so the present E^Z is E of [S90, (7.9)] times a positive rational number, which may or may not be 1. Thus, in order to have the first two equalities of the above theorem, we have to take this constant into account, though it does not play any essential role in practically all applications.

All these were formulated for functions on $\Gamma\backslash\mathcal{H}$ or on $\Gamma\backslash G_{\mathbf{a}}$ of compact support. To state a sufficient convergence condition in a more general case, we first note that given $f \in C_\rho(\Gamma)$, the function $\tilde{f}$ on $G_{\mathbf{a}}$ defined by $\tilde{f}(g) = \rho\bigl(M(g, \mathbf{o})^{-1}\bigr)f(g\mathbf{o})$ for $g \in G_{\mathbf{a}}$ is left Γ-invariant, where $\mathbf{o} = (\mathbf{o}_v)$ is a point of $\mathcal{H}$ such that $\bigl\{\, \alpha \in G_v \bigm| \alpha\mathbf{o}_v = \mathbf{o}_v \bigr\}$ is the standard maximal compact subgroup of G_v with which the results of [S90] are formulated. To be explicit, we take $\mathbf{o}_v = 0$ for Types AB and CB, and $\mathbf{o}_v = i1_n$ for Types AT and CT. Let $\mathfrak{g}_v$ be the Lie algebra of G_v and $\mathfrak{g}_v = \mathfrak{k}_v + \mathfrak{p}_v$ its Cartan decomposition. Given $\{\rho, X\}$, $\{\rho', X'\}$, $f \in C_\rho(\Gamma)$, $h \in C_{\rho'}(\Gamma)$, and a positive integer p, we say that (f, h) *is an integrable pair of type* (p, v), if $\psi(Y_1 \cdots Y_\mu \tilde{f})\psi'(Y_1' \cdots Y_\nu' \tilde{h})$ belongs to $L^1(\Gamma\backslash G_{\mathbf{a}})$ for every $\psi \in S_1(X)$, $\psi' \in S_1(X')$, and every Y_i, $Y_j' \in \mathfrak{p}_v$ with $\mu \geq 0$ and $\nu \geq 0$ such that $\mu + \nu = p$ or $\mu + \nu = p - 1$. Now the first resp. second formula of Theorem 12.15 is valid if

(12.36) $\quad (f, g)$ *resp.* (f, h) *is an integrable pair of type* (p, v).

The reason for this is explained in [S94b, p.173]. As noted in [S90, p.257], the formulas are valid if $\tilde{f}, \tilde{g}, \tilde{h}$ are C^∞ vectors in $L^2(\Gamma\backslash G_{\mathbf{a}})$. Another sufficient condition is that all the holomorphic and anti-holomorphic derivatives of f are rapidly decreasing and g, h are slowly increasing at the cusps of G. Sufficient conditions for the last four formulas of Theorem 12.15 can be given in a similar manner or in the style of (12.36), since they are straightforward consequences of the first two formulas.

The relationship between the formulation on H_v and that on G_v is explained in [S90, §7] and [S94b, p.150]. See also Section A8. of the present book.

12.16. Corollary. *Put* $L_{\rho,v} = -\theta D_{\rho\otimes\sigma_v^1,v} E_v$ *for each* $v \in \mathbf{a}'$ *and put also* $\Lambda = \sum_{v\in\mathbf{a}'} c_v L_{\rho,v}$ *with some fixed positive real numbers* c_v, *where* $\mathbf{a}'$ *is defined by (12.5). Suppose* $f \in C_\rho$ *and* $(f, E_v f)$ *is an integrable pair of type* $(1, v)$ *for every* $v \in \mathbf{a}'$. *Then* f *is holomorphic if and only if* $\Lambda f = 0$.

PROOF. If f is holomorphic, then $E_v f = 0$, and so $\Lambda f = 0$. Conversely, if $\Lambda f = 0$, then by the second equality of Theorem 12.15 we have $\sum_{v\in\mathbf{a}'} c_v \langle E_v f, E_v f \rangle = \langle f, \Lambda f \rangle = 0$ under the given integrability condition. Thus $E_v f = 0$ for every $v \in \mathbf{a}'$, and hence f is holomorphic.

12.17. Let us now show that in the one-dimensional case the operators of this section have simple expressions. Thus take $G_{\mathbf{a}} = SL_2(\mathbf{R})$, $H = \bigl\{\, z \in \mathbf{C} \bigm| \mathrm{Im}(z) > 0 \bigr\}$, $\mathfrak{K}_0 = \mathbf{C}^\times$, and $T = \mathbf{C}$. Then for $z = x + iy \in T$ we have $\xi(z) = \eta(z) = 2y$. We first define operators ε, δ_k, and L_k on H formally by

$$\varepsilon f = 4y^2\partial f/\partial\overline{z}, \qquad \delta_k f = y^{-k}(\partial f/\partial z)(y^k f)\,, \tag{12.37}$$

$$L_k = -\delta_{k-2}\varepsilon = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + 2iky\frac{\partial}{\partial\overline{z}}\,, \tag{12.38}$$

and define also δ_k^p for $0 \le p \in \mathbf{Z}$ inductively by

$$\delta_k^{p+1} = \delta_{k+2p}\delta_k^p, \quad \delta_k^1 = \delta_k, \quad \delta_k^0 = 1. \tag{12.39}$$

We now consider $\{\rho, \mathbf{C}\}$ with $\rho(a) = a^k$ with $k \in \mathbf{Z}$. We can put $S_p(T) = \mathbf{C}$ by identifying an element h of $S_p(T)$ with $h(1)$. Then, dropping the subscript v, we have $Df = \partial f/\partial z$, $\overline{D}f = \partial f/\partial\overline{z}$, and (12.18) becomes $D_\rho f = y^{-k}(\partial f/\partial z)(y^k f)$, which is exactly $\delta_k f$ of (12.37). Similarly $Ef = 4y^2\partial f/\partial\overline{z} = \varepsilon f$ with the above ε. Since $(\rho\otimes\tau^p)(a) = a^{k+2p}$, from Proposition 12.10 (1) we see that D_ρ^p coincides with the above δ_k^p; thus Proposition 10.2 (2) specialized to the present case gives

$$(\delta_k^p f)\|_{k+2p}\,\alpha = \delta_k^p(f\|_k\,\alpha), \quad (\varepsilon f)\|_{k-2}\,\alpha = \varepsilon(f\|_k\,\alpha) \qquad (\alpha \in SL_2(\mathbf{R})). \tag{12.40}$$

Also we have $Z = S_p(T) = \mathbf{C}$, and L_ρ^Z of (12.31) for $Z = S_1(T)$ is exactly L_k of (12.38), which has the property $(L_k f)\|_k\,\alpha = L_k(f\|_k\,\alpha)$ for every $\alpha \in SL_2(\mathbf{R})$.

13. Nearly holomorphic functions

We start with an easy lemma.

13.1. Lemma. *Let $X_1, \dots, X_n$ be mutually commutative C^∞ vector fields on an N-dimensional C^∞ manifold U and $r_1, \dots, r_n$ be elements of $C^\infty(U)$ such that the $n\times n$-matrix $(X_j r_k)$ is everywhere invertible; we assume that $n \le N$. (Here the X_k and the r_k are complex-valued.) Define vector fields $Y_1, \dots, Y_n$ by $Y_i = \sum_{j=1}^n b_{ij}X_j$ with the functions b_{ij} determined by $\sum_{j=1}^n b_{ij}X_j r_k = \delta_{ik}$. Then the Y_i are mutually commutative.*

PROOF. Since $X_iX_j = X_jX_i$, we can easily verify that $Y_pY_q - Y_qY_p = \sum_j c_j^{pq}X_j$ with $c_j^{pq} \in C^\infty(U)$. Now $Y_pY_qr_k = Y_p\sum_{j=1}^n b_{qj}X_jr_k = Y_p\delta_{qk} = 0$ for every p and q, and hence $\sum_j c_j^{pq}X_jr_k = 0$ for every k. Since $\det(X_jr_k) \ne 0$, we have $c_j^{pq} = 0$, so that $Y_pY_q - Y_qY_p = 0$ as expected.

13.2. We now consider an n-dimensional complex manifold W, and take n elements $r_1, \dots, r_n$ of $C^\infty(W)$ with the following property:

(13.1) *Every point of W has a small neighborhood U on which there exist local complex coordinate functions $z_1, \dots, z_n$ such that $(\partial r_k/\partial\overline{z}_j)_{j,k=1}^n$ is invertible everywhere on U.*

Then we can define $2n$ vector fields $\partial/\partial r_k$ and $\partial/\partial\overline{r}_k$ for $1 \le k \le n$ by the relations

$$\partial/\partial\overline{z}_j = \sum_{k=1}^n(\partial r_k/\partial\overline{z}_j)\partial/\partial r_k, \qquad \partial/\partial z_j = \sum_{k=1}^n(\partial\overline{r}_k/\partial z_j)\partial/\partial\overline{r}_k. \tag{13.2}$$

It can easily be seen that these vector fields do not depend on the choice of local coordinates, and so they are meaningful on the whole W. By the above lemma each set of n vector fields $\partial/\partial r_1, \dots, \partial/\partial r_n$ or $\partial/\partial\overline{r}_1, \dots, \partial/\partial\overline{r}_n$ are mutually commutative; however, $\partial/\partial\overline{r}_i$ and $\partial/\partial r_j$ do not necessarily commute, as can easily be seen by a counterexample, even when $n = 1$. In view of the commutativity, we can naturally speak of $\partial^m/\partial r_{i_1}\cdots\partial r_{i_m}$ and $\partial^m/\partial\overline{r}_{i_1}\cdots\partial\overline{r}_{i_m}$ with no ambiguity.

The reason for choosing the symbols $\partial/\partial r_i$ and $\partial/\partial \overline{r}_i$ for these vector fields will be explained by the first assertion of the following lemma.

13.3. Lemma. (1) *Let $g(z, w)$ be a C^∞ function of $(z, w) \in W \times V$ with an open subset V of $\mathbf{C}^n$, holomorphic in w, and let $h(z, w) = \partial^e g/\partial w_{i_1} \cdots \partial w_{i_e}$. If g is holomorphic in z, then $h\big(z, r(z)\big) = (\partial^e/\partial r_{i_1} \cdots \partial r_{i_e})g\big(z, r(z)\big)$ whenever $g\big(z, r(z)\big)$ is meaningful. Similarly, if $g(z, w)$ is antiholomorphic in $z \in W$, then $h\big(z, \overline{r}(z)\big) = (\partial^e/\partial \overline{r}_{i_1} \cdots \partial \overline{r}_{i_e})g\big(z, \overline{r}(z)\big)$.*

(2) *The r_i are algebraically independent over the field of all meromorphic functions on W.*

(3) *A C^∞ function f on W is a polynomial in $r_1, \ldots, r_n$ of degree $< e$ with holomorphic functions as coefficients if and only if all the derivatives of the form $\partial^e f/\partial r_{i_1} \cdots \partial r_{i_e}$ are 0.*

PROOF. Since $\partial/\partial r_i$ (resp. $\partial/\partial \overline{r}_i$) annihilates holomorphic (resp. antiholomorphic) functions, assertion (1) can be verified easily. To prove (2), let $X = (X_1, \ldots, X_n)$ be a set of indeterminates, and $Q(X)$ a polynomial of degree $\leq m$ whose coefficients are holomorphic functions on W. We prove by induction on m that $Q = 0$ if $Q(r) = 0$. This is trivial if $m = 0$. Suppose $Q(r) = 0$ with $m > 0$. Then by (1), $(\partial Q/\partial X_i)(r) = (\partial/\partial r_i)Q(r) = 0$, and hence $\partial Q/\partial X_i = 0$ by the induction assumption. Thus Q is a constant, and must be 0. This proves (2). We prove the "if"-part of (3) by induction on e. The case $e=1$ is obvious. Suppose all the derivatives of the form $\partial^e f/\partial r_{i_1} \cdots \partial r_{i_e}$ are 0 with some $e > 1$. Then by the induction assumption for $e-1$, we can find, for each i, a polynomial $Q_i(X)$ of degree $<e-1$ whose coefficients are holomorphic functions on W such that $\partial f/\partial r_i = Q_i(r)$. Put $Q_{ij} = \partial Q_i/\partial X_j$. By (1), we have $Q_{ij}(r) = \partial^2 f/\partial r_i \partial r_j = Q_{ji}(r)$, and hence $Q_{ij} = Q_{ji}$ by (2). Therefore we can find a polynomial $P(X)$ of degree $< e$ whose coefficients are holomorphic functions on W such that $\partial P/\partial X_i = Q_i$ for every i. (Indeed, for $x = (x_i) \in \mathbf{R}^n$ and $a \in \mathbf{R}^n$ define $P(a) = \int_c \sum_{i=1}^n Q_i(x)dx_i$ with any path c connecting 0 to a. Then $\partial P/\partial x_i = Q_i(x)$. Since $P(x) = \int_0^1 \sum_{i=1}^n Q_i(tx)x_i dt$, we easily see that P is a polynomial of degree $<e$ with holomorphic functions on W as coefficients.) Then $(\partial/\partial r_i)\big(f - P(r)\big) = 0$, and hence $f - P(r)$ is holomorphic. This completes our induction. The "only if"-part of (3) follows immediately from (1).

13.4. Let W be a complex Kähler manifold of complex dimension N with a fundamental 2-form Ω; let $\Omega = i\sum_{p,q=1}^N h_{pq} dz_q \wedge d\overline{z}_p$ with local complex coordinate functions $z_1, \ldots, z_N$ in a coordinate neighborhood U. We can then define N vector fields $R_1, \ldots, R_N$ on U by the relations

$$\partial/\partial \overline{z}_p = \sum_{q=1}^N h_{pq} R_q \qquad (1 \leq p \leq N). \tag{13.3}$$

It is well-known that Ω, for a sufficiently small U, can be given in the form $\Omega = i\sum_{p,q=1}^N \partial^2\varphi/\partial z_q \partial \overline{z}_p \cdot dz_q \wedge d\overline{z}_p$ with a real-valued function φ on U. Define N functions r_q on U by $r_q = \partial\varphi/\partial z_q$. Then $h_{pq} = \partial r_q/\partial \overline{z}_p$, and therefore we can apply the principle of §13.2 to the present case to find that $R_q = \partial/\partial r_q$.

Now the R_p may depend on the choice of the z_p. However, if $w_1, \ldots, w_N$ are coordinate functions in a coordinate neighborhood U' and vector fields $S_1, \ldots, S_N$ are defined relative to the w_p in the same manner, then $R_p = \sum_q \partial z_p/\partial w_q \cdot S_q$ on $U \cap U'$. Therefore if we denote by $\mathcal{N}^{e-1}(U)$ the set of all elements of $C^\infty(U)$

annihilated by $X_{\nu_1} \cdots X_{\nu_e}$ for all $(\nu_1, \dots, \nu_e) \in \{1, \dots, N\}^e$, this is well-defined independently of the choice of the z_p. Then, for every open subset V of W let $\mathcal{N}^e(V)$ denote the set of all $f \in C^\infty(V)$ such that the restriction of f to any coordinate neighborhood U belongs to $\mathcal{N}^e(U)$, and let $\mathcal{N}(V) = \bigcup_{e=0}^{\infty} \mathcal{N}^e(V)$. We call an element of $\mathcal{N}(V)$ a *nearly holomorphic function* on V relative to Ω. If the r_p are defined on U as above, then Lemma 13.3 (3) shows that $\mathcal{N}^e(U)$ consists of all the polynomials in the r_p whose coefficients are holomorphic functions on U. Clearly $\mathcal{N}^0(U)$ is the set of all holomorhic functions on U. In our applications W is a hermitian symmetric space, and we can define the functions r_p on the whole W, so that the last statement is applicable to $\mathcal{N}^e(W)$.

13.5. We now consider the space $\mathcal{H}$ of (12.4a). We first define a matrix-valued function r_v and a scalar-valued function δ_v on H_v as follows:

$$r_v(z) = -\xi(z)^{-1}\overline{z} = -\overline{z} \cdot {}^t\eta(z)^{-1} \qquad \text{(Types AB, CB)}, \tag{13.4a}$$

$$r_v(z) = ({}^tz - \overline{z})^{-1} \qquad \text{(Types AT, CT)}, \tag{13.4b}$$

$$\delta_v(z) = \begin{cases} \det\left[2^{-1}\eta(z)\right] & \text{(Types AT, CT)}, \\ \det\left[\eta(z)\right] & \text{(Types AB, CB)}. \end{cases} \tag{13.5}$$

The function δ_v is the same as what was defined in (3.21). We then put $r(z) = \big(r_v(z_v)\big)_{v\in\mathbf{a}}$ and $\delta(z) = \big(\delta_v(z_v)\big)_{v\in\mathbf{a}}$ for $z = (z_v)_{v\in\mathbf{a}} \in \mathcal{H}$; we do not define r_v if G_v is compact. We fix one $v \in \mathbf{a}$ and consider the behavior of a function on $\mathcal{H}$ only with respect to z_v. For simplicity let us drop temporarily the subscript v from the objects T_v, D_v, $\overline{D}_v$, ξ_v, etc. Then for $u \in T$ we have

$$\eta^{-1}(D\eta)(u) = {}^tru, \tag{13.6a}$$

$$\xi^{-1}(D\xi)(u) = r \cdot {}^tu, \tag{13.6b}$$

$$(Dr)(u) = -r \cdot {}^tur, \tag{13.6c}$$

$$(\overline{D}r)(u) = -\xi^{-1}u \cdot {}^t\eta^{-1}. \tag{13.6d}$$

The first two formulas can be verified in a straightforward way. As for (13.6d) for Type AB, for example, we have $\xi r = -\overline{z}$, and so $(\overline{D}\xi)(u)r + \xi(\overline{D}r)(u) = -u$. Thus $(\overline{D}r)(u) = -\xi^{-1}\big(u + (\overline{D}\xi)(u)r\big) = -\xi^{-1}(u - u \cdot {}^tzr) = -\xi^{-1}u({}^t\eta + {}^tz\overline{z}) \cdot {}^t\eta^{-1} = -\xi^{-1}u \cdot {}^t\eta^{-1}$. All the remaining cases can be verified in the same fashion.

13.6. Lemma. *Let ζ and ψ_Z be as in Theorem 12.13; let $s \in \mathbf{C}$. Then*

$$\zeta(\mathcal{D})\delta(z)^s = \psi_Z(s)\delta(z)^s\zeta\big(r(z)\big). \tag{13.7}$$

Proof. For Types AB and CB we have $\delta(z) = \det(1_n - z^*z)$. Taking $(-z^*, 1)$ to be (c, d) in Theorem 12.13, we obtain the desired formula. The other cases can be handled in the same manner.

13.7. We now note three basic formulas:

$$D(\delta^s)(u) = s \cdot \delta^s \mathrm{tr}({}^tru), \tag{13.8a}$$

$$(D \log \delta)(u) = \mathrm{tr}({}^tru), \tag{13.8b}$$

$$(\overline{D}D \log \delta)(u, v) = -\mathrm{tr}({}^tu\xi^{-1}v \cdot {}^t\eta^{-1}) \qquad (u, v \in T). \tag{13.8c}$$

Indeed, take $Z = S_1(T)$ and $\zeta(x) = \mathrm{tr}({}^t ux)$ with $u \in T$ in Lemma 13.6. Then $\zeta(\mathcal{D}_v)f = (Df)(u)$, and so we obtain (13.8a). The second equality follows immediately from the first one. Combining it with (13.6d), we obtain (13.8c).

Since ξ and η are hermitian and positive definite, we see that H is a Kähler manifold with $i\overline{\partial}\partial \log \delta$ as its fundamental 2-form. For the product space $\mathcal{H} = \prod_{v \in \mathbf{a}} H_v$, we have to take $i\sum_{v\in\mathbf{a}} \overline{\partial}\partial \log \delta_v(z_v)$. Then (13.8b) shows that the entries of the functions $(r_v)_{v\in\mathbf{a}}$ are exactly the r_p discussed in §13.4.

13.8. Take an $\mathbf{R}$-rational basis $\{\varepsilon_\nu\}_{\nu\in N}$ of T_v over $\mathbf{C}$ and put $z_v = \sum_{\nu\in N} z_{v\nu}\varepsilon_\nu$ and $r_v = \sum_{\nu\in N} r_{v\nu}\varepsilon_\nu$. Following the general principle of §13.2, define vector fields $\partial/\partial r_{v\nu}$ and $\partial/\partial \overline{r}_{v\nu}$ by

$$(13.9)\qquad \partial/\partial z_{v\mu} = \sum_{\nu\in N} (\partial \overline{r}_{v\nu}/\partial z_{v\mu})\partial/\partial \overline{r}_{v\nu}, \quad \partial/\partial \overline{z}_{v\mu} = \sum_{\nu\in N} (\partial r_{v\nu}/\partial \overline{z}_{v\mu})\partial/\partial r_{v\nu}.$$

These are well-defined in view of (13.6d). Now, for $u = \sum_{\nu\in N} u_{v\nu}\varepsilon_\nu \in T_v$, we have

$$(13.10)\qquad (C_v f)(u) = -\sum_{\nu\in N} u_{v\nu}\partial f/\partial \overline{r}_{v\nu}, \quad (E_v f)(u) = -\sum_{\nu\in N} u_{v\nu}\partial f/\partial r_{v\nu}.$$

Indeed, $(D_v f)(u) = \sum_{\nu\in N} u_{v\nu}\partial f/\partial z_{v\nu} = \sum_{\nu\in N}(D_v\overline{r}_{v\nu})(u)\partial f/\partial \overline{r}_{v\nu}$, and therefore $(D_v f)({}^t\xi_v u\eta_v) = \sum_{\nu\in N}(D_v\overline{r}_{v\nu})({}^t\xi_v u\eta_v)\partial f/\partial \overline{r}_{v\nu}$, which together with (13.6d) and (12.12b) proves the first equality of (13.10); the second one can be proved similarly.

Since the vector fields $\partial/\partial \overline{r}_{v\nu}$ mutually commute, from (13.10) we see that the values of $C_v^e f$ are symmetric elements of $Ml_e(T_v, X)$; the same is true for $E_v^e f$ because of the commutativity of the $\partial/\partial r_{v\nu}$.

Taking the basis $\{\varepsilon'_\nu\}_{\nu\in N}$ of T_v dual to $\{\varepsilon_\nu\}_{\nu\in N}$, we define symbols $\partial/\partial r_v$ and $\partial/\partial \overline{r}_v$ by

$$(13.11)\qquad \partial/\partial r_v = \sum_{\nu\in N} \varepsilon'_\nu \partial/\partial r_{v\nu}, \qquad \partial/\partial \overline{r}_v = \sum_{\nu\in N} \varepsilon'_\nu \partial/\partial \overline{r}_{v\nu}.$$

These are independent of the choice of bases of T_v. Given $g \in S_p(T_v)$, we define $g(\partial/\partial r_v)$ by

$$(13.12)\quad g(\partial/\partial r_v) = g_*(\partial/\partial r_v, \ldots, \partial/\partial r_v) = \sum g_*(\varepsilon'_{\nu_1}, \ldots, \varepsilon'_{\nu_p})\partial^p/\partial r_{v\nu_1}\cdots\partial r_{v\nu_p},$$

where $(\nu_1, \ldots, \nu_p)$ runs over N^p, and define $g(\partial/\partial \overline{r}_v)$ similarly. Then we have

$$(13.13)\qquad g(\partial/\partial \overline{r}_v)f = (-1)^p[g, C_v^p f], \qquad g(\partial/\partial r_v)f = (-1)^p[g, E_v^p f].$$

These follow from (12.8) and (13.10) immediately.

The notation being the same as in (12.20) and (12.23), we have

$$(13.14a)\qquad (D_\rho^Z f)(\zeta) = (-1)^p \rho(\Xi)^{-1}\zeta'(\partial/\partial \overline{r}_v)\big(\rho(\Xi)f\big),$$

$$(13.14b)\qquad (E^Z f)(\zeta) = (-1)^p \zeta(\partial/\partial r_v)f$$

for every $\zeta \in Z$, where $\zeta' = \sigma_v^p(\Xi)\zeta$. Indeed, combining (12.23) with (12.17) and (12.9), we obtain

$$(D_\rho^Z f)(\zeta) = \big[\zeta, (\rho\otimes\tau_v^p)(\Xi)^{-1}C_v^p\big(\rho(\Xi)f\big)\big] = \rho(\Xi)^{-1}\big[\sigma_v^p(\Xi)\zeta, C_v^p\big(\rho(\Xi)f\big)\big].$$

Applying (13.13) to the last quantity, we obtain (13.14a). Formula (13.14b) follows directly from (12.23) and (13.13).

13.9. Lemma. *Let* $\rho(a, b) = \det(b)^k$ *and* $f(z) = \big(\prod_{v\in\mathbf{a}} \delta_v(z_v)^s\big) \|_k \alpha$ *with* $k \in \mathbf{Z}^{\mathbf{a}}$, $\alpha \in G_{\mathbf{a}}$, *and* $s \in \mathbf{C}$; *let* Z *be an irreducible subspace of* $S_p(T_v)$ *and* ψ_Z *be as in Theorem 12.13. Then for every* $\zeta \in Z$ *we have*

$$(D_\rho^Z f)(\zeta) = \begin{cases} i^p \psi_Z(-k_v - s)\zeta\big((\xi^{-1}\lambda_\alpha^* \cdot {}^t\mu_\alpha^{-1})_v\big) f & \text{(Types AT, CT)}, \\ \psi_Z(-k_v - s)\zeta\big((\xi^{-1}\lambda_\alpha^* \cdot \overline{\alpha z} \cdot {}^t\mu_\alpha^{-1})_v\big) f & \text{(Types AB, CB)}, \end{cases}$$

$$(E^Z f)(\zeta) = \begin{cases} (-i)^p \psi_Z(-s)\zeta\big(({}^t\lambda_\alpha \widehat{\mu}_\alpha \eta)_v\big) f & \text{(Types AT, CT)}, \\ \psi_Z(-s)\zeta\big(({}^t(\alpha z)\lambda_\alpha \widehat{\mu}_\alpha \eta)_v\big) f & \text{(Types AB, CB)}. \end{cases}$$

PROOF. Clearly it is sufficient to consider only $\delta_v(z_v)^s$; so we take $\alpha \in G_v$ and drop the subscript v. Put $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with d of size n. For Types AT and CT we have $\overline{r} = (z^* - z)^{-1}$, and therefore

$$\rho(\Xi)f = 2^{nk}\delta^{k+s} j_\alpha^{-k-s}(\overline{j}_\alpha)^{-s} = 2^{-ns} i^{n(k+s)}(\overline{j}_\alpha)^{-s} \det\big((cz+d)\overline{r}\big)^{-k-s}.$$

Since $z = z^* - \overline{r}^{-1}$, we have $(cz+d)\overline{r} = (cz^* + d)\overline{r} - c$. Now $\zeta'(\partial/\partial\overline{r})g = 0$ for every anti-holomorphic g, so that for every $\zeta' \in Z$,

$$(*) \qquad \zeta'(\partial/\partial\overline{r})\big(\rho(\Xi)f\big) = 2^{-ns} i^{n(k+s)}(\overline{j}_\alpha)^{-s}\zeta'(\partial/\partial\overline{r})\det(A\overline{r} + B)^{-k-s},$$

where $(A,\, B) = (cz^* + d,\, -c)$. In view of Lemma 13.3 (1) we can apply Theorem 12.13 with $\partial/\partial\overline{r}$ as $\mathcal{D}$ there. (Notice that $A \cdot {}^tB = B \cdot {}^tA$ for Type CT.) Thus the quantity of (*) equals

$$(**) \qquad \psi_Z(-k-s)\rho(\Xi)f \cdot \zeta'\big({}^tA \cdot {}^t(A\overline{r}+B)^{-1}\big).$$

Now we employ (13.14a) in the present setting. Then our task is to calculate $\zeta'\big({}^tA \cdot {}^t(A\overline{r}+B)^{-1}\big)$ with $\zeta'(u) = \zeta(\xi^{-1}u \cdot {}^t\eta^{-1})$. Since ${}^tA \cdot {}^t(A\overline{r}+B)^{-1} = {}^t(cz^* + d) \cdot {}^t(cz+d)^{-1}\widehat{r}$, we obtain the formula for $(D_\rho^Z f)(\zeta)$ for Types AT and CT.

The formula for $(E^Z f)(\zeta)$ is simpler. Since $\overline{z} = {}^tz - r^{-1}$, we see that $\delta^s\|_k\, \alpha$ is a holomorphic factor times $\det\big[(\overline{c}\cdot{}^tz + \overline{d})r - \overline{c}\big]^{-s}$. Applying $\zeta(\partial/\partial r)$ to this, from (13.14b) and Theorem 12.13 we obtain the desired formula.

The argument for Types AB and CB is similar but requires modifications. We have $1 - z^*\overline{r} = (\eta + z^*z)\eta^{-1} = \eta^{-1}$, and so $(cz+d)\eta^{-1} = cz\eta^{-1} + d\eta^{-1} = -c\overline{r} + d(1 - z^*\overline{r}) = (-c - dz^*)\overline{r} + d$, and so

$$\rho(\Xi)f = \delta^{k+s} j_\alpha^{-k-s}(\overline{j}_\alpha)^{-s} = (\overline{j}_\alpha)^{-s}\det(A\overline{r} + B)^{-k-s}$$

with $(A,\, B) = (-c-dz^*,\, d)$. By Theorem 12.13, $\zeta'(\partial/\partial\overline{r})\big(\rho(\Xi)f\big)$ equals (**). From (3.14) we see that $A = -(\alpha z)^*\overline{\lambda_\alpha(z)}$, and hence $-\xi^{-1} \cdot {}^tA \cdot {}^t(A\overline{r} + B)^{-1} \cdot {}^t\eta^{-1} = \xi^{-1}\lambda_\alpha^* \cdot (\overline{\alpha z}) \cdot {}^t\mu_\alpha^{-1}$, which gives the desired formula for $(D_\rho^Z f)(\zeta)$. As for $(E^Z f)(\zeta)$, we observe that $1 - {}^tzr = {}^t\eta^{-1}$, and reduce the problem to $\zeta(\partial/\partial r)\det\big[(-\overline{c} - \overline{d}\cdot {}^tz)r + \overline{d}\big]^{-s}$. Then the formula can be obtained in a similar fashion.

In the above proof for Types AB and CB we obtained $c+dz^*$. This is not a factor of automorphy, but may be called a quasi-factor of automorphy for the following reason. Put $\kappa_\alpha(z) = \overline{c} + \overline{d} \cdot {}^tz$. Since $\overline{\kappa_\alpha(z)}$ is the lower left $n \times m$-block of $\alpha B(z)$ with $B(z)$ of (3.10), we can easily verify that

$$(13.15) \qquad \kappa_\alpha(z) = {}^t(\alpha z)\lambda_\alpha(z), \qquad \kappa_{\alpha\beta}(z) = \kappa_\alpha(\beta z)\lambda_\beta(z).$$

Also we can put $\zeta\big((\xi^{-1}\lambda_\alpha^* \cdot \overline{\alpha z} \cdot {}^t\mu_\alpha^{-1})_v\big) = \zeta\big((\xi^{-1}\kappa_\alpha^* \cdot {}^t\mu_\alpha^{-1})_v\big)$. See [S86, §§5, 6, (5.5) and (5.6) in particular] for the formulas essentially of the same nature given in a somewhat different manner.

13.10. Lemma. *The difference* $r_v(\alpha_v z_v) - \lambda(\alpha_v, z_v)r_v(z_v) \cdot {}^t\mu(\alpha_v, z_v)$ *is holomorphic in* z *for every* $\alpha \in G_{\mathbf{a}}$.

PROOF. Dropping the subscript v for simplicity and denoting by 1_T the identity map $T \to T$, we see from (13.10) that $Er = -1_T$. Define $\rho : K^c \to GL(T)$ by $\rho(a, b)u = au \cdot {}^tb$. Then $r\|_\rho\alpha = \lambda(\alpha, z)^{-1}r(\alpha z) \cdot {}^t\mu(\alpha, z)^{-1}$. Clearly $\big[(\rho \otimes \sigma^1)(a, b)(-1_T)\big] = -1_T$, and hence by Proposition 12.10 (2) we have $E(r\|_\rho\alpha) = (Er)\|_{\rho\otimes\sigma^1}\alpha = Er$. Thus $E(r\|_\rho\alpha - r) = 0$, so that $r\|_\rho\alpha - r$ is holomorphic as expected.

13.11. Given $p = (p_v)_{v\in\mathbf{a}} \in \mathbf{Z}^{\mathbf{a}}$ with $p_v \geq 0$ for every v and a representation $\{\rho, X\}$ of $\mathfrak{K}_0$, we denote by $\mathcal{N}^p(\mathcal{H}, X)$ the set of all $f \in C^\infty(\mathcal{H}, X)$ which are polynomials in the components of $r = (r_v)_{v\in\mathbf{a}}$, of degree $\leq p_v$ in r_v, with holomorphic maps of $\mathcal{H}$ into X as coefficients. We naturally take $p_v = 0$ if $v \notin \mathbf{a}'$. Then, (13.10) together with Lemma 13.3 (3) shows that $\mathcal{N}^p(\mathcal{H}, X)$ consists of all $f \in C^\infty(\mathcal{H}, X)$ such that $E_v^{p_v+1}f = 0$ for every $v \in \mathbf{a}'$. Moreover, by Lemma 13.3 (2) the components of r_v for all v are algebraically independent over the field of all meromorphic functions on $\mathcal{H}$. For example, if we view an element f of $\mathcal{N}^p(\mathcal{H}, X)$ as a function of z_v and suppress other variables $z_{v'}$ for $v' \in \mathbf{a}, \neq v$, then

$$f(z) = \sum_{i=0}^{p_v} g_i\big(z_v, r_v(z_v)\big) \tag{13.16}$$

with a holomorphic map $g_i : H_v \to S_i(T_v, X)$ for each i, where $g_i(z_v, u)$ means the element $g_i(z_v)$ of $S_i(T_v, X)$ evaluated at $u \in T_v$. From Proposition 12.10 (2) we see that $\mathcal{N}^p(\mathcal{H}, X)$ is stable under the maps $f \mapsto f\circ\alpha$ and $f \mapsto f\|_\rho\alpha$ for every $\alpha \in G_{\mathbf{a}}$. The elements of $\bigcup_p \mathcal{N}^p(\mathcal{H}, X)$ are called (X-valued) *nearly holomorphic* functions on $\mathcal{H}$, as defined at the beginning of this section. We note here

(13.17) *With* g_i *as in (13.16) we have* $E_v^{p_v}f = (-1)^{p_v}p_v!g_{p_v}$, *that is,* $(E_v^{p_v}f)(u) = (-1)^{p_v}p_v!g_{p_v}(z_v, u)$ *for* $u \in T_v$.

Indeed, $E_v^{p_v}$ kills $g_i\big(z_v, r_v(z_v)\big)$ for $i < p_v$. Now, by (13.13), $\big[h, E_v^{p_v}g_{p_v}(z_v, r_v(z_v))\big] = (-1)^{p_v}h(\partial/\partial r_v)g_{p_v}(z_v, r_v(z_v))$ for every $h \in S_{p_v}(T_v)$. By (12.28) and Lemma 13.3 the last quantity is $(-1)^{p_v}p_v!\big[h, g_{p_v}(z_v, *)\big]$, which proves (13.17).

For a congruence subgroup Γ of G we denote by $\mathcal{N}_\rho^p(\Gamma)$ the subset of $C_\rho(\Gamma)\cap \mathcal{N}^p(\mathcal{H}, X)$ consisting of the functions satisfying the *cusp condition*, which is required only when G is isogenous to $SL_2(\mathbf{Q})$. (For the precise statement of the cusp condition see §13.12 below.) We then denote by $\mathcal{N}_\rho^p$ the union of $\mathcal{N}_\rho^p(\Gamma)$ for all Γ. Clearly $\mathcal{N}_\rho^p\|_\rho\alpha = \mathcal{N}_\rho^p$ for every $\alpha \in G$. Since $\mathcal{N}^0(\mathcal{H}, X)$ consists of all holomorphic maps of $\mathcal{H}$ to X, we see that $\mathcal{N}_\rho^0(\Gamma) = \mathcal{M}_\rho(\Gamma)$ and $\mathcal{N}_\rho^0 = \mathcal{M}_\rho$ with the symbol $\mathcal{M}_\rho$ of Section 5.

In this book we consider almost exclusively nearly holomorphic functions on $\mathcal{H}$ of the above type, which form the most important class from the number-theoretical viewpoint. However, we can also determine such functions on hermitian symmetric spaces of compact type. For details, the reader is referred to [S87a].

13.12. Continuing the discussion of §12.17, let us now consider nearly holomorphic functions on the upper half plane H. In this case $r = (2iy)^{-1}$, and so $\mathcal{N}^p(H, \mathbf{C})$ consists of the functions of the form $\sum_{\nu=0}^p y^{-\nu}g_\nu(z)$ with holomorphic functions g_ν on H, as we already mentioned in the introduction. Since $\partial r/\partial\bar{z} = -(4y^2)^{-1}$, we have

(13.18) $$\partial/\partial r = -4y^2\partial/\partial\overline{z}, \qquad \partial/\partial\overline{r} = -4y^2\partial/\partial z.$$

We say that an element f of $\mathcal{N}^p(H, \mathbf{C}) \cap C_\rho(\Gamma)$, for $\Gamma \subset SL_2(\mathbf{Q})$ and $\rho(x) = x^k$ with $k \in \mathbf{Z}$, satisfies the cusp condition if it satisfies

(13.18a) *For every* $\alpha \in SL_2(\mathbf{Z})$ *we have*
$$(f\|_k\,\alpha)(z) = \textstyle\sum_{\nu=0}^{p}(\pi y)^{-\nu}\sum_{n=0}^{\infty} c_{\alpha\nu n}\exp(2\pi i n z/N_\alpha)$$
with $c_{\alpha\nu n} \in \mathbf{C}$ *and* $0 < N_\alpha \in \mathbf{Z}$.

13.13. Let $\{\rho, X\}$ be a representation of $\mathfrak{K}_0$. So far we considered $D^p_{\rho,v}$ and D^Z_ρ for a fixed $v \in \mathbf{a}$, $0 \le p \in \mathbf{Z}$, and $Z \subset S_p(T_v)$. We now generalize this by considering the derivatives with respect to the variables on the whole $\mathcal{H}$. We put $T = \prod_{v\in\mathbf{a}} T_v$ and $T^e = \prod_{v\in\mathbf{a}} T_v^{e_v}$ for $e = (e_v)_{v\in\mathbf{a}} \in \mathbf{Z}^{\mathbf{a}}$ with $e_v \ge 0$; we take $e_v = 0$ if $v \notin \mathbf{a}'$. We then denote by $Ml_e(T, X)$ the vector space of all $\mathbf{C}$-multilinear maps of T^e (that is, $\mathbf{C}$-linear on each single factor T_v of T^e) into X, and by $S_e(T, X)$ the vector space of all polynomial maps of T into X homogeneous of degree e_v in the variable on T_v for each v. In particular we put $S_e(T) = S_e(T, \mathbf{C})$. Given $h \in S_e(T, X)$, there exists a unique element h_* of $Ml_e(T, X)$ which is symmetric on $T_v^{e_v}$ for every $v \in \mathbf{a}$, and such that
$$h(y) = h_*\big(\overbrace{y_v, \ldots, y_v}^{e_v}, \overbrace{y_w, \ldots, y_w}^{e_w}, \ldots\ \ldots\big)$$
for $y = (y_v)_{v\in\mathbf{a}} \in T$. This is an easy generalization of Lemma 12.4 (2). For $g \in S_e(T)$ and $g \in S_e(T, X)$ we define $[g, h] \in X$ by an obvious generalization of (12.8); also we define representations $\{\rho \otimes \tau^e, S_e(T, X)\}$ and $\{\rho \otimes \sigma^e, S_e(T, X)\}$ of $\mathfrak{K}_0$ by

(13.19a) $$\big[(\rho \otimes \tau^e)(a, b)h\big](u) = \rho(a, b)h\big(({}^t a_v u_v b_v)_{v\in\mathbf{a}}\big),$$

(13.19b) $$\big[(\rho \otimes \sigma^e)(a, b)h\big](u) = \rho(a, b)h\big((a_v^{-1}u_v \cdot {}^t b_v^{-1})_{v\in\mathbf{a}}\big),$$

for $(a, b) \in \mathfrak{K}_0$, $h \in S_e(T, X)$, and $u \in T$. We write these representations simply τ^e and σ^e if $X = \mathbf{C}$ and ρ is trivial.

Now we define operators D^e, $\overline{D}^e$, C^e, and E^e acting on C^∞ functions on $\mathcal{H}$ by

(13.20) $$D^e = \textstyle\prod_{v\in\mathbf{a}} D_v^{e_v}, \quad \overline{D}^e = \prod_{v\in\mathbf{a}} \overline{D}_v^{e_v}, \quad C^e = \prod_{v\in\mathbf{a}} C_v^{e_v}, \quad E^e = \prod_{v\in\mathbf{a}} E_v^{e_v}.$$

These send $C^\infty(\mathcal{H}, X)$ into $C^\infty\big(\mathcal{H}, S_e(T, X)\big)$. We then define an operator D^e_ρ by

(13.21) $$D^e_\rho f = (\rho \otimes \tau^e)(\Xi)^{-1}C^e[\rho(\Xi)f].$$

Since $S_e(T)$ is isomorphic to $\bigotimes_{v\in\mathbf{a}} S_{e_v}(T_v)$, every irreducible subspace of $S_e(T)$ has multilicity 1. Therefore, for every $\mathfrak{K}_0$-stable subspace Z of $S_e(T)$ we can define a projection map φ_Z of $S_e(T) \otimes X$ onto $Z \otimes X$. Then we identify $S_e(T, X)$ with $S_e(T) \otimes X$ by the generalization of (12.19), and define $D^Z_\rho f$ and E^Z by

(13.22) $$D^Z_\rho f = \varphi_Z D^e_\rho f, \qquad E^Z f = \varphi_Z E^e f.$$

Denoting by τ_Z and σ_Z the restrictions of τ^e and σ^e to Z, we have obvious generalizations of (12.21), (12.22), (12.23) and (12.24a, b, c).

For $a, b \in \mathbf{Z}^{\mathbf{a}}$ let us write $a \le b$ if $a_v \le b_v$ for every $v \in \mathbf{a}$, and $a < b$ if $a \le b$ and $a \ne b$. Then, as a generalization of (13.16), for $f \in \mathcal{N}^p(\mathcal{H}, X)$ we can put

(13.23) $$f(z) = \sum_{a\le p} g_a\big(z, r(z)\big)$$

with a holomorphic map $g_a : \mathcal{H} \to S_a(T, X)$ for each a, where $g_a(z, u)$ means the element $g_a(z)$ of $S_a(T, X)$ evaluated at $u \in T$. Then (13.17) has the following generalization:

$$(E^p f)(u) = \Bigl(\prod_{v \in \mathbf{a}} (-1)^{p_v} p_v!\Bigr) g_p(z, u) \qquad (u \in T). \tag{13.24}$$

13.14. Lemma. *Let G be a Lie subgroup of $GL_n(\mathbf{C})$ and L the Lie algebra of G; let f be a C^∞ map of an interval (a, b) into G and $\rho : G \to GL_m(\mathbf{C})$ be an analytic homomorphism. Then $f(t)^{-1}\, df/dt \in L$ and*

$$(d/dt)\rho[f(t)] = \rho[f(t)]d\rho[f(t)^{-1}\, df/dt].$$

PROOF. For a fixed $t_0 \in (a, b)$ and a small $\varepsilon \in \mathbf{R}, > 0$, define a C^∞ map $g : (-\varepsilon, \varepsilon) \to L$ so that $g(0) = 0$ and $\exp\bigl(g(h)\bigr) = f(t_0)^{-1} f(t_0 + h)$. Then

$$f(t_0)^{-1} f'(t_0) = (d/dh) \exp\bigl(g(h)\bigr)_{h=0} = \lim_{h\to 0}\bigl[\exp\bigl(g(h)\bigr) - 1\bigr]/h = g'(0) \in L,$$

since g has values in the vector space L. Next,

$$\rho\bigl(f(t_0 + h)\bigr) = \rho\bigl(f(t_0)\bigr)\rho\bigl(\exp(g(h))\bigr) = \rho\bigl(f(t_0)\bigr) \exp\bigl(d\rho(g(h))\bigr),$$

and hence, similarly,

$$\frac{d}{dt}\bigl(\rho(f(t))\bigr)_{t=t_0} = \rho\bigl(f(t_0)\bigr)\frac{d}{dh} \exp\bigl(d\rho(g(h))\bigr)_{h=0} = \rho\bigl(f(t_0)\bigr)d\rho\bigl(g'(0)\bigr),$$

which combined with the equality $f(t_0)^{-1} f'(t_0) = g'(0)$ gives the desired result.

13.15. Proposition. (1) *Let p and e be elements of $\mathbf{Z}^{\mathbf{a}}$ with nonnegative components such that $p_v = e_v = 0$ for $v \notin \mathbf{a}'$. If $f \in \mathcal{N}^p(\mathcal{H}, X)$, then $E^e f \in \mathcal{N}^q\bigl(\mathcal{H}, S_e(T, X)\bigr)$ with $q_v = \mathrm{Max}(p_v - e_v, 0)$, and $D^e_\rho f \in \mathcal{N}^{p+e}\bigl(\mathcal{H}, S_e(T, X)\bigr)$.*

(2) *In particular, if f is holomorphic, then*

$$(D^e_\rho f)(z)(u) = \sum_{a<e} h_a(z)(u) + P^e_\rho\bigl(r(z), u\bigr) f(z) \qquad (u \in T) \tag{13.25}$$

with $h_a \in \mathcal{N}^a\bigl(\mathcal{H}, S_e(T, X)\bigr)$ and a map $P^e_\rho : T \times T \to \mathrm{End}(X)$ which is determined by ρ and e independently of f, and which as a function of $(u', u) \in T \times T$ is homogeneous of degree e_v in u_v and also in u'_v for every $v \in \mathbf{a}$.

(3) *Define $p^e_\rho(r) \in S_e\bigl(T, \mathrm{End}(X)\bigr)$ by $p^e_\rho(r)(u) = P^e_\rho(r, u)$ for $u \in T$. Then*

$$\begin{aligned} P^{e+v}_\rho(r, u)x = {}& \bigl[D_v\bigl(P^e_\rho(r, u)x\bigr)\bigr](u_v) + P^v_\rho(r, u)P^e_\rho(r, u)x \\ & + \bigl[d\tau^e(r_v \cdot {}^t u_v, {}^t r_v u_v)\bigl(p^e_\rho(r)x\bigr)\bigr](u) \quad \text{for} \quad x \in X, \end{aligned}$$

where we view v as an element of $\mathbf{Z}^{\mathbf{a}}$ such that $(v)_v = 1$ and $(v)_{v'} = 0$ for $v' \neq v$, and $p^e_\rho(r)x$ is the element of $S_e(T, X)$ such that $\bigl(p^e_\rho(r)x\bigr)(u) = p^e_\rho(r)(u)x = P^e_\rho(r, u)x$.

(4) $\quad P^e_\rho(au' \cdot {}^t b, {}^t a^{-1} u b^{-1})\rho(a, b) = \rho(a, b)P^e_\rho(u', u)$ *for every* $(a, b) \in \mathfrak{K}_0$.

PROOF. The assertion concerning $E^e f$ is obvious in view of (13.10) and Lemma 13.3 (1). Next, take Ξ as f in Lemma 13.14. Since $\sum_{\nu \in N} u_{v\nu}\partial/\partial z_{v\nu}$ can be written $\sum_k c_k \partial/\partial t_k$ with $c_k \in \mathbf{C}$ and real parameters t_k, from that lemma we obtain

$$\rho(\Xi)^{-1} D_v \rho(\Xi)(u') = d\rho\bigl(\xi^{-1}(D_v\xi)(u'), \eta^{-1}(D_v\eta)(u')\bigr) = d\rho(r_v \cdot {}^t u', {}^t r_v u') \tag{13.26}$$

for $u' \in T_v$, in view of (13.6a, b). Define $P^v_\rho : T \times T \to \mathrm{End}(X)$ by

$$P^v_\rho(r, u) = d\rho(r_v \cdot {}^t u_v, {}^t r_v u_v). \tag{13.27}$$

Then (13.26) together with (12.18) shows that

$$(13.28)\qquad (D_{\rho,v}f)(u') = (D_v f)(u') + P_\rho^v(r,\, u')f \qquad (u' \in T_v).$$

If f is a polynomial in r of degree $\leq p$ with holomorphic coefficients, then (13.6c) shows that $D_v f$ is a polynomial in r of degree $\leq p+v$. The same is true for $D_{\rho,v}f$, since $P_\rho^v(r,\, u)$ is linear in r. By Proposition 12.10 (1) and induction, we see that $D_\rho^e f \in \mathcal{N}^{p+e}(\mathcal{H},\, S_e(T,\, X))$. We prove (2) by induction on $\sum_{v\in\mathbf{a}} e_v$. Our assertion is trivial if $e = 0$. Now fix one v, and assume (13.25) for some e; put $h = \sum_{a<e} h_a$. and define $p_\rho^e(r)$ as in (3). By Proposition 12.10, $D_\rho^{e+v}f = D_{\rho\otimes\tau^e,\,v}(D_\rho^e f)$, and so

$$(D_\rho^{e+v}f)(u') = (D_{\rho\otimes\tau^e,\,v}h)(u') + \big[D_{\rho\otimes\tau^e,\,v}\big(p_\rho^e(r)f\big)\big](u').$$

Here we view $p_\rho^e(r)f$ as an element of $C^\infty(\mathcal{H},\, S_e(T,\, X))$ by the rule $\big(p_\rho^e(r)f\big)(u) = p_\rho^e(r)(u)f$. Now $D_{\rho\otimes\tau^e,\,v}h \in \sum_{b<e+v}\mathcal{N}^b(\mathcal{H},\, S_1(T_v,\, S_e(T,\, X)))$ by (1). By (13.28) we have

$$\begin{aligned}\big[D_{\rho\otimes\tau^e,\,v}\big(p_\rho^e(r)f\big)\big](u') &= D_v\big(p_\rho^e(r)f\big)(u') + P_{\rho\otimes\tau^e}^v(r,\, u')\big(p_\rho^e(r)f\big)\\ &= p_\rho^e(r)(D_v f)(u') + Q(r,\, u')f\end{aligned}$$

for $u' \in T_v$ with an element $Q(r,\, u')$ of $S_1(X,\, S_e(T,\, X))$ given by $Q(r,\, u')x = D_v\big(p_\rho^e(r)x\big)(u') + P_{\rho\otimes\tau^e}^v(r,\, u')\big(p_\rho^e(r)x\big)$ for $x \in X$. Since $p_\rho^e(r)$ is homogeneous of degree e in r, from (13.6c) we see that $D_v\big(p_\rho^e(r)\big)$ is homogeneous of degree $e+v$ in r. Also, $P_{\rho\otimes\tau^e}^v(r,\, u')$ is bilinear in $(r,\, u')$. Thus we obtain (13.25) for $D_\rho^{e+v}f$ with $P_\rho^{e+v}(r,\, u)x = \big(Q(r,\, u_v)x\big)(u)$, that is,

$$P_\rho^{e+v}(r,\, u)x = \big[D_v\big(P_\rho^e(r,\, u)x\big)\big](u_v) + \big[P_{\rho\otimes\tau^e}^v(r,\, u)\big(p_\rho^e(r)x\big)\big](u) \qquad (u\in T,\, x\in X).$$

This proves (2). To prove (3), take $h \in S_e(T,\, X)$; then

$$\begin{aligned}&\big[d(\rho\otimes\tau^e)(A,\, B)h\big](u)\\ &\quad= (d/dt)_{t=0}\big\{\rho\big(\exp(tA),\, \exp(tB)\big)h\big({}^t\exp(tA)\cdot u\cdot \exp(tB)\big)\big\}\\ &\quad= d\rho(A,\, B)\cdot h(u) + \big[d\tau^e(A,\, B)h\big](u).\end{aligned}$$

Take $h = p_\rho^e(r)x$. Then

$$\begin{aligned}&\big[P_{\rho\otimes\tau^e}^v(r,\, u)\big(p_\rho^e(r)x\big)\big](u)\\ &\qquad= P_\rho^v(r,\, u)\big(p_\rho^e(r)x\big)(u) + \big[d\tau^e(r_v\cdot {}^tu_v,\, {}^tr_v u_v)\big(p_\rho^e(r)x\big)\big](u).\end{aligned}$$

This proves (3). Finally, combining (13.24) and (13.25), we obtain $(E^e D_\rho^e f)(u,\, w) = cP_\rho^e(w,\, u)f(z)$ with $c = \prod_{v\in\mathbf{a}}(-1)^{e_v}e_v!$. Now replace f by $f\|_\rho\alpha$ with $\alpha \in G_{\mathbf{a}}$. By Proposition 12.10 (2), $(E^e D_\rho^e f)\|_{\rho\otimes\tau^e\otimes\sigma^e}\,\alpha = E^e D_\rho^e(f\|_\rho\,\alpha)$, and hence

$$P_\rho^e(w,\, u)(f\|_\rho\,\alpha) = \rho(\lambda_\alpha,\, \mu_\alpha)^{-1}P_\rho^e(\lambda_\alpha w\cdot {}^t\mu_\alpha,\, {}^t\lambda_\alpha^{-1}u\mu_a^{-1})(f\circ\alpha)$$

for every holomorphic map $f : \mathcal{H} \to X$. Therefore

$$P_\rho^e(w,\, u)\rho(\lambda_\alpha,\, \mu_\alpha)^{-1} = \rho(\lambda_\alpha,\, \mu_\alpha)^{-1}P_\rho^e(\lambda_\alpha w\cdot {}^t\mu_\alpha,\, {}^t\lambda_\alpha^{-1}u\mu_a^{-1}),$$

from which we obtain (4).

13.16. Lemma. *Given $\{\rho,\, X\}$ as before and $f \in C^\infty(\mathcal{H},\, X)$, for $u,\, u' \in T_v$ with a fixed $v \in \mathbf{a}'$, we have*

$$\begin{gathered}\big[(D_{\rho\otimes\sigma_v,v}E_v - E_v D_{\rho,v})f\big](u,\, u') = P_\rho^v(u',\, u)f,\\ (L_{\rho,v} - M_{\rho,v})f = B_{\rho,v}f\end{gathered}$$

with a constant element $B_{\rho,v}$ of $\mathrm{End}(X)$ *depending only on ρ and v, where $L_{\rho,v} = -\theta D_{\rho\otimes\sigma_v,v}E_v$ and $M_{\rho,v} = -\theta E_v D_{\rho,v}$ with θ of (12.33), and u (resp. u') corresponds to the operator D_v (resp. E_v).*

PROOF. By (12.12b) and (12.18) we have

$$\begin{aligned}(D_{\rho\otimes\sigma_v,v}E_v f)(u,\, u') &= \big\{(\rho\otimes\sigma_v)(\Xi)^{-1}D_v\big[\rho(\Xi)\overline{D}_v f(u')\big]\big\}(u)\\ &= \rho(\Xi)^{-1}D_v\big(\rho(\Xi)\big)(u)(\overline{D}_v f)(\xi u'\cdot{}^t\eta) + (D_v\overline{D}_v f)(\xi u'\cdot{}^t\eta,\, u).\end{aligned}$$

By (13.26) the first term of the last line equals $P^v_\rho(r,\, u)(E_v f)(u')$. On the other hand, by (13.28),

$$(E_v D_{\rho,v} f)(u,\, u') = E_v\big[(D_v f)(u)\big](u') + E_v\big[P^v_\rho(r,\, u)f\big](u').$$

By (12.12b) the first term on the right-hand side is $(\overline{D}_v D_v f)(u,\, \xi u'\cdot{}^t\eta)$, and by (13.10) the second term equals $-P^v_\rho(u',\, u)f + P^v_\rho(r,\, u)(E_v f)(u')$. Since $(\overline{D}_v D_v f)(u,\, u') = (D_v\overline{D}_v f)(u',\, u)$, we obtain the first formula, which immediately implies the second formula with $B_{\rho,v} = -\theta P^v_\rho$.

13.17. Lemma. *Let $\{\rho, X\}$ and $\{\sigma, Y\}$ be representations of $\mathfrak{K}_0$, and let $f \in C^\infty(\mathcal{H},\, X)$ and $g \in C^\infty(\mathcal{H},\, Y)$. Then for $u \in T$ we have*

$$D^e_{\rho\otimes\sigma}(f\otimes g)(u) = \sum_{a+b=e}\binom{e}{a}(D^a_\rho f)(u)\otimes(D^b_\sigma g)(u), \tag{13.29}$$

where the sum is extended over all $a,\, b \in \mathbf{Z}^{\mathbf{a}}$ with nonnegative components such that $a + b = e$, and $\binom{e}{a} = \prod_{v\in\mathbf{a}}\dfrac{e_v!}{a_v!(e_v - a_v)!}$. Similarly if $X = Y = \mathrm{End}(W)$, $\rho(a,\, b)x = \rho_0(a,\, b)x$ for $x \in \mathrm{End}(W)$ with a representation $\{\rho_0,\, W\}$, and σ is trivial, then the above formula holds with $\otimes$ replaced by multiplication in $\mathrm{End}(W)$.

PROOF. Since C_v is given by (13.10), for $f_1 \in C^\infty(\mathcal{H},\, X)$ and $g_1 \in C^\infty(\mathcal{H},\, Y)$ we have $C_v(f_1\otimes g_1)(u) = (C_v f_1)(u)\otimes g_1 + f_1\otimes(C_v g_1)(u)$ for $u \in T_v$, and

$$\begin{aligned}C_{v'}C_v(f_1\otimes g_1)(u,\, u') &= (C_{v'}C_v f_1)(u,\, u')\otimes g_1 + (C_v f_1)(u)\otimes(C_{v'}g_1)(u')\\ &\quad + (C_{v'}f_1)(u')\otimes(C_v g_1)(u) + f_1\otimes(C_{v'}C_v g_1)(u,\, u')\end{aligned}$$

for $u \in T_v$ and $u' \in T_{v'}$. Applying the C_v successively, for $u \in T$ we find that

$$C^e(f_1\otimes g_1)(u) = \sum_{a+b=e}\binom{e}{a}(C^a f_1)(u)\otimes(C^b g_1)(u).$$

By (13.21), $D^e_{\rho\otimes\sigma}(f\otimes g)(u) = (\rho\otimes\sigma)(\Xi)^{-1}C^e\big(\rho(\Xi)f\otimes\sigma(\Xi)g\big)({}^t\xi^{-1}u\eta^{-1})$, and therefore taking $f_1 = \rho(\Xi)f$ and $g_1 = \sigma(\Xi)g$, we obtain (13.29). The case $X = Y = \mathrm{End}(W)$ can be proved in the same manner.

14. Arithmeticity of nearly holomorphic functions

14.0. Before discussing our problems in the general case, let us first illustrate the main ideas by sketching the proof of the following statement in which we take $G = SL_2(\mathbf{Q})$:

Let $f \in \mathcal{M}_k(\mathbf{Q}_{\mathrm{ab}})$ and $g \in \mathcal{A}_{k+2p}(\mathbf{Q}_{\mathrm{ab}})$ with positive integers k and p. Let K be an imaginary quadratic field embedded in $\mathbf{C}$, and let $\tau \in H \cap K$; suppose that g has neither pole nor zero at τ. Then $(\pi^{-p}\delta^p_k f/g)(\tau)$ belongs to K_{ab}, where δ^p_k is the operator of (12.39), which equals D^p_ρ with $\rho(x) = x^k$ as explained in §12.17.

This is one of the easiest cases of (0.6) in the introduction, formulated with K_{ab} instead of $\overline{\mathbf{Q}}$. Our proof is by induction on p. We may assume that $f(\tau) \neq 0$. Indeed, if $f(\tau) = 0$, then we take $f_1 \in \mathcal{M}_k(\mathbf{Q}_{\mathrm{ab}})$ so that $f_1(\tau) \neq 0$, and put $f_2 = f + f_1$. Then our assertion on $(\pi^{-p}\delta_k^p f/g)(\tau)$ follows from that on $(\pi^{-p}\delta_k^p f_\nu/g)(\tau)$ for $\nu = 1, 2$. We first observe that the group $\{\alpha \in SL_2(\mathbf{Q}) \mid \alpha(\tau) = \tau\}$ contains an element, say α, of infinite order. Assuming $f(\tau) \neq 0$, put $h = (f\|_k \alpha)/f$. Then h is a $\mathbf{Q}_{\mathrm{ab}}$-rational modular function, so that $h(\tau) \in K_{\mathrm{ab}}$ by classical theory of complex multiplication, which is actually a special case of Theorem 9.6. Now we easily see that $\delta_k(rs) = r's + r\delta_k s$ for any meromorphic functions r and s, where $r' = dr/dz$. Applying this principle to the equality $f\|_k \alpha = hf$, we obtain $(\delta_k f)\|_{k+2}\alpha = \delta_k(f\|_k \alpha) = \delta_k(hf) = h'f + h\delta_k f$. Put $\lambda = j_\alpha(\tau)$. Then λ is an eigenvalue of α, so that λ is an element of K that is not a root of unity. We have also $(f\|_k \alpha)(\tau) = \lambda^{-k} f(\tau)$, $h(\tau) = \lambda^{-k}$, and $\big((\delta_k f)\|_{k+2}\alpha\big)(\tau) = \lambda^{-k-2}(\delta_k f)(\tau)$. Therefore $(\lambda^{-2}-1)(\delta_k f)(\tau) = \lambda^k h'(\tau) f(\tau)$. Now $\pi^{-1}h' \in \mathcal{A}_2(\mathbf{Q}_{\mathrm{ab}})$, and hence $(\pi^{-1}h'f/g) \in \mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$, if $g \in \mathcal{A}_{k+2}(\mathbf{Q}_{\mathrm{ab}})$ and $g(\tau) \neq 0$. We have thus $(\pi^{-1}(\delta_k f)/g)(\tau) = \lambda^k(\lambda^{-2} - 1)^{-1}(\pi^{-1}h'f/g)(\tau) \in K_{\mathrm{ab}}$, which proves the case $p = 1$.

If $p > 1$, we first observe that $\delta_k^p(rs) = \sum_{a=0}^p c_a(\delta_0^a r)(\delta_k^{p-a} s)$ with $c_a = p!/\big[a!(p-a)!\big]$, and so

$$(\delta_k^p f)\|_{k+2p}\,\alpha = \delta_k^p(f\|_k \alpha) = \delta_k^p(hf) = h\delta_k^p f + \textstyle\sum_{a=1}^p c_a(\delta_0^a h)(\delta_k^{p-a} f).$$

Evaluating this equality at τ, we obtain

$$(\delta_k^p f)(\tau)(\lambda^{-2p} - 1)\lambda^{-k} = \textstyle\sum_{a=1}^p c_a(\delta_0^a h)(\tau) \cdot (\delta_k^{p-a} f)(\tau).$$

Take $q_a \in \mathcal{A}_{2a}(\mathbf{Q}_{\mathrm{ab}})$ so that $q_a(\tau) \neq 0$. Then

$$\pi^{-p}(\delta_k^p f/g)(\tau) = (\lambda^{-2p} - 1)^{-1}\lambda^k \textstyle\sum_{a=1}^p c_a(\pi^{-a}\delta_0^a h/q_a)(\tau) \cdot (\pi^{a-p} q_a \delta_k^{p-a} f/g)(\tau).$$

Since $g/q_a \in \mathcal{A}_{k-a+2p}(\mathbf{Q}_{\mathrm{ab}})$, our induction shows that $(\pi^{a-p} q_a \delta_k^{p-a} f/g)(\tau) \in K_{\mathrm{ab}}$. As for the other factor, we have $\pi^{-a}\delta_0^a h = \pi^{1-a}\delta_2^{a-1}(\pi^{-1}h')$, and $\pi^{-1}h' \in \mathcal{A}_2(\mathbf{Q}_{\mathrm{ab}})$. Therefore our induction is applicable to that factor. We can thus conclude that $\pi^{-p}(\delta_k^p f/g)(\tau) \in K_{\mathrm{ab}}$.

We shall prove in Theorem 14.7 below a generalization of the above statement, essentially by the same idea, and then in Theorem 14.9 a similar but much stronger result. Since we have to deal with vector-valued or matrix-valued functions, our treatment will become more involved than the above proof. The generalization of the fact $\pi^{-1}h' \in \mathcal{A}_2(\mathbf{Q}_{\mathrm{ab}})$, given in Proposition 14.5, is highly nontrivial in Case UB.

14.1. Thus our principal interest in this section is in the nature of the value of $D_\rho^Z f$ at a CM-point of $\mathcal{H}$. We start with some results without arithmeticity.

Let $\{\rho, X\}$ be a representation of $\mathfrak{K}_0$. Given $h \in S_e\big(T, S_e(T, X)\big)$ and $(u, w) \in T \times T$, we define $h_u, h^w \in S_e(T, X)$ by $h_u(w) = h^w(u) = h(u, w)$. Then we obtain $p, q \in S_e\big(T, Ml_e(T, X)\big)$ defined by $p(u) = (h_u)_*$ and $q(w) = (h^w)_*$. Moreover, p_* and q_* are meaningful as elements of $Ml_e\big(T, Ml_e(T, X)\big)$. We can easily verify that $p_* = q_*$; we then denote this same element of $Ml_e\big(T, Ml_e(T, X)\big)$ by h_{**}, and define $\theta_X^e : S_e\big(T, S_e(T, X)\big) \to X$ by

$$\theta_X^e h = \textstyle\sum h_{**}(a_{\nu_1}, a_{\nu_2}, \ldots ; b_{\nu_1}, b_{\nu_2}, \ldots). \tag{14.1}$$

Here we write $T^e = \prod_{v\in\mathbf{a}} T_v^{e_v} = \prod_{i=1}^{|e|} T_i$, $|e| = \sum_{v\in\mathbf{a}} e_v$, where each T_i is identified with some T_v; $\{a_{\nu_i}\}$ and $\{b_{\nu_i}\}$ are dual bases of T_i; the summation is over all the elements $\{(a_{\nu_i}, b_{\nu_i})\}_{i=1}^{|e|}$. Then we can easily verify that

(14.2) $\quad \theta_X^e \circ (\rho\otimes\tau^e\otimes\sigma^e)(\alpha) = \theta_X^e \circ (\rho\otimes\sigma^e\otimes\tau^e)(\alpha) = \rho(\alpha)\circ\theta_X^e \quad$ *for every* $\alpha \in \mathfrak{K}_0$.

Given $g \in C^\infty(\mathcal{H}, S_e(T, X))$, we can view $D^e_{\rho\otimes\sigma^e} g$ as an element of $C^\infty(\mathcal{H}, S_e(T, S_e(T, X)))$. Thus $\theta_X^e D^e_{\rho\otimes\sigma^e} g$ is meaningful as an element of $C^\infty(\mathcal{H}, X)$.

14.2. Proposition. *Let $\{\rho, X\}$ and $\{\rho_0, X\}$ be representations of $\mathfrak{K}_0$ such that $\rho(a, b) = \det(b)^k\rho_0(a, b)$ for $(a, b) \in \mathfrak{K}_0$ with $k \in \mathbf{Z}^{\mathbf{a}}$. Let f be an element of $\mathcal{N}^p(\mathcal{H}, X)$ such that $f\|_\rho\gamma = f$ for every γ in a subgroup Γ of $G_{\mathbf{a}}$. Then there exists an integer $N(\rho_0, p)$ that depends only on ρ_0 and p with the following property: if $k_v > N(\rho_0, p)$ for every $v \in \mathbf{a}'$, then*

$$f = \sum_{e\le p} \theta_X^e D^e_{\rho\otimes\sigma^e} g_e$$

with holomorphic maps $g_e : \mathcal{H} \to S_e(T, X)$ such that $g_e\|_{\rho\otimes\sigma^e}\gamma = g_e$ for every $\gamma \in \Gamma$, where e runs over the elements of $\mathbf{Z}^{\mathbf{a}}$ such that $0 \le e_v \le p_v$ for every $v \in \mathbf{a}$. Here $\mathbf{a}' = \{ v \in \mathbf{a} \,|\, G_v$ is not compact$\}$ as already defined in (12.5).

PROOF. We prove this by induction on $|p|$. Our assertion is trivial if $p = 0$. Fixing e, put $\pi_0 = \rho_0 \otimes \sigma^e$ and $\pi(a, b) = \det(b)^k\pi_0(a, b)$, and $Y_e = S_e(T, X)$. Then $d\pi(x, y) = d\pi_0(x, y) + \sum_{v\in\mathbf{a}} k_v \cdot \mathrm{tr}(y_v)1_Y$, where 1_Y is the identity element of $\mathrm{End}(Y_e)$. Define P_ρ^e and P_ρ^v by (13.25) and (13.27) (with Y_e as X there). Then $P_\pi^v(r, u) = P_{\pi_0}^v(r, u) + \sum_v k_v \cdot \mathrm{tr}({}^t r_v u_v)1_Y$. Now we have

(14.3) $\quad P_\pi^e(w, u) = \sum_{\nu\le e} k^\nu q_\nu(w, u) \quad$ *with* $\quad q_\nu \in S_e(T, S_e(T, \mathrm{End}(Y_e)))$
depending only on ρ_0 *and* e; *moreover* $q_e(w, u) = \mathrm{tr}({}^t wu)^e 1_Y$.

We prove this by induction on $|e| = \sum_{v\in\mathbf{a}} e_v$. If $|e| = 1$, we have $P_\pi^e(w, u) = P_\pi^v(w, u)$ with some v, and $q_e(w, u) = \mathrm{tr}({}^t w_v u_v)1_Y$; thus (14.3) is true if $|e| = 1$. Assuming (14.3) for some e, define $p_\pi^e(w)$, $q_\nu(w) \in S_e(T, \mathrm{End}(Y_e))$ by $p_\pi^e(w)(u) = P_\pi^e(w, u)$ and $q_\nu(w)(u) = q_\nu(w, u)$ for $w, u \in T$. Then, by Proposition 13.15 (3), for $y \in Y_e$ we have

$$\begin{aligned} P_\pi^{e+v}(r, u)y = {} & \bigl[D_v\bigl(\textstyle\sum_{\nu\le e} k^\nu q_\nu(r, u)y\bigr)\bigr](u_v) + P_\pi^v(r, u)\textstyle\sum_{\nu\le e} k^\nu q_\nu(r, u)y \\ & + \bigl[d\tau^e(r_v \cdot {}^t u_v, {}^t r_v u_v)\bigl(\textstyle\sum_{\nu\le e} k^\nu q_\nu(r)y\bigr)\bigr](u). \end{aligned}$$

Thus we easily see that $P_\pi^{e+v}(r, u) = \sum_{\nu\le e+v} k^\nu s_\nu(r, u)$ with $s_\nu \in S_e(T, S_e(T, \mathrm{End}(Y_e)))$ depending only on ρ_0 and e. Moreover $s_{e+v}(r, u) = \mathrm{tr}({}^t r_v u_v)q_e(r, u)$. Thus we obtain (14.3) for P_π^{e+v}.

Next, define A_e, $B_{e.\nu} \in \mathrm{End}(Y_e)$ by

$$(A_e\varphi)(w) = \theta_X^e\bigl(\bigl[P_\pi^e(w, *)\varphi\bigr](*)\bigr) \quad\text{and}\quad (B_{e.\nu}\varphi)(w) = \theta_X^e\bigl(\bigl[q_\nu(w, *)\varphi\bigr](*)\bigr)$$

for $\varphi \in Y_e = S_e(T, X)$ and $w \in T$, where $[q(w, *)\varphi](*)$ is an element of $S_e(T, Y_e) = S_e(T, S_e(T, X))$ whose value at $(u_1, u_2) \in T \times T$ is $[q(w, u_1)\varphi](u_2)$. We have $q_e(w, u) = \mathrm{tr}({}^t wu)^e 1_Y$, so that (12.10) implies that $(B_{e.e}\varphi)(w) = \varphi(w)$, that is, $B_{e.e} = 1_Y$. Thus $A_e = k^e 1_Y + \sum_{\nu<e} k^\nu B_{e.\nu}$. Observe that A_e is invertible if $k_v > M(\rho_0, e)$ for every $v \in \mathbf{a}'$ with an integer $M(\rho_0, e)$ that depends only on ρ_0 and e. Observe also that if a map $g : \mathcal{H} \to Y_e$ is holomorphic, then $D_\pi^e g = \sum_{a\le e} h_a(r)$ with holomorphic maps $h_a : \mathcal{H} \to S_a(T, S_e(T, Y_e))$, and $\theta_X^e D_\pi^e g = \sum_{a\le e} \theta_X^e h_a(r)$. By

(13.25), $h_e(r) = P_\pi^e(r, *)g$, and so $\theta_X^e h_e(r) = (A_e g)(r)$. Now we consider a given element f of $\mathcal{N}_\rho^p$ of our proposition. Then $f = \sum_{a \le p} \ell_a(r)$ with holomorphic maps $\ell_a : \mathcal{H} \to S_a(T, X)$ as noted in (13.23). By (13.24), $\ell_p = cE^p f$ with $c \in \mathbf{Q}^\times$, and so $\ell_p \|_{\rho\otimes\sigma^p} \gamma = \ell_p$ for every $\gamma \in \Gamma$ by Proposition 12.10 (2). Define an integer $N(\rho_0, p)$ by $N(\rho_0, p) = \mathrm{Max}_{e \le p}\{M(\rho_0, e)\}$. Suppose $k_v > N(\rho_0, p)$ for every $v \in \mathbf{a}'$ and put $g = A_p^{-1}\ell_p$. (So we take p as the above e and consider P_π^p with $\pi = \rho \otimes \sigma^p$.) From Proposition 13.15 (4) and (14.2) we see that A_p commutes with $(\rho\otimes\sigma^p)(\mathfrak{K}_0)$, and hence g is a holomorphic map of $\mathcal{H}$ into $S_p(T, X)$ such that to $g\|_{\rho\otimes\sigma^p}\gamma = g$ for every $\gamma \in \Gamma$. Now the p-th degree part of $\theta_X^p D_{\rho\otimes\sigma^p}^p g$ is $(A_p g)(r) = \ell_p(r)$, so that $f - \theta_X^p D_{\rho\otimes\sigma^p}^p g \in \sum_{a<p} \mathcal{N}^a(\mathcal{H}, X)$. Applying our induction to the last difference, we complete the proof.

14.3. Lemma. *$\mathcal{N}_\rho^p(\Gamma)$ is finite-dimensional over $\mathbf{C}$ for every congruence subgroup Γ of G.*

PROOF. We can find a positive integer κ and a nonzero element h of $\mathcal{M}_{\kappa\mathbf{a}}(\Gamma')$ with a congruence subgroup Γ' contained in Γ. Such an h can be obtained by $h = \prod_{v\in\mathbf{a}} \det(Q_v)$ with Q_v of Proposition 11.14. Take a positive integer m so that $m\kappa > N(\rho, p)$ with $N(\rho, p)$ as in Proposition 14.2. Given $f \in \mathcal{N}_\rho^p(\Gamma)$, we have $h^m f \in \mathcal{N}_{\rho'}^p(\Gamma')$, where $\rho'(a, b) = \rho(a, b)\det(b)^{m\kappa\mathbf{a}}$. By Proposition 14.2, $h^m f = \sum_{e\le p} \theta_X^e D_{\rho'\otimes\sigma^e}^e g_e$ with $g_e \in \mathcal{M}_{\rho'\otimes\sigma^e}(\Gamma')$. Since $\mathcal{M}_{\rho'\otimes\sigma^e}(\Gamma')$ is finite-dimensional, we see that $h^m f$ belongs to a finite-dimensional vector space over $\mathbf{C}$, which implies our lemma.

14.4. To consider arithmeticity, we go back to the setting of Sections 5 and 11, and note that T_v has a natural $\overline{\mathbf{Q}}$-structure. In each case the structure can be obtained by taking a natural coordinate system of T_v determined by the matrix entries. In Case UB, as already noted in the proof of Lemma 4.13, the action of $\widetilde{G}_+$ on $\mathcal{H}$ is $\overline{\mathbf{Q}}$-rational.

Let us now take a $\overline{\mathbf{Q}}$-rational representation $\{\omega, X\}$ of $\mathfrak{K}$, where

$$\mathfrak{K} = \prod_{v\in\mathbf{b}} GL_{n_v}(\mathbf{C}). \tag{14.4}$$

Since $\mathfrak{K}_0 \subset \mathfrak{K}$, we can speak of the restriction of ω to $\mathfrak{K}_0$. Taking this restriction to be ρ, we can define various objects with respect to ρ. To define arithmeticity, however, we have to consider them relative to ω. First of all, we can express an element of $\mathfrak{K}$ in the form (a, b) with $a \in \prod_{v\in\mathbf{a}} GL_{m_v}(\mathbf{C})$ and $b \in \prod_{v\in\mathbf{a}} GL_{n_v}(\mathbf{C})$; for Type C we have $\mathfrak{K} = \mathfrak{K}_0$ and $\omega = \rho$, and we take $a = b$ and $m_v = n_v = n$; for Type A if $\alpha = (\alpha_v)_{v\in\mathbf{b}} \in \mathfrak{K}$, then $a = (\alpha_{v\rho})_{v\in\mathbf{a}}$ and $b = (\alpha_v)_{v\in\mathbf{a}}$; see §5.1 for our convention. We can then define $\omega\otimes\tau^e$ and $\omega\otimes\sigma^e$ by (13.19a, b) with ω in place of ρ. We denote the symbols D_ρ^e, D_ρ^Z, $\mathcal{N}_\rho^p$, etc. by D_ω^e, D_ω^Z, $\mathcal{N}_\omega^p$, etc. Then (12.21) is true for α in $G = U(T)$ or $G = U(\eta_n)$ in the unitary case with $Z \subset S_e(T)$ and ω in place of ρ. This is because (12.1a, b) are true for such an α, and so the proof of Proposition 12.10 is valid. If $\omega(x) = \det(x)^k$ with $k \in \mathbf{Z}^\mathbf{b}$, then we replace the subscript ω by k.

Let $f \in C^\infty(\mathcal{H}, X)$ and $w \in \mathcal{H}_{\mathrm{CM}}$. Then we say that f is *ω-arithmetic* (or simply *arithmetic*, if ω is clear from the context) at w if $\mathfrak{P}_\omega(w)^{-1}f(w)$ is $\overline{\mathbf{Q}}$-rational. Then we denote by $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ (resp. $\mathcal{N}_\omega^p(\Gamma, \overline{\mathbf{Q}})$) the set of the elements of $\mathcal{N}_\omega^p$ (resp. $\mathcal{N}_\omega^p(\Gamma)$) that are ω-arithmetic at every point of $\mathcal{H}_{\mathrm{CM}}$. If $\{\omega, X\}$ is clear from the context, we simply call an element of $\mathcal{N}_\omega^p$ *arithmetic* if it belongs to $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. From Proposition 11.5 (2) we see that if $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ and $\alpha \in \widetilde{G}_+$, then

$f\|_\omega \alpha \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. For Type C we have $\omega = \rho$, and so $\mathcal{N}_\rho^p(\overline{\mathbf{Q}})$ is meaningful. For Type A, however, we cannot speak of $\mathcal{N}_\rho^p(\overline{\mathbf{Q}})$ for the reasons explained in Theorem 11.17.

14.5. Proposition. *For each $v \in \mathbf{a}'$ define a representation $\{\tau_v, S_1(T_v)\}$ of $\mathfrak{K}$ by $[\tau_v(a, b)h](u) = h({}^t a_v u b_v)$ for $h \in S_1(T_v)$ and $u \in T_v$. (This is a special case of (12.7a).) Then $\pi^{-1}D_v f \in \mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$ for every $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$.*

PROOF. By (12.14a), $D_v f \in \mathcal{A}_{\tau_v}$. Thus the main point of our lemma is the $\overline{\mathbf{Q}}$-rationality. In Cases SP and UT we can put $f = g_1/g_0$ with $g_0, g_1 \in \mathcal{M}_{\mu\mathbf{a}}(\overline{\mathbf{Q}})$, $0 < \mu \in \mathbf{Z}$, since $\mathcal{A}_0(\mathbf{Q}_{\mathrm{ab}})$ is generated by the quotients of (7.8). Let z_{jk}^v be the (j, k)-entry of z_v. Then $\pi^{-1}g_0^2 \partial f/\partial z_{jk}^v$ has a Fourier expansion with coefficients in $\overline{\mathbf{Q}}$, as we already observed in (9.9) in a similar setting. This means that $\pi^{-1}g_0^2 D_v f \in \mathcal{M}_\xi(\overline{\mathbf{Q}})$ with $\xi(a, b) = \det(b)^{2\mu\mathbf{a}}\tau_v(a, b)$. Thus $\pi^{-1}D_v f \in \mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$.

In Case UB the matter is not so simple. We employ the notation of §11.6; in particular we consider the embedding $\varepsilon : \mathcal{H} \to \mathfrak{H}_d$ of (11.8). Also, let Γ, $(A_z, \mathcal{C}_z)$, and $\mathfrak{F}_z$ be as in §11.1. Take a point z_0 of $\mathcal{H}$ generic for the elements of $\mathcal{A}_0(\Gamma, \overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$ in the sense of Section 7. Then $\mathfrak{F}_{z_0}$ is algebraic over the field of moduli of $(A_{z_0}, \mathcal{C}_{z_0})$, which can be generated by $\mathfrak{f}(\varepsilon(z_0))$ for all $\mathfrak{f} \in \mathfrak{A}_0(\Gamma^1, \mathbf{Q})$ finite at $\varepsilon(z_0)$, where $\Gamma^1 = Sp(d, \mathbf{Z})$. Therefore we can find elements $\mathfrak{f}_1, \dots, \mathfrak{f}_p$ of $\mathfrak{A}_0(\Gamma^1, \mathbf{Q})$ such that $\mathfrak{f}_1 \circ \varepsilon, \dots, \mathfrak{f}_p \circ \varepsilon$ are algebraically independent, where p is the complex dimension of $\mathcal{H}$. Put $g_j = \mathfrak{f}_j \circ \varepsilon$. By (11.12), $g_j \in \mathcal{A}_0(\overline{\mathbf{Q}})$. Then $\partial/\partial g_1, \dots, \partial/\partial g_p$ are well-defined derivations of $\mathcal{A}_0(\overline{\mathbf{Q}})$, and for $f \in \mathcal{A}_0(\overline{\mathbf{Q}})$ we have $D_v f = \sum_{j=1}^p (\partial f/\partial g_j) D_v g_j$. Since $\partial f/\partial g_j \in \mathcal{A}_0(\overline{\mathbf{Q}})$, our task is to show that $D_v g_j \in \mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$, or rather $D_v(\mathfrak{f} \circ \varepsilon) \in \mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$ for every $\mathfrak{f} \in \mathfrak{A}_0(\overline{\mathbf{Q}})$ for which $\mathfrak{f} \circ \varepsilon$ is meaningful. Fixing v, put $m = m_v$ and $n = n_v$. We may assume that $\mathcal{T} = \mathrm{diag}[\zeta_1, \dots, \zeta_r]$ with $\zeta_\mu \in K^\times$ such that $\zeta_\mu^\rho = -\zeta_\mu$. Our v defines an embedding of F into $\mathbf{R}$, and in §3.5 we fixed an embedding of K into $\mathbf{C}$ that extends v. For $x \in K$ denote by x_v the image of x by that embedding. By (4.17) we may assume that $i\zeta_{\mu v} > 0$ for $\mu \le m$ and $i\zeta_{\mu v} < 0$ for $\mu > m$. Take real numbers $s_1, \dots, s_r$ so that $s_\mu^2 = i\zeta_{\mu v}$ for $\mu \le m$ and $s_\mu^2 = -i\zeta_{\mu v}$ for $\mu > m$. Then we can take $\mathrm{diag}[s_1, \dots, s_r]$ as Q_v of (3.34). Put $[F : \mathbf{Q}] = t$. For each μ take $\mathbf{Q}$-bases $\{a_{\mu j}\}_{j=1}^t$ and $\{a'_{\mu j}\}_{j=1}^t$ of F so that

$$\mathrm{Tr}_{F/\mathbf{Q}}(\zeta_\mu^2 a_{\mu j} a'_{\mu k}) = \delta_{jk}/2. \tag{14.5}$$

Let $\{\widetilde{e}_\mu\}_{\mu=1}^r$ be the standard basis of K_r^1; put $h_{\mu j} = a_{\mu j}\widetilde{e}_\mu$ and $h'_{\mu j} = \zeta_\mu a'_{\mu j}\widetilde{e}_\mu$. Then the elements

$$h_{11}, \dots, h_{1t}, \cdots\cdots, h_{r1}, \dots, h_{rt}, h'_{11}, \dots, h'_{1t}, \cdots\cdots, h'_{r1}, \dots, h'_{rt}$$

form a $\mathbf{Q}$-basis of K_r^1. Define $g : K_r^1 \to \mathbf{Q}_{2d}^1$ so that the image of these by g is the standard basis $\{e_k\}_{k=1}^{2d}$ of $\mathbf{Q}_{2d}^1$. Then (11.5) is satisfied. Let p_z and $\kappa(z)$ be as in (11.9) and (11.7). From (4.19) and (4.22) we see that

$$\begin{bmatrix} p_z(\widetilde{e}_1)_v & \cdots & p_z(\widetilde{e}_r)_v \end{bmatrix} = \begin{bmatrix} b_v & z_v c_v \\ {}^t z_v b_v & c_v \end{bmatrix},$$

where $b = \mathrm{diag}[s_1, \dots, s_m]$ and $c = \mathrm{diag}[s_{m+1}, \dots, s_r]$. Now we have $\mathbf{C}^d = (\mathbf{C}^r)^{\mathbf{a}}$, and we identify $\mathbf{a}$ with $\{1, \dots, t\}$ so that our fixed v corresponds to the index 1. Focusing our attention on the first r components of the vectors $p_z(g^{-1}(e_k))$ and employing (4.10), (4.12), (4.18), (11.6), and (11.7), we see that

$$\kappa(z)\varepsilon(z) = \begin{bmatrix} \beta & z_v\gamma \\ {}^tz_v\beta & \gamma \\ * & * \end{bmatrix}, \qquad \kappa(z) = \begin{bmatrix} \beta' & z_v\gamma' \\ -{}^tz_v\beta' & -\gamma' \\ * & * \end{bmatrix}$$

with $(m\times mt)$-matrices β, β' and $(n\times nt)$-matrices γ, γ' whose entries can be given explicitly in terms of the conjugates of s_μ, $a_{\mu j}$, $a'_{\mu j}$, and ζ_μ. Now for a function ψ on $\mathcal{H}$ denote by $d_v\psi$ the holomorphic v-part of the 1-form $d\psi$; in other words, $d_v\psi = (D_v\psi)(dz_v)$ (see (12.13)). Then

$$\begin{aligned} \kappa\cdot d_v\varepsilon\cdot {}^t\kappa &= d_v(\kappa\varepsilon)\cdot {}^t\kappa - d_v\kappa\cdot {}^t(\kappa\varepsilon) \\ &= \begin{bmatrix} dz_v(\gamma\cdot{}^t\gamma' - \gamma'\cdot{}^t\gamma)\cdot{}^tz_v & -dz_v(\gamma\cdot{}^t\gamma' + \gamma'\cdot{}^t\gamma) & * \\ {}^tdz_v(\beta\cdot{}^t\beta' + \beta'\cdot{}^t\beta) & {}^tdz_v(\beta'\cdot{}^t\beta - \beta\cdot{}^t\beta')z_v & * \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Since this matrix is symmetric, using the explicit forms of β, β', γ, γ' and (14.5), we find that

$$\kappa\cdot d_v\varepsilon\cdot{}^t\kappa = i\begin{bmatrix} 0 & dz_v & 0 \\ {}^tdz_v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Defining D on $\mathfrak{H}_d$ by $d\mathfrak{f} = (D\mathfrak{f})(dZ)$ for the variable Z on $\mathfrak{H}_d$, we have $d_v(\mathfrak{f}\circ\varepsilon) = \bigl((D\mathfrak{f})\circ\varepsilon\bigr)(d_v\varepsilon)$. Put $\mathfrak{T} = \bigl\{U\in\mathbf{C}_d^d \,\big|\, {}^tU = U\bigr\}$ and view it as the (holomorphic) tangent space of $\mathfrak{H}_d$. Define $S_1(\mathfrak{T})$-valued function Y on $\mathfrak{H}_d$ by $Y(U) = \pi^{-1}(D\mathfrak{f})({}^tP^{-1}UP^{-1})$ for $U\in\mathfrak{T}$ with P as in §11.8. Then $Y(U)$ for an algebraic U is an element of $\mathfrak{A}_0(\overline{\mathbf{Q}})$. Let S_v, R_v, and W be as in (11.13); put $A(z)(U) = Y\bigl(\varepsilon(z)\bigr)\bigl({}^tW(z)UW(z)\bigr)$ and $\mathfrak{R} = \mathrm{diag}[S_v, R_v]_{v\in\mathbf{a}}$. Then

$$\begin{aligned} \pi^{-1}D_v(\mathfrak{f}\circ\varepsilon)(dz_v) &= \pi^{-1}d_v(\mathfrak{f}\circ\varepsilon) = A\bigl({}^tW^{-1}\cdot{}^t(P\circ\varepsilon)d_v\varepsilon(P\circ\varepsilon)W^{-1}\bigr) \\ &= A\bigl({}^t\mathfrak{R}\cdot\kappa\cdot d_v\varepsilon\cdot{}^t\kappa\cdot\mathfrak{R}\bigr) = iA\left(\begin{bmatrix} 0 & {}^tS_vdz_vR_v & 0 \\ {}^tR_v\cdot{}^tdz_vS_v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right). \end{aligned}$$

Since W has entries in $\mathcal{A}_0(\overline{\mathbf{Q}})$, $A(w)(U)$ for $w\in\mathcal{H}_{\mathrm{CM}}$ and algebraic U is algebraic. Put $B(z)(u) = A(z)\left(\begin{bmatrix} 0 & u & 0 \\ {}^tu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$ for $u\in T_v$. Then $\pi^{-1}D_v(\mathfrak{f}\circ\varepsilon)(u) = iB({}^tS_vuR_v)$. Thus for $w\in\mathcal{H}_{\mathrm{CM}}$ we have

$$(*)\quad \pi^{-1}\bigl(\mathfrak{P}_{\tau_v}(w)^{-1}D_v(\mathfrak{f}\circ\varepsilon)(w)\bigr)(u) = iB(w)\bigl({}^tS_v(w)\cdot{}^t\mathfrak{p}_{v\rho}(w)^{-1}u\mathfrak{p}_v(w)^{-1}R_v(w)\bigr).$$

We saw in (11.16) that $\mathfrak{p}_v(w)^{-1}R_v(w)$ and $\mathfrak{p}_{v\rho}(w)^{-1}S_v(w)$ are algebraic for every $w\in\mathcal{H}_{\mathrm{CM}}$ where $P\circ\varepsilon$ and W are finite and invertible. This means that the quantity of $(*)$ for algebraic u is algebraic. This proves that $\pi^{-1}D_v(\mathfrak{f}\circ\varepsilon)\in\mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$ as expected.

14.6. Lemma. *Let $\{\omega, X\}$ and $\{\zeta, Y\}$ be $\overline{\mathbf{Q}}$-rational representations of $\mathfrak{K}$, and let φ be a $\mathbf{C}$-linear map of Y into X such that $\varphi\zeta(\alpha) = \omega(\alpha)\varphi$ for every $\alpha\in\mathfrak{K}$. Define a map $\varphi_e : Ml_e(T, Y)\to Ml_e(T, X)$ by $\varphi_e(h) = \varphi\circ h$ for $h\in Ml_e(T, Y)$. Then the following assertions hold:*

(i) *$\varphi_e\circ(\zeta\otimes\tau^e)(\alpha) = (\omega\otimes\tau^e)(\alpha)\circ\varphi_e$ and $\varphi_e\circ(\zeta\otimes\sigma^e)(\alpha) = (\omega\otimes\sigma^e)(\alpha)\circ\varphi_e$ for every $\alpha\in\mathfrak{K}$.*

(ii) *$\varphi_eC^eg = C^e(\varphi g)$, $\varphi_eE^eg = E^e(\varphi g)$, and $\varphi_eD_\zeta^eg = D_\omega^e(\varphi g)$ for every $g\in C^\infty(\mathcal{H}, Y)$.*

(iii) $\varphi C_\zeta \subset C_\omega$, $\varphi \mathcal{M}_\zeta \subset \mathcal{M}_\omega$, *and* $\varphi \mathcal{A}_\zeta \subset \mathcal{A}_\omega$.

(iv) *If* φ *is* $\overline{\mathbf{Q}}$*-rational,* $f \in C_\zeta$, *and* f *is* ζ*-arithmetic at* w, *then* φf *is* ω*-arithmetic at* w.

PROOF. Assertion (i) follows from (13.19a, b), and the first two equalities of (ii) from (13.10). Employing these and (12.17) (or (13.21)), we obtain the last equality of (ii). Assertions (iii) and (iv) follow immediately from our definition.

14.7. Theorem. *Let* Z *be a* $\mathfrak{K}$*-stable subspace of* $S_e(T)$*. If* $f \in \mathcal{A}_\omega(\overline{\mathbf{Q}})$*, then* $\pi^{-|e|}D_\omega^Z f$ *is arithmetic at every CM-point where* f *is finite.*

PROOF. Taking φ_Z as φ of Lemma 14.6, we see that it is sufficient to prove the case $Z = S_e(T)$, that is, the case $D_\omega^Z f = D_\omega^e f$. We take $X = \mathbf{C}^m$. Let g be an $m \times m$-matrix whose columns are all equal to f. Define a representation $\{\psi, \mathrm{End}(X)\}$ of $\mathfrak{K}$ by $\psi(x)y = \omega(x)y$ for $y \in \mathrm{End}(X)$. Then our question can be reduced to the nature of $D_\psi^Z g$. Now given $w \in \mathcal{H}_{\mathrm{CM}}$ where f is finite, take $R = (R_v)_{v\in\mathbf{a}}$ as in Proposition 11.14 (resp. Proposition 9.11) in Case UB (resp, Cases SP and UT), so that the R_v are finite and invertible at w; here we take the columns of R_v in $\mathcal{A}_{\sigma_v}(\overline{\mathbf{Q}})$ with $\sigma_v(x) = x_v$. Put $A_1 = \omega(R)$ and $A_2 = cA_1 + g$ with $c \in \mathbf{Q}$. Then A_1 is finite and invertible at w; we can also take c so that A_2 is invertible at w. Since $g = A_2 - cA_1$, it is sufficient to prove the arithmeticity of $D_\psi^Z A$ at w for every $A \in \mathcal{A}_\psi(\overline{\mathbf{Q}})$ that is finite and invertible at w. Fix such an A and take (Y, h) as in §4.11 such that w is the fixed point of $h(Y^u)$; take also $\beta \in Y^u$ as in Lemma 4.12 and put $\gamma = h(\beta)$. Now our method of proof is an adaptation of that of §14.0. Put $B = A^{-1}(A\|_\psi \gamma)$. Then $A\|_\psi \gamma = AB$ and B has entries in $\mathcal{A}_0(\overline{\mathbf{Q}})$, and so $B(w)$ is algebraic. Since $\gamma w = w$, we have

$$B(w) = A(w)^{-1}\omega\big(M_\gamma(w)^{-1}\big)A(w). \tag{14.6}$$

By (12.21) we have $(D_\omega^e A)\|_{\psi\otimes\tau^e}\gamma = D_\omega^e(A\|_\psi \gamma) = D_\omega^e(AB)$, and hence, by Lemma 13.17, for $u \in T$ we have

$$\omega(M_\gamma^{-1})(D_\omega^e A)({}^t\lambda_\gamma^{-1}u\mu_\gamma^{-1}) = \sum_{a+b=e}\binom{e}{a}(D_\omega^a A)(u)(D_\iota^b B)(u), \tag{14.7}$$

where ι denotes the trivial representation of $\mathfrak{K}$. Put

$$R_a(u) = \pi^{-|a|}A(w)^{-1}(D_\omega^a A)(w)({}^t\mathfrak{p}_{v\rho}^{-1}u\mathfrak{p}_v^{-1}), \quad S_b(u) = \pi^{-|b|}(D_\iota^b B)(w)({}^t\mathfrak{p}_{v\rho}^{-1}u\mathfrak{p}_v^{-1}),$$

where ${}^t\mathfrak{p}_{v\rho}^{-1}u\mathfrak{p}_v^{-1} = \big({}^t\mathfrak{p}_{v\rho}(w)^{-1}u_v\mathfrak{p}_v(w)^{-1}\big)_{v\in\mathbf{a}}$ for $u = (u_v)_{v\in\mathbf{a}} \in T$. Then $B(w)^{-1}$ times (14.7) gives

$$R_e({}^t\zeta' u\zeta) - B(w)^{-1}R_e(u)B(w) = \sum_{a+b=e,\, a\neq e}\binom{e}{a}B(w)^{-1}R_a(u)S_b(u). \tag{14.8}$$

Here $\zeta = \big(\mathfrak{p}_v(w)^{-1}\mu_v(\gamma, w)^{-1}\mathfrak{p}_v(w)\big)_{v\in\mathbf{a}}$ and $\zeta' = \big(\mathfrak{p}_{v\rho}(w)^{-1}\lambda_v(\gamma, w)^{-1}\mathfrak{p}_{v\rho}(w)\big)_{v\in\mathbf{a}}$. Since $A \in \mathcal{A}_\psi(\overline{\mathbf{Q}})$, we see that $\mathfrak{P}_\omega(w)^{-1}A(w)$ is algebraic. Therefore our task is to prove the algebraicity of $R_e(u)$ for algebraic u. We are going to prove this by induction on $|e|$. Since $\gamma w = w$, Proposition 11.5 (2) shows that ζ and ζ' are algebraic. Also, the linear endomorphism $R(u) \mapsto R({}^t\zeta' u\zeta)$ of $S_e\big(T, \mathrm{End}(X)\big)$ commutes with another endomorphism $R \mapsto B(w)^{-1}RB(w)$. Now we can choose γ so that $(\zeta'\otimes\zeta)^{\otimes|e|}$ and $B(w)^{-1}\otimes B(W)$ have no common eigenvalue, as will be shown at the end of the proof. Take any algebraic u. By the induction alssumption, $R_a(u)$ on the right-hand side of (14.8) is algebraic. As for $S_b(u)$, we have $b \neq 0$, so that we have $D_\iota^b B = D_{\tau_v}^c(D_v B)$ for some c and some $v \in \mathbf{a}$. By Proposition 14.5, the entries of $\pi^{-1}D_v B$ belong to $\mathcal{A}_{\tau_v}(\overline{\mathbf{Q}})$. Since $|c| < |e|$, the induction assumption implies the algebraicity of $S_b(u)$. Thus the sum on the right-hand side of (14.8) is algebraic. Viewing (14.8) as a system of linear equations with $R_e(u)$ as the unknown, we obtain the desired algebraicity of $R_e(u)$. Notice that if $|e| = 1$, the right-hand side of (14.8) consists of a single term $\pi^{-1}B(w)^{-1}(D_v B)(w)\big({}^t\mathfrak{p}_{v\rho}(w)^{-1}u\mathfrak{p}_v(w)^{-1}\big)$ with $u \in T_v$ for some v. Therefore Proposition 14.5 gives its algebraicity.

To prove the existence of $\gamma = h(\beta)$ with the desired property on eigenvalues, consider Φ and (K_i, Φ_i) defined for the present Y as in §4.11; put $[Y : \mathbf{Q}] = 2d$. Then there exist d maps $y \mapsto y_i$ for $1 \le i \le d$ of Y into $\mathbf{C}$ such that $y_1, \ldots, y_d$ are the eigenvalues of $\Phi(y)$; moreover these maps together with their complex conjugates form exactly the set J_Y of §11.3. By (4.37) and (4.40) we see that $\alpha_1, \ldots, \alpha_d$ are exactly the eigenvalues of $M\big(h(\alpha), w\big)$ for every $\alpha \in Y^u$. Clearly we may assume that ω is irreducible. Then we can find a homogeneous polynomial representation σ of $\mathfrak{K}$ such that $\sigma(a, b) = \det(a)^\mu \det(b)^\nu \omega(a, b)$ with $\mu, \nu \in \mathbf{Z}^{\mathbf{a}}$. From (14.6) we see that $B(w)^{-1} \otimes B(w)$ has the same eigenvalues as $\sigma\big(M_\gamma(w)\big)^{-1} \otimes \sigma\big(M_\gamma(w)\big)$. Therefore each eigenvalue of $B(w)^{-1} \otimes B(w)$ is of the form $\prod_{i=1}^d \beta_i^{\lambda_i}$ with integers λ_i such that $\sum_{i=1}^d \lambda_i = 0$. On the other hand each eigenvalue of $(\zeta' \otimes \zeta)^{\otimes|e|}$ is of the form $\prod_{i=1}^d \beta_i^{-\mu_i}$ with nonnegative integers μ_i such that $\sum_{i=1}^d \mu_i = 2|e|$. (In Case SP we have $\lambda_v(\gamma, w) = \mu_v(\gamma, w)$ and $\zeta' = \zeta$, but still our statements are valid.) Suppose that $B(w)^{-1} \otimes B(w)$ and $(\zeta' \otimes \zeta)^{\otimes|e|}$ have a common eigenvalue. Then there exist d integers κ_i such that $\prod_{i=1}^d \beta_i^{\kappa_i} = 1$, $\sum_{i=1}^d \kappa_i = 2|e|$, and $|\kappa_i| \le N$ with a positive integer N depending on ω and e. Now the map $\alpha \mapsto (\alpha_1, \ldots, \alpha_d)$ sends Y^u into a dense subset of $\mathbf{T}^d$, since Y is dense in $\mathbf{C}^d$ and $x^\rho/x \in Y^u$ for every $x \in Y^\times$. Therefore we can find an element β of Y^u such that $\prod_{i=1}^d \beta_i^{\kappa_i} \ne 1$ for every $\{\kappa_i\}$ as above. This proves the desired fact, and our proof is now complete.

14.8. Proposition. *Let (V, φ) be a model of $\Gamma/\mathcal{H}$ with the properties given in Theorem 9.1 in Case UB; suppose that $\Gamma \subset SU(\mathcal{T})$. Then the following assertions hold:*

(1) *If $0 \ne h \in \mathcal{A}_{\kappa\mathbf{b}}(\Gamma, \overline{\mathbf{Q}})$ with $\kappa \in \mathbf{Z}$, then* $\operatorname{div}(h)$ *considered on V is $\overline{\mathbf{Q}}$-rational.*

(2) *There exist a positive integer m and a nonzero element $g \in \mathcal{A}_{m\mathbf{b}}(\Gamma, \overline{\mathbf{Q}})$ such that* $\operatorname{div}(g)$ *considered on V is $\overline{\mathbf{Q}}$-rational.*

PROOF. Let p be the complex dimension of $\mathcal{H}$. Take p algebraically independent functions $h_1, \ldots, h_p$ in $\mathcal{A}_0(\Gamma, \overline{\mathbf{Q}})$. Let z_{ab}^v be the (a, b)-entry of the matrix z_v which is the v-th component of the variable $z = (z_v)_{v \in \mathbf{a}} \in \mathcal{H}$. Put

$$q = \pi^{-p}\partial(h_1, \ldots, h_p)/\partial(z_1, \ldots, z_p),$$

where $z_1, \ldots, z_p$ are an arbitrarily fixed arrangement of the variables z_{ab}^v for all $v \in \mathbf{a}'$ and all (a, b). From Lemma 3.4 (2) we see that $q^2 \in \mathcal{A}_\zeta(\Gamma)$, where $\zeta(x) = \det(x)^{r\mathbf{b}}$. Now we consider $D_v h_i$ for $v \in \mathbf{a}'$. By Proposition 14.5, $\pi^{-1}(D_v h_i)(w)$ $\big({}^t\mathfrak{p}_{v\rho}(w)^{-1}u\mathfrak{p}_v(w)^{-1}\big)$ is $\overline{\mathbf{Q}}$-rational for every $\overline{\mathbf{Q}}$-rational $u \in T_v$ and for every $w \in \mathcal{H}_{\mathrm{CM}}$ where h_i is finite. From this we see that $q^2 \in \mathcal{A}_\xi(\Gamma, \overline{\mathbf{Q}})$ with $\xi(x) = \prod_{v \in \mathbf{a}'} \det(x_{v\rho})^{2n_v} \det(x_v)^{2m_v}$. By Proposition 11.17 (2), $\mathcal{A}_\xi(\overline{\mathbf{Q}}) = \mathfrak{q} \cdot \mathcal{A}_\zeta(\overline{\mathbf{Q}})$ with a certain constant $\mathfrak{q}$. Thus $\mathfrak{q}^{-1}q^2 \in \mathcal{A}_{r\mathbf{b}}(\Gamma, \overline{\mathbf{Q}})$. Now $\operatorname{div}(q)$ considered on V is the divisor of $dh_1 \wedge \cdots \wedge dh_p$, which is $\overline{\mathbf{Q}}$-rational. Taking $g = \mathfrak{q}^{-1}q^2$, we obtain (2). Then assertion (1) can be proved in exactly the same fashion as in the proof of Proposition 9.8 (1).

PROOF OF (11.19). Let $h = \prod_{v\in\mathbf{b}} \det(R_v)^\kappa$ with R_v as in Proposition 11.14. Then $0 \neq h \in \mathcal{A}_{\kappa\mathbf{b}}(\Gamma, \overline{\mathbf{Q}})$ with a suitable Γ as in Proposition 14.8. Then (11.19) can be proved by the same technique as in the proof of Theorem 9.9 (1), (2).

14.9. Theorem. (1) *Let $\mathcal{W}$ be a dense subset of $\mathcal{H}$ contained in $\mathcal{H}_{\mathrm{CM}}$ (see §11.1), and $\{\omega, X\}$ a $\overline{\mathbf{Q}}$-rational representation of $\mathfrak{K}$. If an element f of $\mathcal{N}_\omega^p$ is ω-arithmetic at every point of $\mathcal{W}$, then $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$.*

(2) *Let Z be an irreducible subspace of $S_e(T)$. Then $\pi^{-|e|}D_\omega^Z f \in \mathcal{N}_{\omega\otimes\tau_Z}^{p+e}(\overline{\mathbf{Q}})$ and $\pi^{|e|}E^Z f \in \mathcal{N}_{\omega\otimes\sigma_Z}^{p'}(\overline{\mathbf{Q}})$ for every $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$, where $|e| = \sum_{v\in\mathbf{a}} e_v$ and $p'_v = \mathrm{Max}(p_v - e_v, 0)$.*

PROOF. Fix p; let $\mathcal{N}'_\omega$ be the set of all $f \in \mathcal{N}_\omega^p$ that are ω-arithmetic at every point of $\mathcal{W}$. Put $\zeta(\alpha) = \det(\alpha)^{\kappa\mathbf{b}}\omega(\alpha)$ for $\alpha \in \mathfrak{K}$ with $0 < \kappa \in \mathbf{Z}$. Let $\mathcal{N}_\zeta^*$ (resp. $\mathcal{N}_\zeta^*(\overline{\mathbf{Q}})$) be the set of all $h \in \mathcal{N}_\zeta^p$ of the form

$$h = \textstyle\sum_{s\leq p} \pi^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s\, g_s \tag{14.9}$$

with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}$ (resp. $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$). By Proposition 14.2, $\mathcal{N}_\zeta^p = \mathcal{N}_\zeta^*$ if κ is sufficiently large for every $v \in \mathbf{a}'$. Fix such a κ. By Proposition 11.15 we see that $\mathcal{N}_\zeta^p$ is spanned by $\mathcal{N}_\zeta^*(\overline{\mathbf{Q}})$ over $\mathbf{C}$. By Theorem 14.7 and Lemma 14.6 (iv), $\pi^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s\, g_s$ is ζ-arithmetic at every point of $\mathcal{H}_{\mathrm{CM}}$ for every $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$, and hence $\mathcal{N}_\zeta^*(\overline{\mathbf{Q}}) \subset \mathcal{N}_\zeta^p(\overline{\mathbf{Q}}) \subset \mathcal{N}'_\zeta$. Let us now prove that $\mathcal{N}'_\zeta \subset \mathcal{N}_\zeta^*(\overline{\mathbf{Q}})$. Take a basis B of $\mathbf{C}$ over $\overline{\mathbf{Q}}$ including 1; let $h \in \mathcal{N}'_\zeta$. Then $h = \sum_{c\in C} ck_c$ with a finite subset C of B and $k_c \in \mathcal{N}_\zeta^*(\overline{\mathbf{Q}})$. For every $w \in \mathcal{W}$ we have $\mathfrak{P}_\zeta(w)^{-1}h(w) = \sum_c c\mathfrak{P}_\zeta(w)^{-1}k_c(w)$. Since $\mathfrak{P}_\zeta(w)^{-1}h(w)$ and $\mathfrak{P}_\zeta(w)^{-1}k_c(w)$ are algebraic, we have $k_c(w) = 0$ for $c \neq 1$, and hence $h(w) = k_1(w)$ for every $w \in \mathcal{W}$. Since $\mathcal{W}$ is dense in $\mathcal{H}$, we have $h = k_1$. This proves that $\mathcal{N}'_\zeta = \mathcal{N}_\zeta^*(\overline{\mathbf{Q}}) = \mathcal{N}_\zeta^p(\overline{\mathbf{Q}})$. To show that $\mathcal{N}'_\omega \subset \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$, take any $w_0 \in \mathcal{H}_{\mathrm{CM}}$ and take $q \in \mathcal{M}_{\kappa_0\mathbf{b}}(\overline{\mathbf{Q}})$ with some $\kappa_0 \in \mathbf{Z}$ so that $q(w_0) \neq 0$. Such a q in Cases SP and UT is obtained in Lemma 6.17; in Case UB take $q = \prod_{v\in\mathbf{b}} \det(Q_v)$ with Q_v of Proposition 11.14. Changing q for its suitable power, we may take $\kappa = \kappa_0$. Let $f \in \mathcal{N}'_\omega$. Clearly $qf \in \mathcal{N}'_\zeta = \mathcal{N}_\zeta^p(\overline{\mathbf{Q}})$. We can take $q(w_0)\mathfrak{P}_\omega(w_0)$ as $\mathfrak{P}_\zeta(w_0)$. Then $\mathfrak{P}_\omega(w_0)^{-1}f(w_0) = \mathfrak{P}_\zeta(w_0)^{-1}(qf)(w_0)$, which is algebraic. Since this is so for every $w_0 \in \mathcal{H}_{\mathrm{CM}}$, we see that $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. This proves (1).

Next, by Lemma 14.6, in order to prove that $\pi^{-|e|}D_\omega^Z f \in \mathcal{N}_{\omega\otimes\tau_Z}^{p+e}(\overline{\mathbf{Q}})$, it is sufficient to show that $\pi^{-|e|}D_\omega^e f \in \mathcal{N}_{\omega\otimes\tau^e}^{p+e}(\overline{\mathbf{Q}})$. Take w_0 and q again; then we can put $qf = \sum_{s\leq p} \ell_s$, $\ell_s = \pi^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s\, g_s$ with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$. Thus our question is the arithmeticity of $\pi^{-|e|}D_\omega^e(q^{-1}\ell_s)$. First, Lemma 13.17 shows that

$$D_\omega^e(q^{-1}\ell_s)(u) = \sum_{a+b=e} \binom{e}{a} \big(D_{-\kappa}^a(q^{-1})\big)(u)(D_\zeta^b\ell_s)(u)$$

for $u \in T$, where $-\kappa$ stands for $-\kappa\mathbf{b}$. Put $\mathfrak{p}_1 = \big(\mathfrak{p}_{v\rho}(w_0)\big)_{v\in\mathbf{a}}$ and $\mathfrak{p}_2 = \big(\mathfrak{p}_v(w_0)\big)_{v\in\mathbf{a}}$. Then we can write $\mathfrak{p}(w_0) = (\mathfrak{p}_1, \mathfrak{p}_2)$. For $\psi \in S_e(T, X)$ we have $\big[\mathfrak{P}_{\omega\otimes\tau^e}(w_0)^{-1}\psi\big](u) = \mathfrak{P}_\omega(w_0)^{-1}\psi({}^t\mathfrak{p}_1^{-1}u\mathfrak{p}_2)$. Now $\pi^{-|a|}D_{-\kappa}^a(q^{-1})$ is arithmetic by Theorem 14.7. On the other hand, by Lemma 14.6 (ii) we have $\pi^{-|b|}D_\zeta^b\ell_s = \pi^{-|s|-|b|}D_\zeta^b\theta_X^s D_{\zeta\otimes\sigma^s}^s\, g_s = \pi^{-|s|-|b|}(\theta_X^s)_b D_{\zeta\otimes\sigma^s}^{s+b}\, g_s$. Hence by Theorem 14.7 and Lemma 14.6 (i, iv) $\pi^{-|b|}D_\zeta^b\ell_s$ is arithmetic. Thus

$$\mathfrak{P}_{\omega\otimes\tau^e}(w_0)^{-1}\pi^{-|e|}D_\omega^e(q^{-1}\ell_s)(u)$$

$$= \sum_{a,\,b} \binom{e}{a} q(w_0)\bigl[\pi^{-|a|}\bigl(D_{-\kappa}^{a}(q^{-1})\bigr)\bigr]({}^t\mathfrak{p}_1^{-1}u\mathfrak{p}_2)\mathfrak{P}_\zeta(w_0)^{-1}\bigl[\pi^{-|b|}(D_\zeta^b\ell_s)\bigr]({}^t\mathfrak{p}_1^{-1}u\mathfrak{p}_2),$$

which is algebraic for every algebraic $u \in T$. This completes the proof of the part of (2) concerning D_ω^Z.

Finally, to deal with E^Z, it is sufficient to show that $\pi^{|e|}E^e f \in \mathcal{N}_{\omega\otimes\sigma^e}^{p'}(\overline{\mathbf{Q}})$ if $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. Clearly it is sufficient to prove it when $|e| = 1$, that is, $E^e = E_v$ with some v. We prove this by induction on $|p|$. If $p = 0$, then f is holomorphic and $E_v f = 0$. Thus we take $|p| > 0$. First we assume that $f = \pi^{-|p|}D_\psi^p g$ with some $g \in \mathcal{M}_\psi(\overline{\mathbf{Q}})$ and $\overline{\mathbf{Q}}$-trational ψ. Then $D_\psi^p g = D_{\psi',v'}D_\psi^{p-v'}g$ with some ψ' and v'. If $v \neq v'$, then $\pi E_v f = \pi^{-1}D_{\psi'',v'}\pi^{2-|p|}E_v D_\psi^{p-v'}g$ with some ψ''. Now $\pi^{1-|p|}D_\psi^{p-v'}g \in \mathcal{N}_{\psi'}^{p-v'}(\overline{\mathbf{Q}})$ by Theorem 14.7, and so $\pi^{2-|p|}E_v D_\psi^{p-v'}g$ is arithmetic by our induction. Then, by what we already proved, $\pi^{-1}D_{\psi'',v'}\pi^{2-|p|}E_v D_\psi^{p-v'}g$ is arithmetic. Next, assume $v = v'$. Then, by Lemma 13.16,

$$\pi E_v f = \pi^{1-|p|}E_v D_{\psi',v}D_\psi^b g = \pi^{1-|p|}D_{\psi'\otimes\sigma_v,v}E_v D_\psi^b g - A\pi^{-|b|}D_\psi^b g$$

with $b = p - v$ and $A \in S_1(T_v, S_1(T_v, \mathrm{End}(Y)))$ defined by $A(u, u') = P_{\psi\otimes\tau^b}^v(u', u)$ for $u, u' \in T_v$, where Y is the representation space of $\psi\otimes\tau^b$. For the same reason as in the case $v \neq v'$, the first term on the right-hand side and $\pi^{-|b|}D_\psi^b g$ are arithmetic. To deal with the whole second term, put $h = \pi^{-|b|}D_\psi^b g$ and $\varphi = \psi\otimes\tau^b$; let $w \in \mathcal{H}_{\mathrm{CM}}$. Then $\bigl[\mathfrak{P}_{\varphi\otimes\tau_v\otimes\sigma_v}(w)^{-1}Ah\bigr](u, u') = A(u, u')\mathfrak{P}_\varphi(w)^{-1}h$ by Proposition 13.15 (4). Since A is $\overline{\mathbf{Q}}$-rational, we can thus establish the arithmeticity of $\pi E_v f$ for $f = \pi^{-|p|}D_\psi^p g$.

Now take an arbitrary element $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. We again employ the expression $qf = \sum_s \pi^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s$ with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$. Since q is holomorphic, we have $q\cdot\pi E_v f = \pi E_v(qf) = \sum_s \pi^{1-|s|}E_v\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s$. By what we proved about $E_v\pi^{-|p|}D_\psi^p g$, each term of the last sum is arithmetic. Since q is arithmetic, $\pi E_v f$ is arithmetic at every CM-point where q is not zero. By (1) this implies that $E_v f$ is arithmetic. This completes the proof.

We note here a simple fact:

(14.9a) *The symbol ζ being as above, let $\mathcal{N}_\zeta^*(\Gamma, \overline{\mathbf{Q}})$ denote the set of elements h of the form (14.9) with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Gamma, \overline{\mathbf{Q}})$. Then $\mathcal{N}_\zeta^*(\Gamma, \overline{\mathbf{Q}}) = \mathcal{N}_\zeta^p(\Gamma, \overline{\mathbf{Q}})$.*

We have seen that $\mathcal{N}_\zeta^*(\Gamma, \overline{\mathbf{Q}}) \subset \mathcal{N}_\zeta^p(\Gamma, \overline{\mathbf{Q}})$. Let $h \in \mathcal{N}_\zeta^p(\Gamma, \overline{\mathbf{Q}})$. By Proposition 14.2 we have (14.9) with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Gamma)$. Applying Proposition 11.15 to the last set, we have $g_s = \sum_{b\in B} bg_{s,b}$ with $g_{s,b} \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Gamma, \overline{\mathbf{Q}})$. In this way we can put $h = \sum_{c\in C} ck_c$ with a finite subset C of B and $k_c \in \mathcal{N}_\zeta^*(\Gamma, \overline{\mathbf{Q}})$. We have seen in the above proof that $h = k_1$. This proves (14.9a).

14.10. Proposition. *$\mathcal{N}_\omega^p = \mathcal{N}_\omega^p(\overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$ and $\mathcal{N}_\omega^p(\Gamma) = \mathcal{N}_\omega^p(\Gamma, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.*

PROOF. Let $f_1, \ldots, f_\mu$ be elements of $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ linearly independent over $\overline{\mathbf{Q}}$. Suppose $\sum_{i=1}^\mu a_i f_i = 0$ with $a_i \in \mathbf{C}$. Let $\{b_i\}_{i=1}^m$ be a $\overline{\mathbf{Q}}$-basis of $\sum_{i=1}^\mu \overline{\mathbf{Q}}a_i$ and let $a_i = \sum_j c_{ij}b_j$ with $c_{ij} \in \overline{\mathbf{Q}}$. Then $\sum_j b_j \sum_i c_{ij}f_i = 0$, and so $\sum_j b_j \sum_i c_{ij}\mathfrak{P}_\omega(w)^{-1}\cdot f_i(w) = 0$ for every $w \in \mathcal{H}_{\mathrm{CM}}$. Since $\mathfrak{P}_\omega(w)^{-1}f_i(w)$ is algebraic, we have $\sum_i c_{ij}\cdot f_i(w) = 0$ for every j and every $w \in \mathcal{H}_{\mathrm{CM}}$. Therefore $\sum_i c_{ij}f_i = 0$ for every j, so that $c_{ij} = 0$. Thus $a_i = 0$, which shows that the f_i are linearly independent over $\mathbf{C}$. To prove that $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ spans $\mathcal{N}_\omega^p$ over $\mathbf{C}$, take ζ, B, and q as in the proof of Theorem 14.9. We have seen that $\mathcal{N}_\zeta^p(\overline{\mathbf{Q}})$ spans $\mathcal{N}_\zeta^p$ over $\mathbf{C}$. Let $f \in \mathcal{N}_\omega^p$; then $qf \in \mathcal{N}_\zeta^p$, and so $qf = \sum_{c\in B} ct_c$ with $t_c \in \mathcal{N}_\zeta^p(\overline{\mathbf{Q}})$. Thus $f = \sum_{c\in B} ct_c/q$. If we change q for q' and $q'f = \sum_{c\in B} ct'_c$ with $t'_c \in \mathcal{N}_\zeta^p(\overline{\mathbf{Q}})$, then $q't_c = qt'_c$. Given an arbitrary point w_0 of $\mathcal{H}$, we can choose a $\overline{\mathbf{Q}}$-rational q so that $q(w_0) \neq 0$. Then t_c/q is nearly holomorphic of degree $\leq p$ in a neighborhood of w_0. Since $t_c/q = t'_c/q'$, this shows that $t_c/q \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. This proves the first equality. Next let $f \in \mathcal{N}_\omega^p(\Gamma)$. Then $f = \sum_{c\in B} cg_c$ with $g_c \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$, and $\sum_c c(g_c - g_c\|\gamma) = f - f\|\gamma = 0$ for every $\gamma \in \Gamma$. Since $g_c\|\gamma \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$, we obtain $g_c\|\gamma = g_c$, so that $g_c \in \mathcal{N}_\omega^p(\Gamma, \overline{\mathbf{Q}})$. This proves the second equality.

14.11. We now restrict our treatment to Cases SP and UT. Let $\{\omega, X\}$ be a representation of $\mathfrak{K}$. If $f \in \mathcal{N}_\omega^p(\Gamma)$, then $f(z) = \sum_{e\leq p} g_e(z)\bigl(r(z)\bigr)$ with holomorphic maps $g_e : \mathcal{H} \to S_e(T, X)$ as noted in (13.23). Recall that $r_v(z) = ({}^t z_v - \overline{z}_v)^{-1}$. Therefore we easily see that $g_e(z+b) = g_e(z)$ if $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \Gamma$, and hence $g_e(z) = \sum_{h\in L} c_e(h)\mathbf{e}_\mathbf{a}^n(hz)$ with $c_e(h) \in S_e(T, X)$ and a lattice L in the vector space S of (5.17). Thus we can put

$$f(z) = \sum_{h\in L} s_h\bigl(r(z)\bigr)\mathbf{e}_\mathbf{a}^n(hz) \tag{14.10}$$

with $s_h \in \bigcup_{e\leq p} S_e(T, X)$. Now, for $\gamma = \mathrm{diag}[a, \widehat{a}] \in \Gamma$ we have $\omega({}^t a, a^*)f(aza^*) = f(z)$ by (5.19). Since $r_v(aza^*) = {}^t a_v^{-1} r_v(z)\overline{a}_v^{-1}$, we easily see that $g_e(z) = (\omega\otimes\sigma^e)({}^t a, a^*)g_e(aza^*)$. Thus g_e satisfies (5.21) for a suitable U with $\omega\otimes\sigma^e$ in place of ω. Suppose now $n > 1$ or $F \neq \mathbf{Q}$. Then from Proposition 5.7 we can conclude that $c_e(h) \neq 0$ only if $h_v \geq 0$ for every $v \in \mathbf{a}$. Consequently *s_h of (14.10) is not zero only if $h_v \geq 0$ for every $v \in \mathbf{a}$.* If $n = 1$ and $F = \mathbf{Q}$, we need the cusp condition (13.18a).

To speak of the rationality of f over a number field, we assume that $\{\omega, X\}$ has a $\mathbf{Q}$-rational structure, and write the expansion of (14.10) in the form

$$f(z) = \sum_{h\in L} q_h\bigl(\pi^{-1}i\cdot r(z)\bigr)\mathbf{e}_\mathbf{a}^n(hz) \tag{14.11}$$

with polynomials q_h belonging to $\bigcup_{e\leq p} S_e(T, X)$. Notice that in Case SP, $\pi^{-1}ir(z) = \bigl(2\pi\cdot\mathrm{Im}(z)\bigr)^{-1}$. Given $\varepsilon \in \mathrm{Aut}(\mathbf{C})$, for $u \in T = \prod_{v\in\mathbf{a}} T_v$ we define $u^{[\varepsilon]} \in T$ by

$$(u^{[\varepsilon]})_v = \begin{cases} u_{v'} & \text{if } v\varepsilon = v' \text{ on } K, \\ {}^t u_{v'} & \text{if } v\varepsilon = v'\rho \text{ on } K, \end{cases} \tag{14.12}$$

where ρ is complex conjugation. In Case UT we are viewing each v as an embedding of K into $\mathbf{C}$ (see §3.5). For $q \in S_e(T, X)$ we define $q^\varepsilon \in S_e(T, X)$ with respect to the natural $\mathbf{Q}$-rational structure of $S_e(T, X)$. Then for f as in (14.11) we define f^ε formally by

$$f^\varepsilon(z) = \sum_{h\in L} q_h^\varepsilon\bigl(\pi^{-1}i\cdot r(z)^{[\varepsilon]}\bigr)\mathbf{e}_\mathbf{a}^n(hz). \tag{14.13}$$

This includes, as a special case, what we defined in §5.10. Given a subfield W of $\mathbf{C}$, we say that f is *W-rational* if q_h is W-rational for every h. We denote by $\mathcal{N}_\omega^p(W)$ the set of all W-rational elements of $\mathcal{N}_\omega^p$, and put $\mathcal{N}_\omega^p(\Gamma, W) = \mathcal{N}_\omega^p(\Gamma)\cap\mathcal{N}_\omega^p(W)$. If

$W = \overline{\mathbf{Q}}$, this is consistent with what was defined in §14.4; see (1) of the following proposition.

Now for $e \in \mathbf{Z}^{\mathbf{a}}$ we define $e\varepsilon \in \mathbf{Z}^{\mathbf{a}}$ by $(e\varepsilon)_{v'} = e_v$ if $v' = v\varepsilon$ on F. (In other words, $e\varepsilon$ is defined so that $a^{e\varepsilon} = (a^e)^\varepsilon$ for $a \in F^\times$.)

14.12. Theorem (Cases SP and UT). (1) *The set* $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ *defined in §14.4 consists of all the* $\overline{\mathbf{Q}}$*-rational elements of* $\mathcal{N}_\omega^p$ *in the above sense.*

(2) For every $\varepsilon \in \mathrm{Aut}(\mathbf{C})$ and $f \in \mathcal{N}_\omega^p$ the formal series f^ε defined by (14.13) is indeed an element of $\mathcal{N}_{\omega^\varepsilon}^{p\varepsilon}$.

(3) We have $\bigl((\pi i)^{-|e|}D_\omega^e f\bigr)^\varepsilon(u^{[\varepsilon]}) = \bigl((\pi i)^{-|e|}D_{\omega^\varepsilon}^{e\varepsilon} f^\varepsilon\bigr)(u)$ and $\bigl((\pi i)^{|e|}E^e f\bigr)^\varepsilon(u^{[\varepsilon]}) = \bigl((\pi i)^{|e|}E^{e\varepsilon} f^\varepsilon\bigr)(u)$ for every $f \in \mathcal{N}_\omega^p$ and $u \in T$.

(4) Let Z be a $\mathfrak{K}$-stable subspace of $S_e(T)$, and W a subfield of $\mathbf{C}$ containing the Galois closure of K over $\mathbf{Q}$. Then $(\pi i)^{-|e|}D_\omega^Z$ (resp. $(\pi i)^{|e|}E^Z$) sends $\mathcal{N}_\omega^p(W)$ into $\mathcal{N}_{\omega\otimes\tau_Z}^{p+e}(W)$ (resp. $\mathcal{N}_{\omega\otimes\sigma_Z}^{p'}(W)$), where $p'_v = \mathrm{Max}(0, p_v - e_v)$.

PROOF. We first prove (3) in a formal sense when $|e| = 1$. By (13.28) we have, for $u \in T$,

$$(*)\qquad (\pi i)^{-1}D_{\omega,v}\bigl(q_h(\pi^{-1}ir)\mathbf{e}_{\mathbf{a}}^n(hz)\bigr)(u) = (\pi i)^{-1}\mathbf{e}_{\mathbf{a}}^n(hz)\bigl\{D_v\bigl(q_h(\pi^{-1}ir)\bigr)(u) + 2\pi i\cdot \mathrm{tr}(h_v u_v)q_h(\pi^{-1}ir) + P_\omega^v(r, u)q_h(\pi^{-1}ir)\bigr\}.$$

By (13.6c) we have

$$(\pi i)^{-1}D_{\omega,v}\bigl(q_h(\pi^{-1}ir)\bigr)(u) = \textstyle\sum_\mu(\partial q_h/\partial z_{v\mu})(\pi^{-1}ir)\cdot(\pi^{-1}ir\cdot {}^tu\cdot\pi^{-1}ir)_{v\mu},$$

where $u_v = \sum_\mu u_{v\mu}a_\mu$ for $u \in T$ with a $\mathbf{Q}$-rational $\mathbf{C}$-basis $\{a_\mu\}$ of T_v, and in particular $z_v = \sum_\mu z_{v\mu}a_\mu$. Therefore the right-hand side of (*) can be written $\mathbf{e}_{\mathbf{a}}^n(hz)\ell_h(\pi^{-1}ir)(u)$ with a polynomial ℓ_h of degree $\le p+v$ with values in $S_1(T_v, X)$. Write ω in the form $\omega(x) = \otimes_{v\in\mathbf{a}}\omega_v(x_{v\rho}, x_v)$, where $x_{v\rho} = x_v$ in Case SP. From (13.27) we easily see that $P_{\omega^\varepsilon}^{v'}(r, u) = P_\omega^v(r^{[\varepsilon]}, u^{[\varepsilon]})$ if $v\varepsilon = v'$ on F (even if $v\varepsilon = v'\rho$ on K). Thus if we replace $\bigl(\omega, v, q_h(\pi^{-1}ir)\bigr)$ by $\bigl(\omega^\varepsilon, v', q_h^\varepsilon(\pi^{-1}ir^{[\varepsilon]})\bigr)$, then $\ell_h(\pi^{-1}ir)(u)$ is replaced by $\ell_h^\varepsilon(\pi^{-1}ir^{[\varepsilon]})(u^{[\varepsilon]})$, as can easily be verified. This proves (3) for $D_{\omega,v}$. The general case of (3) can be proved by induction on $|e|$, which is not completely trivial. First, for $h \in S_e(T, X)$ put $g(u) = h(u^{[\varepsilon]})$. Then $g \in S_{e\varepsilon}(T, X)$ and

$$(14.14)\qquad \bigl[(\omega^\varepsilon\otimes\tau^{e\varepsilon})(\alpha)g\bigr](u) = \bigl[(\omega\otimes\tau^e)^\varepsilon(\alpha)h\bigr](u^{[\varepsilon]})\ \text{ for every } \alpha \in \mathfrak{K}.$$

Here we can also take σ in place of τ. Notice that $(\omega^\varepsilon\otimes\tau^{e\varepsilon})(\mathfrak{K})$ acts on $S_{e\varepsilon}(T, X)$, but $(\omega\otimes\tau^e)^\varepsilon(\mathfrak{K})$ acts on $S_e(T, X)$ (see §9.10). Now take $h \in C^\infty\bigl(\mathcal{H}, S_e(T, X)\bigr)$ and define $g \in C^\infty\bigl(\mathcal{H}, S_{e\varepsilon}(T, X)\bigr)$ by $g(u) = h(u^{[\varepsilon]})$. Then from (14.14) we can easily derive

$$(14.15)\qquad \bigl(D_{\omega^\varepsilon\otimes\tau^{e\varepsilon}}^a g\bigr)(u, w) = \bigl(D_{(\omega\otimes\tau^e)^\varepsilon}^a h\bigr)(u^{[\varepsilon]}, w)\qquad (u, w \in T).$$

Once this is established, our induction can be done in a straightforward way. This proves (3) for D_ω^e. The statement concerning E^e can be proved by using (13.10) in a similar and simpler way. Clearly our argument proves (4) for $Z = S_e(T)$. For Z of a general type, we only have to observe that φ_Z is $\mathbf{Q}$-rational, and so it sends $\mathcal{N}_{\omega\otimes\tau^e}^e(W)$ (resp. $\mathcal{N}_{\omega\otimes\sigma^e}^e(W)$) into $\mathcal{N}_{\omega\otimes\tau_Z}^e(W)$ (resp. $\mathcal{N}_{\omega\otimes\sigma_Z}^e(W)$).

To prove (1) and (2), we use the symbols κ, ζ, B, and q in the proof of Theorem 14.9. By Lemma 6.17 we may assume that $q \in \mathcal{M}_{\kappa\mathbf{b}}(\mathbf{Q})$. Let $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. In that proof we showed that $qf = \sum_{s\le p}\pi^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s$ with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$. Then for every $\sigma \in \mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$ we have $g_s^\sigma = g_s$, and hence $qf^\sigma = (qf)^\sigma = qf$ by (3). Thus $f^\sigma = f$, which means that f is a $\overline{\mathbf{Q}}$-rational element of $\mathcal{N}_\omega^p$. Conversely let f be a $\overline{\mathbf{Q}}$-rational element of $\mathcal{N}_\omega^p$. By Proposition 14.10 we have $f = \sum_{c\in B} cf_c$ with $f_c \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. Let q_h and q_{ch} be the polynomials defined for f and f_c by (14.11). Then $q_h = \sum_{c\in B} cq_{ch}$. We have shown that f_c is $\overline{\mathbf{Q}}$-rational, so that q_{ch} is $\overline{\mathbf{Q}}$-rational. Then $q_h = q_{1h}$, and hence $f = f_1 \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. This proves (1). To prove (2), we employ the expression $qf = \sum_{s\le p}(\pi i)^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s$ with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}$. Since (14.15) holds with σ in place of τ, we obtain, by (3),

$$\begin{aligned}\left[(\pi i)^{-|s|}D_{\zeta\otimes\sigma^s}^s g_s\right]^\varepsilon(u^{[\varepsilon]}, w^{[\varepsilon]}) &= \left[(\pi i)^{-|s|}D_{(\zeta\otimes\sigma^s)^\varepsilon}^{s\varepsilon} g_s^\varepsilon\right](u^{[\varepsilon]}, w)\\ &= \left[(\pi i)^{-|s|}D_{\zeta^\varepsilon\otimes\sigma^{s\varepsilon}}^{s\varepsilon} \ell_s\right](u, w),\end{aligned}$$

where ℓ_s is defined by $\ell_s(u) = g_s^\varepsilon(u^{[\varepsilon]})$, and hence

$$\left[(\pi i)^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s\right]^\varepsilon = (\pi i)^{-|s|}\theta_X^{s\varepsilon} D_{\zeta^\varepsilon\otimes\sigma^{s\varepsilon}}^{s\varepsilon}\ell_s. \tag{14.16}$$

This shows that $(qf)^\varepsilon$ is meaningful as an element of $\mathcal{N}_{\zeta^\varepsilon}^{p\varepsilon}$. If we take another $q' \in \mathcal{M}_{\kappa\mathbf{b}}(\mathbf{Q})$, then $q(q'f)^\varepsilon = (qq'f)^\varepsilon = q'(qf)^\varepsilon$, so that $(qf)^\varepsilon/q = (q'f)^\varepsilon/q'$. Define a function f^* by $f^* = (qf)^\varepsilon/q$. This is defined where q is not zero. Given $z_0 \in \mathcal{H}$, we can find $q' \in \mathcal{M}_{\kappa\mathbf{b}}(\mathbf{Q})$ so that $q'(z_0) \ne 0$. Since $f^* = (q'f)^\varepsilon/q'$, we see that f^* is defined as a C^∞ function on $\mathcal{H}$. Besides, the equality $f^* = (qf)^\varepsilon/q$ shows that $f^* \in \mathcal{N}_{\omega^\varepsilon}^{p\varepsilon}$, and $qf^* = (qf)^\varepsilon$. Now $(qf)^\varepsilon$ and qf^ε coincide as formal series, and hence qf^* coincides with qf^ε as a formal series. As shown in §5.10, q is not a zero-divisor in the ring of formal series defined there. Therefore, expressing f^ε and f^* as polynomials in r whose coefficients are elements of the ring, we see that f^ε is the expansion of f^* in the sense of (14.11). This proves (2).

14.13. Proposition (Cases SP and UT). *Let Φ be the Galois closure of K over $\mathbf{Q}$. Then the following assertions hold:*

(1) $\mathcal{N}_\omega^p = \mathcal{N}_\omega^p(\Phi)\otimes_\Phi \mathbf{C}$ and $\mathcal{N}_\omega^p(\Gamma) = \mathcal{N}_\omega^p(\Gamma, \Phi)\otimes_\Phi \mathbf{C}$ for every $\Gamma \subset G$ as in Theorem 10.4 such that $\mathcal{M}_{\mu\mathbf{b}}(\Gamma, \overline{\mathbf{Q}}) \ne \{0\}$ for some $\mu \in \mathbf{Z}, > 0$.

(2) *If W is a subfield of $\overline{\mathbf{Q}}$ containing Φ and $\mathbf{Q}_{\mathrm{ab}}$, then $\mathcal{N}_\omega^p(W)$ is stable under the map $f \mapsto f\|_\omega \alpha$ for every $\alpha \in G$.*

PROOF. Take an arbitrary subfield Ψ of $\overline{\mathbf{Q}}$ containing Φ; let B be a Ψ-basis of $\mathbf{C}$ and M a $\mathbf{Q}$-rational $\mathbf{C}$-basis of $\bigcup_{e\le p} S_e(T, X)$. Suppose $0 = \sum_{c\in A} cf_c$ with a finite subset A of B and $f_c \in \mathcal{N}_\omega^p(\Psi)$. Put $f_c = \sum_{\ell\in M}\sum_{h\in S} a(c, \ell, h)\ell(\pi^{-1}i\cdot r(z))\mathbf{e}_{\mathbf{a}}^n(hz)$ with $a(c, \ell, h) \in \Psi$. Then $0 = \sum_{c\in A} ca(c, \ell, h)$ for every (ℓ, h), and so $a(c, \ell, h) = 0$ for every (c, ℓ, h), that is, $f_c = 0$ for every c. This result is of course applicable to the case $\Psi = \Phi$. Thus, to prove (1), it is sufficient to prove, in view of Proposition 14.10, that $\mathcal{N}_\omega^p(\Phi)$ spans $\mathcal{N}_\omega^p(\overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$. Let $f \in \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. We use the expression $qf = \sum_{s\le p}(\pi i)^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_s$ with $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\overline{\mathbf{Q}})$ already employed in the proof of Theorem 14.12. By Theorem 10.4 (4) we have $g_s = \sum_{c\in A} cg_{c,s}$ with a finite subset A of $\overline{\mathbf{Q}}$, linearly independent over Φ, and with $g_{c,s} \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Phi)$. Then $qf = \sum_{c\in A} ck_c$ with $k_c = \sum_{s\le p}(\pi i)^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s g_{c,s}$. By Theorem 14.12 (4), $k_c \in \mathcal{N}_\zeta^p(\Phi)$. If we change q for another q', then we have a similar expression $q'f = \sum_{c\in A} ck'_c$. Then $0 = \sum_{c\in A} c(q'k_c - qk'_c)$, and hence $q'k_c = qk'_c$. Thus $q^{-1}k_c = q'^{-1}k'_c$. Since we can choose $q \in \mathcal{M}_{\kappa\mathbf{a}}(\mathbf{Q})$ so that q does not vanish at any given point of $\mathcal{H}$, we can define a C^∞ function f_c on $\mathcal{H}$ by $f_c = q^{-1}k_c$. Clearly $f_c \in \mathcal{N}_\omega^p(\Phi)$, and $f = \sum_{c\in A} cf_c$. This proves the first part of (1). Suppose

$f \in \mathcal{N}_\omega^p(\Gamma, \overline{\mathbf{Q}})$. Taking q to be a power of an element of $\mathcal{M}_{\mu\mathbf{b}}(\Gamma, \overline{\mathbf{Q}})$, we may assume, by (14.9a), that $g_s \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Gamma, \overline{\mathbf{Q}})$; thus, by Theorem 10.4 (4) we may assume that $g_{c,s} \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Gamma, \Phi)$. Then $k_c \in \mathcal{N}_\zeta^p(\Gamma, \Phi)$, and hence $f_c \in \mathcal{N}_\omega^p(\Gamma, \Phi)$, which, together with the second part of Proposition 14.10 proves the second part of (1).

To prove (2), we again take $f \in \mathcal{N}_\omega^p(W)$ and apply the above procedure to f. We can take A to be the set $\{\, ab \mid a \in A_1,\, b \in A_2 \,\}$, where A_1 is a finite subset of W including 1 and linearly independent over Φ, and A_2 is a finite subset of $\overline{\mathbf{Q}}$ including 1 and linearly independent over W. Then, as shown above, $f = \sum_{a\in A_1}\sum_{b\in A_2} abf_{ab}$ with $f_{ab} \in \mathcal{N}_\omega^p(\Phi)$. Since $f \in \mathcal{N}_\omega^p(W)$, what we said at the beginning shows that $f_{ab} = 0$ if $b \neq 1$, and so $f = \sum_{a\in A_1} af_a$. Now we obtained f_a in the form $f_a = q^{-1}\sum_{s\le p}(\pi i)^{-|s|}\theta_X^s D_{\zeta\otimes\sigma^s}^s\, g_{a,s}$ with $g_{a,s} \in \mathcal{M}_{\zeta\otimes\sigma^s}(\Phi)$. Apply $\|_\omega\alpha$ to this equality with $\alpha \in G$. Then from Theorem 9.13 (3), (12.21), (14.2), and Theorem 14.12 (4) we see that $f_a\|_\omega\alpha \in \mathcal{N}_\omega^p(\Phi\mathbf{Q}_{\mathrm{ab}})$. This completes the proof.

14.14. Let us now extend our results of Sections 12 through 14 to the case of modular forms of half-integral weight. We naturally restrict our discussion to Type CT (that is, Case SP), employing the symbols introduced in §10.6; in particular we take a symbol ψ, which we called a quasi-representation of $\mathfrak{K}$ in §10.6, given by

$$\psi(x) = \det(x)^{\mathbf{a}/2}\omega(x) \tag{14.17}$$

with a $\mathbf{Q}$-rational representation $\{\omega, X\}$ of $GL_n(\mathbf{C})^{\mathbf{a}}$. (If $\omega(x) = \det(x)^m$ with $m \in \mathbf{Z}^{\mathbf{a}}$, then we can put $\psi(x) = \det(x)^k$ with $k_v = m_v + (1/2)$; see §6.10.) We also consider the group $\mathcal{G}$ consisting of all couples (α, p) formed by $\alpha \in G$ and a holomorphic function p on $\mathcal{H}$ such that $p(z)^2/j_\alpha^{\mathbf{a}}$ is a root of unity, the group-law being defined by $(\alpha, p)(\alpha', p') = \bigl(\alpha\alpha', p(\alpha' z)p'(z)\bigr)$. For γ in the group Γ^θ of (6.30) we put $\widetilde{\gamma} = (\gamma, h_\gamma)$ with h_γ of Theorem 6.8. Then $\gamma \mapsto \widetilde{\gamma}$ is an injective homomorphism of Γ^θ into $\mathcal{G}$. We call a subgroup Δ of $\mathcal{G}$ a *congruence subgroup* of $\mathcal{G}$ if the projection map of $\mathcal{G}$ onto G gives an isomorphism of Δ onto a congruence subgroup Γ of G, and the inverse of this isomorphism coincides with the map $\gamma \mapsto \widetilde{\gamma}$ on a congruence subgroup of $\Gamma^\theta \cap \Gamma$. Any conjugate $\xi\Delta\xi^{-1}$ of such a Δ with $\xi \in \mathcal{G}$ is also a congruence subgroup, by virtue of Theorem 6.9 (1). We shall often view a congruence subgroup Γ of Γ^θ as a congruence subgroup of $\mathcal{G}$ by identifying it with its image in $\mathcal{G}$ through the map $\gamma \mapsto \widetilde{\gamma}$.

Now for $\alpha = (\alpha_0, p) \in \mathcal{G}$ and $f \in \mathbf{C}^\infty(\mathcal{H}, X)$ we put

$$(f\|_\psi\, \alpha)(z) = p(z)^{-1}(f\|_\omega\, \alpha_0)(z). \tag{14.18a}$$

If $\omega(x) = \det(x)^m$ and $\psi(x) = \det(x)^k$ with $k = m + \mathbf{a}/2$ as above, then we write $f\|_k\, \alpha$ for $f\|_\psi\, \alpha$; then

$$(f\|_k\, \alpha)(z) = p(z)^{-1}(f\|_m\, \alpha_0)(z). \tag{14.18b}$$

For simplicity we put $M_\alpha(z) = M_{\alpha_0}(z)$ and $\alpha z = \alpha_0 z$. We also define quasi-representations $\psi\otimes\tau^e$ and $\psi\otimes\sigma^e$ by

$$(\psi\otimes\tau^e)(x) = \det(x)^{\mathbf{a}/2}(\omega\otimes\tau^e)(x), \quad (\psi\otimes\sigma^e)(x) = \det(x)^{\mathbf{a}/2}(\omega\otimes\sigma^e)(x). \tag{14.19}$$

We define $D_{\psi,v}^e$, $D_{\psi,v}$, D_ψ^e, and D_ψ^Z by (12.17), (12.18), (13.21) and (13.22) with ψ in place of ρ, in which we take $\psi(\Xi) = \det(\eta)^{\mathbf{a}/2}\omega(\Xi)$ with positive $\det(\eta)^{\mathbf{a}/2}$. Then Proposition 12.10 and (12.21) are true with ψ in place of ρ and with $\alpha \in \mathcal{G}$. The proof of Proposition 12.10 is valid for ψ if we put

$$\psi(M_\alpha) = p \cdot \omega(M_\alpha). \tag{14.20}$$

Lemma 13.9 is valid also for half-integral k and $\alpha \in \mathcal{G}$.

We already defined $\mathcal{M}_\psi$, $\mathcal{S}_\psi$, and $\mathcal{A}_\psi$ in §10.6. We denote by $C_\psi(\Delta)$ the set of all $f \in C^\infty(\mathcal{H}, X)$ such that $f\|_\psi \gamma = f$ for every $\gamma \in \Delta$, and by C_ψ the union of $C_\psi(\Delta)$ for all congruence subgroups Δ of $\mathcal{G}$. We denote by $\mathcal{N}_\psi^p(\Delta)$ the subset of $\mathcal{N}^p(\mathcal{H}, X) \cap C_\psi(\Delta)$ consisting of the functions satisfying the cusp condition, which is required only when G is isogenous to $SL_2(\mathbf{Q})$, and which is an obvious modification of (13.18a). We then denote by $\mathcal{N}_\psi^p$ the union of $\mathcal{N}_\psi^p(\Delta)$ for all such Δ. The inner product $\langle f, g \rangle$ for $f, g \in C_\psi$ can be defined by (12.35a) with ψ in place of ρ. Then Theorem 12.15 is valid with ψ in place of ρ, since the problem can be formulated on a suitable covering group of $G_{\mathbf{a}}$, and can be reduced to the results of [S90] by virtue of the principle of [S94b, Proposition 2.2].

Now ψ can be viewed as a local homomorphism of $\mathfrak{K}$ into $GL(X)$, and so $d\psi : (\mathbf{C}_n^n)^{\mathbf{a}} \to \mathrm{End}(X)$ is meaningful. Therefore Lemma 13.14 and Proposition 13.15 are valid with ψ in place of ρ.

We say that an element f of $\mathcal{N}_\psi^p$ is *arithmetic* at $w \in \mathcal{H}_{\mathrm{CM}}$ if $\mathfrak{P}_\psi(w)^{-1}f(w)$ is $\overline{\mathbf{Q}}$-rational, where we naturally put $\mathfrak{P}_\psi(w) = \prod_{v\in\mathbf{a}} \det(\mathfrak{p}_v(w))^{1/2}\mathfrak{P}_\omega(w)$. Since we are interested only in the algebraicity, the choice of square roots does not matter. We then dfine $\mathcal{N}_\psi^p(\overline{\mathbf{Q}})$ to be the set of all $f \in \mathcal{N}_\psi^p$ that are arithmetic at every $w \in \mathcal{H}_{\mathrm{CM}}$. Now for $f \in \mathcal{N}_\psi^p$ we have an expansion of type (14.10), and so we define f^ε by (14.13) and the rationality of f over a subfield of $\mathbf{C}$ in the same way as for the elements of $\mathcal{N}_\omega^p$. Then we can verify that the results up to Proposition 14.13 are all valid with ψ in place of ω.

14.15. Let us now specialize our discussion to the Hilbert modular case, by taking $G = SL_2(F)$ and $X = \mathbf{C}$. Then we can put $T = \mathbf{C}^{\mathbf{a}}$, $S_e(T, X) = \mathbf{C}$ by identifying $h \in S_e(T, X)$ with $h(1, \ldots, 1)$, and $\omega(x) = x^k$ for $x \in (\mathbf{C}^\times)^{\mathbf{a}}$ with an integral or a half-integral weight k (see §§6.10 and 14.13). Therefore, rewriting the expansion of (14.11), we see that every element of $\mathcal{N}_k^p$ is of the form

$$f(z) = \sum_{h\in F} \sum_{0\le e\le p} c(h, e)(\pi y)^{-e}\mathbf{e}_{\mathbf{a}}(hz) \qquad (z \in \mathfrak{H}_1^{\mathbf{a}}) \tag{14.21}$$

with $c(h, e) \in \mathbf{C}$, where $y = \mathrm{Im}(z)$; $c(h, e) \neq 0$ only if $h = 0$ or $h \gg 0$. Then

$$f^\varepsilon(z) = \sum_{h\in F} \sum_{0\le e\le p} c(h, e)^\varepsilon(\pi y)^{-e\varepsilon}\mathbf{e}_{\mathbf{a}}(hz). \tag{14.22}$$

Thus $f \in \mathcal{N}_k^p(W)$ with a subfield W of $\mathbf{C}$ if and only if $c(h, e) \in W$ for every h and e. Write D_k^e for D_ω^e. Then $(\pi i)^{-|e|}D_k^e = (\pi i)^{-|e|}\prod_{v\in\mathbf{a}} \delta_{k_v}^{e_v}$ with the symbol δ of (12.39). Also, we can ignore (u) and $(u^{[\varepsilon]})$ in Theorem 14.12 (3).

We note that the space $\mathcal{N}_k^p$ can be completely determined; see [S87, Theorem 5.2]. Also a generalization of Theorem 10.9 (1) can be given as follows:

14.16. Theorem. *Let Y, h, w, and Y^* be as in §9.4; suppose $G = SL_2(F)$. (Thus Y is a CM-field and h is an F-linear ring-injection of Y into F_2^2.) Let $f \in \mathcal{M}_k(\overline{\mathbf{Q}})$ and $p \in \mathcal{M}_{k+2e}(\overline{\mathbf{Q}})$ with $k \in \mathbf{Z}^{\mathbf{a}}$ and $0 \le e \in \mathbf{Z}^{\mathbf{a}}$. Given $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, let b be an element of $(Y^*)_{\mathbf{A}}$ such that $\sigma = [b, Y^*]$ on Y_{ab}^* and let $r = h(g(b)^{-1})$ with g of (9.3). Then*

$$\{p^{-1}(\pi i)^{-|e|}D_k^e f\}(w)^\sigma = p^{(r,\sigma)}(w)^{-1}(\pi i)^{-|e|}\big(D_k^e f^{(r,\sigma)}\big)(w)$$

for every CM-point w *such that* $p(w) \neq 0$.

For the proof, see [S75, Main Theorem III]. Notice that $p^{(r,\sigma)}(w) \neq 0$ by Theorem 10.9 (3).

The arithmeticity of $D_\omega^e f$ at CM-points was first obtained in [S75] for $\widetilde{G} = GL_2(F)$ and in [S80] for orthogonal and unitary groups. A general framework of the arithmeticity problems of zeta functions and Eisenstein series was presented in [S78c]. The notion of near holomorphy was introduced in [S86], and further developed in [S87a]. Most of the results in Sections 13 and 14 were essentially given in those two papers. A notable exception is Theorem 14.12, which was not given there. As explained in [S87a, Section 3], we can develop the theory axiomatically so that all the results for other types of groups can be proved in the same manner, once a few basic facts such as the result corresponding to Proposition 14.5 in each case is established. Such a result was given in [S78c, Theorem 4] for a certain quaternion unitary group belonging to Type C, and also in [S80, Proposition 6.6] in the orthogonal case. We can also characterize the elements of $\mathcal{M}_\omega(\overline{\mathbf{Q}})$ in Case UB by the properties of the theta functions that appear as the Fourier coefficients of a given automorphic form; see [S78a] and [S78b, §7]. If $\dim(H_v) \leq 1$ for every $v \in \mathbf{a}$ (which is the case if $G = SL_2(F)$), then we can completely determine the structure of $\mathcal{N}_k^p$, so that we can state Proposition 14.2 in a much stronger form; for details, see [S87a, Theorems 5.2 and 5.5].

15. Holomorphic projection

15.1. Our next aim is to find a certain projection map $\mathcal{N}_\omega^p \to \mathcal{M}_\omega$. It is necessary to consider $\theta h = \sum_{\nu \in N} h(a_\nu, b_\nu)$ defined by (12.33) with any pair of dual bases $\{a_\nu\}$ and $\{b_\nu\}$ of T for several specific $h \in Ml_2(T, X) = S_1(T, S_1(T, X))$. (We again fix one $v \in \mathbf{a}$, and drop the subscript v from the objects T_v, τ_v^p, etc.) For example, for $h(x, y) = {}^t xy$ we can easily verify that

$$\theta h = \sum_{\nu \in N} {}^t a_\nu b_\nu = \lambda(T) 1_n, \tag{15.1}$$

where $\lambda(T) = m$ for Type A and $\lambda(T) = (n+1)/2$ for Type C.

We now define a $\mathbf{C}$-linear endomorphism ψ of $S_p(T)$ by

$$\psi = 0 \quad \textit{if} \quad p = 1, \tag{15.2a}$$

$$(\psi h)(x) = \textstyle\sum_{\nu \in N} h_*(a_\nu,\, x \cdot {}^t b_\nu x,\, x,\, \ldots,\, x) \qquad (p > 1), \tag{15.2b}$$

for $h \in S_p(T)$ and $x \in T$. We can easily verify that this is independent of the choice of dual bases, and $\psi \tau^p(\alpha) = \tau^p(\alpha)\psi$ for every $\alpha \in K^c$, and hence, for each irreducible subspace Z of $S_p(T)$ there is a constant c_Z such that $\psi h = c_Z h$ for every $h \in Z$. Thus $c_Z = 0$ if $p = 1$.

15.2. Lemma. *The constant* c_Z *is a rational number such that* $-1 \leq c_Z \leq 1$ *for Type A and* $-1/2 \leq c_Z \leq 1$ *for Type C. Moreover,* $c_Z = 1$ *if* Z *contains the element* h *of* $S_p(T)$ *given by* $h(x) = x_{11}^p$, *and* $c_Z < 1$ *otherwise. In particular,* $c_Z = -1$ *(resp.* $c_Z = -1/2$*) for Type A (resp. Type C) if* Z *contains the element* h *of* $S_2(T)$ *given by* $h(x) = \det_2(x)$.

PROOF. Type A. Let $\ell = \mathrm{Min}(m, n)$ and let $\zeta = \sum_{i=1}^{\ell} e_{ii}$ $(\in T)$ with the standard matrix units e_{ij}. By Theorem 12.7 we can take a highest weight vector of

Z in the form $h(x) = \prod_{i=1}^{\ell} \det_i(x)^{c_i}$ with $0 \le c_i \in \mathbf{Z}$. Take $k \in Ml_p(T, \mathbf{C})$ so that $h(x) = k(x, \ldots, x)$ by the rule which can be illustrated by the following example: if $p = 9$, $c_1 = c_2 = 2$, $c_3 = 1$, then

$$k(r, s, t, u, v, w, x, y, z) = r_{11}s_{11} \begin{vmatrix} t_{11} & u_{12} \\ t_{21} & u_{22} \end{vmatrix} \cdot \begin{vmatrix} v_{11} & w_{12} \\ v_{21} & w_{22} \end{vmatrix} \cdot \begin{vmatrix} x_{11} & y_{12} & z_{13} \\ x_{21} & y_{22} & z_{23} \\ x_{31} & y_{32} & z_{33} \end{vmatrix}.$$

Then $p!h_* = \sum_\pi k_\pi$ with $k_\pi(x_1, \ldots, x_p) = k(x_{\pi(1)}, \ldots, x_{\pi(p)})$. Since $h(\zeta) = 1$, we have $c_Z = (\psi h)(\zeta) = (1/p!) \sum_\pi \sum_\nu k_\pi(a_\nu, \zeta \cdot {}^t b_\nu \zeta, \zeta, \ldots, \zeta)$. Each $k_\pi(x, y, \zeta, \ldots, \zeta)$ belongs to the following three types of functions: $x_{ii}y_{ii}$, $x_{ii}y_{jj}$ $(i \ne j)$, and $x_{ii}y_{jj} - x_{ji}y_{ij}$ $(i \ne j)$. The value $\sum_\nu k_\pi(a_\nu, \zeta \cdot {}^t b_\nu \zeta, \zeta, \ldots, \zeta)$ is 1, 0, and -1, respectively. Therefore we obtain $-1 \le c_Z \le 1$. If $h(x) = x_{11}^p$, then the only possible type is $x_{11}y_{11}$, and hence $c_Z = 1$. If $\det_i(x)$ with $i > 1$ is involved in h, then $x_{11}y_{22} - x_{21}y_{12}$ can always occur, and hence $c_Z < 1$. In particular, if $h(x) = \det_2(x)$, then that is the only possible type, so that $c_Z = -1$.

Type C. We can employ the same technique as for Type A with 1_n in place of ζ, $h(x) = \prod_{i=1}^{n} \det_i(x)^{c_i}$ and the same k. Then $p!c_Z = \sum_\pi \sum_\nu k_\pi(a_\nu, b_\nu, 1, \ldots, 1)$. Each $k_\pi(x, y, 1, \ldots, 1)$ belongs to the following three types of functions: $x_{ii}y_{ii}$, $x_{ii}y_{jj}$ $(i \ne j)$, and $x_{ii}y_{jj} - x_{ij}y_{ij}$ $(i \ne j)$. The value $\sum_\nu k_\pi(a_\nu, b_\nu, 1, \ldots, 1)$ is 1, 0, and $-1/2$, respectively, and hence $-1/2 \le c_Z \le 1$. The remaining part concerning c_Z for each Z can be proved in the same manner as for Type A.

Before stating the next proposition, we define an operator $L_{\omega,v}$ by

$$L_{\omega,v} = -\theta D_{\omega \otimes \sigma_v^1,\, v} E_v. \tag{15.3}$$

This is a special case of (12.31), which we already mentioned in Corollary 12.16 and Lemma 13.16, and $L_{\omega,v}(f\|_\omega \alpha) = (L_{\omega,v}f)\|_\omega \alpha$ for every $\alpha \in G_{\mathbf{a}}$.

15.3. Proposition. *Let $0 \ne p = (p_v)_{v \in \mathbf{a}} \in \mathbf{Z}^{\mathbf{a}}$ with $p_v \ge 0$ for every $v \in \mathbf{a}$ and $\{\omega, X\}$ a representation or a quasi-representation (in the sense of §14.13) of $\mathfrak{K}$. Suppose that for every $v \in \mathbf{a}$ such that H_v is nontrivial we have $\omega(a_v, b_v) = \det(b_v)^{k_v}$ with $k_v \in (1/2)\mathbf{Z}$. For an irreducible subspace Z of $S_i(T_v)$, put*

$$\alpha_Z = i\{k_v - \kappa + (1-i)c_Z\},$$

where $\kappa = m + n$ or $n + 1$ according as G is of Type A or C, where we understand that $\kappa = 2n$ for Type AT. Suppose that for each v such that $p_v > 0$ the number k_v satisfies the following inequalities:

$$k_v > m + n + p_v - 1 \quad \text{or} \quad k_v < m + n + 1 - p_v \quad \text{if } G \text{ is of Type A},$$
$$k_v > n + p_v \quad \text{or} \quad k_v < n + (3 - p_v)/2 \quad \text{if } G \text{ is of Type C}.$$

Put $A_v^i = \prod_Z \left(1 - \alpha_Z^{-1} L_{\omega,v}\right)$ for $0 < i \le p_v$, where Z runs over all the irreducible subspaces of $S_i(T_v)$, and $\mathfrak{A} = \prod_{v \in \mathbf{a}^*} \prod_{i=1}^{p_v} A_v^i$, where $\mathbf{a}^* = \{ v \in \mathbf{a} \mid p_v > 0 \}$. (Notice that the estimate of c_Z given in Lemma 15.2 shows that $\alpha_Z \ne 0$.) Let $f \in \mathcal{N}_\omega^p$. Then $\mathfrak{A}f \in \mathcal{M}_\omega$ and $f = \mathfrak{A}f + \sum_{v \in \mathbf{a}^*} L_{\omega,v} t_v$ with $t_v \in \mathcal{N}_\omega^p$.

PROOF. We fix one v and consider f as a function of z_v, suppressing the remaining variables. By (13.16) we have $f = \sum_{i=0}^q f_i$, $f_i(z_v) = g_i\big(r_v(z_v)\big)$, $q = p_v$, with a holomorphic map $g_i : H_v \to S_i(T_v)$. Then $E_v f = \sum_{i=1}^q E_v f_i$, and $E_v f_i$ is of degree $i - 1$ in r_v. To study the highest term $E_v f_q$, write simply g for g_q. Then g_* is a holomorphic map of H_v into $Ml_q(T_v, X)$. Let us now write simply T, r, k, D, E, etc. for T_v, r_v, k_v, D_v, E_v, etc. By (13.10), for $u \in T$ we have

$$(Ef_q)(u) = -\sum_{\nu\in N} u_\nu(\partial/\partial r_\nu)g_*(r, \dots, r) = -qg_*(u, r, \dots, r).$$

By (12.7b) and (12.18) we see that $(D_{\omega\otimes\sigma^1}Ef_q)(u, v) = \det(\eta)^{-k}Y(\xi u\cdot {}^t\eta, v)$ with $Y(u, v) = D\{\det(\eta)^k(Ef_q)(\xi^{-1}u\cdot {}^t\eta^{-1})\}(v)$ for $u, v \in T$. Then by (13.8a) and (13.6a, b) we have $D(\det(\eta)^k)(v) = k\cdot\det(\eta)^k\mathrm{tr}({}^trv)$, and

$$D(\xi^{-1}u\cdot{}^t\eta^{-1})(v) = -r\cdot{}^tv\xi^{-1}u\cdot{}^t\eta^{-1} - \xi^{-1}u\cdot{}^t\eta^{-1}\cdot{}^tvr.$$

Therefore

$$\begin{aligned}-q^{-1}(D_{\omega\otimes\sigma^1}Ef_q)(u, v) &= k\cdot\mathrm{tr}({}^trv)g_*(u, r, \dots, r) - g_*(w, r, \dots, r)\\ &\quad + (q-1)g_*(u, (Dr)(v), r, \dots, r) + \sum_{\nu\in N} v_\nu(\partial g_*/\partial z_\nu)(u, r, \dots, r),\end{aligned}$$

where $w = r\cdot{}^tvu + u\cdot{}^tvr$. Applying θ to this equality, we obtain $kg(r)$ from the first term on the right-hand side and, by (15.1), $-\kappa g(r)$ from the second term with κ given as in our proposition. Now $(Dr)(v)$ is given by (13.6c), and therefore θ times the third term is $(1-q)(\psi g)(r)$. The last sum $\sum_\nu$ is of degree at most $q-1$ in r_v. Thus we obtain

$$L_{\omega,v}f \equiv L_{\omega,v}f_q \equiv p_v\{k_v - \kappa + (1-p_v)\psi\}g(r) \pmod{\mathcal{N}^{p'}},$$

where $p'_v = p_v - 1$ and $p'_t = p_t$ for $v \neq t \in \mathbf{a}$. (This is true even if $p_v = 1$.) Let φ_Z be the projection map $S_q(T_v) \to Z$. Then $p_v\{k_v - \kappa + (1-p_v)\psi\} = \sum_Z \alpha_Z\varphi_Z$. Now $L_{\omega,v}$ maps $\mathcal{N}_\omega^p$ into itself. Therefore, for an irreducible subspace W of $S_q(T_v)$ we have

$$(1-\alpha_W^{-1}L_{\omega,v})f \equiv \sum_Z(1-\alpha_W^{-1}\alpha_Z)\varphi_Z g \equiv \sum_{Z\neq W}(a_Z\varphi_Z g)(r) \pmod{\mathcal{N}^{p'}}$$

with $a_Z = 1-\alpha_W^{-1}\alpha_Z$. Call the last sum g'. Taking $(1-\alpha_W^{-1}L_{\omega,v})f$ and g' in place of f and g, we obtain, for another irreducible subspae Y of $S_q(T_v)$,

$$(1-\alpha_Y^{-1}L_{\omega,v})(1-\alpha_W^{-1}L_{\omega,v})f \equiv \textstyle\sum_{Z\notin\{Y,W\}}(b_Z\varphi_Z g')(r) \pmod{\mathcal{N}^{p'}}$$

with $b_Z \in \mathbf{Q}$. Repeating this procedure, we find that $A_v^q f \in \mathcal{N}_\omega^{p'}$ with A_v^q defined in our proposition. Therefore if $h = \mathfrak{A}f$, then $h \in \mathcal{N}_\omega^0 = \mathcal{M}_\omega$. Since $\prod_i A_v^i$ is a polynomial in $L_{\omega,v}$ whose constant term is 1, we obtain the desired expression for f.

15.4. Corollary. *Let $f \in \mathcal{N}_\omega^p$ and $h = \mathfrak{A}f$ with $\mathfrak{A}$ as in Proposition 15.3. Then we have $\langle\varphi, f\rangle = \langle\varphi, h\rangle$ for $\varphi \in \mathcal{M}_\omega$, provided either φ is a cusp form, or $\Gamma\backslash\mathcal{H}$ is compact for a congruence subgroup Γ.*

PROOF. We have $f = h + \sum_{v\in\mathbf{a}'} L_{\omega,v}t_v$ with $t_v \in \mathcal{N}_\omega^p$. By Theorem 12.15,

$$\langle\varphi, L_{\omega,v}t_v\rangle = \langle\varphi, -\theta D_{\omega\otimes\sigma_v^1, v}E_v t_v\rangle = \langle E_v\varphi, E_v t_v\rangle = 0,$$

since $E_v\varphi = 0$ because of the holomorphy of φ. This proves the desired equality. We need a suitable convergence condition, which is certainly satisfied for φ or Γ as above.

15.5. In Cases SP and UT, for $f \in \mathcal{N}_\omega^p$ let $q_h(u, f)$ denote the polynomial in $u \in T$ which appears as a Fourier coefficient of f as in (14.11). Then we put

$$\mathcal{R}_\omega^p = \{f \in \mathcal{N}_\omega^p \mid q_h(u, f\|\alpha) = 0 \text{ for every } \alpha \in G \text{ and every } h \text{ such that } \det(h) = 0\}, \tag{15.4}$$

$$\mathcal{T}_\omega^p = \{f \in \mathcal{R}_\omega^p \mid \langle f, \mathcal{S}_\omega\rangle = 0\}, \tag{15.5}$$

$$\mathcal{R}_\omega^p(\Gamma) = \mathcal{R}_\omega^p \cap \mathcal{N}_\omega^p(\Gamma), \quad \mathcal{T}_\omega^p(\Gamma) = \mathcal{T}_\omega^p \cap \mathcal{N}_\omega^p(\Gamma). \tag{15.6}$$

Here ω is either a representation or a quasi-representation of $\mathfrak{K}$, and Γ is a congruence subgroup of $G = U(\eta_n)$ or $\mathcal{G}$ (see §14.13). Clearly $\mathcal{S}_\omega = \mathcal{M}_\omega \cap \mathcal{R}_\omega^p$. Notice that $\langle f, g \rangle$ is meaningful for $f, g \in \mathcal{R}_\omega^p$, since the elements of $\mathcal{R}_\omega^p$ are rapidly decreasing at the cusps.

15.6. Proposition (Cases SP and UT). *The notation being as above, the following assertions hold:*

(1) $\mathcal{T}_\omega^p(\Gamma) = \{ f \in \mathcal{R}_\omega^p(\Gamma) \mid \langle f, \mathcal{S}_\omega(\Gamma)\rangle = 0 \}$.

(2) $\mathcal{R}_\omega^p(\Gamma) = \mathcal{S}_\omega(\Gamma) \oplus \mathcal{T}_\omega^p(\Gamma)$ *and* $\mathcal{R}_\omega^p = \mathcal{S}_\omega \oplus \mathcal{T}_\omega^p$.

(3) *Let* $\mathfrak{p} : \mathcal{R}_\omega^p \to \mathcal{S}_\omega$ *be the projection map obtained from the last direct sum decomposition of (2). Then* $(\mathcal{R}_\omega^p)^\varepsilon = \mathcal{R}_{\omega^\varepsilon}^{p\varepsilon}$ *and* $\mathfrak{p}(f)^\varepsilon = \mathfrak{p}(f^\varepsilon)$ *for every* $\varepsilon \in \mathrm{Aut}(\mathbf{C})$, *and* $\langle f, g \rangle = \langle \mathfrak{p}(f), g \rangle$ *for every* $f \in \mathcal{R}_\omega^p$ *and every* $g \in \mathcal{S}_\omega$.

PROOF. Clearly the left-hand side of the equality of (1) is contained in the right-hand side. Let f be an element of the right-hand side and let $g \in \mathcal{S}_\omega$. Take a subgroup Γ' of Γ of finite index so that $g \in \mathcal{S}_\omega(\Gamma')$ and take a set of representatives A for $\Gamma' \backslash \Gamma$. Then $\sum_{\alpha \in A} g\|\alpha \in \mathcal{S}_\omega(\Gamma)$, so that $0 = \sum_{\alpha \in A}\langle f, g\|\alpha \rangle = \sum_{\alpha \in A}\langle f\|\alpha^{-1}, g \rangle = [\Gamma : \Gamma']\langle f, g \rangle$. Thus $f \in \mathcal{T}_\omega^p(\Gamma)$. This proves (1). By Lemma 14.3 , $\mathcal{R}_\omega^p(\Gamma)$ is finite-dimensional over $\mathbf{C}$, and hence from (1) we obtain the first equality of (2), which clearly implies the second equality. To prove (3), take $f \in \mathcal{N}_\omega^p$; modifying the expression for qf in the proof of Theorem 14.9, we can put $qf = \sum_s (\pi i)^{-|s|}\theta_X^s D_{\zeta \otimes \sigma^s}^s g_s$ with $g_s \in \mathcal{M}_{\zeta \otimes \sigma^s}$. By (14.16) we have $q^\varepsilon f^\varepsilon = (qf)^\varepsilon = \sum_s (\pi i)^{-|s|}\theta_X^{s\varepsilon} D_{\zeta^\varepsilon \otimes \sigma^{s\varepsilon}}^{s\varepsilon} \ell_s$. If ω is a representation of $\mathfrak{K}$, then given $\alpha \in G$, take β as in Lemma 10.5. Then by that lemma and Proposition 12.10 (2), $(qf)^\varepsilon\|_{\zeta^\varepsilon} \alpha = \big((qf)\|_\zeta \beta\big)^\varepsilon$ and $q^\varepsilon\|_{\kappa\mathbf{b}}\alpha = (q\|_{\kappa\mathbf{b}}\beta)^\varepsilon$, and hence $f^\varepsilon\|_{\omega^\varepsilon}\alpha = (f\|_\omega \beta)^\varepsilon$. Suppose $f \in \mathcal{R}_\omega^p$; then from the last equality we see that $f^\varepsilon \in \mathcal{R}_{\omega^\varepsilon}^p$. This shows that $(\mathcal{R}_\omega^p)^\varepsilon \subset \mathcal{R}_{\omega^\varepsilon}^p$. Considering the action of ε^{-1} similarly, we obtain $(\mathcal{R}_\omega^p)^\varepsilon = \mathcal{R}_{\omega^\varepsilon}^p$. If ω is a quasi-representation, given $\alpha \in \mathcal{G}$, we can find $\beta \in \mathcal{G}$ with which we can make a similar type of argument, as shown in the proof of Theorem 10.8. Then we obtain the desired result. Let $\Lambda = \sum_{v \in \mathbf{a}} L_{\omega,v}$ and $\Lambda' = \sum_{v \in \mathbf{a}} L_{\omega^\varepsilon,v}$ (see Corollary 12.16). From the proof of Theorem 14.12 we easily see that Λ maps $\mathcal{R}_\omega^p$ into itself. Then Theorem 12.15 shows that Λ is a hermitian operator on $\mathcal{R}_\omega^p(\Gamma)$, so that $\mathcal{R}_\omega^p(\Gamma)$ is a finite direct sum $\bigoplus_\mu \mathcal{E}_\mu$, where $\mathcal{E}_\mu = \{ f \in \mathcal{R}_\omega^p(\Gamma) \mid \Lambda f = \mu f \}$. By Corollary 12.16 we have $\mathcal{S}_\omega(\Gamma) = \mathcal{E}_0$, so that $\mathcal{T}_\omega^p(\Gamma) = \sum_{\mu \neq 0} \mathcal{E}_\mu$ by (1). Let $f \in \mathcal{R}_\omega^p(\Gamma)$; then $f = g + \sum_{\mu \neq 0} h_\mu$ with $g \in \mathcal{S}_\omega(\Gamma)$ and $h_\mu \in \mathcal{E}_\mu$, and $f^\varepsilon = g^\varepsilon + \sum_{\mu \neq 0} h_\mu^\varepsilon$. By Theorem 10.8 (1), $g^\varepsilon \in \mathcal{S}_{\omega^\varepsilon}$. Now, for a fixed $h \in \mathcal{N}_\omega^p$, take $\pi i E_v h$ and ω as g_s and ζ in (14.16). By Theorem 14.12 (3) we have $\ell_s = \pi i E_{v\varepsilon} h^\varepsilon$, and hence (14.16) shows that $(L_{\omega,v}h)^\varepsilon = L_{\omega^\varepsilon, v\varepsilon} h^\varepsilon$. Thus $(\Lambda h)^\varepsilon = \Lambda' h^\varepsilon$, so that we have $\Lambda' h_\mu^\varepsilon = \mu^\varepsilon h_\mu^\varepsilon$ and hence $\langle \mathcal{S}_{\omega^\varepsilon}, h_\mu^\varepsilon \rangle = 0$. Thus $\sum_{\mu \neq 0} h_\mu^\varepsilon \in \mathcal{T}_{\omega^\varepsilon}^p$. This shows that $g^\varepsilon = \mathfrak{p}(f^\varepsilon)$, which is the second equality of (3). Since $f - \mathfrak{p}(f) \in \mathcal{T}_\omega^p$, the last equality of (3) follows from (15.5).

15.7. Proposition (Case UB). *Suppose that* $\Gamma \backslash \mathcal{H}$ *is compact for a congruence subgroup* Γ *of* G. *Put* $\mathcal{T}_\omega^p = \{ f \in \mathcal{N}_\omega^p \mid \langle f, \mathcal{S}_\omega \rangle = 0 \}$ *and* $\mathcal{T}_\omega^p(\overline{\mathbf{Q}}) = \mathcal{T}_\omega^p \cap \mathcal{N}_\omega^p(\overline{\mathbf{Q}})$. *Then the following assertions hold:*

(1) $\mathcal{T}_\omega^p(\Gamma) = \{ f \in \mathcal{N}_\omega^p(\Gamma) \mid \langle f, \mathcal{S}_\omega(\Gamma)\rangle = 0 \}$.

(2) $\mathcal{N}_\omega^p(\Gamma) = \mathcal{S}_\omega(\Gamma) \oplus \mathcal{T}_\omega^p(\Gamma)$ *and* $\mathcal{N}_\omega^p = \mathcal{S}_\omega \oplus \mathcal{T}_\omega^p$.

(3) $\mathcal{N}_\omega^p(\overline{\mathbf{Q}}) = \mathcal{S}_\omega(\overline{\mathbf{Q}}) \oplus \mathcal{T}_\omega^p(\overline{\mathbf{Q}})$ *and* $\mathcal{T}_\omega^p = \mathcal{T}_\omega^p(\overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.

PROOF. Under our assumption, $\langle f, g \rangle$ is meaningful for every $f, g \in \mathcal{N}_\omega^p$, and so (1) and (2) can be proved in the same manner as for Proposition 15.6. To prove (3), put $\Lambda = \sum_{v\in\mathbf{a}'} L_{\omega,v}$. Then, for the same reason as in the proof of Proposition 15.6, $\mathcal{N}_\omega^p(\Gamma) = \bigoplus_\mu \mathcal{E}_\mu$ with $\mathcal{E}_\mu = \{\, f \in \mathcal{N}_\omega^p(\Gamma) \mid \Lambda f = \mu f \,\}$, $\mathcal{S}_\omega(\Gamma) = \mathcal{E}_0$, and $\mathcal{T}_\omega^p(\Gamma) = \sum_{\mu\neq 0} \mathcal{E}_\mu$. By Theorem 14.9 (2) and Lemma 14.6 (iv), Λ maps $\mathcal{N}_\omega^p(\Gamma, \overline{\mathbf{Q}})$ into itself. Therefore, in view of the last equality of Proposition 14.10 we see that $\mu \in \overline{\mathbf{Q}}$ and $\mathcal{E}_\mu = \big(\mathcal{E}_\mu \cap \mathcal{N}_\omega^p(\overline{\mathbf{Q}})\big) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$, from which we obtain (3) in a straightforward way.

15.8. Lemma (Cases SP and UT). *Let W be a subfield of $\mathbf{C}$ containing the Galois closure of K in $\mathbf{C}$ over $\mathbf{Q}$; let $g \in \mathcal{M}_l(W)$ and $h \in \mathcal{M}_{l'}(W)$ with $l, l' \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$; further define the operator $\Delta_{l'}^p$ by $\Delta_{l'}^p h = (D_{\rho'}^Z h)(\psi)$ with $\rho'(a, b) = \det(b)^{l'}$, $Z = \bigotimes_{v\in\mathbf{a}} Z_v = \mathbf{C}\psi \subset S_{np}(T)$, where $Z_v = \mathbf{C}\psi_v \subset S_{np_v}(T_v)$, $\psi = \prod_{v\in\mathbf{a}} \psi_v$, and $\psi_v = \det(x)^{p_v}$. Put $l_0 = \mathrm{Min}\{l_v, l'_v \mid v \in \mathbf{a}\}$ and $k = l + l' + 2p$. Suppose that $l_0 \geq n/2$ in Case SP and $l_0 \geq n$ in Case UT. Then there exists an element q of $\mathcal{M}_k(W)$ such that $(\pi i)^{-n|p|}\langle g\Delta_{l'}^p h, f\rangle = \langle q, f\rangle$ for every $f \in \mathcal{S}_k$.*

PROOF. From (12.24c) and Theorem 14.12 (4) we see that $(\pi i)^{-n|p|}\Delta_{l'}^p$ sends $\mathcal{M}_{l'}(W)$ into $\mathcal{N}_{l'+2p}^{np}(W)$. Therefore, if g or h is a cusp form, then $g\Delta_{l'}^p h$ belongs to $\mathcal{R}_\omega^{np}$ with $\omega(a, b) = \det(b)^k$, and hence our assertion follows from Proposition 15.6 (3). If neither g nor h is a cusp form, then $l = \mu\mathbf{a}$ and $l' = \nu\mathbf{a}$ with $\mu, \nu \in 2^{-1}\mathbf{Z}$ by virtue of [S97, Proposition 10.6 (3)]. This case will be proven in §A8.8.

If $G = SL_2(F)$, we can prove a result stronger than Proposition 15.3 and Proposition 15.6. In fact, for every weight $k \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$, there exists a $\mathbf{C}$-linear map $\mathfrak{p}_k : \bigcup_p \mathcal{N}_k^p \to \mathcal{S}_k$ with the following properties:

(15.7a) $\langle f, h\rangle = \langle \mathfrak{p}_k(f), h\rangle$ *for every* $f \in \bigcup_p \mathcal{N}_k^p$ *and* $h \in \mathcal{S}_k$;

(15.7b) $\mathfrak{p}_k(f)^\sigma = \mathfrak{p}_{k\sigma}(f^\sigma)$ *for every* $\sigma \in \mathrm{Aut}(\mathbf{C})$.

See [S87b, Proposition 9.4] for the proof.

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CHAPTER IV

EISENSTEIN SERIES OF SIMPLER TYPES

16. Eisenstein series on $U(\eta_n)$

16.1. This section concerns Cases SP and UT. Thus $G = Sp(n, F)$ or $G = U(\eta_n)$ as in §3.5, and $G_1 = G \cap SL_{2n}(K)$. We retain the convention that $K = F$, $\mathfrak{r} = \mathfrak{g}$, and $\rho = \mathrm{id}_F$ in Case SP. We start with preliminary discussions on certain infinite series that appear as nonarchimedean factors of a Fourier coefficient of an Eisenstein series. We first put

(16.1a) $$S = \{\, h \in K_n^n \mid h^* = h \,\},$$

(16.1b) $$S(\mathfrak{a}) = S \cap (\mathfrak{r}\mathfrak{a})_n^n, \qquad S_{\mathbf{h}}(\mathfrak{a}) = \textstyle\prod_{v\in\mathbf{h}} S(\mathfrak{a})_v,$$

(16.1c) $$\widetilde{S} = \widetilde{S}^n = S_{\mathbf{a}} \times \prod_{v\in\mathbf{h}} \widetilde{S}_v \ (\subset S_{\mathbf{A}}), \quad \widetilde{S}_v = \widetilde{S}_v^n = \{\, x \in S_v \mid \mathrm{tr}\big(x \cdot S(\mathfrak{r})_v\big) \subset \mathfrak{g}_v \,\}.$$

Here $\mathfrak{a}$ is a fractional ideal in F or K. We need the symbols $\mathbf{e}$, $\mathbf{e_h}$, $\mathbf{e_a}$, and $\mathbf{e_A}$ introduced in §1.6; we also recall that $\mathbf{e}_{\mathbf{a}}^n(X) = \exp\big(2\pi i \sum_{v\in\mathbf{a}} \mathrm{tr}(X)\big)$ for $X \in (\mathbf{C}_n^n)^{\mathbf{a}}$ and $\mathbf{e_a}(x) = \mathbf{e}_{\mathbf{a}}^1(x)$ for $x \in \mathbf{C}^{\mathbf{a}}$, as defined in (5.15) and (5.16). Similarly we put

(16.2) $$\mathbf{e}_{\mathbf{x}}^n(W) = \mathbf{e}_{\mathbf{x}}\big(\mathrm{tr}(W)\big) \ \textit{for}\ W \in (K_{\mathbf{x}})_n^n \ \textit{such that}\ \mathrm{tr}(W) \in F_{\mathbf{x}},$$

where $\mathbf{x}$ is one of the symbols v, $\mathbf{h}$, and $\mathbf{A}$. For example, $\mathbf{e}_v^n(YZ)$ is meaningful for $Y, Z \in S_v$ in both Cases SP and UT.

For $\sigma \in S_{\mathbf{A}}$ we define an integral $\mathfrak{r}$-ideal $\nu_0(\sigma)$ and a positive integer $\nu(\sigma)$ by (1.20) and (1.21). Since $\sigma^* = \sigma$, we can show (see [S97, §13.4]) that

(16.3) $$\nu_0(\sigma) = \big(\mathfrak{g} \cap \nu_0(\sigma)\big)\mathfrak{r}.$$

We then put

(16.4) $$\nu[\sigma] = N\big(\mathfrak{g} \cap \nu_0(\sigma)\big) \qquad (\sigma \in S_{\mathbf{A}}).$$

Notice that $\nu(\sigma) = \nu[\sigma]^{[K:F]}$. For $s \in S_{\mathbf{A}}$ in Case SP we put

(16.5) $$\gamma(s) = \prod_{v\in\mathbf{h}} \gamma_v(s), \quad \gamma_v(s) = \int_{L_v} \mathbf{e}_v(xs \cdot {}^t x/2)dx, \quad L_v = (\mathfrak{g}_v)_n^1,$$

(16.6) $$\omega(s) = \gamma(s)/|\gamma(s)|,$$

where dx is the Haar measure of $(F_v)_n^1$ such that $\int_{L_v} dx = 1$, and we assume that $\gamma(s) \neq 0$ in (16.6). Clearly $\gamma_v(s) = 1$ for almost all v, and so the product over all $v \in \mathbf{h}$ is meaningful. Since we view S_v as a subset of $S_{\mathbf{A}}$, we can speak of $\gamma(s)$ and $\omega(s)$ for $s \in S_v$. We shall show in Lemma A1.6 that $\gamma_v(s) \neq 0$ if $v \nmid 2$.

Let $\mathfrak{d}$ be the different of F relative to $\mathbf{Q}$. Take an element δ of $F_{\mathbf{h}}^{\times}$ such that $\mathfrak{d} = \delta\mathfrak{g}$. Given $\zeta \in \widetilde{S}$, we put

(16.7a) $\alpha_{\mathfrak{c}}^0(\zeta, s) = \prod_{v \nmid \mathfrak{c}} \alpha_v(\zeta, s), \quad \alpha_v^0(\zeta, s) = \sum_{\sigma \in S_v/S(\mathfrak{r})_v} \mathbf{e}_v^n(-\delta_v^{-1}\zeta\sigma)\nu[\sigma]^{-s},$

(16.7b) $\alpha_{\mathfrak{c}}^1(\zeta, s) = \prod_{v \nmid \mathfrak{c}} \alpha_v^1(\zeta, s), \quad \alpha_v^1(\zeta, s) = \sum_{\sigma \in S_v/S(\mathfrak{g})_v} \omega(\delta_v^{-1}\sigma)\mathbf{e}_v^n(-\delta_v^{-1}\zeta\sigma)\nu(\sigma)^{-s}.$

The series $\alpha_{\mathfrak{c}}^0$ and α_v^0 are defined in both Cases SP and UT, but $\alpha_{\mathfrak{c}}^1$ and α_v^1 are defined only in Case SP under the condition that $v \nmid 2$ and $\mathfrak{c} \subset 4\mathfrak{g}$. Clearly $\gamma_v(s) = \gamma_v(s+b)$ if $b \prec (2/\delta_v)\mathfrak{g}_v$, and hence $\omega_v(s) = \omega_v(s+b)$ for such a b. Since $\nu_0(\sigma)$ depends only on σ modulo $(\mathfrak{r}_v)_n^n$, we see that the series of (16.7a, b) are formally well-defined. These series appear as nonarchimedean factors of the Fourier coefficients of Eisenstein series, as will be shown in Proposition 16.9 below.

We call an element ψ of $\widetilde{S}_v$ *regular* if:

$K = F$, *n is even, and* $\det(2\psi) \in \mathfrak{g}_v^\times$; *or*

$K = F$, *n is odd, and* $\det(2\psi) \in 2\mathfrak{g}_v^\times$; *or*

$K \neq F$, *n is even, and* $\det(\varepsilon\psi) \in \mathfrak{r}_v^\times$, *where* ε *is an element of* $K_v^\times$ *such that* $\varepsilon^{-1}\mathfrak{r}_v = \{ x \in K_v \mid \mathrm{Tr}_{K/F}(x\mathfrak{r}_v) \subset \mathfrak{g}_v \}$; *or*

$K \neq F$, *v is unramified in K, n is odd, and* $\det(\psi) \in \mathfrak{g}_v^\times$.

In Case SP, for $\xi \in \widetilde{S}_v^r \cap GL_r(F_v)$ we define $\lambda(\xi)$ as follows: Put $h = (-1)^{r/2}\det(\xi)$ if r is even, and $h = 2(-1)^{(r-1)/2}\det(\xi)$ if r is odd; then $\lambda(\xi) = 1$ if $F_v(h^{1/2}) = F_v$, $\lambda(\xi) = -1$ if $F_v(h^{1/2})$ is an unramified quadratic extension of F_v, and $\lambda(\xi) = 0$ if $F_v(h^{1/2})$ is ramified over F_v.

16.2. Theorem. *Let* $\zeta \in \widetilde{S}_v^n$, $v \in \mathbf{h}$, *and* $\mathrm{rank}(\zeta) = r$; *suppose* $\zeta = 0$ *or* $\zeta = \mathrm{diag}[\xi, 0]$ *with* $\xi \in \widetilde{S}_v^r \cap GL_r(K_v)$; *put* $q = |\pi_v|^{-1}$ *with a prime element* π_v *of* F_v. *Define power series* $A_\zeta^0(t)$ *and* $A_\zeta^1(t)$ *in an indeterminate* t *by*

$$A_\zeta^0(t) = \sum_{\sigma \in S_v/S(\mathfrak{r})_v} \mathbf{e}_v^n(-\delta_v^{-1}\zeta\sigma)t^{e(\sigma)},$$

$$A_\zeta^1(t) = \sum_{\sigma \in S_v/S(\mathfrak{g})_v} \omega(\delta_v^{-1}\sigma)\mathbf{e}_v^n(-\delta_v^{-1}\zeta\sigma)t^{e(\sigma)},$$

where $e(\sigma)$ *is the integer defined by* $\nu[\sigma] = q^{e(\sigma)}$. *(This means that* $A_\zeta^i(q^{-s}) = \alpha_v^i(\zeta, s)$.*) Then* $A_\zeta^0 = f_\zeta^0 g_\zeta^0$ *and* $A_\zeta^1(t) = f_\zeta^1(t)g_\zeta^1(q^{-1/2}t)$ *with polynomials* g_ζ^i *with coefficients in* $\mathbf{Z}$ *whose constant terms are 1 and rational functions* f_ζ^i *given as follows:*

$$f_\zeta^0(t) = \frac{(1-t)\prod_{i=1}^{[n/2]}(1-q^{2i}t^2)}{(1-\lambda q^{(2n-r)/2}t)\prod_{i=1}^{[(n-r)/2]}(1-q^{2n-r-2i+1}t^2)} \qquad \text{(Case SP, } r \in 2\mathbf{Z}\text{)},$$

$$f_\zeta^0(t) = \frac{(1-t)\prod_{i=1}^{[n/2]}(1-q^{2i}t^2)}{\prod_{i=1}^{[(n-r+1)/2]}(1-q^{2n-r-2i+2}t^2)} \qquad \text{(Case SP, } r \notin 2\mathbf{Z}\text{)},$$

$$f_\zeta^0(t) = \frac{\prod_{i=1}^{n}(1-\tau^{i-1}q^{i-1}t)}{\prod_{i=1}^{n-r}(1-\tau^{n+i}q^{n+i-1}t)} \qquad \text{(Case UT)},$$

$$f_\zeta^1(t) = \frac{\prod_{i=1}^{[(n+1)/2]}(1-q^{2i-1}t^2)}{(1-\lambda q^{(2n-r)/2}t)\prod_{i=1}^{[(n-r)/2]}(1-q^{2n+1-r-2i}t^2)} \qquad (r \notin 2\mathbf{Z}),$$

$$f_\zeta^1(t) = \frac{\prod_{i=1}^{[(n+1)/2]}\left(1-q^{2i-1}t^2\right)}{\prod_{i=1}^{[(n-r+1)/2]}\left(1-q^{2n+2-r-2i}t^2\right)} \qquad (r \in 2\mathbf{Z}).$$

Here $\lambda=1$ *if* $\zeta=0$ *and* $\lambda=\lambda(\xi)$ *otherwise;* τ^i *is the symbol defined as follows:*

$$(16.8) \qquad \tau^i = \begin{cases} 1 & \text{if } i \text{ is even or } v \text{ splits in } K, \\ -1 & \text{if } i \text{ is odd and } v \text{ remains prime in } K, \\ 0 & \text{if } i \text{ is odd and } v \text{ is ramified in } K. \end{cases}$$

Moreover, $A_\zeta^0 = f_\zeta^0$ *if* ξ *is regular or* $\zeta = 0$, *except when* $K \neq F$, v *is ramified in* K, *and* $\zeta \neq 0$; $A_\zeta^1 = f_\zeta^1$ *if* $\xi \in GL_r(\mathfrak{g}_v)$ *or* $\zeta = 0$.

The assertion concerning A_ζ^0 and the formulas for f_ζ^0 were given in [S97, Theorem 13.6]. We shall prove the part concerning A_ζ^1 and f_ζ^1 in §A1.9.

16.3. By a *Hecke character* of an algebraic number field K we understand a continuous homomorphism ψ of $K_\mathbf{A}^\times$ into $\mathbf{T}$ such that $\psi(K^\times) = 1$. For such a ψ we denote by ψ_v, $\psi_\mathbf{a}$, and $\psi_\mathbf{h}$ its restrictions to $K_v^\times$, $K_\mathbf{a}^\times$, and $K_\mathbf{h}^\times$, respectively. Also we denote by ψ^* the ideal character associated with ψ. We put $\psi^*(\mathfrak{a}) = 0$ if $\mathfrak{a}$ is not prime to the conductor of ψ. In the setting of §16.1, we have $\psi_\mathbf{a}(x) = x_\mathbf{a}^h|x_\mathbf{a}|^{i\kappa-h}$ with $h \in \mathbf{Z}^\mathbf{a}$ and $\kappa \in \mathbf{R}^\mathbf{a}$. In this book we always assume that ψ is *normalized* in the sense that $\sum_{v\in\mathbf{a}}\kappa_v = 0$.

Given a Hecke character ψ of K, we define the L-function $L(s, \psi)$ as usual, and its partial series $L_\mathfrak{c}(s, \psi)$ for an integral ideal $\mathfrak{c}$ in F by

$$(16.9) \qquad L_\mathfrak{c}(s, \psi) = \prod_{\mathfrak{p}\nmid\mathfrak{c}}\left[1-\psi^*(\mathfrak{p})N(\mathfrak{p})^{-s}\right]^{-1} = L(s, \psi)\prod_{\mathfrak{p}|\mathfrak{c}}\left[1-\psi^*(\mathfrak{p})N(\mathfrak{p})^{-s}\right].$$

Here $\mathfrak{p}$ denotes a prime ideal in K. We also put $\psi_\mathfrak{c} = \prod_{v|\mathfrak{c}}\psi_v$.

16.4. For $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_{2n}(K)_\mathbf{A}$ with $a \in (K_\mathbf{A})_n^n$, we write $a = a_x$, $b = b_x$, $c = c_x$, and $d = d_x$. We let $G_\mathbf{A}$ act on $\mathcal{H}$ through the projection map $G_\mathbf{A} \to G_\mathbf{a}$ (see §3.5), and define $j_\alpha(z) = j(\alpha, z)$ for $\alpha \in G_\mathbf{A}$ and $z \in \mathcal{H}$ by (5.3). We put

$$(16.10) \qquad P = \{\, \xi \in G \mid c_\xi = 0 \,\},$$

$$(16.11) \qquad Q = \{\, \xi \in P \mid b_\xi = 0 \,\}, \quad R = \{\, \xi \in P \mid a_\xi = 1 \,\}.$$

We already introduced modular forms of half-integral weight in §6.10. To deal with them adelically we need to consider the metaplectic group $M_\mathbf{A}$ and its subset $\mathfrak{M}$ defined by

$$(16.12) \qquad \mathfrak{M} = \{\, \sigma \in M_\mathbf{A} \mid \mathrm{pr}(\sigma) \in P_\mathbf{A}C^\theta \,\},$$

where pr is the map of (16.13) below and C^θ is defined by (6.29) (see also (A2.12a, b)). We need the following facts ($1 \sim 4$) which will be explained in detail in §A2 in the Appendix I:

(16.13) *There is a surjective homomorphism* $\mathrm{pr} : M_\mathbf{A} \to G_\mathbf{A} = Sp(n, F_\mathbf{A})$ *whose kernel is* $\mathbf{T}$ *(viewed as a subgroup of* $M_\mathbf{A}$ *in a certain way).*

(16.14) *There is a lift* $G \to M_\mathbf{A}$ *which combined with* pr *gives the identity map on* G.

(16.15) *There is also a lift* $r_P : P_\mathbf{A} \to M_\mathbf{A}$ *in the same sense that coincides with the lift of (16.14) on* P.

For $\alpha = \mathrm{pr}(\sigma)$ with $\sigma \in M_{\mathbf{A}}$ and $z \in \mathcal{H}$ we put $a_\sigma = a_\alpha$, $b_\sigma = b_\alpha$, $c_\sigma = c_\alpha$, $d_\sigma = d_\alpha$, $\sigma(z) = \sigma z = \alpha z$, and $j_\sigma(z) = j_\alpha(z)$.

(16.16) *For every* $\sigma \in \mathfrak{M}$ *there is a holomorphic function* $h_\sigma(z) = h(\sigma, z)$ *of* $z \in \mathcal{H}$ *with the following properties:*

(16.16a) $h(\sigma, z)^2 = \zeta \cdot j\big(\mathrm{pr}(\sigma), z\big)^{\mathbf{a}}$ *with* $\zeta \in \mathbf{T}$; $h(\sigma, z) \in \mathbf{T}$ *if* $\mathrm{pr}(\sigma)_{\mathbf{a}} = 1$;

(16.16b) $h(t \cdot r_P(\gamma), z) = t^{-1}|\det(d_\gamma)_{\mathbf{a}}|_{\mathbf{A}}^{1/2}$ *if* $t \in \mathbf{T}$ *and* $\gamma \in P_{\mathbf{A}}$;

(16.16c) $h(\rho\sigma\tau, z) = h(\rho, z)h(\sigma, \tau z)h(\tau, z)$ *if* $\mathrm{pr}(\rho) \in P_{\mathbf{A}}$ *and* $\mathrm{pr}(\tau) \in C^\theta$;

(16.16d) $h(\sigma, z)$ *coincides with* h *of Theorem 6.8 if* σ *belongs to* Γ^θ *of (6.30).*

We shall always view G and its subgroups as subgroups of $M_{\mathbf{A}}$ by means of the lift of (16.14). In §14.14 we defined a group $\mathcal{G}$ consisting of all (α, p) such that $p^2/j_\alpha^{\mathbf{a}}$ is a root of unity. We can view $\mathcal{G}$ as a subgroup of $M_{\mathbf{A}}$ by identifying (α, p) with the element σ of $M_{\mathbf{A}}$ such that $\mathrm{pr}(\sigma) = \alpha_{\mathbf{a}}$ and $h_\sigma = p$.

Let k be an integral or a half-integral weight. Here an integral weight means an element of $\mathbf{Z}^{\mathbf{b}}$; a half-integral weight occurs only in Case SP; see §6.10. We define $\mathcal{M}_k$ as in §§5.5 and 6.10. Now we define a factor of automorphy j^k by

$$j_\sigma^k(z) = j^k(\sigma, z) = \begin{cases} j(\sigma, z)^k & (\sigma \in G_{\mathbf{A}},\ k \in \mathbf{Z}^{\mathbf{b}}), \\ h_\sigma(z) j_\sigma(z)^{[k]} & (\sigma \in \mathfrak{M},\ k \notin \mathbf{Z}^{\mathbf{b}}), \end{cases} \tag{16.17}$$

where $[k] = (k_v - 1/2)_{v \in \mathbf{a}}$, and we are employing the notation of (5.4a). We put $[k] = k$ if $k \in \mathbf{Z}^{\mathbf{b}}$. Then, given $f : \mathcal{H} \to \mathbf{C}$ and $\sigma \in G_{\mathbf{A}}$ or $\sigma \in \mathfrak{M}$, we define $f\|_k\sigma : \mathcal{H} \to \mathbf{C}$ by

$$(f\|_k\,\sigma)(z) = j_\sigma^k(z)^{-1} f(\sigma z). \tag{16.18}$$

For $k \in \mathbf{Z}^{\mathbf{b}}$ this is consistent with (5.7). If k is half-integral, $\sigma \in G \cap \mathfrak{M}$, and we identify σ with the element (σ, h_σ) of $\mathcal{G}$ of §14.14, then (16.18) is consistent with (14.18b); notice that $f\|_k(\sigma\tau) = (f\|_k\sigma)\|_k\tau$ if $\mathrm{pr}(\sigma) \in P_{\mathbf{A}}$ or $\mathrm{pr}(\tau) \in C^\theta$. From (16.16b) and (16.17) we obtain

$$j^k\big(r_P(p), z\big) = |\det(d_p)|^{k-[k]}\det(d_p)^{[k]} \quad \text{if} \quad p \in P_{\mathbf{A}}. \tag{16.19}$$

16.5. For two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F such that $\mathfrak{x}\mathfrak{y} \subset \mathfrak{g}$ we put

$$D[\mathfrak{x}, \mathfrak{y}] = \big\{ x \in G_{\mathbf{A}} \,\big|\, a_x \prec \mathfrak{r},\ b_x \prec \mathfrak{r}\mathfrak{x},\ c_x \prec \mathfrak{r}\mathfrak{y},\ d_x \prec \mathfrak{r} \big\}, \tag{16.20a}$$

$$D_0[\mathfrak{x}, \mathfrak{y}] = \{ x \in D[\mathfrak{x}, \mathfrak{y}] \mid x_{\mathbf{a}}(\mathbf{i}) = \mathbf{i} \}, \tag{16.20b}$$

$$\mathbf{i} = \mathbf{i}_n = (i1_n, \ldots, i1_n). \tag{16.21}$$

Notice that $D[\mathfrak{x}, \mathfrak{y}] = G_{\mathbf{A}} \cap C[\mathfrak{r}\mathfrak{x}, \mathfrak{r}\mathfrak{y}]$ with $C[\ ,\ ]$ of type (1.17).

We now take a fractional ideal $\mathfrak{b}$ in F, and recall that

$$G_{\mathbf{A}} = P_{\mathbf{A}} D_0[\mathfrak{b}^{-1}, \mathfrak{b}]. \tag{16.22}$$

We already stated the equality $G_{\mathbf{A}} = P_{\mathbf{A}} D[\mathfrak{b}^{-1}, \mathfrak{b}]$ at the end of Lemma 1.9, but we can take D_0 in place of D, since the equality at any archimedean prime holds by virtue of [S97, Propositions 6.13, 7.2, and 7.12]. Thus every element x of $G_{\mathbf{A}}$ belongs to $pD_0[\mathfrak{b}^{-1}, \mathfrak{b}]$ for some $p \in P_{\mathbf{A}}$.

Now we fix a weight k which may be integral or half-integral, and make the following convention: pr means the identity map of $G_{\mathbf{A}}$ onto itself if k is integral; otherwise it is the map of (16.13). We are going to define various functions on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$ according as k is integral or half-integral. For example, we define a real positive number $\varepsilon(x)$ and an ideal $\mathrm{il}_{\mathfrak{b}}(x)$ by

(16.23) $\varepsilon(x) = |\det(d_p d_p^*)|_{\mathbf{A}}$ *and* $\mathrm{il}_{\mathfrak{b}}(x) = \det(d_p)\mathfrak{r}$ *if* $\mathrm{pr}(x) \in pD_0[\mathfrak{b}^{-1}, \mathfrak{b}]$ *with* $p \in P_{\mathbf{A}}$,

where $|\ |_{\mathbf{A}}$ is the idele norm on $F_{\mathbf{A}}^{\times}$. These are well-defined, and

(16.23a) $\varepsilon(x_{\mathbf{a}}) = |j_x(\mathbf{i})|^{2\mathbf{a}}, \qquad \varepsilon(x_{\mathbf{h}}) = N\big(\mathrm{il}_{\mathfrak{b}}(x)\big)^{-2/[K:F]} \qquad (x \in G_{\mathbf{A}}).$

For these see [S97, Lemma 18.5]. Clearly $\varepsilon(\pi x) = \varepsilon(x)$ for $\pi \in P$.

In addition to $\mathfrak{b}$ and k, we take an integral ideal $\mathfrak{c}$ in F and a Hecke character χ of K satisfying the following conditions:

(16.24a) $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{i\kappa-\ell}$ *with* $\kappa \in \mathbf{R}^{\mathbf{a}}$ *such that* $\sum_{v\in\mathbf{a}} \kappa_v = 0$ *and*
$$\ell = \begin{cases} [k] & \text{(Case SP)}, \\ (k_v - k_{v\rho})_{v\in\mathbf{a}} & \text{(Case UT)}; \end{cases}$$

(16.24b) $\chi_v(a) = 1$ *if* $v \in \mathbf{h}$, $a \in \mathfrak{r}_v^{\times}$, *and* $a - 1 \in \mathfrak{r}_v\mathfrak{c}_v$;

(16.24c) $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$ *if* k *is half-integral.*

Thus $\mathfrak{c} \subset 4\mathfrak{g}$ if k is half-integral. Hereafter until the end of this section we put

(16.25a) $\widetilde{D} = D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ *and* $\widetilde{D}_0 = D_0[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ *if* $k \in \mathbf{Z}^{\mathbf{b}}$,

(16.25b) $\widetilde{D} = \{ x \in M_{\mathbf{A}} \mid \mathrm{pr}(x) \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \}$ and
$\widetilde{D}_0 = \{ x \in \widetilde{D} \mid \mathrm{pr}(x) \in D_0[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \}$ if $k \notin \mathbf{Z}^{\mathbf{b}}$,

We view $P_{\mathbf{A}}$ as a subgroup of $M_{\mathbf{A}}$ by the map r_P of (16.15). Then $P_{\mathbf{A}}\widetilde{D}$ is a subset of $M_{\mathbf{A}}$ if $k \notin \mathbf{Z}^{\mathbf{b}}$. Notice that $P_{\mathbf{A}}\widetilde{D}_0 = P_{\mathbf{A}}\widetilde{D} \subset \mathfrak{M}$, and so j_{α}^k is meaningful for $\alpha \in P_{\mathbf{A}}\widetilde{D}$.

Next we define a function μ on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$ as follows:

(16.26a) $\mu(x) = 0$ *if* $x \notin P_{\mathbf{A}}\widetilde{D}$,

(16.26b) $\mu(x) = \chi_{\mathbf{h}}\big(\det(d_p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1}j_x^k(\mathbf{i})^{-1}|j_x(\mathbf{i})|^{m-i\kappa}$
if $x = pw$ *with* $p \in P_{\mathbf{A}}$ *and* $w \in \widetilde{D}$,

where $m = k$ in Case SP and $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ in Case UT. Our Eisenstein series $E_{\mathbf{A}}(x, s)$ is defined for $(x, s) \in G_{\mathbf{A}} \times \mathbf{C}$ or $(x, s) \in M_{\mathbf{A}} \times \mathbf{C}$ by

(16.27) $$E_{\mathbf{A}}(x, s) = E_{\mathbf{A}}(x, s; \chi, \widetilde{D}) = \sum_{\alpha\in A} \mu(\alpha x)\varepsilon(\alpha x)^{-s}, \quad A = P\backslash G.$$

This is formally well-defined, since we can easily verify, employing (16.19), (16.16c), and (16.24a), that $\mu(\pi x) = \mu(x)$ for every $\pi \in P$. We investigated this series for integral k in [S97, Sections 18, 19]. Our principal aim of this section is to treat the case of half-integral k.

16.6. Define an element ζ of $Sp(n, F)_{\mathbf{A}}$ by

(16.28) $$\zeta_{\mathbf{a}} = 1, \quad \zeta_{\mathbf{h}} = \begin{bmatrix} 0 & -\delta^{-1}1_n \\ \delta 1_n & 0 \end{bmatrix}$$

with an element δ of $F_{\mathbf{h}}^{\times}$ such that $\delta\mathfrak{g} = \mathfrak{d}$. From (6.29) we easily see that

(16.29) $\zeta D[2\mathfrak{d}^{-1}, 2\mathfrak{d}] = D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]\zeta, \qquad \zeta D[2\mathfrak{d}^{-1}, 2\mathfrak{d}] \cup D[2\mathfrak{d}^{-1}, 2\mathfrak{d}] \subset C^{\theta}.$

We then define an element $\widetilde{\zeta}$ of $M_{\mathbf{A}}$ by

(16.30) $\mathrm{pr}(\widetilde{\zeta}) = \zeta$ *and* $h(\widetilde{\zeta}, z) = 1.$

Notice that the condition $\mathrm{pr}(\widetilde{\zeta}) = \zeta$ implies that $\widetilde{\zeta} \in \mathfrak{M}$, and so $h(\widetilde{\zeta}, z)$ is meaningful; then in view of (16.16a), the condition $h(\widetilde{\zeta}, z) = 1$ determines $\widetilde{\zeta}$ uniquely.

Now let C be an open subgroup of $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$, and φ a function on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$ such that $\varphi(\alpha x w) = j_w^k(\mathbf{i})^{-1}\varphi(x)$ if $\alpha \in G$, $\mathrm{pr}(w) \in C$, and $w(\mathbf{i}) = \mathbf{i}$. Define a function g on $\mathcal{H}$ by $g\big(x(\mathbf{i})\big) = j_x^k(\mathbf{i})\varphi(x)$ for every x such that $\mathrm{pr}(x) \in C$. This is well-defined independently of C and $g\|_k\gamma = g$ for every $\gamma \in G \cap C$. In Case SP we can show that the correspondence $\varphi \mapsto g$ gives a bijection from the set of all such φ's (with a fixed C) onto the set of all functions g on $\mathcal{H}$ such that $g\|_k\gamma = g$ for every $\gamma \in G \cap C$. In Case UT, in order to obtain a bijection, we have to associate several functions on $\mathcal{H}$ to φ; for details, see [S97, Lemma 10.8]; see also §20.1 below.

Assuming k to be integral, define φ' on $G_{\mathbf{A}}$ by $\varphi'(x) = \varphi(x\omega^{-1})$ for $x \in G_{\mathbf{A}}$ with $\omega \in (G_1)_{\mathbf{h}}$. By strong approximation in G_1, we can find an element $\alpha \in G \cap C\omega$. Define g' on $\mathcal{H}$ corresponding to φ' by the above principle. Then we can easily verify that

$$g' = g\|_k\,\alpha. \tag{16.31a}$$

Next, assuming k to be half-integral, by strong approximation we can find an element ζ_0 of G such that $\zeta_0 \in G \cap C\zeta^{-1}$. Clearly $\zeta_0 \in C^\theta$. Define φ' on $M_{\mathbf{A}}$ by $\varphi'(x) = \varphi(x\widetilde{\zeta})$ for $x \in M_{\mathbf{A}}$ and define g' on $\mathcal{H}$ corresponding to φ' by the above principle. Then we have

$$g' = g\|_k\,\zeta_0. \tag{16.31b}$$

To show this, take $x \in M_{\mathbf{A}}$ so that $\mathrm{pr}(x) \in C \cap \zeta C\zeta^{-1}$; put $y = \widetilde{\zeta}^{-1}x\widetilde{\zeta}$. Then $(g'\|_k\,x)(\mathbf{i}) = \varphi'(x) = \varphi(x\widetilde{\zeta}) = \varphi(\widetilde{\zeta}y) = \varphi(\zeta_0\widetilde{\zeta}y) = (g\|_k\zeta_0\widetilde{\zeta}y)(\mathbf{i})$ since $\mathrm{pr}(\zeta_0\widetilde{\zeta}y) \in C$. Now $j^k(\zeta_0\widetilde{\zeta}y,\, z) = j^k(\zeta_0 x\widetilde{\zeta},\, z) = j^k(\zeta_0 x,\, z)j^k(\widetilde{\zeta},\, z) = j^k(\zeta_0 x,\, z)$, since $j^k(\widetilde{\zeta},\, z) = 1$. Then $(g\|_k\zeta_0\widetilde{\zeta}y)(\mathbf{i}) = (g\|_k\zeta_0 x)(\mathbf{i})$, which proves (16.31b).

16.7. Returning to $E_{\mathbf{A}}$ of (16.27), we note, in both Cases SP and UT, that

$$E_{\mathbf{A}}(\alpha x w,\, s) = \chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1} j_w^k(\mathbf{i})^{-1} E_{\mathbf{A}}(x,\, s) \quad \textit{if} \;\; \alpha \in G \;\; \textit{and} \;\; w \in \widetilde{D}_0. \tag{16.32}$$

In Case SP we now define a function $E_{\mathbf{A}}^*(x, s)$ on $G_{\mathbf{A}} \times \mathbf{C}$ or $M_{\mathbf{A}} \times \mathbf{C}$ by

$$E_{\mathbf{A}}^*(x,\, s) = \chi(\delta)^{-n} \cdot \begin{cases} E_{\mathbf{A}}(x\zeta,\, s) & (x \in G_{\mathbf{A}},\ k \in \mathbf{Z}^{\mathbf{a}}), \\ E_{\mathbf{A}}(x\widetilde{\zeta},\, s) & (x \in M_{\mathbf{A}},\ k \notin \mathbf{Z}^{\mathbf{a}}). \end{cases} \tag{16.33}$$

This is independent of the choice of δ in (16.28). We are going to study the Fourier expansion of $E_{\mathbf{A}}^*(x, s)$. This was essentially done for integral k in [S97, Section 18] (see Remark 16.12 below), and so we consider here only Case SP, putting our emphasis on the case of half-integral k. First, by the principle of §16.7 we define functions $E(z, s)$ and $E^*(z, s)$ of $(z, s) \in \mathcal{H} \times \mathbf{C}$ so that

$$E\big(x(\mathbf{i}),\, s\big) = j_x^k(\mathbf{i})E_{\mathbf{A}}(x,\, s), \quad E^*\big(x(\mathbf{i}),\, s\big) = j_x^k(\mathbf{i})E_{\mathbf{A}}^*(x,\, s) \quad \textit{if} \;\; \mathrm{pr}(x) \in G_{\mathbf{a}}. \tag{16.34}$$

We consider $E(z, s)$ in both Cases SP and UT; as for E^*, we define it, for the moment, only in Case SP. By (16.31a, b) we have

$$E^*(z,\, s) = \chi(\delta)^{-n} j_{\zeta_0}^k(z)^{-1} E(\zeta_0 z,\, s). \tag{16.35}$$

Now we have

$$E(z,\, s) = \sum_{\alpha \in A_0} N\big(\mathrm{il}_{\mathfrak{b}}(\alpha)\big)^{us} \chi[\alpha]\delta(z)^{s\mathbf{a} - (m - i\kappa)/2}\|_k\alpha, \quad A_0 = P\backslash(G \cap P_{\mathbf{A}}\widetilde{D}), \tag{16.36}$$

Here m is as in (16.26b); $u = 2$ in Case SP and $u = 1$ in Case UT; $\delta(z) = \big(\det\big((i/2)(z^* - z)\big)_v\big)_{v\in\mathbf{a}}$, which is consistent with (3.21) and (13.5); and $\chi[\alpha]$ is an element of $\mathbf{T}$ defined for $\alpha \in G \cap P_{\mathbf{A}}\widetilde{D}$ by

$$\chi[\alpha] = \begin{cases} \chi_{\mathbf{a}}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}_{\mathfrak{b}}(\alpha)^{-1}\big) & \textit{if}\ \ \mathfrak{c} \neq \mathfrak{g}, \\ \chi^*\big(\mathrm{il}_{\mathfrak{b}}(\alpha)\big)^{-1} & \textit{if}\ \ \mathfrak{c} = \mathfrak{g}. \end{cases} \tag{16.37}$$

(There are two δ's. However, δ in (16.35) never appears together with z, and there will be no fear of confusion.) Notice that by Lemma 1.11 (3), $\det(d_\alpha) \neq 0$ and $\det(d_\alpha)\mathrm{il}_{\mathfrak{b}}(\alpha)^{-1}$ is prime to $\mathfrak{c}$ if $\mathfrak{c} \neq \mathfrak{g}$. To prove (16.36), take x in $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$ so that $\mathrm{pr}(x) \in G_{\mathbf{a}}$ and $x(\mathbf{i}) = z$; let $\alpha \in G$. We are going to calculate each term of (16.27). By (16.26a), $\mu(\alpha x) \neq 0$ only if $\alpha \in P_{\mathbf{A}}\widetilde{D}$. Thus A of (16.27) can be replaced by A_0 of (16.36). Put $\alpha x = pw$ with $p \in P_{\mathbf{A}}$ and $w \in \widetilde{D}$. By (16.23a), $\varepsilon(\alpha x) = \varepsilon\big((\alpha x)_{\mathbf{h}}\big)\varepsilon\big((\alpha x)_{\mathbf{a}}\big)$, $\varepsilon\big((\alpha x)_{\mathbf{h}}\big) = N\big(\mathrm{il}_{\mathfrak{b}}(\alpha)\big)^{-u}$, and $\varepsilon\big((\alpha x)_{\mathbf{a}}\big) = |j_{\alpha x}(\mathbf{i})|^{2\mathbf{a}}$. Thus, by (16.26b),

$$\begin{aligned} &\mu(\alpha x)\varepsilon(\alpha x)^{-s} \\ &= N\big(\mathrm{il}_{\mathfrak{b}}(\alpha)\big)^{us}\chi_{\mathbf{h}}\big(\det(d_p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1}|j_{\alpha x}(\mathbf{i})|^{m-i\kappa-2s\mathbf{a}}j_{\alpha x}^k(\mathbf{i})^{-1}. \end{aligned} \tag{16.37a}$$

Assuming $\mathfrak{c} \neq \mathfrak{g}$, we see that d_α is invertible and $(d_\alpha)_{\mathbf{h}} = (d_p d_w)_{\mathbf{h}}$, and hence $\det(d_\alpha)\mathrm{il}_{\mathfrak{b}}(\alpha)^{-1} = \det(d_\alpha)\det(d_p)^{-1}\mathfrak{g} = \det(d_w)\mathfrak{g}$. Therefore $\chi^*\big(\det(d_\alpha)\mathrm{il}_{\mathfrak{b}}(\alpha)^{-1}\big) = \big(\chi_{\mathbf{h}}/\chi_{\mathfrak{c}}\big)\big(\det(d_w)\big)$, so that

$$\chi[\alpha] = \chi_{\mathbf{h}}\big(\det(d_p d_w)\big)^{-1}\big(\chi_{\mathbf{h}}/\chi_{\mathfrak{c}}\big)\big(\det(d_w)\big) = \chi_{\mathbf{h}}\big(\det(d_p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1}.$$

Now $j_\alpha^k(z)j_x^k(\mathbf{i}) = j_{\alpha x}^k(\mathbf{i})$ and $\delta(\alpha z) = |j_{\alpha x}(\mathbf{i})|^{-2}$. Combining all these and (16.34), we obtain (16.36) when $\mathfrak{c} \neq \mathfrak{g}$. The case $\mathfrak{c} = \mathfrak{g}$ can be handled in the same manner.

Next put $\Gamma = G \cap D[\mathfrak{b}^{-1}, \mathfrak{bc}]$. By [S97, Lemma 18.7 (4)] there exists a subset B of $G \cap \big(\prod_{v\nmid\mathfrak{c}} Q_v \cdot D[\mathfrak{b}^{-1}, \mathfrak{bc}]\big)$ such that

$$G \cap P_{\mathbf{A}}\widetilde{D} = \bigsqcup_{\beta\in B} P\beta\Gamma. \tag{16.38}$$

Let $\mathcal{R}_\beta$ be a complete set of representatives for $(P \cap \beta\Gamma\beta^{-1})\backslash\beta\Gamma$. Clearly $P\backslash(G \cap P_{\mathbf{A}}\widetilde{D})$ can be given by $\bigsqcup_{\beta\in B}\mathcal{R}_\beta$. Since $\mathrm{il}_{\mathfrak{b}}(\alpha) = \mathrm{il}_{\mathfrak{b}}(\beta)$ if $\alpha \in \beta\Gamma$, we have

$$E(z,\, s) = \sum_{\beta\in B} N\big(\mathrm{il}_{\mathfrak{b}}(\beta)\big)^{us}\sum_{\alpha\in\mathcal{R}_\beta}\chi[\alpha]\delta(z)^{s\mathbf{a}-(m-i\kappa)/2}\|_k\alpha. \tag{16.39}$$

Recall that in Case SP the ideals $\mathrm{il}_{\mathfrak{b}}(\beta)$ for $\beta \in B$ form a complete set of representatives for the ideal classes of F (see [S97, Lemma 18.7 (5)]).

Take $F = \mathbf{Q}$, for example; then $\kappa = 0$ and $k \in 2^{-1}\mathbf{Z}$; we can take $B = \{1\}$, and $\mathfrak{c} = c\mathbf{Z}$ with $0 < c \in \mathbf{Z}$. Thus, as a special case of (16.39), we have

$$E(z,\, s) = \sum_{\gamma\in(P\cap\Gamma)\backslash\Gamma}\chi_c\big(\det(d_\gamma)\big)^{-1}\delta(z)^{s-k/2}\|_k\gamma, \tag{16.40}$$

where we understand that $\chi_c\big(\det(d_\alpha)\big) = 1$ if $c = 1$. Thus χ_c may be viewed as a Dirichlet character which is even or odd according as $[k]$ is even or odd.

Now the convergence of the series of (16.27) can be reduced to that of $\sum_{\alpha\in\mathcal{R}_\beta}$ of (16.39), and to that of $\sum_{\alpha\in\mathcal{R}}|\delta(\alpha z)^{s\mathbf{a}}|$, where $\mathcal{R} = (P \cap \Gamma')\backslash\Gamma'$ with a congruence subgroup Γ' of G. As noted in [S97, Proposition A3.7 and §A3.9], the last series is convergent for $\mathrm{Re}(s) > n$ in Case UT and $\mathrm{Re}(s) > (n+1)/2$ in Case SP. Thus the series for $E(x,\, s)$ is convergent in that domain.

16.8. Since j_w^k for $w \in C^\theta$ is a factor of automorphy in the ordinary sense, from (16.32) we can easily derive that

$$E_{\mathbf{A}}^*(\alpha x w,\, s) = \chi_{\mathfrak{c}}\big(\det(a_w)\big)^{-1} j_w^k(\mathbf{i})^{-1} E_{\mathbf{A}}^*(x, s) \qquad \text{if } \alpha \in G \text{ and } \mathrm{pr}(w) \in D_0[\mathfrak{d}^{-2}\mathfrak{b}\mathfrak{c}, \mathfrak{d}^2\mathfrak{b}^{-1}], \tag{16.41}$$

and so it has a Fourier expansion of the form

$$E_{\mathbf{A}}^*\left(r_P\begin{pmatrix} q & \sigma\widehat{q} \\ 0 & \widehat{q}\end{pmatrix}\right) = \sum_{h \in S} c(h,\, q,\, s)\mathbf{e}_{\mathbf{A}}^n(h\sigma) \qquad (q \in GL_n(F_{\mathbf{A}}),\ \sigma \in S_{\mathbf{A}}) \tag{16.42}$$

with $c(h,\, q,\, s) \in \mathbf{C}$. The principle of such an expansion is stated in [S97, Proposition 18.3] for integral k, but the case of half-integral k is similar. (Cf. also Proposition 20.2 below.) Now write an element z of $\mathfrak{H}^{\mathbf{a}}$ in the form $z = x + iy$ with $x, y \in S_{\mathbf{a}}$ with $y_v > 0$ for every $v \in \mathbf{a}$. Take q and σ in (16.42) so that $q_{\mathbf{h}} = 1$, $q_{\mathbf{a}} = y^{1/2}$, $\sigma_{\mathbf{h}} = 0$, and $\sigma_{\mathbf{a}} = x$. Write simply $y^{1/2}$ for such a q. Then from (16.34) we easily obtain

$$E^*(x + iy,\, s) = \det(y)^{-k/2} \sum_{h \in S} c\big(h,\, y^{1/2},\, s\big)\mathbf{e}_{\mathbf{a}}^n(hx). \tag{16.43}$$

To obtain the explicit form of $c(h,\, q,\, s)$, we first put

$$\xi(g,\, h;\, s,\, s') = \int_{S_v} \mathbf{e}_v^n(-hx)\det(x + ig)^{-s}\det(x - ig)^{-s'}\,dx \qquad (s,\, s' \in \mathbf{C};\ 0 < g \in S_v,\ h \in S_v,\ v \in \mathbf{a}), \tag{16.44}$$

$$\Xi(y,\, w;\, t,\, t') = \prod_{v \in \mathbf{a}} \xi(y_v,\, w_v;\, t_v,\, t_v') \qquad (t,\, t' \in \mathbf{C}^{\mathbf{a}},\ y \in S_{\mathbf{a}},\ y_v > 0,\ w \in S_{\mathbf{a}}).$$

Here for $s \in \mathbf{C}$ and $z \in \mathfrak{H}_n$ we choose the branches of $\det(z)^s$ and $\overline{\det(z)}^{\,s}$ so that their values at $z = i1_n$ are i^{ns} and i^{-ns}, respectively, where i^α is defined by

$$i^\alpha = \exp(\pi i\alpha/2) \qquad (\alpha \in \mathbf{C}). \tag{16.45}$$

The function ξ was investigated in [S82]. We note here only that the integral of (16.44) is convergent for sufficiently large $\mathrm{Re}(s + s')$, and can be continued as a meromorphic function of $(s,\, s')$ to the whole $\mathbf{C}^2$. We also put, for $\tau \in \widetilde{S}$,

$$\alpha_{\mathfrak{c}}^0(\tau,\, s,\, \chi) = \prod_{v \nmid \mathfrak{c}} \sum_{\sigma \in S_v/S(\mathfrak{g})_v} \mathbf{e}_v^n(-\delta_v^{-1}\tau\sigma)\chi^*\big(\nu_0(\sigma)\big)\nu(\sigma)^{-s},$$

$$\alpha_{\mathfrak{a}}^1(\tau,\, s,\, \chi) = \prod_{v \nmid \mathfrak{c}} \sum_{\sigma \in S_v/S(\mathfrak{g})_v} \mathbf{e}_v^n(-\delta_v^{-1}\tau\sigma)\chi^*\big(\nu_0(\sigma)\big)\omega\big(\delta_v^{-1}\sigma\big)\nu(\sigma)^{-s}.$$

16.9. Proposition (Case SP). *Suppose that $\mathfrak{c} \neq \mathfrak{g}$ and $\det(q_v) > 0$ for every $v \in \mathbf{a}$; let $y = {}^t q_{\mathbf{a}} q_{\mathbf{a}}$. Then $c(h,\, q,\, s) \neq 0$ only if $({}^t qhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v\widetilde{S}_v$ with $\widetilde{S}_v$ of (16.1c) for every $v \in \mathbf{h}$, in which case*

$$c(h,\, q,\, s) = C \cdot \chi_{\mathbf{h}}\big(\det(-q)\big)^{-1}|\det(q)_{\mathbf{h}}|_{\mathbf{A}}^{n+1-2s}|D_F|^{-2ns+3n(n+1)/4}N(\mathfrak{b}\mathfrak{c})^{-n(n+1)/2}$$
$$\cdot \det(y)^{s\mathbf{a}+i\kappa/2}\,\Xi\big(y,\, h;\, s\mathbf{a} + (k + i\kappa)/2,\, s\mathbf{a} - (k - i\kappa)/2\big)$$
$$\cdot\ \alpha_{\mathfrak{c}}^e\big(\varepsilon_b^{-1} \cdot {}^t qhq,\, 2s,\, \chi\big),$$

where $C = 1$ and $e = 0$ if $k \in \mathbf{Z}^{\mathbf{a}}$, and $C = \mathbf{e}(n[F : \mathbf{Q}]/8)$ and $e = 1$ if $k \notin \mathbf{Z}^{\mathbf{a}}$; ε_b is an element of $F_{\mathbf{h}}^\times$ such that $\varepsilon_b\mathfrak{g} = \mathfrak{b}^{-1}\mathfrak{d}$ if $k \in \mathbf{Z}^{\mathbf{a}}$, and $\varepsilon_b = 1$ if $k \notin \mathbf{Z}^{\mathbf{a}}$; D_F is the discriminant of F.

If $k \notin \mathbf{Z}^{\mathbf{a}}$, our assumption (16.24c) implies that $\mathfrak{b}_v = \mathfrak{d}_v$ for $v \nmid \mathfrak{c}$. Therefore $(\varepsilon_b^{-1} \cdot {}^t qhq)_v \in \widetilde{S}_v$ if $v \nmid \mathfrak{c}$ for both integral and half-integral k. The proof will be given in §A2.13.

16.10. Proposition (Case SP). *With h, q such that $c(h, q, s) \neq 0$ and ε_b as in Proposition 16.9, put $r = \operatorname{rank}(h)$ and ${}^t ghg = \operatorname{diag}[h', 0]$ with $g \in GL_n(F)$ and $h' \in S^r$. Let ρ_h be the Hecke character corresponding to $F(c^{1/2})/F$, where $c = (-1)^{[r/2]} \det(2h')$, if $r > 0$; let $\rho_h = 1$ if $r = 0$. Then*

$$\alpha_{\mathfrak{c}}^e(\varepsilon_b^{-1} \cdot {}^t qhq,\, 2s,\, \chi) = \Lambda_{\mathfrak{c}}(s)^{-1}\Lambda_h(s) \prod_{v \in \mathbf{c}} f_{h,q,v}\left(\chi(\pi_v)|\pi_v|^{2s+e/2}\right) \tag{16.46}$$

with a finite subset $\mathbf{c}$ of $\mathbf{h}$, polynomials $f_{h,q,v}$ with coefficients in $\mathbf{Z}$ independent of χ, and functions $\Lambda_{\mathfrak{c}}$ and Λ_h given as follows:

$$\Lambda_{\mathfrak{c}}(s) = \Lambda_{\mathfrak{c}}^n(s) = \begin{cases} L_{\mathfrak{c}}(2s,\, \chi) \displaystyle\prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(4s - 2i,\, \chi^2) & \text{if } k \in \mathbf{Z}^{\mathbf{a}}, \\ \displaystyle\prod_{i=1}^{[(n+1)/2]} L_{\mathfrak{c}}(4s - 2i + 1,\, \chi^2) & \text{if } k \notin \mathbf{Z}^{\mathbf{a}}, \end{cases}$$

$$\Lambda_h(s) = L_{\mathfrak{c}}(2s - n + r/2,\, \chi\rho_h) \prod_{i=1}^{[(n-r)/2]} L_{\mathfrak{c}}(4s - 2n + r + 2i - 1,\, \chi^2) \qquad \text{if } \ 2k_v + r \in 2\mathbf{Z},$$

$$\Lambda_h(s) = \prod_{i=1}^{[(n-r+1)/2]} L_{\mathfrak{c}}(4s - 2n + r + 2i - 2,\, \chi^2) \qquad \text{if } \ 2k_v + r \notin 2\mathbf{Z}.$$

The set $\mathbf{c}$ is determined as follows: $\mathbf{c} = \varnothing$ if $r = 0$. If $r > 0$, take $a \in \prod_{v \nmid \mathfrak{c}} GL_n(\mathfrak{g}_v)$ so that $(\varepsilon_b^{-1} \cdot {}^t a \cdot {}^t qhqa)_v = \operatorname{diag}[\tau_v, 0]$ with $\tau_v \in \widetilde{S}_v^r$ for every $v \nmid \mathfrak{c}$. (Such an a is guaranteed by [S97, Lemma 13.3].) Then $\mathbf{c}$ consists of all the primes v not dividing $\mathfrak{c}$ such that τ_v is not regular in the sense of §16.1.

This follows immediately from Theorem 16.2.

Next, to state a theorem concerning the analytic nature of $E(z, s)$, we first put, for $\iota = 1$ or 2, and $s \in \mathbf{C}$,

$$\Gamma_0^\iota(s) = 1, \qquad \Gamma_n^\iota(s) = \pi^{\iota n(n-1)/4} \prod_{\nu=0}^{n-1} \Gamma\big(s - (\iota\nu/2)\big) \quad (n > 0). \tag{16.47}$$

16.11. Theorem (Case SP). *Write simply Γ_n for Γ_n^1; define $\gamma(s, h)$ and $\mathcal{G}(s)$ as follows:*

$$\mathcal{G}(s) = \mathcal{G}_{k,\kappa}^n(s) = \prod_{v \in \mathbf{a}} \gamma\big(s + i\kappa_v/2,\, |k_v|\big),$$

$$\gamma(s,\, h) = \begin{cases} \Gamma\left(s + \dfrac{h}{2} - \left[\dfrac{2h+n}{4}\right]\right) \Gamma_n\left(s + \dfrac{h}{2}\right) & (n/2 \le h \in \mathbf{Z},\ n \text{ even}), \\ \Gamma_n\left(s + \dfrac{h}{2}\right) & (n/2 < h \in \mathbf{Z},\ n \text{ odd}), \\ \Gamma_{2h+1}\left(s + \dfrac{h}{2}\right) \displaystyle\prod_{i=h+1}^{[n/2]} \Gamma(2s - i) & (0 \le h < n/2,\ h \in \mathbf{Z}), \end{cases}$$

$$\gamma(s,\,h) = \begin{cases} \Gamma\left(s+\dfrac{h-1}{2}-\left[\dfrac{2h+n-2}{4}\right]\right)\Gamma_n\left(s+\dfrac{h}{2}\right) & (n/2 < h \notin \mathbf{Z},\ n\ odd), \\ \Gamma_n\left(s+\dfrac{h}{2}\right) & (n/2 < h \notin \mathbf{Z},\ n\ \text{even}), \\ \Gamma_{2h+1}\left(s+\dfrac{h}{2}\right)\displaystyle\prod_{i=[h]+1}^{[(n-1)/2]}\Gamma\left(2s-\dfrac{1}{2}-i\right) & (0 < h \le n/2,\ h \notin \mathbf{Z}). \end{cases}$$

Put $\mathcal{P}(s) = \mathcal{G}(s)\Lambda_{\mathfrak{c}}^{n}(s)E(z,\,s)$ with E of (16.36). Then $\mathcal{P}(s)$ is a meromorphic function in s on the whole $\mathbf{C}$ with only finitely many poles, and each pole is simple. In particular, $\mathcal{P}$ is an entire function of s if $\chi^2 \neq 1$. If $\chi^2 = 1$, the poles of $\mathcal{P}$ are determined as follows:

(I) $\chi^2 = 1$ *and* $\mathfrak{c} \neq \mathfrak{g}$: *Let $m = Max_{v\in\mathbf{a}}|k_v|$. If $m > n/2$, $\mathcal{P}$ has no pole except for a possible pole at $s = (n+2)/4$ which occurs only if $2|k_v| - n \in 4\mathbf{Z}$ for every v such that $2|k_v| > n$. If $m \le n/2$, $\mathcal{P}$ has possible poles only in the set*

(i) $\quad \{\, j/2 \mid j \in \mathbf{Z},\ [(n+3)/2] \le j \le n+1-m \,\}$ *if* $k \in \mathbf{Z}^{\mathbf{a}}$,

(ii) $\quad \{\, (2j+1)/4 \mid j \in \mathbf{Z},\ 1+[n/2] \le j \le n+(1/2)-m \,\}$ *if* $k \notin \mathbf{Z}^{\mathbf{a}}$.

(II) $\chi^2 = 1$, $\mathfrak{c} = \mathfrak{g}$, *and* $k \in \mathbf{Z}^{\mathbf{a}}$: *In this case each pole belongs to the set of poles described in (I) or to*

(iii) $\quad \{\, j/2 \mid j \in \mathbf{Z},\ 0 \le j \le [n/2] \,\}$,

where $j = 0$ is unnecessary if $\chi \neq 1$.

We can derive this from Propositions 16.9 and 16.10 by means of the principle of [S97, Proposition 19.1] and the procedure described in [S97, §§19.4~19.6] in Case UT. In fact, the detailed discussion was given in [S94a, pp.565-571].

16.12. Remark. (I) In [S97, §18.6], we considered $E(x,\,s)$ for integral k in Cases SP and UT. In Case UT we have $j_x^k = \det(x)^\nu j_x^m$ with $\nu = (-k_{v\rho})_{v\in\mathbf{a}}$ and $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$, and hence the present j_x^k can be written $j_x^{m,\nu}$ in the notation of [S97, (10.4.3)]. Thus our $E(x,\,s)$ coincides with E of [S97, (18.6.1)] if we take $j_x^{k,\nu}$ there to be the present $j_x^{m,\nu}$. Also we considered there $E^*(x,\,s) = E_{\mathbf{A}}(x\eta_{\mathbf{h}}^{-1},\,s)$ in both Cases SP and UT, instead of (16.33), and obtained its Fourier expansion in [S97, Proposition 18.14]. The function is different from the present $E_{\mathbf{A}}^*$, but the nature is the same. At any rate, both functions can be handled by the same methods.

(II) We can define $E_{\mathbf{A}}(x,\,s)$ also on $(G_1)_{\mathbf{A}}$ with $G_1 = SU(\eta_n)$ in Case UT and the corresponding $E(z,\,s)$. This was done in [S97, Section 18]. In this case we take $k \in \mathbf{Z}^{\mathbf{a}}$. By Lemma 1.3 (2), $\det(d_\alpha) \in F$ for every $\alpha \in G_1$. Therefore we put $\mathrm{il}_{\mathfrak{b}}(\alpha) = \det(d_p)\mathfrak{g}$ in (16.23), and take χ to be a Hecke character of F satisfying $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^k|x_{\mathbf{a}}|^{i\kappa-k}$. To obtain the formula corresponding to (16.39) in this case, we take G_1 and P_1 in place of G and P, and take B so that $G_1 \cap (P_1)_{\mathbf{A}}D[\mathfrak{b}^{-1},\,\mathfrak{bc}]_1 = \bigsqcup_{\beta\in B} P_1\beta\Gamma_1$, where $X_1 = (G_1)_{\mathbf{A}} \cap X$ for any subgroup X of $G_{\mathbf{A}}$. With this choice of B the ideals $\mathrm{il}_{\mathfrak{b}}(\beta)$ for $\beta \in B$ form a complete set of repressentatives for the ideal classes of F; see [S97, Lemma 18.7 (5)]. Then with $\mathcal{R}_\beta = (P_1 \cap \beta\Gamma_1\beta^{-1})\backslash\beta\Gamma_1$ we have

$$E(z,\,s) = \sum_{\beta\in B} N\big(\mathrm{il}_{\mathfrak{b}}(\beta)\big)^{2s} \sum_{\alpha\in\mathcal{R}_\beta} \chi[\alpha]\delta(z)^{s\mathbf{a}-(k-i\kappa)/2}\|_k\alpha. \tag{16.48}$$

(III) Let $L = \sum_{i=1}^{n}(\mathfrak{r}e_i + \mathfrak{b}^{-1}e_{n+i})$ with the standard basis $\{e_i\}_{i=1}^{2n}$ of K_{2n}^1 in Case UT. Take $\zeta \in K^\times$ such that $\zeta^\rho = -\zeta$. Then we can easily verify that L is a maximal lattice with respect to the hermitian form $\zeta\eta$ in the sense of [S97, §4.7]. Clearly $D[\mathfrak{b}^{-1}, \mathfrak{b}] = \{\alpha \in G_{\mathbf{A}} \mid L\alpha = L\}$. Therefore the results of [S97, Section 5] are applicable to L_v and $D[\mathfrak{b}^{-1}, \mathfrak{b}] \cap G_v$ for every $v \in \mathbf{h}$.

17. Arithmeticity and near holomorphy of Eisenstein series

17.1. The purpose of this section is to study the nature of $E(z, s)$ at suitably chosen points s_0 belonging to an interval. We are going to show that the value or the residue of E at s_0 is nearly holomorphic, sometimes holomorphic, and arithmetic up to a power of π. Throughout this section we put

$$d = [F : \mathbf{Q}]. \tag{17.1}$$

We return to the setting of §§16.5 and 16.7 in Cases SP and UT. However, in Case UT we take $G_1 = SU(\eta_n)$ instead of $G = U(\eta_n)$. Therefore, in this section until Theorem 17.9, we speak of Case SU instead of Case UT. We shall return to Case UT in §17.10. The notation being as in §16.7, let Γ be a congruence subgroup of G_1 or $\mathcal{G}$ according as k is integral or half-integral, where $\mathcal{G}$ is the group defined in §14.14. As we said there, we view every congruence subgroup of Γ^θ as a congruence subgroup of $\mathcal{G}$. In Case SU we take $k \in \mathbf{Z}^{\mathbf{a}}$. We also take an element κ of $\mathbf{R}^{\mathbf{a}}$ such that $\sum_{v\in\mathbf{a}} \kappa_v = 0$. For $\xi = (\alpha, p) \in \mathcal{G}$ we put $a_\xi = a_\alpha$, $b_\xi = b_\alpha$, $c_\xi = c_\alpha$, and $d_\xi = d_\alpha$. Then we put

$$\Gamma^P = \{\gamma \in \Gamma \mid c_\gamma = 0\}, \tag{17.2}$$

$$E(z, s; k, \kappa, \Gamma) = \sum_{\alpha \in \Gamma^P \backslash \Gamma} \delta(z)^{s\mathbf{a}-(k-i\kappa)/2} \|_k \alpha \qquad ((z, s) \in \mathcal{H} \times \mathbf{C}). \tag{17.3}$$

$$E(z, s; k, \Gamma) = E(z, s; k, 0, \Gamma). \tag{17.3a}$$

To make the sum at least formally meaningful, we have to assume

$$|j_\gamma(z)|^{i\kappa-k} j_\gamma^k(z) = 1 \quad \textit{for every} \quad \gamma \in \Gamma^P. \tag{17.4}$$

The series is convergent for $\mathrm{Re}(s) > (n+1)/2$ in Case SP and $\mathrm{Re}(s) > n$ in Case UT, as noted in [S97, Proposition A3.7 and §A3.9]. If Γ_1 is a congruence subgroup contained in Γ, then we can easily verify that

$$[\Gamma^P : \Gamma_1^P] E(z, s; k, \kappa, \Gamma) = \sum_{\alpha \in \Gamma_1 \backslash \Gamma} E(z, s; k, \kappa, \Gamma_1) \|_k \alpha. \tag{17.5}$$

To study the properties of $E(z, s; k, \kappa, \Gamma)$ for an arbitrary Γ, we introduce special types of congruence subgroups as follows:

$$\Gamma_0(\mathfrak{c}) = G_1 \cap D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}], \tag{17.6a}$$

$$\Gamma(\mathfrak{c}) = \{\alpha \in \Gamma_0(\mathfrak{c}) \mid a_\gamma - 1 \prec \mathfrak{r}\mathfrak{c},\ b_\alpha \prec \mathfrak{r}\mathfrak{b}^{-1}\mathfrak{c}\}, \tag{17.6b}$$

$$\Gamma_u(\mathfrak{c}) = \{\alpha \in \Gamma_0(\mathfrak{c}) \mid \det(d_\alpha) \in \mathfrak{g}^\times W\}, \tag{17.6c}$$

$$W = \{x \in F^\times \mid x - 1 \in \mathfrak{c}_v \ \textit{ for every } \ v|\mathfrak{c}\}. \tag{17.6d}$$

Here we fix $\mathfrak{b}$ and assume (16.24c). We also denote by $E(z, s; k, \chi, \mathfrak{c})$ the series of (16.36) in Case SP and that of (16.48) in Case SU; though E depends also on $\mathfrak{b}$, we suppress it, since it has no essential effect on the nature of E.

17.2. Lemma (Cases SP and SU). (1) $\Gamma_u(\mathfrak{c}) = \Gamma_u(\mathfrak{c})^P \Gamma(\mathfrak{c})$.

(2) *Let X be the set of all Hecke characters of F satisfying (16.24a, b) with a fixed κ. Let Γ be a congruence subgroup of G_1 or $\mathcal{G}$ which satisfies (17.4) and contains $\Gamma(\mathfrak{c})$. Then $X \neq \varnothing$ and*

$$\#(X)[\Gamma^P : \Gamma(\mathfrak{c})^P]E(z,\ s;\ k,\ \kappa,\ \Gamma) = \sum_{\chi \in X} \sum_{\alpha \in \Gamma(\mathfrak{c}) \backslash \Gamma} E(z,\ s;\ k,\ \chi,\ \mathfrak{c})\|_k\, \alpha.$$

(3) *If $\mathfrak{a}$ is a multiple of $\mathfrak{c}$, then*

$$E(z,\ s;\ k,\ \chi,\ \mathfrak{c}) = \sum_{\xi \in \Gamma_0(\mathfrak{a}) \backslash \Gamma_0(\mathfrak{c})} \chi_{\mathfrak{c}}\bigl(\det(d_\xi)\bigr)^{-1} E(z,\ s;\ k,\ \chi,\ \mathfrak{a})\|_k\, \xi.$$

(4) *Let $\lambda_n = (n+1)/2$ in Case SP and $\lambda_n = n$ in Case UT. Then $E(z,\ s;\ k,\ \kappa,\ \Gamma)$ can be continued to a meromorphic function on the whole s-plane which is holomorphic for $\mathrm{Re}(s) > \lambda_n$. Moreover it has a pole at $s = \lambda_n$ only if $k = \kappa = 0$, in which case it has a simple pole at $s = \lambda_n$ with a positive real number as its residue.*

PROOF. To prove (1), take $\alpha \in \Gamma_u(\mathfrak{c})$. Then $\det(d_\alpha) \in eW$ with $e \in \mathfrak{g}^\times$. By strong approximation on $SL_n(K)$, we can find an element q of $GL_n(\mathfrak{r})$ such that $q - d_\alpha \prec \mathfrak{c}\mathfrak{r}$ and $\det(q) = e$. Put

$$\beta = \begin{bmatrix} q^* & -d_\alpha^* b_\alpha q^{-1} \\ 0 & q^{-1} \end{bmatrix}.$$

Then $\beta \in \Gamma_u(\mathfrak{c})^P$ and $\beta\alpha \in \Gamma(\mathfrak{c})$, which proves (1). Next, since $\Gamma(\mathfrak{c}) \subset \Gamma$, from (16.19) and (17.4) we see that $e^{[k]}|e|^{i\kappa - [k]} = 1$ for every $e \in \mathfrak{g}^\times$ such that $e - 1 \in \mathfrak{c}$. Thus $X \neq \varnothing$ by [S97, Lemma 11.14]. Take any $\chi_0 \in X$; observe that, for a $\mathfrak{g}$-ideal $\mathfrak{a}$ prime to $\mathfrak{c}$, we have $\sum_{\chi \in X} \chi^*(\mathfrak{a}) = \#(X)\chi_0(\mathfrak{a})$ if $\mathfrak{a} = a\mathfrak{g}$ with $a \in W$; otherwise the sum is 0. Now take the sum of (16.39) or (16.48) for all $\chi \in X$. For $\alpha \in \mathcal{R}_\beta \subset \beta\Gamma_0(\mathfrak{c})$ we have $\sum_{\chi \in X} \chi[\alpha] \neq 0$ only when $\det(d_\alpha)\mathrm{il}_{\mathfrak{b}}(\alpha)^{-1} = a\mathfrak{g}$ with $a \in W$, so that $\mathrm{il}_{\mathfrak{b}}(\beta)$ is principal. Since the $\mathrm{il}_{\mathfrak{b}}(\beta)$ for all $\beta \in B$ represent the ideal classes of F, we can take $\beta = 1$, and hence $\alpha \in \Gamma_0(\mathfrak{c})$ and $\det(d_\alpha) \in W\mathfrak{g}^\times$, that is, $\alpha \in \Gamma_u(\mathfrak{c})$. Also $\chi[\alpha] = \chi_{\mathbf{a}}\bigl(\det(d_\alpha)\bigr)\chi^*\bigl(\det(d_\alpha)\mathfrak{g}\bigr) = \chi_{\mathfrak{c}}\bigl(\det(d_\alpha)\bigr)^{-1}$. Since $\Gamma_0(\mathfrak{c})^P = \Gamma_u(\mathfrak{c})^P$, we have

$$(*) \qquad \sum_{\chi \in X} E(z,\ s;\ k,\ \chi,\ \mathfrak{c}) = \#(X) \sum_{\alpha \in \mathcal{R}} (\chi_0)_{\mathfrak{c}}\bigl(\det(d_\alpha)\bigr)^{-1} \delta(x)^{s\mathbf{a} - (k - i\kappa)/2}\|_k\, \alpha$$

with $\mathcal{R} = \Gamma_u(\mathfrak{c})^P \backslash \Gamma_u(\mathfrak{c})$. By (1) we can take $\Gamma(\mathfrak{c})^P \backslash \Gamma(\mathfrak{c})$ as $\mathcal{R}$. Then the right-hand side of (*) is $\#(X)E\bigl(z,\ s;\ k,\ \Gamma(\mathfrak{c})\bigr)$. Combining this with (17.5), we obtain (2). To prove (3), put $\Gamma = \Gamma_0(\mathfrak{c})$ and $\Gamma' = \Gamma_0(\mathfrak{a})$; denote by $\widetilde{D}'$ the group of (16.25a, b) defined with $\mathfrak{a}$ in place of $\mathfrak{c}$. By strong approximation we have $\widetilde{D} \subset \widetilde{D}'G_1$, and so $\widetilde{D} = \widetilde{D}'\Gamma$. Let $T = \Gamma' \backslash \Gamma$, $A_0 = P\backslash(G \cap P_{\mathbf{A}}\widetilde{D})$, and $A_0' = P\backslash(G \cap P_{\mathbf{A}}\widetilde{D}')$. Since $P_{\mathbf{A}} \cap \widetilde{D} = P_{\mathbf{A}} \cap \widetilde{D}'$, we easily see that $P_{\mathbf{A}}\widetilde{D} = \bigsqcup_{\xi \in T} P_{\mathbf{A}}\widetilde{D}'\xi$. Therefore A_0 can be given by $\bigsqcup_{\xi \in T} A_0'\xi$. From this and (16.36) or (16.48) we obtain (3), since $\chi[\alpha\xi] = \chi[\alpha]\chi_{\mathfrak{c}}\bigl(\det(d_\xi)\bigr)^{-1}$ for $\xi \in \Gamma$.

As for (4), the holomorphy for $\mathrm{Re}(s) > \lambda_n$ follows from the convergence of our series in that domain. Now $E(z,\ s;\ k,\ \chi,\ \mathfrak{c})$ has meromorphic continuation to the whole s-plain by Theorem 16.11 in Case SP and by [S97, Theorem 19.7] in Case SU. Also, from these theorems we see that $E(z,\ s;\ k,\ \chi,\ \mathfrak{c})$ is holomorphic at $s = \lambda_n$ except when $k = 0$ and $\chi = 1$, in which case it has a simple pole at $s = \lambda_n$ with a positive number as its residue. Combining this with the equality of (2), we obtain (4).

17.3. Hereafter we assume that $\kappa = 0$, which means that a Hecke character χ of F satisfying (16.24a) is of finite order. Let us now put

$$D(z,\, s;\, k,\, \chi,\, \mathfrak{c}) \tag{17.7}$$
$$= E(z,\, s;\, k,\, \chi,\, \mathfrak{c}) \cdot \begin{cases} L_{\mathfrak{c}}(2s,\, \chi) \prod_{i=1}^{[n/2]} L_{\mathfrak{c}}(4s-2i,\, \chi^2) & (\text{Case SP},\ k \in \mathbf{Z}^{\mathbf{a}}), \\ \prod_{i=1}^{[(n+1)/2]} L_{\mathfrak{c}}(4s-2i-1,\, \chi^2) & (\text{Case SP}, k \notin \mathbf{Z}^{\mathbf{a}}), \\ \prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s-i,\, \chi\theta^i) & (\text{Case SU}), \end{cases}$$

where θ is the quadratic Hecke character of F corresponding to K/F.

We begin our investigation of E at special values of s by considering the special case in which $k = \mu\mathbf{a}$ with $\mu \in 2^{-1}\mathbf{Z}$; naturally $\mu \in \mathbf{Z}$ in Case SU. We then define $E^*(z,\, s)$ in Case SP as in §16.7, and in Case SU we put

$$E^*(z,\, s) = E(z,\, s;\, \mu\mathbf{a},\, \chi,\, \mathfrak{c})\|_{\mu\mathbf{a}}\, \eta \qquad (\text{Case SU}). \tag{17.8}$$

Here we assume that $\mathfrak{c} \neq \mathfrak{g}$ and $\chi_{\mathbf{a}}(x) = \operatorname{sgn}(x_{\mathbf{a}})^{[\mu]\mathbf{a}}$. We have a Fourier expansion

$$E^*(x+iy,\, s) = \sum_{h \in S} c_h(y,\, s)\mathbf{e}_{\mathbf{a}}^n(hx) \tag{17.9}$$
$$\text{with} \quad c_h(y,\, s) = \det(y)^{-\mu\mathbf{a}/2} c(h,\, y^{1/2},\, s)$$

as observed in (16.43) and [S97, Lemma 18.7 (2)]. The formula for $c(h, \cdots)$ in Case SU is given in [S97, Propositions 18.14 and 19.2]; these correspond to Propositions 16.9 and 16.10.

Now, if the series $E(z,\, s)$ of (16.36) or (17.3) with $k = \mu\mathbf{a}$ is convergent at $s = \mu/2$, then clearly $E(z,\, \mu/2)$ is holomorphic in z, and so it belongs to $\mathcal{M}_{\mu\mathbf{a}}$. It can happen that $E(z,\, s)$ is not convergent at $s = \mu/2$, but that analytic continuation allows us to speak of $E(z,\, \mu/2)$. Then we can ask the following questions:

(R1) *When is $E(z,\, \mu/2)$ meaningful? If $E(z,\, \mu/2)$ is meaningful, is it holomorphic in z? If so, is it an element of $\mathcal{M}_{\mu\mathbf{a}}(\overline{\mathbf{Q}})$?*

(R2) *If $E(z,\, \mu/2)$ is not holomorphic in z, can we say something about its analytic nature? What happens if we take a more general k?*

(R3) *Are there more values of s for which we can describe the nature of $E(z,\, s)$?*

We can ask similar questions by taking D of (17.7) in place of E. We shall answer these questions in Theorems 17.7, 17.8, and 17.9. In particular, we shall see that $E(z,\, \mu/2)$ can be nonholomorphic but nearly holomorphic in certain cases.

17.4. We are going to consider the behavior of $c_h(y,\, s)$ at $s = \mu/2$. To recall some of the properties of the function ξ of (16.44), we first put

$$\lambda = \lambda_n = (n+1)/2, \quad \iota = 1 \qquad (\text{Case SP}), \tag{17.10a}$$
$$\lambda = \lambda_n = n, \quad \iota = 2 \qquad (\text{Case SU}). \tag{17.10b}$$

In [S82] we obtained a function $\omega(g,\, h;\alpha,\, \beta)$ defined for $(g,\, h;\alpha,\, \beta)$ as in (16.44), holomorphic in $(\alpha,\, \beta) \in \mathbf{C}^2$, with which we have

$$\begin{aligned}(17.11)\qquad \xi(g, h; \alpha, \beta) = {}& i^{n\beta-n\alpha}2^{\tau}\pi^{\varepsilon}\Gamma_t^{\iota}(\alpha+\beta-\lambda)\Gamma_{n-q}^{\iota}(\alpha)^{-1}\Gamma_{n-p}^{\iota}(\beta)^{-1}\\ &\cdot \det(g)^{\lambda-\alpha-\beta}\delta_+(hg)^{\alpha-\lambda+\iota q/4}\delta_-(hg)^{\beta-\lambda+\iota p/4}\\ &\cdot \omega(2\pi g, h; \alpha, \beta).\end{aligned}$$

Here p (resp. q) is the number of positive (resp. negative) eigenvalues of h and $t = n - p - q$; $\delta_+(x)$ is the product of all positive eigenvalues of x and $\delta_-(x) = \delta_+(-x)$; Γ_n^{ι} is defined by (16.47);

$$\begin{aligned}\tau &= (2p-n)\alpha + (2q-n)\beta + n + t\lambda + (\iota pq/2),\\ \varepsilon &= p\alpha + q\beta + t + (\iota/2)\{t(t-1) - pq\}.\end{aligned}$$

In particular we have

$$(17.12)\qquad \xi(g, h; \alpha, 0) = 2^{1-\lambda}i^{-n\alpha}(2\pi)^{n\alpha}\Gamma_n^{\iota}(\alpha)^{-1}\det(h)^{\alpha-\lambda}\mathbf{e}^n(igh) \quad \textit{if} \quad h>0,$$

$$(17.13)\qquad \xi(g, 0; \alpha, \beta) = i^{n\beta-n\alpha}2^{n(\lambda+1-\alpha-\beta)}\pi^{n\lambda}\frac{\Gamma_n^{\iota}(\alpha+\beta-\lambda)}{\Gamma_n^{\iota}(\alpha)\Gamma_n^{\iota}(\beta)}\det(g)^{\lambda-\alpha-\beta},$$

$$\begin{aligned}(17.14)\qquad &\lim_{s\to 0}\xi(g, h; \lambda+s, s) = 2^{\sigma}i^{-n\lambda}\pi^{n\lambda}\Gamma_n^{\iota}(\lambda)^{-1}\mathbf{e}^n(igh) \quad \textit{if} \quad q=0\\ &\qquad \textit{with} \quad \sigma = [(n+p)/2] \textit{ in Case SP and } \sigma = p \textit{ in Case SU.}\end{aligned}$$

For these see [S82, (4.34.K), (4.35.K), (1.31)]. To state one more formula, we take an indeterminate T and define polynomial functions $\varphi_r(X)$ of $X \in \mathbf{C}_n^n$ by

$$(17.15)\qquad \det(T1_n - X) = \sum_{r=0}^{n}(-1)^r\varphi_r(X)T^{n-r}.$$

Notice that $\varphi_\nu(\mathrm{diag}[X, 0]) = \varphi_\nu(X)$ for $\nu \le n$. Now we have

$$\begin{aligned}(17.16)\qquad \omega(2\pi y, h; \lambda+1, 0) = {}& 2^{-p\lambda}\pi^{\iota p(n-p)/2}\mathbf{e}^n(ihy)\delta_+(4\pi hy)^{-1}\\ &\cdot\sum_{\nu=0}^{p}(-1)^{\nu}b_\nu(\iota(n-p)/2)\varphi_{p-\nu}(4\pi hy) \quad \textit{if} \quad q=0, \textit{ where}\\ & b_0(\alpha) = 1 \textit{ and } b_\nu(\alpha) = \prod_{m=0}^{\nu-1}(\alpha + (\iota m/2)) \quad \textit{if} \quad \nu>0.\end{aligned}$$

This was proved in [S93, Lemma 9.2 (iv)]. Put $X = \pi hy$ in (17.15) and multiply by $(T\pi y)^{-1}$; then we obtain $\det\left[(\pi y)^{-1} - T^{-1}h\right] = \sum_{r=0}^{n}\det(\pi y)^{-1}\varphi_r(\pi hy)(-T)^{-r}$, which shows that

(17.17) $\det(\pi y)^{-1}\varphi_r(\pi hy)$ *is a polynomial of degree* $\le n-r$ *in the entries of* $(\pi y)^{-1}$ *with coefficients in the field generated over* $\mathbf{Q}$ *by the entries of* h.

17.5. Lemma. *Define* $L_{\mathfrak{c}}(s, \psi)$ *by (16.9) with a Hecke character* ψ *of* F.

(1) *If* $0 \ge m \in \mathbf{Z}$, $\psi_v(x_v) = \mathrm{sgn}(x_v)^m$ *for some* $v \in \mathbf{a}$, *and* $\mathfrak{c} \ne \mathfrak{g}$, *then* $L_{\mathfrak{c}}(m, \psi) = 0$.

(2) *If* $\psi_{\mathbf{a}}(x) = \mathrm{sgn}(x_{\mathbf{a}})^{m\mathbf{a}}$ *with* $0 < m \in \mathbf{Z}$, *then* $L_{\mathfrak{c}}(m, \psi) \in \pi^{dm}\mathbf{Q}_{\mathrm{ab}}$ *and* $L_{\mathfrak{c}}(1-m, \psi) \in \mathbf{Q}_{\mathrm{ab}}$.

(3) *If* $\psi_{\mathbf{a}}(x) = \mathrm{sgn}(x_{\mathbf{a}})^{m\mathbf{a}}$ *with* $0 \le m \in \mathbf{Z}$, *then* $L_{\mathfrak{c}}(s, \psi)$ *at* $s = -m$ *has a zero of order at least* d *except when* $m = 0$, $\psi = 1$, *and* $\mathfrak{c} = \mathfrak{g}$, *in which case the order of zero is* $d-1$.

PROOF. Take $t \in \mathbf{Z}^{\mathbf{a}}$ so that $0 \le t_v \le 1$ for every $v \in \mathbf{a}$ and $m - t \in 2\mathbf{Z}^{\mathbf{a}}$. Then $L(s, \psi)\prod_{v\in\mathbf{a}}\Gamma((s+t_v)/2)$ is holomorphic on $\mathbf{C}$ except for possible simple poles at $s = 0$ and $s = 1$, which occur if and only if $\psi = 1$. Assertions (1) and (3) can easily be derived from this fact. Assertion (2) is included in a result which we prove as Theorem 18.12 in the next section.

17.6. Proposition. *Let E^* be defined as above with $\kappa = 0$, $\mathfrak{c} \neq \mathfrak{g}$, and $k = \mu\mathbf{a}$, $\mu \in 2^{-1}\mathbf{Z}$; suppose $\mu \geq \lambda$; let $h \in S$ and $r = \mathrm{rank}(h)$. In Case SU let θ be the quadratic Hecke character of F corresponding to K/F. Then $c_h(y, s)$ is finite at $s = \mu/2$. Moeover, $c_h(y, \mu/2) \neq 0$ only in the following cases, and in each case, except in case (viii), the value can be described as follows:*

(I) $h = 0$.

(i) $\mu = \lambda:\quad c_0(y, \mu/2) \in \mathbf{Q}_{\mathrm{ab}}$.

(ii) (Case SP) $\mu = (n+2)/2$, $F = \mathbf{Q}$, *and* $\chi^2 = 1:$
$c_0(y, \mu/2) = c\det(y)^{-1/2}$ *with* $c \in \mathbf{C}$; $c \in \pi^{-\lambda}\mathbf{Q}_{\mathrm{ab}}$ *if* $\mu \notin \mathbf{Z}$.

(iii) (Case SP) $2 < \mu = (n+3)/2$, $F = \mathbf{Q}$, and $\chi^2 = 1:$
$c_0(y, \mu/2) = c\pi^{-n}\det(y)^{-1}$ with $c \in \mathbf{Q}_{\mathrm{ab}}$.

(iv) (Case SP) $\mu = 2$, $n = 1$, $F = \mathbf{Q}$, and $\chi = 1:$
$c_0(y, \mu/2) = c\pi^{-1}y^{-1}$ with $c \in \mathbf{Q}_{\mathrm{ab}}$.

(v) (Case SU) $\mu = n+1$, $F = \mathbf{Q}$, and $\chi = \theta^{n-1}:$
$c_0(y, \mu/2) = c\pi^{-n}\det(y)^{-1}$ with $c \in \mathbf{Q}_{\mathrm{ab}}$.

(II) $h \neq 0$.

(vi) $\mu > \lambda$ *and* $h_v > 0$ *for every* $v \in \mathbf{a}:$
$c_h(y, \mu/2) = c\mathbf{e}_{\mathbf{a}}^n(ihy)$ *with* $c \in \mathbf{Q}_{\mathrm{ab}}$.

(vii) $\mu = \lambda$ *and* $h_v \geq 0$ *for every* $v \in \mathbf{a}:$
$c_h(y, \mu/2) = c\mathbf{e}_{\mathbf{a}}^n(ihy)$ *with* $c \in \mathbf{Q}_{\mathrm{ab}}$.

(viii) (Case SP) $\mu = (n+2)/2$, $F = \mathbf{Q}$, $\chi^2 = 1$.

(ix) (Case SP) $\mu = (n+3)/2$, $F = \mathbf{Q}$, $\chi^2 = 1$, $0 < r < n$, *and* $h \geq 0:$
$c_h(y, \mu/2) = c\pi^e\det(\pi y)^{-1}\delta_+(\pi hy)\omega(2\pi y,\, h;\, \lambda+1,\, 0)$ *with* $c \in \mathbf{Q}_{\mathrm{ab}}$ *and* $e = r(r-n)/2$.

(x) (Case SU) $\mu = n+1$, $F = \mathbf{Q}$, $\chi = \theta^{n-1}$, $0 < r < n$, *and* $h \geq 0:$
$c_h(y, \mu/2) = c\pi^e\det(\pi y)^{-1}\delta_+(\pi hy)\omega(2\pi y,\, h;\, n+1,\, 0)$ *with* $c \in \mathbf{Q}_{\mathrm{ab}}$ *and* $e = r(r-n)$.

PROOF. By Propositions 16.9 and 16.10, $c_h(y, s)$ in Case SP is easy finite non-vanishing factors times

$$(*)\qquad f(s)\Lambda_{\mathfrak{c}}(s)^{-1}\Lambda_h(s)\prod_{v\in\mathbf{a}}\xi(y_v,\, h_v;\, s+\mu/2,\, s-\mu/2),$$

where we denote by $f(s)$ the product $\prod_{v\in\mathbf{c}}$ of (16.46). Since we know the explicit forms of $\Lambda_{\mathfrak{c}}$ and Λ_h, we can derive our assertions from their properties and the formulas concerning ξ in §17.4, combined with Lemma 17.5. The verification is fairly straightforward, but lengthy, as there are many combinations of n, μ, χ, and h, which produce different results. Therefore, we discuss here only a few typical cases in Case SP, leaving the remaining cases to the reader. Case SU can be handled in a similar and much simpler way by employing the formulas of [S97, Propositions 18.14 and 19.2].

First of all, since $\chi_{\mathbf{a}}(x) = \mathrm{sgn}(x_{\mathbf{a}})^{\mu\mathbf{a}}$, from Lemma 17.5 (2) we see that

$$(17.18\mathrm{a})\qquad 0 \neq \Lambda_{\mathfrak{c}}(\mu/2) \in \pi^{df}\mathbf{Q}_{\mathrm{ab}},\quad \textit{where}$$
$$f = \begin{cases} \mu(2m+1) - m(m+1) \ \textit{with}\ m = [n/2] \ \textit{if}\ \mu \in \mathbf{Z}, \\ m(2\mu - m) \ \textit{with}\ m = [(n+1)/2] \ \textit{if}\ \mu \notin \mathbf{Z}. \end{cases}$$

We also observe, for $(n+1)/2 \le \mu \in 2^{-1}\mathbf{Z}$, that

$$\text{(17.18b)} \qquad \Gamma_n^1(\mu) \in \begin{cases} \pi^{[n^2/4]}\mathbf{Q}^\times & \text{if } \mu n \in \mathbf{Z}, \\ \pi^{(n^2+1)/4}\mathbf{Q}^\times & \text{if } \mu n \notin \mathbf{Z}. \end{cases}$$

(I) Suppose $h = 0$ and $\mu \in \mathbf{Z}$; then in (*) we have $f = 1$ and $\xi(y, 0; \cdots)$ is given by (17.13). Thus (*) takes the form

$$\text{(**)} \qquad \Lambda_\mathfrak{c}(s)^{-1}L_\mathfrak{c}(2s-n, \chi)\prod_{i=1}^{[n/2]} L_\mathfrak{c}(4s-2n+2i-1, \chi^2)$$
$$\cdot\, a \cdot 2^{-dns}\pi^{dn\lambda}\det(y)^{(\lambda-2s)\mathbf{a}}\Gamma_n^1(2s-\lambda)^d\Gamma_n^1(s+\mu/2)^{-d}\Gamma_n^1(s-\mu/2)^{-d}$$

with $a \in \mathbf{Q}_{\mathrm{ab}}$. Take $\mu = (n+2)/2$ with even n, for example. By Lemma 17.5 (3), $L_\mathfrak{c}(2s-n, \chi)$ has a zero of order at least d at $s = \mu/2$ for any χ. Now $\prod_{i=1}^{[n/2]}$ is finite everywhere except when $\chi^2 = 1$, in which case the factor for $i = n/2$ has a pole of order 1 at $s = \mu/2$; the other factors belong to $\mathbf{Q}_{\mathrm{ab}}$. Now, from (17.18b) we easily see that the product of the last gamma factors of (**) at $s = \mu/2$ produces a rational number times $\pi^{n^2/4}$. Thus the whole product is 0 if $\chi^2 \ne 1$ or $F \ne \mathbf{Q}$. If $\chi^2 = 1$ and $F = \mathbf{Q}$, we obtain a constant times $\det(y)^{-1/2}$, as stated in (ii).

Still with $\mu = (n+2)/2$, assume that $n \notin 2\mathbf{Z}$. Then the first line of (**) has a different expression, but the second line is the same. This time $\Gamma_n^1(2s-\lambda)^d\Gamma_n^1(s-\mu/2)^{-d}$ has a zero of order d at $s = \mu/2$. Then we obtain (ii) in this case in a similar manner.

Next suppose $\mu = (n+3)/2$ with odd n. Again we see that the nonvanishing can occur only if $F = \mathbf{Q}$ and $\chi^2 = 1$, in which case

$$\Lambda_0(s) = L_\mathfrak{c}(2s-n, \chi)\prod_{i=1}^{[n-1)/2} L_\mathfrak{c}(4s-2n+2i-1, 1).$$

Suppose $n = 1$; then $\mu = 2$ and we easily see that the nonvanishing can occur only when $\chi = 1$, and the result is as stated in (iv). If $n > 1$, we have $\mu < n$, and so $L_\mathfrak{c}(\mu-n, \chi) \in \mathbf{Q}_{\mathrm{ab}}$; also $\prod_{i=1}^{(n-1)/2}$ has a pole of order 1 at $s = \mu/2$ and its residue is a rational number times $\prod_{i=1}^{(n-3)/2} L_\mathfrak{c}(1-2i, 1)$, which belongs to $\mathbf{Q}_{\mathrm{ab}}$. In this way we obtain (iii) and (iv).

(II) Suppose $h \ne 0$; define ρ_h as in Proposition 16.10 and put $r = \mathrm{rank}(h)$; let p_v (resp. q_v) be the number of positive (resp. negative) characteristic roots of h_v. Then $(\chi\rho_h)_v(x) = \mathrm{sgn}(x_v)^{[\mu]+[r/2]+q_v}$ for $v \in \mathbf{a}$. First suppose $r = n$ and $2\mu + n \notin 2\mathbf{Z}$; then $\Lambda_h = 1$. From (17.11) we see that the last factor $\prod_{v\in\mathbf{a}}$ of (*) is finite at $s = \mu/2$, and is nonzero only if $q_v = 0$ for every v, in which case (17.12) shows that $c_h(y, \mu/2)$ is an element of $\mathbf{Q}_{\mathrm{ab}}$ times $\Lambda_\mathfrak{c}(\mu/2)^{-1}\Gamma_n^1(\mu)^{-d}\pi^{dn\mu}\mathbf{e}_\mathbf{a}^n(ihy)$. Employing (17.18a, b), we see that $c_h(y, \mu/2) = c\mathbf{e}_\mathbf{a}^n(ihy)$ with $c \in \mathbf{Q}_{\mathrm{ab}}$.

The case $0 < r < n$ is more complicated. Suppose $\mu \notin \mathbf{Z}$, $r < n \in 2\mathbf{Z}$, and $r \notin 2\mathbf{Z}$, for example; then

$$\text{(***)} \qquad \Lambda_h(s) = L_\mathfrak{c}(2s-n+r/2, \chi\rho_h)\prod_{i=1}^{[(n-r)/2]} L_\mathfrak{c}(4s-2n+r+2i-1, \chi^2).$$

By (17.11), $\prod_{v\in\mathbf{a}}\xi(\cdots)$ of (*) is a finite factor times $\prod_{v\in\mathbf{a}}\Gamma_t^1(2\sigma+\mu-\lambda)\Gamma_{t+q_v}^1(\sigma)^{-1}$ with $\sigma = s - \mu/2$, where $t = n - r$. First suppose $\mu = \lambda$; then t is odd and (***) is finite at $s = \mu/2$; the last $\prod_{v\in\mathbf{a}}$ is 0 at $\sigma = 0$ if $q_v \ge 2$ for some $v \in \mathbf{a}$. Thus we may assume that $q_v \le 1$ for every $v \in \mathbf{a}$. Suppose $q_v = 1$ for some v. Then

$[\mu] + [r/2] + q_v = (n + r + 1)/2 \equiv \mu - n + r/2 \pmod 2$. Therefore the first factor of (***) is 0 at $s = \mu/2$ by Lemma 17.5 (1). Thus the nonvanishing occurs only when $q_v = 0$ for every $v \in \mathbf{a}$, that is, when h is totally nonnegative. Then employing (17.14), we obtain the desired formula as given in (vii).

Next suppose $\mu = \lambda + 1$. Then (***) is finite if $\chi^2 \neq 1$; it has a pole of order at most 1 at $s = \mu/2$ if $\chi^2 = 1$. This time the product of gamma functions has a zero of order at least d at $\sigma = 0$. Therefore the nonvanishing can occur only if $\chi^2 = 1$, $F = \mathbf{Q}$, and Λ_h has a pole of order 1 at $s = \mu/2$. In that special case suppose $q > 1$; then $\Gamma_t^1(2\sigma + 1)/\Gamma_{t+q}^1(\sigma)$ has a zero of order at least 2 at $\sigma = 0$, so that the nonvanishing can occur only if $q \leq 1$. If $r = n-1$, then $\chi = \rho_h$ and so the signature formula for $\chi\rho_h$ says that $q \in 2\mathbf{Z}$, a contradiction. Thus we may assume that $q \leq 1$ and $r \leq n - 3$. Suppose $q = 1$. Then $[\mu] + [r/2] + q = (n + 3 + r)/2 \equiv \mu - n + r/2 \pmod 2$. Therefore $L_{\mathfrak{c}}(2s - n + r/2, \chi\rho_h)$ is 0 at $s = \mu/2$ by Lemma 17.5 (1). Thus we may assume $q = 0$. Then from (17.11) we obtain the desired formula as given in (ix). All the remaining cases can be handled more or less in the same manner.

To make our statements shorter, we make the following convention: whenever we speak of a function $f(z, \mu/2)$ belonging to a set, it means that $f(z, s)$ is finite at $s = \mu/2$, and the value as a function of z belongs to the set in question; whenever we speak of $E(z, s; \nu\mathbf{a}, \dots)$ or $D(z, s; \nu\mathbf{a}, \dots)$, we assume that $\nu \in 2^{-1}\mathbf{Z}$ in Case SP and $\nu \in \mathbf{Z}$ in case SU. In §14.11 we defined $\mathcal{N}_\omega^p(W)$ for $p \in \mathbf{Z}^{\mathbf{a}}$, a representation ω of $GL_n(\mathbf{C})^{\mathbf{b}}$, and a subfield W of $\mathbf{C}$. In Case SU, viewing $\mathbf{Z}^{\mathbf{a}}$ as a submodule of $\mathbf{Z}^{\mathbf{b}}$ in an obvious way, we can speak of $\mathcal{N}_k^p(W)$ for $k \in \mathbf{Z}^{\mathbf{a}}$.

17.7. Theorem (Cases SP and SU). (i) *If $\mu \geq \lambda$, then $E(z, \mu/2; \mu\mathbf{a}, \Gamma)$ belongs to $\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ except when $F = \mathbf{Q}$ and $\lambda + (1/2) \leq \mu \leq \lambda + 1$.*

(ii) *If $F = \mathbf{Q}$ and $\mu = \lambda + 1$, then $E(z, \mu/2; \mu\mathbf{a}, \Gamma)$ belongs to $\mathcal{N}_\mu^n(\mathbf{Q}_{\mathrm{ab}})$.*

(iii) *If $\mu \geq \lambda$, then $E(z, \mu/2; \mu\mathbf{a}, \chi, \mathfrak{c})$ belongs to $\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ except in the following four cases:*

(A) Case SP: $\mu = (n + 2)/2$, $F = \mathbf{Q}$, *and* $\chi^2 = 1$;
(B) Case SP: $n = 1$, $\mu = 2$, $F = \mathbf{Q}$, *and* $\chi = 1$;
(C) Case SP: $n > 1$, $\mu = (n + 3)/2$, $F = \mathbf{Q}$, *and* $\chi^2 = 1$;
(D) Case SU: $\mu = n + 1$, $F = \mathbf{Q}$, *and* $\chi = \theta^{n+1}$.

(iv) *In Cases (B), (C), (D), $E(z, \mu/2; \mu\mathbf{a}, \chi, \mathfrak{c})$ belongs to $\mathcal{N}_\mu^n(\mathbf{Q}_{\mathrm{ab}})$.*

(v) *Taking an integer $\mu \leq \lambda$, put $\nu = 2\lambda - \mu$ and*

$$e = \begin{cases} n(n+2)/4 & (\text{Case SP, } n \in 2\mathbf{Z}), \\ (n+1)\lambda/2 & (\text{all othe cases}). \end{cases}$$

Then $D(z, \mu/2; \nu\mathbf{a}, \chi, \mathfrak{c})$ belongs to $\pi^{de}\mathcal{M}_{\nu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ except in the following three cases:

(E) Case SP: $\mu = 0$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi = 1$;
(F) Case SP: $0 < \mu \leq n/2$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi^2 = 1$;
(G) Case SU: $0 \leq \mu < n$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi = \theta^\mu$.

(vi) *If $n = 1$ and $\mathfrak{c} = \mathfrak{g}$, then $D(z, 0; 2\mathbf{a}, \chi, \mathfrak{c})$ belongs to $\pi^d\mathcal{M}_{2\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ except when $F = \mathbf{Q}$, in which case it belongs to $\pi\mathcal{N}_2^1(\mathbf{Q}_{\mathrm{ab}})$.*

PROOF. We first prove (iii) and (iv) when $\mathfrak{c} \neq \mathfrak{g}$. By Theorem 7.11, $\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ is stable under $f \mapsto f\|\alpha$ for every $\alpha \in G_1$ or $\alpha \in \mathcal{G}$. As for $\mathcal{N}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$, we need it only when $F = \mathbf{Q}$. Therefore, by Theorem 14.13, it is stable under $f \mapsto f\|\alpha$ for every such α (see §14.14). Thus it is sufficient to prove (iii) and (iv) for $E^*(z, s)$ instead of $E(z,\, s;\, \cdots)$. Now the desired facts for E^* follow immediately from Proposition 17.6. The case in which $\mu > \lambda+1$ is the easiest: $E^*(z,\, \mu/2) = \sum_{h\in S} b_h \mathbf{e}_{\mathbf{a}}^n(hz)$ with $b_h \neq 0$ only if h is totally positive and $b_h \in \mathbf{Q}_{\mathrm{ab}}$. All other cases are similar. Special care must be taken when $\mu = \lambda + 1$ and $F = \mathbf{Q}$. Here let us discuss only the case in which $n > 1$ in Case SP. We assume $\chi^2 = 1$, since the function is holomorphic otherwise. By (iii) and (vi) of Proposition 17.6, $c_0(y,\, \mu/2) = a \det(\pi y)^{-1}$ with $a \in \mathbf{Q}_{\mathrm{ab}}$ and $c_h(y,\, \mu/2) = b\mathbf{e}_{\mathbf{a}}^n(ihy)$ with $b \in \mathbf{Q}_{\mathrm{ab}}$ if $\det(h) \neq 0$. For $0 < \mathrm{rank}(h) = r < n$, $c_h(y,\, \mu/2)$, if nonzero, can be given by the formula of (ix) of Proposition 17.6. From (17.16) and (17.17) we easily see that $c_h(y,\, \mu/2) = q\bigl((\pi y)^{-1}\bigr)\mathbf{e}_{\mathbf{a}}^n(ihy)$ with a $\mathbf{Q}_{\mathrm{ab}}$-rational polynomial q of degree $\leq n - r$. Thus $E^*(z,\, \mu/2)$ in this case belongs to $\mathcal{N}_{\mu\mathbf{a}}^n(\mathbf{Q}_{\mathrm{ab}})$.

Next suppose $\mathfrak{c} = \mathfrak{g}$; then $\mu \in \mathbf{Z}$. Fix an arbitrary prime ideal $\mathfrak{p}$ of F. By Lemma 17.2 (3), $E(z,\, s;\, k,\, \chi,\, \mathfrak{g}) = \sum_{\alpha\in A} E(z,\, s;\, k,\, \chi,\, \mathfrak{p})\|_k\, \alpha$ with a finite subset A of $\Gamma_0(\mathfrak{c})$. Therefore we obtain (iii) and (iv) in the case $\mathfrak{c} = \mathfrak{g}$ from those in the case $\mathfrak{c} \neq \mathfrak{g}$.

Once (iii) and (iv) are established, (i) and (ii) can be obtained by combining (iii) and (iv) with Lemma 17.2 (2), because of the stability of $\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ and $\mathcal{N}_{\mu}^n(\mathbf{Q}_{\mathrm{ab}})$ under $\|_{\mu\mathbf{a}}\, \alpha$ as mentioned at the beginning.

As for (v), the method of proof is the same. In Case SU, $\Lambda_\mathfrak{c}$ is given in [S97, Proposition 19.2], which is the same as the product of L-functions in (17.7). If $\mathfrak{c} \neq \mathfrak{g}$, we again reduce the problem to $\Lambda_\mathfrak{c}(s)E^*(z,\, s;\, \nu\mathbf{a})$. Thus the question is the behavior of $\Lambda_\mathfrak{c}(s)c_h(y,\, s)$ at $s = \mu/2$ with $k = \nu\mathbf{a}$. The analysis of the value is similar to that of $c_h(y,\, s)$ at $s = \mu/2$ in the proof of Proposition 17.6, and so we do not go into details here. We note only that we need tha value $\xi(y,\, h;\, \lambda,\, \mu - \lambda)$, which can be obtained from [S82, (4.34.K), (4.35.K)]. If $\mathfrak{c} = \mathfrak{g}$, with any fixed prime ideal $\mathfrak{p}$ we have

$$B(s)D(z,\, s;\, k,\, \chi,\, \mathfrak{g}) = \sum_{\alpha\in A} D(z,\, s;\, k,\, \chi,\, \mathfrak{p})\|_k\, \alpha$$

$$\text{with}\quad B(s) = \begin{cases} \bigl(1 - \chi^*(\mathfrak{p})N(\mathfrak{p})^{-2s}\bigr) \displaystyle\prod_{i=1}^{[n/2]} \bigl(1 - \chi^*(\mathfrak{p})^2 N(\mathfrak{p})^{2i-4s}\bigr) & \text{(Case SP)}, \\ \displaystyle\prod_{i=0}^{n-1} \bigl(1 - (\chi\theta^i)^*(\mathfrak{p})N(\mathfrak{p})^{i-2s}\bigr) & \text{(Case SU)}. \end{cases}$$

Observe that we can choose $\mathfrak{p}$ so that $B(\mu/2) \neq 0$ if we exclude Cases (E), (F), and (G). Therefore we can derive the desired conclusion of (v) for $\mathfrak{c} = \mathfrak{g}$ from that for $\mathfrak{c} = \mathfrak{p}$.

We shall prove (vi) in §A2.14.

17.8. Theorem (Cases SP and SU). *Suppose $0 < \mu < \lambda$; put $s_\mu = \lambda - (\mu/2)$. Then $E(z,\, s;\, \mu\mathbf{a},\, \chi,\, \mathfrak{c})$ has at most a simple pole at $s = s_\mu$, which occurs only if $\chi^2 = 1$ in Case SP and $\chi = \theta^\mu$ in Case SU. The residue at $s = s_\mu$ is of the form $\pi^{-d\gamma}A \cdot g(z)$ with an element g of $\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ and constants γ and A given by*

$$\gamma = \begin{cases} n(n/2 - \mu) & (\text{Case SP},\ \mu \in \mathbf{Z},\ n \in 2\mathbf{Z}), \\ (n-1)(n/2 - \mu) - 1 & (\text{Case SP},\ \mu \in \mathbf{Z},\ n \notin 2\mathbf{Z}), \\ (n+1-2\mu)[(n+1)/2)] & (\text{Case SP},\ \mu \notin \mathbf{Z}), \\ n(n-\mu) & (\text{Case SU}), \end{cases}$$

$$A = R_F \cdot \begin{cases} L(n+1-\mu,\,\chi) \displaystyle\prod_{i=1}^{[n/2-\mu]} \zeta_F(2i+1) & (\text{Case SP},\ \mu \in \mathbf{Z}), \\ \displaystyle\prod_{i=1}^{[n/2-\mu]} \zeta_F(2i+1) & (\text{Case SP},\ \mu \notin \mathbf{Z}), \\ \displaystyle\prod_{i=2}^{n-\mu} L(i,\,\theta^{i-1}) & (\text{Case SU}), \end{cases}$$

where R_F is the regulator of F, and ζ_F is the Dedekind zeta function of F. Moreover, $g(z) = \sum_h a(h)\mathbf{e}_{\mathbf{a}}^n(hz)$ with $a(h) = 0$ whenever $\operatorname{rank}(h) > 2\mu$ *in Case SP and* $\operatorname{rank}(h) > \mu$ *in Case SU.*

PROOF. The method of proof is the same as in Theorem 17.7. Namely, we reduce the problem to $E^*(z, s)$, and study the behavior of the Fourier coefficients at $s = s_\mu$, employing the explicit form of $c(h, q, s)$ given in Proposition 16.9. In particular, we have to analyze $\xi(g, h;\, s+\mu/2,\, s-\mu/2)$ at $s = s_\mu$, for which we need (17.11) and [S82, (4.35.K)]. The details may be left to the reader. There is one more nontrivial point: that the property $a(h) = 0$ for $\operatorname{rank}(h) > t$ with a fixed t can be preserved by the transformation $g \mapsto g\|\alpha$ for every $\alpha \in G_1$. For this, see [S94b, (5.14)].

17.9. Theorem (Cases SP and SU). *Let Φ be the Galois closure of K over $\mathbf{Q}$ and let k be a weight (which means that $k \in \mathbf{Z}^{\mathbf{a}}$ in Case SU); suppose that $k_v \geq \lambda$ for every $v \in \mathbf{a}$ and $k_v - k_{v'} \in 2\mathbf{Z}$ for every $v, v' \in \mathbf{a}$.*

(ia) *Let μ be an element of $2^{-1}\mathbf{Z}$ such that $\lambda \leq \mu \leq k_v$ and $\mu - k_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Then $E(z, \mu/2;\, k, \Gamma)$ and $E(z, \mu/2;\, k, \chi, \mathfrak{c})$ belong to $\pi^\alpha \mathcal{N}_k^t(\Phi\mathbf{Q}_{\mathrm{ab}})$ except when $F = \mathbf{Q}$ and $\mu = (n+2)/2$ in Case SP, where $\alpha = (n/2)\sum_{v\in\mathbf{a}}(k_v - \mu)$ and*

$$t = \begin{cases} n(k-\mu+2)/2 & \textit{if}\ \ \mu = \lambda+1 \ \ \textit{and}\ \ F = \mathbf{Q}, \\ n(k-\mu\mathbf{a})/2 & \textit{otherwise}. \end{cases}$$

(ib) *For μ and α as above, $E(z, \mu/2;\, k, \chi, \mathfrak{c})$ belongs to $\pi^\alpha \mathcal{N}_k^u(\Phi\mathbf{Q}_{\mathrm{ab}})$ except in Case (A) of Theorem 17.7, where*

$$u = \begin{cases} n(k-\mu+2)/2 & \textit{in Cases (B), (C), (D) of Theorem 17.7}, \\ n(k-\mu\mathbf{a})/2 & \textit{otherwise}. \end{cases}$$

(ii) *Let $\mu \in 2^{-1}\mathbf{Z}$; suppose that $2\lambda - k_v \leq \mu \leq k_v$ and $|\mu-\lambda| + \lambda - k_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Suppose also that μ, F, χ, and $\mathfrak{c}$ do not fall into Cases (A), (E), (F), and (G) of Theorem 17.7. Then $D(z, \mu/2;\, k, \chi, \mathfrak{c})$ belongs to $\pi^\beta \mathcal{N}_k^r(\Phi\mathbf{Q}_{\mathrm{ab}})$, where*

$$r = \begin{cases} n(k-\mu+2)/2 & \textit{in Cases (B), (C), (D)}, \\ (n/2)(k - |\mu-\lambda|\mathbf{a} - \lambda\mathbf{a}) & \textit{otherwise}, \end{cases}$$

and $\beta = (n/2)\sum_{v\in\mathbf{a}}(k_v + \mu) - de$ with

$$e = \begin{cases} \big[(n+1)^2/4\big] - \mu & (\text{Case SP: } 2\mu+n \in 2\mathbf{Z} \text{ and } \mu \geq \lambda), \\ [n^2/4] & (\text{Case SP: } 2\mu+n \notin 2\mathbf{Z} \text{ or } \mu < \lambda), \\ n(n-1)/2 & (\text{Case SU}). \end{cases}$$

(iii) *Suppose $n = 1$, $\mu = 0$, and $2 \leq k_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Then $D(z, 0;\, k, \chi, \mathfrak{c})$ belongs to $\pi^\beta \mathcal{N}_k^r(\Phi\mathbf{Q}_{\mathrm{ab}})$ with $r = (k-2\mathbf{a})/2$ and $\beta = \sum_{v\in\mathbf{a}} k_v/2$,* *except when* $F = \mathbf{Q}$ *and* $\mathfrak{c} = \mathbf{Z}$. *If* $F = \mathbf{Q}$ *and* $\mathfrak{c} = \mathbf{Z}$, *then it belongs to* $\pi^{k/2}\mathcal{N}_k^{k/2}(\mathbf{Q}_{\mathrm{ab}})$.

Assertion (iii) means that if $n=1$ we have results even in Cases (E) and (G). It is conjecturable that similar results hold in Cases (E), (F), and (G) even for $n > 1$.

PROOF. For $p \in \mathbf{Z}^{\mathbf{a}}$ and a weight q define the operator Δ_q^p by $\Delta_q^p f = (D_\rho^Z f)(\psi)$ with $\rho(a, b) = \det(b)^q$, $Z = \bigotimes_{v\in\mathbf{a}} Z_v = \mathbf{C}\psi \subset S_{np}(T)$, where $Z_v = \mathbf{C}\psi_v \subset S_{np_v}(T_v)$, $\psi = \prod_{v\in\mathbf{a}} \psi_v$, and $\psi_v = \det(x)^{p_v}$. Then, by (12.24c) and Theorem 14.12 (4),

$$\Delta_q^p \mathcal{N}_q^t(\Phi\mathbf{Q}_{\mathrm{ab}}) \subset \pi^{n|p|}\mathcal{N}_{q+2p}^{t+np}(\Phi\mathbf{Q}_{\mathrm{ab}}). \tag{17.19}$$

(See §14.14 if $q \notin \mathbf{Z}^{\mathbf{a}}$.) Let simply $E(z, s; k)$ denote $E(z, s; k, \Gamma)$ or $E(z, s; k, \chi, \mathfrak{c})$. We now apply Lemma 13.9 to each term of E with $\zeta(x) = \det(x)^{p_v}$. Employing the formula for ψ_Z given in Theorem 12.13, we find that

$$\Delta_q^p E(z, s; q) = c_q^p(s)(i/2)^{n|p|}E(z, s; q + 2p) \tag{17.20}$$

$$\text{with}\quad c_q^p(s) = \begin{cases} \displaystyle\prod_{v\in\mathbf{a}}\prod_{a=1}^{n}\prod_{b=1}^{p_v}\{-s-(q_v/2)-b+(a+1)/2\} & \text{(Case SP)},\\ \displaystyle\prod_{v\in\mathbf{a}}\prod_{a=1}^{n}\prod_{b=1}^{p_v}\{-s-(q_v/2)-b+a\} & \text{(Case SU)}.\end{cases}$$

This is so even when $q \notin \mathbf{Z}^{\mathbf{a}}$; see §14.14. Now, given μ as in (ia), put $p = (k - \mu\mathbf{a})/2$. Then $0 \le p \in \mathbf{Z}^{\mathbf{a}}$, and (17.20) with $q = \mu\mathbf{a}$ yields

$$\Delta_{\mu\mathbf{a}}^p E(z, \mu/2; \mu\mathbf{a}) = c_{\mu\mathbf{a}}^p(\mu/2)(i/2)^{n|p|}E(z, \mu/2; k). \tag{17.21}$$

Observing that $c_{\mu\mathbf{a}}^p(\mu/2) \neq 0$, we obtain our assertions of (ia) and (ib) from (i) and (ii) of Theorem 17.7 and (17.19).

Next, let μ be given as in (ii). If $\mu \ge \lambda$, the above proof is applicable also to this case, and we obtain the desired result from (iii) and (iv) of Theorem 17.7, except that we have to multiply the functions by the product of certain values of L-functions, and hence a power of π appears as stated. If $\mu < \lambda$, we put $\nu = 2\lambda - \mu$ and $p = (k - \nu\mathbf{a})/2$. Then $\nu > \lambda$, $0 \le p \in \mathbf{Z}^{\mathbf{a}}$, and

$$\Delta_{\nu\mathbf{a}}^p D(z, \mu/2; \nu\mathbf{a}, \chi, \mathfrak{c}) = (i/2)^{n|p|}c_{\nu\mathbf{a}}^p(\mu/2)D(z, \mu/2; k, \chi, \mathfrak{c}). \tag{17.22}$$

Observing that $c_{\nu\mathbf{a}}^p(\mu/2) \neq 0$, we obtain our assertion of (ii) from (v) of Theorem 17.7 and (17.19). Assertion (iii) follows from (vi) of Theorem 17.7 in the same manner.

17.10. In our later investigations we need the series $E_{\mathbf{A}}$ of (16.27) in Case UT. We put $E^*(x, s) = E_{\mathbf{A}}(x\eta_{\mathbf{h}}^{-1}, s)$. Given $r \in G_{\mathbf{h}}$ in Case UT, we define functions $E_r(z, s)$ and $E_r^*(z, s)$ of $(z, s) \in \mathcal{H} \times \mathbf{C}$ by

$$E_r\big(x(\mathbf{i}), s\big) = E_r\big(x(\mathbf{i}), s; k, \chi, \mathfrak{c}\big) = j_x^k(\mathbf{i})E_{\mathbf{A}}(rx, s), \tag{17.23a}$$

$$E_r^*\big(x(\mathbf{i}), s\big) = E_r^*\big(x(\mathbf{i}), s; k, \chi, \mathfrak{c}\big) = j_x^k(\mathbf{i})E_{\mathbf{A}}^*(rx, s) \tag{17.23b}$$

for $x \in G_{\mathbf{a}}$. Then we put

$$D_r(z, s; k, \chi, \mathfrak{c}) = E_r(z, s; k, \chi, \mathfrak{c})\prod_{i=0}^{n-1} L_{\mathfrak{c}}(2s - i, \chi_1\theta^i), \tag{17.24}$$

where χ_1 is the restriction of χ to $F_{\mathbf{A}}^\times$. Here recall that in Case UT, χ is a Hecke character of K.

17.11. Lemma. *Let χ be a Hecke character of K satisfying (16.24a) with $\ell \in \mathbf{Z}^{\mathbf{a}}$ and $\kappa = 0$. Then $\chi(c)$ for every $c \in K_{\mathbf{h}}^{\times}$ and $\chi^*(\mathfrak{a})$ for every $\mathfrak{r}$-ideal $\mathfrak{a}$ are algebraic.*

PROOF. Since $\chi^*(\mathfrak{a})$ is 0 or $\chi(c)$ for some $c \in K_{\mathbf{h}}^{\times}$, it is sufficient to treat $\chi(c)$. Given c, we can find $\alpha \in K^{\times}$ and a positive integer m such that $c^m\mathfrak{r} = \alpha\mathfrak{r}$. Then $c^m/\alpha = ef$ with $e \in \prod_{v\in\mathbf{h}} \mathfrak{r}_v^{\times}$ and $f \in K_{\mathbf{a}}^{\times}$. We can find an integral $\mathfrak{g}$-ideal $\mathfrak{c}$ with which (16.24b) is satisfied. We can also find a positive integer ν such that $e_v^{\nu} - 1 \in \mathfrak{r}_v\mathfrak{c}_v$ for every $v|\mathfrak{c}$. Then $\chi(e^{\nu}) = 1$, and hence $\chi(e)$ is a root of unity. Now $f = \alpha_{\mathbf{a}}^{-1}$, so that $\chi(f) = |\alpha|^{\ell}\alpha^{-\ell} \in \overline{\mathbf{Q}}$. Since $\chi(\alpha) = 1$, we have $\chi(c)^m = \chi(e)\chi(f) \in \overline{\mathbf{Q}}$, which proves our lemma.

It should be noted that given $\ell \in \mathbf{Z}^{\mathbf{a}}$, a Hecke character χ as in the above lemma always exists; see [S97, Lemma 11.14 (3)].

17.12. Theorem (Case UT). *Let $k \in \mathbf{Z}^{\mathbf{b}}$ and $\mu \in \mathbf{Z}$; put $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$. Let χ be a Hecke character of K satisfying (16.24a, b) with $\kappa = 0$, and χ_1 the restriction of χ to $F_{\mathbf{A}}^{\times}$. Further let K' be the reflex field (in the sense of §9.4) of the CM-type $(K, \{\tau_v\})$ we fixed in §3.5, and K_{χ} the field generated over K' by the values $\chi(c)$ for all $c \in K_{\mathbf{h}}^{\times}$. (Then $K_{\chi} \subset \overline{\mathbf{Q}}$ by Lemma 17.11.) Then the following assertions hold:*

(i) *Suppose $m_v = \mu \geq n$ for every $v \in \mathbf{a}$; then $E_r(z, \mu/2; k, \chi, \mathfrak{c})$ belongs to $\mathcal{M}_k(K_{\chi}\mathbf{Q}_{\mathrm{ab}})$ except when $\mu = n + 1$, $F = \mathbf{Q}$, and $\chi_1 = \theta^{n+1}$.*

(ii) *Let k and μ be as in (i). If $\mu = n + 1$, $F = \mathbf{Q}$, and $\chi_1 = \theta^{n+1}$, then $E_r(z, \mu/2; k, \chi, \mathfrak{c})$ belongs to $\mathcal{N}_k^n(K_{\chi}\mathbf{Q}_{\mathrm{ab}})$.*

(iii) *Let k and μ be as in (i); put $\nu = 2n - \mu$ and $e = n(n + 1)/2$. Then $D_r(z, \nu/2; k, \chi, \mathfrak{c})$ belongs to $\pi^{de}\mathcal{M}_k(K_{\chi}\mathbf{Q}_{\mathrm{ab}})$ except when $0 \leq \nu < n$, $\mathfrak{c} = \mathfrak{g}$, and $\chi_1 = \theta^{\mu}$.*

(iv) *Let Φ be the Galois closure of K over $\mathbf{Q}$. Suppose $n \leq \mu \leq m_v$ and $\mu - m_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Then $E_r(z, \mu/2; k, \chi, \mathfrak{c})$ belongs to $\pi^{\alpha}\mathcal{N}_k^t(\Phi K_{\chi}\mathbf{Q}_{\mathrm{ab}})$, where $\alpha = (n/2)\sum_{v\in\mathbf{a}}(m_v - \mu)$ and*

$$t = \begin{cases} n(m - \mu + 2)/2 & \text{if } \ \mu = n + 1, \ F = \mathbf{Q}, \ \text{ and } \ \chi_1 = \theta^{n+1}, \\ n(m - \mu\mathbf{a})/2 & \text{otherwise.} \end{cases}$$

(v) *Suppose $2n - m_v \leq \mu \leq m_v$ and $m_v - \mu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Then $D_r(z, \mu/2; k, \chi, \mathfrak{c})$ belongs to $\pi^{\beta}\mathcal{N}_k^t(\Phi K_{\chi}\mathbf{Q}_{\mathrm{ab}})$, except when $0 \leq \mu < n$, $\mathfrak{c} = \mathfrak{g}$, and $\chi_1 = \theta^{\mu}$, where $\beta = (n/2)\sum_{v\in\mathbf{a}}(m_v + \mu) - dn(n-1)/2$ and*

$$t = \begin{cases} n(m - \mu + 2)/2 & \text{if } \ \mu = n + 1, \ F = \mathbf{Q}, \ \text{ and } \ \chi_1 = \theta^{n+1}, \\ (n/2)(m - |\mu - n|\mathbf{a} - n\mathbf{a}) & \text{otherwise.} \end{cases}$$

REMARK. (1) If $k_{v\rho} = k_v$ for every $v \in \mathbf{a}$, then $\chi_{\mathbf{a}} = 1$, so that χ is of finite order. Therefore $K_{\chi} \subset \mathbf{Q}_{\mathrm{ab}}$ in that case.

(2) We can define $E(z, s; k, \Gamma)$ for $k \in \mathbf{Z}^{\mathbf{a}}$ and a congruence subgroup Γ of $U(\eta_n)$ in Case UT by (17.2), (17.3), and (17.3a). Then (17.5) holds for $\Gamma_1 \subset \Gamma \subset U(\eta_n)$. Now we can take Γ_1 in $SU(\eta_n)$. Therefore the nature of $E(z, \mu/2; k, \Gamma)$ for $\Gamma \subset U(\eta_n)$ can be reduced to the case $\Gamma \subset SU(\eta_n)$, which is given in Theorems 17.7 and 17.9.

PROOF. We first note that $E_{\mathbf{A}}^*$ has a Fourier expansion of the form

$$(17.25)\qquad E_{\mathbf{A}}^{*}\left(\begin{bmatrix} q & \sigma\widehat{q} \\ 0 & \widehat{q} \end{bmatrix}\right)=\sum_{h\in S} c(h,\, q,\, s)\mathbf{e}_{\mathbf{A}}^{n}(h\sigma) \qquad (q\in GL_n(K)_{\mathbf{A}},\, \sigma\in S_{\mathbf{A}})$$

with $c(h,\, q,\, s) \in \mathbf{C}$. We assume $\mathfrak{c} \neq \mathfrak{g}$ and take k and μ as in (i). Given $r \in G_{\mathbf{h}}$, by [S97, Lemma 9.8 (3)] we can put $r\eta_{\mathbf{h}} = \alpha^{-1}tw$ with $\alpha \in G$, $w \in D[\mathfrak{b}\mathfrak{c},\, \mathfrak{b}^{-1}]$, and $t = \mathrm{diag}[q_1,\, \widehat{q}_1]$, $q_1 \in GL_n(K)_{\mathbf{h}}$. Then, by [S97, Lemma 18.7 (1), (2)] we have

$$(17.26\mathrm{a})\qquad E_r(z,\, s) = \chi_{\mathfrak{c}}\big(\det(a_w)\big)^{-1} j_\alpha^k(z)^{-1} E_t^*(\alpha z,\, s),$$

$$(17.26\mathrm{b})\qquad E_t^*(z,\, s) = \sum_{h\in S} \det(y)^{-\mu\mathbf{a}/2} c(h,\, q,\, s)\mathbf{e}_{\mathbf{a}}^{n}(hx),$$

where $q_{\mathbf{h}} = q_1$ and $q_{\mathbf{a}} = y^{1/2}$. Now $c(h,\, q,\, s)$ is not much different from $c_h(y,\, s)$ of (17.9) in Case SU. Indeed, using the symbols of [S97, Proposition 18.14], put

$$X = c(S)N(\mathfrak{b}\mathfrak{c})^{-n^2}|\det(y)|^{s\mathbf{a}}\varXi\big(y,\, h;\, (s+\mu/2)\mathbf{a},\, (s-\mu/2)\mathbf{a}\big).$$

Then that proposition says that $c(h,\, q,\, s) \neq 0$ only if $(q^*hq)_v \in \big(\mathfrak{b}\mathfrak{c}\mathfrak{d}\big)_v^{-1}\widetilde{S}_v$ for every $v \in \mathbf{h}$, in which case

$$c(h,\, q,\, s) = \chi\big(\det(q_1)\big)|\det(q_1)|_K^{n-s}\alpha_{\mathfrak{c}}^0\big(\omega q^*hq,\, 2s,\, \chi_1\big)X,$$

where ω is an element of $F_{\mathbf{h}}^{\times}$ such that $\omega\mathfrak{g} = \mathfrak{b}\mathfrak{d}$, and $c_h(y,\, s) = \det(y)^{-\mu\mathbf{a}/2}$ $\cdot c(h,\, y^{1/2},\, s)$, where $c(h,\, y^{1/2},\, s)$ is a special case of $c(h,\, q,\, s)$ with $q_1 = 1$. Notice that the quantity X stays the same. Now $\alpha_{\mathfrak{c}}^0\big(\omega q^*hq,\, 2s,\, \chi_1\big)$ is given in [S97, Proposition 19.2], and its nature does not depend on q. Therefore the analysis of $c(h,\, q,\, s)$ at $s = \mu/2$ is practically the same as that of $c_h(y,\, s)$ in Case SU; the only essential difference is the additional factor $\chi\big(\det(q_1)\big)|\det(q_1)|_K^{n-\mu/2}$. Combining this with (17.26a, b), we see that

$$\chi\big(\det(q_1^{-1})\det(a_w)_{\mathfrak{c}}\big)|\det(q_1)|_K^{\mu/2-n}E_r(z,\, \mu/2)$$

is of the same nature as $E(z,\, \mu/2;\mu\mathbf{a},\, \chi_1,\, \mathfrak{c})$ in Case SU as stated in Theorem 17.7, (iii), (iv), except that we have to consider $\|_k\alpha$ with α in G, not necessarily in G_1. Therefore by Theorem 7.11 we obtain (i), (ii), and (iii) when $\mathfrak{c} \neq \mathfrak{g}$. To deal with the case $\mathfrak{c} = \mathfrak{g}$, we use the equality

$$E_r(z,\, s;\, k,\, \chi,\, \mathfrak{g}) = \textstyle\sum_b E_{rb}(z,\, s;\, k,\, \chi,\, \mathfrak{p})$$

of [S97, (19.6.1)], where $\mathfrak{p}$ is a prime ideal and b runs over a finite subset of $G_{\mathbf{h}}$. Thus the results for $\mathfrak{c} = \mathfrak{g}$ follow from those for $\mathfrak{c} \neq \mathfrak{g}$.

Given k and μ as in (iv), put $p = (m-\mu\mathbf{a})/2$ and define $k' \in \mathbf{Z}^{\mathbf{b}}$ by $k'_{v\rho} = k_{v\rho} - p_v$ and $k'_v = k_v - p_v$ for $v \in \mathbf{a}$; define also $\varDelta_{\mu\mathbf{a}}^p$ as in the proof of Theorem 17.9. Then clearly $k'_{v\rho} + k'_v = \mu$ for every $v \in \mathbf{a}$ and moreover

$$(17.27)\qquad \varDelta_{\mu\mathbf{a}}^p E_r(z,\, \mu/2;\, k',\, \chi,\, \mathfrak{c}) = (i/2)^{n|p|}c_{\mu\mathbf{a}}^p(\mu/2)E_r(z,\, \mu/2;\, k,\, \chi,\, \mathfrak{c}).$$

To show this, we first observe that $\varDelta_{\mu\mathbf{a}}^p(f\|_{k'}\alpha) = (\varDelta_{\mu\mathbf{a}}^p f)\|_k\alpha$ for every $\alpha \in G$. Indeed, if $\alpha \in (G_1)_{\mathbf{a}}$, this is a special case of (12.21), since $\|_k\alpha = \|_m\alpha$ and $\|_{k'}\alpha = \|_{\mu\mathbf{a}}\alpha$. If $\alpha \in G_{\mathbf{a}}$, then $\alpha = c\beta$ with $\beta \in (G_1)_{\mathbf{a}}$ and $c \in \mathbf{C}^{\mathbf{a}}$ such that $|c_v| = 1$ for every $v \in \mathbf{a}$, and the equality for α can be reduced to that for β (see also §14.4). Now, given r, by [S97, Lemma 9.8 (3)] we can put $r = \alpha^{-1}fu$ with $\alpha \in G$, $f = \mathrm{diag}[\widehat{g},\, g]$, $g \in GL_n(K)_{\mathbf{h}}$, and $u \in D[\mathfrak{b}\mathfrak{c},\, \mathfrak{b}^{-1}]$. Then $\alpha_{\mathbf{a}} = u_{\mathbf{a}}$. For $z = x(\mathbf{i})$ with $x \in G_{\mathbf{a}}$ we have

$$\begin{aligned} E_r(z,\, s)j_x^k(\mathbf{i})^{-1} &= E_{\mathbf{A}}(rx,\, s) = E_{\mathbf{A}}(fux,\, s) = \chi_{\mathfrak{c}}\big(\det(a_u)\big)^{-1}E_{\mathbf{A}}(fu_{\mathbf{a}}x,\, s) \\ &= \chi_{\mathfrak{c}}\big(\det(a_u)\big)^{-1}E_f(\alpha x(\mathbf{i}),\, s)j_{\alpha x}^k(\mathbf{i})^{-1} = \chi_{\mathfrak{c}}\big(\det(a_u)\big)^{-1}E_f(\alpha z,\, s)j_\alpha^k(z)^{-1}j_x^k(\mathbf{i})^{-1}. \end{aligned}$$

Thus we obtain

$$E_r(z,\, s) = \chi_{\mathfrak{c}}\big(\det(a_u)\big)^{-1} E_f(\alpha z,\, s) j_\alpha^k(z)^{-1}. \tag{17.28a}$$

In [S97, Lemma 18.7 (3)] we showed that

$$E_f(z,\, s) = \chi\big(\det(g)^{-1}\big)|\det(g)|_K^{-s} \sum_{\alpha\in A'} N\big(\mathfrak{a}_f(\alpha)\big)^s \chi[\alpha]_f \delta(z)^{su-m/2}\|_k\, \alpha, \tag{17.28b}$$

with $A' = P\backslash\big(G\cap P_{\mathbf{A}} f D[\mathfrak{b}^{-1},\, \mathfrak{bc}]f^{-1}\big)$ and $\mathfrak{a}_f(\alpha)$, $\chi[\alpha]_f$ as described there. These symbols are independent of k. Therefore term-wise application of the differential operator shows that (17.27) is true for E_f in place of E_r for the same reason as in (17.20). Then from (17.28a) we obtain (17.27). Now we have

$$\Delta_{\mu\mathbf{a}}^p \mathcal{N}_{k'}^t(W) \subset \pi^{n|p|}\mathcal{N}_k^{t+np}(W). \tag{17.29}$$

for a subfield W of $\mathbf{C}$ containing Φ. Therefore we obtain (iv) from (i), (ii), and (17.27), since $c_{\mu\mathbf{a}}^p(\mu/2) \neq 0$ as observed in the proof of Theorem 17.9. Finally, given μ as in (v), we can repeat the above proof with D_r in place of E_r if $\mu \geq n$. If $\mu < n$, we put $\nu = 2n-\mu$, $p = (m-\nu\mathbf{a})/2$, and $k'' = (k_{v\rho}-p_v,\, k_v-p_v)_{v\in\mathbf{a}}$. Then $k_v''+k_{v\rho}'' = \nu$ for every $v\in\mathbf{a}$, and

$$\Delta_{\nu\mathbf{a}}^p D_r(z,\, \mu/2;\, k'',\, \chi,\, \mathfrak{c}) = (i/2)^{n|p|} c_{\nu\mathbf{a}}^p(\mu/2) D_r(z,\, \mu/2;\, k,\, \chi,\, \mathfrak{c}). \tag{17.30}$$

Thus we obtain (v) for the same reason as above.

17.13. Lemma. *Define $E_r(z,\, s)$ by (17.23a) for $r\in G_{\mathbf{h}}$ in Cases SP and UT with $k\in\mathbf{Z}^{\mathbf{b}}$ and $\kappa = 0$. Then there is a finite sum expression*

$$E_r(z,\, s) = \sum_{i=1}^m b_i c_i^s E(z,\, s;\, k, \Gamma_i)\|_k \alpha_i \tag{17.31}$$

with $b_i\in\overline{\mathbf{Q}}$, $0 < c_i \in \mathbf{R}\cap\overline{\mathbf{Q}}$, congruence subgroups Γ_i of G, and $\alpha_i\in G$. Moreover, in Case UT we can take Γ_i in $SU(\eta_n)$.

PROOF. In Case UT we have (17.28a, b) with $f = \mathrm{diag}[\hat{g},\, g]$. By [S97, Lemma 18.7 (4)], A' of (17.28b) can be given by $\bigsqcup_{\beta\in B} S_\beta\beta$ with a finite subset B of G and $S_\beta = (P\cap\beta\Gamma\beta^{-1})\backslash\beta\Gamma\beta^{-1}$, where $\Gamma = G\cap fD[\mathfrak{b}^{-1},\, \mathfrak{bc}]f^{-1}$. For $\gamma\in S_\beta$ we have $\mathfrak{a}_f(\gamma\beta) = \mathfrak{a}_f(\beta)$ and $\chi[\gamma\beta]_f = c_\beta\chi_{\mathfrak{c}}\big(\det(d_\gamma)\big)$ with a constant c_β independent of γ. Therefore, taking a suitable congruence subgroup of $\beta\Gamma\beta^{-1}$, we obtain expression (17.31) for E_f, which combined with (17.28a) proves (17.31) in the general case. Now (17.5) is valid in Case UT, and in that formula we can take Γ_1 in $SU(\eta_n)$. This proves our lemma in Case UT. Case SP is simpler, since we have $G_{\mathbf{A}} = GD[\mathfrak{b}^{-1},\, \mathfrak{b}]$, so that (17.28a) holds with $f = 1$, and $E_1(z,\, s)$ is given by (16.39).

18. Eisenstein series in the Hilbert modular case

18.1. In this section we consider Eisenstein series when the group G is $SL_2(F)$, and as applications we treat critical values of certain L-functions of a totally imaginary quadratic extension of F. Though our results on Eisenstein series are essentially special cases of what was obtained in Sections 16 and 17, we present them in a somewhat different fashion. We start with some elementary facts on L-functions of F stated in the form suitable for our later applications. Throughout this section we denote by $\mathfrak{d}$, D_F, and R_F the different of F relative to $\mathbf{Q}$, the discriminant of F, and the regulator of F. We put $[F:\mathbf{Q}] = e$ and $N(x) = N_{F/\mathbf{Q}}(x)$ for $x\in F$.

Then $N(x\mathfrak{g}) = |N(x)| = |x|^{\mathbf{a}}$ for $x \in F^\times$. For $a_0 \in F$ and a $\mathfrak{g}$-ideal $\mathfrak{a}$ we put $a_0 + \mathfrak{a} = \{ a_0 + a \,|\, a \in \mathfrak{a} \}$.

Given $\kappa \in \mathcal{S}(F_{\mathbf{h}})$ and $k \in \mathbf{Z}^{\mathbf{a}}$, we can find a subgroup U of $\mathfrak{g}^\times$ of finite index such that $\kappa(ud) = \kappa(d)$ and $u^{-k}|u|^k = 1$ for every $u \in U$ and every $d \in F$. As explained in §1.6, we view κ as a function on F. With such a U we put

$$D_k(s,\,\kappa) = [\mathfrak{g}^\times : U]^{-1} \sum_{d \in F^\times/U} \kappa(d) d^{-k} |d|^{k - s\mathbf{a}} \qquad (s \in \mathbf{C}^\times). \tag{18.1}$$

Clearly this does not depend on the choice of U, and is convergent for $\mathrm{Re}(s) > 1$.

18.2. Lemma. *Let $t_v = 0$ or 1 according as k_v is even or odd. Put $R_t(s,\,\kappa) = \Gamma_t(s) D_k(s,\,\kappa)$ with $\Gamma_t(s) = \prod_{v \in \mathbf{a}} \pi^{-(s+t_v)/2} \Gamma\big((s+t_v)/2\big)$. Then $R_t(s,\,\kappa)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$, and satisfies $R_t(1-s,\,\kappa) = R_t(s,\,\kappa_*)$ with the element κ_* of $\mathcal{S}(F_{\mathbf{h}})$ given by*

$$\kappa_*(x) = i^{-|t|} \int_{F_{\mathbf{h}}} \kappa(y) \mathbf{e}_{\mathbf{h}}(-xy)\,dy,$$

where $|t| = \sum_{v \in \mathbf{a}} t_v$ and dy is the Haar measure of $F_{\mathbf{h}}$ such that $\prod_{v \in \mathbf{h}} \mathfrak{g}_v$ has measure $D_F^{-1/2}$. Moreover, $R_t(s,\,\kappa)$ is entire except when $t = 0$, in which case $R_t(s,\,\kappa)$ is holomorphic on $\mathbf{C}$ except for possible simple poles at $s = 0$ and $s = 1$ with residues $-2^{e-1}\kappa(0) R_F$ and $2^{e-1}\kappa_*(0) R_F$, respectively.

Though this can be proved by the standard method by taking a suitable zeta integral on $F_{\mathbf{A}}^\times$, we shall derive it from a more general result concerning the Mellin transform of a Hilbert modular form in §A7.3.

Take κ to be the characteristic function of the set $a_0 + \mathfrak{a}$ as above, for example. Then $\kappa_*(0) = i^{-|t|} D_F^{-1/2} N(\mathfrak{a})^{-1}$, and so if $t = 0$, (that is, if $k \in 2\mathbf{Z}^{\mathbf{a}}$,) then $D_k(s,\,\kappa)$ has a simple pole at $s = 1$ with residue $2^{e-1} D_F^{-1/2} N(\mathfrak{a})^{-1} R_F$.

18.3. We put $P = \{ \alpha \in G \,|\, c_\alpha = 0 \}$ and $\mathcal{H} = \mathfrak{H}_1^{\mathbf{a}}$ as before. In addition, we put $H = F_2^1 - \{0\}$ and

$$j_h(z) = (c_v z_v + d_v)_{v \in \mathbf{a}} \quad \text{for} \quad h = (c,\, d) \in H \quad \text{and} \quad z \in \mathcal{H}. \tag{18.2}$$

We easily see that $j_{h\gamma}(z) = j_h(\gamma z) j_\gamma(z)$ for $h \in H$ and $\gamma \in G$. Given $k \in \mathbf{Z}^{\mathbf{a}}$ and $\lambda \in \mathcal{S}\big((F_2^1)_{\mathbf{h}}\big)$, we can find a subgroup U of $\mathfrak{g}^\times$ of finite index such that $u^k|u|^{-k} = 1$ and $\lambda(uh) = \lambda(h)$ for every $u \in U$ and every $h \in F_2^1$. Then we put, for $(z,\, s) \in \mathcal{H} \times \mathbf{C}$,

$$\begin{aligned} E_k(z,\, s;\, \lambda) &= [\mathfrak{g}^\times : U]^{-1} \sum_{h \in H/U} \lambda(h) j_h(z)^{-k} y^{s\mathbf{a} - k/2} |j_h(z)|^{k - 2s\mathbf{a}} \\ &= [\mathfrak{g}^\times : U]^{-1} \sum_{(c,d) \in H/U} \lambda(c,\, d)(cz+d)^{-k} y^{s\mathbf{a} - k/2} |cz+d|^{k - 2s\mathbf{a}}, \end{aligned} \tag{18.3}$$

where of course $y = \mathrm{Im}(z)$. The sum is formally well-defined, independently of the choice of U. Defining λ^γ for $\gamma \in G$ by $\lambda^\gamma(h) = \lambda(h\gamma^{-1})$, we can easily verify that .

$$E_k(z,\, s;\, \lambda^\gamma) = E_k(\gamma z,\, s;\, \lambda) j_\gamma(z)^{-k} \quad \text{for every } \gamma \in G. \tag{18.4}$$

If we put $\Gamma = \{ \gamma \in G \,|\, \lambda^\gamma = \lambda \}$, then clearly Γ is a congruence subgroup of G, and $E_k(z,\, s;\, \lambda)\|_k \gamma = E_k(z,\, s;\, \lambda)$ for every $\gamma \in \Gamma$.

Let us now show that $E_k(z,\, s;\, \lambda)$ is a finite "linear combination" of transforms of functions of type (17.3). We first put $x_0 = [0 \;\; 1]$ and observe that $x_0 \alpha = [c_\alpha \;\; d_\alpha]$

for $\alpha \in G$, $P = \{\alpha \in G \mid x_0\alpha \in Fx_0\}$, and the map $\alpha \mapsto x_0\alpha$ gives a bijection of $P\backslash G$ onto $F^\times\backslash H$. Then we easily see that the map $(d, \alpha) \mapsto dx_0\alpha$ for $(d, \alpha) \in F^\times \times G$ gives a bijection of $(F^\times/U) \times (P\backslash G)$ onto H/U.

Given $\lambda \in \mathcal{S}((F_2^1)_{\mathbf{h}})$, take U and Γ as above, and take also a complete set of representatives B for $P\backslash G/\Gamma$, which is finite by [S97, Lemma 9.8 (3)]. Then $P\backslash G$ can be given by $\bigsqcup_{\beta\in B}\mathcal{R}_\beta\beta$ with $\mathcal{R}_\beta = (P \cap \beta\Gamma\beta^{-1})\backslash\beta\Gamma\beta^{-1}$. Write an element h of H/U in the form $h = dx_0\gamma\beta$ with $d \in F^\times/U$, $\beta \in B$, and $\gamma \in \mathcal{R}_\beta$. Since $\lambda(dx_0\gamma\beta) = \lambda(dx_0\beta\beta^{-1}\gamma\beta) = \lambda(dx_0\beta)$, we have

$$(18.5)\quad E_k(z, s; \lambda) = [\mathfrak{g}^\times : U]^{-1}\sum_{\beta\in B}\sum_{d\in F^\times/U}\lambda(dx_0\beta)d^{-k}|d|^{k-2s\mathbf{a}}\sum_{\gamma\in\mathcal{R}_\beta} y^{s\mathbf{a}-k/2}\|_k\gamma\beta$$
$$= \sum_{\beta\in B} D_k(2s, \kappa_\beta)E(z, s; k, \beta\Gamma\beta^{-1})\|_k\beta,$$

where κ_β is defined by $\kappa_\beta(d) = \lambda(dx_0\beta)$ for $d \in F$ and $E(z, s; k, \beta\Gamma\beta^{-1})$ by (17.3a). Since (17.3) is convergent for $\mathrm{Re}(s) > 1$ and $D_k(2s, \kappa_\beta)$ is holomorphic for $\mathrm{Re}(s) > 1/2$, we see that $E_k(z, s; \lambda)$ is meaningful as a holomorphic function of s at least for $\mathrm{Re}(s) > 1$. Also, from Lemmas 17.2 and 18.2 we see that $E_k(z, s; \lambda)$ can be continued as a meromorphic function of s to the whole $\mathbf{C}$. We can actually derive a stronger result from the explicit Fourier expansion of $E_k(z, s; \lambda)$, which is our next problem. For that purpose we need

18.4. Lemma. *For a $\mathfrak{g}$-ideal $\mathfrak{m}$ we have*

$$D_F^{1/2}N(\mathfrak{m})\sum_{a\in\mathfrak{m}}(z + a)^{-\alpha}(\overline{z} + a)^{-\beta} = \sum_{b\in(\mathfrak{d}\mathfrak{m})^{-1}}\mathbf{e}_{\mathbf{a}}(bx)\Xi(y, b; \alpha, \beta).$$

Here $z = x + iy \in \mathcal{H}$, $\alpha \in \mathbf{C}^{\mathbf{a}}$, $\beta \in \mathbf{C}^{\mathbf{a}}$, $\mathrm{Re}(\alpha_v + \beta_v) > 1$ for every $v \in \mathbf{a}$, and $\Xi(y, b; \alpha, \beta) = \prod_{v\in\mathbf{a}}\xi(y_v, b_v; \alpha_v, \beta_v)$ with

$$\xi(g, h; s, s') = \int_{\mathbf{R}}\mathbf{e}(-ht)(t + ig)^{-s}(t - ig)^{-s'}\,dt \quad (s, s' \in \mathbf{C};\ 0 < g \in \mathbf{R},\ h \in \mathbf{R}).$$

PROOF. Put $f(x) = (x + iy)^{-\alpha}(x - iy)^{-\beta}$ for $x \in \mathbf{R}^{\mathbf{a}}$ with a fixed $y \in \mathbf{R}^{\mathbf{a}}$, $\gg 0$, and let $\hat{f}(x)$ be its Fourier transform. (That f is L^1 will be shown later.) Then $\hat{f}(x) = \Xi(y, x; \alpha, \beta)$. By the Poisson summation formula we obtain $\mathrm{vol}(F_{\mathbf{a}}/\mathfrak{m}) \cdot \sum_{a\in\mathfrak{m}} f(x + a) = \sum_{b\in(\mathfrak{d}\mathfrak{m})^{-1}}\mathbf{e}_{\mathbf{a}}(bx)\hat{f}(b)$, which gives the desired equality, and which holds if the left-hand side is convergent and defines a C^∞ function of x. To see the last point, we take an easy equality

$$|x + iy|^{-2r} = \prod_{v\in\mathbf{a}}\Gamma(r_v)^{-1}\int_0^\infty e^{-t|z_v|^2}t^{r_v-1}dt,$$

valid for $z = x + iy \in \mathbf{C}^{\mathbf{a}}$ and $r \in \mathbf{R}^{\mathbf{a}}$, $\gg 0$. Then we can easily verify that

$$\int_{\mathbf{R}^{\mathbf{a}}}|x + iy|^{-2r}dx = \pi^{e/2}y^{\mathbf{a}-2r}\prod_{v\in\mathbf{a}}\Gamma(r_v - 1/2)\Gamma(r_v)^{-1}$$

if $2r - \mathbf{a} \gg 0$. Let T be a compact subset of $\mathcal{H}$. By a well-known principle, there exists an open subset U of $\mathcal{H}$ containing T, whose closure is compact and contained in $\mathcal{H}$, and a constant C_1 such that

$$(*)\qquad |h(w)| \le C_1\int_U |h(z)|dv(z)$$

for every $w \in T$ and every holomorphic function h on $\mathcal{H}$, where $dv(z)$ is the Lebesgue measure on $\mathbf{C}^{\mathbf{a}}$. Take a compact subset X of $R^{\mathbf{a}}$ and a compact subset Y

of $\{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}$ so that $U \subset X + iY$. Let $M = \#\{a \in \mathfrak{m} \mid (a + X) \cap X \neq \varnothing\}$. Then, for any finite subset A of $\mathfrak{m}$, we have

$$\int_Y \int_X \sum_{a \in A} |x + iy + a|^{-2r} dx dy \leq M \int_Y \int_{\mathbf{R}^{\mathbf{a}}} |x + iy|^{-2r} dx dy \leq M C_2 \int_Y y^{\mathbf{a}-2r} dy \leq M C_3$$

if r belongs to a compact subset Z of $\{r \in \mathbf{R}^{\mathbf{a}} \mid 2r - \mathbf{a} \gg 0\}$, where C_2 and C_3 are constants depending only on Y and Z. By (*) we have $\sum_{a \in A} |z + a|^{-2r} \leq C_4$ for every $z \in T$ and every $r \in Z$ with a constant C_4 depending only on X, Y, and Z. Now $|w^{-\alpha}\overline{w}^{-\beta}| \leq C_5 |w|^{-\mathrm{Re}(\alpha+\beta)}$ for every $w \in \mathcal{H}$ if (α, β) belongs to a compact subset S of $\mathbf{C}^{\mathbf{a}} \times \mathbf{C}^{\mathbf{a}}$, where C_5 is a constant depending only on S. Thus f belongs to $L^1(\mathbf{R}^{\mathbf{a}})$ and $\sum_{a \in \mathfrak{m}} f(x + a)$ is locally uniformly convergent for (x, a, β) if $\mathrm{Re}(\alpha + \beta) \gg \mathbf{a}$. Since $(\partial/\partial x_v)(z^{-\alpha}\overline{z}^{-\beta}) = -\alpha_v z^{-\alpha-v}\overline{z}^{-\beta} - \beta_v z^{-\alpha}\overline{z}^{-\beta-v}$, we see that $\sum_{a \in \mathfrak{m}} (\partial/\partial x_v) f(x + a)$ is locally uniformly convergent if $\mathrm{Re}(\alpha + \beta) \gg \mathbf{a}$. The same is true for derivatives of any order. This proves that $\sum_{a \in \mathfrak{m}} f(x + a)$ converges to a C^∞ function if $\mathrm{Re}(\alpha + \beta) \gg \mathbf{a}$, which completes the proof.

Notice that the above ξ is exactly the function of (16.44) with $n = 1$. We insert here an easy fact:

18.5. Lemma. *Let $f(z) = \sum_{h \in F} b(h)\mathbf{e}_{\mathbf{a}}(hz)$ and $g(z) = \sum_{h \in F} c(h)\mathbf{e}_{\mathbf{a}}(hz)$ be elements of $\mathcal{M}_{\mu\mathbf{a}}$ with $0 < \mu \in 2^{-1}\mathbf{Z}$, and let $\sigma \in \mathrm{Aut}(\mathbf{C})$. If $b(h)^\sigma = c(h)$ for every $h \in F, \neq 0$, then $b(0)^\sigma = c(0)$. Consequently $b(0)$ is contained in the field generated over $\mathbf{Q}$ by the $b(h)$ for all $h \neq 0$.*

PROOF. By Theorem 9.9 (4) or Theorem 10.7 (5), $f^\sigma \in \mathcal{M}_{\mu\mathbf{a}}$, and hence $b(0)^\sigma - c(0) = f^\sigma - g \in \mathcal{M}_{\mu\mathbf{a}}$. Since $\mu > 0$, we have $b(0)^\sigma - c(0) = 0$, which proves our lemma.

18.6. We now take λ in (18.3) to be the characteristic function of $(a_0 + \mathfrak{a}) \times (b_0 + \mathfrak{b})$ with $a_0, b_0 \in F$ and $\mathfrak{g}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$. Then we can take

$$U = \{u \in \mathfrak{g}^\times \mid u \gg 0,\ (u - 1)a_0 \in \mathfrak{a},\ (u - 1)b_0 \in \mathfrak{b}\}.$$

Notice that $\mathcal{S}((F_2^1)_{\mathbf{h}})$ is spanned by such λ's. Let $A = [(a_0 + \mathfrak{a}) \cap F^\times]/U$ and $B = [(b_0 + \mathfrak{b}) \cap F^\times]/U$; also let κ_A resp. κ_B be the characteristic function of $a_0 + \mathfrak{a}$ resp. $b_0 + \mathfrak{b}$. Then

$$[\mathfrak{g}^\times : U] y^{k/2 - s\mathbf{a}} E_k(z, s; \lambda) = \varepsilon(a_0, \mathfrak{a}) \sum_{d \in B} d^{-k} |d|^{k - 2s\mathbf{a}}$$
$$+ \sum_{c \in A} \sum_{d \in b_0 + \mathfrak{b}} (cz + d)^{-k} |cz + d|^{k - 2s\mathbf{a}},$$

where $\varepsilon(a_0, \mathfrak{a})$ is 1 or 0 according as $a_0 \in \mathfrak{a}$ or $a_0 \notin \mathfrak{a}$. For a fixed $c \in A$ we have

$$\sum_{d \in b_0 + \mathfrak{b}} (cz+d)^{-k} |cz+d|^{k-2s\mathbf{a}} = c^{-k}|c|^{k-2s\mathbf{a}} \sum_{a \in c^{-1}\mathfrak{b}} (z + c^{-1}b_0 + a)^{-k} |z + c^{-1}b_0 + a|^{k-2s\mathbf{a}}.$$

By Lemma 18.4 this equals

$$c^{-k}|c|^{k-2s\mathbf{a}} D_F^{-1/2} N(c\mathfrak{b}^{-1}) \sum_{b \in c\mathfrak{b}^{-1}\mathfrak{d}^{-1}} \mathbf{e}_{\mathbf{a}}(bx + bc^{-1}b_0) \Xi(y, b;\ s\mathbf{a} + k/2,\ s\mathbf{a} - k/2).$$

Thus the Fourier expansion of E_k can be given by

$$(18.6) \qquad E_k(z, s; \lambda) = \varepsilon(a_0, \mathfrak{a}) y^{s\mathbf{a} - k/2} D_k(2s, \kappa_B)$$
$$+ D_F^{-1/2} N(\mathfrak{b})^{-1} y^{s\mathbf{a} - k/2} \Xi(y, 0;\ s\mathbf{a} + k/2,\ s\mathbf{a} - k/2) D_k(2s - 1, \kappa_A)$$

$$+[\mathfrak{g}^\times : U]^{-1} D_F^{-1/2} N(\mathfrak{b})^{-1} y^{s\mathbf{a}-k/2} \sum_{b\in F^\times} \mathbf{e}_{\mathbf{a}}(bx)\Xi(y,\, b;\, s\mathbf{a}+k/2,\, s\mathbf{a}-k/2)\varphi(b,\, s)$$

$$\text{with}\quad \varphi(b,\, s) = \textstyle\sum_{a,\, c} \mathbf{e}_{\mathbf{a}}(ab_0)c^{-k}|c|^{k+(1-2s)\mathbf{a}},$$

where the last sum is taken over all $(a,\, c) \in \mathfrak{b}^{-1}\mathfrak{d}^{-1} \times A$ such that $ac = b$. Notice that it is a finite sum.

18.7. Proposition. *The product* $E_k(z,\, s;\, \lambda)\prod_{v\in\mathbf{a}} \Gamma\big(s + |k_v|/2\big)$ *can be continued as a meromorphic function of* s *to the whole* $\mathbf{C}$, *which is holomorphic except for possible simple poles at* $s = 0$ *and* $s = 1$. *The pole at* $s = 1$ *occurs if and only if* $k = 0$, *and the residue is* $2^{e-2}\pi^e N(\mathfrak{a}\mathfrak{b}\mathfrak{d})^{-1}R_F$; *the pole at* $s = 0$ *occurs if and only if* $k=0$, $a_0 \in \mathfrak{a}$, *and* $b_0 \in \mathfrak{b}$, *and the residue is* $-2^{e-2}R_F$.

PROOF. Put $\Delta(s) = \prod_{v\in\mathbf{a}} \Gamma\big(s+|k_v|/2\big)$ and $E_k(z,\, s;\, \lambda) = \sum_{b\in F} c_b(y,\, s)\mathbf{e}_{\mathbf{a}}(bx)$. We already know meromorphic continuation of $E_k(z,\, s;\, \lambda)$ to the whole $\mathbf{C}$ (which can actually be derived from the above Fourier expansion). Thus our task is to study $\Delta(s)c_b(y,\, s)$ for each b. From (17.11) and the above formula for $\varphi(b,\, s)$, we see that $\Delta(s)c_b(y,\, s)$ is holomorphic everywhere if $b \neq 0$. Now, from (17.13) we obtain

$$\xi(g,\, 0;\, s + k_v/2;\, s - k_v/2) = i^{-k_v} 2^{2-2s}\pi g^{1-2s} \frac{\Gamma(2s-1)}{\Gamma(s+k_v/2)\Gamma(s-k_v/2)}. \tag{18.7}$$

For $b = 0$, $\Delta(s)c_0(y,\, s)$ consists of two terms: one involving $\Delta(s)D_k(2s,\, \kappa_B)$ and the other involving $\Delta(s)D_k(2s-1,\, \kappa_A)$. From Lemma 18.2 we easily see that the former has possible simple poles at $s=0$ and $s=1/2$; similarly, by (18.7), the latter has possible simple poles at $s=1/2$ and $s=1$. Therefore $\Delta(s)E_k(z,\, s;\, \lambda)$ may have poles only at these points, and the residue at $s=0$, $1/2$, or 1 is a constant times $y^{-k/2}$, $y^{(\mathbf{a}-k)/2}$, or $y^{-k/2}$, respectively. However, by (18.4), these must be invariant under the action of $\|_k\gamma$ for every $\gamma \in \Gamma$, which is possible only when $s \neq 1/2$ and $k=0$, and the residue is a constant. Therefore $\Delta(s)E_k(z,\, s;\, \lambda)$ is entire if $k \neq 0$. If $k=0$, it has possible simple poles at $s = 0$ and $s = 1$. The residue at $s = 0$ is that of $\varepsilon(a_0,\, \mathfrak{a})R_0(2s,\, \kappa_B)$ at $s = 0$. By Lemma 18.2 this is nonzero if and only if $a_0 \in \mathfrak{a}$ and $b_0 \in \mathfrak{b}$. Similarly, by (18.7) and Lemma 18.2, the residue at $s=1$ is a nonzero constant times the residue of $R_0(2s-1,\, \kappa_A)$ at $s=1$, which is nonzero. A simple calculation gives each residue as stated in our proposition.

18.8. We now assume that $k = \mu\mathbf{a}$ with $0 < \mu \in \mathbf{Z}$, and take the value of E_k at $s = \mu/2$. From (17.12) and (18.7) we obtain

$$\xi(g,\, h;\, \mu,\, 0) = \begin{cases} (-2\pi i)^\mu \Gamma(\mu)^{-1} h^{\mu-1}\mathbf{e}(igh) & \text{if } h > 0, \\ 0 & \text{if } h < 0, \end{cases}$$

$$\lim_{\sigma\to 0} \xi(g,\, 0;\, \sigma+\mu,\, \sigma) = \begin{cases} -\pi i & \text{if } \mu = 1, \\ 0 & \text{if } \mu > 1, \end{cases}$$

$$\lim_{\sigma\to 0} \xi(g,\, 0;\, \sigma+2,\, \sigma)/\sigma = -\pi g^{-1}.$$

Putting $k = \mu\mathbf{a}$ and $s = \mu/2$ in (18.6), we have

$$\begin{aligned} &D_F^{1/2} N(\mathfrak{b})(-2\pi i)^{-\mu e}\Gamma(\mu)^e E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda) \\ &\qquad = [\mathfrak{g}^\times : U]^{-1} \sum_{0\ll b\in F} \varphi(b)\mathbf{e}_{\mathbf{a}}(bz) + \mathcal{D}_1(\mu) + \mathcal{D}_2(0) + C \end{aligned} \tag{18.8}$$

$$\text{with}\quad \varphi(b) = \sum_{0\neq a\in\mathfrak{b}^{-1}\mathfrak{d}^{-1},\, a^{-1}b\in A} N(a)^\mu |N(a)|^{-1}\mathbf{e}_{\mathbf{a}}(ab_0),$$

$$\mathcal{D}_1(s) = \varepsilon(a_0,\, \mathfrak{a})D_F^{1/2}N(\mathfrak{b})(-2\pi i)^{-\mu e}\Gamma(\mu)^e D_{\mu\mathbf{a}}(s,\, \kappa_B),$$
$$\mathcal{D}_2(s) = \begin{cases} 2^{-e}D_{\mathbf{a}}(s,\, \kappa_A) & \text{if } \mu = 1, \\ 0 & \text{if } \mu > 1, \end{cases}$$
$$C = \begin{cases} (8\pi y)^{-1}N(\mathfrak{a})^{-1} & \text{if } F = \mathbf{Q} \text{ and } \mu = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\Xi(y,\, 0;\, (s+2)\mathbf{a},\, s\mathbf{a})$ has a zero of order e at $s = 0$, which is why $C = 0$ if $F \neq \mathbf{Q}$.

Next let us consider $E_{\mu\mathbf{a}}(z,\, \nu/2;\, \lambda)$ for $\nu = 2 - \mu$. We note that

$$\xi(g,\, h;\, 1,\, 1-\mu) = \begin{cases} (-2i)^\mu \pi g^{\mu-1}\mathbf{e}(igh) & \text{if } h > 0, \\ (-2i)^\mu 2^{-1}\pi g^{\mu-1} & \text{if } h = 0, \\ 0 & \text{if } h < 0. \end{cases}$$

These can be obtained from (17.11) and the formula $\omega(g,\, h; 1,\, \beta) = 2^{-1}\mathbf{e}(igh)$ given in [S82, (4.35K)]. If $\mu = 1$, then $\nu = 1$ and $E_{\mu\mathbf{a}}(z,\, \nu/2;\, \lambda) = E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda)$, which is already given. Therefore we assume $\mu > 1$, so that $\nu \leq 0$. Then

$$\begin{aligned} (18.9)\qquad & D_F^{1/2}N(\mathfrak{b})(-2i)^{-\mu e}\pi^{-e}E_{\mu\mathbf{a}}(z,\, \nu/2;\, \lambda) \\ &= [\mathfrak{g}^\times : U]^{-1} \sum_{0 \ll b \in F} \varphi'(b)\mathbf{e}_{\mathbf{a}}(bz) + 2^{-e}D_{\mu\mathbf{a}}(\nu - 1,\, \kappa_A) + C' \end{aligned}$$

$$\text{with } \varphi'(b) = \sum_{c \in A,\, c^{-1}b \in \mathfrak{b}^{-1}\mathfrak{d}^{-1}} |N(c)|^{-1}N(c)^\mu \mathbf{e}_{\mathbf{a}}(c^{-1}bb_0),$$

$$C' = \begin{cases} \varepsilon(a_0,\, \mathfrak{a})\varepsilon(b_0,\, \mathfrak{b})N(\mathfrak{b})(4\pi y)^{-1} & \text{if } F = \mathbf{Q} \text{ and } \mu = 2, \\ 0 & \text{otherwise.} \end{cases}$$

18.9. Theorem. *Let λ be a $\mathbf{Q}_{\rm ab}$-valued element of $\mathcal{S}\big((F_2^1)_{\mathbf{h}}\big)$ and let $0 < \mu \in \mathbf{Z}$. Then the following assertions hold:*

(1) $E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda)$ belongs to $\pi^{\mu e}\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\rm ab})$ except when $F = \mathbf{Q}$ and $\mu = 2$, in which case if belongs to $\pi^2\mathcal{N}_2^1(\mathbf{Q}_{\rm ab})$.

(2) $E_{\mu\mathbf{a}}(z,\, 1-\mu/2;\, \lambda)$ belongs to $\pi^e\mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\rm ab})$ except when $F = \mathbf{Q}$ and $\mu = 2$, in which case it belongs to $\pi\mathcal{N}_2^1(\mathbf{Q}_{\rm ab})$.

PROOF. We may assume that λ is the characteristic function of $(a_0 + \mathfrak{a}) \times (b_0 + \mathfrak{b})$. From (18.4) and (18.8) we see that $E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda)$ belongs to $\mathcal{M}_{\mu\mathbf{a}}$ except when $F = \mathbf{Q}$ and $\mu = 2$. Since $\varphi(b) \in \mathbf{Q}_{\rm ab}$, Lemma 18.5 shows that $\pi^{-\mu e}E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda) \in \mathcal{M}_{\mu\mathbf{a}}(\mathbf{Q}_{\rm ab})$. Suppose $F = \mathbf{Q}$ and $\mu = 2$. Then (18.4) and (18.8) show that $E_{2\mathbf{a}}(z,\, 1;\, \lambda)$ satisfies condition (13.18a) with $p = 1$. Thus $E_{2\mathbf{a}}(z,\, 1;\, \lambda) \in \mathcal{N}_2^1$. Now $\varphi(b) \in \mathbf{Q}_{\rm ab}$, and from that fact we can conclude that $E_{2\mathbf{a}}(z,\, 1;\, \lambda) \in \pi^2\mathcal{N}_2^1(\mathbf{Q}_{\rm ab})$ for the reason which will be explained in the proof of the following proposition. This proves (1). Assertion (2) can be proved in the same manner.

REMARK. In (18.3) take $F = \mathbf{Q}, U = \{\pm 1\}$, $k = 2$, and take λ to be the characteristic function of $\prod_p \mathbf{Z}_p$, which means that $\mathfrak{a} = \mathfrak{b} = \mathbf{Z}$ and $a_0 = b_0 = 0$. Then $D_{2\mathbf{a}}(s,\, \kappa_A) = \zeta(s)$, and (18.8) with $\mu = 2$ shows that

$$(2\pi i)^{-2}E_2(z,\, 1/2;\, \lambda) = (8\pi y)^{-1} - (1/24) + \textstyle\sum_{b=1}^\infty \mathbf{e}_{\mathbf{a}}(bz)\sum_{0<a|b} a,$$

which is exactly twice formula (0.7) of the introduction. This type of nearly holomorphic Eisenstein series occurs in Cases SP and UT if $F = \mathbf{Q}$, as shown in Theorem

17.7 (iv). For further investigations of the series of this nature, see [S83, §9] and [S85a, pp.290-291].

18.10. Proposition. *For $0 < \mu \in \mathbf{Z}$, $b_0 \in F$, and a $\mathfrak{g}$-ideal $\mathfrak{b}$ put $D_\mu(s;\, b_0,\, \mathfrak{b}) = D_{\mu\mathbf{a}}(s,\, \kappa)$, where κ is the characteristic function of $b_0 + \mathfrak{b}$; put also $Q(\mu;\, b_0,\, \mathfrak{b}) = (2\pi i)^{-\mu e} D_F^{1/2} D_\mu(\mu;\, b_0,\, \mathfrak{b})$. Then the following assertions hold:*

(1) $Q(\mu;\, b_0,\, \mathfrak{b}) \in \mathbf{Q}_{\mathrm{ab}}$. Moreover, let $\sigma = [t,\, \mathbf{Q}]$ with $t \in \mathbf{Z}_{\mathbf{h}}^\times$ (see §8.1). Then $Q(\mu;\, b_0,\, \mathfrak{b})^\sigma = Q(\mu;\, b_1,\, \mathfrak{b})$ with an element $b_1 \in F$ such that $(tb_1 - b_0)_v \in \mathfrak{b}_v$ for every $v \in \mathbf{h}$.

(2) $D_\mu(1-\mu;\, b_0,\, \mathfrak{b}) \in \mathbf{Q}$.

PROOF. Take $E_{\mu\mathbf{a}}(z,\, \mu/2;\, \lambda)$ as in §18.8 with $a_0 = 0$ and $\mathfrak{a} = \mathfrak{g}$. Then $\varphi(b) \in \mathbf{Q}_{\mathrm{ab}}$ and for σ and b_1 as above, $\varphi(b)^\sigma$ is the quantity obtained from the formula for $\varphi(b)$ with b_0 replaced by b_1. Suppose $\mu > 1$; exclude the case in which $F = \mathbf{Q}$ and $\mu = 2$. Then $\mathcal{D}_2 = C = 0$. Since $\mathcal{D}_1(\mu) = N(\mathfrak{b})(-1)^{\mu e}\Gamma(\mu)^e Q(\mu;\, b_0,\, \mathfrak{b})$, Lemma 18.5 shows tht $Q(\mu;\, b_0,\, \mathfrak{b}) \in \mathbf{Q}_{\mathrm{ab}}$ and $Q(\mu;\, b_0,\, \mathfrak{b})^\sigma = Q(\mu;\, b_1,\, \mathfrak{b})$. If $F = \mathbf{Q}$ and $\mu = 2$, then $E_{2\mathbf{a}}(z,\, 1;\, \lambda) \in \mathcal{N}_2^1$, and so we need an analogue of Lemma 18.5 for the elements of $\mathcal{N}_2^1$. For $f(z) = c(\pi y)^{-1} + \sum_{h\in\mathbf{Q}} a(h)\mathbf{e}(hz) \in \mathcal{N}_2^1$ and $\tau \in \mathrm{Aut}(\mathbf{C})$ we have $f^\tau(z) = c^\tau(\pi y)^{-1} + \sum_{h\in\mathbf{Q}} a(h)^\tau \mathbf{e}(hz) \in \mathcal{N}_2^1$ (see §14.15). Now $\mathcal{N}_2^1$ contains no element, other than 0, of the form $p(\pi y)^{-1} + q$ with $p,\, q \in \mathbf{C}$, as can easily be verified. Therefore, modifying the proof of Lemma 18.5, we obtain (1) for $\mu > 1$ even when $F = \mathbf{Q}$ and $\mu = 2$. Then, from (18.8) we see that $E_{2\mathbf{a}}(z,\, 1;\, \lambda)$ belongs to $\pi^2\mathcal{N}_2^1(\mathbf{Q}_{\mathrm{ab}})$ in that case. To prove (2) for $\mu > 1$, take $E_{\mu\mathbf{a}}(z,\, \nu/2;\, \lambda)$ with $b_0 = 0$. Then $\varphi'(b) \in \mathbf{Q}$, and hence $D_{\mu\mathbf{a}}(\nu - 1;\, \kappa_A) \in \mathbf{Q}$ by Lemma 18.5 (with the modification when $F = \mathbf{Q}$ and $\mu = 2$, mentioned above). This proves (2) for $\mu > 1$.

Suppose $\mu = 1$. Take $a_0 \in \mathfrak{a}$ and put $\mathfrak{a}^{-1}\mathfrak{d}^{-1} = \mathfrak{c}$; denote by κ' the characteristic function of $\mathfrak{c}$; define κ_* as in Lemma 18.2 with κ_A as κ. Then $\kappa_* = i^{-e}N(\mathfrak{a})^{-1}D_F^{-1/2}\kappa'$. Now the above reasoning applied to the case $\mu = 1$ with $a_0 \in \mathfrak{a}$ shows that

$$D_F^{1/2}N(\mathfrak{b})(-2\pi i)^{-e}D_{\mathbf{a}}(1,\, \kappa_B) + 2^{-e}D_{\mathbf{a}}(0,\, \kappa_A) \in \mathbf{Q}_{\mathrm{ab}}.$$

In particular, this quantity belongs to $\mathbf{Q}$ if $b_0 \in \mathfrak{b}$. From the functional equation of Lemma 18.2 we obtain $D_{\mathbf{a}}(0,\, \kappa_A) = \pi^{-e}D_{\mathbf{a}}(1,\, \kappa_*) = (\pi i)^{-e}D_F^{1/2}N(\mathfrak{c})D_{\mathbf{a}}(1,\, \kappa')$. Take $b_0 \in \mathfrak{b}$ and $\mathfrak{c} = g\mathfrak{b}$ with $g \in F^\times$. Then $D_{\mathbf{a}}(1,\, \kappa') = g^{-\mathbf{a}}D_{\mathbf{a}}(1,\, \kappa_B)$, and so

$$D_F^{1/2}N(\mathfrak{b})(-2\pi i)^{-e}D_{\mathbf{a}}(1,\, \kappa_B)\{1 + (-g)^{-\mathbf{a}}|g|^{\mathbf{a}}\} \in \mathbf{Q}.$$

Choosing a suitable g, we find that $Q(1;\, b_0,\, \mathfrak{b}) \in \mathbf{Q}$ if $b_0 \in \mathfrak{b}$. Taking $\mathfrak{c}$ to be $\mathfrak{b}$, we find that $D_{\mathbf{a}}(0,\, \kappa_A) \in \mathbf{Q}$ if $a_0 \in \mathfrak{a}$. Then applying σ to the Fourier coefficients, we obtain $Q(1;\, b_0,\, \mathfrak{b})^\sigma = Q(1;\, b_1,\, \mathfrak{b})$ for the same reason as in the case $\mu > 1$. This proves (1) for $\mu = 1$. As for assertion (2) for $\mu = 1$, we have seen that $D_1(0;\, a_0,\, \mathfrak{a}) \in \mathbf{Q}$ if $a_0 \in \mathfrak{a}$. If $a_0 \notin \mathfrak{a}$, then taking $b_0 = 0$, we immediately see from Lemma 18.5 that $D_{\mathbf{a}}(0,\, \kappa_A) = 2^e\mathcal{D}_2(0) \in \mathbf{Q}$. This proves (2) for $\mu = 1$.

That $Q(\mu;\, b_0,\, \mathfrak{b}) \in \mathbf{Q}_{\mathrm{ab}}$ for $\mu > 1$ was proven by Klingen [K] in a somewhat different formulation.

18.11. Let ψ be a Hecke character of F, and $\mathfrak{f}$ the conductor of ψ. We understand that the symbol $\psi_{\mathbf{a}}(x)\psi^*(x\mathfrak{b})$ for $x = 0$ and a $\mathfrak{g}$-ideal $\mathfrak{b}$ denotes $\psi^*(\mathfrak{b})$ or 0 according as $\mathfrak{f} = \mathfrak{g}$ or $\mathfrak{f} \neq \mathfrak{g}$. Then, for any fixed $\mathfrak{g}$-ideal $\mathfrak{a}$, $x \mapsto \psi_{\mathbf{a}}(x)\psi^*(x\mathfrak{a}^{-1})$ defines a function of $x \in \mathfrak{a}/\mathfrak{a}\mathfrak{f}$. Indeed, we easily see that the property that $x\mathfrak{a}^{-1}$ is prime to $\mathfrak{f}$ depends only on $x + \mathfrak{a}\mathfrak{f}$. Take $a \in F_{\mathbf{h}}^\times$ so that $\mathfrak{a} = a\mathfrak{g}$. If $x \in \mathfrak{a}$ and $x\mathfrak{a}^{-1}$

is prime to $\mathfrak{f}$, then $\psi_{\mathbf{a}}(x)\psi^*(x\mathfrak{a}^{-1}) = \psi_{\mathbf{a}}(x)(\psi_{\mathbf{h}}/\psi_{\mathfrak{f}})(xa^{-1}) = \psi_{\mathfrak{f}}(ax^{-1})\psi(a^{-1})$, and clearly this depends only on $x + \mathfrak{a}\mathfrak{f}$.

We now define the Gauss sum $\mathbf{g}(\psi)$ of ψ by

$$\mathbf{g}(\psi) = \begin{cases} \psi^*(\mathfrak{d}) & \text{if } \mathfrak{f} = \mathfrak{g}, \\ \displaystyle\sum_{t\in(\mathfrak{f}\mathfrak{d})^{-1}/\mathfrak{d}^{-1}} \psi_{\mathbf{a}}(t)\psi^*(t\mathfrak{f}\mathfrak{d})\mathbf{e}_{\mathbf{a}}(t) & \text{if } \mathfrak{f} \neq \mathfrak{g}. \end{cases} \tag{18.10}$$

We note that $|\mathbf{g}(\psi)|^2 = N(\mathfrak{f})$ (see [S97, (A6.3.2)]).

18.12. Theorem. *Let $0 < \mu \in \mathbf{Z}$ and let ψ be a Hecke character of F such that $\psi_{\mathbf{a}}(x) = x^{\mu\mathbf{a}}|x|^{-\mu\mathbf{a}}$. For any $\mathfrak{g}$-ideal $\mathfrak{c}$ put*

$$P_{\mathfrak{c}}(\mu, \psi) = \mathbf{g}(\psi)^{-1}(2\pi i)^{-\mu e}D_F^{1/2}L_{\mathfrak{c}}(\mu, \psi).$$

Then the following assertions hold:

(1) *$L(1-\mu, \psi)$ is $P_{\mathfrak{g}}(\mu, \overline{\psi})$ times an element of $\mathbf{Q}^{\times}$.*

(2) *Both $L_{\mathfrak{c}}(1-\mu, \psi)$ and $P_{\mathfrak{c}}(\mu, \psi)$ belong to the field generated over $\mathbf{Q}$ by the values of ψ, (which is contained in $\mathbf{Q}_{\mathrm{ab}}$, since ψ is of finite order).*

(3) *For every $\sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$ we have $P_{\mathfrak{c}}(\mu, \psi)^{\sigma} = P_{\mathfrak{c}}(\mu, \psi^{\sigma})$ and $L_{\mathfrak{c}}(1-\mu, \psi)^{\sigma} = L_{\mathfrak{c}}(1-\mu, \psi^{\sigma})$.*

PROOF. Put $R(s, \psi) = N(\mathfrak{d}\mathfrak{f})^{s/2}\Gamma_{(t)}(s)L(s, \psi)$ with $\Gamma_{(t)}$ of Lemma 18.2, where $t = t_0\mathbf{a}$ with an integer t_0 such that $0 \le t_0 \le 1$ and $t_0 - \mu \in 2\mathbf{Z}$. Then $R(s, \psi) = W_{\psi}R(1-s, \overline{\psi})$ with $W_{\psi} = (-i)^{t_0 e}N(\mathfrak{f})^{-1/2}\mathbf{g}(\psi)$ (see [S97, Theorem A6.2, (A6.3.3)]). Therefore (1) can be verified by a straightforward calculation. Since $L_{\mathfrak{c}}(s, \psi) = L(s, \psi)\prod_{\mathfrak{p}|\mathfrak{c}}\left[1 - \psi^*(\mathfrak{p})N(\mathfrak{p})^{-s}\right]$, it is sufficient to prove (2) and (3) for $\mathfrak{c} = \mathfrak{g}$. Clearly (2) follows from (3). Thus, in view of (1), we only have to prove that $L(1-\mu, \psi)^{\sigma} = L(1-\mu, \psi^{\sigma})$ for every $\sigma \in \mathrm{Gal}(\mathbf{Q}_{\mathrm{ab}}/\mathbf{Q})$. Take a complete set of representatives $\{\mathfrak{a}_i\}_{i=1}^{n}$ for the ideal group of F modulo the subgroup $\{p\mathfrak{g} \mid p \in F,\ N(p) > 0\}$. Put $U_0 = \{u \in \mathfrak{g}^{\times} \mid N(u) > 0\}$ and $U = U_0 \cap (1+\mathfrak{f})$. Then the ideals $x\mathfrak{a}_{\nu}^{-1}$ for all ν and all $x \in F^{\times}/U_0$ cover the ideal group of F exactly twice, and hence

$$\begin{aligned} 2[U_0 : U]L(s, \psi) &= \sum_{\nu=1}^{m} \sum_{y\in(F^{\times}\cap\mathfrak{a}_{\nu})/U} \psi^*(y\mathfrak{a}_{\nu}^{-1})N(y\mathfrak{a}_{\nu}^{-1})^{-s} \\ &= \sum_{\nu=1}^{m} N(\mathfrak{a}_{\nu})^{s} \sum_{x\in\mathfrak{a}_{\nu}/\mathfrak{a}_{\nu}\mathfrak{f}} \psi^*(x\mathfrak{a}_{\nu}^{-1}) \sum_{d\in(x+\mathfrak{a}_{\nu}\mathfrak{f})/U} \psi^*(x^{-1}d\mathfrak{g})N(d\mathfrak{g})^{-s}. \end{aligned}$$

If $d \in x + \mathfrak{a}_{\nu}\mathfrak{f}$ and $\psi^*(x^{-1}d\mathfrak{g}) \neq 0$, then $\psi^*(x^{-1}d\mathfrak{g}) = \psi_{\mathbf{a}}(d^{-1}x)$, and hence

$$2[U_0 : U]L(s, \psi) = \sum_{\nu=1}^{m} N(\mathfrak{a}_{\nu})^{s} \sum_{x\in\mathfrak{a}_{\nu}/\mathfrak{a}_{\nu}\mathfrak{f}} \psi_{\mathbf{a}}(x)\psi^*(x\mathfrak{a}_{\nu}^{-1}) \sum_{d\in(x+\mathfrak{a}_{\nu}\mathfrak{f})/U} \psi_{\mathbf{a}}(d)N(d\mathfrak{g})^{-s}.$$

Thus

$$2L(1-\mu, \psi) = [\mathfrak{g}^{\times} : U_0]\sum_{\nu=1}^{m} N(\mathfrak{a}_{\nu})^{1-\mu} \sum_{x\in\mathfrak{a}_{\nu}/\mathfrak{a}_{\nu}\mathfrak{f}} \psi_{\mathbf{a}}(x)\psi^*(x\mathfrak{a}_{\nu}^{-1})D_{\mu}(1-\mu;\, x,\, \mathfrak{a}_{\nu}\mathfrak{f}).$$

Applying σ to this equality, we obtain $L(1-\mu, \psi)^{\sigma} = L(1-\mu, \psi^{\sigma})$ as expected.

18.13. For $v \in \mathbf{a}$, $0 < m \in \mathbf{Z}$, $k \in \mathbf{Z}^{\mathbf{a}}$, and $0 \le p \in \mathbf{Z}^{\mathbf{a}}$ we define differential operators δ_m^v and δ_k^p on $\mathcal{H}$ by

$$\delta_m^v f = y^{-m}\frac{\partial}{\partial z_v}(y_v^m f) = \left(\frac{m}{2iy_v} + \frac{\partial}{\partial z_v}\right)f, \tag{18.11}$$

$$\delta_k^p = \prod_{v\in\mathbf{a}} \delta_{k_v+2p_v-2}^v \cdots \delta_{k_v+2}^v\delta_{k_v}^v, \tag{18.12}$$

where f is a function on $\mathcal{H}$. As explained in §12.17, δ_k^p is a special case of D_ρ^p, and maps $\mathcal{N}_k^q$ into $\mathcal{N}_{k+2p}^{q+p}$ for $0 \le q \in \mathbf{Z}^\mathbf{a}$. Moreover, $(\pi i)^{-|p|}\delta_k^p$ maps $\mathcal{N}_k^q(W)$ into $\mathcal{N}_{k+2p}^{q+p}(W)$ for every subfield W of $\mathbf{C}$ containing the Galois closure of F over $\mathbf{Q}$. This is a special case of Theorem 14.12 (4), and in fact, can be verified by a direct calculation employing the expression (14.21) for the elements of $\mathcal{N}_k^q$. Now we have

$$\delta_k^p(f\|_k\,\alpha) = (\delta_k^p f)\|_{k+2p}\,\alpha \quad \textit{for every} \quad \alpha\in G, \tag{18.13}$$

$$\delta_k^p(y^r) = (2i)^{-|p|}y^{r-p}\prod_{v\in\mathbf{a}}\frac{\Gamma(k_v+r_v+p_v)}{\Gamma(k_v+r_v)} \qquad (r\in\mathbf{C}^\mathbf{a}). \tag{18.14}$$

The former is a special case of Proposition 12.10 (2) as noted in §12.17; the latter can be verified by a direct calculation. Next, for h in the set H of §18.3 we have

$$\begin{aligned}&\delta_k^p\{j_h(z)^{-k}|j_h(z)|^{k-2s\mathbf{a}}y^{s\mathbf{a}-k/2}\}\\ &\quad= (2i)^{-|p|}y^{s\mathbf{a}-\ell/2}j_h(z)^{-\ell}|j_h(z)|^{\ell-2s\mathbf{a}}\prod_{v\in\mathbf{a}}\frac{\Gamma(s+k_v/2+p_v)}{\Gamma(s+k_v/2)}\end{aligned} \tag{18.15}$$

with $\ell = k+2p$. Indeed, we can find $\alpha\in G$ such that $j_\alpha = j_h$. Then the left-hand side of (18.15) is $\delta_k^p(y^{s\mathbf{a}-k/2}\|_k\,\alpha)$, which equals $(\delta_k^p y^{s\mathbf{a}-k/2})\|_{k+2p}\,\alpha$ by (18.13). From this and (18.14) we obtain (18.15). Thus, if $\mathrm{Re}(s)$ is sufficiently large,

$$\delta_k^p E_k(z,\,s;\,\lambda) = (2i)^{-|p|}E_{k+2p}(z,\,s;\,\lambda)\prod_{v\in\mathbf{a}}\frac{\Gamma(s+k_v/2+p_v)}{\Gamma(s+k_v/2)}. \tag{18.16}$$

By analytic continuation, the equality holds for every s.

18.14. Theorem. *Let Φ be the Galois closure of F over $\mathbf{Q}$, and λ a $\mathbf{Q}_{\mathrm{ab}}$-valued element of $\mathcal{S}\big((F_2^1)_\mathbf{h}\big)$; let k be an element of $\mathbf{Z}^\mathbf{a}$ such that $k_v\ge 1$ for every $v\in\mathbf{a}$ and $k_v - k_{v'}\in 2\mathbf{Z}$ for every $v,\,v'\in\mathbf{a}$. Then, for every $\mu\in\mathbf{Z}$ such that $\mu-k_v\in 2\mathbf{Z}$ and $2-k_v\le\mu\le k_v$ for every $v\in\mathbf{a}$, the function $E_k(z,\,\mu/2;\,\lambda)$ belongs to $\pi^\alpha\mathcal{N}_k^t(\Phi\mathbf{Q}_{\mathrm{ab}})$, where $\alpha = (1/2)\sum_{v\in\mathbf{a}}(k_v+\mu)$, and $t = (k-|\mu-1|\mathbf{a}-\mathbf{a})/2$, except when $F=\mathbf{Q}$ and $|\mu-1|=1$, in which case $t=k/2$.*

PROOF. First suppose that $\mu>0$; put $p=(k-\mu\mathbf{a})/2$. Then $0\le p\in\mathbf{Z}^\mathbf{a}$, and (18.16) shows that

$$\delta_{\mu\mathbf{a}}^p E_{\mu\mathbf{a}}(z,\,\mu/2;\,\lambda) = (2i)^{-|p|}cE_k(z,\,\mu/2;\,\lambda)$$

with $c\in\mathbf{Q}^\times$. Therefore our assertion follows from Theorem 18.9 (1) combined with the fact that $\pi^{-|p|}\delta_{\mu\mathbf{a}}^p$ sends $\mathcal{N}_{\mu\mathbf{a}}^q(\Phi\mathbf{Q}_{\mathrm{ab}})$ into $\mathcal{N}_k^{q+p}(\Phi\mathbf{Q}_{\mathrm{ab}})$. Next, take an integer ν such that $2-k_v\le\nu\le 0$ and $\nu-k_v\in 2\mathbf{Z}$ for every $v\in\mathbf{a}$. Put $\mu=2-\nu$ and $p=(k-\mu\mathbf{a})/2$. Then $0\le p\in\mathbf{Z}^\mathbf{a}$, and (18.16) shows that

$$\delta_{\mu\mathbf{a}}^p E_{\mu\mathbf{a}}(z,\,\nu/2;\,\lambda) = (2i)^{-|p|}c'E_k(z,\,\nu/2;\,\lambda)$$

with $c'\in\mathbf{Q}^\times$. By virtue of Theorem 18.9 (2), we obtain the desired conclusion for the same reason as in the case $\mu>0$.

18.15. We now take a totally imaginary quadratic extension K of F, and denote by $\mathfrak{r}$ the maximal order of K. We consider the L-function $L(s,\,\chi)$ of a Hecke character χ of K such that $\chi_\mathbf{a}(x) = \prod_{\sigma\in J_K}\big(x^\sigma/|x^\sigma|\big)^{\mu_\sigma}$ with $\mu\in I_K$, where J_K

and I_K are as in §11.3. Then we can easily find a CM-type $\tau = \sum_{v\in\mathbf{a}} \tau_v$ of K such that

$$\chi_{\mathbf{a}}(x) = \prod_{v\in\mathbf{a}} \left(x^{\tau_v}/|x^{\tau_v}|\right)^{m_v} \tag{18.17}$$

with $0 \le m_v \in \mathbf{Z}$. Fixing a CM-type (K, τ) and $m \in \mathbf{Z}^{\mathbf{a}}$, ≥ 0, put

$$L_m(s, \ell) = [\mathfrak{r}^\times : U]^{-1} \sum_{\alpha\in K^\times/U} \ell(\alpha)\alpha^{-m}|\alpha|^{m-2s\mathbf{a}}. \tag{18.18}$$

Here $\ell \in \mathcal{S}(K_{\mathbf{h}})$, $\alpha^{-m}|\alpha|^{m-2s\mathbf{a}} = \prod_{v\in\mathbf{a}}(\alpha^{\tau_v})^{-m_v}|\alpha^{\tau_v}|^{m_v-2s}$, U is a subgroup of $\mathfrak{r}^\times$ of finite index such that $\ell(u\alpha) = \ell(\alpha)$ and $u^{-m}|u|^m = 1$ for every $u \in U$. Clearly such a U exists, and $L_m(s, \ell)$ does not depend on the choice of U; we easily see that the sum is convergent for $\mathrm{Re}(s) > 1$.

18.16. Theorem. *The notation being as above, suppose $m \ne 0$; then the following assertions hold:*

(1) *$L_m(s, \ell)\prod_{v\in\mathbf{a}} \Gamma(s + m_v/2)$ can be continued as an entire function of s to the whole $\mathbf{C}$.*

(2) *Suppose that ℓ is $\overline{\mathbf{Q}}$-valued and $m_v - m_{v'} \in 2\mathbf{Z}$ for every $v, v' \in \mathbf{a}$; let μ be an integer such that $2 - m_v \le \mu \le m_v$ and $\mu - m_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Then $L_m(\mu/2, \ell) \in \pi^\alpha p_K(\sum_{v\in\mathbf{a}} m_v\tau_v, \tau)\overline{\mathbf{Q}}$, where $\alpha = (1/2)\sum_{v\in\mathbf{a}}(m_v + \mu)$ and p_K is the symbol of §11.3.*

(3) *If χ is a Hecke character of type (18.17), then $L(\mu/2, \chi) \in \pi^\alpha p_K(\sum_{v\in\mathbf{a}} m_v\tau_v, \tau)\overline{\mathbf{Q}}$ for μ and α as in (2).*

PROOF. Assertion (1) will be proven in §A7.4. To prove (2), we first assume that $\mu \ge 1$. Take $w_0 \in K$ so that $\mathrm{Im}(w_0^{\tau_v}) > 0$ for every $v \in \mathbf{a}$, and put $w = (w_0^{\tau_v})_{v\in\mathbf{a}}$. By Proposition 4.14, w is a CM-point of $\mathcal{H}$. Define an F-linear bijection $g : F_2^1 \to K$ by $g(c, d) = cw_0 + d$, and put $\lambda = \ell \circ g$. Then $\lambda \in \mathcal{S}\big((F_2^1)_{\mathbf{h}}\big)$ and $j_h(w)^m = g(h)^m$ for $h \in F_2^1$, $\ne 0$, and hence

$$L_m(s, \ell) = [\mathfrak{r}^\times : U]^{-1} \sum_{h\in H/U} \lambda(h)j_h(w)^{-m}|j_h(w)|^{m-2s\mathbf{a}}$$
$$= [\mathfrak{r}^\times : \mathfrak{g}^\times]^{-1}y^{m/2-s\mathbf{a}}E_m(w, s; \lambda),$$

where $y = \mathrm{Im}(w)$. (Notice that we may take $U \subset \mathfrak{g}^\times$ in (18.18).) Given μ and α as in (2), we have $E_m(z, \mu/2; \lambda) \in \pi^\alpha\mathcal{N}_m^t(\overline{\mathbf{Q}})$ by Theorem 18.14, and hence its value at w belongs to $\pi^\alpha\mathfrak{P}_m(w)\overline{\mathbf{Q}}$. Since y is algebraic and $\mathfrak{P}_m(w) = p_K(\sum_{v\in\mathbf{a}} m_v\tau_v, \tau)$ by Proposition 11.18, we obtain (2). To prove (3), let $\mathfrak{h}$ be the conductor of χ, and A a complete set of representatives for the ideal classes of K modulo $\mathfrak{h}$, consisting of integral ideals prime to $\mathfrak{h}$. Then

$$L(s, \chi) = \sum_{\mathfrak{a}\in A} \chi^*(\mathfrak{a})^{-1}N(\mathfrak{a})^s \sum_{\alpha\in W_\mathfrak{a}/U_\mathfrak{h}} \chi^*(\alpha\mathfrak{r})N(\alpha\mathfrak{r})^{-s}$$

with $W_\mathfrak{a} = \mathfrak{a} \cap (1 + \mathfrak{h}) \cap K^\times$, $1 + \mathfrak{h} = \{1+x \mid x \in \mathfrak{h}\}$, and $U_\mathfrak{h} = \mathfrak{r}^\times \cap (1 + \mathfrak{h})$. Let $\ell_\mathfrak{a}$ be the characteristic function of $\mathfrak{a} \cap (1 + \mathfrak{h})$. Since $\chi^*(\alpha\mathfrak{r}) = \chi_{\mathbf{a}}(\alpha)^{-1} = \alpha^{-m}|\alpha|^m$ for $\alpha \in 1 + \mathfrak{h}$, we have $u^{-m}|u|^m = 1$ for $u \in U_\mathfrak{h}$, and hence

$$L(s, \chi) = [\mathfrak{r}^\times : U_\mathfrak{h}]\sum_{\mathfrak{a}\in A} \chi^*(\mathfrak{a})^{-1}N(\mathfrak{a})^sL_m(s, \ell_\mathfrak{a}).$$

Now $\chi^*(\mathfrak{a}) \in \overline{\mathbf{Q}}$ by Lemma 17.11. Therefore (3) follows from (2).

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CHAPTER V

ZETA FUNCTIONS ASSOCIATED WITH HECKE EIGENFORMS

19. Formal Euler products and generalized Möbius functions

19.1. Our setting is the same as in Section 16; thus we consider only Cases SP and UT. We do not consider $G_1 = SU(\eta_n)$ with $K \neq F$ in this section. Let us first make the following notational convention: For $v \in \mathbf{h}$ and a subgroup X of $G_{\mathbf{A}}$ (resp. $GL_n(K)_{\mathbf{A}}$) we put $X_v = X \cap G_v$ (resp. $X_v = X \cap GL_n(K_v)$). In fact, whenever this notation is used, X_v is the projection of X to G_v. We now take a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ in F, and consider the subgroup $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$ of $G_{\mathbf{A}}$ defined in §16. We also take a divisor $\mathfrak{e}$ of $\mathfrak{c}$ such that $\mathfrak{e}^{-1}\mathfrak{c} + \mathfrak{e} = \mathfrak{g}$, and throughout this section we denote by C the subgroup of $G_{\mathbf{A}}$ defined by

$$C = \left\{ x \in D[\mathfrak{b}^{-1}\mathfrak{e}, \mathfrak{bc}] \,\middle|\, a_x - 1 \prec \mathfrak{re} \right\}. \tag{19.1}$$

Recall the equality $G_{\mathbf{A}} = P_{\mathbf{A}} D[\mathfrak{b}^{-1}, \mathfrak{b}]$ as stated in (16.10). It may also be emphasized that $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{e}$ are in F.

To define Hecke operators, put

$$E = \textstyle\prod_{v \in \mathbf{h}} GL_n(\mathfrak{r}_v), \qquad B = \left\{ x \in GL_n(K)_{\mathbf{h}} \,\middle|\, x \prec \mathfrak{r} \right\}, \tag{19.2a}$$

$$B' = \left\{ x \in B \,\middle|\, x - 1 \prec \mathfrak{re} \right\}, \qquad E' = B' \cap E, \tag{19.2b}$$

$$\mathfrak{X} = CQ(\mathfrak{e})C, \qquad Q(\mathfrak{e}) = \left\{ \mathrm{diag}[\widehat{r}, r] \,\middle|\, r \in B' \right\}. \tag{19.2c}$$

By [S97, Proposition 5.10 or 7.8] (see also Remark 16.12 (III)) we have

$$G_{\mathbf{A}} = D[\mathfrak{b}^{-1}, \mathfrak{b}]Q(\mathfrak{g})D[\mathfrak{b}^{-1}, \mathfrak{b}]. \tag{19.3}$$

Thus $G_v = C_v Q(\mathfrak{e})_v C_v$ for every $v \nmid \mathfrak{c}$, and so $\mathfrak{X}_v = G_v$ for such a v; clearly $Q(\mathfrak{e})_v \subset C_v$ and $\mathfrak{X}_v = C_v$ for $v|\mathfrak{e}$. We are primarily interested in the cases $\mathfrak{e} = \mathfrak{g}$ and $\mathfrak{e} = \mathfrak{c}$; if $\mathfrak{e} = \mathfrak{c}$, the group is essentially a principal congruence subgroup; if $\mathfrak{e} = \mathfrak{g}$, it is an analogue of the group $\Gamma_0(N)$ in the elliptic modular case.

19.2. Lemma. (1) $\mathfrak{X} \subset CP_{\mathbf{h}}$.

(2) *Let* $J_v = C_v \cap P_v$. *If* $v|\mathfrak{c}$ *and* $\sigma \in Q(\mathfrak{e})_v$, *then* $C_v\sigma C_v \cap P_v = J_v \sigma J_v$ *and* $C_v \sigma C_v = C_v \sigma J_v$.

(3) *Let* $\sigma = \mathrm{diag}[\widehat{q}, q] \in Q(\mathfrak{e})_v$ *with* $v|\mathfrak{c}$, $v \nmid \mathfrak{e}$. *Then* $C_v\sigma C_v = \bigsqcup_{d,b} C_v \begin{bmatrix} \widehat{d} & \widehat{d}b \\ 0 & d \end{bmatrix}$ *with* $d \in E_v \backslash E_v q E_v$ *and* $b \in S(\mathfrak{b}^{-1})_v / d^* S(\mathfrak{b}^{-1})_v d$, *where* $S(\ )$ *is as in (16.1b).*

PROOF. To prove (1), it is sufficient to show that $\sigma C_v \subset C_v P_v$ for every $\sigma = \mathrm{diag}[\widehat{q}, q] \in Q(\mathfrak{e})_v$, $v \in \mathbf{h}$. Since $C_v P_v = G_v$ if $v \nmid \mathfrak{c}$ and $\mathfrak{X}_v = C_v$ if $v|\mathfrak{e}$, we may assume that $v|\mathfrak{c}$ and $v \nmid \mathfrak{e}$. Put $A_v = D[\mathfrak{b}^{-1}, \mathfrak{b}]_v$. Let $\alpha \in C_v$. Since $G_v = A_v P_v$, we have $\sigma\alpha = \beta^{-1}\pi$ with $\beta \in A_v$ and $\pi \in P_v$. Since $v|\mathfrak{c}$, both a_α and d_α belong to $GL_n(\mathfrak{r}_v)$. Also $0 = c_\pi = c_\beta\widehat{q}a_\alpha + d_\beta q c_\alpha$, we have $c_\beta = -d_\beta q c_\alpha a_\alpha^{-1}q^* \prec \mathfrak{r}_v\mathfrak{b}_v\mathfrak{c}_v$, and hence $\beta \in C_v$, which proves (1). To prove (2), we may again assume that $v \nmid \mathfrak{e}$, since $C_v\sigma C_v = C_v$ and $\sigma \in J_v$ if $v|\mathfrak{e}$. Given $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in C_v$, $v|\mathfrak{c}$, put $\beta = \begin{bmatrix} d^* & -b^* \\ 0 & d^{-1} \end{bmatrix}$. Then $\beta \in J_v$ and $\beta\alpha = \begin{bmatrix} 1 & 0 \\ s & 1 \end{bmatrix}$ with $s \in S_v$. Now let $\sigma = \mathrm{diag}[\widehat{q},\, q] \in Q(\mathfrak{e})_v$ and $\pi = \alpha_1\sigma\alpha_2^{-1} \in P_v$ with $\alpha_i \in C_v$. Applying the result just proved to α_i, we obtain elements $\beta_i \in J_v$ such that $\beta_i\alpha_i = \begin{bmatrix} 1 & 0 \\ s_i & 1 \end{bmatrix}$ with $s_i \in S_v$. Then $\beta_1\pi\beta_2^{-1} = \begin{bmatrix} \widehat{q} & 0 \\ e & q \end{bmatrix}$ with some $e \in (K_v)_n^n$. Since $\beta_1\pi\beta_2^{-1} \in P_v$, we see that $e=0$, which shows that $\pi \in J_v\sigma J_v$, and hence $C_v\sigma C_v \cap P_v = J_v\sigma J_v$. Next, take any $\gamma,\, \varepsilon \in C_v$. By (1), $\gamma\sigma\varepsilon = \xi\zeta$ with $\xi \in C_v$ and $\zeta \in P_v$. Then $\zeta \in C_v\sigma C_v \cap P_v = J_v\sigma J_v$, and hence $\gamma\sigma\varepsilon \in \xi J_v\sigma J_v \subset C_v\sigma J_v$, from which we obtain $C_v\sigma C_v = C_v\sigma J_v$. As for (3), since $C_v\sigma C_v = C_v\sigma J_v$, each coset of $C_v\backslash C_v\sigma C_v$ has a representative in σJ_v. Then our assertion can be verified in a straightforward way.

19.3. Lemma. *The set $\mathfrak{X}$ is closed under multiplication.*

PROOF. It is sufficient to prove that $\mathfrak{X}_v$ is closed under multiplication for every $v \in \mathbf{h}$; again we may assume that $v|\mathfrak{c}$ and $v \nmid \mathfrak{e}$. Let $C_v\sigma C_v = \bigsqcup_{d,\, b} C_v[[d,\, b]]$ with $\sigma \in Q(\mathfrak{e})_v$ as in Lemma 19.2 (3), where $[[d,\, b]] = \begin{bmatrix} \widehat{d} & \widehat{d}b \\ 0 & d \end{bmatrix}$; similarly let $C_v\tau C_v = \bigsqcup_{e,\, g} C_v[[e,\, g]]$ for $\tau \in Q(\mathfrak{e})_v$. Then $C_v\sigma C_v\tau C_v = \bigsqcup_{d,\, b,\, e,\, g} C_v[[d,\, b]][[e,\, g]] = \bigsqcup_{d,\, b,\, e,\, g} C_v\mathrm{diag}[\widehat{de},\, de][[1,\, x]]$, where $x = g + e^*be$. Since $[[1,\, x]] \in C_v$ and $\mathrm{diag}[\widehat{de},\, de] \in Q(\mathfrak{e})_v$, we obtain the desired result.

19.4. Lemma. *Employing the symbols of (16.1a, b, c), put $L_0 = \mathfrak{r}_1^n$ and*

$$W = \{\, (g,\, h) \in B' \times B' \mid gL_0 + hL_0 = L_0,\ h_v \in E'_v \text{ for every } v|\mathfrak{c} \,\},$$
$$S' = \{\, \sigma \in S_{\mathbf{h}} \mid \sigma_v \in S(\mathfrak{b}^{-1}\mathfrak{e})_v \text{ for every } v|\mathfrak{c} \,\}.$$

Then a complete set of representatives for $C\backslash\mathfrak{X}$ can be given by

$$\left\{ \begin{bmatrix} g^{-1}h & g^{-1}\sigma\widehat{h} \\ 0 & g^*\widehat{h} \end{bmatrix} \,\middle|\, (g,\, h) \in E'\backslash W/(E' \times 1),\ \sigma \in S'/gS_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})g^* \right\},$$

where we let $e \in E'$ and $(f,\, 1) \in E' \times 1$ act on W by $e(g,\, h)(f,\, 1) = (egf,\, eh)$.

PROOF. This is essentially the local problem of finding $C_v\backslash\mathfrak{X}_v$. This is trivial if $v|\mathfrak{e}$. If $v \nmid \mathfrak{c}$, then the question is about $C_v\backslash G_v$. Since $G_v = C_vP_v$, we can take representatives from $(C_v \cap P_v)\backslash P_v$. Now $(C_v \cap P_v)\backslash P_v$ can be given by $\begin{bmatrix} a & s\widehat{a} \\ 0 & \widehat{a} \end{bmatrix}$ with $a \in E_v\backslash GL_n(K_v)$ and $s \in S_v/S(\mathfrak{b}^{-1})_v$, as noted in [S97, p.131, last 4 lines]. Also in [S97, Lemma 16.4] we showed that the map $(g,\, h) \mapsto g^{-1}h$ gives a bijection of $E_v\backslash W_v$ onto $GL_r(K_v)$. Therefore the desired fact can easily be verified. If $v|\mathfrak{c}$ and $v \nmid \mathfrak{e}$, then by Lemma 19.2 (3), $C_v\backslash\mathfrak{X}_v$ can be given by $\begin{bmatrix} \widehat{d} & \widehat{d}b \\ 0 & d \end{bmatrix}$ with $d \in E_v\backslash B_v$ and $b \in S(\mathfrak{b}^{-1})_v/d^*S(\mathfrak{b}^{-1})_vd$. This time $W_v = B_v \times E_v$ and $E_v\backslash W_v/(E_v \times 1) = (B_v/E_v) \times 1$, and so the same conclusion holds in a much simpler way.

19.5. We shall state some of our results in terms of formal Dirichlet series of the form $\sum_{\mathfrak{a}\subset\mathfrak{r}} c(\mathfrak{a})[\mathfrak{a}]$. Here $c(\mathfrak{a}) \in \mathbf{C}$ and $\{[\mathfrak{a}]\}$ is a system of formal multiplicative symbols defined for the fractional ideals $\mathfrak{a}$ in K as follows: the $[\mathfrak{p}]$ for the prime ideals $\mathfrak{p}$ are independent indeterminates; $[\mathfrak{r}] = 1$ and $[\mathfrak{a}\mathfrak{b}] = [\mathfrak{a}][\mathfrak{b}]$. Substituting $\varphi(\mathfrak{a})N(\mathfrak{a})^{-s}$ with an ideal-character φ for $[\mathfrak{a}]$, we obtain an ordinary Dirichlet series. Hereafter we denote by $\sum_{\mathfrak{a}}$ the sum over all the integral ideals $\mathfrak{a}$ in K.

Given $\xi \in G_{\mathbf{A}}$, take $q \in B$ so that $\xi \in D[\mathfrak{b}^{-1}, \mathfrak{b}]\mathrm{diag}[q^{-1}, q^*]D[\mathfrak{b}^{-1}, \mathfrak{b}]$, which is feasible by virtue of (19.3). Then we put

$$\nu_{\mathfrak{b}}(\xi) = \det(q)\mathfrak{r}. \tag{19.4}$$

If $\beta = \mathrm{diag}[1_n, b_0^{-1}1_n]$ with an lement $b_0 \in F_{\mathbf{h}}^{\times}$ such that $\mathfrak{b} = b_0\mathfrak{g}$, then

$$\nu_{\mathfrak{b}}(\xi) = \nu_0(\beta\xi\beta^{-1}) \tag{19.5}$$

with ν_0 defined in §1.10. Let $\mathfrak{d}$, δ, and $\widetilde{S}$ be as in §16.1. Given $\zeta \in \widetilde{S}$, we define a formal Dirichlet series $\alpha_{\mathfrak{c}}^0(\zeta)$ by

$$\alpha_{\mathfrak{c}}^0(\zeta) = \prod_{v\nmid\mathfrak{c}} \alpha_v^0(\zeta), \quad \alpha_v^0(\zeta) = \sum_{\tau\in S_v/S(\mathfrak{r})_v} \mathbf{e}_v^n(-\delta_v^{-1}\zeta\tau)\bigl[\nu_0(\tau)\bigr]. \tag{19.6}$$

This can be obtained by substituting $\bigl(\tau,\ [\nu_0(\tau)]\bigr)$ for $\bigl(\sigma,\ \nu[\sigma]^{-s}\bigr)$ in (16.7a). Clearly

$$\alpha_{\mathfrak{c}}^0(c\gamma\zeta\gamma^*) = \alpha_{\mathfrak{c}}^0(\zeta) \quad \textit{if} \quad c \in \textstyle\prod_{v\in\mathbf{h}} \mathfrak{g}_v^{\times} \quad \textit{and} \quad \gamma \in E. \tag{19.7}$$

19.6. Lemma. *Let S' be as in Lemmas 19.4, and b_0 be as above. Let $\zeta \in S_{\mathbf{h}}$ and $g \in B$; suppose $g^*\zeta g \in \mathfrak{b}\mathfrak{e}^{-1}\widetilde{S}$. Then*

$$\sum_{\tau\in X} \mathbf{e}_{\mathbf{h}}^n(-\delta^{-1}\zeta\tau)[\nu_0(b_0\tau)] \neq 0 \quad \textit{for} \quad X = S'/gS_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})g^*$$

only if $\zeta \in \mathfrak{b}\mathfrak{e}^{-1}\widetilde{S}$, in which case the sum equals $|\det(g)|_K^{-\kappa}\alpha_{\mathfrak{c}}^0(b_0^{-1}\zeta)$, where $|x|_K$ denotes the idele norm of $x \in K_{\mathbf{A}}^{\times}$, and $\kappa = n+1$ in Case SP and $\kappa = n$ in Case UT.

PROOF. Recall that $\nu_0(\tau)$ depends only on τ mod $S_{\mathbf{h}}(\mathfrak{r})$. Change τ for $\tau + \gamma$ with $\gamma \in S_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})$. Then the sum in question is multiplied by the factor $\mathbf{e}_{\mathbf{h}}^n(-\delta^{-1}\zeta\gamma)$, which is nonzero for some such γ if $\zeta \notin \mathfrak{b}\mathfrak{e}^{-1}\widetilde{S}$. Thus the sum is nonzero only if $\zeta \in \mathfrak{b}\mathfrak{e}^{-1}\widetilde{S}$, in which case the sum is $\alpha_{\mathfrak{c}}^0(b_0^{-1}\zeta)$ times $[S_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e}) : gS_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})g^*]$. The last number is $|\det(g)|_K^{-\kappa}$ as noted in [S97, Lemma 13.2]. This proves our lemma.

19.7. We consider the $\mathbf{Q}$-algebra $\mathfrak{R}(C, \mathfrak{X})$ spanned by the $C\xi C$ for all $\xi \in \mathfrak{X}$ over $\mathbf{Q}$, with the law of multiplication defined as usual (see [S97, Section 11]). This is meaningful because of Lemma 19.3. Similarly we can consider $\mathfrak{R}(C_v, \mathfrak{X}_v)$ for each $v \in \mathbf{h}$. These algebras are commutative. Indeed, the commutativity of $\mathfrak{R}(C, \mathfrak{X})$ can be reduced to that of $\mathfrak{R}(C_v, \mathfrak{X}_v)$ as can easily be seen. If $v|\mathfrak{e}$, then $\mathfrak{R}(C_v, \mathfrak{X}_v)$ is just $\mathbf{Q}\cdot C_v 1 C_v$. The commutativity of $\mathfrak{R}(C_v, \mathfrak{X}_v)$ for $v\nmid\mathfrak{e}$ follows from the existence of an injection ω_v into a commutative ring as will be shown in Theorem 19.8 below.

We now define formal Dirichlet series $\mathfrak{T}$ and $\mathfrak{T}_v$ with coefficients in $\mathfrak{R}(C, \mathfrak{X})$ and $\mathfrak{R}(C_v, \mathfrak{X}_v)$ by

$$\mathfrak{T} = \sum_{\xi\in C\backslash\mathfrak{X}/C} C\xi C\bigl[\nu_{\mathfrak{b}}(\xi)\bigr], \qquad \mathfrak{T}_v = \sum_{\xi\in C_v\backslash\mathfrak{X}_v/C_v} C_v\xi C_v\bigl[\nu_{\mathfrak{b}}(\xi)\bigr]. \tag{19.8}$$

If $\xi = \mathrm{diag}[r^{-1}, r^*]$ with $r \in B'$, then looking at the elementary divisors of ξ and r, we easily see that $C\xi C$ determines $E'rE'$ and vice versa. Therefore we have

$$\mathfrak{T} = \sum_{r \in E' \backslash B'/E'} C\mathrm{diag}[r^{-1}, r^*]C\big[\det(r)\mathfrak{r}\big]. \tag{19.9}$$

For an integral $\mathfrak{r}$-ideal $\mathfrak{a}$ we denote by $T(\mathfrak{a})$ the sum of all the different $C\xi C$ with $\xi \in \mathfrak{X}$ such that $\nu_{\mathfrak{b}}(\xi) = \mathfrak{a}$. Then clearly

$$\mathfrak{T} = \prod_{v \in \mathbf{h}} \mathfrak{T}_v = \sum_{\mathfrak{a}} T(\mathfrak{a})[\mathfrak{a}]. \tag{19.10}$$

We have $\mathfrak{T}_v = 1$ if $v|\mathfrak{e}$, since $C_v = \mathfrak{X}_v$ for such a v.

19.8. Theorem. *Let $t_1, \ldots, t_m$ be m indeterminates, where $m = 2n$ if $K \neq F$ and v splits in K, and $m = n$ otherwise. Then for each $v \in \mathbf{h}$ prime to $\mathfrak{e}$ there exists a $\mathbf{Q}$-linear ring-injection*

$$\omega : \mathfrak{R}(C_v, \mathfrak{X}_v) \to \mathbf{Q}[t_1, \ldots, t_m, t_1^{-1}, \ldots, t_m^{-1}] \tag{19.11}$$

such that $\omega(\mathfrak{T}_v)$ has the following expressions:

(I) $v \nmid \mathfrak{c}$.

$$\omega(\mathfrak{T}_v) = \frac{1 - [\mathfrak{p}]}{1 - q^n[\mathfrak{p}]} \prod_{i=1}^{n} \frac{1 - q^{2i}[\mathfrak{p}^2]}{(1 - q^n t_i[\mathfrak{p}])(1 - q^n t_i^{-1}[\mathfrak{p}])} \qquad \text{(Case SP)},$$

$$\omega(\mathfrak{T}_v) = \frac{\prod_{i=1}^{2n}(1 - (-q)^{i-1}[\mathfrak{q}])}{\prod_{i=1}^{n}(1 - q^{2n-2}t_i[\mathfrak{q}])(1 - q^{2n}t_i^{-1}[\mathfrak{q}])} \qquad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}),$$

$$\omega(\mathfrak{T}_v) = \frac{\prod_{i=0}^{n-1}(1 - q^{2i}[\mathfrak{q}^2])}{\prod_{i=1}^{n}(1 - q^{n-1}t_i[\mathfrak{q}])(1 - q^{n}t_i^{-1}[\mathfrak{q}])} \qquad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}^2),$$

$$\omega(\mathfrak{T}_v) = \prod_{i=1}^{2n} \frac{1 - q^{i-1}[\mathfrak{q}_1\mathfrak{q}_2]}{(1 - q^{2n}t_i^{-1}[\mathfrak{q}_1])(1 - q^{-1}t_i[\mathfrak{q}_2])} \qquad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2).$$

(II) $v|\mathfrak{c}$.

$$\omega(\mathfrak{T}_v) = \prod_{i=1}^{n}(1 - q^n t_i[\mathfrak{p}])^{-1} \qquad \text{(Case SP)},$$

$$\omega(\mathfrak{T}_v) = \prod_{i=1}^{n}(1 - q^{k(n-1)}t_i[\mathfrak{q}])^{-1} \qquad (\text{Case UT},\ \mathfrak{p}^k\mathfrak{r} = \mathfrak{q}^2),$$

$$\omega(\mathfrak{T}_v) = \prod_{i=1}^{n}(1 - q^{n-1}t_i[\mathfrak{q}_1])^{-1}(1 - q^{n-1}t_{n+i}[\mathfrak{q}_2])^{-1} \qquad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2).$$

Here $\omega(\mathfrak{T}_v) = \sum_{\xi \in X} \omega(C_v\xi C_v)\big[\nu_{\mathfrak{b}}(\xi)\big]$, $X = C_v \backslash \mathfrak{X}_v / C_v$; $\mathfrak{p}$ is the prime ideal in F at v and $q = N(\mathfrak{p})$; in Case UT, $\mathfrak{q}$ and $\mathfrak{q}_i$ are prime ideals in K; $\mathfrak{q}_1 \neq \mathfrak{q}_2$.

PROOF. For $v \nmid \mathfrak{c}$ the formulas were given in [S97, Theorem 16.16 and (16.17.5)]; as for the injectivity of ω, see [S97, Proposition 16.14]. Suppose $\mathfrak{p}|\mathfrak{c}$ and $\mathfrak{p} \nmid \mathfrak{e}$ in Case UT. We first consider the case $\mathfrak{p}\mathfrak{r} = \mathfrak{q}^{2/k}$ with $k = 1$ or 2; then $N(\mathfrak{q}) = q^k$. Given a coset $E_v d$ with $d \in B_v$, we can find an upper triangular g such that $E_v d = E_v g$; we may assume that the diagonal elements of g are of the forms $\pi^{e_1}, \ldots, \pi^{e_n}$ with $e_i \in \mathbf{Z}$, where π is a prime element of K_v. We then put $\omega_0(E_v d) = \prod_{i=1}^{n}(q^{-ik}t_i)^{e_i}$. Next, given $C_v\sigma C_v$ with $\sigma \in Q(\mathfrak{e})_v$, we take a decomposition $C_v\sigma C_v = \bigsqcup_{\xi} C_v\xi$ with $\xi \in P_v$ and put $\omega(C_v\sigma C_v) = \sum_{\xi} \omega_0(E_v d_{\xi})$.

We then extend ω to $\mathfrak{R}(C_v, \mathfrak{X}_v)$ $\mathbf{Q}$-linearly. It can easily be verified that ω is a ring-homomorphism. (This is similar to what was done in [S97, §16.13].) Its injectivity can be proved in the same manner as in [S97, Proposition 16.14]. By Lemma 19.2 (3) we have

$$\begin{aligned}\omega(\mathfrak{T}_v) &= \sum_{d\in E_v\backslash B_v} \omega_0(E_v d)\big[S(\mathfrak{b}^{-1})_v : d^*S(\mathfrak{b}^{-1})_v d\big]\big[\det(d)\mathfrak{r}\big]\\ &= \sum_{d\in E_v\backslash B_v} \omega_0(E_v d)|\det(d)|^{-n}\big[\det(d)\mathfrak{r}\big],\end{aligned}$$

since $\big[S(\mathfrak{b}^{-1})_v : d^*S(\mathfrak{b}^{-1})_v d\big] = |\det(d)|^{-n}$ by [S97, Lemma 13.2], where $|\ |$ is the valuation of K_v such that $|\pi| = q^{-k}$. The last sum is essentially the series $\mathcal{B}$ of [S97, Lemma 16.3], and so using the formula for $\mathcal{B}$ given there, we obtain the desired formula for $\omega(\mathfrak{T}_v)$ in the present case. Case SP can be handled in the same manner. (This is actually done in [S94, Theorem 2.9].) Finally consider the case $\mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2$. In this case $GL_n(K_v)$ can be identified with $GL_n(F_v) \times GL_n(F_v)$, and E_v with $E_v^0 \times E_v^0$, where $E_v^0 = GL_n(\mathfrak{g}_v)$. Let $E_v d = E_v^0 a \times E_v^0 b$ with $d \in GL(K_v)$ and upper triangular matrices a and b in $GL_n(F_v)$ whose diagonal elements are $\pi^{e_1}, \dots, \pi^{e_n}$ and $\pi^{e_{n+1}}, \dots, \pi^{e_{2n}}$, respectively, where π is a prime element of F_v. Putting $\omega_0(E_v d) = \prod_{i=1}^n (q^{-i}t_i)^{e_i}(q^{-i}t_{n+i})^{e_{n+i}}$, we define $\omega(C_v\sigma C_v)$ in the same manner as in the above case, and repeat the calculation with these modifications to obtain the desired result.

19.9. Lemma. *With a fixed $v \in \mathbf{h}$ prime to $\mathfrak{c}$, put $\omega(\mathfrak{T}_v) = \mathcal{T}_v(t_1, \dots, t_m)$ with a rational expression $\mathcal{T}_v$ defined for each fixed v as in Theorem 19.8. Let $\mathfrak{R}^v$ be the subalgebra of $\mathfrak{R}(C_v, \mathfrak{X}_v)$ generated over $\mathbf{Q}$ by $T_v(\mathfrak{a})$ for all integral $\mathfrak{r}_v$-ideals $\mathfrak{a}$, where $T_v(\mathfrak{a})$ is the sum of all $C_v\xi C_v$ with $\xi \in \mathfrak{X}_v$ such that $\nu_\mathfrak{b}(\xi) = \mathfrak{a}$. Let λ be a $\mathbf{Q}$-linear ring-homomorphism of $\mathfrak{R}^v$ into $\mathbf{C}$ that maps the identity element to 1. Then there exist m elements μ_i of $\mathbf{C}$ such that $\mu_i \neq 0$ if $v \nmid \mathfrak{c}$ and that the series $\sum_\mathfrak{a} \lambda(T_v(\mathfrak{a}))[\mathfrak{a}]$ coincides with the expression $\mathcal{T}_v(\mu_1, \dots, \mu_m)$.*

PROOF. First assume that $K \neq F$, $v \nmid \mathfrak{c}$, and $\mathfrak{p}\mathfrak{r} = \mathfrak{q}$. Let $P(X)$ be the polynomial in an indeterminate X such that $P([\mathfrak{q}])$ coincides with the denominator of $\omega(\mathfrak{T}_v)$ given in Theorem 19.8. Then we see that P has coefficients in $\omega(\mathfrak{R}^v)$ and hence t_i and t_i^{-1} are integral over $\omega(\mathfrak{R}^v)$. Identify $\mathfrak{R}^v$ with $\omega(\mathfrak{R}^v)$. Then the integrality guarantees that $\lambda : \omega(\mathfrak{R}^v) \to \mathbf{C}$ can be extended to a homomorphism λ' of $\mathbf{Q}[t_1, \dots, t_n, t_1^{-1}, \dots, t_n^{-1}]$ into $\mathbf{C}$. Putting $\mu_i = \lambda'(t_i)$ and applying λ' to $\omega(\mathfrak{T}_v)$, we obtain our assertion in the present situation. If $v \nmid \mathfrak{c}$ and $\mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2$, then the denominator can be written $P_1([\mathfrak{q}_1])P_2([\mathfrak{q}_2])$ with polynomials P_i, and we again see that t_i and t_i^{-1} are integral over $\omega(\mathfrak{R}^v)$, and hence we obtain the same conclusion. All the remaining cases can be handled in the same manner, except that if $v|\mathfrak{c}$, then the quantities t_i^{-1} do not appear, so that we cannot say that $\mu_i \neq 0$ for $v|\mathfrak{c}$.

We now present a lemma concerning a generalized Möbius function μ defined on the set of $\mathfrak{r}$-submodules of a torsion $\mathfrak{r}$-module, which will play an essential role in the next section. Here $\mathfrak{r}$ is the ring of algebraic integers in an arbitrary algebraic number field K. We denote by $\mathbf{k}$ the set of all nonarchimedean primes of K.

19.10. Lemma. *To every finitely generated torsion $\mathfrak{r}$-module A we can uniquely assign an integer $\mu(A)$ so that*

$$\sum_{B\subset A} \mu(B) = \begin{cases} 1 & \text{if } A = \{0\}, \\ 0 & \text{if } A \neq \{0\}. \end{cases} \tag{19.12a}$$

Moreover μ *has the following properties:*

$$\sum_{B\subset A}\mu(A/B)=\begin{cases}1 & \text{if } A=\{0\},\\ 0 & \text{if } A\neq\{0\}.\end{cases} \tag{19.12b}$$

(19.12c) $\mu(A\oplus B)=\mu(A)\mu(B)$ *if* $\mathfrak{a}A=\mathfrak{b}B=\{0\}$ *with relatively prime integral ideals* $\mathfrak{a}$ *and* $\mathfrak{b}$.

(19.12d) $\mu\big((\mathfrak{r}/\mathfrak{p})^r\big)=(-1)^rN(\mathfrak{p})^{r(r-1)/2}$ *if* $0\le r\in\mathbf{Z}$ *and* $\mathfrak{p}$ *is a prime ideal in* K.

(19.12e) $\mu(A)\neq 0$ *if and only if* A *is annihilated by a squarefree integral ideal.*

PROOF. We can define $\mu(A)$ inductively by $\mu(A)=-\sum_{B\subsetneqq A}\mu(B)$, starting from $\mu(\{0\})=1$, which shows also the uniqueness. To prove (19.12b), we may assume that $A=L/N$ with two $\mathfrak{r}$-lattices L and N in K^n. For every $\mathfrak{r}$-lattice X in K^n put $X'=\{\,y\in K^n\,|\,{}^tyX\subset\mathfrak{r}\,\}$. Given an $\mathfrak{r}$-submodule B of A, take an $\mathfrak{r}$-lattice M so that $N\subset M\subset L$ and $B=M/N$. Put $\psi(B)=\varphi(M'/L')$ with any fixed $\mathfrak{r}$-isomorphism φ of N'/L' onto A. Then ψ gives a one-to-one map of the set of all $\mathfrak{r}$-submodules of A onto itself, and $\psi(B)\cong A/B$ and $A/\psi(B)\cong B$. Therefore $\sum_{B\subset A}\mu(A/B)=\sum_{B\subset A}\mu(\psi(B))=\sum_{C\subset A}\mu(C)$, which combined with (19.12a) gives (19.12b). Next, if A and B are as in (19.12c), then every $\mathfrak{r}$-submodule of $A\oplus B$ is of the form $A'\oplus B'$ with $\mathfrak{r}$-submodules A' of A and B' of B. Then (19.12c) can be derived from the relation $\mu(A)=-\sum_{C\subsetneqq A}\mu(C)$ by induction. The formula of (19.12d) follows from the well-known equality $\sum_{r=0}^{n}(-1)^rN(\mathfrak{p})^{r(r-1)/2}c_r^n=0$ which holds for $n>0$, where c_r^n denotes the number of $\mathfrak{r}$-submodules of $(\mathfrak{r}/\mathfrak{p})^n$ isomorphic to $(\mathfrak{r}/\mathfrak{p})^r$. To prove (19.12e), we first observe that $\mu(\mathfrak{r}/\mathfrak{p}^2)=0$ for every prime ideal $\mathfrak{p}$. Given A, let C be the maximum $\mathfrak{r}$-submodule of A that is annihilated by a squarefree integral ideal. Suppose $A\neq C$; then $C\neq\{0\}$ and $-\mu(A)=\sum_{D\subset C}\mu(D)+\sum_{B\not\subset C,\,B\subsetneqq A}\mu(B)$. The first sum on the right-hand side is 0. Therefore we obtain $\mu(A)=0$ by induction. The converse part follows from (19.12c, d).

19.11. Let $\mathcal{L}$ denote the set of all $\mathfrak{r}$-lattices in K^n. For $L\in\mathcal{L}$ and $y\in GL_n(K)_{\mathbf{A}}$ we denote by yL the $\mathfrak{r}$-lattice in K^n such that $(yL)_v=y_vL_v$ for every $v\in\mathbf{k}$. For L and M in $\mathcal{L}$ we define a fractional ideal $\{L/M\}$ and a multiplicative symbol $[L/M]$ (in the sense of §19.5) by

$$\{L/M\}=\det(y)\mathfrak{r},\qquad [L/M]=[\{L/M\}]=[\det(y)\mathfrak{r}] \tag{19.13}$$

with any $y\in GL_n(K)_{\mathbf{A}}$ such that $M=yL$. These are well-defined. Clearly we have $[L/M][M/N]=[L/N]$. If $L,\ M\in\mathcal{L}$ and $M\subset L$, we can speak of $\mu(L/M)$. Moreover, for each $v\in\mathbf{k}$ we can speak of $\mu(L_v/M_v)$ either by viewing L_v/M_v as an $\mathfrak{r}$-module, or by defining μ for $\mathfrak{r}_v$-modules, which makes no difference. From (19.12c) we easily obtain

$$\mu(L/M)=\textstyle\prod_{v\in\mathbf{k}}\mu(L_v/M_v). \tag{19.14}$$

We now take a subset Λ of $\mathcal{L}$ satisfying the following condition: *if* $L\subset H\subset M$, $L,\ M\in\Lambda$, *and* $H\in\mathcal{L}$, *then* $H\in\Lambda$. Fixing an integral ideal $\mathfrak{c}$, we write $L<M$ and $M>L$ if $L\subset M$ and $M_v=L_v$ for every $v|\mathfrak{c}$.

19.12. Lemma. *For two functions* α *and* β *defined on* Λ *with values in a* $\mathbf{Z}$*-module, we have*

$$\alpha(L)=\sum_{L<M\in\Lambda}\beta(M) \text{ for every } L\in\Lambda$$
$$\iff \beta(L)=\sum_{L<M\in\Lambda}\mu(M/L)\alpha(M) \text{ for every } L\in\Lambda,$$
$$\alpha(L)=\sum_{L>M\in\Lambda}\beta(M) \text{ for every } L\in\Lambda$$
$$\iff \beta(L)=\sum_{L>M\in\Lambda}\mu(L/M)\alpha(M) \text{ for every } L\in\Lambda.$$

Here and in Lemma 19.14 below each sum may be an infinite sum, and so we have to assume that it is convergent in a suitable sense, or it is a formal sum.

PROOF. Assume the first equality. Then, for a fixed $L\in\Lambda$, we have

$$\sum_{L<M\in\Lambda}\mu(M/L)\alpha(M)=\sum_{L<M\in\Lambda}\mu(M/L)\sum_{M<H\in\Lambda}\beta(H)$$
$$=\sum_{L<H\in\Lambda}\beta(H)\sum_{L<M<H}\mu(M/L).$$

The condition $L<M<H$ can be changed into $L\subset M\subset H$. Applying (19.12a) with H/L as A to the last sum, we find that the last double sum equals $\beta(L)$, which proves the first $\Rightarrow$. All the other cases can be proved similarly by employing (19.12a) or (19.12b).

Notice that if $\mathfrak{r}=\mathbf{Z}$, then $n\mapsto\mu(\mathbf{Z}/n\mathbf{Z})$ is the classical Möbius function, and the first half of Lemma 19.12 is exactly the classical Möbius inversion formula if we take $\Lambda=\{\,n\mathbf{Z}\mid 0<n\in\mathbf{Z}\,\}$ and consider $\alpha(n\mathbf{Z})$ a function of n.

19.13. Lemma. *For any fixed $L\in\mathcal{L}$ we have*

$$\sum_{L\subset M\in\mathcal{L}}[M/L]=\sum_{L\supset M\in\mathcal{L}}[L/M]=\sum_{x\in B/E}[\det(x)\mathfrak{r}]=\prod_{i=1}^{n}\prod_{v\in\mathbf{k}}\left(1-q_v^{i-1}[\mathfrak{q}_v]\right)^{-1},$$
$$\sum_{L\subset M\in\mathcal{L}}\mu(M/L)[M/L]=\sum_{L\supset M\in\mathcal{L}}\mu(L/M)[L/M]=\prod_{i=1}^{n}\prod_{v\in\mathbf{k}}\left(1-q_v^{i-1}[\mathfrak{q}_v]\right),$$

where $\mathfrak{q}_v$ is the prime ideal at v and $q_v=N(\mathfrak{q}_v)$.

PROOF. Putting $M={}^tx^{-1}L$ or $M=xL$ with $x\in B$, we see that the first two sums equal to the third sum, which is clealy the product of $\sum_{x\in B_v/E_v}\left[\det(x)\mathfrak{r}_v\right]$ for all $v\in\mathbf{k}$. Each such sum is determined by [S97, Lemma 3.13], and so we obtain the first line of equalities. Notice that the sums are independent of L. Next, the product of the first sums of the two lines (for a fixed L) is

$$\sum_{L\subset M\in\mathcal{L}}\mu(M/L)[M/L]\sum_{M\subset N\in\mathcal{L}}[N/M]=\sum_{L\subset N\in\mathcal{L}}[N/L]\sum_{L\subset M\subset N}\mu(M/L).$$

Applying (19.12a) to the last sum, we see that the double sum is 1. Similarly the product of the second sums of the two lines is 1. Therefore we obtain the second line of equalities.

19.14. Lemma. *Let α and γ be functions defined on Λ with values in a $\mathbf{Z}$-module Y, and let δ be a function on Λ with values in* $\mathrm{End}(Y)$. *If*

$$\alpha(L)=\sum_{L<N\in\Lambda}\delta(N)\sum_{H\in\Lambda,\,L+H=N}\gamma(H) \quad \text{for every } L\in\Lambda,$$

then

$$\sum_{L<M\in\Lambda} \mu(M/L)\alpha(M) = \sum_{L<M\in\Lambda} \mu(M/L)\delta(M) \sum_{L\supset H\in\Lambda} \gamma(H) \quad \textit{for every } L \in \Lambda.$$

PROOF. For fixed L and N in Λ we have

$$\sum_{H\in\Lambda,\, L+H=N} \gamma(H) = \sum_{N\supset H\in\Lambda} \Bigl(\sum_{L+H\subset M\subset N} \mu(N/M) \Bigr)\gamma(H) = \sum_{L\subset M\subset N} \mu(N/M)\varepsilon(M),$$

where $\varepsilon(M) = \sum_{M\supset H\in\Lambda} \gamma(H)$, and hence

$$\begin{aligned} \alpha(L) &= \sum_{L<N\in\Lambda} \delta(N) \sum_{L\subset M\subset N} \mu(N/M)\varepsilon(M) \\ &= \sum_{L<M\in\Lambda} \sum_{M<N\in\Lambda} \mu(N/M)\delta(N)\varepsilon(M) = \sum_{L<M\in\Lambda} \beta(M), \end{aligned}$$

where $\beta(M) = \sum_{M<N\in\Lambda} \mu(N/M)\delta(N)\varepsilon(M)$. Therefore Lemma 19.12 gives the desired conclusion.

20. Dirichlet series obtained from Hecke eigenvalues and Fourier coefficients

20.1. Let us now take the ideals $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{e}$ as in §19.1 and the group C as in (19.1); in addition we take a Hecke character ψ of K such that

(20.1) $\psi_v(a) = 1$ *for every* $a \in \mathfrak{r}_v^\times$, $v \in \mathbf{h}$, *such that* $a - 1 \in \mathfrak{r}_v\mathfrak{c}_v$.

Let k be an integral or a half-integral weight (see §16.4). We assume that

(20.2) $\qquad \mathfrak{b}^{-1} \subset 2\mathfrak{d}^{-1}$ *and* $\mathfrak{b}\mathfrak{c} \subset 2\mathfrak{d}$ *if* k *is half-integral.*

This means that $C \subset D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$ and $\mathfrak{c} \subset 4\mathfrak{g}$ if k is half-integral, and so j_α^k is meaningful if $\alpha \in \mathrm{pr}^{-1}(C)$. Then we denote by $\mathcal{M}_k(C, \psi)$ for integral k (resp. half-integral k) the set of all functions $\mathbf{f} : G_{\mathbf{A}} \to \mathbf{C}$ (resp. $\mathbf{f} : M_{\mathbf{A}} \to \mathbf{C}$) satisfying the following two conditions:

(20.3a) $\mathbf{f}(\alpha x w) = \psi_{\mathfrak{c}}\bigl(\det(a_w)\bigr)^{-1} j_w^k(\mathbf{i})^{-1}\mathbf{f}(x)$ *if* $\alpha \in G$, $w \in C$, *(resp.* $\mathrm{pr}(w) \in C$*) and* $w(\mathbf{i}) = \mathbf{i}$, *where* $\psi_{\mathfrak{c}} = \prod_{v|\mathfrak{c}} \psi_v$.

(20.3b) *For every* $p \in G_{\mathbf{h}}$ *(resp.* $p \in M_{\mathbf{A}}$ *such that* $\mathrm{pr}(p) \in G_{\mathbf{h}}$*) there exists an element* f_p *of* $\mathcal{M}_k$, *called the* p*-component of* $\mathbf{f}$, *such that* $\mathbf{f}(py) = (f_p\|_k y)(\mathbf{i})$ *for every* $y \in G_{\mathbf{a}}$ *(resp.* $y \in M_{\mathbf{A}}$ *such that* $\mathrm{pr}(y) \in G_{\mathbf{a}}$*).*

Clearly f_p is uniquely determined by $\mathbf{f}$ and p. In general a_w for an arbitrary $w \in C$ may not be an element of $GL_n(K)_{\mathbf{A}}$, but $(a_w)_v \in GL_n(\mathfrak{r}_v)$ for $v|\mathfrak{c}$, so that $\psi_{\mathfrak{c}}\bigl(\det(a_w)\bigr)$ is meaningful (see (1.18)). Also, if k is half-integral, $x \in M_{\mathbf{A}}$, and $\mathrm{pr}(x) \in C$, then $x \in \mathfrak{M}$, so that $j_x^k(\mathbf{i})$ and $\|_k x$ are meaningful (see §16.4). Put $\Gamma^p = G \cap pCp^{-1}$ and

(20.4) $\mathcal{M}_k(\Gamma^p, \psi) = \bigl\{ f \in \mathcal{M}_k \bigm| f\|_k\gamma = \psi_{\mathfrak{c}}\bigl(\det(a(p^{-1}\gamma p))\bigr) f$ *for every* $\gamma \in \Gamma^p \bigr\}$.

Here $a(x)$ is the a-block of x; we consider only the case $p = 1$ if $k \notin \mathbf{Z}^{\mathbf{b}}$; notice that $\Gamma^1 \subset \Gamma^\theta$ by (20.2), so that $f\|_k\gamma$ is meaningful for $\gamma \in \Gamma^1$. It can easily be verified that f_p of (20.3b) belongs to $\mathcal{M}_k(\Gamma^p, \psi)$. Now, by [S97, Lemma 8.12] we can take a finite subset $\mathcal{B}$ of $G_{\mathbf{h}}$ so that

(20.5) $$G_{\mathbf{A}} = \bigsqcup_{q\in\mathcal{B}} GqC \quad \textit{and } q_v = 1 \textit{ for every } q \in \mathcal{B} \textit{ and every } v|\mathfrak{c}.$$

Then for integral k we can show that the map $\mathbf{f} \mapsto (f_q)_{q\in\mathcal{B}}$ is a bijection of $\mathcal{M}_k(C, \psi)$ onto $\prod_{q\in\mathcal{B}} \mathcal{M}_k(\Gamma^q, \psi)$ (see [S97, Lemma 10.8]). In this situation we write $\mathbf{f} \leftrightarrow (f_q)_{q\in\mathcal{B}}$. We call $\mathbf{f}$ a *cusp form* if $f_p \in \mathcal{S}_k$ for every p, which is the case if $f_q \in \mathcal{S}_k$ for every $q \in \mathcal{B}$. We denote by $\mathcal{S}_k(C, \psi)$ the set of all cusp forms contained in $\mathcal{M}_k(C, \psi)$. If ψ is the trivial character, then we denote $\mathcal{M}_k(C, \psi)$ and $\mathcal{S}_k(C, \psi)$ by $\mathcal{M}_k(C)$ and $\mathcal{S}_k(C)$. Clearly $\mathcal{M}_k(C, \psi) = \mathcal{M}_k(C)$ and $\mathcal{S}_k(C, \psi) = \mathcal{S}_k(C)$ for any ψ if $\mathfrak{e} = \mathfrak{c}$. Notice that $\mathcal{M}_k(\Gamma^p, \psi) \neq \{0\}$ only if

(20.6) $\psi_{\mathbf{a}}(\zeta^n) = \zeta^k$ *for every root of unity* $\zeta \in K$ *such that* $\zeta - 1 \in \mathfrak{r}\mathfrak{e}$.

In Case SP we have $G_{\mathbf{A}} = GC$ because of strong approximation in $G = Sp(n, F)$, and so we can take $\mathcal{B} = \{q\}$ with any $q \in G_{\mathbf{h}}$, and hence $\mathbf{f} \mapsto f_q$ is a bijection of $\mathcal{M}_k(C, \psi)$ onto $\mathcal{M}_k(\Gamma^q, \psi)$. The same conclusion holds for half-integral k with $q = 1$. Indeed, we first observe that $M_{\mathbf{A}} = G \cdot \mathrm{pr}^{-1}(C)$. Given $f \in \mathcal{M}_k(\Gamma^1, \psi)$, define $\mathbf{f} : M_{\mathbf{A}} \to \mathbf{C}$ by $\mathbf{f}(\alpha w) = \psi_{\mathfrak{c}}\big(\det(a_w)\big)^{-1}(f\|_k w)(\mathbf{i})$ for $\alpha \in G$ and $w \in \mathrm{pr}^{-1}(C)$. Then this is well-defined and satisfies (20.3a). Now given $p \in \mathrm{pr}^{-1}(G_{\mathbf{h}})$ and $y \in \mathrm{pr}^{-1}(G_{\mathbf{a}})$, take $\alpha \in G$ and $x \in \mathrm{pr}^{-1}(C)$ so that $p = \alpha x$. Then $\mathbf{f}(py) = \psi_{\mathfrak{c}}\big(\det(a_x)\big)^{-1}(f\|_k xy)(\mathbf{i})$. Since $\mathrm{pr}(x)_{\mathbf{a}} = \alpha^{-1}$, if we choose a suitable element $\xi = (\alpha^{-1}, t(z))$ in the group $\mathcal{G}$ of §14.14, then $f\|_k x = f\|_k \xi \in \mathcal{M}_k$. Thus $\mathbf{f}$ satisfies (20.3b). Clearly $f_1 = f$. This proves the surjectivity of the map. The injectivity follows immediately from (20.3b).

Now, with S as in (16.1a), put

(20.7) $$S_+ = \{\, \xi \in S \mid \xi_v \geq 0 \quad \textit{for every} \quad v \in \mathbf{a} \,\}.$$

20.2. Proposition. *Given* $\mathbf{f} \in \mathcal{M}_k(C, \psi)$, *there is a complex number* $c(\tau, q; \mathbf{f})$, *written also* $c_{\mathbf{f}}(\tau, q)$, *determined for* $\tau \in S_+$ *and* $q \in GL_n(K)_{\mathbf{A}}$, *such that*

(20.8) $$\mathbf{f}\left(r_P\begin{bmatrix} q & s\widehat{q} \\ 0 & \widehat{q}\end{bmatrix}\right) = \det(q)_{\mathbf{a}}^{[k]\rho}|\det(q)_{\mathbf{a}}|^{k-[k]} \sum_{\tau\in S_+} c(\tau, q; \mathbf{f})\mathbf{e}_{\mathbf{a}}^n(\mathbf{i}q^*\tau q)\mathbf{e}_{\mathbf{A}}^n(\tau s),$$

for every $s \in S_{\mathbf{A}}$, *where* r_P *should be ignored if* k *is integral, and* $[k]$ *is the integral part of* k *as defined in* §16.4. *Moreover* $c_{\mathbf{f}}(\tau, q)$ *has the following properties:*

(20.9a) $c_{\mathbf{f}}(\tau, q) \neq 0$ *only if* $\mathbf{e}_{\mathbf{h}}^n(q^*\tau q s) = 1$ *for every* $s \in S_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})$;

(20.9b) $c_{\mathbf{f}}(\tau, q) = c_{\mathbf{f}}(\tau, q_{\mathbf{h}})$;

(20.9c) $c_{\mathbf{f}}(b^*\tau b, q) = \det(b)^{[k]\rho}|\det(b)|^{k-[k]}c_{\mathbf{f}}(\tau, bq)$ *for every* $b \in GL_n(K)$;

(20.9d) $\psi_{\mathbf{h}}\big(\det(e)\big)c_{\mathbf{f}}(\tau, qe) = c_{\mathbf{f}}(\tau, q)$ *for every* $e \in E'$.

Furthermore, if $\beta \in G \cap \mathrm{diag}[r, \widehat{r}]Cp^{-1}$ *with* $r \in GL_n(K)_{\mathbf{h}}$ *and* $p \in G_{\mathbf{h}}$, *then*

(20.9e) $$j^k(\beta, \beta^{-1}z)f_p(\beta^{-1}z) = \psi_{\mathfrak{c}}\big(\det(a_{\beta p}^{-1}r)\big)\textstyle\sum_{\tau\in S_+} c_{\mathbf{f}}(\tau, r)\mathbf{e}_{\mathbf{a}}^n(\tau z),$$

where f_p *is the* p*-component of* $\mathbf{f}$ *and* $a_{\beta p}$ *is the* a*-block of* βp. *Here we take* $p = 1$ *if* $k \notin \mathbf{Z}^{\mathbf{b}}$.

REMARK. Taking $\beta = 1$ and $p = \mathrm{diag}[r, \widehat{r}]$ in (20.9e), we obtain

(20.9f) $$f_p(z) = \textstyle\sum_{\tau\in S_+} c_{\mathbf{f}}(\tau, r)\mathbf{e}_{\mathbf{a}}^n(\tau z) \quad \textit{if} \quad p = \mathrm{diag}[r, \widehat{r}].$$

If $k \notin \mathbf{Z}^{\mathbf{b}}$ and $p = 1$, then β of (20.9e) belongs to $\mathfrak{M}$, so that j_β^k is meaningful.

PROOF. We first consider the case of integral k. Let $x = \begin{bmatrix} q & s\widehat{q} \\ 0 & \widehat{q}\end{bmatrix}$ with $q \in GL_n(K)_{\mathbf{A}}$ and $s \in S_{\mathbf{A}}$; put $p = x_{\mathbf{h}}$, define f_p by (20.3b), and put $f_p(z) =$

$\sum_{\tau\in S_+} c(\tau)\mathbf{e}_{\mathbf{a}}^n(\tau z)$. Then $x(\mathbf{i}) = iq_{\mathbf{a}}q_{\mathbf{a}}^* + s_{\mathbf{a}}$. Since $c(\tau)$ depends only on p, we can put

$$\mathbf{f}(x) = \mathbf{f}(px_{\mathbf{a}}) = (f_p\|x)(\mathbf{i}) = \det(q^*)^k \sum_{\tau\in S_+} c(\tau,\, q,\, s)\mathbf{e}_{\mathbf{a}}^n(iq^*\tau q)\mathbf{e}_{\mathbf{A}}^n(\tau s)$$

with $c(\tau, q, s) = \mathbf{e}_{\mathbf{h}}^n(-\tau s)c(\tau)$, which depends only on $q_{\mathbf{h}}$, $s_{\mathbf{h}}$, and τ. Now $\mathbf{f}(\alpha x w) = \mathbf{f}(x)$ for every $\alpha \in R$ and $w \in R_{\mathbf{h}} \cap C$, so that we easily see that $c(\tau, q, s+h) = c(\tau, q, s)$ for every $h \in S + \prod_{v\in\mathbf{h}} M_v$ with an $\mathfrak{r}$-lattice M in S (depending on q). Thus $c(\tau, q, s)$ is independent of s. Therefore we obtain (20.8) and (20.9b). Now given β, r, and p as in the last part of our proposition, take $y \in P_{\mathbf{a}}$ so that $z = y(\mathbf{i})$. Then we can put $\beta^{-1}\mathrm{diag}[r, \widehat{r}]y = pw$ with $w \in C$. By (20.3a, b), $\mathbf{f}(\mathrm{diag}[r, \widehat{r}]y) = \mathbf{f}(pw) = \psi_{\mathfrak{c}}(\det(a_w))^{-1}\mathbf{f}(pw_{\mathbf{a}}) = \psi_{\mathfrak{c}}(\det(a_w))^{-1}(f_p\|w)(\mathbf{i})$. Since $(\beta^{-1}y)_{\mathbf{a}} = w_{\mathbf{a}}$, we have $f_p\|w = f_p\|(\beta^{-1}y)$. Observing that $\psi_{\mathfrak{c}}(\det(a_w)) = \psi_{\mathfrak{c}}(\det(a_{\beta p}^{-1}r))$, we have $(f_p\|(\beta^{-1}y))(\mathbf{i}) = \psi_{\mathfrak{c}}(\det(a_{\beta p}^{-1}r))\mathbf{f}(\mathrm{diag}[r, \widehat{r}]y)$. Applying (20.8) to $\mathbf{f}(\mathrm{diag}[r, \widehat{r}]y)$, we obtain (20.9e). The remaining properties of $c_{\mathbf{f}}(\tau, q)$ can easily be verified by means of (20.3a).

As for half-integral k, identify every element of $P_{\mathbf{A}}$ with its image under r_P. Then we can repeat the above argument to obtain our assertion in the same manner, except (20.9e); the only necessary modification is that we have to take $|\det(q)|^{k-[k]}\cdot\det(q)^{[k]}$ instead of $\det(q^*)^k$, in view of (16.19). To prove (20.9e), we put $\mathrm{diag}[r, \widehat{r}]y = \beta w$ with y as before and $w \in \mathrm{pr}^{-1}(C)$. Then $\beta^{-1}z = w(\mathbf{i})$ and we have again $\mathbf{f}(\mathrm{diag}[r, \widehat{r}]y) = \psi_{\mathfrak{c}}(\det(a_w))^{-1}(f_1\|w)(\mathbf{i})$. By (16.16c) and (16.19),

$$(*)\qquad j_\beta^k(w(\mathbf{i}))j_w^k(\mathbf{i}) = j_{\beta w}^k(\mathbf{i}) = \det(d_y)^{[k]}|\det(d_y)|^{k-[k]}.$$

Now $j_w^k(\mathbf{i})^{-1}f_1(\beta^{-1}z) = (f_1\|w)(\mathbf{i})$. Therefore, employing (*) and applying (20.8) to $\mathbf{f}(\mathrm{diag}[r, \widehat{r}]y)$, we obtain (20.9e).

20.3. Assuming k to be integral, we now define the action of $\mathfrak{R}(C, \mathfrak{X})$ on $\mathcal{M}_k(C, \psi)$; the case of half-integral k will be explained in Section 21. We first make the following observation. If ξ belongs to the set $\mathfrak{X}$ of (19.2c), then $(a_\xi)_v$ is invertible for every $v|\mathfrak{c}$, and so $\psi_{\mathfrak{c}}(\det(a_\xi))$ is meaningful. Put $\varphi(\xi) = \psi_{\mathfrak{c}}(\det(a_\xi))$. Then $\varphi(\alpha\xi\beta) = \varphi(\alpha)\varphi(\xi)\varphi(\beta)$ for $\alpha, \beta \in C$.

Now, given $\xi \in \mathfrak{X}$ and $\mathbf{f} \in \mathcal{M}_k(C, \psi)$, take a finite subset Y of $G_{\mathbf{h}}$ so that $C\xi C = \bigsqcup_{\eta\in Y} Cy$ and define $\mathbf{f}|C\xi C : G_{\mathbf{A}}^{\varphi} \to \mathbf{C}$ by

$$(20.10)\qquad (\mathbf{f}|C\xi C)(x) = \sum_{y\in Y} \psi_{\mathfrak{c}}(\det(a_y))^{-1}\mathbf{f}(xy^{-1}) \qquad (x \in G_{\mathbf{A}}^{\varphi}).$$

We can easily verify that this does not depend on the choice of Y, and also that $\mathbf{f}|C\xi C \in \mathcal{M}_k(C, \psi)$. This action can be extended linearly to the whole $\mathfrak{R}(C, \mathfrak{X})$. We easily see that it defines a ring-homomorphism of $\mathfrak{R}(C, \mathfrak{X})$ into $\mathrm{End}(\mathcal{M}_k(C, \psi))$. We then define a formal Dirichlet series $\mathbf{f}|\mathfrak{T}$ with coefficients in $\mathcal{M}_k(C, \psi)$ by

$$(20.11)\qquad \mathbf{f}|\mathfrak{T} = \sum_{\xi\in C\backslash\mathfrak{X}/C} (\mathbf{f}|C\xi C)[\nu_{\mathfrak{b}}(\xi)].$$

where $\nu_{\mathfrak{b}}(\xi)$ is defined by (19.4). Clearly $\mathbf{f}|\mathfrak{T} = \sum_{\mathfrak{a}}(\mathbf{f}|T(\mathfrak{a}))[\mathfrak{a}]$. Notice also that $(\mathbf{f}|C\xi C)(x) = \sum_{y\in Y}\mathbf{f}(xy^{-1})$ if $\mathfrak{e} = \mathfrak{c}$.

As in §19.11 let $\mathcal{L}$ denote the set of all $\mathfrak{r}$-lattices in K_1^n. We put $L_0 = \mathfrak{r}_1^n$ and we shall often express an element L of $\mathcal{L}$ in the form $L = yL_0$ with $y \in GL_n(K)_{\mathbf{h}}$. Let $\mathfrak{d}$, δ, and $\widetilde{S}$ be as in §16.1. For $\tau \in S$ put

$$(20.12)\qquad \mathcal{L}_\tau = \{\, L \in \mathcal{L} \mid \ell^*\tau\ell \in \mathfrak{b}\mathfrak{e}^{-1}\mathfrak{d}^{-1} \quad \textit{for every} \quad \ell \in L \,\}.$$

We easily see that $\mathcal{L}_\tau$ consists of all the $\mathfrak{r}$-lattices yL_0 with $y \in GL_n(K)_\mathbf{h}$ such that $y^*\tau y \in \mathfrak{b}\mathfrak{e}^{-1}\mathfrak{d}^{-1}\widetilde{S}$. Notice that if $L \in \mathcal{L}$ and $L \subset H \in \mathcal{L}_\tau$, then $L \in \mathcal{L}_\tau$; moreover, if $\det(\tau) \neq 0$, the set $\{M \in \mathcal{L}_\tau \mid L \subset M\}$ for a fixed L is finite. For L and M in $\mathcal{L}$ let us write $L < M$ if $L \subset M$ and $L_v = M_v$ for every $v|\mathfrak{c}$.

We consider the Fourier expansion of Proposition 20.2 and investigate their relationship with the formal series $\mathfrak{T}$. By (20.9a), for $y \in GL_n(F)_\mathbf{h}$ we have

$$(20.13) \qquad c(\tau,\, y;\, \mathbf{f}) \neq 0 \implies y^*\tau y \in \mathfrak{b}\mathfrak{e}^{-1}\mathfrak{d}^{-1}\widetilde{S} \iff yL_0 \in \mathcal{L}_\tau.$$

Now our first main result of this section can be stated as follows:

20.4. Theorem. *Given $\tau \in S_+$, $L \in \mathcal{L}_\tau$, and $\mathbf{f} \in \mathcal{M}_k(C,\, \psi)$, take $q \in GL_n(K)_\mathbf{h}$ so that $L = qL_0$ and define formal Dirichlet series $D(\tau,\, q;\, \mathbf{f})$, $a(\tau,\, L)$, and $A(\tau,\, L)$ by*

$$D(\tau,\, q;\, \mathbf{f}) = \sum_{x \in B'/E'} \psi_\mathfrak{c}\big(\det(qx)\big)|\det(x)|_K^{-\kappa} c(\tau,\, qx;\, \mathbf{f})[\det(x)\mathfrak{r}],$$

$$A(\tau,\, L) = |\det(q)|_K^{-\kappa}[\det(qq^*)\mathfrak{r}] \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L) a(\tau,\, M),$$

$$a(\tau,\, L) = |\det(q)|_K^{\kappa}[\det(qq^*)^{-1}\mathfrak{r}]\alpha_\mathfrak{c}^0(\varepsilon_\mathfrak{b} q^*\tau q).$$

Here $|w|_K$ denotes the idele norm of $w \in K_\mathbf{A}^\times$; $\kappa = n+1$ in Case SP and $\kappa = n$ in Case UT; $\mu(M/L)$ is the Möbius function introduced in Section 19; $\varepsilon_\mathfrak{b}$ is an element of $F_\mathbf{h}^\times$ such that $\varepsilon_\mathfrak{b}\mathfrak{g} = \mathfrak{b}^{-1}\mathfrak{d}$; $\alpha_\mathfrak{c}^0$ is the series of (19.6). Then

$$\big[\det(\widehat{q})\mathfrak{r}\big] A(\tau,\, L) D(\tau,\, q;\, \mathbf{f}) = \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)\psi_\mathfrak{c}\big(\det(y)\big)[\det(\widehat{y})\mathfrak{r}] c(\tau,\, y;\, \mathbf{f}|\mathfrak{T}),$$

$$\big[\det(\widehat{q})\mathfrak{r}\big]\psi_\mathfrak{c}\big(\det(q)\big) c(\tau,\, q;\, \mathbf{f}|\mathfrak{T}) = \sum_{L < M \in \mathcal{L}_\tau} [\det(\widehat{y})\mathfrak{r}] A(\tau,\, M) D(\tau,\, y;\, \mathbf{f}),$$

where y in the last two sums is an element of $GL_n(K)_\mathbf{h}$ chosen for each M so that $M = yL_0$ and $y^{-1}q \in B'$. In particular, if $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ with $\lambda(\mathfrak{a}) \in \mathbf{C}$ for every integral $\mathfrak{r}$-ideal $\mathfrak{a}$, then

$$A(\tau,\, L) D(\tau,\, q;\, \mathbf{f}) = \sum_\mathfrak{a} \lambda(\mathfrak{a})[\mathfrak{a}] \sum_{L < M \in \mathcal{L}_\tau} \mu(M/L)\psi_\mathfrak{c}\big(\det(y)\big)[\det(q^*\widehat{y})\mathfrak{r}] c_\mathbf{f}(\tau,\, y),$$

$$\psi_\mathfrak{c}\big(\det(q)\big) c(\tau,\, q;\, \mathbf{f}) \sum_\mathfrak{a} \lambda(\mathfrak{a})[\mathfrak{a}] = \sum_{L < M \in \mathcal{L}_\tau} [\det(q^*\widehat{y})\mathfrak{r}] A(\tau,\, M) D(\tau,\, y;\, \mathbf{f}).$$

REMARK. The series $a(\tau,\, L)$ and $A(\tau,\, L)$ are defined independently of the choice of q, as can be seen from (19.7). $D(\tau,\, q;\, \mathbf{f})$ depends only on $(\tau,\, qE',\, \mathbf{f})$. Thus, if $\mathfrak{e} = \mathfrak{g}$, it is independent of the choice of q, and we can put $D(\tau,\, q;\, \mathbf{f}) = D(\tau,\, L;\, \mathbf{f})$. Then we can write $D(\tau,\, M;\, \mathbf{f})$ for $D(\tau,\, y;\, \mathbf{f})$ in the above formulas. The sum $\sum_{L < M \in \mathcal{L}_\tau}$ is a finite sum if $\det(\tau) \neq 0$. In general it may be an infinite sum. We can show, however, that in all cases $A(\tau,\, L)$ can be expressed as an easy Euler product times a finite sum which is essentially a lower-dimensional version of $A(\tau,\, L)$ (see [S94a, Proposition 5.4]).

PROOF. Taking a subset $\mathcal{R}$ of $G_\mathbf{h}$ that represents $C\backslash\mathfrak{X}$, from (20.10) and (20.11) we obtain

$$(20.14) \qquad (\mathbf{f}|\mathfrak{T})(x) = \sum_{y \in \mathcal{R}} \psi_\mathfrak{c}\big(\det(a_y)\big)^{-1}\mathbf{f}(xy^{-1})[\nu_\mathfrak{b}(y)] \qquad (x \in G_\mathbf{A}).$$

By Lemma 19.4 we can take $\mathcal{R}$ to be the set of all $y = \begin{bmatrix} g^{-1}h & g^{-1}\sigma\widehat{h} \\ 0 & g^*\widehat{h} \end{bmatrix}$ with

$(g, h) \in E' \backslash W/(E' \times 1)$ and $\sigma \in S'/gS_{\mathbf{h}}(\mathfrak{b}^{-1}\mathfrak{e})g^*$. Applying [S97, Proposition 3.9] to $\beta^{-1}y\beta$ with β as in (19.5), we obtain $\nu_{\mathfrak{b}}(y) = \det(gh^*)\nu_0(b_0\sigma)$. Thus

$$(\mathbf{f}|\mathfrak{T})(x) = \sum \psi_{\mathfrak{c}}\big(\det(h^{-1}g)\big)\mathbf{f}(xy^{-1})[\det(gh^*)\nu_0(b_0\sigma)].$$

Substituting $\begin{bmatrix} q & s\widehat{q} \\ 0 & \widehat{q} \end{bmatrix}$ for x and making a straightforward calculation, we find, for every $q \in GL_n(K)_{\mathbf{h}}$, that

$$c(\tau,\, q;\, \mathbf{f}|\mathfrak{T}) = \sum_{g,\, h,\, \sigma} \psi_{\mathfrak{c}}\big(\det(h^{-1}g)\big)c_{\mathbf{f}}(\tau,\, qh^{-1}g)\mathbf{e}^n_{\mathbf{h}}(-\widehat{h}q^*\tau qh^{-1}\sigma)[\det(gh^*)\nu_0(b_0\sigma)].$$

By (20.13) we may assume that $g^*\widehat{h}q^*\tau qh^{-1}g \in \mathfrak{b}\mathfrak{e}^{-1}\mathfrak{d}^{-1}\widetilde{S}$. Therefore, by Lemma 19.6 we have, with ε_b defined as in our theorem,

$$c(\tau,\, q;\, \mathbf{f}|\mathfrak{T}) = \sum_{g,\, h} \psi_{\mathfrak{c}}\big(\det(h^{-1}g)\big)|\det(g)|_K^{-\kappa}[\det(gh^*)\mathfrak{r}] \cdot c(\tau,\, qh^{-1}g;\, \mathbf{f})\alpha^0_{\mathfrak{c}}\big(\varepsilon_b\widehat{h}q^*\tau qh^{-1}\big), \tag{20.15}$$

where (g, h) runs over $E' \backslash W/(E' \times 1)$ under the condition that $\widehat{h}q^*\tau qh^{-1} \in \mathfrak{b}\mathfrak{e}^{-1}\mathfrak{d}^{-1}\widetilde{S}$, which is so, by (20.13), if and only if $qh^{-1}L_0 \in \mathcal{L}_\tau$. We now fix $\tau \in S_+$ and $p \in GL_n(K)_{\mathbf{h}}$, and put

$$\Lambda = \big\{\, M \in \mathcal{L}_\tau \,\big|\, M_v = (pL_0)_v \;\; \textit{for every} \;\; v|\mathfrak{e} \,\big\},$$
$$X = \big\{\, y \in GL_n(K)_{\mathbf{h}} \,\big|\, y_v \in E'_v \;\; \textit{for every} \;\; v|\mathfrak{e} \,\big\}.$$

Now take $L = qL_0 \in \Lambda$ with $q \in pX$. (If we start from a given q, then we define Λ with q as p.) Then it can easily be seen that $(g, h) \mapsto (qh^{-1}L_0,\, qh^{-1}gL_0)$ gives a one-to-one map of the set of all such (g, h) onto the set of all (N, H) in $\Lambda \times \Lambda$ such that $L + H = N$ and $L < N$. Given $M \in \Lambda$, we can choose $y \in pX$ so that $M = yL_0$. Then, for a fixed τ put

$$c(M) = \psi_{\mathfrak{c}}\big(\det(y)\big)c(\tau,\, y;\, \mathbf{f})|\det(y)|_K^{-\kappa}[\det(y)\mathfrak{r}],$$
$$c'(M) = \psi_{\mathfrak{c}}\big(\det(y)\big)c(\tau,\, y;\, \mathbf{f}|\mathfrak{T})[\det(\widehat{y})\mathfrak{r}].$$

These are well-defined because of (20.9d). Therefore (20.15) can be written

$$c'(L) = \sum_{L<N\in\Lambda} a(\tau,\, N) \sum_{L+H=N,\, H\in\Lambda} c(H).$$

By Lemma 19.14 we obtain

$$\sum_{L<M\in\Lambda} \mu(M/L)c'(M) = \sum_{L<M\in\Lambda} \mu(M/L)a(\tau,\, M) \sum_{L\supset H\in\Lambda} c(H)$$

for every $L \in \Lambda$. Now the condition $L < M \in \Lambda$ is equivalent to $L < M \in \mathcal{L}_\tau$, since $M \in \Lambda$ if $L \in \Lambda$ and $L < M$. Therefore we obtain the first equality of our theorem. The second equality follows immediately from this and Lemma 19.12. The last two equalities are immediate consequences of the first two.

20.5. Lemma. *Let $\tau \in S_+ \cap GL_n(K)$ and $L = qL_0 \in \mathcal{L}_\tau$; let $\mathbf{b}$ be the set of all primes $v \in \mathbf{h}$ prime to $\mathfrak{c}$ such that $\varepsilon_b q^*\tau q$ is not regular in the sense of §16.1. Then*

$$A(\tau,\, L) = \begin{cases} \displaystyle\prod_{v\in\mathbf{b}} g_v([\mathfrak{p}]) \prod_{v\nmid\mathfrak{c}} h_v([\mathfrak{p}])^{-1}(1-[\mathfrak{p}]) \prod_{i=1}^{[n/2]} (1 - N(\mathfrak{p})^{2i}[\mathfrak{p}]^2) & \text{(Case SP)}, \\ \displaystyle\prod_{v\in\mathbf{b}} g_v([\mathfrak{p}\mathfrak{r}]) \prod_{v\nmid\mathfrak{c}} \prod_{i=1}^{n} (1 - (\theta^{i-1})^*(\mathfrak{p})N(\mathfrak{p})^{i-1}[\mathfrak{p}\mathfrak{r}]) & \text{(Case UT)}. \end{cases}$$

Here $\mathfrak{p}$ is the prime ideal of F at v and g_v is a polynomial with constant term 1 and with coefficients in $\mathbf{Z}$; $h_v = 1$ if n is odd, and $h_v(t) = 1 - \rho_\tau^*(\mathfrak{p})N(\mathfrak{p})^{n/2}t$ with the Hecke character ρ_τ of F corresponding to $F(c^{1/2})/F$, $c = (-1)^{n/2}\det(\tau)$, if n is even; θ is the Hecke character of F corresponding to K/F.

PROOF. If $L < M \in \mathcal{L}_\tau$, then we can put $M = q\widehat{x}L_0$ with $x \in B'$. Therefore $A(\tau, L) = \prod_{v\nmid \mathfrak{c}} A_v(\tau, L)$ with

$$A_v(\tau, L) = \sum_x \mu(L_0/x^*L_0)|\det(x)|_K^{-\kappa}[\det(xx^*)\mathfrak{r}]\alpha_v^0\big(x^{-1}(\varepsilon_b q^*\tau q)_v\widehat{x}\big), \tag{20.16}$$

where x runs over B_v/E_v under the condition that $x^{-1}(\varepsilon_b q^*\tau q)_v\widehat{x} \in \widetilde{S}_v$. If $v \notin \mathbf{b}$, then $A_v(\tau, L) = \alpha_v^0\big((\varepsilon_b q^*\tau q)_v\big)$, which is given by Theorem 16.2; if $v \in \mathbf{b}$, then $A_v(\tau, L)$ is a finite sum, and each $\alpha_v^0\big(x^{-1}(\varepsilon_b q^*\tau q)_v\widehat{x}\big)$ is a polynomial times a rational expression given in that theorem. (Notice that, in view of (16.7a), $\alpha_v^0(\zeta) = A_\zeta^0([\mathfrak{p}\mathfrak{r}])$ with A_ζ^0 of that theorem.) Therefore we obtain our lemma.

20.6. Suppose now $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for every $\mathfrak{a}$ as in the last part of Theorem 20.4. We naturally assume that $\mathbf{f} \neq 0$. By Lemma 19.9, for each $v \in \mathbf{h}$ we can determine complex numbers $\lambda_{v,i}$ so that $\sum_\mathfrak{a} \lambda(\mathfrak{a})[\mathfrak{a}] = \prod_{v\in\mathbf{h}} \mathcal{T}_v(\lambda_{v,1}, \ldots, \lambda_{v,m})$ with $\mathcal{T}_v$ in that lemma. Let Z_v^{-1} denote the denominator of the expression for $\mathcal{T}_v(\lambda_{v,1}, \ldots, \lambda_{v,m})$ obtained from the expression for $\omega(\mathfrak{T}_v)$ given in Theorem 19.8. Namely, denoting by $\mathfrak{p}$ the prime ideal in F at v, we have:

(I) $v\nmid\mathfrak{c}$.

$$Z_v = \big(1-N(\mathfrak{p})^n[\mathfrak{p}]\big)^{-1}\prod_{i=1}^n\Big\{\big(1-N(\mathfrak{p})^n\lambda_{v,i}[\mathfrak{p}]\big)\big(1-N(\mathfrak{p})^n\lambda_{v,i}^{-1}[\mathfrak{p}]\big)\Big\}^{-1} \quad \text{(Case SP)},$$

$$Z_v = \prod_{i=1}^n\Big\{\big(1-N(\mathfrak{q})^{n-1}\lambda_{v,i}[\mathfrak{q}]\big)\big(1-N(\mathfrak{q})^n\lambda_{v,i}^{-1}[\mathfrak{q}]\big)\Big\}^{-1} \quad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}^e),$$

$$Z_v = \prod_{i=1}^{2n}\Big\{\big(1-N(\mathfrak{q}_1)^{2n}\lambda_{v,i}^{-1}[\mathfrak{q}_1]\big)\big(1-N(\mathfrak{q}_2)^{-1}\lambda_{v,i}[\mathfrak{q}_2]\big)\Big\}^{-1} \quad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2).$$

(II) $v|\mathfrak{c}$, $v\nmid\mathfrak{e}$.

$$Z_v = \prod_{i=1}^n\big(1-N(\mathfrak{p})^n\lambda_{v,i}[\mathfrak{p}]\big)^{-1} \quad \text{(Case SP)},$$

$$Z_v = \prod_{i=1}^n\big(1-N(\mathfrak{q})^{n-1}\lambda_{v,i}[\mathfrak{q}]\big)^{-1} \quad (\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}^e),$$

$$Z_v = \prod_{i=1}^n\Big\{\big(1-N(\mathfrak{q}_1)^{n-1}\lambda_{v,i}[\mathfrak{q}_1]\big)\big(1-N(\mathfrak{q}_2)^{n-1}\lambda_{v,n+i}[\mathfrak{q}_2]\big)\Big\}^{-1}$$
$$(\text{Case UT},\ \mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2).$$

(III) $v|\mathfrak{e}$. $\quad Z_v = 1 \quad$ (Cases SP and UT).

Then from Theorem 19.8 we obtain

$$\mathfrak{L}\cdot\sum_\mathfrak{a}\lambda(\mathfrak{a})[\mathfrak{a}] = \prod_{v\in\mathbf{h}} Z_v, \tag{20.17}$$

where $\mathfrak{L}$ is a formal Dirichlet series given by

$$\mathfrak{L} = \prod_{\mathfrak{p}\nmid\mathfrak{c}} \left\{ (1 - [\mathfrak{p}]) \prod_{i=1}^{n} (1 - N(\mathfrak{p})^{2i}[\mathfrak{p}]^2) \right\}^{-1} \qquad \text{(Case SP)},$$

$$\mathfrak{L} = \prod_{\mathfrak{p}\nmid\mathfrak{c}} \prod_{i=1}^{2n} \left(1 - (\theta^{i-1})^*(\mathfrak{p}) N(\mathfrak{p})^{i-1} [\mathfrak{p}\mathfrak{r}] \right)^{-1} \qquad \text{(Case UT)},$$

where $(\theta^{i-1})^*$ is the same as in Lemma 20.5, and $\mathfrak{p}$ runs over all the prime $\mathfrak{g}$-ideals prime to $\mathfrak{c}$. We define also a similar formal Dirichlet series $\mathfrak{L}_0$ by

$$\mathfrak{L}_0 = \prod_{\mathfrak{p}\nmid\mathfrak{c}} \prod_{i=1}^{[(n+1)/2]} \left(1 - N(\mathfrak{p})^{2n+2-2i} [\mathfrak{p}]^2 \right)^{-1} \qquad \text{(Case SP)},$$

$$\mathfrak{L}_0 = \prod_{\mathfrak{p}\nmid\mathfrak{c}} \prod_{i=1}^{n} \left(1 - (\theta^{n+i-1})^*(\mathfrak{p}) N(\mathfrak{p})^{n+i-1} [\mathfrak{p}\mathfrak{r}] \right)^{-1} \qquad \text{(Case UT)}.$$

20.7. Theorem. *Let* $\mathbf{f}$ *and* Z_v *be as above and let* $\tau \in S_+ \cap GL_n(K)$ *and* $L = qL_0 \in \mathcal{L}_\tau$ *with* $q \in GL_n(K)_{\mathbf{h}}$. *Then we have*

$$D(\tau,\, q;\, \mathbf{f}) \cdot \mathfrak{L}_0 \cdot \prod_{v\in\mathbf{b}} g_v([\mathfrak{p}\mathfrak{r}]) \cdot \prod_{v\nmid\mathfrak{c}} h_v([\mathfrak{p}\mathfrak{r}])^{-1}$$
$$= \prod_{v\in\mathbf{h}} Z_v \cdot \sum_{L<M\in\mathcal{L}_\tau} \mu(M/L)\psi_\mathfrak{c}(\det(y))[\det(q^*\widehat{y})\mathfrak{r}]c_\mathbf{f}(\tau,\, y),$$

where $M = yL_0$ *as in Theorem 20.4, and* $\mathbf{b}$, g_v, *and* h_v *are determined for* τ *and* q *as in Lemma 20.5; we put* $h_v = 1$ *in Case UT.*

PROOF. This follows immediately from (20.17), the third equality of Theorem 20.4 concerning $A(\tau,\, L)D(\tau,\, q;\, \mathbf{f})$, and Lemma 20.5.

20.8. Lemma. *Let* $0 \neq \mathbf{f} \in \mathcal{M}_k(C,\, \psi)$ *with integral or half-integral* k *as in §20.1. Then the following three conditions are mutually equivalent:*

(1) Case SP: $k_v \geq n/2$ *for some* $v \in \mathbf{a}$; Case UT: $k_v + k_{v\rho} \geq n$ *for some* $v \in \mathbf{a}$.

(2) Case SP: $k_v \geq n/2$ *for every* $v \in \mathbf{a}$; Case UT: $k_v + k_{v\rho} \geq n$ *for every* $v \in \mathbf{a}$.

(3) $c_\mathbf{f}(\tau,\, r) \neq 0$ *for some* $\tau \in S_+ \cap GL_n(K)$ *and some* $r \in GL_n(K)_\mathbf{h}$.

Moreover, these conditions are satisfied if $\mathbf{f}$ *is a cusp form.*

PROOF. We first note that $G_\mathbf{A} = \bigsqcup_{r\in\{r\}} G \cdot \mathrm{diag}[r,\, \widehat{r}]C$ with a finite subset $\{r\}$ of $GL_n(K)_\mathbf{h}$. This is trivial in Case SP since $G_\mathbf{A} = GC$; in Case UT the fact is included in [S97, Lemma 9.8 (3)]. Given $\mathbf{f} \neq 0$, take $p \in G_\mathbf{h}$ so that the p-component of $\mathbf{f}$ is nonzero; we may assume that $p = \mathrm{diag}[r,\, \widehat{r}]$ with $r \in \{r\}$. Then, in view of (20.9f), the mutual equivalence of (1), (2), and (3) follow from Proposition 6.16. The last assertion follows from (6.42).

20.9. Theorem. *Let* $0 \neq \mathbf{f} \in \mathcal{M}_k(C,\, \psi)$ *with integral* k; *suppose that the conditions of Lemma 20.8 are satisfied. Then there exist* $\tau \in S_+ \cap GL_n(K)$ *and* $r \in GL_n(K)_\mathbf{h}$ *such that*

$$0 \neq \psi_\mathfrak{c}(\det(r))c(\tau,\, r;\, \mathbf{f}|\mathfrak{T}) = A(\tau,\, rL_0)D(\tau,\, r;\, \mathbf{f}). \tag{20.18}$$

Suppose in particular that $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *as above. Then* $c_\mathbf{f}(\tau,\, r) \neq 0$ *and*

$$\psi_\mathfrak{c}(\det(r))c_\mathbf{f}(\tau,\, r)\prod_{v\in\mathbf{h}} Z_v = D(\tau,\, r;\, \mathbf{f}) \cdot \mathfrak{L}_0 \cdot \prod_{v\nmid\mathfrak{c}} h_v([\mathfrak{p}\mathfrak{r}])^{-1} \cdot \prod_{v\in\mathbf{b}} g_v([\mathfrak{p}\mathfrak{r}]) \tag{20.19}$$

with the symbols as in Lemma 20.5 and Theorem 20.7.

PROOF. We have $c_{\mathbf{f}}(\tau, q) \neq 0$ for some $\tau \in S_+ \cap GL_n(K)$ and some $q \in GL_n(K)_{\mathbf{h}}$. Then $D(\tau, q; \mathbf{f}) \neq 0$. Let $\mathcal{L}'_\tau$ be the set of all lattices $M \in \mathcal{L}_\tau$ such that $qL_0 < M$ and $M = rL_0$ with some $r \in GL_n(K)_{\mathbf{h}}$ for which $D(\tau, r; \mathbf{f}) \neq 0$. Since $\mathcal{L}'_\tau$ is a finite set containing qL_0, it has a maximal element. Writing it rL_0 with r such that $D(\tau, r; \mathbf{f}) \neq 0$, we obtain (20.18) and (20.19) from Theorems 20.4 and 20.7.

20.10. Lemma. *Let $\mathcal{B}$ be as in (20.5); let $\mathbf{f} \in \mathcal{M}_k(C, \psi)$; also let $\mathbf{f} \leftrightarrow (f_p)_{p\in\mathcal{B}}$ and $\mathbf{f}|\mathfrak{T} \leftrightarrow (g_q)_{q\in\mathcal{B}}$ in the sense of §20.1. Further let R^p be a complete set of representatives for $\Gamma^p\backslash(\mathfrak{X} \cap G)$. Then*

$$g_q = \sum_{p\in\mathcal{B}} \sum_{\gamma\in R^p} \psi_{\mathfrak{c}}\big(\det(a(p^{-1}\gamma q))\big)^{-1}(f_p\|\gamma)[\nu_{\mathfrak{b}}(p^{-1}\gamma q)].$$

This was given in [S97, (11.9.1) and (11.11.3)] for forms on a unitary group of a general type. The proof given in [S97, Lemma 11.8, §§11.9 and 11.11] is applicable to the present case. In Case SP the matter is simpler, since we can take $\mathcal{B} = \{1\}$.

20.11. We now assume our $\mathbf{f}$ to be a cusp form. Thus, given an eigenform $\mathbf{f} \in \mathcal{S}_k(C, \psi)$ as in §20.6 and a Hecke character χ of K, we put

$$\Lambda^m_{\mathfrak{c}}(s, \chi) = \begin{cases} L_{\mathfrak{c}}(2s, \chi) \displaystyle\prod_{i=1}^{[m/2]} L_{\mathfrak{c}\mathfrak{h}}(4s-2i, \chi^2) & \text{(Case SP)}, \\ \displaystyle\prod_{i=1}^{m} L_{\mathfrak{c}\mathfrak{h}}(2s-i+1, \chi_1\theta^{i-1}) & \text{(Case UT)}, \end{cases} \tag{20.20}$$

$$\mathcal{Z}(s, \mathbf{f}, \chi) = \prod_{v\in\mathbf{h},\, v\nmid\mathfrak{h}} Z_v\big(\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s}\big), \tag{20.21}$$

$$\mathfrak{T}(s, \mathbf{f}, \chi) = \sum_{\mathfrak{a}+\mathfrak{r}\mathfrak{h}=\mathfrak{r}} \chi^*(\mathfrak{a})\lambda(\mathfrak{a})N(\mathfrak{a})^{-s}. \tag{20.22}$$

Here $L_{\mathfrak{c}}$ is defined by (16.9), χ_1 is the restriction of χ to $F^\times_{\mathbf{A}}$, $\mathfrak{h} = \mathfrak{g}\cap$(the conductor of χ), θ is the Hecke character of F corresponding to K/F, and $Z_v\big(\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s}\big)$ should be understood as follows: In Case SP, it is the expression obtained from Z_v of §20.6 by substituting $\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s}$ for $[\mathfrak{p}]$; similarly in Case UT, it is obtained from Z_v by substituting $\big(\chi^*(\mathfrak{q}_1)N(\mathfrak{q}_1)^{-s}, \chi^*(\mathfrak{q}_2)N(\mathfrak{q}_2)^{-s}\big)$ or $\chi^*(\mathfrak{q})N(\mathfrak{q})^{-s}$ for $([\mathfrak{q}_1], [\mathfrak{q}_2])$ or $[\mathfrak{q}]$ according as $\mathfrak{p}\mathfrak{r} = \mathfrak{q}_1\mathfrak{q}_2$ or $\mathfrak{p}\mathfrak{r} = \mathfrak{q}^e$. From (20.17) we obtain, at least formally,

$$\mathcal{Z}(s, \mathbf{f}, \chi) = \Lambda^{2n}_{\mathfrak{c}}(s/u, \chi)\mathfrak{T}(s, \mathbf{f}, \chi), \quad u = 2/[K:F]. \tag{20.23}$$

20.12. Lemma. (1) *The series of (20.22) and the product of (20.21) are absolutely convergent for* $\mathrm{Re}(s) > 2n+1$ *in Case SP and* $\mathrm{Re}(s) > 2n$ *in Case UT. (This is preliminary to the stronger result given in the following theorem.)*

(2) *If ψ is of finite order and the conditions of Lemma 20.8 are satisfied, then the eigenvalues $\lambda(\mathfrak{a})$ generate a finite algebraic extension of* $\mathbf{Q}$ *that depends on* $\mathbf{f}$. *(As to the nature of this extension, see Lemma 23.15 below.)*

(3) *If $\mathfrak{e} = \mathfrak{c}$, the space $\mathcal{S}_h(C)$ is spanned by eigenfunctions $\mathbf{f}$ of the above type.*

PROOF. As explained in [S97, §20.13], the convergence can be reduced to that of $\sum_{\tau\in C\backslash\mathfrak{X}} |N\big(\mu_{\mathfrak{b}}(\tau)\big)^{-s}|$. Also, as explained in the proof of [S97, Lemma 20.11], $\sum_{\tau\in C\backslash\mathfrak{X}} N\big(\mu_{\mathfrak{b}}(\tau)\big)^{-s}$ has an Euler product, whose Euler factor in Case SP can be obtained by substituting $(N(\mathfrak{p})^{-s}, N(\mathfrak{p})^i)$ for $([\mathfrak{p}], t_i)$ in the expression for $\omega(\mathfrak{T}_v)$ in Theorem 19.8. Then we find that the series is convergent for $\mathrm{Re}(s) > 2n+1$ in Case SP. Similarly, in Case UT, it is convergent for $\mathrm{Re}(s) > 2n$, as already noted in [S97, Proposition 20.4 (3)]. To prove (2), assuming that ψ is of finite order and the conditions of Lemma 20.8 are satisfied, denote by D the field generated over $\mathbf{Q}$ by the values of ψ and the conjugates of K; let $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$. In Lemma 23.14 below we shall establish a nonzero element $\mathbf{f}^\sigma$ of $\mathcal{M}_k(C, \psi)$ such that $\mathbf{f}^\sigma | T(\mathfrak{a}) = \lambda(\mathfrak{a})^\sigma \mathbf{f}^\sigma$ for every $\mathfrak{a}$. Since $\mathcal{M}_k(C, \psi)$ is finite-dimensional, the $\lambda(\mathfrak{a})$ must belong to a finite algebraic extension of D. As for (3), with $\mathfrak{X}$ as in (19.2c) we see that $\tau^{-1} \in \mathfrak{X}$ for every $\tau \in \mathfrak{X}$ if $\mathfrak{e} = \mathfrak{c}$. Since $\mathfrak{R}(C, \mathfrak{X})$ is commutative, [S97, Proposition 11.7] shows that $C\tau C$ for every $\tau \in \mathfrak{X}$ defines a normal operator on $\mathcal{S}_h(C)$. Thus we obtain (3).

We can now state our main theorems about the above Euler product.

20.13. Theorem. *The function $\mathcal{Z}(s, \mathbf{f}, \chi)$ can be continued to a meromorphic function on the whole s-plane. Moreover, the Euler product on the right-hand side of (20.21) is convergent, and consequently $\mathcal{Z}(s, \mathbf{f}, \chi) \neq 0$, at least for* $\mathrm{Re}(s) > (3n/2) + 2 - [K : F]$.

The proof will be completed in §22.9. We shall also show in Theorem 22.11 that the bound $(3n/2) + 2 - [K : F]$ is best possible in general.

If $\mathfrak{e} = \mathfrak{c}$, we can state analytic properties of $\mathcal{Z}$ in a better form; for some technical reasons we denote the weight of $\mathbf{f}$ by h instead of k.

20.14. Theorem. *Let $\mathbf{f}$ be an eigenform contained in $\mathcal{S}_h(C)$ with C as above; suppose that $\mathfrak{e} = \mathfrak{c}$ and $\chi_\mathbf{a}(x) = x_\mathbf{a}^\ell |x_\mathbf{a}|^{i\kappa - \ell}$ with $\ell \in \mathbf{Z}^\mathbf{a}$ and $\kappa \in \mathbf{R}^\mathbf{a}$, $\sum_{v \in \mathbf{a}} \kappa_v = 0$; suppose also that $0 \le h_v - \ell_v \le 1$ for every $v \in \mathbf{a}$ in Case SP. Put*

$$\mathcal{R}(s, \mathbf{f}, \chi) = \mathcal{Z}(s, \mathbf{f}, \chi) \prod_{v \in \mathbf{a}} \Gamma_v^{h,\ell}\big((s/u) + (i\kappa_v/2)\big)$$

with $u = 2/[K : F]$ and $\Gamma_v^{h,\ell}$ defined as follows:

Case SP. $\quad \Gamma_v^{h,\ell}(s) = \Gamma_n^1\big(s + h_v - (\ell_v + n + 1)/2\big) g^n(s, \ell_v),$

$$g^n(s, a) = \begin{cases} \Gamma_n^1\left(s + \dfrac{a-n}{2}\right) \Gamma\left(s + \dfrac{a}{2} - \left[\dfrac{a+n}{2}\right]\right) & \text{if } a \ge n, \\ \Gamma_{2a+2-n}^1\left(s + \dfrac{a-n}{2}\right) \Gamma\left(s - \dfrac{a}{2}\right) \displaystyle\prod_{i=a+2}^{n} \Gamma(2s - i) & \text{if } (n-2)/2 \le a < n; \end{cases}$$

Case UT. $\quad \Gamma_v^{h,\ell}(s) = p_v(s) q^{2n}(s, |2h_{v\rho} + \ell_v|)$

$$\cdot \Gamma_n^2\left(s - n + \frac{h_v + h_{v\rho} + |d_v|}{2}\right) \Gamma_n^2\left(s - n + \frac{|2h_{v\rho} + \ell_v|}{2}\right),$$

$$p_v(s) = \begin{cases} \Gamma_n^2\left(s + \dfrac{|2h_{v\rho} + \ell_v|}{2}\right) \Gamma_n^2\left(s + \dfrac{2h_{v\rho} + \ell_v}{2}\right)^{-1} & \text{if } d_v \ge 0, \\ \Gamma_n^2\left(s - \dfrac{2h_v - \ell_v}{2}\right) \Gamma_n^2\left(s - \dfrac{2h_{v\rho} + \ell_v}{2}\right)^{-1} & \text{if } d_v < 0, \end{cases}$$

$$q^t(s, a) = \prod_{i=1}^{t-a-1} \Gamma\left(s - \frac{a}{2} - \left[\frac{i}{2}\right]\right) \Gamma\left(s - \frac{a}{2} - i\right)^{-1} \qquad (0 < t \in \mathbf{Z}).$$

Here Γ_n^ι is defined by (16.47); $d_v = h_v - h_{v\rho} - \ell_v$ for $v \in \mathbf{a}$ in Case UT. Then $\mathcal{R}(s, \mathbf{f}, \chi)$ can be continued to the whole s-plane as a meromorphic function with *finitely many poles, which are all simple. The set of poles of* $\mathcal{R}(us,\, \mathbf{f},\, \chi)$ *is contained in the set of poles of the function* $\mathcal{P}(s)$ *defined as follows: in Case SP,* $\mathcal{P}$ *is the product of Theorem 16.11 defined with* $\{2n,\, \ell,\, \chi\}$ *as* $\{n,\, k,\, \chi\}$ *there; in Case UT,* $\mathcal{P}$ *is the product given in [S97, Theorem 19.3] defined with* $\{2n,\, 2h_{v\rho} + \ell_v,\, \chi\}$ *as* $\{n,\, k_v,\, \chi\}$ *there.*

Some more precise results concerning the poles of $\mathcal{Z}$ and $\mathcal{R}$ in Case SP are given in [S96, Theorems B1 and B2]. The proof of the above theorem will be completed in Section 25. Notice that $p_v(s)$ and $q^t(s,\, \ell)$ are polynomials in s; in particular, $p_v = 1$ if $0 \le d_v \le h_v + h_{v\rho}$ and $q^t(s,\, a) = 1$ if $a \ge t-1$. By (6.42) we may assume that $h_v \ge n/2$ for every $v \in \mathbf{a}$ in Case SP, which is why $g^n(s,\, a)$ is defined only for $a \ge (n-2)/2$. (Correction to [S97, Theorem 19.3], lines 3$\sim$4 from the bottom: Read "the set described in Case I" for "the set of (19.3.1).")

21. The Euler products for the forms of half-integral weight

21.1. Let us now briefly indicate that the analogues of the theorems of Section 20 can be proved for the forms of half-integral weight. We content ourselves only with giving definitions and making statements without proofs, since such require lengthy calculations; the reade is referred to [S95b] for details. We first put

$$U = \{\, \alpha \in M_{\mathbf{A}} \,|\, \mathrm{pr}(\alpha) \in D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}] \,\}, \tag{21.1}$$

$$\mathcal{Z} = \{\, \alpha \in M_{\mathbf{A}} \,|\, \mathrm{pr}(\alpha) \in \mathfrak{X}_0 \,\}, \quad \mathfrak{X}_0 = D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]Q(\mathfrak{g})D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}], \tag{21.2}$$

where $Q(\mathfrak{g})$ is defined by (19.2c). Let k be a half-integral weight. Given an element $\alpha = \xi_1\sigma\xi_2 \in \mathcal{Z}$ with $\xi_i \in U$ and $\mathrm{pr}(\sigma) \in \mathfrak{X}_0$, we put

$$J^k(\alpha,\, z) = j^k(\xi_1\xi_2,\, z) \qquad (z \in \mathcal{H}) \tag{21.3}$$

with j^k of (16.17).

21.2. Lemma. (1) *$J^k(\alpha,\, z)$ is well-defined.*

(2) *$J^k(\xi,\, z) = j^k(\xi,\, z)$ if $\xi \in U$.*

(3) *$J^k(\xi\alpha\eta,\, z) = J^k(\xi,\, \alpha\eta z)J^k(\alpha,\, \eta z)J^k(\eta,\, z)$ if $\alpha \in \mathcal{Z}$ and $\xi,\, \eta \in U$.*

(4) *$J^k(\alpha,\, z) = j^{[k]}(\mathrm{pr}(\alpha),\, z)J^{k-[k]}(\alpha,\, z)$.*

The proof of (1) requires a nontrivial fact [S95b, Lemma 2.2]; see the first paragraph of [S95b, p.32]. Once this is established, the remaining assertions follow immediately from (21.3).

21.3. We now take C, B', and E' as in §19.1, and put $\Gamma = G \cap C$. This is Γ^1 of §20.1, and $\mathcal{M}_k(\Gamma,\, \psi)$ is meaningful. As noted there, $\mathcal{M}_k(C,\, \psi)$ is isomorphic to $\mathcal{M}_k(\Gamma,\, \psi)$. We put $\mathbf{f} = f_{\mathbf{A}}$ if an element $\mathbf{f}$ of $\mathcal{M}_k(C,\, \psi)$ corresponds to $f \in \mathcal{M}_k(\Gamma,\, \psi)$. Given $q \in B'$, we take a decomposition $G \cap \bigl(C\mathrm{diag}[\widehat{q},\, q]C\bigr) = \bigsqcup_{\alpha \in R} \Gamma\alpha$ with $R \subset G \cap \mathfrak{X}_0$. Then we define $f|T_q$ and $\mathbf{f}|T_q$ by

$$\mathbf{f}|T_q = \bigl(f|T_q\bigr)_{\mathbf{A}}, \quad (f|T_q)(z) = \sum_{\alpha \in R} \psi_{\mathfrak{c}}\bigl(\det(a_\alpha)\bigr)^{-1} J^k(\alpha,\, z)^{-1} f(\alpha z). \tag{21.4}$$

It can easily be seen that $f|T_q$ is well-defined and belongs to $\mathcal{M}_k(\Gamma,\, \psi)$. For an integral $\mathfrak{g}$-ideal $\mathfrak{a}$ we denote by $T(\mathfrak{a})$ the sum of T_q for all $E'qE'$ such that $\det(q)\mathfrak{g} = \mathfrak{a}$, and put $\mathbf{f}|\mathfrak{T} = \sum_{\mathfrak{a}} \bigl(\mathbf{f}|T(\mathfrak{a})\bigr)[\mathfrak{a}]$. By [S95b, Theorem 4.4] we can establish the analogues of $\mathfrak{R}(C_v,\, \mathfrak{X}_v)$ and ω_v in the present case so that $\omega(\mathfrak{T}_v) = 1$ for $v|\mathfrak{c}$ and

(21.5a) $$\omega(\mathfrak{T}_v) = \prod_{i=1}^{n} \frac{1 - q^{2i-1}[\mathfrak{p}^2]}{(1 - q^n t_i[\mathfrak{p}])(1 - q^n t_i^{-1}[\mathfrak{p}])} \qquad (v \nmid \mathfrak{c}),$$

(21.5b) $$\omega(\mathfrak{T}_v) = \prod_{i=1}^{n} (1 - q^n t_i[\mathfrak{p}])^{-1} \qquad (v | \mathfrak{e}^{-1}\mathfrak{c}),$$

where $q = N(\mathfrak{p})$ and $\mathfrak{p}$ is the prime ideal at v. Assuming $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ for every $\mathfrak{a}$, we have complex numbers $\lambda_{v,i}$ with which we have

(21.6a) $$\mathfrak{L} \cdot \sum_{\mathfrak{a}} \lambda(\mathfrak{a})[\mathfrak{a}] = \prod_{v \in \mathbf{h}} Z_v,$$

(21.6b) $$\mathfrak{L} = \prod_{\mathfrak{p} \nmid \mathfrak{c}} \prod_{i=1}^{n} \left(1 - N(\mathfrak{p})^{2i-1}[\mathfrak{p}]^2\right)^{-1},$$

(21.6c) $$Z_v = \begin{cases} 1 & (v | \mathfrak{e}), \\ \prod_{i=1}^{n} \left(1 - N(\mathfrak{p})^n \lambda_{v,i}[\mathfrak{p}]\right)^{-1}, & (v | \mathfrak{e}^{-1}\mathfrak{c}), \\ \prod_{i=1}^{n} \left\{ \left(1 - N(\mathfrak{p})^n \lambda_{v,i}[\mathfrak{p}]\right)\left(1 - N(\mathfrak{p})^n \lambda_{v,i}^{-1}[\mathfrak{p}]\right) \right\}^{-1} & (v \nmid \mathfrak{c}), \end{cases}$$

where $\mathfrak{p}$ is the prime ideal at v.

Given a Hecke character χ of F of conductor $\mathfrak{h}$, we put

(21.7) $$\mathcal{Z}(s, \mathbf{f}, \chi) = \prod_{v \in \mathbf{h},\, v \nmid \mathfrak{h}} Z_v\left(\chi^*(\mathfrak{p}) N(\mathfrak{p})^{-s}\right),$$

where $Z_v\left(\chi^*(\mathfrak{p})N(\mathfrak{p})^{-s}\right)$ is the expression obtained from Z_v by substituting $\chi^*(\mathfrak{p})$ $N(\mathfrak{p})^{-s}$ for $[\mathfrak{p}]$.

21.4. Theorem. *The formulas and assertions of Theorems 20.4, 20.7, 20.9, and 20.13 and Lemmas 20.5 and 20.12 are true for* $\mathbf{f} \in \mathcal{M}_k(C, \psi)$ *with half-integal* k *if we make the following modifications:* $\mathfrak{L}_0 = \prod_{\mathfrak{p} \nmid \mathfrak{c}} \prod_{i=1}^{[n/2]} \left[1 - N(\mathfrak{p})^{2n+1-2i}[\mathfrak{p}]^2\right]^{-1}$; $\alpha_{\mathfrak{c}}^0(\varepsilon_b q^* \tau q)$ *should be replaced by* $\alpha_{\mathfrak{c}}^1({}^t q \tau q)$, *with* $\alpha_{\mathfrak{c}}^1$ *defined by*

$$\alpha_{\mathfrak{c}}^1(\zeta) = \prod_{v \nmid \mathfrak{c}} \alpha_v^1(\zeta, s), \quad \alpha_v^1(\zeta) = \sum_{\sigma \in S_v / S(\mathfrak{g})_v} \omega(\delta_v^{-1}\sigma)\mathbf{e}_v^n(-\delta_v^{-1}\zeta_v\sigma)[\nu_0(\sigma)] \qquad (\zeta \in \widetilde{S}),$$

where $\omega(\delta_v^{-1}\sigma)$ *is as in (16.7b);*

$$A(\tau, L) = \prod_{v \in \mathbf{b}} g_v\left(N(\mathfrak{p})^{-1/2}[\mathfrak{p}]\right) \prod_{v \nmid \mathfrak{c}} \left\{ h_v([\mathfrak{p}])^{-1} \prod_{i=1}^{[(n+1)/2]} \left(1 - N(\mathfrak{p})^{2i-1}[\mathfrak{p}]^2\right) \right\};$$

$\mathbf{b}$ *is the set of all* $v \in \mathbf{h}$ *such that* $v \nmid \mathfrak{c}$ *and* $({}^t q \tau q)_v \notin E_v$; $h_v = 1$ *if* n *is even, and* $h_v(t) = 1 - \rho_\tau^*(\mathfrak{p}) N(\mathfrak{p})^{n/2} t$ *with the Hecke ideal character* ρ_τ^* *of* F *corresponding to* $F(c^{1/2})/F$, $c = (-1)^{(n-1)/2}\det(2\tau)$, *if* n *is odd; in Lemma 20.12 (2),* $\lambda(\mathfrak{a})$ *must be replaced by* $N(\mathfrak{a})^{1/2}\lambda(\mathfrak{a})$.

We add a remark about the last point. Put $T'(\mathfrak{a}) = N(\mathfrak{a})^{1/2}T(\mathfrak{a})$ and $\lambda'(\mathfrak{a}) = N(\mathfrak{a})^{1/2}\lambda(\mathfrak{a})$; take $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$ with D of the proof of Lemma 20.12. Then in Lemma 23.15 below we find an element $\mathbf{f}^\sigma$ of $\mathcal{M}_k(C, \psi)$ such that $\mathbf{f}^\sigma | T'(\mathfrak{a}) = \lambda'(\mathfrak{a})^\sigma \mathbf{f}^\sigma$. Therefore the assertion must be formulated in terms of $N(\mathfrak{a})^{1/2}\lambda(\mathfrak{a})$.

21.5. Theorem. *Let* $\mathbf{f}$ *be an eigenform contained in* $\mathcal{S}_k(C)$ *with* C *as above; suppose that* $\mathfrak{e} = \mathfrak{c}$ *and* $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{[k]-e}|x_{\mathbf{a}}|^{i\kappa-[k]+e}$ *with* $e \in \mathbf{Z}^{\mathbf{a}}$ *and* $\kappa \in \mathbf{R}^{\mathbf{a}}$ *such that* $0 \le e_v \le 1$ *for every* $v \in \mathbf{a}$ *and* $\sum_{v \in \mathbf{a}} \kappa_v = 0$. *Put*

$$\mathcal{R}(s,\,\mathbf{f},\,\chi) = \mathcal{Z}(s,\,\mathbf{f},\,\chi)\prod_{v\in\mathbf{a}} \Gamma_v^{k,e}\big((s+i\kappa_v)/2\big)$$

with $\Gamma_v^{k,e}$ defined as follows:

$$\Gamma_v^{k,e}(s) = \Gamma_n^1\big(s+(k_v+e_v-n-1)/2\big)g\big(s,\,k_v-e_v\big),$$

$$g(s,\,a) = \begin{cases} \Gamma_n^1\left(s+\dfrac{a-n}{2}\right) & \text{if } \ a>n, \\ \Gamma_{2a+1-n}^1\left(s+\dfrac{a-n}{2}\right)\displaystyle\prod_{i=[a]+1}^{n-1}\Gamma\left(2s-a-\frac{1}{2}\right) & \text{if } \ (n-2)/2<a\le n, \\ \Gamma\left(s-\dfrac{n-2}{4}\right)\displaystyle\prod_{i=(n+1)/2}^{n-1}\Gamma\left(2s-a-\frac{1}{2}\right) & \text{if } \ a=(n-2)/2. \end{cases}$$

Then $\mathcal{R}(s,\,\mathbf{f},\,\chi)$ can be continued meromorphically to the whole s-plane with finitely many poles. Moreover each pole is simple. In particular, $\mathcal{R}$ is an entire function of s if $\chi^2\neq 1$. If $\chi^2=1$, the poles are determined as follows: Let $\ell = \mathrm{Max}_{v\in\mathbf{a}}\{\,k_v-e_v\,\}$. If $\ell>n$, $\mathcal{R}$ has no pole. If $\ell<n$, $\mathcal{R}$ has possible poles only in the set $\{\,(2j+1)/2\,|\,j\in\mathbf{Z},\ n+1\le j\le 2n+(1/2)-\ell\,\}$.

22. The largest possible pole of $\mathcal{Z}(s,\,\mathbf{f},\,\chi)$

22.1. Put $\lambda_n=(n+1)/2$ in Case SP and $\lambda_n=n$ in Case UT. We consider the set S of (16.1a) and put

(22.1a) $$S^+ = \{\,\xi\in S\,|\,\xi_v>0 \quad \textit{for every} \quad v\in\mathbf{a}\,\},$$

(22.1b) $$S_{\mathbf{a}}^+ = \prod_{v\in\mathbf{a}} S_v^+, \qquad S_v^+ = \{\,\xi\in S_v\,|\,\xi>0\,\}.$$

Given a subgroup U of $GL_n(K)$, we define an equivalence relation $\sim$ in S^+ with respect to U by: $\sigma\sim\sigma'$ if and only if $\sigma'=a^*\sigma a$ with $a\in U$. We then denote by S^+/U the set of all equivalence classes in this sense.

Take $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{e}$, and C as in §19.1; define E' by (19.2b). For $q\in GL_n(K)_{\mathbf{h}}$ and $\sigma\in S^+$ put

(22.2a) $$U_q = GL_n(K)\cap qE^*q^{-1}, \qquad E^* = GL_n(K)_{\mathbf{a}}E',$$

(22.2b) $$\nu_{\sigma,q} = [U_{\sigma,\,q}:1]^{-1}, \qquad U_{\sigma,\,q} = \{\,a\in U_q\,|\,a^*\sigma a=\sigma\,\}.$$

Let $\mathbf{f}\in\mathcal{S}_k(C,\,\psi)$ and $\mathbf{g}\in\mathcal{M}_l(C',\,\psi')$ with integral or half-integral k and l. Here C' is a group of the same type as C with possibly different $\mathfrak{b}$, $\mathfrak{c}$, and $\mathfrak{e}$; ψ and ψ' are Hecke characters of K. We consider the Fourier coefficients $c_{\mathbf{f}}$ and $c_{\mathbf{g}}$ in the sense of Proposition 20.2. We assume that

(22.3a) $$\psi'_{\mathbf{h}}\big(\det(a)\big)c_{\mathbf{g}}(\sigma,\,qa) = c_{\mathbf{g}}(\sigma,\,q) \quad \textit{for every } a\in E',$$

(22.3b) $$(\psi/\psi')_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t}|x_{\mathbf{a}}|^{t-i\kappa} \quad \textit{with} \quad \kappa\in\mathbf{R}^{\mathbf{a}},\ \textstyle\sum_{v\in\mathbf{a}}\kappa_v=0, \text{ where}$$

(22.3c) $$t = \begin{cases} [l]-[k] & \text{(Case SP)}, \\ (l_v-l_{v\rho}-k_v+k_{v\rho})_{v\in\mathbf{a}} & \text{(Case UT)}. \end{cases}$$

Notice that (20.9d) impies (22.3a) with $\mathbf{f}$ and ψ in place of $\mathbf{g}$ and ψ'. We then define, for $s\in\mathbf{C}$,

(22.4) $$D_{q,\kappa}(s;\,\mathbf{f},\,\mathbf{g}) = \sum_{\sigma\in S^+/U_q}\nu_{\sigma,q}c_{\mathbf{f}}(\sigma,\,q)\overline{c_{\mathbf{g}}(\sigma,\,q)}\det(\sigma)^{-s\mathbf{a}-h}, \quad \text{where}$$

$$\text{(22.4a)} \qquad h = \begin{cases} (1/2)(k+l-i\kappa) & \text{(Case SP)}, \\ (1/2)(k_v + k_{v\rho} + l_v + l_{v\rho} - i\kappa_v)_{v\in\mathbf{a}} & \text{(Case UT)}. \end{cases}$$

This is well-defined. Indeed, if $a \in U_q$, then $\det(a) \in \mathfrak{r}^\times$ and $c_\mathbf{f}(a^*\tau a, q) = \det(a)^{[k]\rho} \cdot c_\mathbf{f}(\tau, aq)$ by (20.9b, c). Since $aq = qq^{-1}aq$ and $q^{-1}aq \in E^*$, by (20.9b, d) we have

$$\text{(22.5)} \qquad c_\mathbf{f}(a^*\tau a, q) = \psi_\mathbf{a}(\det(a)) \det(a)^{[k]\rho} c_\mathbf{f}(\tau, q) \ \text{ for every } a \in U_q.$$

A similar relation holds for $c_\mathbf{g}$. Combining these, we easily see that each term of (22.4) depends only on the equivalence class of σ under U_q. Now the right-hand side of (22.4) is convergent for sufficiently large $\mathrm{Re}(s)$. This will be proven in §A6.7.

22.2. Proposition. *Define $m, m' \in \mathbf{Z}^\mathbf{a}$ as follows: $m = k$ and $m' = l$ in Case SP; $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ and $m' = (l_v + l_{v\rho})_{v\in\mathbf{a}}$ in Case UT. Then the following assertions hold:*

(1) *$D_{q,\kappa}(s;\, \mathbf{f},\, \mathbf{g})$ can be continued to a meromorphic function on the whole plane, which is holomorphic for $\mathrm{Re}(s) > 0$. Moreover it is holomorphic at $s = 0$ if $m \neq m'$ or $\kappa \neq 0$.*

(2) *The right-hand side of (22.4) is absolutely convergent for $\mathrm{Re}(s) > 0$ if $\mathbf{g}$ is a cusp form.*

(3) *Let $p = \mathrm{diag}[q, \hat{q}]$; let f_p and g_p be the p-components of $\mathbf{f}$ and $\mathbf{g}$, respectively. If $m = m'$ and $\kappa = 0$, then $D_{q,0}(s;\, \mathbf{f},\, \mathbf{g})$ has at most a simple pole at $s = 0$ whose residue is a positive number times $\langle g_p, f_p \rangle$.*

The proof will be completed in §22.4.

22.3. Let f_p and g_p be as in (3) above; take $\beta = 1$ and $r = q$ in (20.9e). Then

$$f_p(z) = \sum_{\sigma\in S_+} c_\mathbf{f}(\sigma, q)\mathbf{e}^n_\mathbf{a}(\sigma z), \qquad g_p(z) = \sum_{\sigma\in S_+} c_\mathbf{g}(\sigma, q)\mathbf{e}^n_\mathbf{a}(\sigma z).$$

Take a congruence subgroup Γ of G_1 so that $f_p \in \mathcal{M}_k(\Gamma)$ and $g_p \in \mathcal{M}_l(\Gamma)$; take also an $\mathfrak{r}$-lattice L in S so that $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \in \Gamma$ for every $b \in L$. Let $X = S_\mathbf{a}/L$ and $Y = S^+_\mathbf{a}/U_q$, the latter being defined modulo the map $y \mapsto a\sigma a^*$ for $a \in U_q$. Write the variable z on $\mathcal{H}^n$ in the form $z = x + iy$ with $x \in S_\mathbf{a}$ and $y \in S^+_\mathbf{a}$; define a measure dx on $S_\mathbf{a}$ by $dx = \prod_{v\in\mathbf{a}} dx_v$ with dx_v defined on S_v as in §5.12. Clearly

$$\text{(22.6)} \qquad \int_X f_p(x+iy)\overline{g_p(x+iy)}dx = \mathrm{vol}(X) \sum_{\sigma\in S^+} c_\mathbf{f}(\sigma, q)\overline{c_\mathbf{g}(\sigma, q)}\mathbf{e}^n_\mathbf{a}(2i\sigma y).$$

We recall a well-known formula

$$\text{(22.7)} \qquad \int_{S^+_v} e^{-\mathrm{tr}(y\tau)} \det(y)^{s-\lambda_n} dy = \Gamma^\iota_n(s) \det(\tau)^{-s} \quad (\mathrm{Re}(s) > \lambda_n - 1; \tau \in S^+_v),$$

where Γ^ι_n is defined by (16.47). Put

$$H(y) = \sum_{\sigma\in\mathcal{T}} \nu_{\sigma,q} M(\sigma,\, y), \quad M(\sigma,\, y) = c_\mathbf{f}(\sigma, q)\overline{c_\mathbf{g}(\sigma, q)}\mathbf{e}^n_\mathbf{a}(2i\sigma y) \det(y)^{s\mathbf{a}+h},$$

where $\mathcal{T}$ is a fixed complete set of representatives for S^+/U_q. Since $M(a^*\sigma a,\, y) = M(\sigma,\, aya^*)$ for every $a \in U_q$ by (22.5), putting $d'y = \det(y)^{-\lambda_n\mathbf{a}}dy$, we have

$$\int_Y \sum_{\sigma\in S^+} M(\sigma,\, y)d'y = \int_Y \sum_{a\in U_q} \sum_{\sigma\in\mathcal{T}} \nu_{\sigma,q} M(a^*\sigma a, y)d'y = \int_Y \sum_{a\in U_q} H(aya^*)d'y$$

$$= \nu \int_{S_{\mathbf{a}}^+} H(y)d'y = \nu \sum_{\sigma \in \mathcal{T}} \nu_{\sigma,q} \int_{S_{\mathbf{a}}^+} M(\sigma, y)d'y = \nu \cdot \Gamma((s))D_{q,\kappa}(s; \mathbf{f}, \mathbf{g}),$$
$$\text{where } \Gamma((s)) = \prod_{v \in \mathbf{a}} \Gamma_n^\iota(s + h_v)(4\pi)^{-n(s+h_v)}$$

and ν is the number of the roots of unity ζ in K such that $\zeta - 1 \in \mathfrak{e}$; we employed (22.7) in the last step. Termwise integration can be justified for sufficiently large $\mathrm{Re}(s)$ because of the convergence of (22.4). Combining this with (22.6), we obtain

$$\nu \cdot \mathrm{vol}(X)\Gamma((s))D_{q,\kappa}(s; \mathbf{f}, \mathbf{g}) = \int_X \int_Y f_p(x+iy)\overline{g_p(x+iy)}\delta(z)^{h+s\mathbf{a}-\lambda_n\mathbf{a}}dxdy$$

for sufficiently large $\mathrm{Re}(s)$, where $\delta(x+iy) = \big(\det(y_v)\big)_{v\in\mathbf{a}}$. Put $\Gamma^P = \Gamma \cap P$ as in (17.2). Then $\begin{bmatrix} a & b \\ 0 & \widehat{a} \end{bmatrix} \in \Gamma^P$ for every $b \in L$ and every a in a suitable subgroup U' of U_q of finite index. Therefore $X \times Y$ is "commensurable with" $\Gamma^P\backslash\mathcal{H}$. Now we have $\delta^{-2\lambda_n\mathbf{a}}dxdy = c\mathbf{d}z = c\prod_{v\in\mathbf{a}}\mathbf{d}z_v$ with $\mathbf{d}z_v$ as in Lemma 3.4; $c = 1$ in Case SP and $c = 2^{n(1-n)[F:\mathbf{Q}]}$ in Case UT as explained in §5.12. Thus the last double integral is a positive rational number times

$$(22.8) \qquad \int_Z f_p(z)\overline{g_p(z)}\delta(z)^{h+s\mathbf{a}+\lambda_n\mathbf{a}}\mathbf{d}z \qquad (Z = \Gamma^P\backslash\mathcal{H}^n).$$

Clearly $j_\gamma^{k-l}j_\gamma^l = j_\gamma^k$ if both k and l are integral, but the case of half-integral weights is not so simple. However, by (16.17) and Theorem 6.8 (5), the equality always holds for γ in a suitable congruence subgroup Γ of G_1. With this choice of Γ, we have $j_\gamma^k\overline{j_\gamma^l} = (\overline{j_\gamma^{k-l}})^{-1}|j_\gamma^k|^2$ for $\gamma \in \Gamma$. Thus, putting $\mathcal{F}(z) = f_p(z)\overline{g_p(z)}\delta^{h+s'\mathbf{a}}$ with $s' = s + \lambda_n$ and defining m and m' as in our proposition, we find that

$$\mathcal{F} \circ \gamma = \mathcal{F} \cdot |j_\gamma|^{2m-2h-2s'\mathbf{a}}\overline{(j_\gamma^{k-l})}^{\,-1} \text{ for every } \gamma \in \Gamma.$$

Let $\mathfrak{D} = \Gamma\backslash\mathcal{H}^n$ and $\mathcal{R} = \Gamma^P\backslash\Gamma$. Choosing Γ suitably, we may assume that $\Gamma \cap K^\times = \{1\}$. Then (22.8) equals

$$\int_Z \mathcal{F}(z)\mathbf{d}z = \int_{\mathfrak{D}} \sum_{\gamma\in\mathcal{R}} \mathcal{F}(\gamma(z))\mathbf{d}z = \int_{\mathfrak{D}} \mathcal{F}(z)\Big\{\sum_{\gamma\in\mathcal{R}} |j_\gamma|^{2m-2h-2s'\mathbf{a}}\overline{(j_\gamma^{k-l})}^{\,-1}\Big\}\mathbf{d}z.$$

The last sum over $\mathcal{R}$ can be written $\delta^{m-h-s'\mathbf{a}}\overline{E(z, \bar{s}+\lambda_n; m-m', \kappa, \Gamma)}$ with $E(\cdots)$ of (17.3). Thus

$$(22.9) \quad \Gamma((s))D_{q,\kappa}(s; \mathbf{f}, \mathbf{g}) = A\int_{\mathfrak{D}} f_p(z)\overline{g_p(z)}\,\overline{E(z, \bar{s}+\lambda_n; m-m', \kappa, \Gamma)}\delta(z)^m\,\mathbf{d}z.$$

with a constant A, which is a positive rational number times $\mathrm{vol}(X)^{-1}$.

22.4. *Proof of Proposition 22.2.* By Lemma 17.2 (4), $E(z, s; m-m', \kappa, \Gamma)$ is holomorphic for $\mathrm{Re}(s) > \lambda_n$. Moreover, by a well-known principle, if it has a pole of order t at s_0, then $(s-s_0)^tE$ as a function of z is slowly increasing at every cusp. If $\mathbf{f} \neq 0$, (6.42) shows that $m_v > \lambda_n - 1$ for every $v \in \mathbf{a}$, and hence $\prod_{v\in\mathbf{a}}\Gamma_n^\iota(m_v)^{-1} \neq 0$. Therefore (1) and (3) follow immediately from (22.9) combined with Lemma 17.2 (4). To prove (2), we observe that if the series for $D_{q,0}(s; \mathbf{f}, \mathbf{f})$ and $D_{q,0}(s; \mathbf{g}, \mathbf{g})$ for real s are convergent, then the Cauchy-Schwarz inequality gives

$$\sum_{\sigma\in S^+/U_q} \nu_{\sigma,q}\big|c_{\mathbf{f}}(\sigma, q)c_{\mathbf{g}}(\sigma, q)\det(\sigma)^{-s\mathbf{a}-h}\big| \le \big[D_{q,0}(s; \mathbf{f}, \mathbf{f})D_{q,0}(s; \mathbf{g}, \mathbf{g})\big]^{1/2},$$

and therefore assertion (2) in the general case follows from the special case $\mathbf{f} = \mathbf{g}$. Now we know that $D_{q,0}(s; \mathbf{f}, \mathbf{f})$ is holomorphic for $\mathrm{Re}(s) > 0$. Since all its coefficients are nonnegative, the series defining $D_{q,0}(s;\, \mathbf{f},\, \mathbf{f})$ must be convergent for $\mathrm{Re}(s) > 0$. This completes the proof.

22.5. Now given $\mathbf{f} \in \mathcal{S}_k(C, \psi)$, a Hecke character χ of K, $\tau \in S^+$, and $r \in GL_n(K)_\mathbf{h}$, we define a formal Dirichlet series $\mathcal{D}(\mathbf{f}, \chi)$ and an ordinary Dirichlet series $D(s, \mathbf{f}, \chi)$ by

$$\mathcal{D}(\mathbf{f}, \chi) = \sum_{x \in B'/E'} \psi\bigl(\det(rx)\bigr)\chi^*\bigl(\det(x)\mathfrak{r}\bigr)c_\mathbf{f}(\tau, rx)|\det(xx^*)|_F^{-\lambda_n}[\det(x)\mathfrak{r}], \tag{22.10}$$

$$D(s, \mathbf{f}, \chi) = \sum_{x \in B'/E'} \psi\bigl(\det(rx)\bigr)\chi^*\bigl(\det(x)\mathfrak{r}\bigr)c_\mathbf{f}(\tau, rx)|\det(xx^*)|_F^{-\lambda_n}|\det(x)|_K^s, \tag{22.11}$$

where B' and E' are as in (19.2b), and $|\ \ |_F$ is the idele norm in $F_\mathbf{A}^\times$. These depend on r and τ, but we fix them in the following treatment. Let

$$GL_n(K)_\mathbf{A} = \bigsqcup_{q \in Q} GL_n(K)qE^* \tag{22.12}$$

with a finite subset Q of $GL_n(K)_\mathbf{h}$; put then

$$X_q = GL_n(K) \cap rB^*q^{-1}, \quad X_{\sigma,q} = \{\, \xi \in X_q \mid \sigma = \xi^*\tau\xi \,\} \quad (q \in Q,\ \sigma \in S^+),$$
$$B^* = B' \cdot GL_n(K)_\mathbf{a}.$$

Given $x \in B^*$, take $\xi \in GL_n(K)$ and $q \in Q$ so that $rx \in \xi qE^*$. Then $x \in r^{-1}\xi qE^*$ and $\xi \in X_q$. From this we easily see that B'/E' can be given by $\bigsqcup_{q\in Q}\{\, r^{-1}\xi_\mathbf{h} q \mid \xi \in X_q/U_q \,\}$. Therefore we have

$$\mathcal{D}(\mathbf{f}, \chi) = \sum_{q \in Q} |\det(r^{-1}\widehat{r}qq^*)|_F^{-\lambda_n} \cdot \sum_{\xi \in X_q/U_q} c_\mathbf{f}(\tau, \xi q)\psi_\mathbf{h}\bigl(\det(\xi q)\bigr)\chi^*\bigl(\det(r^{-1}\xi q)\mathfrak{r}\bigr)|\det(\xi)|^{2\lambda_n\mathbf{a}}[\det(r^{-1}\xi q)\mathfrak{r}]. \tag{22.13}$$

22.6. Take $\mu \in \mathbf{Z}^\mathbf{b}$ under the following conditions: $\mu_v \geq 0$ for every $v \in \mathbf{b}$; $\mu_v \leq 1$ for every $v \in \mathbf{a}$ in Case SP; $\mu_v\mu_{v\rho} = 0$ for every $v \in \mathbf{a}$ in Case UT. Given χ as above, let $\mathfrak{f}$ be the conductor of χ. We define a weight l and a Hecke character ψ' of K by

$$l = \begin{cases} \mu + (n/2)\mathbf{a} & \text{(Case SP)}, \\ \mu + n\mathbf{a} & \text{(Case UT)}, \end{cases} \tag{22.14a}$$

$$\psi' = \begin{cases} \chi^{-1}\rho_\tau & \text{(Case SP)}, \\ \chi^{-1}\varphi^{-n} & \text{(Case UT)}. \end{cases} \tag{22.14b}$$

Here ρ_τ is the Hecke character of F corresponding to the extension $F(c^{1/2})/F$ with $c = (-1)^{[n/2]}\det(2\tau)$; φ is a Hecke character of K such that $\varphi(y) = y^{-\mathbf{a}}|y|^\mathbf{a}$ for $y \in K_\mathbf{a}^\times$ and the restriction of φ to $F_\mathbf{A}^\times$ is the Hecke character of F corresponding to K/F. Such a φ always exists, but not necessarily unique; see Lemma A5.1. We fix one such φ in the following treatment.

As will be shown in §A5.5, there exists an element $\mathbf{g} \in \mathcal{M}_l(C', \psi')$ such that

$$c_\mathbf{g}(\sigma, q) = |\det(q)|_K^{n/2}\psi'\bigl(\det(q)\bigr)^{-1} \cdot \sum_{\xi \in X_{\sigma,q}} \overline{\chi}_\mathbf{a}\bigl(\det(\xi)\bigr)\overline{\chi}^*\bigl(\det(r^{-1}\xi q)\mathfrak{r}\bigr)\det(\xi)^{\mu\rho} \tag{22.15}$$

for $\sigma \in S^+$, where

$$C' = \{\, \alpha \in D[\mathfrak{b}_1^{-1}\mathfrak{e}, \mathfrak{b}_1\mathfrak{c}_1] \mid a_\alpha - 1 \prec \mathfrak{r}\mathfrak{e} \,\}$$

with the same $\mathfrak{e}$ as before and some $\mathfrak{b}_1$ and $\mathfrak{c}_1$; $\mathfrak{c}_1$ is divisible by $\mathfrak{e}$, $\mathfrak{f}$, and the conductor of ρ_τ or φ.

We take this $\mathbf{g}$ to be that in (22.4); naturally (22.3b, c) must be satisfied, which is so only for a suitable choice of μ. In fact, given $\mathbf{f} \in \mathcal{S}_k(C, \psi)$ and χ, define ψ' as above, and put $(\psi\chi)_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'-i\kappa}$ with $t' \in \mathbf{Z}^{\mathbf{a}}$ and $\kappa \in \mathbf{R}^{\mathbf{a}}$. Then $(\psi/\psi')_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t}|x_{\mathbf{a}}|^{t-i\kappa}$ with $t = t' + [n/2]\mathbf{a}$ in Case SP and $t = t' + n\mathbf{a}$ in Case UT. Now take $\mu \in \mathbf{Z}^{\mathbf{b}}$ as follows: In Case SP define μ_v by the conditions $0 \le \mu_v \le 1$ and $\mu - [k] - t' \in 2\mathbf{Z}^{\mathbf{a}}$. In Case UT, put

$$\text{(22.15a)} \qquad \mu_v = t'_v - k_{v\rho} + k_v \quad \text{and} \quad \mu_{v\rho} = 0 \quad \text{if} \quad t'_v \ge k_{v\rho} - k_v,$$

$$\text{(22.15b)} \qquad \mu_v = 0 \quad \text{and} \quad \mu_{v\rho} = k_{v\rho} - k_v - t'_v \quad \text{if} \quad t'_v < k_{v\rho} - k_v.$$

We consider $\mathbf{g}$ with this μ. Then (22.3b, c) are satisfied.

Let $\mathcal{D}_q(\mathbf{f}, \mathbf{g})$ denote the formal Dirichlet series obtained from the right-hand side of (22.4) with $\det(\sigma)^{-h}[\det(\sigma)\mathfrak{r}]$ in place of $\det(\sigma)^{-s\mathbf{a}-h}$. Then

$$\text{(22.16)} \quad \det(\tau)^h[\det(\tau)\mathfrak{r}]^{-1}|\det(q)|_K^{-n/2}\psi'\big(\det(q)^{-1}\big)\mathcal{D}_q(\mathbf{f}, \mathbf{g})$$
$$= \sum_{\xi \in X_q/U_q} \psi_{\mathbf{h}}\big(\det(\xi)\big)\chi^*\big(\det(r^{-1}\xi q)\mathfrak{r}\big)c_{\mathbf{f}}(\tau, \xi q)|\det(\xi)|^{-(n/u)\mathbf{a}}[\det(\xi\xi^*)\mathfrak{r}],$$

where $u = 2/[K : F]$. To see this, take complete sets of representatives R and $Y_{\sigma,q}$ for S^+/U_q and $X_{\sigma,q}/U_{\sigma,q}$, respectively. Then we easily see that $\bigsqcup_{\sigma\in R} Y_{\sigma,q}$ gives X_q/U_q. In Case UT, for $x \in K^\times$ we can easily verify that

$$\text{(22.17)} \qquad (\psi\chi)_{\mathbf{a}}(x)^{-1}x^{-k\rho} = x^{t'-k\rho}|x|^{i\kappa-t'} = x^\mu|x|^{n\mathbf{a}-2h}.$$

Here we have to remember that $k\rho$ and h belong to $\mathbf{Z}^{\mathbf{b}}$; t' and h ar elements of $\mathbf{Z}^{\mathbf{a}}$ viewed also as elements of $\mathbf{Z}^{\mathbf{b}}$. To make our formulas short, let us put ${}^d\xi = \det(\xi)$ temporarily. Then the sum over X_q/U_q in (22.16) in Case UT equals $\sum_{\sigma\in R}\nu_{\sigma,q}A_\sigma$ with

$$\begin{aligned} A_\sigma &= \textstyle\sum_{\xi\in X_{\sigma,q}} \psi_{\mathbf{a}}({}^d\xi)^{-1}c_{\mathbf{f}}(\tau, \xi q)|{}^d\xi|^{-n\mathbf{a}}\chi^*\big({}^d(r^{-1}\xi q)\mathfrak{r}\big)[{}^d(\xi\xi^*)\mathfrak{r}] \\ &= \textstyle\sum_\xi \psi_{\mathbf{a}}({}^d\xi)^{-1}c_{\mathbf{f}}(\sigma, q)({}^d\xi)^{-k\rho}|{}^d\xi|^{-n\mathbf{a}}\chi^*\big({}^d(r^{-1}\xi q)\mathfrak{r}\big)[{}^d(\xi\xi^*)\mathfrak{r}] \\ &= c_{\mathbf{f}}(\sigma, q)\textstyle\sum_\xi (\psi\chi)_{\mathbf{a}}({}^d\xi)^{-1}({}^d\xi)^{-k\rho}|{}^d\xi|^{-n\mathbf{a}}\chi_{\mathbf{a}}({}^d\xi)\chi^*\big({}^d(r^{-1}\xi q)\mathfrak{r}\big)[{}^d(\xi\xi^*)\mathfrak{r}] \\ &= |{}^d(\tau^{-1}\sigma)|^{-h}\big[{}^d(\tau^{-1}\sigma)\mathfrak{r}]c_{\mathbf{f}}(\sigma, q)\textstyle\sum_\xi \chi_{\mathbf{a}}({}^d\xi)\chi^*\big({}^d(r^{-1}\xi q)\mathfrak{r}\big)({}^d\xi)^\mu, \end{aligned}$$

where we employed (22.17) in the last step. By (22.15) the last sum over ξ equals $|\det(q)|_K^{-n/2}\psi'\big(\det(q)\big)^{-1}\overline{c_{\mathbf{g}}(\sigma, q)}$. Therefore we obtain (22.16) in Case UT. Case SP can be handled in a similar and simpler way.

Now (22.16) together with (22.13) gives

$$\text{(22.18a)} \qquad \mathcal{D}\big(N(\mathfrak{a})^{-s_0}[\mathfrak{a}\mathfrak{a}^\rho]\big)$$
$$= \det(\tau)^h|\det(r)|_K^{-n/2}\sum_{q\in Q}(\psi/\psi')\big(\det(q)\big)[\det(r^{-1}\widehat{r}qq^*\tau^{-1})\mathfrak{r}]\mathcal{D}_q(\mathbf{f}, \mathbf{g}),$$

$$\text{(22.18b)} \qquad D(us + s_0, \mathbf{f}, \chi)$$
$$= \det(\tau)^{s\mathbf{a}+h}|\det(r)|_K^{-us-n/2}\sum_{q\in Q}(\psi/\psi')\big(\det(q)\big)|\det(qq^*)|_F^s D_{q,\kappa}(s; \mathbf{f}, \mathbf{g}),$$

where $u = 2/[K : F]$, $s_0 = (3n/2) + u - 1$, and the left-hand side of (22.18a) means the formal series obtained from $\mathcal{D}(\mathbf{f}, \chi)$ by substituting $N(\mathfrak{a})^{-s_0}[\mathfrak{a}\mathfrak{a}^\rho]$ for $[\mathfrak{a}]$.

22.7. Lemma. *Given a formal Dirichlet series $\sum_{\mathfrak{a}\subset\mathfrak{r}} c_{\mathfrak{a}}[\mathfrak{a}]$ with $c_{\mathfrak{a}} \in \mathbf{C}$, suppose that $\sum_{\mathfrak{a}\subset\mathfrak{r}} \left|c_{\mathfrak{a}}N(\mathfrak{a})^{-s}\right| < \infty$ for $\mathrm{Re}(s) > \alpha$ and also that it can be decomposed formally into an Euler product in the sense that $\sum_{\mathfrak{a}} c_{\mathfrak{a}}[\mathfrak{a}] = \prod_{\mathfrak{p}} V_{\mathfrak{p}}\big([\mathfrak{p}]\big)^{-1}$ with complex polynomials $V_{\mathfrak{p}}(x)$ such that $V_{\mathfrak{p}}(0) = 1$ defined for all prime ideals $\mathfrak{p}$ in K. Then the infinite product $\prod_{\mathfrak{p}} V_{\mathfrak{p}}\big(\varphi(\mathfrak{p})N(\mathfrak{p})^{-s}\big)^{-1}$, for any $\mathbf{T}$-valued ideal character φ, is convergent to the Dirichlet series $\sum_{\mathfrak{a}\subset\mathfrak{r}} c_{\mathfrak{a}}\varphi(\mathfrak{a})N(\mathfrak{a})^{-s}$ and nonvanishing for $\mathrm{Re}(s) > \alpha$.*

PROOF. We can formally put $V_{\mathfrak{p}}(x)^{-1} = 1 + \sum_{n=1}^{\infty} b_{\mathfrak{p},n}x^n$ with $b_{\mathfrak{p},n} \in \mathbf{C}$. Then $\sum_{\mathfrak{p}}\sum_{n=1}^{\infty} b_{\mathfrak{p},n}N(\mathfrak{p})^{-ns}$ is a partial series of $\sum_{\mathfrak{a}} c_{\mathfrak{a}}N(\mathfrak{a})^{-s}$ and therefore $\sum_{\mathfrak{p}}\sum_{n=1}^{\infty} |b_{\mathfrak{p},n}\varphi(\mathfrak{p})^nN(\mathfrak{p})^{-ns}| < \infty$ for $\mathrm{Re}(s) > \alpha$. Thus the infinite product must be convergent. Clearly each factor $V_{\mathfrak{p}}\big(\varphi(\mathfrak{p})N(\mathfrak{p})^{-s}\big)^{-1}$ is nonvanishing, and hence we obtain our lemma.

22.8. Lemma. *Given (ψ, τ) as above and an integral ideal $\mathfrak{x}$ in F, there exists a Hecke character χ and $\mu \in \mathbf{Z}^{\mathbf{b}}$ with the following properties:* (i) *our discussion of §22.6 is applicable to (χ, μ);* (ii) *$\mu \neq 0$;* (iii) *the conductor of χ is prime to $\mathfrak{x}$.*

PROOF. In Case SP take $\mu \in \mathbf{Z}^{\mathbf{a}}$, $\neq 0$, so that $0 \le \mu_v \le 1$, and put $t' = \mu - [k]$; in Case UT take $\mu \in \mathbf{Z}^{\mathbf{b}}$, $\neq 0$, so that $\mu_v \ge 0$ and $\mu_{v\rho} = 0$ for every $v \in \mathbf{a}$, and put $t' = (\mu_v - k_v + k_{v\rho})_{v\in\mathbf{a}}$. We can find an integral $\mathfrak{r}$-ideal $\mathfrak{a}$ prime to $\mathfrak{x}$ so that if ζ is a root of unity in K and $\zeta - 1 \in \mathfrak{a}$, then $\zeta = 1$. Put $W = \{e \in \mathfrak{r}^\times \mid e - 1 \in \mathfrak{a}\}$. Then the map $e \mapsto (\log|e_v|)_{v\in\mathbf{a}}$ gives an isomorphism of W onto a $\mathbf{Z}$-lattice in $\{x \in \mathbf{R}^{\mathbf{a}} \mid \sum_{v\in\mathbf{a}} x_v = 0\}$. Now $e \mapsto \psi_{\mathbf{a}}(e)^{-1}|e|^{t'}e^{-t'}$ defines a $\mathbf{T}$-valued character of W. Therefore we can easily find $\kappa \in \mathbf{R}^{\mathbf{a}}$ such that $\sum_{v\in\mathbf{a}} \kappa_v = 0$ and $\psi_{\mathbf{a}}(e)^{-1}|e|^{t'}e^{-t'} = |e|^{i\kappa}$ for every $e \in W$. Now $\psi_{\mathbf{a}}(x) = x^c|x|^{-c-i\kappa'}$ with $c \in \mathbf{Z}^{\mathbf{a}}$ and $\kappa' \in \mathbf{R}^{\mathbf{a}}$, $\sum_{v\in\mathbf{a}} \kappa'_v = 0$. Therefore, by [S97, Lemma 11.14 (1)] there exists a Hecke character χ of K whose conductor divides $\mathfrak{a}$ and such that $\chi_{\mathbf{a}}(x) = \psi_{\mathbf{a}}(x)^{-1}x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'-i\kappa}$. This proves our lemma.

22.9. We are now ready to prove Theorem 20.13. Given an eigenform $\mathbf{f} \in \mathcal{S}_k(C, \psi)$, take μ and χ as in Lemma 22.8 with an arbitrary choice of $\mathfrak{x}$. Since $\mathbf{f}$ is a cusp form, by Lemma 20.8 and Theorem 20.9 we can find $r \in GL_n(K)_{\mathbf{h}}$ and $\tau \in S^+$ such that $c_{\mathbf{f}}(\tau, r) \neq 0$ and (20.19) holds. Then we consider (22.18a, b) for these τ, r, χ. Observe that $D(s, \mathbf{f}, \chi)$ can be obtained from $D(\tau, r; \mathbf{f})$ of Theorem 20.4 by substituting $\chi^*(t\mathfrak{r})(\psi/\psi_{\mathfrak{c}})(t)|t|_K^s$ for $[t\mathfrak{r}]$, $t \in K_{\mathbf{h}}^\times$, and multiplying by $(\psi/\psi_{\mathfrak{c}})\big(\det(r)\big)$. Let $\mathcal{Z}'(s)$ denote the Euler product obtained from $\mathcal{Z}(s, \mathbf{f}, \chi)$ of (20.21) by substituting $(\psi/\psi_{\mathfrak{c}})(\pi_{\mathfrak{P}})N(\mathfrak{P})^{-s}$ for $N(\mathfrak{P})^{-s}$ for each prime ideal $\mathfrak{P}$ in K, where $\pi_{\mathfrak{P}}$ is the prime element of $K_{\mathfrak{P}}$. Then (20.19) gives an equality between formal Dirichlet series which leads to

$$\psi_{\mathfrak{c}}\big(\det(r)\big)c_{\mathbf{f}}(\tau, r)\mathcal{Z}'(s) = D(s, \mathbf{f}, \chi)P(s)\Lambda(s). \tag{22.19}$$

Here P is a finite Dirichlet series with constant term 1, and Λ is the product of certain L-functions obtained from $\mathfrak{L}_0\prod_{v\nmid\mathfrak{c}} h_v([\mathfrak{p}\mathfrak{r}])^{-1}$ of (20.19) by substituting $\chi^*(\mathfrak{p}\mathfrak{r})\psi^*(\mathfrak{p}\mathfrak{r})N(\mathfrak{p}\mathfrak{r})^{-s}$ for $[\mathfrak{p}\mathfrak{r}]$ for each prime ideal $\mathfrak{p}$ in F.

Put $\mathcal{D}_q(\mathbf{f}, \mathbf{g}) = \sum_{\sigma\in R} a_\sigma^q[\det(\sigma)\mathfrak{r}]$ and $\mathcal{D}(\mathbf{f}, \chi) = \sum_{\mathfrak{a}} b_{\mathfrak{a}}[\mathfrak{a}]$ with $a_\sigma^q, b_{\mathfrak{a}} \in \mathbf{C}$. Then $D(s, \mathbf{f}, \chi) = \sum_{\mathfrak{a}} b_{\mathfrak{a}}N(\mathfrak{a})^{-s}$, and (22.18a) shows that $\sum_{\mathfrak{a}} b_{\mathfrak{a}}N(\mathfrak{a})^{-s_0}[\mathfrak{a}\mathfrak{a}^\rho] = \sum_{q\in Q} d_q\sum_{\sigma\in R} a_\sigma^q[\det(\sigma)\mathfrak{r}]$ with $d_q \in \mathbf{C}$. Our $\mathbf{g}$ will be given in §A5.5 as a certain theta series $\theta(z, \lambda)$. Since $\mu \neq 0$, $\mathbf{g}$ is a cusp form by (A3.16) or (A5.5). Therefore

Proposition 22.2 (2) applied to $D_{q,\kappa}(s;\, \mathbf{f},\, \mathbf{g})$ in (22.18b) shows that $\sum_{\mathfrak{a}} |b_{\mathfrak{a}} N(\mathfrak{a})^{-s}| < \infty$ for $\mathrm{Re}(s) > s_0$. The same holds obviously for P, and also for Λ, as can easily be seen from the explicit forms of $\mathfrak{L}_0$ and $h_v([\mathfrak{p}\mathfrak{r}])$. Thus the Dirichlet series expressing $\mathcal{Z}'(s)$ has the same property, and clearly the same is true for $\mathcal{Z}(s, \mathbf{f}, \chi)$. Now given an arbitrary Hecke character ω of K, Lemma 22.7 shows that the Euler product for $\mathcal{Z}(s, \mathbf{f}, \omega)$ is absolutely convergent and nonvanishing for $\mathrm{Re}(s) > s_0$, if we remove the Euler $\mathfrak{P}$-factors for $\mathfrak{P}$ dividing the conductor of χ. By Lemma 22.8, we can choose χ so that the conductor of χ is prime to any integral ideal in F, which means that the Euler $\mathfrak{P}$-factor of $\mathcal{Z}(s, \mathbf{f}, \omega)$ for an arbitrary $\mathfrak{P}$ is nonvanishing for $\mathrm{Re}(s) > s_0$. Consequently the whole Euler product for $\mathcal{Z}(s, \mathbf{f}, \omega)$ is nonvanishing for $\mathrm{Re}(s) > s_0$. This proves the second part of Theorem 20.13.

To prove the first part concerning meromorphic continuation, observe that the above $\mathcal{Z}'$ coincides with $\mathcal{Z}(s, \mathbf{f}, \chi\psi)$ if we remove, if necessary, finitely many Euler factors. Here we can take an arbitrary χ; we do not have to assume $\mu \neq 0$; still (22.19) holds. Therefore meromorphic continuation of $\mathcal{Z}(s, \mathbf{f}, \chi)$ for an arbitrary χ follows from Proposition 22.2 (1), (22.18b), and (22.19).

In the above proof we employed formal Dirichlet series and stated equality (22.18a) in order to emphasize that (22.18b) is not merely an equality between two functions, but also the equalities between the formal Dirichlet series, which is essential for the desired convergence of the series.

Theorem 20.13 concerns integral k. As noted in Theorem 21.4, the result is valid also for half-integral k. The proof is the same as above; indeed, our discussions of §§21.1∼8 include that case.

22.10. To show that the bound s_0 is best possible, we denote by $\mathcal{T}_l$ the vector space consisting of the functions g of the form

$$g(z) = \sum_{\xi \in V} \lambda(\xi) \det(\xi)^{\mu\rho} \mathbf{e}_{\mathbf{a}}^{n}(\xi^* \tau \xi z) \qquad (\lambda \in \mathcal{S}(V_{\mathbf{h}}),\ z \in \mathcal{H}). \tag{22.20}$$

Here $V = K_n^n$, $\mu \in \mathbf{Z}^{\mathbf{b}}$; $l = \mu + (n/2)\mathbf{a}$ in Case SP and $l = \mu + n\mathbf{a}$ in Case UT; we assume $\mu_v \geq 0$ and $\mu_v \mu_{v\rho} = 0$ for every $v \in \mathbf{a}$ in Case UT and $0 \leq \mu_v \leq 1$ for every $v \in \mathbf{a}$ in Case SP. As will be shown in §§A3.16 and A5.5, $\mathcal{T}_l$ is contained in $\mathcal{M}_l$ and stable under $g \mapsto g\|_l\, \alpha$ for $\alpha \in G$. Let $\mathcal{T}_l' = \mathcal{T}_l \cap \mathcal{S}_l$ and let $\mathcal{U}_l$ denote the orthogonal complement of $\mathcal{T}_l'$ in $\mathcal{S}_l$. Then we see that both $\mathcal{T}_l'$ and $\mathcal{U}_l$ satisfy [S97, (10.7.1, 2, 3)]. Therefore we can speak of $\mathcal{T}_l'(C)$ and $\mathcal{U}_l(C)$ for a subgroup C of $G_{\mathbf{A}}$ of the above type. In fact, $\mathbf{g}$ of §22.6 belongs to $\mathcal{T}_l(C', \psi')$, as its construction in §A5.5 shows. As noted in [S97, §10.9], for every $s \in G_{\mathbf{h}}$ the map $\mathbf{f} \mapsto \mathbf{f}(xs)$ sends $\mathcal{T}_l'(C)$ and $\mathcal{U}_l(C)$ onto $\mathcal{T}_l'(sCs^{-1})$ and $\mathcal{U}_l(sCs^{-1})$. Now we have an inner product on $\mathcal{S}_l(C)$ defined by [S97, (10.9.6)]. Then $\mathcal{U}_l(C)$ is the orthogonal complement of $\mathcal{T}_l'(C)$ in $\mathcal{S}_l(C)$. Indeed, that $\mathcal{T}_l'(C)$ and $\mathcal{U}_l(C)$ are orthogonal is obvious. Suppose $\mathbf{f} \in \mathcal{S}_l(C)$ and $\langle \mathbf{f},\, \mathcal{T}_l'(C)\rangle = 0$. Given $g \in \mathcal{T}_l'$, we can define $\mathbf{g} \in \mathcal{T}_l(C_1)$ with a sufficiently small open subgroup C_1 of C so that g is one of the components of $\mathbf{g}$. Put $C = \bigsqcup_{y \in Y} yC_1$ with $Y \subset G_{\mathbf{h}}$ and $\mathbf{g}_1(x) = \sum_{y \in Y} \mathbf{g}(xy)$. Then $\mathbf{g}_1 \in \mathcal{T}_l'(C)$ and $0 = \langle \mathbf{f},\, \mathbf{g}_1 \rangle = \#(Y)\langle \mathbf{f},\, \mathbf{g}\rangle$ by [S97, (10.9.8)]. From this we easily see that every component of $\mathbf{f}$ is orthogonal to $\mathcal{T}_l'$, that is, $\mathbf{f} \in \mathcal{U}_l(C)$ as expected.

22.11. Theorem. *Let $s_0 = (3n/2)+2-[K : F]$; let $\mathbf{f}$ be an eigenform belonging to $\mathcal{S}_k(C, \psi)$ with integral or half-integral k as in §20.11 and let χ be a Hecke character of K. Then $\mathcal{Z}(s, \mathbf{f}, \chi)$ has a pole at $s = s_0$ only if the components of $\mathbf{f}$ in the sense of (20.3b) belong to $\mathcal{T}_l'$ with $l = \mu + (n/2)\mathbf{a}$ or $l = \mu + n\mathbf{a}$ with*

$\mu \in \mathbf{Z}^{\mathbf{b}}$ as in §22.10. Moreover, for such l and μ and an arbitrary χ, there exists an eigenform $\mathbf{f} \in \mathcal{T}_l'(C')$ for some C' such that $\mathcal{Z}(s, \mathbf{f}, \chi)$ has a pole at $s = s_0$, provided $\mu \neq 0$.

PROOF. Since every Euler factor of $\mathcal{Z}(s, \mathbf{f}, \chi)$ is nonvanishing at s_0, we may remove any finite number of Euler factors by assuming $\mathfrak{c} = \mathfrak{e}$ and changing $\mathfrak{c}$ for its suitable multiple. Then $\mathbf{f} \in \mathcal{S}_k(C)$ and we may assume ψ to be trivial; thus we have (22.19) with $\mathcal{Z}'(s) = \mathcal{Z}(s, \mathbf{f}, \chi)$. We can even assume that $P = 1$ with a suitable choice of $\mathfrak{c}$, since only finitely many ideals are involved in P. Then $\mathcal{Z}(s, \mathbf{f}, \chi)$ has a pole at $s = s_0$ if and only if $D(s, \mathbf{f}, \chi)$ has a pole at $s = s_0$, since Λ is finite and nonzero at s_0. Now, by Proposition 22.2 (1), the right-hand side of (22.18b) has a pole at $s = 0$ only if $k = l$ in Case SP and $k_v + k_{v\rho} = l_v + l_{v\rho}$ for every $v \in \mathbf{a}$ in Case UT. Assuming these conditions on k and l, take C so that $\det(G \cap pCp^{-1}) = 1$ for every $p \in G_{\mathbf{h}}$; such a C always exists in view of (4.33) and (4.34). Then we have $\mathcal{S}_k(C) = \mathcal{S}_l(C)$. In view of our remark about the map $\mathbf{f}(x) \mapsto \mathbf{f}(xs)$ in §22.10, both $\mathcal{T}_l'(C)$ and $\mathcal{U}_l(C)$ are stable under $T(\mathfrak{a})$, and so, by Lemma 20.12 (2), $\mathcal{S}_l(C)$ is spanned by some eigenforms, each of which belongs to either $\mathcal{T}_l'(C)$ or $\mathcal{U}_l(C)$. Suppose $\mathbf{f}$ is an eigenform belonging to $\mathcal{U}_l(C)$. Given χ, define $\mathbf{g}$ as in §22.6. Then $\langle f_p, g_p \rangle = 0$ for every p, and so by Proposition 22.2 (3), (22.18b), and (22.19), $\mathcal{Z}(s, \mathbf{f}, \chi)$ is finite at $s = s_0$. Thus $\mathbf{f} \in \mathcal{T}_l'(C)$ if it has a pole at s_0.

Next, given k as above with $\mu \neq 0$, take a Hecke character χ and define $\mathbf{g}$ as in §22.6 with $r = 1_n$ and an arbitray $\tau \in S^+$. Take also an integral ideal $\mathfrak{a}$ in F so that if $\xi \in GL_n(K)$, $\xi^*\tau\xi = \tau$, and $\xi - 1 \prec \mathfrak{r}\mathfrak{a}$, then $\xi = 1$. Change $\mathfrak{c}$ for $\mathfrak{a}\mathfrak{c}$; then from (22.15) we see that $c_{\mathbf{g}}(\tau, 1) \neq 0$, so that $\mathbf{g} \neq 0$. We have $\mathbf{g} \in \mathcal{T}_l'(C')$ with some C'. Suppose $\mathcal{Z}(s, \mathbf{f}, \chi)$ is finite at $s = s_0$ for every eigenform $\mathbf{f} \in \mathcal{T}_l'(C')$; then we see that $D(s, \mathbf{f}', \chi)$ is finite at $s = s_0$ for every $\mathbf{f}' \in \mathcal{T}_l'(C')$, which is not necessarily an eigenform. In particular $D(s, \mathbf{g}, \chi)$ is finite at $s = s_0$. This is a contradiction, since Proposition 22.2 (3) together with (22.18b) with $\mathbf{f} = \mathbf{g}$ shows that $D(s, \mathbf{g}, \chi)$ has a positive residue at s_0. This completes the proof.

22.12. Remark. (I) In the above we expressed $\mathcal{Z}(s, \mathbf{f}, \chi)$ times some factors as a finite linear combination of the functions $D_{q,\kappa}(s; \mathbf{f}, \mathbf{g})$, each of which can be given by an integral of the form (22.9). In fact, at least if $\mathfrak{e} = \mathfrak{g}$ in Case SP, we can express $\mathcal{Z}(s, \mathbf{f}, \chi)$ itself as a single integral similar to that of (22.9) times some elementary factors. We refer the reader to [S96, (4.1)] and [S94a, (8.11)] for the explicit forms.

(II) If $G = SL_2(F)$, that is, if $n = 1$ in Case SP, then we have a result supplementary to the above theorem. Indeed:

22.13. Theorem. *We have $\mathcal{Z}(s, \mathbf{f}, \chi) \neq 0$ for $\mathrm{Re}(s) \geq 2$ in the following three cases:* (i) $k = [k]$; (ii) $k \neq [k]$ *and* $k_v > 3/2$ *for some* $v \in \mathbf{a}$; (iii) $k = 3\mathbf{a}/2$ *and* $\mathbf{f}$ *is orthogonal to $\mathcal{T}'_{3\mathbf{a}/2}$. Moreover, in these cases $\mathcal{Z}(s, \mathbf{f}, \chi)$ is finite for every $s \in \mathbf{C}$ except that it has a possible simple pole at $s = 1$ and $s = 2$ if $k = [k]$.*

If $k = [k]$, the function $\mathcal{Z}(s, \mathbf{f}, \chi)$ coincides with the function of [S91, (10.3)] as explained at the end of [S95a, §6], and hence nonvanishing follows from [S91, Proposition 3.3 and (10.4)]. The information concerning possible poles is given in [S95a, Theorem 6.4]. As for the case $k \neq [k]$, we refer the reader to [S95b, Proposition 6.2] and the succeeding paragraph. In this case $\mathcal{Z}$ is entire.

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CHAPTER VI

ANALYTIC CONTINUATION AND NEAR HOLOMORPHY OF EISENSTEIN SERIES OF GENERAL TYPES

23. Eisenstein series of general types

23.1. To emphasize the dimensionality, denote the group $U(\eta_n)$ and the space $\mathcal{H}$ in Cases SP and UT by G^n and $\mathcal{H}^n$, and the symbols $\mathcal{M}_k$ and $\mathcal{S}_k$ by $\mathcal{M}_k^n$ and $\mathcal{S}_k^n$; we write $M_{\mathbf{A}}^n$ and $\mathfrak{M}^n$ for $M_{\mathbf{A}}$ and $\mathfrak{M}$. We make the convention that $G^0 = \{1\}$ and $\mathcal{M}_k^0 = \mathcal{S}_k^0 = \mathbf{C}$; also we understand that pr is the identity map of $G_{\mathbf{A}}^n$ onto itself if the weight is integral. We are going to introduce Eisenstein series associated with elements of $\mathcal{S}_k^r$ on subgroups G^r embedded in G^n. These include those of Section 16 as special cases. The series of this type in the unitary case was treated in [S97] and we indicated there that the symplectic case can be handled in the same manner. In this section we present a more detailed treatment in Cases SP and UT, which, in Case UT, is essentially included in what was done in [S97]. Fixing an integer r such that $0 \le r \le n$, we write each element α of $(K_{\mathbf{A}})_{2n}^{2n}$ in the form

$$\alpha = \begin{bmatrix} a_1 & a_2 & b_1 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ c_1 & c_2 & d_1 & d_2 \\ c_3 & c_4 & d_3 & d_4 \end{bmatrix}, \tag{23.1}$$

where x_1 is of size r, and x_4 is of size $n-r$. Then we write $x_i = x_i(\alpha)$ and $x_\alpha = x(\alpha) = \begin{bmatrix} x_1(\alpha) & x_2(\alpha) \\ x_3(\alpha) & x_4(\alpha) \end{bmatrix}$ for $x = a, b, c, d$, and $i = 1, 2, 3, 4$. We understand that $x(\alpha) = x_1(\alpha)$ if $r = n$, and $x(\alpha) = x_4(\alpha)$ if $r = 0$. We define a parabolic subgroup $P^{n,r}$ of G^n by

$$P^{n,r} = \{\, \alpha \in G^n \mid a_2(\alpha) = c_2(\alpha) = 0,\ c_3(\alpha) = d_3(\alpha) = 0,\ c_4(\alpha) = 0 \,\} \quad (0 < r < n), \tag{23.2a}$$

$$P^n = P^{n,0} = \{\, \alpha \in G^n \mid c(\alpha) = 0 \,\}, \qquad P^{n,n} = G^n, \tag{23.2b}$$

and define also maps $\pi_r : (K_{\mathbf{A}})_{2n}^{2n} \to (K_{\mathbf{A}})_{2r}^{2r}$ and $\lambda_r : (K_{\mathbf{A}})_{2n}^{2n} \to K_{\mathbf{A}}$ by

$$\pi_r(\alpha) = \begin{bmatrix} a_1(\alpha) & b_1(\alpha) \\ c_1(\alpha) & d_1(\alpha) \end{bmatrix}, \qquad \lambda_r(\alpha) = \det\bigl(d_4(\alpha)\bigr). \tag{23.3}$$

These define homomorphisms $P_{\mathbf{A}}^{n,r} \to G_{\mathbf{A}}^r$ and $P_{\mathbf{A}}^{n,r} \to K_{\mathbf{A}}^\times$. Clearly $\lambda_0(\alpha) = \det(d_\alpha)$. We understand that $G_{\mathbf{A}}^0 = \pi_0\left(P_{\mathbf{A}}^{n,0}\right) = 1$, $\pi_n(\alpha) = \alpha$, and $\lambda_n(\alpha) = 1$ for $\alpha \in G_{\mathbf{A}}^n$. Notice that $P^{n,r}$ is a parabolic subgroup of G^n in the sense of [S97, Section 2]; see [S97, §2.11] in particular.

Assuming $r > 0$, for $z \in \mathbf{C}_n^n$ we let $\wp_r(z)$ denote the upper left $(r\times r)$-block of z, and use the same letter $\wp_r$ for the map $(\mathbf{C}_n^n)^{\mathbf{a}} \to (\mathbf{C}_r^r)^{\mathbf{a}}$ defined by $\wp_r\big((z_v)_{v\in\mathbf{a}}\big) = \big(\wp_r(z_v)_{v\in\mathbf{a}}\big)$. Then for $\alpha \in P_{\mathbf{A}}^{n,r}$ and $z \in \mathcal{H}^n$ we have

$$\text{(23.4)} \qquad \wp_r(\alpha z) = \pi_r(\alpha)\wp_r(z), \qquad j(\alpha, z) = \lambda_r(\alpha_{\mathbf{a}})j\big(\pi_r(\alpha), \wp_r(z)\big).$$

Here $\lambda_r(\alpha_{\mathbf{a}}) = \big(\lambda_r(\alpha)_v^\rho, \lambda_r(\alpha)_v\big)_{v\in\mathbf{a}} \in \mathbf{C}^{\mathbf{b}}$ in Case UT. For $\beta \in G_{\mathbf{A}}^r$ and $\gamma \in G_{\mathbf{A}}^{n-r}$ we define an element $\beta\times\gamma$ of $G_{\mathbf{A}}^n$ by

$$\text{(23.5)} \qquad \beta\times\gamma = \begin{bmatrix} a_\beta & 0 & b_\beta & 0 \\ 0 & a_\gamma & 0 & b_\gamma \\ c_\beta & 0 & d_\beta & 0 \\ 0 & c_\gamma & 0 & d_\gamma \end{bmatrix}.$$

Writing $\mathcal{G}^n$ for the group $\mathcal{G}$ defined as in §14.14, we put

$$\text{(23.6)} \qquad \mathcal{P}^{n,r} = \big\{ (\alpha, p) \in \mathcal{G}^n \,\big|\, \alpha \in P^{n,r} \big\},$$

and define homomorphisms $\pi_r : \mathcal{P}^{n,r} \to \mathcal{G}^r$ and $\lambda_r : \mathcal{P}^{n,r} \to F_{\mathbf{a}}^\times$ by

$$\text{(23.7a)} \qquad \pi_r\big((\alpha, p)\big) = \big(\pi_r(\alpha), |\lambda_r(\alpha)|^{-\mathbf{a}/2}p'\big) \quad \text{with} \quad p'(z) = p\left(\begin{bmatrix} z & w \\ {}^t w & z' \end{bmatrix}\right),$$

$$\text{(23.7b)} \qquad \lambda_r\big((\alpha, p)\big) = \lambda_r(\alpha_{\mathbf{a}}),$$

where it should be observed that $p'(z)$ does not depend on the choice of w and z'.

23.2. We now fix a weight k as in Section 16, and make the convention that $\mathcal{G}^n = G^n$ and $\mathcal{P}^{n,r} = P^{n,r}$ if k is integral. We put $[k] = k$ if k is integral, and $[k] = (k_v - 1/2)_{v\in\mathbf{a}}$ otherwise; we also put $m = k$ and $\ell = [k]$ in Case SP, and $m = (k_{v\rho} + k_v)_{v\in\mathbf{a}}$ and $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$ in Case UT. Clearly

$$\text{(23.8)} \qquad j^k(\xi, z) = j^k\big(\pi_r(\xi), \wp_r(z)\big)\lambda_r(\xi)^{[k]}|\lambda_r(\xi)|^{k-[k]} \quad \textit{if} \quad \xi \in \mathcal{P}^{n,r}.$$

For $0 \le r \le n$ and a congruence subgroup Γ of $\mathcal{G}^n$ we put

$$\text{(23.9)} \qquad \mathcal{M}_k^r(\Gamma, \mathcal{P}^{n,r}) = \big\{ f \in \mathcal{M}_k^r \,\big|\, f\|_k\pi_r(\gamma) = \lambda_r(\gamma)^\ell|\lambda_r(\gamma)|^{-\ell}f \quad \textit{for every} \quad \gamma \in \Gamma\cap\mathcal{P}^{n,r} \big\},$$

$$\text{(23.9a)} \qquad \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}) = \mathcal{M}_k^r(\Gamma, \mathcal{P}^{n,r}) \cap \mathcal{S}_k^r.$$

These spaces are $\{0\}$ unless the following condition is satisfied:

(23.10) *There is a homomorphism φ of $\pi_r\big(\Gamma\cap\mathcal{P}^{n,r}\big)$ into $\mathbf{T}$ such that $\varphi\big(\pi_r(\gamma)\big) = \lambda_r(\gamma)^\ell|\lambda_r(\gamma)|^{-\ell}$ for every $\gamma \in \Gamma\cap\mathcal{P}^{n,r}$.*

Clearly $\varphi^2 = 1$ in Case SP. As will be shown in the proof of Lemma 23.13 below, φ is of finite order also in Case UT. Under (23.10), $\mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r})$ consists of all $f \in \mathcal{S}_k^r$ such that $f\|_k\varepsilon = \varphi(\varepsilon)f$ for every $\varepsilon \in \pi_r(\Gamma\cap\mathcal{P}^{n,r})$. For $f \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r})$, $z \in \mathcal{H}_n$, and $s \in \mathbf{C}$ we put

$$\text{(23.11)} \qquad \delta(z, s; f) = f\big(\wp_r(z)\big)\big[\delta(z)/\delta\big(\wp_r(z)\big)\big]^{s\mathbf{a}-m/2},$$

where $\delta(z) = \big(\det\big((i/2)(z^* - z)\big)_v\big)_{v\in\mathbf{a}}$; we understand that $\delta\big(\wp_0(z)\big) = 1$. We note that

$$\text{(23.12)} \qquad \delta(z, s; f)\|_k\beta = \delta\big(z, s; f\|_k\pi_r(\beta)\big)\lambda_r(\beta)^{-\ell}|\lambda_r(\beta)|^{\ell-2s\mathbf{a}} \quad \textit{if} \quad \beta \in \mathcal{P}^{n,r},$$

and slso that $a^{-\ell}|a|^{\ell} = a^{-k}|a|^m$ for every $a \in K^\times$ if $k \in \mathbf{Z}^{\mathbf{b}}$.

We now define an Eisenstein series $E_k^{n,r}(z,\ s;\ f,\ \Gamma)$ by

$$E_k^{n,r}(z,\ s;\ f,\ \Gamma) = \sum_{\gamma \in A} \delta(z,\ s;\ f)\|_k\gamma, \qquad A = (\Gamma \cap \mathcal{P}^{n,r})\backslash\Gamma. \tag{23.13}$$

The sum is formally well-defined, since $f \in \mathcal{S}_k^r(\Gamma,\ \mathcal{P}^{n,r})$. It is convergent for $\mathrm{Re}(2s) > n+r+1$ in Case SP and $\mathrm{Re}(s) > n+r$ in Case UT (see [S97, Proposition A3.7 and §A3.9]). For $\Gamma' \subset \Gamma$ we can easily verify that

$$[\Gamma \cap P : \Gamma' \cap P]E_k^{n,r}(z,\ s;\ f,\ \Gamma) = \sum_{\alpha \in \Gamma' \backslash \Gamma} E_k^{n,r}(z,\ s;\ f,\ \Gamma')\|_k\,\alpha. \tag{23.13a}$$

The series $E_k^{n,r}(z,\ s;\ f,\ \Gamma)$ can be defined even when $r = 0$ or n. If $r = n$, we have $\mathcal{S}_k^n(\Gamma,\ \mathcal{P}^{n,n}) = \mathcal{S}_k^n(\Gamma)$ and

$$E_k^{n,n}(z,\ s;\ f,\ \Gamma) = f. \tag{23.14}$$

If $r = 0$, we have $\delta(z,\ s;\ c) = c\delta(z)^{s\mathbf{a}-m/2}$ for a constant $c \in \mathbf{C}$, and

$$\mathcal{S}_k^0(\Gamma,\ \mathcal{P}^{n,0}) = \begin{cases} \mathbf{C} & \textit{if}\ \ \lambda_0(\gamma)^{-\ell}|\lambda_0(\gamma)|^{\ell} = p_\gamma \ \ \textit{for every}\ \ \gamma \in \Gamma \cap \mathcal{P}^{n,0}, \\ \{\,0\,\} & \textit{otherwise}, \end{cases} \tag{23.15}$$

where we understand that $\gamma = \bigl(\mathrm{pr}(\gamma),\ p_\gamma\bigr)$ if $k \neq [k]$ and $p_\gamma = 1$ if $k = [k]$. Thus

$$E_k^{n,0}(z,\ s;\ c,\ \Gamma) = c\sum_{\gamma \in A} \delta^{s\mathbf{a}-m/2}\|_k\,\gamma, \qquad A = (\Gamma \cap \mathcal{P}^{n,0})\backslash\Gamma. \tag{23.16}$$

We can now ask questions similar to (R1, 2, 3) of §17.3 about the nature of $E_k^{n,r}(z,\ s;\ f,\ \Gamma)$ for some values of s. The answer will given in Theorem 23.11.

23.3. To define the adelized version of our Eisenstein series, we take a set of data $(k,\ \mathfrak{b},\ \mathfrak{c},\ \mathfrak{e},\ \chi)$ as in §§16.5 and 19.1; we assume conditions (16.24a, b, c). We denote the subgroup $D[\mathfrak{x},\ \mathfrak{y}]$ of $G_{\mathbf{A}}^n$ by $D^n[\mathfrak{x},\ \mathfrak{y}]$ and similarly C of (19.1) by C^n. For simplicity let us put $D^n = D^n[\mathfrak{b}^{-1},\ \mathfrak{b}\mathfrak{c}]$, $D_0^n = D_0^n[\mathfrak{b}^{-1},\ \mathfrak{b}\mathfrak{c}]$, and

$$C^{n,r} = \bigl\{\, x \in D^n \bigm| a_1(x) - 1 \prec \mathfrak{r}\mathfrak{e},\ a_2(x) \prec \mathfrak{r}\mathfrak{e},\ b_1(x) \prec \mathfrak{r}\mathfrak{b}^{-1}\mathfrak{e} \,\bigr\}, \tag{23.17a}$$

$$C_0^{n,r} = D_0^n \cap C^{n,r}. \tag{23.17b}$$

It can easily be verified that $C^{n,r}$ is indeed a subgroup of $G_{\mathbf{A}}^n$ and that $\pi_r(P^{n,r} \cap C^{n,r}) \subset C^r$. If $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D^n$, we see that $ad^* - 1 \prec \mathfrak{r}\mathfrak{c}$, and hence the first two conditions on a_1 and a_2 can be replaced by $d_1(x) - 1 \prec \mathfrak{r}\mathfrak{e}$ and $d_3(x) \prec \mathfrak{r}\mathfrak{e}$.

For every $g \in G_{\mathbf{h}}^n$ we have

$$G_{\mathbf{A}}^n = P_{\mathbf{A}}^{n,r} g D_0^n[\mathfrak{b}^{-1},\ \mathfrak{b}]g^{-1}. \tag{23.18}$$

We already noted this in (16.22) when $g = 1$ and $r = 0$, but the equality with $g = 1$ is true for an arbitrary r by virtue of [S97, Propositions 6.13, 7.2, and 7.12]. Then the case of more general g follows from that special case by an easy principle; see a remark at the end of [S97, Section 5]. Now, taking $g \in P_{\mathbf{h}}^{n,r}$, we define an $\mathfrak{r}$-ideal $\mathfrak{a}_r^g(x)$ for $x \in G_{\mathbf{A}}^n$ and also an $\mathbf{R}$-valued function ε_r on $G_{\mathbf{A}}^n$ by

$$\mathfrak{a}_r^g(x) = \lambda_r(p)\mathfrak{r} \quad \textit{if}\ \ x \in pgD^n[\mathfrak{b}^{-1},\ \mathfrak{b}]g^{-1} \ \ \textit{with}\ \ p \in P_{\mathbf{A}}^{n,r}, \tag{23.19}$$

$$\varepsilon_r(x) = |\lambda_r(p)\lambda_r(p)^\rho|_F \quad \textit{if}\ \ x \in pD_0^n[\mathfrak{b}^{-1},\ \mathfrak{b}] \ \ \textit{with}\ \ p \in P_{\mathbf{A}}^{n,r}. \tag{23.20}$$

These are well-defined, and depend on $\mathfrak{b}$. If $r = 0$ and $g = 1$, $\mathfrak{a}_r^g(x)$ coincides with $\mathrm{il}_{\mathfrak{b}}(x)$ of (16.23); also ε_0 coincides with ε of (16.23). We have

(23.21a) $\qquad \varepsilon_r(x_{\mathbf{h}}) = N\big(\mathfrak{a}_r^1(x)\big)^{-u}$, *where* $u = 2/[K : F]$,

(23.21b) $\qquad \varepsilon_r(x_{\mathbf{a}}) = \big[\delta(z)/\delta(\wp_r(z))\big]^{-\mathbf{a}}$ *if* $z = x(\mathbf{i})$.

The first equality is obvious; the latter follows easily from (23.4). For $x \in M^n_{\mathbf{A}}$ we put $\mathfrak{a}^g_r(x) = \mathfrak{a}^g_r\big(\mathrm{pr}(x)\big)$ and $\varepsilon_r(x) = \varepsilon_r\big(\mathrm{pr}(x)\big)$.

In Case UT our treatment in this section is a special case of what was done in [S97, §§12.5∼12.9]. This can be seen by taking $(G^n, D[\mathfrak{b}, \mathfrak{b}^{-1}], C^{n,r}; G^r, C^r; P^{n,r}, \pi_r, \lambda_r)$ here to be $(G^\psi, C^\psi, D^\psi; G^\varphi, D^\varphi; P, \pi, \lambda_0)$ there; define also ξ^ψ and ξ^φ there by $\xi^\psi(w) = \chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1}$ for $w \in C^{n,r}$ and $\xi^\varphi(x) = \chi_{\mathfrak{c}}\big(\det(d_x)\big)^{-1}$ for $x \in C^r$. Case SP can be handled in the same manner as noted in [S97, §12.13]; a similar but a somewhat different treatment was given in [S95a].

Now we consider $\mathcal{S}^r_k(C^r, \chi_0)$ as defined in §20.1 with χ_0 given by $\chi_0(a) = \chi(a^\rho)^{-1}$. Taking an element $\mathbf{f}$ of $\mathcal{S}^r_k(C^r, \chi_0)$, we define a function μ on $G^n_{\mathbf{A}}$ or on $M^n_{\mathbf{A}}$ according as k is integral or half-integral as follows: $\mu(x) = 0$ if $\mathrm{pr}(x) \notin P^{n,r}_{\mathbf{A}} C^{n,r}$; if $x = pw$ with $\mathrm{pr}(p) \in P^{n,r}_{\mathbf{A}}$ and $\mathrm{pr}(w) \in C^{n,r}_0$, then

(23.22) $\qquad \mu(x) = \chi\big(\lambda_r(p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1} j^k_w(\mathbf{i})^{-1}\mathbf{f}\big(\pi_r(p)\big).$

This is well-defined. Notice that $b^k = b^\ell$ if $b \in K_{\mathbf{a}}$ and $|b_v| = 1$ for every $v \in \mathbf{a}$ in Case UT. For half-integral k we have to establish $\pi_r(p)$ as an element of $M^r_{\mathbf{A}}$, which will be done in §23.8. We can easily verify that

(23.23) $\qquad \mu(\alpha x w) = \chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1} j^k_w(\mathbf{i})^{-1}\mu(x)$ *if* $\alpha \in P^{n,r}$ *and* $\mathrm{pr}(w) \in C^{n,r}_0$.

Then we define a function $E_{\mathbf{A}}(x, s; \mathbf{f}, \chi, C^{n,r})$ for $s \in \mathbf{C}$ and $x \in G^n_{\mathbf{A}}$ or $x \in M^n_{\mathbf{A}}$ according as k is integral or half-integral by

(23.24) $\qquad E_{\mathbf{A}}(x, s) = E_{\mathbf{A}}(x, s; \mathbf{f}, \chi, C^{n,r}) = \sum_{\alpha \in A} \mu(\alpha x)\varepsilon_r(\alpha x)^{-s}, \quad A = P^{n,r}\backslash G^n.$

This is well-defined. The domain of convergence for (23.24) is the same as that for (23.13), as explained in [S97, §12.11].

23.4. Fix an element $g \in G^r_{\mathbf{h}}$ such that $g_v = 1$ for every $v|\mathfrak{c}$. By [S97, Lemma 9.8 (3)] or by strong approximation on $Sp(n, F)$, we can find a finite subset $\{q\}$ of $P^{n,r}_{\mathbf{h}}$ such that $G^n_{\mathbf{A}} = \bigsqcup_q G^n q C^{n,r}$ and

(23.25) $\qquad q = g \times \mathrm{diag}[\widehat{\varphi}, \varphi]$ *with* $\varphi \in GL_{n-r}(K)_{\mathbf{h}}$, $q_v = 1$ *if* $v|\mathfrak{c}$.

We can take $g = 1$, but for some technical reasons, we consider q with g of a more general type. For such a q we have

(23.25a) $\qquad P^{n,r}_{\mathbf{A}} q C^{n,r} q^{-1} = P^{n,r}_{\mathbf{A}} C^{n,r}.$

Indeed, if $v \nmid \mathfrak{c}$, then $P^{n,r}_v C^{n,r}_v = P^{n,r}_v D^n_v = G^n_v$, and hence $P^{n,r}_v q_v C^{n,r}_v q_v^{-1} = G^n_v$. Since $q_v = 1$ for $v|\mathfrak{c}$, we obtain (23.25a).

Since $E_{\mathbf{A}}(x, s)$ satisfies a formula of type (20.3a), the principle of [S97, (10.7.5)] defines a function $E_q(z) = E_q(z, s; \mathbf{f}, \chi, C)$ for $z \in \mathcal{H}^n$ for each q by

(23.26) $\qquad E_{\mathbf{A}}(qy, s) = \big(E_q\|_k y\big)(\mathbf{i})$ *if* $\mathrm{pr}(y) \in G^n_{\mathbf{a}}$.

The function $E_{\mathbf{A}}$ is completely determined by the functions E_q. In Case SP, E_1 determines $E_{\mathbf{A}}$.

Next, for each fixed q we consider a complete set of representatives $\mathcal{R}_q$ for $P^{n,r}\backslash[G^n \cap P^{n,r}_{\mathbf{A}} q C^{n,r} q^{-1}]$. By [S97, Lemma 9.6(3, 4) and Lemma 9.8(1)] we can find a finite subset $\{\zeta\}$ of $P^{n,r}_{\mathbf{h}}$ such that $G^n \cap \zeta q C^{n,r} q^{-1} \neq \emptyset$, $G^n \cap P^{n,r}_{\mathbf{A}} q C^{n,r} q^{-1} = \bigsqcup_\zeta \big(G^n \cap P^{n,r}\zeta q C^{n,r} q^{-1}\big)$, and each ζ is of the form

$$(23.27)\qquad \zeta = e \times \mathrm{diag}[\widehat{s},\, s] \ \ \textit{with}\ s \in GL_{n-r}(K)_{\mathbf{h}}\ \textit{and}\ e \in \mathcal{B}^r,\ \ \zeta_v = 1\ \ \textit{if}\ v|\mathfrak{c},$$

where $\mathcal{B}^r$ is any fixed subset of $G^r_{\mathbf{h}}$ satisfying (20.5) with r in place of n. We can and do take $\mathcal{B}^r = \{1\}$ in Case SP. Then we can take $\mathcal{R}_q$ to be a subset of $\bigsqcup_\zeta \bigl(G^n \cap \zeta q C^{n,r} q^{-1}\bigr)$. With κ as in (16.24a), we put

$$(23.28)\qquad \delta(z,\, s;\, f,\, \kappa) = f\bigl(\wp_r(z)\bigr)\bigl[\delta(z)/\delta\bigl(\wp_r(z)\bigr)\bigr]^{s\mathbf{a}-(m-i\kappa)/2}.$$

If $\kappa = 0$, this is the same as (23.11).

23.5. Proposition. *Suppose $\mathfrak{e} = \mathfrak{c}$; let q and g be as in (23.25); suppose $q = 1$ in Case SP. Then we have*

$$E_q(z) = \chi_{\mathbf{h}}\bigl(\lambda_r(q)^{-1}\bigr)|\lambda_r(q)|_K^{-s} \cdot \sum_{\alpha\in\mathcal{R}_q} N\bigl(\mathfrak{a}^q_r(\alpha)\bigr)^{us}\chi_{\mathbf{a}}\bigl(\lambda_r(\alpha)\bigr)\chi^*\bigl(\lambda_r(\alpha)\mathfrak{a}^q_r(\alpha)^{-1}\bigr)\delta(z,\, s;\, f_{eg},\, \kappa)\|_k\, \alpha,$$

where u is as in (23.21a), f_{eg} is the eg-component of $\mathbf{f}$ in the sense of (20.3b), and e is the element of $G^r_{\mathbf{h}}$ in (23.27) for ζ such that $\alpha \in \zeta q C^{n,r} q^{-1}$.

In Case UT this is included in [S97, Proposition 12.10] as a special case. The proof there can easily be translated into the symplectic case. However, there is one nontrivial technical point in the case of half-integral k, which will be explained in §23.8.

23.6. Proposition. *Let X be the set of all Hecke characters of K satisfying (16.24a, b) with $\kappa = 0$. Suppose $\mathfrak{e} = \mathfrak{c}$ and $X \neq \varnothing$; put $\Gamma' = \bigl\{\alpha \in G^n \cap D[\mathfrak{b}^{-1}\mathfrak{c},\, \mathfrak{b}\mathfrak{c}] \bigm| a_\alpha - 1 \prec \mathfrak{c}\bigr\}$. Let $f \in \mathcal{S}^r_k(\Gamma,\, P^{n,r})$ with a congruence subgroup Γ of $\mathcal{G}^n$ containing Γ'. Take $\mathcal{B}^r$ as in §23.4 with 1 as one of its members, and take $\mathbf{f} \in \mathcal{S}^r_k(C^r)$ so that $\mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}^r}$ with $f_1 = f$ and $f_b = 0$ for $1 \neq b \in \mathcal{B}^r$. Then*

$$[P^{n,r}\cap\Gamma : P^{n,r}\cap\Gamma']\#(X)E^{n,r}_k(z,\, s;\, f,\, \Gamma) = \sum_{\chi\in X}\ \sum_{\xi\in\Gamma'\backslash\Gamma} E_1(z,\, s;\, \mathbf{f},\, \chi,\, C^{n,r})\|\xi,$$

where $E_1(\cdots)$ denotes $E_q(z)$ of (23.26) with $q = 1$.

Here if k is half-integral, we identify Γ' with its image in $\mathcal{G}^n$ under the map $\gamma \mapsto (\gamma,\, h_\gamma)$, which is meaningful, since $\Gamma' \subset \Gamma^\theta$ by (16.24c).

The proof in Case UT was given in [S97, Proposition 20.10]. The symplectic case can be proven in a similar and simpler way; in fact it is an easy modification of the proof of Lemma 17.2 (2). We insert here an easy fact:

23.7. Lemma. *Let $P' = \bigcap_{r=0}^n P^{n,r}$. Then $P^n_{\mathbf{A}}D^n = P'_{\mathbf{A}}D^n$.*

PROOF. Clearly it is sufficient to prove that $P^n_{\mathbf{A}} \subset P'_{\mathbf{A}}D^n$. Let $\alpha \in P^n_{\mathbf{A}}$. By [S97, Proposition 3.5] we can find an element g of $\prod_{v\in\mathbf{h}} GL_n(\mathfrak{r}_v)$ such that $(d_\alpha g^{-1})_{\mathbf{h}}$ is upper triangular. Put $\gamma = \mathrm{diag}[\widehat{g},\, g]$. Then $\gamma \in D^n$ and $(\alpha\gamma^{-1})_{\mathbf{h}} \in P'_{\mathbf{h}}$, from which we obtain the desired fact.

23.8. The symbol $\pi_r(p)$ in (23.22) is meaningful as an element of $M^r_{\mathbf{A}}$, since there exists a homomorphism $\pi_r : \mathrm{pr}^{-1}(P^{n,r}_{\mathbf{A}}) \to M^r_{\mathbf{A}}$ with the following properties:

(23.29a) $\mathrm{pr}\circ\pi_r = \pi_r\circ\mathrm{pr}$.

(23.29b) *If $\xi \in \mathrm{pr}^{-1}(P^{n,r}_{\mathbf{A}})$ and $\pi_r(\xi) \in \mathfrak{M}^r$, then $\xi \in \mathfrak{M}^n$ and $j^k(\xi,\, z) = \lambda_r(\xi)^{[k]} \cdot |\lambda_r(\xi)|^{k-[k]}j^k\bigl(\pi_r(\xi),\, \wp_r(z)\bigr)$.*

(23.29c) *The restrictions of* π_r *and* λ_r *to* $\mathcal{P}^{n,r}$ *coincide with the maps of (23. 7a, b).*

(23.29d) $\ell_r \circ \pi_r = \pi_r \circ \ell_n$, *where* ℓ_r *denotes the canonical lift* $G^r \to M^r_{\mathbf{A}}$.

For the proof, see [S95a, Lemmas 3.4 and 3.5].

Now as to the proof of Proposition 23.5 for half-integral k, we take $y \in \mathrm{pr}^{-1}(G^n_{\mathbf{a}})$ and put $z = y(\mathbf{i})$. Then $E_1(z) = \sum_{\alpha \in A} \mu(\alpha y)\varepsilon(\alpha y)^{-s} j^k_y(\mathbf{i})$. Since $\mu(\alpha y) \neq 0$ only if $\alpha \in G^n \cap P^{n,r}_{\mathbf{A}} C^{n,r}$, we can take α in $\mathcal{R}_1$. Then $\alpha \in \zeta C^{n,r}$ with some ζ as in (23.27). Clearly $\zeta \in P^{n,0}_{\mathbf{h}} \cap P^{n,r}_{\mathbf{h}}$. Now we can put $\alpha_{\mathbf{a}}\mathrm{pr}(y) = xx'$ with $x \in P^{n,0}_{\mathbf{a}} \cap P^{n,r}_{\mathbf{a}}$ and $x' \in G^n_{\mathbf{a}}$ such that $x'(\mathbf{i}) = \mathbf{i}$. This is well-known, and proved in [S97, p.54, lines 8~9]. Then $\zeta x \in P^{n,0}_{\mathbf{A}} \cap P^{n,r}_{\mathbf{A}}$. Put $\sigma = r_P(\zeta x)$ with r_P of (16.15) and $w = \sigma^{-1}\alpha y$. Then $\mathrm{pr}(w) \in C^{n,r}_0$ and $\mathrm{pr}\big(\pi_r(\sigma)\big)_{\mathbf{h}} = \pi_r\big(\mathrm{pr}(\sigma)\big)_{\mathbf{h}} = \pi_r(\zeta) = 1$, and hence $\pi_r(\sigma) \in \mathfrak{M}^r$. By (16.16c) and (23.29b) we have $j^k_\alpha(z) j^k_y(\mathbf{i}) = j^k_{\alpha y}(\mathbf{i}) = j^k_{\sigma w}(\mathbf{i}) = j^k_\sigma(\mathbf{i}) j^k_w(\mathbf{i}) = j^k\big(\pi_r(\sigma), \mathbf{i}\big)\lambda_r(\sigma)^{[k]}|\lambda_r(\sigma)|^{k-[k]} j^k_w(\mathbf{i})$. Once this is established, we can repeat what was done in [S97, p.98] to obtain the formula of Proposition 23.5

23.9. Theorem. *The notation being the same as in §23.3, suppose that* $\mathfrak{e} = \mathfrak{c}$ *and* $n > r > 0$. *Assuming that* $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *for every* $\mathfrak{a}$, *define* $\mathcal{Z}(s, \mathbf{f}, \chi)$ *by (20.21) or (21.7). Put*

$$\mathcal{F}_q\big(z, s; \mathbf{f}, \chi, C\big) = E_q\big(z, s; \mathbf{f}, \chi, C\big) \cdot \mathcal{Z}(us, \mathbf{f}, \chi) \cdot \begin{cases} \displaystyle\prod_{i=r+1}^{[(n+r)/2]} L_{\mathfrak{c}}(4s - 2i, \chi^2) & (\text{Case SP, } k \in \mathbf{Z}^{\mathbf{a}}), \\ \displaystyle\prod_{i=r+1}^{[(n+r+1)/2]} L_{\mathfrak{c}}(4s - 2i + 1, \chi^2) & (\text{Case SP, } k \notin \mathbf{Z}^{\mathbf{a}}), \\ \displaystyle\prod_{i=2r}^{n+r-1} L_{\mathfrak{c}}(2s - i, \chi_1\theta^i) & (\text{Case UT}), \end{cases}$$

$$\Gamma^{n,r}_{k,\kappa}(s) = \prod_{v \in \mathbf{a}} \Gamma^{\iota}_r\big(s - \lambda_r + (m_v + i\kappa_v)/2\big)\gamma\big(s + (i\kappa_v/2), m_v\big)$$

with $\gamma(s, a)$ *defined in both cases as follows:*

Case SP:

$$\gamma(s, a) = \begin{cases} \Gamma\left(s + \dfrac{a}{2} - \left[\dfrac{2a+n+r}{4}\right]\right)\Gamma^1_n\left(s + \dfrac{a-r}{2}\right) & \left(a \in \mathbf{Z},\ a \geq \dfrac{n+r}{2} \in \mathbf{Z}\right), \\ \Gamma^1_n\big(s + (a-r)/2\big) & (a \in \mathbf{Z},\ a > (n+r)/2 \notin \mathbf{Z}), \\ \Gamma^1_{2a+1-r}\left(s + \dfrac{a-r}{2}\right)\displaystyle\prod_{i=a+1}^{[(n+r)/2]} \Gamma(2s - i) & \left(a \in \mathbf{Z},\ \dfrac{r}{2} \leq a < \dfrac{n+r}{2}\right), \\ \Gamma^1_n\big(s + (a-r)/2\big) & (a \notin \mathbf{Z},\ a > (n+r)/2 \in \mathbf{Z}), \\ \Gamma\left(s + \dfrac{a-1}{2} - \left[\dfrac{2a+n+r-2}{4}\right]\right)\Gamma^1_n\left(s + \dfrac{a-r}{2}\right) & \left(a \notin \mathbf{Z},\ a > \dfrac{n+r}{2} \notin \mathbf{Z}\right), \\ \Gamma^1_{2a+1-r}\left(s + \dfrac{a-r}{2}\right)\displaystyle\prod_{i=[a]+1}^{[(n+r-1)/2]} \Gamma\left(2s - i - \dfrac{1}{2}\right) & \left(a \notin \mathbf{Z},\ \dfrac{r}{2} \leq a \leq \dfrac{n+r}{2}\right); \end{cases}$$

Case UT: $\gamma(s, a) = q^{n+r}(s, a)\Gamma^2_n\big(s - r + (a/2)\big)$.

Here $\iota = [K : F]$, $u = 2/\iota$, $m = k$ *in Case SP,* $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ *in Case UT;* $\lambda_r = (r+1)/2$ *in Case SP and* $\lambda_r = r$ *in Case UT;* θ *and* χ_1 *are the same as in (20.20);* $q^{n+r}(s, a)$ *is defined as in Theorem 20.14 (with* $n + r$ *in place of* t*). Then the product* $\Gamma^{n,r}_{k,\kappa}(s)\mathcal{F}_q(z, s; \mathbf{f}, \chi, C)$ *can be continued to the whole s-plane as a meromorphic function with finitely many poles, which are all simple. The set of poles of this product is contained in the set of poles of the function* $\mathcal{P}(s)$ *defined as follows: in Case SP,* $\mathcal{P}$ *is the product of Theorem 16.11 defined with* $\{n+r, k, \chi\}$ *as* $\{n, k, \chi\}$ *there; in Case UT,* $\mathcal{P}$ *is the product given in [S97, Theorem 19.3] defined with* $\{n+r, m, \chi\}$ *as* $\{n, k, \chi\}$ *there.*

The proof will be completed in §25.6. Notice that $k_v \geq n/2$ in Case SP and $m_v \geq n$ in Case UT for every $v \in \mathbf{a}$ by (6.42).

In the above theorem we assumed $r > 0$. If $r = 0$ and $\mathbf{f}$ is constant 1, then we easily see that the function of (23.24) is exactly that of (16.27), and the conclusion of Theorem 23.9 is reduced to Theorem 16.11 in Case SP and to [S97, Theorem 19.3] in Case UT, if we simply take $\mathcal{Z}(us, \mathbf{f}, \chi)$ to be 1 and $r = 0$ in the definition of various objects.

23.10. Corollary. *The function* $E^{n,r}_k(z, s; f, \Gamma)$ *of (23.13) can be continued as a meromorphic function of* s *to the whole complex plane. In particular, it is holomorphic for* $\mathrm{Re}(4s) > \mathrm{Max}(n+r+2, 3r+2)$ *if* $\mathrm{Max}_{v\in\mathbf{a}} 2|k_v| \geq n+r$ *in Case SP, and for* $\mathrm{Re}(2s) > \mathrm{Max}(n+r, 3r)$ *if* $\mathrm{Min}_{v\in\mathbf{a}}|k_v + k_{v\rho}| \geq n+r$ *in Case UT.*

PROOF. Let X and Γ' be as in Proposition 23.6. Given Γ, we can take $\mathfrak{c}$ (employed in the definition of Γ') so that $\Gamma' \subset \Gamma$; changing $\mathfrak{c}$ for its suitable multiple, we may also assume that $X \neq \varnothing$, by virtue of [S97, Lemma 11.14 (3)]. By Lemma 20.12 (2), $\mathcal{S}^r_k(C^r)$ is spanned by eigenforms. Therefore the desired meromorphic continuation follows from Proposition 23.6 and Theorem 23.9. To obtain $E(z, s; \mathbf{f}, \chi, C)$ from $\mathcal{F}(z, s; \mathbf{f}, \chi, C)$ whose analytic properties are given in Theorem 23.9, we have to divide the latter by $\mathcal{Z}(us, \mathbf{f}, \chi)$ and some L-functions. Employing Theorem 20.13 and the standard fact on the nonvanishing of L-functions, we obtain the last assertion.

For $0 < r \in \mathbf{Z}$ and a weight k we put

$$(23.30)\quad \Lambda(r, k) = \begin{cases} \{x \in \mathbf{R} \mid x \geq 2\} & (\text{Case SP: } r=1;\ k=[k] \text{ or } \mathrm{Max}_{v\in\mathbf{a}} k_v > 3/2), \\ \{x \in \mathbf{R} \mid x > (3r/2)+1\} & (\text{Case SP: all other cases}), \\ \{x \in \mathbf{R} \mid x > 3r\} & (\text{Case UT}). \end{cases}$$

23.11. Theorem. *Suppose that* $n > r > 0$*; put* $\lambda = (n+r+1)/2$ *in Case SP and* $\lambda = n + r$ *in Case UT. Let* k *be a weight; put* $m = k$ *in Case SP and* $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ *in Case UT. Let* $\mathbf{f}$ *and* χ *be as above with* $\kappa = 0$ *in (16.24a); let* $g \in \mathcal{S}^r_k(\Gamma, \mathcal{P}^{n,r})$ *with an arbitrary congruence subgroup* Γ *of* G^n*. In Case UT let* χ_1 *denote the restriction of* χ *to* $F^\times_\mathbf{A}$*. Define* E_q *as in §23.4; define also* $\mathcal{F}_q$ *as in Theorem 23.9 when* $\mathbf{f}$ *is a Hecke eigenform. Let* $\mu \in 2^{-1}\mathbf{Z}$ *in Case SP and* $\mu \in \mathbf{Z}$ *in Case UT.*

(I) *If* $\lambda \leq \mu \leq m_v$ *and* $\mu - m_v \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$*, and* $\mu \in \Lambda(r, k)$*, then* $E^{n,r}_k(z, \mu/2; g, \Gamma)$ *and* $E_q(z, \mu/2; \mathbf{f}, \chi, C)$ *belong to* $\mathcal{N}^t_k$ *except when* $F = \mathbf{Q}$ *and* $\mu = (n+r+2)/2$ *in Case SP, where*

$$t = \begin{cases} (n+r)(m-\mu+2)/2 & \text{if } \mu = \lambda+1 \text{ and } F = \mathbf{Q}, \\ (n+r)(m-\mu\mathbf{a})/2 & \text{otherwise.} \end{cases}$$

(II) *If* μ *is as in (I), then* $E_q(z, \mu/2; \mathbf{f}, \chi, C)$ *belongs to* $\mathcal{N}_k^t$ *with* $t = (n + r)(m - \mu\mathbf{a})/2$ *except in the following two cases:*

(A) Case SP, $F = \mathbf{Q}$, $2\mu \in \{n+r+2, n+r+3\}$, *and* $\chi^2 = 1$,

(B) Case UT, $F = \mathbf{Q}$, $\mu = n+r+1$, *and* $\chi_1 = \theta^\mu$.

(If $\mu=\lambda+1$, *then the statement of (I) is applicable regardless of the nature of* χ.*)*

(III) *Suppose that* $2\lambda - m_v \le \mu \le m_v$ *and* $|\mu - \lambda| + \lambda - m_v \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$. *Then* $\mathcal{F}_q(z, \mu/2; \mathbf{f}, \chi, C)$ *belongs to* $\mathcal{N}_k^t$, *where*

$$t = \begin{cases} (n+r)(k-\mu+2)/2 & (\text{Case SP}, \ \mu = \lambda+1, \ F = \mathbf{Q}, \ \textit{and} \ \chi^2 = 1), \\ (n+r)(m-\mu+2)/2 & (\text{Case UT}, \ \mu = \lambda+1, \ F = \mathbf{Q}, \ \textit{and} \ \chi_1 = \theta^\mu), \\ (n+r)\{m - |\mu-\lambda|\mathbf{a} - \lambda\mathbf{a}\}/2 & \textit{otherwise}, \end{cases}$$

except in the following four cases:

(C) Case SP, $\mu = 0$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi = 1$;

(D) Case SP, $0 < \mu \le (n+r)/2$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi^2 = 1$;

(E) Case SP, $\mu = (n+r+2)/2$, $F = \mathbf{Q}$, *and* $\chi^2 = 1$;

(F) Case UT, $0 \le \mu < n+r$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi_1 = \theta^\mu$.

23.12. Theorem. *Let* n, r, λ, $\mathbf{f}$, χ, *and* χ_1 *be as in Theorem 23.11 and let* k *be a weight. Suppose* $k = \mu\mathbf{a}$ *in Case SP and* $(k_v + k_{v\rho})_{v\in\mathbf{a}} = \mu\mathbf{a}$ *in Case UT with* μ *such that* $0 < \mu < \lambda$; *put* $s_0 = \lambda - (\mu/2)$. *Then* $\mathcal{F}_q\big(z, s; \mathbf{f}, \chi, C\big)$ *has at most a simple pole at* s_0, *which occurs only when* $\chi^2 = 1$ *in Case SP and* $\chi_1 = \theta^\mu$ *in Case UT. Moreover, the residue is an element of* $\mathcal{M}_k$.

These two theorems will be proven in §25.7. We can naturally ask whether the nearly holomorphic functions of Theorem 23.11 are arithmetic up to a well-defined constant. That is indeed so in most cases as will be shown in Theorems 27.16 and 28.9. We end this section by proving three lemmas concerning the rationality of certain automorphic forms and Hecke eigenvalues.

23.13. Lemma. *Put* $\mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D) = \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}) \cap \mathcal{M}_k^r(D)$ *for any subfield* D *of* $\mathbf{C}$. *Let* Φ *be the Galois closure of* K *over* $\mathbf{Q}$ *in* $\mathbf{C}$. *Then* $\mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}) = \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D) \otimes_D \mathbf{C}$ *if* $D \supset \Phi\mathbf{Q}_{\rm ab}$. *In particular, suppose* $k = \kappa\mathbf{a}$ *with* $\kappa \in 2^{-1}\mathbf{Z}$ *in Case SP and* $\kappa \in \mathbf{Z}$ *in Case UT. Then* $\mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}) = \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D) \otimes_D \mathbf{C}$ *if* $D \supset \mathbf{Q}_{\rm ab}$ *in Case SP and* $D \supset K'\mathbf{Q}_{\rm ab}$ *in Case UT, where* K' *is the reflex field defined for* (K, τ) *of §3.5 as in §1.12.*

PROOF. The notation being as in (23.10), let $\varepsilon = \pi_r(\gamma)$ with $\gamma \in \Gamma \cap \mathcal{P}^{n,r}$; then $\lambda_r(\gamma) \in \mathfrak{r}^\times$ and $|\varphi(\varepsilon)| = 1$. Since $\varphi(\varepsilon)^2 = \lambda_r(\gamma)^\ell\overline{\lambda_r(\gamma)}^{-\ell}$, we see that $\varphi(\varepsilon)^2$ is a unit contained in Φ, and $|\varphi(\varepsilon)^{2\sigma}| = 1$ for every isomorphic embedding σ of Φ into $\mathbf{C}$. Therefore $\varphi(\varepsilon)$ is a root of unity whose square belongs to Φ. Thus φ is of finite order. Suppose $D \supset \Phi\mathbf{Q}_{\rm ab}$; let $\Delta = \pi_r(\Gamma \cap \mathcal{P}^{n,r})$ and $f \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r})$. By Theorem 10.8 (2), we can put $f = \sum_{a\in A} ag_a$ with a finite subset A of $\mathbf{C}$ and $g_a \in \mathcal{S}_k^r(D)$. We may assume that A is linearly independent over D. Then for $\varepsilon \in \Delta$ we have $\sum_{a\in A} a\varphi(\varepsilon)g_a = \varphi(\varepsilon)f = f\|\varepsilon = \sum_{a\in A} ag_a\|\varepsilon$. Since $\varphi(\varepsilon) \in \mathbf{Q}_{\rm ab}$ and $g_a\|\varepsilon \in \mathcal{S}_k^r(D)$ by Theorem 9.13 (3), we have $\varphi(\varepsilon)g_a = g_a\|\varepsilon$, that is, $g_a \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D)$. This proves the first assertion. Suppose $k = \kappa\mathbf{a}$ with κ given as above. Let $D' = \mathbf{Q}_{\rm ab}$ in Case SP and $D' = K'\mathbf{Q}_{\rm ab}$ in Case UT. Given $f \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, \overline{\mathbf{Q}})$, take a finite Galois extension E of D' so that f is E-rational and put $G = \mathrm{Gal}(E/D')$. Take γ and ε as above. Then for $\sigma \in G$ we have $\varphi(\varepsilon)f^\sigma = \big(\varphi(\varepsilon)f\big)^\sigma = (f\|\varepsilon)^\sigma =$

$f^\sigma \| \varepsilon$ by Lemma 10.10, and hence $f^\sigma \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, E)$. Therefore $\sum_{\sigma \in G}(bf)^\sigma \in \mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D')$ for every $b \in E$. This shows that f is an E-linear combination of elements of $\mathcal{S}_k^r(\Gamma, \mathcal{P}^{n,r}, D')$. This combined with the first assertion proves the second assertion.

23.14. Lemma. *Let* $\mathbf{f} \in \mathcal{M}_k(C, \psi)$ *as in §20.1 with* ψ *of finite order, and let* $\sigma \in \mathrm{Aut}(\mathbf{C})$; *put* $\mu(\tau, q; \mathbf{f}) = |\det(q)_\mathbf{h}|_\mathbf{A}^{-\nu} c(\tau, q; \mathbf{f})$, *where* $\nu = 0$ *or* $1/2$ *according as* k *is integral or half-integral. Then there exists an element of* $\mathcal{M}_{k^\sigma}(C, \psi^\sigma)$, *which we write* $\mathbf{f}^\sigma$ *and which is uniquely determined by the property* $\mu(\tau, q; \mathbf{f}^\sigma) = \mu(\tau, q; \mathbf{f})^\sigma$ *for every* (τ, q), *where* k^σ *is defined as in Theorem 10.4 (5) or Theorem 10.7 (5). Moreover, if the conditions of Lemma 20.8 are satisfied and* $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *as in §20.6 or §21.3, then* $\mathbf{f}^\sigma|T(\mathfrak{a}) = \lambda_\sigma(\mathfrak{a})\mathbf{f}^\sigma$, *where* $\lambda_\sigma(\mathfrak{a}) = \lambda(\mathfrak{a})^\sigma \big(N(\mathfrak{a})^\nu\big)^\sigma / N(\mathfrak{a})^\nu$.

PROOF. Let f_p be the p-component of $\mathbf{f}$ with $p = \mathrm{diag}[q, \widehat{q}]$, $q \in GL_n(K)_\mathbf{h}$. By Theorem 10.4 (2), Theorem 10.7 (2), and the remark after the proof of Theorem 10.4, we have $(f_p)^\sigma \in \mathcal{M}_{k^\sigma}(\Gamma^p, \psi^\sigma)$, where $\Gamma^p = G \cap pCp^{-1}$ and we take $p = 1$ if $k \notin \mathbf{Z}^\mathbf{b}$. (Take W of Theorem 10.4 to be pCp^{-1}. Then Γ there is Γ^p.) We first consider the case of integral k. By [S97, Lemma 8.8 (3)] there is a finite subset Q of $GL_n(K)_\mathbf{h}$ such that $G_\mathbf{A} = \bigsqcup_{q \in Q} G\mathrm{diag}[q, \widehat{q}]C$; in Case SP strong approximation allows us to take $Q = \{1\}$. Given $\mathbf{f}$, we define $\mathbf{f}^\sigma \in \mathcal{M}_{k^\sigma}(C, \psi^\sigma)$ so that its p-component is $(f_p)^\sigma$ for every $p = \mathrm{diag}[q, \widehat{q}]$ with $q \in Q$. Our task is to show that $c(\tau, q; \mathbf{f}^\sigma) = c(\tau, q; \mathbf{f})^\sigma$ for every (τ, q). Since $\mathcal{M}_k$ is spanned by $\mathcal{M}_k(\overline{\mathbf{Q}})$, it is sufficient to prove the case where $f_p \in \mathcal{M}_k(\overline{\mathbf{Q}})$ for every such p. Given $r \in GL_n(K)_\mathbf{h}$, we can find $\beta \in G$ and $p = \mathrm{diag}[q, \widehat{q}]$ with $q \in Q$ so that $\mathrm{diag}[r, \widehat{r}] \in \beta pC$. Then we have (20.9e). Take $s \in \mathbf{Z}_\mathbf{h}^\times$ so that $[s, \mathbf{Q}] = \sigma$ on $\mathbf{Q}_{\mathrm{ab}}$, and put $x = \mathrm{diag}[1_n, s^{-1}1_n]$ and $\alpha = \beta^{-1}$. Since $\det(x^{-1}\alpha x \alpha^{-1}) = 1$, by strong approximation in G_1 we have $x^{-1}\alpha x \alpha^{-1} \in U^N G_1$ with U^N of (8.5) for any N. Thus we can put $x^{-1}\alpha x = u\varepsilon$ with $\varepsilon \in G_1\alpha$ and $u \in U_N$. Then $u \in G_\mathbf{A}$. Take N so that both f_p and $f_p\|\alpha$ belong to $\mathcal{M}_k(\Gamma^N, \overline{\mathbf{Q}})$ with Γ^N of (7.6). Then $(f_p\|\alpha)^\sigma = (f_p)^{(\alpha x, \sigma)}$ and $(f_p)^\sigma = (f_p)^{(xu, \sigma)}$ by Theorem 10.2 (8). Thus $(f_p\|\alpha)^\sigma = (f_p)^{(\alpha x, \sigma)} = (f_p)^{(xu\varepsilon, \sigma)} = (f_p)^\sigma\|\varepsilon$. Put $\gamma = \varepsilon^{-1}$. Since $xCx^{-1} = C$ and x commutes with p and $\mathrm{diag}[r, \widehat{r}]$, we have $\mathrm{diag}[r, \widehat{r}] \in x^{-1}\beta xpC = \gamma u^{-1}pC$. Now we can choose sufficiently large N so that $U^N \cap G_\mathbf{A} \subset pCp^{-1}$. Then $\mathrm{diag}[r, \widehat{r}] \in \gamma pC$, so that we can write formula (20.9e) with $(\mathbf{f}^\sigma, \psi^\sigma, \gamma)$ as $(\mathbf{f}, \psi, \beta)$. Applying σ to (20.9e), we obtain $(f_p)^\sigma\|\gamma^{-1}$ on the left-hand side. Comparing the two equalities, we find that $c(\tau, q; \mathbf{f}^\sigma) = c(\tau, q; \mathbf{f})^\sigma$, since $u \in U_N$ and so we have $\psi_\mathfrak{c}\big(\det(a_{\beta p}^{-1}a_{\gamma p})\big) = 1$ for a sufficiently large N.

Next let us consider the case of half-integral k. We define $\mathbf{f}^\sigma \in \mathcal{M}_k(C, \psi^\sigma)$ so that its 1-component is $(f_1)^\sigma$. Given $r \in GL_n(K)_\mathbf{h}$, take $\beta \in G$ so that $\mathrm{diag}[r, \widehat{r}] \in \beta C$, and take α, ε, and γ as above. Then both β and γ belong to $\mathfrak{M}$, and so h_β and h_γ are meaningful. Put $\theta(z) = \theta(0, z; \ell)$ with the notation of (A2.23), where we take ℓ to be the characteristic function of $\prod_{v \in \mathbf{h}}(\mathfrak{g}_n^1)_v$. Then $\theta^{-1}f_1 \in \mathcal{A}_{[k]}(\overline{\mathbf{Q}})$, and the above reasoning shows that $\big((\theta^{-1}f_1)\|_{[k]}\beta^{-1}\big)^\sigma = (\theta^{-1}f_1)^\sigma\|_{[k^\sigma]}\gamma^{-1}$. By Proposition A2.5, $h_\beta(\beta^{-1}z)\theta(\beta^{-1}z) = \theta(0, z; {}^\beta\ell)$, and by Theorem A2.4 (6), (8) and (A2.3a) we have $({}^\beta\ell)(y) = e\ell(yr)$, where $e = |\det(r)|_\mathbf{A}^{1/2}$. The same holds with γ in place of β, so that $\big[e^{-1}h_\beta(\beta^{-1}z)\theta(\beta^{-1}z)\big]^\sigma = \theta(0, z; e^{-1}\cdot{}^\beta\ell)^\sigma = \theta(0, z; e^{-1}\cdot{}^\beta\ell) = e^{-1}h_\gamma(\gamma^{-1}z)\theta(\gamma^{-1}z)$, since $e^{-1}\cdot{}^\beta\ell$ is $\mathbf{Q}$-valued. Therefore we obtain

$$\big[e^{-1}j_\beta^k(\beta^{-1}z)f_1(\beta^{-1}z)\big]^\sigma = e^{-1}j_\gamma^{k^\tau}(\gamma^{-1}z)f_1^\sigma(\gamma^{-1}z).$$

Employing (20.9e), we obtain $\big[e^{-1}c(\tau, r; \mathbf{f})\big]^\sigma = e^{-1}c(\tau, r; \mathbf{f}^\sigma)$ as expected.

To prove the assertion concerning $\lambda(\mathfrak{a})$, substitute $N(\mathfrak{a})^{\nu}[\mathfrak{a}]$ for $[\mathfrak{a}]$ in (20.18), and apply σ to the coefficients of the formal Dirichlet series; see Lemma 20.5 and Theorem 21.4 for the explicit forms of $A(\tau, L)$. Comparing the coefficients of $[\mathfrak{a}]$, we obtain the last equality of our proposition.

23.15. Lemma. *The notation and assumption being the same as in Lemma 23.14, suppose that* $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *for every* $\mathfrak{a}$. *Let* W *be the field generated by the* $\lambda(\mathfrak{a})$ *over* $\mathbf{Q}$ *for all* $\mathfrak{a}$ *prime to* $\mathfrak{c}$. *Then* W *is totally real or a CM-field, and the latter can occur only in Case UT.*

PROOF. Denote complex conjugation in $\mathbf{C}$ by ρ. If $T(\mathfrak{a}) = \sum_{\tau} C\tau C$ and $\tau_v = 1$ for every $v|\mathfrak{c}$, then we easily see that $T(\mathfrak{a}^{\rho}) = \sum_{\tau} C\tau^{-1}C$. Therefore from [S97, Proposition 11.7] and [S95b, Lemma 4.5] we can easily derive that $\lambda(\mathfrak{a})^{\rho} = \lambda(\mathfrak{a}^{\rho})$ if $\mathfrak{a}$ is prime to $\mathfrak{c}$. In Case SP this means that $\lambda(\mathfrak{a})$ is real, and hence, by Lemma 23.14, $\lambda_{\sigma}(\mathfrak{a})$ is real for every $\sigma \in \mathrm{Aut}(\mathbf{C})$. Since $\left(N(\mathfrak{a})^{\nu}\right)^{\sigma} = \pm N(\mathfrak{a})^{\nu}$, we see that $\lambda(\mathfrak{a})^{\sigma}$ is real for every σ, so that W is totally real. In Case UT, we have $\lambda(\mathfrak{a})^{\sigma\rho} = \lambda_{\sigma}(\mathfrak{a})^{\rho} = \lambda_{\sigma}(\mathfrak{a}^{\rho}) = \lambda(\mathfrak{a}^{\rho})^{\sigma} = \lambda(\mathfrak{a})^{\rho\sigma}$, so that $\sigma\rho = \rho\sigma$ on W. Thus W is totally real or a CM-field in Case UT.

24. Pullback of Eisenstein series

We fix two integers r and n as in §23.1 and put $t = n - r$ and $N = n + r$. Taking (n, r, N) in place of $(r, n - r, n)$ in (23.5), for $(\beta, \gamma) \in G^n \times G^r$ we can define $\beta \times \gamma$ as an element of G^N and view $G^n \times G^r$ as a subgroup of G^N.

24.1. Lemma. *We have* $G^N = \bigsqcup_{\nu=0}^{r} P^N\tau_{\nu}(G^n \times G^r)$ *with* τ_{ν} *given by*

$$\tau_{\nu} = \begin{bmatrix} 1_N & 0 \\ f_{\nu} & 1_N \end{bmatrix}, \quad f_{\nu} = \begin{bmatrix} 0 & g_{\nu} \\ {}^t g_{\nu} & 0 \end{bmatrix}, \quad g_{\nu} = \begin{bmatrix} 1_{\nu} & 0 \\ 0 & 0 \end{bmatrix} \in K_r^n. \tag{24.1}$$

Moreover $P^N\tau_{\nu}(G^n \times G^r) = \bigsqcup_{\xi,\beta,\gamma} P^N\tau_{\nu}\big((\xi \times 1_{2n-2\nu})\beta \times \gamma\big)$, *where* ξ *runs over* G^{ν}, β *over* $P^{n,\nu}\backslash G^n$, *and* γ *over* $P^{r,\nu}\backslash G^r$. *Furthermore,* $\big(\kappa_{\nu}\pi_{\nu}(\gamma)\kappa_{\nu} \times 1_{2n-2\nu}\big) \times \gamma \in \tau_{\nu}^{-1}P^N\tau_{\nu}$ *for every* $\gamma \in P^{r,\nu}$, *where* $\kappa_{\nu} = \begin{bmatrix} 0 & 1_{\nu} \\ 1_{\nu} & 0 \end{bmatrix}$.

PROOF. This is essentially included in [S97, Propositions 2.4 and 2.7, and Lemma 2.6]. Indeed, put $\varphi = \eta_r$,

$$\omega = \begin{bmatrix} \psi & 0 \\ 0 & -\varphi \end{bmatrix}, \quad \psi = \begin{bmatrix} 0 & 0 & -1_t \\ 0 & \eta_r & 0 \\ 1_t & 0 & 0 \end{bmatrix}, \quad R = \mathrm{diag}\left[1_n, \begin{bmatrix} 0 & 1_r \\ 1_n & 0 \end{bmatrix}, 1_r\right], \tag{24.2}$$

$$T = \begin{bmatrix} 0 & 1_r & 0 \\ 1_t & 0 & 0 \\ 0 & 0 & 1_n \end{bmatrix}, \quad S = \begin{bmatrix} 1_t & 0 & 0 & 0 \\ 0 & 1_{2r} & 0 & -\lambda \\ 0 & 0 & 1_t & 0 \\ 0 & -1_{2r} & 0 & \lambda^* \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0_r & 1_r \\ 0_r & 0_r \end{bmatrix}, \tag{24.3}$$

Denote the group $U(\varphi)$ of (1.7) by $G^{\varphi} = G(\varphi)$ in conformity with the notation of [S97]. We have $G^n = G(\eta_n)$, for example. Then $\eta_r = \lambda^* - \lambda$ and $S\eta_N S^* = \omega$, and so $S^{-1}G^{\omega}S = G^N$; $\psi = {}^tT\eta_n T$, and so $G^{\psi} = T^{-1}G^nT$; also $R \cdot \mathrm{diag}[\alpha, \beta]R^{-1} = \alpha \times \beta$ for $\alpha \in G^n$ and $\beta \in G^r$. Now in [S97, Proposition 2.4] we showed that $P_U^{\omega}\backslash G^{\omega}/\big[G^{\psi} \times G^{\varphi}\big]$ for a certain parabolic subgroup P_U^{ω} of G^{ω} has exactly $r + 1$ orbits, say X_{ν} for $0 \le \nu \le r$, and gave an explicit set of representatives for $P_U^{\omega}\backslash X_{\nu}$. Observing that $P_U^{\omega} = SP^NS^{-1}$ (cf. [S97, (21.1.8)]) and employing the above isomorphisms among the groups involved, we see that $P^N\backslash G^N/\big[G^n \times G^r\big]$

has exactly $r+1$ orbits. To find good representatives, for $x \in G^N$ put $[c_x \;\; d_x] = [a \;\; b \;\; a' \;\; b']$ with $a, a' \in K_n^N$ and $b, b' \in K_r^N$. Clearly the rank of $[a \;\; a']$ depends only on $P^N x(G^n \times G^r)$. Since this rank is $n+\nu$ if $x = \tau_\nu$, we obtain the first asserion of our proposition. Then the second and third assertions can easily be verified by translating (or by modifying) the proof of [S97, Lemma 2.6 and Proposition 2.7]. For a more direct proof in Case SP, see [S95a, pp.556-557].

We now consider C^r and $C^{n,r}$ of §23.3. We put $D^n = D^n[\mathfrak{b}^{-1}, \mathfrak{bc}]$ as we did there; hereafter we assume that $\mathfrak{e} = \mathfrak{c}$. We define a subgroup C' of D^n and an element σ of $G_{\mathbf{h}}^N$ by

$$C' = \{ x \in D^n \mid d_3(x) \prec \mathfrak{rc} \}, \tag{24.4}$$

$$\sigma_v = \begin{cases} \mathrm{diag}[1_n, \theta_v^{-1}1_r, 1_n, \theta_v 1_r]\tau_r & \text{if } v|\mathfrak{c}, \\ \mathrm{diag}[1_n, \theta_v^{-1}1_r, 1_n, \theta_v 1_r] & \text{if } v\nmid\mathfrak{c}, \end{cases} \tag{24.5}$$

where θ is an element of $F_{\mathbf{h}}^\times$ such that $\theta\mathfrak{g} = \mathfrak{b}$. To see that C' is indeed a subgroup, take the homomorphism $x \mapsto \big((d_x)_v\big)_{v|\mathfrak{c}}$ of D^n into $\prod_{v|\mathfrak{c}} GL_n(\mathfrak{r}_v/\mathfrak{r}_v\mathfrak{c}_v)$ noted in (1.18). Then C' is the inverse image of the subgroup of the latter group defined by the vanishing of the lower left $(t \times r)$-block. Notice also that C' can be defined by the condition $a_2(x) \prec \mathfrak{rc}$ instead of $d_3(x) \prec \mathfrak{rc}$.

24.2. Lemma. (1) $P_{\mathbf{A}}^{n,r} C^{n,r} = P_{\mathbf{A}}^{n,r} C'$.

(2) Let q be any fixed element of $G_{\mathbf{h}}^n$ as in (23.25). Then

$$\big(P^N\tau_r(G^n \times G^r)\big) \cap P_{\mathbf{A}}^N D^N \sigma = \bigsqcup_{\xi \in \Xi,\, \beta \in \mathcal{R}_q} P^N \tau_r\big((\xi \times 1_{2t})\beta \times 1_{2r}\big),$$

where $\mathcal{R}_q$ is the subset of $\bigsqcup_\zeta G^n \cap (\zeta q C^{n,r} q^{-1})$ of §23.4, and $\Xi = G^r \cap \mathfrak{X}$ with $\mathfrak{X}$ of (19.2c) defined with $\mathfrak{e} = \mathfrak{c}$ and r in place of n there; we take $\mathcal{R}_q = \{1\}$ if $r = n$.

PROOF. To prove (1), by mens of the map $x \mapsto \varepsilon x \varepsilon^{-1}$ with $\varepsilon = \mathrm{diag}[1_n, \theta 1_n]$, we may assume that $\mathfrak{b} = \mathfrak{g}$. Given $x \in C'$, put $y = \pi_r(x)$. Then we easily see that $y\eta_r y^* - \eta_r = z - z^*$ with a matrix $z \prec \mathfrak{rc}$. Fix a prime $v|\mathfrak{c}$, assume $K \neq F$, and put $f = e - e^\rho$ with e such that $\mathfrak{r}_v = \mathfrak{g}_v[e]$. Then $(z - z^*)_v \in f\mathfrak{c}_v \widetilde{S}_v$ with $\widetilde{S}_v$ of (16.1c). Then successive approximation produces an element k_v of $G_v^r \cap GL_{2r}(\mathfrak{r}_v)$ such that $k_v^{-1}y_v - 1 \prec \mathfrak{r}_v\mathfrak{c}_v$, as proved in [S97, Lemma 17.2 (2)]. (Take (δ, φ) there to be $(f, f^{-1}\eta_r)$ here.) Let $p = (p_v)$ with $p_v = k_v \times 1_{2t}$ for $v|\mathfrak{c}$ and $p_v = 1$ for all other v's. Then $p \in P_{\mathbf{h}}^{n,r}$ and $p^{-1}x \in C^{n,r}$, and hence $x \in P_{\mathbf{A}}^{n,r} C^{n,r}$, which proves (1) in Case UT, since $C^{n,r} \subset C'$. Case SP, in which f is unnecessary, can be proved in a similar and simpler way.

Next, to prove (2), we first assume $q = 1$. Let $\alpha = (\xi \times 1_{2t})\beta$ with $\xi \in G^r$ and $\beta \in P^{n,r}\backslash G^n$. By Lemma 24.1, $P^N\tau_r(G^n \times G^r)$ is a disjoint union of $P^N\tau_r(\alpha \times 1_{2r})$ with such α's. Thus our task is to determine $P^N\tau_r(\alpha \times 1_{2r})$ contained in $P_{\mathbf{A}}^N D^N \sigma$. Put $\omega = \sigma(\alpha \times 1_{2r})\sigma^{-1}$. Since $G_v^N = P_v^N D_v^N$ for $v \nmid \mathfrak{c}$, and $(\tau_r\sigma^{-1})_v \in P_v^N$ for $v|\mathfrak{c}$, we only have to find those α such that $\omega \in P_{\mathbf{A}}^N D^N$. Clearly $\omega_v = (\alpha \times 1_{2r})_v$ for $v \nmid \mathfrak{c}$. Fix a prime $v|\mathfrak{c}$ and write α in the form (23.1). Then

$$\omega_v = \begin{bmatrix} a_1 & a_2 & -\theta b_1 & b_1 & b_2 & 0 \\ a_3 & a_4 & -\theta b_3 & b_3 & b_4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ c_1 & c_2 & \theta(1-d_1) & d_1 & d_2 & 0 \\ c_3 & c_4 & -\theta d_3 & d_3 & d_4 & 0 \\ \theta(a_1 - 1) & \theta a_2 & -\theta^2 b_1 & \theta b_1 & \theta b_2 & 1 \end{bmatrix}_v . \tag{24.6}$$

By Lemma 1.9, $\omega \in P^N_{\mathbf{A}} D^N$ if and only if $(d_\omega)_v \in GL_N(K_v)$ and $(d_\omega^{-1} c_\omega)_v \prec (\mathfrak{rbc})_v$ for every $v|\mathfrak{c}$, which is so only if $d_\alpha \in GL_n(K_v)$ and $d_\alpha^{-1} c_\alpha \prec (\mathfrak{rbc})_v$ for every $v|\mathfrak{c}$, as can easily be seen from (24.6). Then $\alpha \in P^n_{\mathbf{A}} D^n \subset P^{n,r}_{\mathbf{A}} D^n$ by Lemmas 1.9 and 23.7, and $\beta \in P^{n,r}_{\mathbf{A}} D^n$, since $\xi \times 1 \in P^{n,r}$. Thus we may restrict β to $G^n \cap P^{n,r}_{\mathbf{A}} D^n$. As explained in §23.4, we may assume that $\beta = \zeta w$ with $w \in D^n$ and ζ as in (23.27). If $r = n$, we can take $\beta = 1$, and so can take $\zeta = w = 1$. Assuming $n > r$, we have $\xi \times 1 \in P^n_{\mathbf{A}} D^n$ since $(\xi \times 1)\zeta w = \alpha \in P^n_{\mathbf{A}} D^n$, $\zeta_v = 1$ for $v|\mathfrak{c}$, and $P^n_v D^n_v \neq G^n_v$ only if $v|\mathfrak{c}$. Consequently $d_\xi \in GL_r(K_v)$ and $d_\xi^{-1} c_\xi \prec (\mathfrak{rbc})_v$ for every $v|\mathfrak{c}$ by Lemma 1.9. Write ζw in the form (23.1) and put $p_i = a(\zeta w)_i$, $q_i = b(\zeta w)_i$, $r_i = c(\zeta w)_i$, $s_i = d(\zeta w)_i$; put $e = d_\xi^{-1} c_\xi$. Then

$$[c_\alpha \;\; d_\alpha] = \operatorname{diag}[d_\xi,\, 1_{2t}][r' \;\; s'] \quad \text{with} \quad r' = r + \operatorname{diag}[e,\, 0]p \quad \text{and} \quad s' = s + \operatorname{diag}[e,\, 0]q.$$

Compute c_ω and d_ω with these c_α and d_α. Fix $v|\mathfrak{c}$. Recall that $\zeta_v = 1$ for such a v. Since $e_v \prec (\mathfrak{rbc})_v$, $q_v \prec (\mathfrak{rb}^{-1})_v$ and $s_v \in GL_r(\mathfrak{r}_v)$, we see that $s'_v \in GL_r(\mathfrak{r}_v)$. Focusing our attention on the upper right $(n \times r)$-block of $(d_\omega^{-1} c_\omega)_v$ we see that

$$(*) \qquad s'^{-1}_v \begin{bmatrix} \theta(d_\xi^{-1} - s'_1) \\ -\theta s_3 \end{bmatrix}_v \prec (\mathfrak{rbc})_v.$$

Thus $(s_3)_v \prec (\mathfrak{rc})_v$, and hence $w \in C'$, so that $\beta \in \zeta C' \subset P^{n,r}_{\mathbf{A}} C' = P^{n,r}_{\mathbf{A}} C^{n,r}$ by (1). This means that changing w suitably, we may now assume that $w \in C^{n,r}$. From (24.6) we easily see that $\sigma(w \times 1)\sigma^{-1} \in D^N$, and hence $\sigma\big((\xi \times 1)\zeta \times 1\big)\sigma^{-1} \in P^N_{\mathbf{A}} D^N$ which is true also when $n = r$, since $\zeta = w = 1$. Thus, for $\beta \in \zeta C^{n,r}$ we have $\sigma\big((\xi \times 1)\beta \times 1\big)\sigma^{-1} \in P^N_{\mathbf{A}} D^N$ if and only if $\big[\sigma\big((\xi \times 1) \times 1\big)\sigma^{-1}\big]_v \in P^N_v D^N_v$ for every $v|\mathfrak{c}$, in which case we can repeat the above computation of $d_\omega^{-1} c_\omega$ with $w = 1$. Then in $(*)$ we have $s'_v = 1$, and hence $d_\xi - 1 \prec \mathfrak{r}_v \mathfrak{c}_v$ and $c_\xi \prec (\mathfrak{rbc})_v$. From (24.6) we see that the lower right $(r \times r)$-block of $d_\omega^{-1} c_\omega$ is $-\theta^2 b_\xi d_\xi^{-1}$, and so $b_\xi \prec (\mathfrak{rb}^{-1}\mathfrak{c})_v$. Thus $\xi \in \mathfrak{X}$. Conversely suppose $\xi \in G^r \cap \mathfrak{X}$ and $v|\mathfrak{c}$. Then $\xi_v \times 1 \in C^{n,r}_v$, and so $\big[\sigma\big((\xi \times 1) \times 1\big)\sigma^{-1}\big]_v \in D^N_v$. This proves (2) when $q = 1$. For a more general q, we can repeat our argument with $\beta \in \zeta q C^{n,r} q^{-1}$ since $q_v = 1$ for every $v|\mathfrak{c}$. This completes our proof.

24.3. Lemma. *Define* $\mathrm{il}_\mathfrak{b}$ *on* $GL_{2n+2r}(K)_{\mathbf{A}}$ *by (1.19) (or (16.23)) by taking* N *as both* m *and* n *there. Let* $\alpha = \tau_r\big((\xi \times 1_{2t})\beta \times 1_{2r}\big)$ *with* $\xi \in \Xi$ *and* $\beta \in \zeta q C^{n,r} q^{-1}$, *where* $q = g \times \operatorname{diag}[\widehat{\varphi},\, \varphi]$ *and* $\zeta = e \times \operatorname{diag}[\widehat{s},\, s]$ *as in (23.25) and (23.27); let* $x = \alpha(q \times \varepsilon h \varepsilon)\sigma^{-1}$ *with* $\varepsilon = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ *and* $h \in G^r_\mathbf{h}$ *such that* $h_v = 1$ *for every* $v|\mathfrak{c}$. *Then* $\nu_\mathfrak{b}(h^{-1}\xi e g)$ *is prime to* $\mathfrak{c}$ *and*

$$\mathrm{il}_\mathfrak{b}(x) = \theta^{-r} \det(h) \det(s\varphi) \nu_\mathfrak{b}(h^{-1}\xi e g)^{-1}, \tag{24.7}$$

$$[\det(d_x)^{-1}\theta^{-r}\lambda_r(\beta)]_v \equiv 1 \pmod{\mathfrak{c}_v \mathfrak{r}_v} \quad \textit{for every} \quad v|\mathfrak{c}, \tag{24.8}$$

where $\nu_\mathfrak{b}$ *is defined by (19.5) and* λ_r *is defined by (23.3).*

PROOF. Put $\beta = \zeta q w q^{-1}$ with $w \in C^{n,r}$, $h' = \varepsilon h \varepsilon$, and $f = \tau_r\big((\xi \times 1)\zeta q \times h'\big)\sigma^{-1}$. Then $x = f\sigma(w \times 1)\sigma^{-1}$ and $\sigma(w \times 1)\sigma^{-1} \in D^N$ as seen in the proof of Lemma 24.2. Therefore $\mathrm{il}_\mathfrak{b}(x) = \mathrm{il}_\mathfrak{b}(f)$. If $v|\mathfrak{c}$, then ζ_v, q_v, h_v, e_v, and g_v are all identity matrices (of various sizes), and so $f_v = \big[\tau_r \sigma^{-1} \sigma\big((\xi \times 1) \times 1\big)\sigma^{-1}\big]_v$. Now $\xi_v \times 1 \in C^{n,r}$ since $\xi_v \in C^r$. Therefore $\mathrm{il}_\mathfrak{b}(f)_v = \mathrm{il}_\mathfrak{b}(\tau_r \sigma^{-1})_v = \theta_v^{-r} \mathfrak{r}_v$, since

$$(\tau_r \sigma^{-1})_v = \operatorname{diag}[1_n,\, \theta_v 1_r,\, 1_n,\, \theta_v^{-1} 1_r] \quad \textit{if} \quad v|\mathfrak{c}. \tag{24.9}$$

Also $\nu_{\mathfrak{b}}(\xi_v) = \mathfrak{r}_v$, since $\xi_v \in C^r$. Thus $\nu_{\mathfrak{b}}(h^{-1}\xi eg)$ is prime to $\mathfrak{c}$. Suppose $v \nmid \mathfrak{c}$. Then, putting $\xi eg = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $h' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, by a direct calculation we can show that

$$[\,c_f \quad d_f\,]_v = \begin{bmatrix} c & 0 & \theta a' & d & 0 & \theta^{-1}b' \\ 0 & 0 & 0 & 0 & s\varphi & 0 \\ a & 0 & \theta c' & b & 0 & \theta^{-1}d' \end{bmatrix}_v.$$

Clearly $\mathrm{il}_{\mathfrak{b}}(f)_v = \det(s\varphi)_v \mathrm{il}_{\mathfrak{b}}(z)_v$ with a matrix $z \in GL_{4r}(K_v)$ such that

$$[\,c_z \quad d_z\,]_v = \begin{bmatrix} a & \theta c' & b & \theta^{-1}d' \\ c & \theta a' & d & \theta^{-1}b' \end{bmatrix}_v.$$

Put $p = h\varepsilon$, $\gamma = h^{-1}\xi eg$, and $A = \mathrm{diag}\left[1_r, \begin{bmatrix} 0 & -\theta^{-1}1_r \\ \theta 1_r & 0 \end{bmatrix}, 1_r\right]$. Then

$$p_v^{-1}\,[\,c_z \quad d_z\,]_v A_v = \begin{bmatrix} c_\gamma & \theta d_\gamma & -1_r & 0 \\ a_\gamma & \theta b_\gamma & 0 & \theta^{-1}1_r \end{bmatrix}_v.$$

Since $A \in D^{2r}[\mathfrak{b}^{-1}, \mathfrak{b}]$, by Lemma 1.11 (2) we obtain $\mathrm{il}_{\mathfrak{b}}(z)_v = \det(h)_v \theta_v^{-r} \nu_0(\gamma')_v^{-1}$ with $\gamma' = \begin{bmatrix} -\theta^{-1}c_\gamma & -d_\gamma \\ a_\gamma & \theta b_\gamma \end{bmatrix}$. In view of (19.5) we easily see that $\nu_0(\gamma')_v = \nu_{\mathfrak{b}}(\gamma)_v$. Combining all these, we obtain (24.7).

Next, let $v|\mathfrak{c}$; then $x_v = (\tau_r\sigma^{-1}\omega)_v$ with $\omega = \sigma\bigl((\xi \times 1)\beta \times 1\bigr)\sigma^{-1}$. By (24.9), $\mathrm{diag}[1_n,\, \theta_v 1_r](d_x)_v = (d_\omega)_v$, and $(d_\omega)_v$ can be obtained from (24.6) by taking α there to be $(\xi \times 1)\beta$ here. Since $\beta_v = w_v \in C_v^{n,r}$ and $\xi_v \in C_v^r$, we see that

$$(d_\omega)_v \equiv \begin{bmatrix} 1_r & y_1 & 0 \\ 0 & d_4(\beta)_v & 0 \\ y_2 & y_3 & 1_r \end{bmatrix} \pmod{\mathfrak{r}_v\mathfrak{c}_v}$$

with matrices y_i with entries in $\mathfrak{r}_v$. Now the right-hand side belongs to $GL_N(\mathfrak{r}_v)$. Therefore, taking the determinant, we obtain (24.8).

If $r = n$, we can take $\beta = 1$ and $\zeta = 1$. In this case formula (24.7) takes a simpler form:

$$\mathrm{il}_{\mathfrak{b}}\bigl(\alpha(q \times \varepsilon h\varepsilon)\sigma^{-1})\bigr) = \theta^{-n}\det(h)\nu_{\mathfrak{b}}(h^{-1}\xi g)^{-1}. \tag{24.10}$$

24.4. Lemma. *Let $w,\, w' \in \mathcal{H}^r$ and $z \in \mathcal{H}^n$; let $\xi \in G^r$ and $\beta \in G^n$; further let τ_ν be defined by (24.1). Then we have the following formulas:*

$$j\bigl(\tau_\nu,\, \mathrm{diag}[z,\, w]\bigr) = \det\bigl[1_\nu - \wp_\nu(w)\wp_\nu(z)\bigr], \tag{24.11a}$$

$$j\bigl(\tau_r,\, \mathrm{diag}[z,\, w]\bigr) = \det\bigl[1_r - w\cdot\wp_r(z)\bigr] = j(\eta_r^{-1},\, w)\det\bigl[\eta_r^{-1}w + \wp_r(z)\bigr], \tag{24.11b}$$

$$\begin{aligned} &j\bigl(\tau_r\bigl((\xi \times 1_{2t})\beta \times 1_{2r}\bigr),\, \mathrm{diag}[z,\, w]\bigr) \\ &\quad = j_\beta(z)j_\xi\bigl(\wp_r(\beta z)\bigr)j(\eta_r^{-1},\, w)\det\bigl[\eta_r^{-1}w + \xi\wp_r(\beta z)\bigr]. \end{aligned} \tag{24.12}$$

PROOF. The first two formulas can be verified by a straightforward calculation. Now, ptting $y = \wp_r(\beta z)$, we have $\wp_r\bigl((\xi \times 1)\beta z\bigr) = \xi y$ and

$$\begin{aligned} &j\bigl(\tau_r\bigl((\xi \times 1_{2t})\beta \times 1_{2r}\bigr),\, \mathrm{diag}[z,\, w]\bigr) \\ &\quad = j\bigl(\tau_r,\, \mathrm{diag}\bigl[(\xi \times 1)\beta z,\, w\bigr]\bigr)j\bigl((\xi \times 1_{2t})\beta \times 1,\, \mathrm{diag}[z,\, w]\bigr) \\ &\quad = \det[1 - w\cdot(\xi y)]j_{\xi\times 1}(\beta z)j_\beta(z) = \det[\eta^{-1}w + \xi y]j(\eta^{-1},\, w)j_\xi(y)j_\beta(z). \end{aligned}$$

by (23.4) and (24.11b). This proves (24.12).

We now define functions $\delta(w)$ and $\delta(w',\, w)$ for $w,\, w' \in \mathcal{H}^r$ by

$$(24.13)\qquad \delta(w) = \delta(w,\, w), \quad \delta(w',\, w) = \bigl(\det\bigl[(i/2)(w^* - w')_v\bigr]\bigr)_{v\in\mathbf{a}} \quad (\in \mathbf{C}^{\mathbf{a}}).$$

The symbol $\delta(w)$ is consistent with δ of (16.36). By [S97, (6.6.9) and (7.14.7)],

$$(24.14)\qquad \delta(\gamma w',\, \gamma w)^m \overline{j_\gamma^k(w)} j_\gamma^k(w') = \delta(w',\, w)^m \quad \textit{for every } \gamma \in G^r,$$

if k and m are as in §23.2, since $j_\gamma^k = j_\gamma^m \prod_{v\in\mathbf{a}} \det(\alpha)_v^{-k_{v\rho}}$.

24.5. Lemma. *Let $h \in \mathbf{R}^{\mathbf{a}}$ and $\mathbf{s} = (s_v)_{v\in\mathbf{a}} \in \mathbf{C}^{\mathbf{a}}$; suppose that $h_v \geq 0$, $\mathrm{Re}(s_v) \geq 0$, and $\mathrm{Re}(s_v) + (h_v/2) > 2\lambda_r - 1$ for every $v \in \mathbf{a}$, where $\lambda_r = (r+1)/2$ in Case SP and $\lambda_r = r$ in Case UT. Then for every holomorphic function f on $\mathcal{H}^r$ such that $\delta(w)^{h/2} f(w)$ is bounded, we have*

$$c_h(\mathbf{s})\delta(w')^{-\mathbf{s}} f(w') = \int_{\mathcal{H}^r} \delta(w',\, w)^{-h} |\delta(w',\, w)|^{-2\mathbf{s}} \delta(w)^{h+\mathbf{s}} f(w) \mathbf{d}w$$

$$\textit{with}\quad c_h(\mathbf{s}) = 2^a\, \pi^b \prod_{v\in\mathbf{a}} \Gamma_r^\iota\bigl(s_v + h_v - \lambda_r\bigr) \Gamma_r^\iota(s_v + h_v)^{-1},$$

where $\mathbf{d}w = \prod_{v\in\mathbf{a}} \mathbf{d}w_v$ with $\mathbf{d}w_v$ defined as in Lemma 3.4, $a = r(r+3)[F:\mathbf{Q}]/2$ in Case SP, $a = 2r^2[F:\mathbf{Q}]$ in Case UT, $b = r\lambda_r[F:\mathbf{Q}]$ in both cases, and Γ_r^ι is defined by (16.47) with $\iota = [K:F]$.

This is a restatement of [S97, Propositions A2.9 and A2.11]. The expression for the exponent a can be given uniformly $a = r\bigl(1 + \lambda_r\bigr)[F:\mathbf{Q}]$ in both cases, if we take the measure on $\mathcal{H}_r$ in Case UT to be $\det(y)^{-2r} dx dy$ described in §5.12.

24.6. We now consider $E_{\mathbf{A}}$ of (16.27) with the present G^N as G there. We assume k to be integral; we shall make comments in the case of half-integral k in §25.6. Recall that $E_{\mathbf{A}}$ is determined by the set of data $\{k,\, \mathfrak{b},\, \mathfrak{c},\, \chi\}$ satisfying (16.24a, b, c). We put $m = k$ in Case SP and $m = (k_{v\rho} + k_v)_{v\in\mathbf{a}}$ in Case UT as we did in the previous sections. We are interested in $E_{\mathbf{A}}(xq_1\sigma^{-1},\, s)$ with the elements σ and q_1 of $G_{\mathbf{h}}^N$ given by (24.5) and $q_1 = q \times \varepsilon h\varepsilon$ as in Lemma 24.3. To be explicit, we have

$$(24.15)\qquad E_{\mathbf{A}}(xq_1\sigma^{-1},\, s) = \sum_{\alpha\in A} \mu(\alpha x q_1\sigma^{-1})\varepsilon(\alpha x q_1 \sigma^{-1})^{-s}, \quad A = P^N\backslash G^N.$$

The function μ is given by (16.26a, b) with P^N and D^N as P and $\widetilde{D}$ there. For some practical reasons, we hereafter use the letter a instead of h; thus

$$(24.15\text{a})\qquad q_1 = q \times \varepsilon a\varepsilon, \quad a \in G_{\mathbf{h}}^r, \quad a_v = 1 \ \text{ for every } v|\mathfrak{c}.$$

Now, from $E_{\mathbf{A}}(xq_1\sigma^{-1},\, s)$ we obtain a function $H_{q,a}(\mathfrak{z},\, s)$ of $(\mathfrak{z},\, s) \in \mathcal{H}^N \times \mathbf{C}$ by the standard principle, that is, $H_{q,a}\bigl(y(\mathbf{i}),\, s\bigr) = E_{\mathbf{A}}(q_1\sigma^{-1}y,\, s) j_y^k(\mathbf{i})$ for every $y \in G_{\mathbf{a}}^N$. Suppressing the variable s, we write the function simply $H_{q,a}(\mathfrak{z})$. Then putting $\mathfrak{z} = y(\mathbf{i})$, we have $H_{q,a}(\mathfrak{z}) = \sum_{\alpha\in A} p_\alpha(\mathfrak{z})$ with

$$p_\alpha(\mathfrak{z}) = \mu(\alpha q_1\sigma^{-1}y)\varepsilon(\alpha q_1\sigma^{-1}y)^{-s} j_y^k(\mathbf{i}).$$

From (16.23a) and (16.26b) we easily see that

$$(24.16)\qquad p_\alpha(\mathfrak{z}) = \mu(\alpha_{\mathbf{h}} q_1\sigma^{-1})\varepsilon(\alpha_{\mathbf{h}} q_1\sigma^{-1})^{-s} \delta(\mathfrak{z})^{s\mathbf{a} - (m - i\kappa)/2} \|_k\, \alpha,$$

where $\delta(\mathfrak{z}) = \bigl(\det\bigl((i/2)(\mathfrak{z}^* - \mathfrak{z})_v\bigr)\bigr)_{v\in\mathbf{a}}$. By Lemma 24.1, $P^N\backslash G^N$ can be given as $\bigsqcup_{\nu=0}^r A_\nu$ with $A_\nu = P^N\backslash P^N\tau_\nu(G^n \times G^r)$. Put

$$(24.17)\qquad \mathcal{E}_\nu(\mathfrak{z}) = \sum_{\alpha\in A_\nu} p_\alpha(\mathfrak{z}), \quad \mathcal{E}_\nu(z,\, w) = \mathcal{E}_\nu\bigl(\mathrm{diag}[z,\, w]\bigr) \qquad (z \in \mathcal{H}^n,\, w \in \mathcal{H}^r).$$

We have then $H_{q,a} = \sum_{\nu=0}^{r} \mathcal{E}_\nu$. The functions $H_{q,a}$ and $\mathcal{E}_\nu$ involve s, but for the moment we suppress it. From (24.16) we easily see that

$$p_\alpha \| \alpha' = p_{\alpha\alpha'} \text{ for every } \alpha' \in G^N \cap D' \tag{24.18}$$

with a suitable open subgroup D' of D^N independent of α. Take a congruence subgroup Γ of G^r such that $1_{2n} \times \Gamma \subset D'$. Then, from (24.17) and (24.18) we obtain

$$\mathcal{E}_\nu \| (1_{2n} \times \gamma) = \mathcal{E}_\nu \text{ for every } \gamma \in \Gamma. \tag{24.19}$$

Our next task is to obtain an explicit form of $\mathcal{E}_r(z, w)$. Given $y \in G_{\mathbf{a}}^N$ and $\alpha \in A_r$, we have $\mu(\alpha y q_1 \sigma^{-1}) \neq 0$ if and only if $\alpha y q_1 \sigma^{-1} \in P_{\mathbf{A}}^N D^N$. Since $(q_1)_v = 1$ for every $v|\mathfrak{c}$ and $P_v^N D_v^N = G_v$ for $v \nmid \mathfrak{c}$, we have $P_{\mathbf{A}}^N D^N \sigma q_1^{-1} = P_{\mathbf{A}}^N D^N \sigma$. Therefore by Lemma 24.2 (2) we can replace A_r by the set of elements α of the form $\alpha = \tau_r\big((\xi \times 1)\beta \times 1\big)$ with $\xi \in \Xi$ and $\beta \in \mathcal{R}_q$; we have $\beta \in \zeta q C^{n,r} q^{-1}$ where ζ and q are as in (23.25) and (23.27). Changing the notation, we use the letter b instead of g in (23.25); thus $q = b \times \operatorname{diag}[\widehat{\varphi}, \varphi]$ with $b \in G_{\mathbf{h}}^r$, $b_v = 1$ for every $v|\mathfrak{c}$. Put $x = \alpha y q_1 \sigma^{-1}$ with such an α and $y \in G_{\mathbf{a}}^N$ as above. Since $x \in P_{\mathbf{A}}^N D^N$, we can put $x = pw$ with $p \in P_{\mathbf{A}}^N$ and $w \in D^N$. Then we have $d_x = d_p d_w$ and $\det(d_p)\mathfrak{r} = \mathrm{il}_{\mathfrak{b}}(x)$, and so by (24.7) and (24.8),

$$\begin{aligned}
&\chi_{\mathbf{h}}\big(\det(d_p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_w)\big)^{-1} = (\chi_{\mathbf{h}}/\chi_{\mathfrak{c}})\big(\det(d_p)\big)^{-1}\chi_{\mathfrak{c}}\big(\det(d_x)\big)^{-1} \\
&\quad = \chi^*\big(\nu_{\mathfrak{b}}(a^{-1}\xi e b)\big)(\chi_{\mathbf{h}}/\chi_{\mathfrak{c}})\big(\theta^{-r}\det(a)\det(s\varphi)\big)^{-1}\chi_{\mathfrak{c}}\big(\theta^r \lambda_r(\beta)^{-1}\big) \\
&\quad = \chi_{\mathbf{h}}\big(\theta^{-r}\det(a)\det(\varphi)\big)^{-1}\chi^*\big(\nu_{\mathfrak{b}}(a^{-1}\xi e b)\big)\chi_{\mathbf{a}}\big(\lambda_r(\beta)\big)(\chi_{\mathbf{h}}/\chi_{\mathfrak{c}})\big(\det(s)^{-1}\lambda_r(\beta)\big).
\end{aligned}$$

24.7. To simplify our notation, we put $\eta = \eta_r$, $\wp = \wp_r$, and

$$\mathbf{s} = s\mathbf{a} - (m - i\kappa)/2 \quad (\in \mathbf{C}^{\mathbf{a}}), \tag{24.20a}$$

$$\nu_1(\xi) = \nu_{\mathfrak{b}}(a^{-1}\xi e b), \quad N_1(\xi) = N\big(\nu_1(\xi)\big) \qquad (\xi \in \Xi), \tag{24.20b}$$

where $N(\mathfrak{x})$ denotes the absolute norm of an $\mathfrak{r}$-ideal $\mathfrak{x}$. These are temporary and will be discarded soon. It should be remembered that ν_1 depends on $\mathfrak{b}$, a, b, and e, among which a, b, and $\mathfrak{b}$ can be fixed, but e depends on β. Now, by (16.23a) and (24.7),

$$\varepsilon(\alpha_{\mathbf{h}} q_1 \sigma^{-1}) = \varepsilon(x_{\mathbf{h}}) = N\big(\mathrm{il}_{\mathfrak{b}}(x)\big)^{-u} = N_1(\xi)^u |\theta^{-r}\det(s\varphi)|_K^u,$$

where $u = 2/[K : F]$. (Notice that $|\det(a)|_K = 1$, since $\det(a)\det(a)^\rho = 1$.) Since $\det(s)\mathfrak{r} = \mathfrak{a}_r^q(\beta)$, the last factor of the equality at the end of §24.6 can be written $\chi^*\big(\lambda_r(\beta)\mathfrak{a}_r^q(\beta)^{-1}\big)$. Combining all these formulas with (16.26b) and (24.16), we obtain

$$\begin{aligned}
p_\alpha(\mathfrak{z}) &= \chi_{\mathbf{h}}\big(\theta^{-r}\det(a)\det(\varphi)\big)^{-1}|\theta^{-r}\det(\varphi)|_K^{-us} \\
&\quad \cdot \chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us}N\big(\mathfrak{a}_r^q(\beta)\big)^{us}\chi[\beta]\big(\delta^{\mathbf{s}}\|_k\,\alpha\big)(\mathfrak{z}),
\end{aligned} \tag{24.21}$$

where $\chi[\beta] = \chi_{\mathbf{a}}\big(\lambda_r(\beta)\big)\chi^*\big(\lambda_r(\beta)\mathfrak{a}_r^q(\beta)^{-1}\big)$. Put

$$M(w', w) = \delta(w)^{m+\mathbf{s}}\det(w' - w^*)^{-m}|\det(w' - w^*)|^{-2\mathbf{s}} \quad (w, w' \in \mathcal{H}^r), \tag{24.22a}$$

$$\mathcal{A}(s) = \chi\big(\theta^{-r}\det(a)\det(\varphi)\big)^{-1}|\theta^{-r}\det(\varphi)|_K^{-us}. \tag{24.22b}$$

Now $\big(\delta^{\mathbf{s}}\|_k\,\alpha\big)(\mathfrak{z}) = j_\alpha^k(\mathfrak{z})^{-1}|j_\alpha(\mathfrak{z})|^{-2\mathbf{s}}\delta(\mathfrak{z})^{\mathbf{s}}$. By (24.12) and (24.13) we obtain

$$(\delta^s\|_k\,\alpha)\bigl(\mathrm{diag}[z,\,w]\bigr) = j_\xi^k\bigl(\wp(\beta z)\bigr)^{-1}\left|j_\xi\bigl(\wp(\beta z)\bigr)\right|^{-2s}\bigl(\delta(w)^s\|_k\,\eta^{-1}\bigr)$$
$$\cdot\det\bigl[\xi\wp(\beta z)+\eta^{-1}w\bigr]^{-m}\left|\det\bigl[\xi\wp(\beta z)+\eta^{-1}w\bigr]\right|^{-2s}\bigl(\delta(z)^s\|_k\,\beta\bigr).$$

Transforming our functions by η, we put

$$\mathcal{F}(z,\,w) = H_{q,a}\bigl(\mathrm{diag}[z,\,\eta w]\bigr)j_\eta^k(w)^{-1},\quad \mathcal{F}_\nu(z,\,w) = \mathcal{E}_\nu(z,\,\eta w)j_\eta^k(w)^{-1}. \tag{24.23}$$

Change w for $-w^*$; then from the above calculations we obtain

$$\delta(w)^m\mathcal{F}_r(z,\,-w^*) = \mathcal{A}(s)\sum_{\beta\in\mathcal{R}_q}\sum_{\xi\in\Xi}\chi^*\bigl(\nu_1(\xi)\bigr)N_1(\xi)^{-us}j_\xi^k\bigl(\wp(\beta z)\bigr)^{-1}$$
$$\cdot\left|j_\xi\bigl(\wp(\beta z)\bigr)\right|^{-2s}\chi[\beta]N\bigl(\mathfrak{a}_r^q(\beta)\bigr)^{us}M\bigl(\xi\wp(\beta z),\,w\bigr)\bigl(\delta(z)^s\|\beta\bigr). \tag{24.24}$$

We note here an easy fact: If $f\in\mathcal{S}_k^r(\Gamma)$ and $\gamma\in\Gamma$, then the expression

$$M(\xi w',w)j_\xi^k(w')^{-1}|j_\xi(w')|^{-2s}f(w)$$

is invariant under $(w,\,\xi)\mapsto(\gamma w,\,\gamma\xi)$ for every $\gamma\in\Gamma$. This follows immediately from (24.14).

24.8. We take $\mathcal{B}$ as in (20.5) with r in place of n, which we denoted by $\mathcal{B}^r$ in §23.4; we assume that $c_v=1$ for every $c\in\mathcal{B}$ and every $v|\mathfrak{c}$; we take $\mathcal{B}=\{1\}$ in Case SP; also, in Case UT we take each c in the form $c=\mathrm{diag}[\widehat{d},\,d]$ with $d\in GL_r(K)_{\mathbf{h}}$ (see [S97, Lemma 9.8 (3)].) We put $\Gamma^c = G^r\cap cC^rc^{-1}$ for each $c\in\mathcal{B}$. Let $\mathbf{f}\in\mathcal{S}_k^r(C^r)$. For each $c\in\mathcal{B}$ define f_c as in (20.3b). For every function ψ on $\mathcal{H}^r$ we define a function $\psi^\sim=\widetilde{\psi}$ on $\mathcal{H}^r$ by $\widetilde{\psi}(w)=\overline{\psi(-w^*)}$. Notice that $\widetilde{\psi}\in\mathcal{S}_{k\rho}^r(\Gamma^c)$ if $\psi\in\mathcal{S}_k^r(\Gamma^c)$. This is easy in Case SP; we need (5.34) in Case UT.

We now assume that $n>r$, and consider

$$\int_{\mathfrak{D}}\mathcal{F}(z,\,w)\overline{\psi_a(w)}\delta(w)^m\mathbf{d}w\qquad(\mathfrak{D}=\Gamma_0\backslash\mathcal{H}^r) \tag{24.25}$$

with a congruence subgroup Γ_0 of G^r contained in Γ^a, where $\psi_a=(f_a)^\sim$ and $\mathbf{d}w$ is as in Lemma 24.5. This can be written $\sum_{\nu=0}^r I_\nu$ with

$$I_\nu = \int_{\mathfrak{D}}\mathcal{F}_\nu(z,\,w)\overline{\psi_a(w)}\delta(w)^m\mathbf{d}w. \tag{24.26}$$

By (24.19), these integrals are (at least formally) meaningful for a suitable choice of Γ_0. As explained in [S97, §22.12], integrals of this type converge locally uniformly on $\mathcal{H}^n\times\{\,s\in\mathbf{C}\,|\,\mathrm{Re}(s)>\sigma_0\,\}$ for some $\sigma_0\in\mathbf{R}$. We shall also show in §24.10 that $I_\nu=0$ for $0\le\nu<r$.

To compute I_r, we first note

$$I_r=\int_{\mathfrak{D}}\mathcal{F}_r(z,\,-w^*)f_a(w)\delta(w)^m\mathbf{d}w.$$

Now $\nu_1(\xi)$ and $N_1(\xi)$ depend only on $\Gamma^a\xi$, and so, in view of the remark at the end of §24.7, each term of (24.24) times $f_a(w)$ is invariant under $(w,\,\xi)\mapsto(\gamma w,\,\gamma\xi)$ for every $\gamma\in\Gamma^a$. Let R^a be a complete set of representatives for $\Gamma^a\backslash\Xi$, and let $\mathcal{F}'(z,\,-w^*)$ denote the function such that $\delta(w)^m\mathcal{F}'(z,\,-w^*)$ is the right-hand side of (24.24) with $\sum_{\xi\in R^a}$ in place of $\sum_{\xi\in\Xi}$. Then

$$\mathcal{F}_r(z,\,-w^*)\delta(w)^mf_a(w)=\sum_{\gamma\in\Gamma^a}\mathcal{F}'\bigl(z,\,-(\gamma w)^*\bigr)(\delta^mf_a)(\gamma w),$$

so that

$$I_r=\mu\int_{\mathcal{H}^r}\mathcal{F}'(z,\,-w^*)\delta(w)^mf_a(w)\mathbf{d}w,$$

where $\mu = [\Gamma^a : \Gamma_0][\Gamma_0 \cap \mathfrak{r}^\times : 1]$. For $c \in G_\mathbf{h}^r$ put

(24.27)
$$f_a|\mathfrak{T}_c^a = \sum_{\xi \in R^a} \chi^*\big(\nu_\mathfrak{b}(a^{-1}\xi c)\big) N\big(\nu_\mathfrak{b}(a^{-1}\xi c)\big)^{-us} f_a\|_k\xi.$$

Then $f_a|\mathfrak{T}_{eb}^a = \sum_{\xi \in R^a} \chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us} f_a\|_k\xi$. By Lemma 24.5, termwise integration yields

$$\begin{aligned} I_r &= \mu c_m(\mathbf{s})\mathcal{A}(s)\sum_\beta \sum_{\xi \in R^a} \chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us} \\ &\qquad\qquad \cdot (f_a\|\xi)\big(\wp(\beta z)\big)\delta\big(\wp(\beta z)\big)^{-\mathbf{s}}\chi[\beta]N\big(\mathfrak{a}_r^q(\beta)\big)^{us}\delta^{\mathbf{s}}\|\,\beta \\ &= \mu c_m(\mathbf{s})\mathcal{A}(s)\sum_\beta (f_a|\mathfrak{T}_{eb}^a)\big(\wp(\beta z)\big)\chi[\beta]N\big(\mathfrak{a}_r^q(\beta)\big)^{us}\big(\delta/(\delta\circ\wp)\big)^{\mathbf{s}}\|\,\beta \\ &= \mu c_m(\mathbf{s})\mathcal{A}(s)\sum_\beta \chi[\beta]N\big(\mathfrak{a}_r^q(\beta)\big)^{us}\delta(z,\, s;\, f_a|\mathfrak{T}_{eb}^a,\, \kappa)\|\,\beta \end{aligned}$$

with c_m of Lemma 24.5 and $\delta(\cdots)$ of (23.28). Put

(24.28)
$$\mathbf{f}|_\chi\mathfrak{T} = \sum_{\tau \in C\backslash\mathfrak{X}/C} (\mathbf{f}|C\tau C)\chi^*\big(\nu_\mathfrak{b}(\tau)\big)N\big(\nu_\mathfrak{b}(\tau)\big)^{-us} \qquad (C = C^r).$$

This is obtained from (20.11) by substituting $\chi^*\big(\nu_\mathfrak{b}(\tau)\big)N\big(\nu_\mathfrak{b}(\tau)\big)^{-us}$ for $[\nu_\mathfrak{b}(\tau)]$. Let $\mathbf{f}|_\chi\mathfrak{T} \leftrightarrow (f_b')_{b\in\mathcal{B}}$. By Lemma 20.10 we have $f_{eb}' = \sum_{a\in\mathcal{B}} f_a|\mathfrak{T}_{eb}^a$.

Define $E(x,\, s;\, \mathbf{f}|_\chi\mathfrak{T},\, \chi,\, C^{n,r})$ by (23.24) with $\mathbf{f}|_\chi\mathfrak{T}$ in place of $\mathbf{f}$; associate a function E_q to this by (23.26) for q as in (23.25), and denote it by $E_q(z,\, s;\, \mathbf{f}|_\chi\mathfrak{T})$. Combining the above calculation with Proposition 23.5, we obtain a fundamental formula

(24.29)
$$\begin{aligned} &\mu c_m(\mathbf{s})\chi(\theta)^r N(\mathfrak{b}\mathfrak{r})^{-rus}E_q(z,\, s;\, \mathbf{f}|_\chi\mathfrak{T}) \\ &\quad = \sum_{a\in\mathcal{B}} \chi_\mathbf{h}\big(\det(a)\big)\int_\mathfrak{D} J_{q,a}(z,\, -w^*;\, s)f_a(w)\delta(w)^m\mathbf{d}w, \end{aligned}$$

where $J_{q,a}(z,\, w;\, s) = H_{q,a}\big(\mathrm{diag}[z,\, \eta w]\big)j_\eta^k(w)^{-1}$. (We can take μ to be the same for all $a \in \mathcal{B}$, since $[\Gamma^a : \Gamma_0]$ does not depend on a; see [S97, Lemma 8.15].) Now, our termwise integration can be justified for sufficiently large $\mathrm{Re}(s)$, because of the validity of Lemma 24.5 for such an s and of the convergence of (24.25) and the series expressing E_q in (23.26). Assuming that $\mathbf{f}$ is an eigenform, multiply both sides of (24.29) by $\Lambda_\mathfrak{c}^{n+r}(s,\, \chi)$. Then we obtain

(24.29a)
$$\begin{aligned} &\mu c_m(\mathbf{s})\chi(\theta)^r N(\mathfrak{b}\mathfrak{r})^{-rus}\mathcal{F}_q(z,\, s;\, \mathbf{f},\, \chi,\, C) \\ &\quad = \sum_{a\in\mathcal{B}} \chi_\mathbf{h}\big(\det(a)\big)\int_\mathfrak{D} \Lambda_\mathfrak{c}^{n+r}(s,\, \chi)J_{q,a}(z,\, -w^*;\, s)f_a(w)\delta(w)^m\mathbf{d}w. \end{aligned}$$

24.9. So far we have assumed that $n > r$ in the above treatment, but everything is meaningful even if $n = r$. Suppose $n = r$; then the sum $\sum_\beta$ in §24.8 consists of a single term for $\beta = 1$; $\wp(z) = z$ for $z \in \mathcal{H}^n = \mathcal{H}^r$, $q = b$, $\zeta = 1$, $e = 1$, and so we have $I_r = \mu c_m(\mathbf{s})\mathcal{A}(s)f_a|\mathfrak{T}_b^a$. Then (24.29) can be written

(24.30)
$$\begin{aligned} &\mu c_m(\mathbf{s})\chi(\theta)^n N(\mathfrak{b}\mathfrak{r})^{-nus}f_b'(z,\, s) \\ &\quad = \sum_{a\in\mathcal{B}} \chi_\mathbf{h}\big(\det(a)\big)\int_\mathfrak{D} J_{b,a}(z,\, -w^*;\, s)f_a(w)\delta(w)^m\mathbf{d}w, \end{aligned}$$

where $f_b'(z,\, s)$ is the b-component of $\mathbf{f}|\chi\mathfrak{T}$. In particular, if $\mathbf{f}$ is an eigenfunction as in §20.6, we have $f_b'(z,\, s) = \mathfrak{T}(us,\, \mathbf{f},\, \chi)f_b(z)$.

24.10. Let us now prove that $I_\nu = 0$ for $0 \le \nu < r$. Fixing ν, take a complete set of representatives $\mathcal{R}^n$ (resp. $\mathcal{R}^r$) for $P^{n,\nu}\backslash G^n$ (resp. $P^{r,\nu}\backslash G^r$), and put $p_{\xi,\beta,\gamma} = p_\alpha$ for $\alpha = \tau_\nu\big((\xi \times 1_{2n-2\nu})\beta \times \gamma\big)$ with $\xi \in G^\nu$, $\beta \in G^n$, and $\gamma \in G^r$. By Lemma 24.1, $\mathcal{E}_\nu = \sum_{\xi\in G^\nu}\sum_{\beta\in\mathcal{R}^n}\sum_{\gamma\in\mathcal{R}^r} p_{\xi,\beta,\gamma}$. Let Γ be a congruence subgroup of G^r such that $1_{2n} \times \Gamma \subset D'$ with D' as in (24.18). Take a finite subset T of G^r so that $G^r = \bigsqcup_{\tau\in T} P^{r,\nu}\tau\Gamma$ (see [S97, Lemma 9.8]). Then we can take $\mathcal{R}^r = \bigsqcup_{\tau\in T}\mathcal{R}_\tau$ with $\mathcal{R}_\tau = (P^{r,\nu}\cap\tau\Gamma\tau^{-1})\backslash\tau\Gamma$. Put $g_\tau = \sum_{\xi\in G^\nu}\sum_{\beta\in\mathcal{R}^n} p_{\xi,\beta,\tau}\|(1\times\tau^{-1})$. If $\varepsilon \in \tau\Gamma$, then $1 \times \tau^{-1}\varepsilon \in D'$, and so $p_{\xi,\beta,\tau}\|(1\times\tau^{-1}\varepsilon) = p_{\xi,\beta,\varepsilon}$ by (24.18). Thus $\mathcal{E}_\nu = \sum_{\tau\in T}\sum_{\varepsilon\in\mathcal{R}_\tau} g_\tau\|(1\times\varepsilon)$. Now, given $\gamma \in P^{r,\nu}\cap\tau\Gamma\tau^{-1}$, put $\omega = \kappa_\nu\pi_\nu(\gamma)^{-1}\kappa_\nu$ with κ_ν of Lemma 24.1. Then by that lemma we have

$$\tau_\nu\big((\xi\times 1_{2n-2\nu})\beta\times\tau\big)(1_{2n}\times\tau^{-1}\gamma\tau) \in P_N\tau_\nu\big((\omega\xi\times 1_{2n-2\nu})\beta\times\tau\big),$$

and hence $p_{\xi,\beta,\tau}\|(1\times\tau^{-1}\gamma\tau) = p_{\omega\xi,\beta,\tau}$, since p_α depends only on $P^N\alpha$. This shows that $g_\tau\|(1\times\gamma) = g_\tau$ for such a γ. Next, put

$$p^\circ(z,\,w) = p\big(\mathrm{diag}[z,\,w]\big) \qquad (z\in\mathcal{H}^n,\ w\in\mathcal{H}^r) \tag{24.31}$$

for a function p on $\mathcal{H}^N$; let R be the unipotent radical of $P^{r,\nu}$ and let $\zeta \in R_{\mathbf{a}}$. Since $\wp_\nu(\zeta w) = \wp_\nu(w)$ for every $w \in \mathcal{H}^r$ and $j_\zeta = 1$, from (24.11a) we see that $\big[\delta^s\|\big(\tau_\nu(1_{2n}\times\zeta)(\beta\times\gamma)\big)\big]^\circ = \big[\delta^s\|\tau_\nu(\beta\times\gamma)\big]^\circ$. Now, for $\alpha = \tau_\nu\alpha'$ with $\alpha' = (\xi\times 1_{2n-2\nu})\beta\times\tau$ we have $p_\alpha\|(1\times\tau^{-1}) = c_\alpha\delta^s\|\big(\tau_\nu\alpha'(1\times\tau^{-1})\big)$ with a constant c_α, and hence $\big[p_\alpha\|(1\times\tau^{-1})(1\times\zeta)\big]^\circ = c_\alpha\big[\delta^s\|\big(\tau_\nu(1\times\zeta)\alpha'(1\times\tau^{-1})\big)\big]^\circ = c_\alpha\big[\delta^s\|\big(\tau_\nu\alpha'(1\times\tau^{-1})\big)\big]^\circ = \big[p_\alpha\|(1\times\tau^{-1})\big]^\circ$. Thus $\big[g_\tau\|(1\times\zeta)\big]^\circ = (g_\tau)^\circ$ for every $\zeta\in R_{\mathbf{a}}$. Put $g(w) = (g_\tau)^\circ(z,\,w)$ with fixed τ and z. We have shown that $g\|\gamma = g$ for every $\gamma\in P^{r,\nu}\cap\tau\Gamma\tau^{-1}$ and $g\|\zeta = g$ for every $\zeta\in R_{\mathbf{a}}$. Therefore, by [S97, Lemma A3.8], $\langle\psi, \sum_{\varepsilon\in\mathcal{R}_\tau} g\|\varepsilon\rangle = 0$ for every $\psi\in\mathcal{S}_k^r$ if $r > \nu$. Consequently $\langle\psi(w),\,\mathcal{E}_\nu(z,\,w)\rangle = 0$ for every such ψ, at least for sufficiently large $\mathrm{Re}(s)$. Tranforming $\mathcal{E}_\nu$ by η, we obtain $I_\nu = 0$.

24.11. Lemma. *Let k be an integral or a half-integral weight, and Ψ a subfield of $\mathbf{C}$ containing the Galois closure of K over $\mathbf{Q}$. In order to emphasize the dimensionality, denote by $\mathcal{N}_k^{n,p}(\Psi)$ (resp. $\mathcal{N}_k^{n,p}(\Gamma,\,\Psi)$) the set $\mathcal{N}_k^p(\Psi)$ (resp. $\mathcal{N}_k^p(\Gamma,\,\Psi)$) of §14.11 defined with respect to G^n. If $f\in\mathcal{N}_k^{n+r,p}(\Psi)$, then $f^\circ(z,\,w)$ can be written as a finite sum $f^\circ(z,\,w) = \sum_{a=1}^t g_a(z)h_a(w)$ with $g_a\in\mathcal{N}_k^{n,p}(\Psi)$ and $h_a\in\mathcal{N}_k^{r,p}(\Psi)$. In particular, if $p = 0$, the conclusion holds for every subfield Ψ of $\mathbf{C}$ containing the field Φ_k of Theorem 10.4 (5) or Theorem 10.7 (5).*

PROOF. Take congruence subgroups Γ^i of G^i for $i = n,\ r$, and $n+r$ so that $\Gamma^n\times\Gamma^r\subset\Gamma^{n+r}$ and $f\in\mathcal{N}_k^{n+r,p}(\Gamma^{n+r},\Psi)$. We can take each Γ^i to be a principal congruence subgroup of some level such that $\mathcal{N}_k^{i,p}(\Gamma^i) = \mathcal{N}_k^{i,p}(\Gamma^i,\,\Psi)\otimes_\Psi\mathbf{C}$ as in Proposition 14.13 (1). Let $\{h_a\}_{a=1}^t$ be a Ψ-basis of $\mathcal{N}_k^{r,p}(\Gamma^r,\,\Psi)$. We easily see that $f^\circ(z,\,w)$ as a function of z (resp. w) belongs to $\mathcal{N}_k^{n,p}(\Gamma^n)$ (resp. $\mathcal{N}_k^{r,p}(\Gamma^r)$). (If k is half-integral, Proposition A2.12 is essential.) Therefore for each fixed z we have $f^\circ(z,\,w) = \sum_{a=1}^t g_a(z)h_a(w)$ with complex numbers $g_a(z)$ uniquely determined by z and a. Since $\bigcap_w\big\{x\in\mathbf{C}^t \bigm| \sum_{a=1}^t x_a h_a(w)\big\} = \{0\}$, we can find t points $w_1,\,\ldots,\,w_t$ of $\mathcal{H}^r$ such that $\det\big(h_a(w_b)\big)_{a,b=1}^t \ne 0$. Solving the linear equations $f^\circ(z,\,w_b) = \sum_{a=1}^t g_a(z)h_a(w_b)$, we find that $g_a\in\mathcal{N}_k^{n,p}$. Let $\sigma\in\mathrm{Aut}(\mathbf{C}/\Psi)$; then $f^\sigma = f$ and $(h_a)^\sigma = h_a$. Comparing the Fourier coefficients of f with those of g_a and h_a, and applying σ to them, we can easily verify that $f^\circ(z,\,w) =$

$\sum_{a=1}^{t}(g_a)^\sigma(z)h_a(w)$, and hence $(g_a)^\sigma = g_a$, that is, $g_a \in \mathcal{N}_k^{n,p}(\Psi)$. This proves our lemma in the general case. In the case $p = 0$, we have $\mathcal{M}_k(\Gamma^r) = \mathcal{M}_k(\Gamma^r, \Phi_k) \otimes_{\Phi_k} \mathbf{C}$ by Theorem 10.4 (5) or Theorem 10.7 (5), and hence the above argument is valid if $\Phi_k \subset \Psi$.

25. Proof of Theorems in Sections 20 and 23

25.1. Our computation of Section 24 is sufficient for the proof of Theorem 23.9 and a special case of Theorem 20.14. To prove the most general case of the latter theorem, we have to introduce certain differential operators. They are necessary only in the case $n = r$, and so we speak of G^{2n} instead of G^N. Returning to the setting of Section 12, we take T and $S_p(T)$ as in §13.13 and also $\mathfrak{K}$ as in (14.4), all with $2n$ instead of n there. Thus $T_v = \mathbf{C}_{2n}^{2n}$ in Case UT and $T_v = \{ z \in \mathbf{C}_{2n}^{2n} \mid {}^t z = z \}$ in Case SP; $\mathfrak{K} = GL_{2n}(\mathbf{C})^{\mathbf{b}}$. We shall simply write τ for the representation τ^p of §§13.13 and 14.4.

Given a representation $\{\omega, X\}$ of $\mathfrak{K}$, an irreducible subspace Z of $S_p(T)$ as in §13.13, and $\zeta \in Z$, we define differential operators B_ζ and C_ζ on $\mathcal{H}^{2n}$ by

$$B_\zeta f = (D_\omega^Z f)(\zeta), \qquad C_\zeta f = (E^Z f)(\zeta) \tag{25.1}$$

for $f \in C^\infty(\mathcal{H}^{2n}, X)$, where D_ω^Z and E^Z are defined in (13.22) and §14.4. Then

$$(B_\zeta f)\|_\omega\, \alpha = B_{\psi_1}(f\|_\omega\, \alpha) \ \text{ with } \ \psi_1(u) = \zeta\bigl(\lambda_\alpha(\mathfrak{z})u \cdot {}^t\mu_\alpha(\mathfrak{z})\bigr), \tag{25.2a}$$

$$(C_\zeta f)\|_\omega\, \alpha = C_{\psi_2}(f\|_\omega\, \alpha) \ \text{ with } \ \psi_2(u) = \zeta\bigl({}^t\lambda_\alpha(\mathfrak{z})^{-1}u\mu_\alpha(\mathfrak{z})^{-1}\bigr), \tag{25.2b}$$

where $u \in T$ and $\mathfrak{z}$ is the variable on $\mathcal{H}^{2n}$. Indeed, put $\lambda = \lambda_\alpha(\mathfrak{z})$, $\mu = \mu_\alpha(\mathfrak{z})$, $g = D_\omega^Z f$, and define ψ_1 as above for a given $\zeta \in Z$. Then, by (12.21) and (12.24a), or rather by their generalizations mentioned in §13.13, we have

$$\begin{aligned} B_{\psi_1}(f\|_\omega\, \alpha) &= \bigl[D_\omega^Z(f\|_\omega\, \alpha)\bigr](\psi_1) = \omega(\lambda, \mu)^{-1}(g \circ \alpha)\bigl(\tau({}^t\lambda, {}^t\mu)^{-1}\psi_1\bigr) \\ &= (g\|_\omega\, \alpha)(\zeta) = g(\zeta)\|_\omega\, \alpha = (B_\zeta f)\|_\omega\, \alpha, \end{aligned}$$

which gives (25.2a). The other formula can be proved in the same way.

Let us now consider the case $\omega(x) = \det(x)^k$ with $k \in \mathbf{Z}^{\mathbf{b}}$ and $Z = \bigotimes_{v\in\mathbf{a}} Z_v$, where Z_v is the irreducible subspace of $S_{ne_v}(T_v)$ whose highest weight vector is $\det_n(u)^{e_v}$ for $u \in T_v$ with $0 \le e_v \in \mathbf{Z}$ (see Theorem 12.7). We assume $e_v \le 1$ in Case SP. For $u \in \mathbf{C}_{2n}^{2n}$ we denote by u_ℓ the lower left $(n \times n)$-block of z. We then define an elements φ of $S_{ne}(T)$ by

$$\varphi(u) = \prod_{v\in\mathbf{a}} \det((u_v)_\ell)^{e_v} \qquad (u \in T). \tag{25.3}$$

Then $\varphi \in Z$. This is clear in Case UT, since we can find an element $(a, b) \in \mathfrak{K}$ such that the upper right $(n \times n)$-block of ${}^t aub$ is u_ℓ. In Case SP, the inclusion $\varphi \in Z$, which holds only under the assumption that $e_v \le 1$, can be seen as follows.

Clearly it is sufficient to consider a single v, and so we drop the subscript v, Using the symbols a_x, b_x, c_x, and d_x for $x \in \mathbf{C}_{2n}^{2n}$, we have $({}^t xux)_\ell = {}^t b_x a_u a_x + Y$ for $(x, x) \in \mathfrak{K}$ and $u \in T$ with a matrix Y that does not involve u_{11}. Thus $\varphi({}^t xux)$ as a function of u is of degree ≤ 1 in u_{11}. Let W be the irreducible subspace of $S_n(T_v)$ contained in the $\tau(\mathfrak{K})$-span of φ. A highest weight vector of W can be given in the form $\prod_\nu \det_\nu(u)^{c_\nu}$ with (c_ν) such that $n = \sum_\nu \nu c_\nu$. Since this must be of degree ≤ 1 in u_{11}, we see that $W = Z$, which proves that $\varphi \in Z$ in Case SP.

25.2. Given $k \in \mathbf{Z}^{\mathbf{b}}$, $\beta, \gamma \in G^n$, and a function g on $\mathcal{H}^n \times \mathcal{H}^n$, we define a function $g\|_k(\beta \times \gamma)$ on $\mathcal{H}^n \times \mathcal{H}^n$ by

$$(25.4)\qquad \big(g\|_k(\beta \times \gamma)\big)(z, w) = j_\beta^k(z)^{-1} j_\gamma^k(w)^{-1} g(\beta z, \gamma w) \qquad (z, w \in \mathcal{H}^n).$$

Fixing k, let us hereafter write B_e and C_e for B_φ and C_φ with φ as above. Employing the symbol p° of (24.31) with $r = n$, for $\beta, \gamma \in G^n$ we have

$$(25.5\text{a})\qquad \big[B_e\big(f\|_k(\beta \times \gamma)\big)\big]^\circ = \det(\gamma)^e (B_e f)^\circ\|_{k+e}(\beta \times \gamma),$$

$$(25.5\text{b})\qquad \big[C_e\big(f\|_k(\beta \times \gamma)\big)\big]^\circ = \det(\gamma)^{-e} (C_e f)^\circ\|_{k-e}(\beta \times \gamma),$$

where we consider e as an element of $\mathbf{Z}^{\mathbf{b}}$ via the natural injection of $\mathbf{Z}^{\mathbf{a}}$ into $\mathbf{Z}^{\mathbf{b}}$. To prove these, take $\omega(x) = \det(x)^k$ and $\zeta = \varphi$ in (25.2a). For $\alpha = \beta \times \gamma$ and $\mathfrak{z} = \mathrm{diag}[z, w]$ we have $\big(\lambda_\alpha(\mathfrak{z})u \cdot {}^t\mu_\alpha(\mathfrak{z})\big)_\ell = \lambda_\gamma(w)u_\ell \cdot {}^t\mu_\beta(z)$, and hence $\psi_1 = j_\gamma^{e\rho}(w) j_\beta^e(z)\varphi$, which gives (25.5a). Formula (25.5b) follows from (25.2b) in a similar way. C_e can be defined in both cases, but actually it is unnecessary in Case SP.

Now take two elements e and e' of $\mathbf{Z}^{\mathbf{a}}$ such that $e_v \geq 0$, $e'_v \geq 0$, and $e_v e'_v = 0$ for every $v \in \mathbf{a}$. Put $D_{e,e'} = B_e C_{e'}$. Then from (25.5a, b) we obtain immediately

$$(25.6)\qquad \big[D_{e,e'}\big(f\|_k(\beta \times \gamma)\big)\big]^\circ = \det(\gamma)^{e-e'} (D_{e,e'} f)^\circ\|_{k+e-e'}(\beta \times \gamma).$$

25.3. Lemma. *Let τ_n be as in Lemma 24.1 with $N = 2n$ and let $\mathrm{s} = (s_v)_{v \in \mathbf{a}} \in \mathbf{C}^{\mathbf{a}}$; let $m = k$ in Case SP and $m = (k_v + k_{v\rho})_{v \in \mathbf{a}}$ in Case UT. Then*

$$\big[D_{e,e'}(\delta^{\mathrm{s}}\|_k \tau_n)\big]^\circ = \Psi(\mathrm{s})\big(\delta^{\mathrm{s}+e'}\|_h \tau_n\big)^\circ$$
$$\text{with}\quad \Psi(\mathrm{s}) = \prod_{e_v > 0} \psi_v(-m_v - s_v) \prod_{e'_v > 0} 2^{2ne'_v}\psi_v(-s_v),$$

where $h = k + e - e'$ in Case SP, $h = (k_{v\rho}, k_v + e_v - e'_v)_{v \in \mathbf{b}}$ in Case UT, and ψ_v is the polynomial ψ_Z of Theorem 12.13 for Z with $\det_n(x)^{e_v}$ or $\det_n(x)^{e'_v}$ as its highest weight vector.

PROOF. Take (τ_n, φ) as (α, ζ) in Lemma 13.9. Focusing our attention on one $v \in \mathbf{a}$, we drop the subscript v. For $\mathfrak{z} = \mathrm{diag}[z, w]$ we have $\lambda_\alpha(\mathfrak{z}) = \begin{bmatrix} 1 & {}^t w \\ {}^t z & 1 \end{bmatrix}$, $\mu_\alpha(\mathfrak{z}) = \begin{bmatrix} 1 & w \\ z & 1 \end{bmatrix}$, and $\mu_\alpha(\mathfrak{z})^{-1} = \begin{bmatrix} 1 - wz & 0 \\ 0 & 1 - zw \end{bmatrix}^{-1} \begin{bmatrix} 1 & -w \\ -z & 1 \end{bmatrix}$. Therefore

$$\big[\xi(\mathfrak{z})^{-1}\lambda_\alpha(\mathfrak{z})^* \cdot {}^t\mu_\alpha(\mathfrak{z})^{-1}\big]_\ell = -i \cdot {}^t(1 - wz)^{-1},$$

and hence $\varphi\big(\xi(\mathfrak{z})^{-1}\lambda_\alpha(\mathfrak{z})^* \cdot {}^t\mu_\alpha(\mathfrak{z})^{-1}\big) = (-i)^{ne} j(\tau_n, \mathfrak{z})^{-e}$ by (24.11b). Similarly, for $e'_v > 0$ we have

$$\big[{}^t\lambda_\alpha(\mathfrak{z})\widehat{\mu}_\alpha(\mathfrak{z})\eta(\mathfrak{z})\big]_\ell = (w - w^*) \cdot {}^t(1 - \overline{wz})^{-1}\eta(z),$$

and hence

$$\varphi\big({}^t\lambda_\alpha(\mathfrak{z})\widehat{\mu}_\alpha(\mathfrak{z})\eta(\mathfrak{z})\big) = (4i)^{ne'}\delta(z)^{e'}\delta(w)^{e'}\overline{j(\tau_n, \mathfrak{z})}^{-e'} = (4i)^{ne'}\big(\delta^{e'}\|_{-e'}\tau_n\big)^\circ(z, w).$$

Therefore we obtain the desired result from the formulas of Lemma 13.9.

25.4. To prove Theorem 20.14, we first consider Case UT. Given h and ℓ as in that theorem, put $d = (d_v)_{v \in \mathbf{a}}$, $d_v = h_v - h_{v\rho} - \ell_v$, $e = (e_v)_{v \in \mathbf{a}}$, $e_v = \mathrm{Max}(d_v, 0)$, $e' = (e'_v)_{v \in \mathbf{a}}$, $e'_v = \mathrm{Max}(-d_v, 0)$, and $k = h - d$. We now consider $E_{\mathbf{A}}$ of (24.15) on $G_{\mathbf{A}}^{2n}$ with this k. Changing $\mathfrak{c}$ for its suitable multiple, we may assume, without changing $\mathcal{Z}(s, \mathbf{f}, \chi)$, that (16.24b) is satisfied. Notice that $d = e - e'$ and $k_v - k_{v\rho} = \ell_v$ for every $v \in \mathbf{a}$, so that condition (16.24a) is consistent with the assumption on χ in Theorem 20.14. We now take $H_{b,a}$ and $\mathcal{E}_\nu$ of §24.6 with $n = r$, and apply $D_{e,e'}$ to them; recall that $q = b$ if $n = r$. For $\nu = n$ we obtain, from (24.21), $\mathcal{E}_n = \mathcal{A}(s)\sum_{\xi\in\Xi}\chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us}\big(\delta^{\mathbf{s}}\|_k\,(\tau_n(\xi\times 1))\big)(\mathfrak{z})$, and hence by (25.6) and Lemma 25.3,

$$\big(D_{e,e'}\mathcal{E}_n\big)^\circ = \mathcal{A}(s)\Psi(\mathbf{s})\sum_{\xi\in\Xi}\chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us}\big(\delta^{\mathbf{s}+e'}\|_h\,\tau_n\big)^\circ\|_h\,(\xi\times 1).$$

Put

$$\mathcal{G}(z,\,w) = \big(D_{e,e'}H_{b,a}\big)^\circ(z,\,\eta w)j_\eta^h(w)^{-1},\quad \mathcal{G}_\nu(z,\,w) = \big(D_{e,e'}\mathcal{E}_\nu\big)^\circ(z,\,\eta w)j_\eta^h(w)^{-1}.$$

Then

$$\delta(w)^{m'}\mathcal{G}_n(z,\,-w^*) = \mathcal{A}(s)\Psi(\mathbf{s})\sum_{\xi\in\Xi}\chi^*\big(\nu_1(\xi)\big)N_1(\xi)^{-us}(\delta^{\mathbf{s}'}\|_h\xi)(z)M'(\xi z,\,-w^*),$$

where $m' = (h_v + h_{v\rho})_{v\in\mathbf{a}}$, $\mathbf{s}' = \mathbf{s} + e'$, and M' is defined by (24.22a) with m' and $\mathbf{s}'$ in place of m and $\mathbf{s}$.

Now $\big(D_{e,e'}\mathcal{E}_\nu\big)^\circ = 0$ if $\nu < n$ and $d \neq 0$. In view of (25.6) this follows from

(25.7) $$\big[D_{e,e'}\big(\delta^{\mathbf{s}}\|_k\,\tau_\nu\big)\big]^\circ = 0 \quad \text{if } \nu < n \text{ and } e + e' \neq 0.$$

To prove this, take g_ν as in Lemma 24.1. Then for $\mathfrak{z} = \mathrm{diag}[z,\,w]$ we have

$$\lambda(\tau_\nu,\,\mathfrak{z}) = \begin{bmatrix} 1_n & g_\nu\cdot{}^t w \\ g_\nu\cdot{}^t z & 1_n \end{bmatrix},\qquad \mu(\tau_\nu,\,\mathfrak{z}) = \begin{bmatrix} 1_n & g_\nu w \\ g_\nu z & 1_n \end{bmatrix},$$

and we can easily verify that $\big[\xi(\mathfrak{z})^{-1}\lambda(\tau_\nu,\,\mathfrak{z})^*\cdot{}^t\mu(\tau_\nu,\,\mathfrak{z})^{-1}\big]_\ell$ has rank $< n$. Therefore, taking $(\tau_\nu,\,\varphi)$ as $(\alpha,\,\zeta)$ in Lemma 13.9, we find that $\big(B_e(\delta^{\mathbf{s}}\|_k\,\tau_\nu)\big)^\circ = 0$ if $e \neq 0$. Similarly $\big[{}^t\lambda(\tau_\nu,\,\mathfrak{z})\cdot{}^t\overline{\mu(\tau_\nu,\,\mathfrak{z})}^{-1}\eta(\mathfrak{z})\big]_\ell$ has rank $< n$, and so we obtain, by the same lemma, $\big(C_{e'}(\delta^{\mathbf{s}}\|_k\,\tau_\nu)\big)^\circ = 0$ if $e' \neq 0$. This proves (25.7).

Thus $\mathcal{G} = \mathcal{G}_n$ if $d \neq 0$. Returning to the setting of §24.8, given $\mathbf{f} \in \mathcal{S}_h^n(C)$, we consider

$$\int_{\mathfrak{D}}\mathcal{G}(z,\,-w^*)f_a(w)\delta(w)^{m'}\mathbf{d}w \qquad (\mathfrak{D} = \Gamma_0\backslash\mathcal{H}^r)$$

with the function f_a associated with $\mathbf{f}$ as before. Repeating the calculations in §§24.8 and 24.9 with $\mathcal{G}$ in place of $\mathcal{F}$, we find that

(25.8) $$\begin{aligned}&\mu c_{m'}(\mathbf{s}')\chi(\theta)^nN(\mathfrak{b}\mathfrak{r})^{-nus}\Psi(\mathbf{s})f_b'(z,\,s)\\ &\quad= \sum_{a\in\mathcal{B}}\chi_{\mathbf{h}}\big(\det(a)\big)\int_{\mathfrak{D}}J'_{b,a}(z,\,-w^*;\,s)f_a(w)\delta(w)^{m'}\mathbf{d}w\end{aligned}$$

for sufficiently large $\mathrm{Re}(s)$, where $J'_{b,a}(z,\,w;\,s) = \big(D_{e,e'}H_{b,a}\big)^\circ(z,\,\eta w)j_\eta^h(w)^{-1}$. If $d = 0$, (25.8) becomes (24.30), and so (25.8) is true for any d.

In Case SP with integral h the matter is simpler. Given h and ℓ as in Theorem 20.14, put $e = h - \ell$ and $k = \ell$. We have again $(B_e\mathcal{E}_\nu)^\circ = 0$ for $\nu < n$ if $e \neq 0$, as the proof of (25.7) is valid in this case too. Then we have (25.8) with $(h,\,\mathbf{s})$ in place of $(m',\,\mathbf{s}')$.

25.5. Let $E_{\mathbf{A}}^N$ denote $E_{\mathbf{A}}$ of (16.27) defined with G^N as G, and $E_t^N(\mathfrak{z})$ denote, for $t \in G_{\mathbf{h}}^N$ and $\mathfrak{z} \in \mathcal{H}^N$, the function defined by formula (17.23a), which is meaningful for integral k in Cases SP and UT. In Case SP, if $t \in \alpha^{-1}D'$ with a sufficiently small open subgroup D' of $G_{\mathbf{A}}^N$, then we easily see that $E_t^N = E_1^N\|\alpha$. Observe that $H_{q,a} = E_t^N$ with $t = q_1\sigma^{-1}$.

Suppose that $\mathbf{f}$ is an eigenform; then $f_b'(z,\,s) = \mathfrak{T}(us,\,\mathbf{f},\,\chi)f_b(z)$ as noted at the end of §24.9, and $\mathcal{Z}(us,\,\mathbf{f},\,\chi) = \Lambda_{\mathfrak{c}}^{2n}(s,\,\chi)\mathfrak{T}(us,\,\mathbf{f},\,\chi)$ by (20.23), and hence from (25.8) we obtain

(25.8a) $$\mu c_{m'}(\mathbf{s}')\chi(\theta)^n N(\mathfrak{br})^{-nus}\Psi(\mathbf{s})\mathcal{Z}(us,\,\mathbf{f},\,\chi)f_b(z) = \sum_{a\in\mathcal{B}}\chi_{\mathbf{h}}\bigl(\det(a)\bigr)\int_{\mathfrak{D}}\Lambda_{\mathfrak{c}}^{2n}(s,\,\chi)J'_{b,a}(z,\,-w^*;\,s)f_a(w)\delta(w)^{m'}\mathbf{d}w$$

Now, for the reason explained in [S97, §23.12], the integral of (25.8a) defines a meromorphic function on the whole s-plane. To make the result more precise, we first consider Case SP, and put $Q(s) = \mathcal{G}^{2n}_{k,\kappa}(s)\Lambda^{2n}_{\mathfrak{c}}$ with the symbols of Theorem 16.11, taking $2n$ in place of n there. Now $Q(s)J'_{b,a}$ is the pullback of $Q(s)D_{e,e'}E^{2n}_t$, and hence it is meromorphic on the whole s-plane with possible poles only in the finite set described in Theorem 16.11. Therefore we can say the same for $\mathcal{G}^{2n}_{k,\kappa}(s)$ times the left-hand side of (25.8a). This means that $\mathcal{G}^{2n}_{k,\kappa}(s)c_h(\mathbf{s})\Psi(\mathbf{s})\mathcal{Z}(2s,\,\mathbf{f},\,\chi)$ can be continued to the whole s-plane with possible poles in the same set. Recall that $\mathbf{s} = (s_v) = s\mathbf{a}-(k-i\kappa)/2$. Therefore our task is to show that $c_h(\mathbf{s})\Psi(\mathbf{s})\mathcal{G}^{2n}_{k,\kappa}(s)$, with s replaced by $s/2$, produces the gamma factors as stated in Theorem 20.14.

For that purpose, let us write $g \sim g'$ for two meromorphic functions g and g' on $\mathbf{C}$ if both g/g' and g'/g are entire. Employing the explicit form of ψ_Z in Theorem 12.13, we find that

$$\Psi(\mathbf{s}) \sim \prod_{e_v>0}\prod_{b=0}^{n-1}\bigl(s + (k_v + i\kappa_v - b)/2\bigr).$$

Then by Lemma 24.5 we have

(*) $$c_h(\mathbf{s})\Psi(\mathbf{s}) \sim \prod_{v\in\mathbf{a}}\Gamma^1_n\bigl(s + e_v - \lambda_n + (k_v + i\kappa_v)/2\bigr)\Gamma^1_n\bigl(s + (k_v + i\kappa_v)/2\bigr)^{-1}.$$

Take $\gamma(s,\,a)$ as in Theorem 16.11 with $2n$ in place of n. If $a \ge n$, we easily see that $\gamma(s,\,a)$ times (*) gives the desired $\Gamma^{h,\ell}_v$ of Theorem 20.14. If $(n-2)/2 \le a < n$, we have

$$\begin{aligned}\gamma(s,\,a) &= \Gamma^1_{2a+1}(s + a/2)\Gamma(2s - a - 1)\textstyle\prod_{b=a+2}^{n}\Gamma(2s - b)\\ &\sim \Gamma^1_{2a+1}(s + a/2)\Gamma\bigl(s - (a+1)/2\bigr)\Gamma(s - a/2)\textstyle\prod_{b=a+2}^{n}\Gamma(2s - b)\\ &\sim \Gamma^1_{2a+2}(s + a/2)\Gamma(s - a/2)\textstyle\prod_{b=a+2}^{n}\Gamma(2s - b).\end{aligned}$$

The last product times $\Gamma^1_n(s + a/2)^{-1}$ gives $g^n(s,\,a)$ of Theorem 20.14 for $a < n$. This completes the proof of Theorem 20.14 in Case SP.

Case UT can be handled in the same manner. In fact the argument was given in [S97, §23.12]. Though the function $J'_{q,a}$ is different from that of [S97, (23.11.3)], the necessary $\Gamma^{h,\ell}_v$ in Case UT in Theorem 20.14 is a special case of $\Gamma^{h,m}_v$ of [S97, Theorem 20.5]. We have to take h and m in [S97, Theorem 20.5] to be $(h_v+h_{v\rho})_{v\in\mathbf{a}}$ and d with the present d and h; also ν, μ, n, r_v there are $-(h_{v\rho})_{v\in\mathbf{a}}$, ℓ, $2n$, n here; k of [S97, §23.10] is m' here. Then $c_h(\mathbf{s}')\Psi(\mathbf{s})$ of [S97, (23.11.3)] coincides with $c_{m'}(\mathbf{s}')\Psi(\mathbf{s})$ of (25.8a). In this sense no new proof is necessary in Case UT.

25.6. We next prove Theorem 23.9. Take integral k in Case SP. Using the notation of Theorem 16.11, multiply both sides of (24.29a) by $\mathcal{G}^{n+r}_{k,\kappa}(s)$ and observe that $\mathcal{G}^{n+r}_{k,\kappa}(s)\Lambda^{n+r}_{\mathfrak{c}}(s,\,\chi)J_{q,a}$ is the pullback of $\mathcal{P}(s)\|\alpha$ with $\mathcal{P}$ of Theorem 16.11 and some $\alpha \in G$. Then we find that $c_k(\mathbf{s})\mathcal{G}^{n+r}_{k,\kappa}(s)\mathcal{F}_q(z,\,s;\,\mathbf{f},\,\chi,\,C)$ can be continued to the whole s-plane as a meromorphic function whose poles are contained in the set described in Theorem 16.11 (with $n+r$ in place of n). Employing the explicit forms of $c_k(\mathbf{s})$ and $\mathcal{G}^{n+r}_{k,\kappa}(s)$, we obtain Theorem 23.9 for integral k in Case SP. Case UT can be treated in a similar way. In fact, Theorem 23.9 in Case UT is a special case of [S97, Theorem 20.7] which is proved in [S97, §23.13].

We have assumed k to be integral in the above. If k is half-integral, we can still make all the arguments in Sections 24 and 25 valid by modifying the formulas suitably, though the analogue of Lemma 24.4, as well as the analysis of $p_\alpha(\mathfrak{z})$, becomes more complicated. We can eventually find the analogues of (24.29), (24.29a), and (25.8) for half-integral k, and obtain the desired results as stated in Theorems 21.4 and 23.9. For details, the reader is referred to [S95b]; to be precise, the formulas for half-integral k corresponding to (24.29) and (25.8) are given in [S95b, (7.22) and (8.4)].

25.7. Let us now prove Theorems 23.11 and 23.12. Let the notation be as in those theorems. We first consider the case of integral k. Observe that the assertions of Theorems 17.7, 17.8, and 17.9 are applicable to $E_t^N = H_{q,a}$ of §25.5 in an obvious sense, since $E_t^N = E_1^N \| \alpha$ as noted there. In Case UT, Theorem 17.12 is applicable. We now evaluate (24.29) and (24.29a) at $s = \mu/2$. Applying Lemma 24.11 to $(H_{q,a})^\circ$ and taking the transform of the result by $1 \times \eta$, we find that

$$J_{q,a}(z,\ w;\ \mu/2) = \textstyle\sum_i g_{ai}(z) h_{ai}(w) \tag{25.9}$$

with holomorphic or nearly holomorphic g_{ai} and h_{ai}, according to the nature of $(n,\ r,\ \mu,\ F,\ \chi)$. Moreover, suppose $\kappa = 0$; let $W = \Phi\mathbf{Q}_{\mathrm{ab}}$ in Case SP and $W = K_\chi \Phi \mathbf{Q}_{\mathrm{ab}}$ in Case UT with the notation of Theorems 17.9 and 17.12; then

(25.10) *$\pi^{-\alpha} g_{ai}$ and h_{ai} are W-rational, where* $\alpha = \sum_{v \in \mathbf{a}} (m_v - \mu)(n+r)/2$.

Suppose $\mathbf{f}$ is an eigenform and $\mu \in \Lambda(r,\ k)$; then $\mathbf{f}|_\chi \mathfrak{T} = \Lambda_{\mathfrak{c}}^{2r}(s,\ \chi)^{-1} \mathcal{Z}(us,\ \mathbf{f},\ \chi)$. By Theorems 20.13 and 22.13, $\mathcal{Z}(us,\ \mathbf{f},\ \chi) \neq 0$ at $s = \mu/2$ if $\mu \in \Lambda(r,\ k)$; $\Lambda_{\mathfrak{c}}^{2r}(s,\ \chi)$ is finite at $s = \mu/2$. Also, from the formula for $c_m(\mathbf{s})$ in Lemma 24.5 we can easily derive that

$$c_m(\mathbf{s}) \in \pi^{r\lambda_r[F:\mathbf{Q}]}\mathbf{Q}^\times \quad \text{at} \quad s = \mu/2 \quad \text{if } \kappa = 0, \tag{25.11}$$

and $c_m(\mathbf{s}) \neq 0$ at $s = \mu/2$ for any κ. Therefore from (24.29) we obtain

$$E_q(z,\ \mu/2;\ \mathbf{f},\ \chi,\ C) = \textstyle\sum_{a,i} \langle h'_{ai},\ f_a \rangle g_{ai}(z), \tag{25.12}$$

where h'_{ai} is a constant multiple of $\overline{h_{ai}(-w^*)}$. Thus $E_q(z,\ \mu/2;\ \mathbf{f},\ \chi,\ C)$ belongs to $\mathcal{M}_k^n$ or $\mathcal{N}_k^{n,p}$ (of Lemma 24.11) with some p, according to the nature of $(n,\ r,\ \mu,\ F,\ \chi)$. The conclusion holds for an arbitrary $\mathbf{f}$ in view of Lemma 20.12 (3). The results concerning $E_k^{n,r}(z,\ \mu/2;\ g,\ \Gamma)$ follow from those for $E_q(z,\ \mu/2;\ \mathbf{f},\ \chi,\ C)$ by Proposition 23.6. As for $\mathcal{F}_q(z,\ \mu/2;\ \cdots)$, we employ (24.29a) instead of (24.29). Since $\Lambda_{\mathfrak{c}}^{n+r}(s,\ \chi) J_{q,a}$ is the pullback of $D(z,\ \mu/2;\ \cdots)\|\alpha$ or $D_t(z,\ \mu/2;\ \cdots)$, the same technique is applicable. In this way we obtain Theorems 23.11 and 23.12 from the theorems of Section 17 mentioned above. Finally, the case of half-integral k can be handled in the same way, since the analogues of (24.29) and (24.29a) can be proved as explained at the end of §25.6; for details the reader is referred to [S95b].

We insert here a lemma which will be needed in Section 28.

25.8. Lemma. *The notation being as in Lemma 24.11, suppose that $n = r$ and $f \in \mathcal{N}_k^{2n,p}(\Psi)$. Then we have a finite sum expression $(\pi i)^{n|e'-e|}(D_{e,e'}f)^\circ\ (z,\ w) = \sum_{a=1}^t g_a(z) h_a(w)$ with $g_a,\ h_a \in \mathcal{N}_{k+e-e'}^{n,q}(\Psi)$, where $q_v = p_v + ne_v$ if $e_v > 0$ and $q_v = Max(0,\ p_v - ne'_v)$ if $e_v = 0$.*

PROOF. From our definition of $D_{e,e'}$ and Theorem 14.12 (4) we see that $D_{e,e'}f$ is a nearly holomorphic function on $\mathcal{H}^{2n}$ of degree q; also, it has Fourier expansion whose coefficients belong to $(\pi i)^{n|e-e'|}\Psi$. We know that $(D_{e,e'}f)^\circ(z, w)$ has the automorphy property of an appropriate type. Therefore we obtain our assertion by the same type of argument as in the proof of Lemma 24.11.

26. Near holomorphy of Eisenstein series in Case UB

26.1. In [S97, Section 12] we defined certain Eisenstein series in Case UB similar to those of Section 23. The main purpose of this section is to prove analogues of Theorems 23.11 and 23.12 for such series. To define Eisenstein series, we have to use, as we did in [S97], certain unbounded forms of the symmetric spaces instead of the bounded domain $\mathfrak{B}_{m,n}$ of (3.7). Let us now recall some basic symbols introduced in [S97, Sections 6, 10, and 12] in the unitary case. We fix a CM-type (K, τ), $\tau = \{\tau_v\}_{v\in\mathbf{a}}$, as in §3.5 throughout this section.

For $\varphi = \varphi^* \in GL_n(K)$ we denote the group $U(\varphi)$ of (1.7) by G^φ in conformity with the notation of [S97]. We put $V = K_n^1$, and speak of the structure (V, φ) and its localizations (V_v, φ_v) for $v \in \mathbf{v}$. For $v \in \mathbf{a}$ let r_v be the dimension of maximal φ_v-isotropic subspaces of V_v, and put $n = 2r_v + t_v$ with $0 \le t_v \in \mathbf{Z}$. Then φ_v has signature $(r_v + t_v, r_v)$ or $(r_v, r_v + t_v)$. We take and fix an element κ of K such that

$$\kappa^\rho = -\kappa \text{ and } i\kappa_v\varphi_v \text{ has signature } (r_v + t_v, r_v) \text{ for every } v \in \mathbf{a}. \tag{26.1}$$

For each $v \in \mathbf{a}$ we fix an element $\sigma_v \in GL_n(\mathbf{C})$, as we did in [S97, §10.3], so that

$$\kappa_v\sigma_v\varphi_v\sigma_v^* = -i\varphi'_v, \qquad \varphi'_v = \begin{bmatrix} 0 & 0 & -i1_{r_v} \\ 0 & \theta_v & 0 \\ i1_{r_v} & 0 & 0 \end{bmatrix} \tag{26.2}$$

with $0 < \theta_v = \theta_v^* \in GL_{t_v}(\mathbf{C})$. In this book we take σ_v from $GL_n(\overline{\mathbf{Q}})$, which is certainly feasible; then $\theta_v \in GL_{t_v}(\overline{\mathbf{Q}})$. This is necessary for our later investigation of arithmeticity problems. We then put

$$\mathfrak{Z}^\varphi = \textstyle\prod_{v\in\mathbf{a}} \mathfrak{Z}_v^\varphi, \qquad \mathfrak{Z}_v^\varphi = \mathfrak{Z}(r_v, \theta_v), \tag{26.3}$$

$$\mathfrak{Z}(r, \theta) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{C}_r^{r+t} \;\middle|\; x \in \mathbf{C}_r^r,\ y \in \mathbf{C}_r^t,\ i(x^* - x) > y^*\theta^{-1}y \right\}, \tag{26.4}$$

where $0 < \theta = \theta^* \in GL_t(\mathbf{C})$. (See [S97, §6.14] if $r_v = 0$.) We define *the origin* $\mathbf{i} = \mathbf{i}^\varphi$ of $\mathfrak{Z}^\varphi$ by

$$\mathbf{i} = \mathbf{i}^\varphi = (\mathbf{i}_v)_{v\in\mathbf{a}}, \quad \mathbf{i}_v = \begin{bmatrix} i1_{r_v} \\ 0 \end{bmatrix} \in \mathfrak{Z}_v^\varphi. \tag{26.5}$$

In [S97, §§6.3 and 10.3] we defined the action of $G_\mathbf{A}^\varphi$ on $\mathfrak{Z}^\varphi$ and also factors of automorphy $\kappa(\alpha, z)$ and $\mu(\alpha, z)$ for $\alpha \in G_\mathbf{A}^\varphi$ and $z \in \mathfrak{Z}^\varphi$. Strictly speaking, these are first defined for $\alpha \in \prod_{v\in\mathbf{a}} U(\varphi'_v)$, and then transferred to $G_\mathbf{A}^\varphi$ via the map $\gamma \mapsto (\sigma_v\gamma_v\sigma_v^{-1})_{v\in\mathbf{a}}$ for $\gamma \in G_\mathbf{A}^\varphi$.

In the present book (in conformity with what we did in Sections 3, 4, and 5) we write $\lambda(\alpha, z)$ for $\kappa(\alpha, z)$, and put $\mu_v(\alpha, z) = \mu(\sigma_v\alpha_v\sigma_v^{-1}, z_v)$, $\mu_{v\rho}(\alpha, z) = \lambda(\sigma_v\alpha_v\sigma_v^{-1}, z_v)$, and $r_{v\rho} = r_v + t_v$ for $v \in \mathbf{a}$; we then put $\mu(\alpha, z) = \bigl(\mu_v(\alpha, z)\bigr)_{v\in\mathbf{b}}$. This is an element of $\prod_{v\in\mathbf{b}} GL_{r_v}(\mathbf{C})$. We also put $j(\alpha, z) = j_\alpha(z) = \bigl(j_v(\alpha, z)\bigr)_{v\in\mathbf{b}}$ with $j_v(\alpha, z) = \det\bigl(\mu_v(\alpha, z)\bigr)$ and $j^k(\alpha, z) = j_\alpha^k(z) = j_\alpha(z)^k$ for $k \in \mathbf{Z}^\mathbf{b}$. We then define the spaces of holomorphic automorphic forms $\mathcal{M}_k$, $\mathcal{M}_k(\Gamma)$, and also the space of cusp forms $\mathcal{S}_k$, $\mathcal{S}_k(\Gamma)$. (For the definition of a cusp form, see [S97, §10.5].) In order to emphasize φ, we denote these by $\mathcal{M}_k^\varphi$, $\mathcal{S}_k^\varphi(\Gamma)$, etc.

If φ is totally definite, we understand that $\mathfrak{Z}^\varphi$ consists of a sigle point, written also $\mathbf{i} = \mathbf{i}^\varphi$, and $\mathcal{M}_k^\varphi = \mathcal{S}_k^\varphi = \mathbf{C}$. For these and other conventions, see [S97, §§6.14, 10.3, and 10.5].

Taking $\kappa\varphi$ to be $\mathcal{T}$ of Sections 3 through 5, we can let G^φ act on $\mathcal{H} = \prod_{v\in\mathbf{a}} \mathfrak{B}(r_v + t_v, r_v)$. Notice that by our choice of κ, the signature $(r_v + t_v, r_v)$ is exactly (m_v, n_v) in those sections. See the following § for the map of $\mathcal{H}$ onto $\mathfrak{Z}^\varphi$.

26.2. Define $\xi(z)$, $\eta(z)$, and $\delta(z)$ for $z \in \mathfrak{Z}(r, \theta)$ as in [S97, (6.1.8) and (6.3.11)]. Then formulas (12.1a, b) are valid as shown in [S97, Section 6]; also, Ξ can be defined by (12.4b). Since our exposition of Sections 12 and 13 is practically axiomatic, all the definitions and formulas there are valid for G^φ and $\mathfrak{Z}^\varphi$, once we have the right definition of $r(z)$. In fact, for $z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{Z}(r, \theta)$ put

$$r(z) = -\begin{bmatrix} i1_r \\ {}^t\theta^{-1}\overline{y} \end{bmatrix} {}^t\eta(z)^{-1}. \tag{26.6}$$

Then $r(z) = \xi(z)^{-1}\begin{bmatrix} -i1_r \\ 0 \end{bmatrix}$; moreover, formulas (13.6a, b, c, d), (13.7), and (13.8a, b, c) are all valid in the present case. (We take $T = \mathbf{C}_r^{r+t}$ in those formulas.) The verification is straightforward. Notice that the entries of η^{-1}, being entries of $ir(z)$, belong to $\mathcal{N}^1(\mathfrak{Z}(r, \theta))$; the same is true for ξ^{-1}, as can be seen from [S97, (6.1.11)]; thus

(26.7) *The entries of* $\xi(z)^{-1}$ *and* $\eta(z)^{-1}$ *belong to* $\mathcal{N}^1(\mathfrak{Z}(r, \theta))$.

Naturally, for $z \in \mathfrak{Z}^\varphi$ we put $r(z) = \big(r_v(z_v)\big)_{v\in\mathbf{a}}$ with r_v denoting the function r on $\mathfrak{Z}_v^\varphi$. Then we can speak of nearly holomorphic functions on $\mathfrak{Z}^\varphi$, and can define $\mathcal{N}_\omega^d$ for $d \in \mathbf{Z}^\mathbf{a}$ and a $\overline{\mathbf{Q}}$-rational representation of $\mathfrak{K} = \prod_{v\in\mathbf{b}} GL_{r_v}(\mathbf{C})$.

In [S97, Lemma A2.3] we defined a holomorphic bijection $\mathfrak{t}$ of $\mathfrak{B} = \mathfrak{B}(r + t, r)$ onto $\mathfrak{Z} = \mathfrak{Z}(r, \theta)$. From the equality on line 8 from the bottom in [S97, p.216] we see that $-2idz = {}^t\kappa(z)d\mathfrak{t}(z)\mu(z)$ for $z \in \mathfrak{B}$ with holomorphic functions κ and μ on $\mathfrak{B}$ with values in $GL_{r+t}(\mathbf{C})$ and $GL_r(\mathbf{C})$. Therefore $-2i\big((Df)\circ\mathfrak{t}\big)(u) = D(f\circ\mathfrak{t})\big({}^t\kappa(z)du\mu(z)\big)$ for $u \in T = \mathbf{C}_r^{r+t}$ and $f \in C^\infty(\mathfrak{Z})$. Let r_0 and δ_0 denote the functions on $\mathfrak{B}$ corresponding to r and δ on $\mathfrak{Z}$ as above. Take $f = \log\delta$. By [S97, (A2.3.2)] we have $f\circ\mathfrak{t} = \log\delta_0 - \log|j(z)|^2 + c$ with a holomorphic function j on $\mathfrak{B}$ and a constant c. Also, by (13.8b), $(D\log\delta_0)(u) = \mathrm{tr}({}^tr_0u)$ and $(D\log\delta)(u) = \mathrm{tr}({}^tru)$. Combining these, we find that

(26.8) $r\big(\mathfrak{t}(z)\big) - (i/2)\kappa(z)r_0(z)\cdot{}^t\mu(z)$ *is holomorphic in* $z \in \mathfrak{B}$.

We can naturally consider the map of $\mathcal{H} = \prod_{v\in\mathbf{a}} \mathfrak{B}(r_v + t_v, r_v)$ onto $\mathfrak{Z}^\varphi$ by taking the above $\mathfrak{t}$ at each $v \in \mathbf{a}$. Denoting the map again by $\mathfrak{t}$, from (26.8) we easily see that $f\circ\mathfrak{t} \in \mathcal{N}^d(\mathcal{H})$ if $f \in \mathcal{N}^d(\mathfrak{Z}^\varphi)$ with $d \in \mathbf{Z}^\mathbf{a}$. If f is an automorphic form on $\mathfrak{Z}^\varphi$, we have to multiply $f\circ\mathfrak{t}$ by a certain factor in order to get an automorphic form on $\mathcal{H}$, but since that factor is holomorphic on $\mathcal{H}$, it does not change the degree of near holomorphy.

26.3. Now we can introduce the notion of CM-point on $\mathfrak{Z}^\varphi$ in exactly the same fashion as on $\mathcal{H}$. To be precise, we consider a CM-algebra Y such that $[Y : K] = n$, and take a K-linear ring-injection $h : Y \to K_n^n$ such that $h(a^\rho) = \varphi h(a)^*\varphi^{-1}$ for every $a \in Y$. Then $h(Y^u)$ is contained in G^φ, and has a unique common fixed point w on $\mathfrak{Z}^\varphi$, which we call a *CM-point* on $\mathfrak{Z}^\varphi$. We can also define the symbols $\mathfrak{p}(w)$

and $\mathfrak{P}_k(w)$ for $k \in \mathbf{Z}^{\mathbf{b}}$ (see (11.17a)). These can be obtained by transferring the corresponding objects on $\mathcal{H}$ to those on $\mathfrak{Z}^{\varphi}$ by the above map $\mathfrak{t}$, or equivalently, by repeating their definitions with the symbols of Section 11 replaced by the corresponding ones on $\mathfrak{Z}^{\varphi}$. Then $\mathcal{M}_k(\overline{\mathbf{Q}})$ and $\mathcal{M}_k(\Gamma, \overline{\mathbf{Q}})$ are meaningful, and all the results of Section 11 are valid in the present case. Also, $\mathcal{N}_\omega^d(\overline{\mathbf{Q}})$ and $\mathcal{N}_\omega^d(\Gamma, \overline{\mathbf{Q}})$ can be defined in the same way as in §14.4, and all the results of Section 14 in Case UB can be translated to the present case. To emphasize φ, we shall often write $\mathcal{M}_k^{\varphi}$ and $\mathcal{N}_\omega^{\varphi,d}$ for $\mathcal{M}_k$ and $\mathcal{N}_\omega^d$. We note a simple fact:

(26.9) *If* $j^k\bigl(h(a), w\bigr) = a^{\varepsilon}$ *for every* $a \in Y^u$ *with* $\varepsilon \in I_Y$, *then* $\mathfrak{P}_k(w) = p_Y(\varepsilon, \Phi)$.

Here Φ and I_Y are as in §11.3. This follows immediately from (11.3a, b), (11.4a, b), and (11.17a).

26.4. With (V, φ) as above, put $(W, \psi) = (V, \varphi) \oplus (H_q, \eta_q')$ (see §1.1) with $H_q = K_{2q}^1$ and $\eta_q' = \begin{bmatrix} 0 & 1_q \\ 1_q & 0 \end{bmatrix}$. We are going to consider Eisenstein series on $G_{\mathbf{A}}^{\psi}$ relative to this decomposition of (W, ψ). For our later purposes, however, it is more natural to start from an arbitrary (W, ψ), and consider such a sum decomposition for various different (V, φ) as follows. Given (W, ψ), let $l(\psi)$ be the dimension of a maximal totally ψ-isotropic subspace of W. We fix a decomposition $(W, \psi) = (Z, \zeta) \oplus (K_{2l}^1, \eta_l')$ with an anisotropic ζ, so that $l = l(\psi)$, take a standard basis $\{g_i\}_{i=1}^{2l}$ of H_l with respect to η_l', and put

(26.10)
$$(V_r, \varphi_r) = (Z, \zeta) \oplus \Bigl(\textstyle\sum_{i=1}^{r}(Kg_{l-r+i} + Kg_{2l-r+i}),\ \eta_r'\Bigr), \qquad \varphi_r = \begin{bmatrix} 0 & 0 & 1_r \\ 0 & \zeta & 0 \\ 1_r & 0 & 0 \end{bmatrix},$$

where η_r' is the restriction of η_l' to $\sum_{i=1}^{r}(Kg_{l-r+i} + Kg_{2l-r+i})$. Put $p = \dim(Z) = t_v + 2s_v$ for $v \in \mathbf{a}$ with $s_v = l(\zeta_v)$. Taking (ζ, s_v, t_v) in place of (φ, r_v, t_v) in (26.2), we choose κ, τ_v, θ_v so that

$$(26.11) \qquad \kappa_v \tau_v \zeta_v \tau_v^* = -i\zeta_v', \qquad \zeta_v' = \begin{bmatrix} 0 & 0 & -i1_{s_v} \\ 0 & \theta_v & 0 \\ i1_{s_v} & 0 & 0 \end{bmatrix}.$$

Fixing an integer r such that $0 \le r \le l$, take φ_r as φ of §26.1; then $r_v = r + s_v$. Put then

$$(26.12) \qquad \varphi_v'' = \begin{bmatrix} 0 & 0 & -i1_r \\ 0 & \zeta_v' & 0 \\ i1_r & 0 & 0 \end{bmatrix}, \quad \varepsilon_v = \operatorname{diag}\left[1_{r_v+t_v},\ \begin{bmatrix} 0 & 1_r \\ 1_{s_v} & 0 \end{bmatrix}\right],$$

$$(26.13) \qquad \xi_v = \operatorname{diag}[1_r,\ \tau_v,\ \kappa_v^{-1}1_r], \quad \sigma_v = \varepsilon_v \xi_v.$$

Then we can easily verify that $\kappa_v \xi_v \varphi_v \xi_v^* = -i\varphi_v''$, and $\varepsilon_v \varphi_v'' \varepsilon_v^*$ coincides with φ_v' of (26.2), so that (26.2) holds with the present σ_v. Thus we let G^{φ_r} act on $\mathfrak{Z}^{\varphi_r}$ as in §26.1 for $0 \le r \le l$; Clearly $(V_0, \varphi_0) = (Z, \zeta)$.

In [S97, §12.1] we put $\psi = \begin{bmatrix} 0 & 0 & 1_m \\ 0 & \varphi & 0 \\ 1_m & 0 & 0 \end{bmatrix}$, and defined the action of G^{ψ} on $\mathfrak{Z}^{\psi}$. Take $m = l - r$ and $\varphi = \varphi_r$; then $\varphi_l = \delta\psi\delta^*$ and $\delta G^{\psi}\delta^{-1} = G^{\varphi_l}$ with $\delta = \operatorname{diag}\left[1_{l+t},\ \begin{bmatrix} 0 & 1_{l-r} \\ 1_r & 0 \end{bmatrix}\right]$. Let σ_v^l denote σ_v of (26.13) defined with $r = l$. Then we can easily verify that $\sigma_v^l\delta$ coincides with τ_v of [S97, (12.1.4)], and therefore the action of $\sigma_v^l(\delta\beta\delta^{-1})_v(\sigma_v^l)^{-1}$ on $\mathfrak{Z}^\psi$ for $\beta \in G^\psi$ is exactly that of β defined in [S97, p.93, line 10]. Therefore we can put $(W, \psi) = (V_l, \varphi_l)$, and identify G^ψ with G^{φ_l}. Then all what we did in [S97, Section 12] is applicable to the present situation.

26.5. Throughout the rest of this section we fix an element k of $Z^{\mathbf{b}}$. Before considering Eisenstein series on $G^\psi_{\mathbf{A}}$, we prove a few basic facts on cusp forms. Fixing a positive integer $q \le l(\psi)$, put $(V, \varphi) = (V_r, \varphi_r)$ with $r = l - q$ and write a point $\mathfrak{z}_v$ of $\mathfrak{Z}^\psi_v$ in the form

$$\mathfrak{z}_v = \begin{bmatrix} z_v & u_{v2} \\ {}^t u_{v1} & w_v \end{bmatrix} \quad \text{with } z_v \in \mathbf{C}^q_q,\ u_{v1} \in \mathbf{C}^q_{r_v+t_v},\ u_{v2} \in \mathbf{C}^q_{r_v},\ w_v \in \mathbf{C}^{r_v+t_v}_{r_v}.$$

Write then $\mathfrak{z} = (z, u, w)$ with $z = (z_v)_{v\in\mathbf{a}}$, $u = (u_v)_{v\in\mathbf{a}}$, $w = (w_v)_{v\in\mathbf{a}}$, where $u_v = [u_{v1}\ \ u_{v2}]$; then $w \in \mathfrak{Z}^\varphi$. In particular, for $u = 0$ we put

$$(26.14) \qquad (z, 0, w) = \mathrm{diag}[z, w] \qquad (z \in \mathcal{H}^{\mathbf{a}}_q,\ w \in \mathfrak{Z}^\varphi),$$

though w_v is not necessarily square.

Let $f \in \mathcal{M}^\psi_k(\Gamma)$. In [S97, (A4.4.1)] we obtained a Fourier expansion

$$(26.15) \qquad f(z, u, w) = \sum_{h\in\Lambda} c^q_h(u, w)\mathbf{e}^q_{\mathbf{a}}(hz) \qquad (0 < q \le l(\psi))$$

with a $\mathbf{Z}$-lattice Λ in $S^q = \{ x \in K^q_q \mid x^* = x \}$ and functions $c^q_h(u, w)$. More precisely, given $w \in \mathfrak{Z}^\varphi$ and $u \in (\mathbf{C}^q_n)^{\mathbf{a}}$, the function f is defined for $i(z^*_v - z_v) > p_v$ with a positive definite hermitian matrix p_v determined by u_v and w_v, and we have (26.15) there. Emphasizing the dependence on f, we put $c^q_h(u, w) = c^q_h(u, w; f)$. (If $r_v = 0$ for every $v \in \mathbf{a}$, then $\mathfrak{Z}^\varphi$ consists of a single point, $u = (u_{v1})_{v\in\mathbf{a}}$, and (26.15) takes the form $f(z, u) = \sum_{h\in\Lambda} c^q_h(u)\mathbf{e}^q_{\mathbf{a}}(hz)$.)

26.6. Proposition. *The notation being as above, the following assertions hold:*

(1) $c^q_h(u, w) \ne 0$ *only if* $h_v \ge 0$ *for every* $v \in \mathbf{a}$.

(2) f *is a cusp form if and only if* $c^l_h(u, w; f\|_k\alpha) = 0$ *for every* $\alpha \in G^\psi$ *and every* h *such that* $\det(h) = 0$, *where* $l = l(\psi)$.

(3) $c^q_0(u, w)$ *does not depend on* u, *and so we can put* $c^q_0(u, w) = c^q_0(w; f)$.

(4) f *is a cusp form if and only if* $c^q_0(w; f\|_k\alpha) = 0$ *for every* $\alpha \in G^\psi$ *and every* q *such that* $0 < q \le l(\psi)$.

(5) f *is a cusp form if and only if* $c^1_0(w; f\|_k\alpha) = 0$ *for every* $\alpha \in G^\psi$.

PROOF. All these assertions except (5) were proven in [S97, Proposition A4.5]. (The case in which $F = \mathbf{Q}$ and $\dim(W) = 2 = 2q$ was excluded in (1). But in that case $SU(\psi)$ is conjugate to $SL_2(\mathbf{Q})$, and so assertion (1) follows from the cusp condition.) To prove (5), let $0 < t < q \le l(\psi)$, $\mathfrak{z} = \mathrm{diag}[z, \mathfrak{z}']$ and $\mathfrak{z}' = \mathrm{diag}[z', w]$ with $z \in \mathcal{H}^{\mathbf{a}}_t$, $z' \in \mathcal{H}^{\mathbf{a}}_{q-t}$, and $w \in \mathfrak{Z}^\varphi$. Then, for $f \in \mathcal{M}^\psi_k$ we have $f(\mathfrak{z}) = \sum_{g\in S^t} c^t_g(0, \mathfrak{z}')\mathbf{e}^t_{\mathbf{a}}(gz)$. On the other hand we have a Fourier expansion

$$f(\mathfrak{z}) = \sum_{g\in S^t} \sum_{h\in S^{q-t}} a_{g,h}(w)\mathbf{e}^t_{\mathbf{a}}(gz)\mathbf{e}^{q-t}_{\mathbf{a}}(hz')$$

with functions $a_{g,h}$ on $\mathfrak{Z}^\varphi$. Then clearly $c^q_0(w; f) = a_{0,0}(w)$ and

$$c^t_0(\mathrm{diag}[z', w]; f) = \sum_{h\in S^{q-t}} a_{0,h}(w)\mathbf{e}^{q-t}_{\mathbf{a}}(hz'),$$

and hence if $c^t_0(\mathfrak{z}'; f) = 0$, then $c^q_0(w; f) = 0$ for every $q > t$. Therefore (5) follows from (4).

26.7. Define a map $\varepsilon_q : \mathcal{H}^{\mathbf{a}}_q \times \mathfrak{Z}^\varphi \to \mathfrak{Z}^\psi$ by $\varepsilon_q(z, w) = \mathrm{diag}[z, w]$. We view $U(\eta_q) \times G^\varphi$ as a subgroup of G^ψ. For $\beta \in U(\eta_q)$ and $\gamma \in G^\psi$ we easily see that

$$(26.16)\qquad (\beta\times\gamma)\varepsilon_q(z,\,w)=\varepsilon_q(\beta' z,\,\gamma w),\qquad j\big(\beta\times\gamma,\,\varepsilon_q(z,\,w)\big)=j(\beta',\,z)j(\gamma,\,w),$$

where $\beta' = \mathrm{diag}[1_q,\,\kappa^{-1}1_q]\beta\mathrm{diag}[1_q,\,\kappa 1_q]$. Now, taking u of (26.15) to be 0, for $f\in\mathcal{M}_k^\psi$ we obtain an expansion of the form

$$(26.17)\qquad f\big(\varepsilon_q(z,\,w)\big) = \sum_{h\in S^q} c_h^q(w;\,f)\mathbf{e}_{\mathbf{a}}^q(hz)\qquad (0<q\le l(\psi))$$

with some functions $c_h^q(w;\,f)$ of $w\in\mathfrak{Z}^\varphi$, which belong to $\mathcal{M}_k^\varphi$ as can easily be seen. Clearly c_0^q coincides with that of Proposition 26.6 (3).

Transferring Proposition 11.15 to $\mathfrak{Z}^\psi$, we have $\mathcal{M}_k^\psi=\mathcal{M}_k^\psi(\overline{\mathbf{Q}})\otimes\mathbf{C}$. Therefore we can let an element σ of $\mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$ act on $\mathcal{M}_k^\psi$ by $\big(\sum_\nu c_\nu g_\nu\big)^\sigma=\sum_\nu c_\nu^\sigma g_\nu$ for $c_\nu\in\mathbf{C}$ and $g_\nu\in\mathcal{M}_k^\psi(\overline{\mathbf{Q}})$. Similarly we can let σ act on $\mathcal{N}_\omega^p$. By Proposition 11.13 (1), $\mathcal{M}_k^\psi(\overline{\mathbf{Q}})$ is stable under $g\mapsto g\|_k\gamma$ for every $\gamma\in G^\psi$. Therefore we have

$$(26.18)\qquad (f\|_k\gamma)^\sigma = f^\sigma\|_k\gamma\quad\textit{for every}\quad \gamma\in G^\psi.$$

It should be noted that f^σ depends on k. In fact, $\mathcal{M}_k=\mathcal{M}_l$ if $k_v+k_{v\rho}=l_v+l_{v\rho}$ for every $v\in\mathbf{a}$, but $\mathcal{M}_k(\overline{\mathbf{Q}})=\mathfrak{t}\mathcal{M}_l(\overline{\mathbf{Q}})$ with a constant $\mathfrak{t}$ which may not be algebraic; see Theorem 11.17 (2). Therefore f^σ defined with k as its weight is $\mathfrak{t}/\mathfrak{t}^\sigma$ times f^σ defined with l as its weight. Thus, whenever we speak of f^σ in Case UB, it should be understood that k is already fixed.

26.8. Proposition. (1) *We have* $c_h^q(w;\,f^\sigma)=c_h^q(w;\,f)^\sigma$ *for every* $f\in\mathcal{M}_k^\psi$ *and every* $\sigma\in\mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$.

(2) *If Γ is a congruence subgroup of G^ψ, then $\mathcal{S}_k^\psi(\Gamma)^\sigma=\mathcal{S}_k^\psi(\Gamma)$ for every $\sigma\in\mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$ and $\mathcal{S}_k^\psi(\Gamma)=\mathcal{S}_k^\psi(\Gamma,\,\overline{\mathbf{Q}})\otimes_{\overline{\mathbf{Q}}}\mathbf{C}$.*

PROOF. Given Γ, we can take congruence subgroups Γ_1 of $U(\eta_q)$ and Γ_2 of $U(\varphi)$ so that $f\big(\varepsilon_q(z,\,w)\big)$ as a function of z (resp. w) for a fixed w (resp. z) belongs to $\mathcal{M}_k^\eta(\Gamma_1)$ (resp. $\mathcal{M}_k^\varphi(\Gamma_2)$) for every $f\in\mathcal{M}_k^\psi(\Gamma)$. Take a $\overline{\mathbf{Q}}$-basis P of $\mathcal{M}_k^\eta(\Gamma_1,\,\overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$. Then P is a $\mathbf{C}$-basis of $\mathcal{M}_k^\eta(\Gamma_1)$. We have $f\big(\varepsilon_q(z,\,w)\big)=\sum_{p\in P}g_p(w)p(z)$ with some functions g_p on $\mathfrak{Z}^\varphi$. Let $\mathcal{X}$ (resp. $\mathcal{Y}$) be the set of all CM-points on $\mathcal{H}_q^{\mathbf{a}}$ (resp. $\mathfrak{Z}^\varphi$). We have

$$\{0\}=\textstyle\bigcap_{x\in\mathcal{X}}\Big\{\,(c_p)\in\mathbf{C}^P\;\big|\;\sum_{p\in P}c_pp(x)=0\,\Big\},$$

since $\mathcal{X}$ is dense in $\mathcal{H}_q^{\mathbf{a}}$. Therefore we can find a finite subset X of $\mathcal{X}$ such that $\#X=\#P$ and $\det\big(p(x)\big)_{p\in P,\,x\in X}\neq 0$. Then, from the equations $f\big(\varepsilon_q(x,\,w)\big)=\sum_{p\in P}g_p(w)p(x)$ for all $x\in X$ we see that $g_p(w)\in\mathcal{M}_k^\varphi(\Gamma_2)$. We can put $p(z)=\sum_h b_h(p)\mathbf{e}_{\mathbf{a}}^q(hz)$ with $b_h(p)\in\overline{\mathbf{Q}}$. Then we have

$$(26.19)\qquad c_h^q(w;\,f)=\textstyle\sum_{p\in P}b_h(p)g_p(w).$$

Now for $x\in\mathcal{X}$ and $y\in\mathcal{Y}$ we easily see that the point $\varepsilon_q(x,\,y)$ is a CM-point of $\mathfrak{Z}^\psi$ and $\mathfrak{P}_k\big(\varepsilon_q(x,\,y)\big)=\mathfrak{P}_k(x)\mathfrak{P}_k(y)$. Therefore, if $f\in\mathcal{M}_k^\psi(\Gamma,\,\overline{\mathbf{Q}})$, then $\sum_{p\in P}\mathfrak{P}_k(y)^{-1}\cdot g_p(y)\mathfrak{P}_k(x)^{-1}p(x)=\mathfrak{P}_k\big(\varepsilon_q(x,\,y)\big)^{-1}f\big(\varepsilon_q(x,\,w)\big)\in\overline{\mathbf{Q}}$ for every $(x,\,y)\in\mathcal{X}\times\mathcal{Y}$. Taking x in X, we see that $\mathfrak{P}_k(y)^{-1}g_p(y)\in\overline{\mathbf{Q}}$ for every $y\in\mathcal{Y}$ and every $p\in P$. Thus $g_p\in\mathcal{M}_k^\varphi(\Gamma_2,\,\overline{\mathbf{Q}})$ for every $p\in P$, and hence $c_h^q(w;\,f)\in\mathcal{M}_k^\varphi(\Gamma_2,\,\overline{\mathbf{Q}})$ if f is $\overline{\mathbf{Q}}$-rational. Next, let f be an element of $\mathcal{M}_k^\psi(\Gamma)$ that is not necessarily $\overline{\mathbf{Q}}$-rational. By Proposition 11.15 we have $f=\sum_{a\in A}af_a$ with a finite subset A of $\mathbf{C}$ and $f_a\in\mathcal{M}_k^\psi(\Gamma,\,\overline{\mathbf{Q}})$. Let $\sigma\in\mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$. Then $c_h^q(w;\,f)=$

$\sum_{a\in A} ac_h^q(w;\, f_a)$ and $c_h^q(w;\, f^\sigma) = \sum_{a\in A} a^\sigma c_h^q(w;\, f_a) = \sum_{a\in A} a^\sigma c_h^q(w;\, f_a)^\sigma = c_h^q(w;\, f)^\sigma$, since $c_h^q(w;\, f)$ is $\overline{\mathbf{Q}}$-rational. This proves (1). Suppose f is a cusp form; then by Proposition 26.6 (4), $c_0^q(w;\, f^\sigma\|\gamma) = c_0^q\bigl(w;\, (f\|\gamma)^\sigma\bigr) = c_0^q(w;\, f\|\gamma)^\sigma = 0$ for every $\gamma \in G^\psi$, which means that f^σ is a cusp form. This proves the first assertion, since $\mathcal{M}_k^\psi(\Gamma)$ is stable under σ, which can be seen from Proposition 11.15. Once this is established, the second assertion can be proved by the same technique as in the proof of Theorem 10.8 (2) by means of Lemma 10.3.

26.9. With $\{g_i\}$ as in §26.4 put $I_q' = \sum_{i=1}^q Kg_i$ and $I_q = \sum_{i=1}^q Kg_{l+i}$, where $q = l - r$, $0 \le r \le l(\psi)$; put also $(V,\, \varphi) = (V_r,\, \varphi_r)$. Then the decomposition on the first line of §26.4 can be written $(W,\, \psi) = (V,\, \varphi) \oplus (I_q' + I_q,\, \eta_q')$. Define a parabolic subgroup P_r^ψ of G^ψ by

$$P = P_r^\psi = \bigl\{\, \alpha \in G^\psi \bigm| I_q\alpha = I_q \,\bigr\}. \tag{26.20}$$

If we represent every element of G^ψ according to the decomposition $W = I_q' \oplus V \oplus I_q$, then P consists of the elements of the form

$$\alpha = \begin{bmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & d \end{bmatrix}, \tag{26.21}$$

where d represents the restriction of α to I_q. We then define a homomorphism $\pi_r : P \to G^\varphi$ and a homomorphism $\lambda_r : P \to K^\times$ by

$$\pi_r(\alpha) = e, \qquad \lambda_r(\alpha) = \det(d). \tag{26.22}$$

For simplicity we shall often write π for π_r. If $r = l(\psi)$, then $I_q = \{0\}$ and $P_r^\psi = G^\psi = G^\varphi$, and we understand that $\pi_r(\alpha) = \alpha$ and $\lambda_r(\alpha) = 1$ for $\alpha \in G^\psi$.

For a congruence subgroup Γ of G^ψ we put

$$\mathcal{M}_k^\varphi(\Gamma,\, P) = \bigl\{ f \in \mathcal{M}_k^\varphi \bigm| f\|_k\pi(\gamma) = \lambda_r(\gamma)^\ell |\lambda_r(\gamma)|^{-\ell} f \ \text{ for every } \gamma \in \Gamma \cap P \bigr\}, \tag{26.23}$$

$$\mathcal{S}_k^\varphi(\Gamma,\, P) = \mathcal{M}_k^\varphi(\Gamma,\, P) \cap \mathcal{S}_k^\varphi, \tag{26.23a}$$

$$\mathcal{S}_k^\varphi(\Gamma,\, P,\, \overline{\mathbf{Q}}) = \mathcal{S}_k^\varphi(\Gamma,\, P) \cap \mathcal{M}_k^\varphi(\overline{\mathbf{Q}}), \tag{26.24}$$

where $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$. Then

$$\mathcal{S}_k^\varphi(\Gamma,\, P) = \mathcal{S}_k^\varphi(\Gamma,\, P,\, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}. \tag{26.25}$$

Indeed, let $f \in \mathcal{S}_k^\varphi(\Gamma,\, P)$. By Proposition 26.8 we can put $f = \sum_{a\in A} ag_a$ with a finite subset A of $\mathbf{C}$ and $g_a \in \mathcal{S}_k^\varphi(\overline{\mathbf{Q}})$; we may assume that A is linearly independent over $\overline{\mathbf{Q}}$. Then, for every $\gamma \in \Gamma$ we have $\sum_a a\lambda_r(\gamma)^\ell|\lambda_r(\gamma)|^{-\ell}g_a = f\|\pi(\gamma) = \sum_a ag_a\|\pi(\gamma)$. Since $g_a\|\pi(\gamma) \in \mathcal{S}_k^\varphi(\overline{\mathbf{Q}})$, we have $g_a\|\pi(\gamma) = \lambda_r(\gamma)^\ell|\lambda_r(\gamma)|^{-\ell}g_a$, that is, $g_a \in \mathcal{S}_k^\varphi(\Gamma,\, P,\, \overline{\mathbf{Q}})$, which proves (26.25).

We now define a holomorphic map

$$\wp_\varphi^\psi : \mathfrak{Z}^\psi \to \mathfrak{Z}^\varphi \tag{26.26}$$

by $\wp_\varphi^\psi(z,\, u,\, w) = w$ for $(z,\, u,\, w) \in \mathfrak{Z}^\psi$ as in §26.5. (If $r_v = 0$, then $\wp(\mathfrak{z})_v = \mathbf{i}_v$.) We write simply $\wp$ for $\wp_\varphi^\psi$ if there is no fear of confusion.

Given $f \in \mathcal{S}_k^\varphi(\Gamma,\, P)$, we put, for $(z,\, s) \in \mathfrak{Z}^\psi \times \mathbf{C}$,

$$\delta_{s,f}(z) = \delta(z,\, s;\, f) = f\bigl(\wp(z)\bigr)\bigl[\delta(z)/\delta\bigl(\wp(z)\bigr)\bigr]^{s\mathbf{a}-m/2}, \quad m = (k_v + k_{v\rho})_{v\in\mathbf{a}}, \tag{26.27}$$

$$E_k^{\psi,\varphi}(z,\, s;\, f,\, \Gamma) = \sum_{\alpha\in A} \delta_{s,f}\|_k\alpha, \quad A = (\Gamma \cap P)\backslash\Gamma, \tag{26.28}$$

where $\delta(z) = \bigl(\delta(z_v)\bigr)_{v\in\mathbf{a}}$ with $\delta(z_v)$ defined by [S97, (6.3.11)]; we understand that $\delta\bigl(\wp(z)\bigr)_v = 1$ if $\mathfrak{Z}_v^\varphi$ is trivial. These are similar to (23.11) and (23.13), and in fact, the series of (26.28) was investigated in [S97, §12.3]. For $\Gamma' \subset \Gamma$ we can easily verify that

$$(26.28\mathrm{a}) \qquad [\Gamma \cap P : \Gamma' \cap P] E_k^{\psi,\varphi}(z,\, s;\, f,\, \Gamma) = \sum_{\alpha\in\Gamma'\backslash\Gamma} E_k^{\psi,\varphi}(z,\, s;\, f,\, \Gamma')\|_k\, \alpha.$$

If $r = l(\psi)$, then $\mathcal{S}_k^\varphi(\Gamma,\, P) = \mathcal{S}_k^\psi(\Gamma)$ and $\delta_{s,f} = f$, and so $E_k^{\psi,\psi}(z,\, s;\, f,\, \Gamma) = f(z)$.

26.10. Let us now recall briefly the notion of automorphic form on $G_\mathbf{A}^\varphi$, the zeta function associated to an eigenform, and also Eisenstein series of the above type formulated as functions on $G_\mathbf{A}^\psi$. We let n denote the size of φ as we did in §26.1. Given an open subgroup D of $G_\mathbf{A}^\varphi$ such that $D \cap G_\mathbf{h}^\varphi$ is compact, we denote by $\mathcal{M}_k^\varphi(D)$ (resp. $\mathcal{S}_k^\varphi(D)$) the set of all functions $\mathbf{f} : G_\mathbf{A}^\varphi \to \mathbf{C}$ satisfying the following conditions:

(26.29a) $\mathbf{f}(\alpha x w) = j_w^k(\mathbf{i})^{-1}\mathbf{f}(x)$ *if* $\alpha \in G^\varphi$, $w \in D$, *and* $w(\mathbf{i}) = \mathbf{i}$.

(26.29b) *For every* $p \in G_\mathbf{h}^\varphi$ *there exists an element* f_p *of* $\mathcal{M}_k^\varphi$ *(resp.* $\mathcal{S}_k^\varphi$*) such that* $\mathbf{f}(py) = (f_p\|_k y)(\mathbf{i})$ *for every* $y \in G_\mathbf{a}^\varphi$.

To find a good D, we first assume that

$$(26.30) \qquad \det(\varphi) \in \mathfrak{g}^\times N_{K/F}(K^\times) \text{ if } n \text{ is odd},$$

which is always satisfied if we change φ for its suitable multiple by an element of $F^\times$. We also fix a $\mathfrak{g}$-maximal $\mathfrak{r}$-lattice M in V and an integral $\mathfrak{g}$-ideal $\mathfrak{c}$, and put

$$(26.31) \qquad \widetilde{M} = \bigl\{ x \in V \bigm| \varphi(x,\, M) \subset \mathfrak{d}(K/F)^{-1} \bigr\},$$

$$(26.32) \qquad D^\varphi = \Bigl\{ \gamma \in G_\mathbf{A}^\varphi \Bigm| M\gamma = M,\ \widetilde{M}_v(\gamma_v - 1) \subset \mathfrak{c}_v M_v \text{ for every } v|\mathfrak{c} \Bigr\},$$

$$(26.33) \qquad \mathfrak{X} = \bigl\{ \xi \in G_\mathbf{A}^\varphi \bigm| \xi_v \in D^\varphi \cap G_v^\varphi \text{ for every } v|\mathfrak{c} \bigr\},$$

where $\mathfrak{d}(K/F)$ is the different of K relative to F. Then we can define the action of $\mathfrak{R}(D^\varphi,\, \mathfrak{X})$ on $\mathcal{S}_k^\varphi(D^\varphi)$ and a formal Dirichlet series $\mathbf{f}|\mathfrak{T}$ for $\mathbf{f} \in \mathcal{S}_k^\varphi(D^\varphi)$ as in [S97, §§11.6-11.11]. They are similar to what was done in §20.3 of the present book. To be precise, $\mathbf{f}|\mathfrak{T} = \sum_\mathfrak{a} \bigl(\mathbf{f}|T(\mathfrak{a})\bigr)[\mathfrak{a}]$, where $T(\mathfrak{a})$ is the sum of all different $D^\varphi\tau D^\varphi$ with $\tau \in \mathfrak{X}$ such that the ideal $\nu^\sigma(\tau)$ defined by [S97, (11.11.1)] coincides with $\mathfrak{a}$. We can also define $\mathcal{Z}(s,\, \mathbf{f},\, \chi)$ for an eigenform $\mathbf{f}$ and prove a theorem similar to Theorem 20.14; see [S97, (20.4.1) and Theorem 20.5]. (Corrections: In [S97, (20.4.1)] the condition $\mathfrak{q} \nmid \mathfrak{c}$ must be replaced by $\mathfrak{q} \nmid \mathfrak{c}\mathfrak{h}$, where $\mathfrak{h} = F \cap$ (the conductor of χ). Also after "$\psi = \varphi$" in [S97, p.196, line 4] insert: "Changing $\mathfrak{c}$ for its suitable multiple, we may assume, without changing $\mathcal{Z}(s,\, \mathbf{f},\, \chi)$, that the conductor of χ divides $\mathfrak{c}$.) Furthermore, in the setting of §26.9, given $\mathbf{f} \in \mathcal{S}_k^\varphi(D^\varphi)$, we can define an Eisenstein series $E_\mathbf{A}(x,\, s;\, \mathbf{f},\, \chi,\, D^\psi)$ for $(x,\, s) \in G_\mathbf{A}^\psi \times \mathbf{C}$, a Hecke character χ of K, and a suitably chosen open subgroup D^ψ of $G_\mathbf{A}^\psi$, and prove its meromorphic continuation. These are similar to $E_\mathbf{A}$ of (23.14) and Theorem 23.9. For details, see [S97, Section 12 and Theorem 20.7]; in particular, D^ψ is given by [S97, (20.6.8)], and we assume that χ satisfies (16.24a, b) with $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$ and $\mathfrak{c}$ is divisible by the ideal $\mathfrak{e}$ of [S97, Lemma 20.2]. (The last assumption is [S97, (20.3.2)].)

Let p be an element of $G_\mathbf{h}^\psi$ of the form $p = \mathrm{diag}[\hat{u},\, b,\, u]$ (with respect to the decomposition $W = I_q' + V + I_q$ of §26.9) with $u \in GL_q(K)_\mathbf{h}$ and $b \in G_\mathbf{h}^\varphi$, such that $p_v = 1$ for every $v|\mathfrak{c}$. Then we define a function $E_p(z,\, s;\, \mathbf{f},\, \chi,\, D^\psi)$ of $(z,\, s) \in \mathfrak{Z}^\psi \times \mathbf{C}$ by

(26.34) $E_p(y(\mathbf{i}), s; \mathbf{f}, \chi, D^\psi)j_y^k(\mathbf{i})^{-1} = E_{\mathbf{A}}(py, s; \mathbf{f}, \chi, D^\psi)$ *for every* $y \in G_{\mathbf{a}}^\psi$.

26.11. We are going to state an analogue of Lemma 24.11. We first put

$$\omega = \begin{bmatrix} \psi & 0 \\ 0 & -\varphi \end{bmatrix}, \quad \eta = \begin{bmatrix} 0 & -1_{q+n} \\ 1_{q+n} & 0 \end{bmatrix} \quad (n = \dim(V)). \tag{26.35}$$

Then in [S97, (21.1.6)] we took an element S of $GL_{2q+2n}(K)$ such that $S\eta S^* = \kappa\omega$. For $(\beta, \gamma) \in G^\psi \times G^\varphi$ we define an element $[\beta, \gamma]_S$ of G^η by

$$[\beta, \gamma]_S = S^{-1}\mathrm{diag}[\beta, \gamma]S. \tag{26.36}$$

In [S97, Proposition 22.2] we defined an embedding $\iota_U : \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi \to \mathcal{H}_{q+n}^{\mathbf{a}}$ which is compatible with (26.36) in the sense that

$$[\beta, \gamma]_S\iota_U(z, w) = \iota_U(\beta z, \gamma w) \qquad (z \in \mathfrak{Z}^\psi, \ w \in \mathfrak{Z}^\varphi). \tag{26.37}$$

We note that $\iota_U(z, w)$ is holomorphic in z and antiholomorphic in w (see [S97, Proposition 6.11 and (22.2.1)]. For a function f on $\mathcal{H}_{q+n}^{\mathbf{a}}$ we define its pullback f° to be a function on $\mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$ given by

$$f^\circ(z, w) = \delta\big(w, \wp(z)\big)^{-m} f\big(\iota_U(z, w)\big) \qquad (z \in \mathfrak{Z}^\psi, \ w \in \mathfrak{Z}^\varphi), \tag{26.38}$$

where $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ and $\delta\big(w, \wp(z)\big)$ is defined by [S97, (6.6.8)].

To study the nature of f° at CM-points, take CM-points z_0 and w_0 on $\mathfrak{Z}^\psi$ and $\mathfrak{Z}^\varphi$ obtained from maps $r : Y \to K_{n+2q}^{n+2q}$ and $s : Z \to K_n^n$ with the properties described in §26.3, where Y and Z are CM-algebras. Define $\ell : Y \oplus Z \to K_{2n+2q}^{2n+2q}$ by $\ell(b, c) = S^{-1}\mathrm{diag}[r(b), s(c)]S$ for $(b, c) \in Y \oplus Z$; put $\mathfrak{z}_0 = \iota_U(z_0, w_0)$. From (26.37) we see that $\mathfrak{z}_0$ is the CM-point fixed by $\ell\big((Y \oplus Z)^u\big)$. Let Φ (resp. Φ' and Ξ) be defined for (Y, q) (resp. (Z, s) and $(Y \oplus Z, \ell)$) as in (4.40).

We are going to show that $\Xi = \Phi + \Phi'\rho$. For that purpose we use the notation of [S97, §6.10 and the proof of Proposition 22.2]. (The factor of automorphy κ there is now written λ, and so we use λ instead of κ.) Put $\alpha = [\beta, \gamma]_S$ with $\beta \in G^\psi$ and $\gamma \in G^\varphi$; put also $\beta'_v = \tau_v\beta_v\tau_v^{-1}$, $\gamma'_v = \sigma_v\gamma_v\sigma_v^{-1}$, and $\varepsilon_v = [\beta'_v, \gamma'_v]_R$. In the first paragraph of that proof, we showed that $\alpha_v = U_v^{-1}\varepsilon_v U_v$. Let $\mathfrak{z}_1 = \iota(z_0, w_0)$; then $\mathfrak{z}_0 = U^{-1}\mathfrak{z}_1$. Take $\beta = r(b)$ and $\gamma = s(c)$ with $b \in Y^u$ and $c \in Z^u$. Then $\varepsilon\mathfrak{z}_1 = \mathfrak{z}_1$ and $\alpha\mathfrak{z}_0 = \mathfrak{z}_0$. Suppressing the subsript v for simplicity, we have

$$\lambda_\alpha(\mathfrak{z}_0) = \lambda(U^{-1}\varepsilon U, U^{-1}\mathfrak{z}_1) = \lambda(U^{-1}, \mathfrak{z}_1)\lambda_\varepsilon(\mathfrak{z}_1)\lambda(U, U^{-1}\mathfrak{z}_1). \tag{26.39}$$

Define $M(w)$ and $N(z)$ as in [S97, (6.11.4)]. By [S97, (6.11.5)],

$$\lambda_\varepsilon(\mathfrak{z}_1) = M(w_0)\mathrm{diag}\big[\lambda_\beta(z_0), \overline{\mu_\gamma(w_0)}\big]M(w_0)^{-1}.$$

Putting $A = \lambda(U^{-1}, \mathfrak{z}_1)M(w_0)$, we obtain the first of the following two equalities:

$$\lambda_\alpha(\mathfrak{z}_0) = A \cdot \mathrm{diag}\big[\lambda_\beta(z_0), \overline{\mu_\gamma(w_0)}\big]A^{-1}, \tag{26.40a}$$

$$\mu_\alpha(\mathfrak{z}_0) = B \cdot \mathrm{diag}\big[\overline{\lambda_\gamma(w_0)}, \mu_\beta(z_0)\big]B^{-1}. \tag{26.40b}$$

The second formula, in which we take $B = \mu(U^{-1}, \mathfrak{z}_1)N(z_0)$, can be proved similarly. In view of the definition of Φ in (4.37) and (4.40), from (26.3ya, b) we easily see that $\Xi = \Phi + \Phi'\rho$. Also, our definition of the symbols $\mathfrak{p}_v$ in §11.4 shows that

$$\mathfrak{p}_{v\rho}(\mathfrak{z}_0) = A_v \cdot \mathrm{diag}\big[\mathfrak{p}_{v\rho}(z_0), \mathfrak{p}_v(w_0)\big]A_v^{-1}, \tag{26.41a}$$

$$\mathfrak{p}_v(\mathfrak{z}_0) = B_v \cdot \mathrm{diag}\big[\mathfrak{p}_{v\rho}(w_0), \mathfrak{p}_v(z_0)\big]B_v^{-1}. \tag{26.41b}$$

Here we used (11.23) to find that $p_Z(\xi\rho, \Phi'\rho) = p_Z(\xi, \Phi')$ for $\xi \in I_Z$.

Our reasoning is valid even when $G_{\mathbf{a}}^{\varphi}$ is compact, in which case $\mathfrak{Z}^{\varphi}$ consists of a single point $\mathbf{i}^{\varphi}$. In fact, we can view $\mathbf{i}^{\varphi}$ as a CM-point in the following way. Take $\zeta \in GL_n(K)$ so that $\zeta\varphi\zeta^*$ is diagonal, and define $s : K^n \to K_n^n$ by $s(a_1, \ldots, a_n) = \zeta^{-1}\mathrm{diag}[a_1, \ldots, a_n]\zeta$; thus $Z = K^n$ in the present case. Then $s(a^\rho) = \varphi s(a)^*\varphi^{-1}$, so that we can take $\mathbf{i}^{\varphi}$ to be the above w_0. (If in addition $\psi = \varphi$, then we take $z_0 = \mathbf{i}^{\varphi}$.) From (4.37), (4.40), and (3.24a, b) we see that (K^n, Φ') in this case is the sum of n copies of $(K, \tau\rho)$, where (K, τ) is the CM-type we fixed in §3.5, and $\mathfrak{p}_{v\rho}(\mathbf{i}^{\varphi}) = p_K(\tau_v\rho, \tau\rho)1_n = p_K(\tau_v, \tau)1_n$ and $\mathfrak{p}_v(\mathbf{i}^{\varphi}) = 1$. Then (26.41a, b) are valid, and $\mathfrak{P}_k(\mathbf{i}^{\varphi}) = p_K\bigl(\sum_{v\in\mathbf{a}} k_{v\rho}\tau_v, n\tau\bigr)$. Therefore, according to our general principle of §11.12, we have

(26.41c) $\mathcal{M}_k(\overline{\mathbf{Q}}) = p_K\bigl(\sum_{v\in\mathbf{a}} k_{v\rho}\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$ for every $k \in \mathbf{Z}^{\mathbf{b}}$ if $G_{\mathbf{a}}^{\varphi}$ is compact.

Notice that this is consistent with Theorem 11.17.

26.12. Lemma. *If $f \in \mathcal{N}_k^{\eta,d}$, then $f^\circ(z, w)$ can be written as a finite sum $f^\circ(z, w) = \sum_{a=1}^{e} g_a(z)\overline{h_a(w)}$ with $g_a \in \mathcal{N}_k^{\psi,d}$ and $h_a \in \mathcal{N}_k^{\varphi,d}$. Moreover, if f is $\overline{\mathbf{Q}}$-rational, we have $\mathfrak{q}^{-1}f^\circ(z, w) = \sum_{a=1}^{e} g_a'(z)\overline{h_a'(w)}$ with $g_a' \in \mathcal{N}_k^{\psi,d}(\overline{\mathbf{Q}})$ and $h_a \in \mathcal{N}_k^{\varphi,d}(\overline{\mathbf{Q}})$, where $\mathfrak{q} = p_K\bigl(\sum_{v\in\mathbf{a}}(k_v - k_{v\rho})\tau_v, \sum_{v\in\mathbf{a}} t_v\tau_v\bigr)$ with t_v as in (26.1). This holds even if $G_{\mathbf{a}}^{\varphi}$ is compact, in which case we understand that $\mathfrak{Z}^{\varphi} = \{\mathbf{i}^{\varphi}\}$ and $\mathcal{N}_k^{\varphi,d}(\overline{\mathbf{Q}}) = p_K\bigl(\sum_{v\in\mathbf{a}} k_{v\rho}\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$. In particular, if $\psi = \varphi$ and $G_{\mathbf{a}}^{\varphi}$ is compact, then $f^\circ(\mathbf{i}^{\varphi}, \mathbf{i}^{\varphi}) \in p_K\bigl(\sum_{v\in\mathbf{a}}(k_v + k_{v\rho})\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$.*

PROOF. Take congruence subgroups Γ^ξ of $SU(\xi)$ for $\xi = \varphi, \psi, \eta$ so that $f \in \mathcal{N}_k^{\eta,d}(\Gamma^\eta)$ and $[\Gamma^\psi, \Gamma^\varphi]_S \subset \Gamma^\eta$. In view of Proposition 14.10, it is sufficient to prove the last assertion for $f \in \mathcal{N}_k^{\eta,d}(\Gamma^\eta, \overline{\mathbf{Q}})$. We first prove that $f^\circ(z, w)$ (resp. $\overline{f^\circ(z, w)}$) as a function of z (resp. w) belongs to $\mathcal{N}_k^{\psi,d}(\Gamma^\psi)$ (resp. $\mathcal{N}_k^{\varphi,d}(\Gamma^\varphi)$). The main point here is near holomorphy, since the desired automorphy property follows from [S97, (22.3.3)]. As for the assertion concerning the functions of z, since $\delta\bigl(w, \wp(z)\bigr)$ is holomorphic in z, it is sufficient to show that $f\bigl(\iota_U(z, w)\bigr)$ as a function of z belongs to $\mathcal{N}^d(\mathfrak{Z}^\psi)$. Now, if $w = \gamma(\mathbf{i}^{\varphi})$ with $\gamma \in G^\varphi$, then $\iota_U(z, w) = \alpha\iota_U(z, \mathbf{i}^{\varphi})$ with $\alpha = [1, \gamma]_S$ by (26.37). Recall also that $\iota_U(z, w) = U^{-1}\iota(z, w)$ with $U \in G_{\mathbf{a}}^{\eta}$ and $\iota(z, w)$ defined by [S97, (6.10.2), (6.14.2), (6.14.3)]. Since $f \circ \alpha \in \mathcal{N}^d$ for every $\alpha \in G_{\mathbf{a}}^{\eta}$, it is sufficient to show that $f\bigl(\iota(z, \mathbf{i}^{\varphi})\bigr)$ as a function of z belongs to $\mathcal{N}^d(\mathfrak{Z}^\psi)$. Now $\iota(z, \mathbf{i}^{\varphi})$ is holomorphic in z, and hence the problem can be reduced to $r(\mathfrak{z})$ for $\mathfrak{z} = \iota(z, \mathbf{i}^{\varphi})$. Focusing our attention on $\mathfrak{Z}_v^\psi$ with a fixed v and suppressing the subscript v, from [S97, (6.10.2)] we obtain, for $\mathfrak{z} = \iota(z, \mathbf{i}^{\varphi})$, $r(\mathfrak{z}) = ({}^t\mathfrak{z} - \bar{\mathfrak{z}})^{-1} = \mathrm{diag}[i\xi(z), 2i1_r]^{-1}$, which combined with (26.7) shows that the entries of $r(\mathfrak{z})$ are nearly holomorphic in z of degree 1. This gives the desired fact concerning z. Similarly $\delta\bigl(w, \wp(z)\bigr)$ and $\iota(z, w)$ are holomorphic in $\overline{w}$; also $r\bigl(\iota(\mathbf{i}^\psi, w)\bigr) = \mathrm{diag}[2i1_{q+r}, i\tau\xi(w)\tau^{-1}]^{-1}$ with some $\tau \in GL_{r+t}(\mathbf{Q})$. Therefore we obtain the desired near holomorphy of $\overline{f^\circ(z, w)}$ in w.

Next, let $\{g_a\}_{a=1}^{e}$ be a $\overline{\mathbf{Q}}$-basis of $\mathcal{N}_k^{\psi,d}(\Gamma^\psi, \overline{\mathbf{Q}})$. For each fixed w we have $f^\circ(z, w) = \sum_{a=1}^{e} g_a(z)\overline{h_a(w)}$ with complex numbers $h_a(w)$ uniquely determined by w and a. Since $\bigcap_z \bigl\{x \in \mathbf{C}^e \bigm| \sum_{a=1}^{e} x_a g_a(z)\bigr\} = \{0\}$, we can find e points $z_1, \ldots, z_e$ of $\mathfrak{Z}^\psi$ such that $\det\bigl(g_a(z_b)\bigr)_{a,b=1}^{e} \neq 0$. Solving the linear equations $\overline{f^\circ(z_b, w)} = \sum_{a=1}^{e} \overline{g_a(z_b)}h_a(w)$, we find that $h_a \in \mathcal{N}_k^{\varphi,d}$.

To prove the $\overline{\mathbf{Q}}$-rationality of the h_a, take z_0, w_0, and $\mathfrak{z}_0$ as in §26.11. From (26.41a, b) we easily see tht $\mathfrak{P}_k(\mathfrak{z}_0) = \mathfrak{P}_k(z_0)\mathfrak{P}_{k\rho}(w_0)$. Now

$$\mathfrak{P}_{k\rho}(w_0)/\mathfrak{P}_k(w_0)$$
$$= \prod_{v\in\mathbf{a}} \bigl[\det(\mathfrak{p}_v(w_0))/\det(\mathfrak{p}_{v\rho}(w_0))\bigr]^{k_{v\rho}-k_v} = p_K\Bigl(\sum_{v\in\mathbf{a}}(k_v - k_{v\rho})\tau_v, \sum_{v\in\mathbf{a}} t_v\tau_v\Bigr)$$

by Theorem 11.17 (1) and the remark after it. Thus $\mathfrak{P}_k(\mathfrak{z}_0) = \mathfrak{P}_k(z_0)\mathfrak{P}_{k_0}(w)\mathfrak{q}$ with $\mathfrak{q}$ of our proposition. By Proposition 11.19 we may assume that every period symbol is real. Therefore

$$(*)\qquad \overline{\delta\bigl(w_0, \wp(z_0)\bigr)}^h\mathfrak{P}_k(\mathfrak{z}_0)^{-1}\overline{f(\mathfrak{z}_0)} = \sum_{a=1}^{e}\mathfrak{P}_k(z_0)^{-1}\overline{g_a(z_0)}\cdot\mathfrak{P}_k(w_0)^{-1}\mathfrak{q}^{-1}h_a(w_0).$$

Since the CM-points on $\mathfrak{Z}^\psi$ form a dense subset of $\mathfrak{Z}^\psi$, we can take the above z_a to be CM-points. In §26.1 we took algebraic σ_v and θ_v, and so the entries of z_v and w_v are algebraic for the same reason as in the proof of Lemma 4.13. Therefore from [S97, (6.6.1) and (6.6.8)] we see that $\delta\bigl(w_0, \wp(z_0)\bigr)$ is algebraic. Since f and g_a are $\overline{\mathbf{Q}}$-rational, $\mathfrak{P}_k(\mathfrak{z}_0)^{-1}\overline{f(\mathfrak{z}_0)}$ and $\mathfrak{P}_k(z_0)^{-1}\overline{g_a(z_0)}$ are algebraic. Taking z_0 of (*) to be those z_a, we see that $\mathfrak{P}_k(w_0)^{-1}\mathfrak{q}^{-1}h_a(w_0)$ is algebraic for every a and every CM-point w_0 on $\mathfrak{Z}^\varphi$, that is, $\mathfrak{q}^{-1}h_a$ is $\overline{\mathbf{Q}}$-rational for every a, which completes the proof. Our argument is valid even for compact $G^\varphi_{\mathbf{a}}$ for the reason explained at the end of §26.11.

26.13. Theorem. *Define E_p by (26.34) for $\mathbf{f} \in \mathcal{S}^\varphi_k(D^\varphi)$; assuming $\mathbf{f}$ to be a Hecke eigenform, put*

$$\mathcal{F}_p(z, s; \mathbf{f}, \chi, D^\psi) = E_p(z, s; \mathbf{f}, \chi, D^\psi)\mathcal{Z}(s, \mathbf{f}, \chi)\prod_{j=n}^{q+n-1} L_\mathfrak{c}(2s-j, \chi_1\theta^j),$$

where θ and χ_1 are as in §20.11 and $n = \dim(V)$; define $E^{\psi,\varphi}_k(z, s; f, \Gamma)$ by (26.28); let $\nu \in \mathbf{Z}$ and $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$. Suppose that κ of (16.24a) is 0. Then the following assertions hold:

(i) *If* $\mathrm{Max}(2n+1, q+n) \le \nu \le m_v$ *and* $m_v - \nu \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$, *then* $E_p(z, \nu/2; \mathbf{f}, \chi, D^\psi)$ *and* $E^{\psi,\varphi}_k(z, \nu/2; f, \Gamma)$ *belong to* $\mathcal{N}^t_k$, *where*

$$t = \begin{cases} (q+n)(m-\nu+2)/2 & \text{if } \nu = q+n+1,\ F=\mathbf{Q},\ \text{and } \chi_1 = \theta^\nu, \\ (q+n)(m-\nu\mathbf{a})/2 & \text{otherwise.}\end{cases}$$

(ii) *If* $2q + 2n - m_v \le \nu \le m_v$ *and* $m_v - \nu \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$, *then* $\mathcal{F}_p(z, \nu/2; \mathbf{f}, \chi, D^\psi)$ *belongs to* $\mathcal{N}^t_k$, *except when* $0 \le \nu < q+n$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi_1 = \theta^\nu$, *where*

$$t = \begin{cases} (q+n)(m-\nu+2)/2 & \text{if } \nu = q+n+1,\ F=\mathbf{Q},\ \text{and } \chi_1 = \theta^\nu, \\ (q+n)\{m - |\nu-q-n|\mathbf{a} - (q+n)\mathbf{a}\}/2 & \text{otherwise.}\end{cases}$$

(iii) *Suppose* $m = \mu\mathbf{a}$ *with an integer* μ *such that* $0 < \mu < q+n$; *put* $s_\mu = q + n - (\mu/2)$. *Then* $\mathcal{F}_p(z, s; \mathbf{f}, \chi, D^\psi)$ *has at most a simple pole at* s_μ, *which occurs only when* $\chi_1 = \theta^\mu$. *Moreover, the residue is an element of* $\mathcal{M}_k$.

PROOF. Our reasoning is the same as in §25.7. For simplicity, we suppress $(\mathbf{f}, \chi, D^\psi)$ in the symbols E_p and $\mathcal{F}_p$. In [S97, (22.6.6)] we showed, for an eigenform $\mathbf{f}$,

$$(26.42)\qquad \mu c_m(\mathbf{s})C'(s)\mathcal{Z}(s, \mathbf{f}, \chi)E_p(z, s)$$
$$= \Lambda^n_\mathfrak{c}(s, \chi)\sum_{a\in\mathcal{B}}\chi_\mathbf{h}\bigl(\det(a)\bigr)\int_{\mathfrak{D}_a}(H_{p.a})^\circ(z, w; s)f_a(w)\delta(w)^m\mathbf{d}w.$$

Here $\mathcal{B}$ is a complete set of representatives for $G^{\varphi}\backslash G^{\varphi}_{\mathbf{A}}/D^{\varphi}$; $\mathbf{f} \leftrightarrow (f_a)_{a\in\mathcal{B}}$ in the sense of [S97, p.80] (which is similar to the notation of §20.1); $\Gamma^a = G^{\varphi} \cap aD^{\varphi}a^{-1}$, $\mathfrak{D}_a = \Gamma^a\backslash\mathfrak{Z}^{\varphi}$, and $\mu = [\Gamma^a \cap \mathfrak{r}^{\times} : 1]$; $c_m(\mathbf{s})$ is given by [S97, (A2.9.2)] with $\mathbf{s} = s\mathbf{a} - m/2$ and $C'(s)$ by [S97, (22.6.5)]; $\Lambda^n_{\mathfrak{c}}(s, \chi)$ is as in (20.20) in Case UT; $H_{p,a}(\mathfrak{z}, s) = E_{q_0}(\mathfrak{z}, s)$ for $\mathfrak{z} \in \mathcal{H}^{\mathbf{a}}_{q+n}$ with E_{q_0} of type (17.23a) obtained from an Eisenstein series on $G^{q+n}_{\mathbf{A}}$ of type (16.27) in Case UT. (See [S97, p.184, line 3]; q_0 denotes $q_1\Sigma_{\mathbf{h}}^{-1}$ there.) From (26.42) we immediately obtain

$$\text{(26.43)} \qquad \mu c_m(\mathbf{s})C'(s)\mathcal{F}_p(z, s) = \Lambda^{q+n}_{\mathfrak{c}}(s, \chi)\sum_{a\in\mathcal{B}}\chi_{\mathbf{h}}\big(\det(a)\big)\int_{\mathfrak{D}_a}(H_{p,a})^{\circ}(z, w;\, s)f_a(w)\delta(w)^m\mathbf{d}w.$$

Now evaluate (26.42) at $s = \nu/2$ with ν as in (i). By Theorem 17.12 (iv), $E_{q_0}(\mathfrak{z}, \nu/2)$ belongs to $\pi^{\alpha}\mathcal{N}^{\eta,\,t}_k(\overline{\mathbf{Q}})$ with $\alpha = (n+q)\sum_{v\in\mathbf{a}}(m_v - \nu)/2$ and t as in (i), and so, by Lemma 26.12, $(H_{p,a})^{\circ}(z, w; \nu/2) = \pi^{\alpha}\mathfrak{q}\sum_{i=1}^{e} g_{ai}(z)\overline{h_{ai}(w)}$ with $g_{ai} \in \mathcal{N}^{\psi,t}_k(\overline{\mathbf{Q}})$ and $h_{ai} \in \mathcal{N}^{\varphi,t}_k(\overline{\mathbf{Q}})$. From [S97, (A2.9.2)] we see that for $s = \nu/2$ with ν as in (I), $c_m(\mathbf{s}) \in \pi^{d_0}\overline{\mathbf{Q}}^{\times}$, where d_0 is the complex dimension of $\mathfrak{Z}^{\varphi}$. Since $\chi_{\mathbf{a}}(x) = x^m_{\mathbf{a}}|x_{\mathbf{a}}|^{-m}$ for $x \in F^{\times}_{\mathbf{a}}$, Lemma 17.5 (2) shows that $\Lambda^n_{\mathfrak{c}}(\nu/2, \chi) \in \pi^{\gamma}\overline{\mathbf{Q}}^{\times}$ with $\gamma = dn\nu - dn(n-1)/2$; also $C'(\nu/2) \in \overline{\mathbf{Q}}^{\times}$. Thus

$$\text{(26.44)} \qquad \mathcal{Z}(\nu/2, \mathbf{f}, \chi)E_p(z, \nu/2) = \pi^{\alpha+\gamma-d_0}\mathfrak{q}\sum_{a,\,i}\operatorname{vol}(\mathcal{D}_a)\langle h_{ai}, f_a\rangle\, g'_{ai}(z)$$

with some $g'_{ai} \in \mathcal{N}^{\psi,t}_k(\overline{\mathbf{Q}})$. By [S97, Proposition 20.4 (3)], $\mathcal{Z}(s, \mathbf{f}, \chi)$ is finite and nonzero for $\mathrm{Re}(s) > n$. Therefore from (26.44) we obtain (i) for $E_p(z, \nu/2)$ when $\mathbf{f}$ is a Hecke eigenform. The result holds also for an arbitrary $\mathbf{f} \in \mathcal{S}^{\varphi}_k(D^{\varphi})$, since the last space is spanned by eigenforms as shown in [S97, Proposition 20.4 (1)]. This proves (i) for $E_p(z, \nu/2)$, which combined with [S97, Proposition 20.10] proves (i) for $E^{\psi,\varphi}_k(z, \nu/2; f, \Gamma)$. To prove (ii), we employ (26.43) and D_{q_0} instead of (26.42) and E_{q_0}, where D_{q_0} is defined by (17.24). Then we obtain (ii) from Theorem 17.12 (v). Similarly (iii) follows from Theorem 17.8, since $\Lambda^{q+n}_{\mathfrak{c}}(s, \chi)$ is finite and nonzero at s_{μ}.

REMARK. For the proof of (i) we employed the nonvanishing of $\mathcal{Z}(s, \mathbf{f}, \chi)$ for $\mathrm{Re}(s) > n$. In fact, it is plausible that such nonvanishing holds for $\mathrm{Re}(s) > 3n/4$. If that is so, we can replace $\mathrm{Max}(2n+1, q+n)$ by $\mathrm{Max}\big((3n/2)+1, q+n\big)$.

We can naturally ask whether the functions of Theorem 26.13 are $\overline{\mathbf{Q}}$-rational. The answers will be given in Theorems 27.16, 29.6, and 29.7 below.

26.14. Lemma. *The notation being as in §26.10, let* $\mathbf{f}$ *be a nonzero element of* $\mathcal{S}^{\varphi}_k(D^{\varphi})$ *such that* $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ *with* $\lambda(\mathfrak{a}) \in \mathbf{C}$ *for every* $\mathfrak{a}$. *Then the eigenvalues* $\lambda(\mathfrak{a})$ *generate an algebraic number field stable under complex conjugation.*

PROOF. Let $\mathcal{S}^{\varphi}_k(D^{\varphi}, \overline{\mathbf{Q}})$ denote the set of all $\overline{\mathbf{Q}}$-rational elements of $\mathcal{S}^{\varphi}_k(D^{\varphi})$. (We call $\mathbf{f}$ as in (26.29a, b) $\overline{\mathbf{Q}}$-*rational* if f_p is $\overline{\mathbf{Q}}$-rational for every p.) By Proposition 11.15 we have $\mathcal{S}^{\varphi}_k(D^{\varphi}) = \mathcal{S}^{\varphi}_k(D^{\varphi}, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$. From Proposition 11.13 and [S97, (11.9.1)] we see that $\mathcal{S}^{\varphi}_k(D^{\varphi}, \overline{\mathbf{Q}})$ is stable under $\mathfrak{R}(D^{\varphi}, \mathfrak{X})$. Since $\mathcal{S}^{\varphi}_k(D^{\varphi}, \overline{\mathbf{Q}})$ is finite-dimensional over $\overline{\mathbf{Q}}$, each $\lambda(\mathfrak{a})$ is algebraic. Now, if $T(\mathfrak{a}) = \sum_{\tau} D^{\varphi}\tau D^{\varphi}$, then we easily see that $T(\mathfrak{a}^{\rho}) = \sum_{\tau} D^{\varphi}\tau^{-1}D^{\varphi}$. By [S97, Proposition 11.7] we have $\langle\, \mathbf{f}|T(\mathfrak{a}), \mathbf{f}\,\rangle = \langle\, \mathbf{f}, \mathbf{f}|T(\mathfrak{a}^{\rho})\,\rangle$, so that $\lambda(\mathfrak{a}^{\rho}) = \overline{\lambda(\mathfrak{a})}$, which gives the desired result.

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CHAPTER VII

ARITHMETICITY OF THE CRITICAL VALUES OF ZETA FUNCTIONS AND EISENSTEIN SERIES OF GENERAL TYPES

27. The spaces of holomorphic Eisenstein series

27.1. Let (W, ψ), (Z, ζ), (V_r, φ_r), and $l = l(\psi)$ be as in §26.4; let $k \in \mathbf{Z}^{\mathbf{b}}$. Our next task is the construction of a certain subspace of $\mathcal{M}_k^\psi$, spanned by holomorphic Eisenstein series, such that $\mathcal{M}_k^\psi$ is the direct sum of $\mathcal{S}_k^\psi$ and that subspace. Put $m = (k_{v\rho} + k_v)_{v\in\mathbf{a}}$ and $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$. By [S97, Proposition 10.6 (3)], $\mathcal{M}_k^\psi = \mathcal{S}_k^\psi$ either if ψ is anisotropic, or if $F \neq \mathbf{Q}$ and $m_v \neq m_{v'}$ for some $v, v' \in \mathbf{a}$. Therefore, our problem is meaningful only when ψ is not anisotropic, that is, $l(\psi) > 0$, and $m = \mu\mathbf{a}$ with $\mu \in \mathbf{Z}$, which we assume throughout this section. We present our results not only in Case UB, but also in Cases SP and UT, but for the most part give the proof in Case UB; in the other two cases we only need minor (or rather, obvious) modifications and the following changes of symbols: G^ψ, G^{φ_r}, P_r^ψ, $\mathfrak{Z}^\psi$, π_r, λ_r, $\wp_r$, l, $\mathcal{M}_k^\psi$, $\mathcal{M}_k^\varphi$, $\mathcal{S}_k^\psi$, $\mathcal{S}_k^\varphi$, $E_k^{\psi,\varphi}$, and $\mathrm{diag}[z, w]$ should be replaced by G^n, G^r, $P^{n,r}$, $\mathcal{H}^n$, π_r, λ_r, $\wp_r$, n, $\mathcal{M}_k^n$, $\mathcal{M}_k^r$, $\mathcal{S}_k^n$, $\mathcal{S}_k^r$, $E_k^{n,r}$ of §§23.1 and 23.2, and $\mathrm{diag}[w, z]$; $Z = \{0\}$ in Case UT. The case of half-integral k can be included if we take $\mathcal{G}^n$ and $\mathcal{P}^{n,r}$ in place of G^n and $P^{n,r}$; of course $\mu - 1/2 \in \mathbf{Z}$ in that case. We put $[k] = k$ if k is integral and $[k] = k - \mathbf{a}/2$ otherwise; also we put $m = k$ and $\ell = [k]$ in Case SP.

We denote by Γ an unspecified congruence subgroup of G^ψ, and by ρ a real variable on $(0, \infty)$. We put $\mathbf{i}_q = i1_q$ and view it as as an element of $\mathcal{H}_q^{\mathbf{a}}$; also we write $\wp_r^\psi$ or $\wp_r$ for $\wp_{\varphi_r}^\psi$ (see §23.1 and (26.26)). We note here two basic formulas (see (23.4), (23.8), and [S97, (6.9.1), (12.3.4)]):

$$\wp_r(\alpha\mathfrak{z}) = \pi_r(\alpha)\wp_r(\mathfrak{z}), \tag{27.1}$$

$$j_\alpha^k(\mathfrak{z}) = \lambda_r(\alpha)^{[k]}|\lambda_r(\alpha)|^{k-[k]}j^k\big(\pi_r(\alpha), \wp_r(\mathfrak{z})\big) \qquad (\alpha \in P_r^\psi,\ \mathfrak{z} \in \mathfrak{Z}^\psi). \tag{27.2}$$

The factor $|\lambda_r(\alpha)|^{k-[k]}$ can be eliminated if k is integral. Since it is cumbersome to have such a factor in each case, we hereafter state our formulas only for integral k. The corresponding formulas for half-integral k can be found in [S95a, Section 8]. We note that

$$a^{-k}|a|^m = a^{-\ell}|a|^\ell \quad \textit{for every} \quad a \in K^\times \quad \textit{if} \ \ k \in \mathbf{Z}^{\mathbf{b}}. \tag{27.3}$$

Given a function $f : \mathfrak{Z}^\psi \to \mathbf{C}$, we define $\varPhi f : \mathfrak{Z}^{\varphi_{l-1}} \to \mathbf{C}$ by

$$(\varPhi f)(w) = \begin{cases} \lim_{\rho\to\infty} f\big(\mathrm{diag}[\rho\mathbf{i}_1, w]\big) & (\text{Case UB},\ w \in \mathfrak{Z}^{\varphi_{l-1}}), \\ \lim_{\rho\to\infty} f\big(\mathrm{diag}[w, \rho\mathbf{i}_1]\big) & (\text{Cases SP, UT},\ w \in \mathcal{H}^{n-1}), \end{cases} \tag{27.4}$$

whenever the limit exists. If φ_{l-1} is totally definite or $\psi = \eta'_1$, we ignore w, and so Φf is a constant. We define Φ^q for $0 \le q \le l(\psi)$ by $\Phi^q = \Phi\Phi^{q-1}$, with the identity map as Φ^0. Then $\Phi^q f$, if meaningful, is a function on $\mathfrak{Z}^{\varphi_r}$, $r = l - q$. If φ_r is totally definite or $\psi = \eta'_q$, then $\Phi^q f$ is a constant.

27.2. Lemma. *For $f \in \mathcal{M}_k^\psi(\Gamma)$, the following assertions hold:*

(1) $(\Phi^q f)(w) = c_0^q(w;\, f) = \lim_{\rho\to\infty} f(\mathrm{diag}[\rho \mathbf{i}_q,\, w])$.

(2) f is a cusp form if and only if $\Phi(f\|_k\alpha) = 0$ for every $\alpha \in G^\psi$.

(3) $\Phi^q(f\|_k\,\alpha) = \lambda_r(\alpha)^{-k}(\Phi^q f)\|_k\pi_r(\alpha)$ *if $\alpha \in P_r^\psi$ and $q = l - r$.*

(4) $\Phi^{l-r} f$ *belongs to $\mathcal{M}_k^{\varphi_r}(\Gamma,\, P_r^\psi)$ of (23.9) or (26.23).*

(5) *In Cases SP and UT if $f(z) = \sum_{h\in S^n} c(h)\mathbf{e}_\mathbf{a}^n(hz)$ for $z \in \mathcal{H}^n$ as in (5.22a), then $\Phi^n f = c(0)$, and $(\Phi^{n-r} f)(w) = \sum_{g\in S^r} c(\mathrm{diag}[g,\, 0_{n-r}])\mathbf{e}_\mathbf{a}^r(gw)$ if $r > 0$.*

PROOF. That $c_0^q(w;\, f) = \lim_{\rho\to\infty} f(\mathrm{diag}[\rho\mathbf{i}_q,\, w])$ can easily be seen from (26.17) and Proposition 26.6 (1). Then we obtain (1) for $q = 1$. Assuming (1) for $q = t$ and taking $q = t + 1$ in the proof of Proposition 26.6, we obtain

$$(\Phi^t f)(\mathrm{diag}[z',\, w]) = c_0^t(\mathrm{diag}[z',\, w];\, f) = \sum_{h\in F} a_{0,h}(w)\mathbf{e}_\mathbf{a}(hz'),$$

so that $(\Phi(\Phi^t f))(w) = a_{0,0}(w) = c_0^{t+1}(w;\, f)$ as shown there. This proves (1) for an arbitrary q by induction. Assertion (5) can be proved in a similar and simpler way. Then (2) follows from Proposition 26.6 (5), or from the definition of a cusp form in §5.8. Next, let $\alpha \in P_r^\psi$, $\beta = \pi_r(\alpha)$ and $q = l - r$. By (27.1) and (27.2), for $\mathfrak{z} = \mathrm{diag}[z,\, w]$ with $z \in \mathcal{H}_q^\mathbf{a}$ and $w \in \mathfrak{Z}^{\varphi_r}$ we have $\alpha(\mathfrak{z}) = (z_1,\, u_1,\, \beta w)$ with some z_1 and u_1, so that

$$\begin{aligned}(f\|_k\,\alpha)(\mathrm{diag}[z,\, w]) &= \lambda_r(\alpha)^{-k} j_\beta^k(w)^{-1} f(z_1,\, u_1,\, \beta w)\\ &= \lambda_r(\alpha)^{-k} j_\beta^k(w)^{-1} \textstyle\sum_h c_h^q(u_1,\, \beta w;\, f)\mathbf{e}_\mathbf{a}^q(hz_1).\end{aligned}$$

Observe that $z_1 = (az+b)a^*$ with $a \in GL_q(\mathbf{C})^\mathbf{a}$ and $b \in (\mathbf{C}_q^q)^\mathbf{a}$ that depend only on α, and u_1 is independent of z, and hence $c_0^q(w;\, f\|\,\alpha) = \lambda_r(\alpha)^{-k} j_\beta^k(w)^{-1} c_0^q(\beta w;\, f)$, which together with (1) proves (3). Since $\lambda_r(\alpha) \in \mathfrak{r}^\times$ for $\alpha \in P_r^\psi \cap \Gamma$ and $|u|^m = |u|^{\mu\mathbf{a}} = 1$ for $u \in \mathfrak{r}^\times$, (4) follows from (3) and (27.3).

27.3. Lemma. *Let X be a complete set of representatives for $P_r^\psi\backslash G^\psi/\Gamma$; let $f \in \mathcal{M}_k^\psi(\Gamma)$ and $q = l - r$. Then $\Phi^q(f\|_k\alpha) = 0$ for every $\alpha \in G^\psi$ if and only if $\Phi^q(f\|_k\xi^{-1}) = 0$ for every $\xi \in X$, in which case $\Phi^{q-1}(f\|_k\alpha)$ is a cusp form for every $\alpha \in G^\psi$.*

PROOF. Given $\alpha \in G^\psi$, we can put $\alpha = \gamma\xi^{-1}\beta$ with $\gamma \in \Gamma$, $\xi \in X$, and $\beta \in P_r^\psi$. By Lemma 27.2 (3), $\Phi^q(f\|\alpha) = \Phi^q(f\|\xi^{-1}\beta) = c\Phi^q(f\|\xi^{-1})\|\pi_r(\beta)$ with a constant c, from which our first assertion follows. Next, given any $\varepsilon \in G^{\varphi_{r+1}}$, we can find $\delta \in G^\psi$ such that $\pi_{r+1}(\delta) = \varepsilon$. Then $\Phi(\Phi^{q-1}(f\|\alpha)\|\varepsilon) = c'\Phi(\Phi^{q-1}(f\|\alpha\delta)) = c'\Phi^q(f\|\alpha\delta)$ with a constant c', again by Lemma 27.2 (3). Therefore, if $\Phi^q(f\|G^\psi) = 0$, then by Lemma 27.2 (2), $\Phi^{q-1}(f\|\alpha)$ must be a cusp form.

27.4. Lemma. *Let $a \in R_p^r$ and $b \in R_q^r$ with $r \le q \le p$, where R is a field with infinitely many elements. If* $\mathrm{rank}[a\ \ b] = r$, *then there exists an element $x \in R_q^p$ such that* $\mathrm{rank}(ax + b) = r$. *Moreover, if R is $\mathbf{R}$ or $\mathbf{C}$, such an x can be found in any nonempty open subset of R_q^p.*

PROOF. We may assume that $0 < \mathrm{rank}(a) < r$, since our assertions are obvious otherwise. Take $u \in GL_r(R)$ and $v \in GL_p(R)$ so that $uav = \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}$ with $0 < n < r$. Changing a and b for uav and ub, we may assume that $a = \begin{bmatrix} 1_n & 0 \\ 0 & 0 \end{bmatrix}$. Put $b = \begin{bmatrix} c \\ d \end{bmatrix}$ with $c \in R_q^n$ and $d \in R_q^{r-n}$; let y be the upper $(n \times q)$-block of x. Then $\mathrm{rank}(d) = r - n$ and $ax + b = \begin{bmatrix} y + c \\ d \end{bmatrix}$. Choosing a suitable y, we have $\mathrm{rank}(ax + b) = r$ as desired.

27.5. Lemma. *For $\alpha \in G^\psi$ and $0 \le r < l$ we have $\varPhi\big((\delta \circ \wp_r \circ \alpha)^{-\mathbf{a}}|j_\alpha|^{-2\mathbf{a}}\big) \neq 0$ if and only if $\alpha \in P_r^\psi P_{l-1}^\psi$, where we understand that $\delta \circ \wp_r \circ \alpha = 1$ if $r = 0$.*

PROOF. Put $q = l-r$. Let Z and g_i be as in §26.4 and let $\{u_1, \dots, u_p\}$ be a K-basis of Z; express α by a matrix with respect to the basis $\{g_1, \dots, g_l, u_1, \dots, u_p, g_{l+1}, \dots, g_{2l}\}$; thus $\dim(Z) = p$ and $\dim(W) = 2l+p$. Write the $q \times (2l+p)$-block of α whose i-th row is the $(l+p+i)$-th row of α for $1 \le i \le q$ in the form $[a \quad b \quad c]$ with $a, c \in K_l^q$ and $b \in K_p^q$. This $q \times (2l+p)$-matrix represents the restriction of α to $I_q = \sum_{i=1}^q Kg_{l+i}$, so that $\alpha \in P_{l-1}^\psi$ if and only if its top row except the upper left entry of c vanishes. Now fix one $v \in \mathbf{a}$; take κ_v and τ_v as in (26.11) and define σ_v by (26.13) by taking r there to be l. Let h be the $q \times (2l+p)$-matrix whose i-th row is the $(l + s_v + t_v + i)$-th row of $(\sigma\alpha\sigma^{-1})_v$. Then $h = [\kappa_v^{-1}a_v \quad b' \quad c_v \quad b'']$, where b' (resp. b'') consists of the first $s_v + t_v$ (resp. the last s_v) rows of $\kappa_v^{-1}b_v\tau_v^{-1}$. Since we are looking at matrices in $\mathfrak{Z}_v^\psi$ and in the group acting on it, hereafter we suppress the subscript v. Recall that $\eta(\mathfrak{z}) = i(\mathfrak{x}^* - \mathfrak{x}) - \mathfrak{y}^*\theta^{-1}\mathfrak{y}$ for $\mathfrak{z} = \begin{bmatrix} \mathfrak{x} \\ \mathfrak{y} \end{bmatrix} \in \mathfrak{Z}^\psi$ (see [S97, (6.1.8)]). Let $\mathfrak{z} = \mathrm{diag}[\rho\mathbf{i}_1, w]$ with $w \in \mathfrak{Z}^{\varphi_{l-1}}$ and let y be the upper left $(q \times q)$-block of $\eta(\alpha\mathfrak{z})_v^{-1}$. Put $a = [a_1 \quad a_2]$ and $c = [c_1 \quad c_2]$, where a_1 (resp. c_1) is the first column of a (resp. c) and $\kappa^{-1}a_1 = a'$ and $\kappa^{-1}a_2 = a''$. Then $h = [a' \quad d \quad c_1 \quad e]$ with $d = [a'' \quad b']$ and $e = [c_2 \quad b'']$. Let us first assume that $\mathfrak{Z}_v^{\varphi_{l-1}}$ is nontrivial. Observe that $\eta\big(\wp_r(\alpha\mathfrak{z})\big)$ is the lower right $r_v \times r_v$-block of $\eta(\alpha\mathfrak{z})$, where $r_v = r + s_v$. Therefore by Lemma 1.3 (1) we have $2^{r_v}\delta\big(\wp_r(\alpha\mathfrak{z})\big) = 2^{q+r_v}\delta(\alpha\mathfrak{z})\det(y) = 2^{q+r_v}|j_\alpha(\mathfrak{z})|^{-2}\delta(\mathfrak{z})\det(y)$, and hence $|j_\alpha(\mathfrak{z})|^2\delta\big(\wp_r(\alpha\mathfrak{z})\big) = 2^q\delta(w)\rho \cdot \det(y)$. Observe that the upper q rows of $\mu(\alpha, \mathfrak{z})$ is $[i\rho a' + c_1 \quad dw + e]$. Since $\eta(\mathfrak{z}) = \mathrm{diag}[2\rho, \eta(w)]$ and $\eta(\alpha\mathfrak{z})^{-1} = \mu(\alpha, \mathfrak{z})\eta(\mathfrak{z})^{-1}\mu(\alpha, \mathfrak{z})^*$, we obtain, by a direct calculation, $y = A+B$ with

$$A = (i\rho a' + c_1)(2\rho)^{-1}(i\rho a' + c_1)^* \quad \text{and} \quad B = (dw + e)\eta(w)^{-1}(dw + e)^*.$$

Thus

$$|j_\alpha(\mathfrak{z})|^2\delta\big(\wp_r(\alpha\mathfrak{z})\big) = \delta(w)\rho \cdot \det(\rho^{-1}c_1c_1^* + \rho C + D)$$

with matrices C and D which do not involve ρ. Since $c_1c_1^*$ is of rank ≤ 1, we easily see that the right-hand side is a polynomial of ρ. (We shall later show that $\big((\delta \circ \wp_r \circ \alpha)|j_\alpha|^2\big)_v$ is a polynomial in ρ even if $\mathfrak{Z}_v^{\varphi_{l-1}}$ is trivial.) Suppose $\varPhi\big((\delta \circ \wp_r \circ \alpha)^{-\mathbf{a}}|j_\alpha|^{-2\mathbf{a}}\big) \neq 0$; then this polynomial (for any fixed $v \in \mathbf{a}$) must be a constant. Since A and B are nonnegative, we have $\rho \cdot \det[A + B] \ge \rho \cdot \det(B)$, and hence $\det(B) = 0$. It follows that $\mathrm{rank}(dw + e) < q$. This is so for every $w \in \mathfrak{Z}_v^{\varphi_{l-1}}$. By Lemma 27.4 we have $\mathrm{rank}[d \quad e] < q$, and hence $\mathrm{rank}[a_2 \quad b \quad c_2] < q$. Thus we can find $u \in SL_q(K)$ such that the top row of $u[a_2 \quad b \quad c_2]$ is 0. Put

$\beta = \mathrm{diag}[\widehat{u},\, 1_{r+p},\, u,\, 1_r;$ then $\beta \in P_r^\psi$, $\pi_r(\beta) = 1$, and $j_\beta = 1$. Change α for $\beta\alpha$ and observe that $\delta\big(\wp_r(\alpha\mathfrak{z})\big)|j_\alpha(\mathfrak{z})|^2$ does not change. Thus we may assume that the top row of $[a_2 \;\; b \;\; c_2]$ is 0. Then the top row of $[d \;\; e]$ is 0, and so the top row of B is 0. Let ν and ν' be the first components of a_1 and c_1. Suppose $\nu \neq 0$. Then the top row of A is 0 for $\rho = i\kappa\nu'/\nu$. Thus $\rho \cdot \det(y) = 0$ for such a ρ, so that the constant $\rho \cdot \det(y)$ must be 0. On the other hand, the quantity is positive for $0 < \rho \in \mathbf{R}$, which is a contradiction. Therefore $\nu = 0$, and hence the top row of $[\, a \;\; b \;\; c_2\,]$ is 0, which means that the present α belongs to P_{l-1}^ψ. Thus the original α belongs to $P_r^\psi P_{l-1}^\psi$ as expected.

We have assumed that $\mathfrak{Z}_v^{\varphi_{l-1}}$ is nontrivial. If $\mathfrak{Z}_v^{\varphi_{l-1}}$ is trivial, then $l = 1$ and $s_v = 0$, and so $r = 0$ and $q = 1$. In this case $\mu(\alpha,\, \mathfrak{z})_v = (\rho a'i + c_1)_v$, and hence $\delta\big(\wp_r(\alpha\mathfrak{z})\big)_v|j_\alpha(\mathfrak{z})_v|^2 = |(\rho a'i + c_1)_v|^2$, which is again a polynomial in ρ. Thus $\varPhi\big((\delta \circ \wp_r \circ \alpha)^{-\mathbf{a}}|j_\alpha|^{-2\mathbf{a}}\big) \neq 0$ only if $a' = 0$, in which case $a = 0$. Since $s_v = 0$, ζ is anisotropic. Now $[a \;\; b \;\; c]$ represents $g_2\alpha$, so that $b\zeta b^* = 0$. Thus $b = 0$, which shows that $\alpha \in P_0^\psi$ as expected. This proves the 'only if'-part.

Conversely, let $\alpha = \gamma\varepsilon$ with $\gamma \in P_r^\psi$ and $\varepsilon \in P_{l-1}^\psi$; put $\xi = \pi_{l-1}(\varepsilon)$. Then by (27.1) and (27.2), for $\mathfrak{z} = \mathrm{diag}[\rho\mathbf{i}_1,\, w]$ we have $\wp_r(\varepsilon\mathfrak{z}) = \wp_r^{\varphi_{l-1}}\big(\wp_{l-1}(\varepsilon\mathfrak{z})\big) = \wp_r^{\varphi_{l-1}}(\xi w)$ and $\delta\big(\wp_r(\alpha\mathfrak{z})\big)|j_\alpha(\mathfrak{z})|^2 = |\lambda_r(\gamma)\lambda_{l-1}(\varepsilon)j_\xi(w)|^2\delta\big(\wp_r^{\varphi_{l-1}}(\xi w)\big)$ which proves the 'if'-part.

27.6. Lemma. *Let $\delta_f^\psi(\mathfrak{z}) = \delta(\mathfrak{z},\, s;\, f)$ with the notation of (26.27) with $f \in \mathcal{S}_k^\varphi$, $\varphi = \varphi_r$, and let $\alpha \in G^\psi$. Then for* $\mathrm{Re}(s) > 0$ *and $l > t \geq r$ we have $\varPhi^{l-t}\big\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}} \cdot [\delta_f^\psi\|_k\, \alpha]\big\} \neq 0$ only if $\alpha = \beta\gamma$ with $\beta \in P_r^\psi$ and $\gamma \in P_t^\psi$, in which case for $\mathfrak{z} =$* $\mathrm{diag}[w',\, w]$ *with $(w',\, w) \in \mathcal{H}_{l-t}^{\mathbf{a}} \times \mathfrak{Z}^\omega$, $\omega = \varphi_t$, we have*

$$\begin{aligned}(27.5)\qquad \delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|_k\alpha] &= \varPhi^{l-t}\big\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|_k\,\alpha]\big\}\\ &= \delta(w)^{m/2-s\mathbf{a}}|\lambda_r(\beta)\lambda_t(\gamma)|^{m-2s\mathbf{a}}\big(\lambda_r(\beta)\lambda_t(\gamma)\big)^{-k}\delta\big(w,\, s;\, f\|_k\pi_r(\beta)\big)\|_k\pi_t(\gamma).\end{aligned}$$

PROOF. Let $\alpha = \beta\gamma$ and $\mathfrak{z} = \mathrm{diag}[w',\, w]$ as above. Put $g = f\|\pi_r(\beta)$. By [S97, (12.3.5)] or (23.12) we have $\delta_f^\psi\|\beta = \lambda_r(\beta)^{-k}|\lambda_r(\beta)|^{m-2s\mathbf{a}}\delta_g^\psi$, and

$$(*)\qquad \delta(\mathfrak{z})^{m/2-s\mathbf{a}} \cdot (\delta_g^\psi\|\gamma) = j_\gamma^k(\mathfrak{z})^{-1}|j_\gamma(\mathfrak{z})|^{m-2s\mathbf{a}}g\big(\wp_r(\gamma\mathfrak{z})\big)\delta\big(\wp_r(\gamma\mathfrak{z})\big)^{m/2-s\mathbf{a}}.$$

Put $\xi = \pi_t(\gamma)$. Then $\wp_r(\gamma\mathfrak{z}) = \wp_r^\omega\big(\wp_t(\gamma\mathfrak{z})\big) = \wp_r^\omega\big(\xi\wp_t(\mathfrak{z})\big) = \wp_r^\omega(\xi w)$, and hence, by (27.2), the quantity of $(*)$ equals $\lambda_t(\gamma)^{-k}|\lambda_t(\gamma)|^{m-2s\mathbf{a}}\delta(w)^{m/2-s\mathbf{a}} \cdot \big(\delta_g^\omega\|\xi\big)$. From these we obtain (27.5). Now by [S97, Proposition 10.6 (1)], $|\delta(z)^{m/2}f(z)| \leq C$ for every $z \in \mathfrak{Z}^\varphi$ with a constant C. Therefore, for $\alpha \in G^\psi$ we have

$$\begin{aligned}\Big|\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|\alpha]\Big| &= \Big|f\big(\wp_r(\alpha\mathfrak{z})\big)\delta\big(\wp_r(\alpha\mathfrak{z})\big)^{m/2-s\mathbf{a}}j_\alpha(\mathfrak{z})^{-2s\mathbf{a}}\Big|\\ &\leq C\Big|\delta\big(\wp_r(\alpha\mathfrak{z})\big)^{-s\mathbf{a}}j_\alpha(\mathfrak{z})^{-2s\mathbf{a}}\Big|.\end{aligned}$$

By Lemma 27.5 the last quantity, with $\mathfrak{z} = \mathrm{diag}[\rho\mathbf{i}_1, \mathfrak{z}_1]$ and $\mathrm{Re}(s) > 0$, tends to 0 as $\rho \to \infty$ if $\alpha \notin P_r^\psi P_{l-1}^\psi$. This proves the case $t = l-1$. Suppose that our lemma is true for $\varPhi^{l-t}$, $t > r$, and $\varPhi^{l-t+1}\big(\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|\alpha]\big) \neq 0$. Then $\varPhi^{l-t}\big(\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|\alpha]\big) \neq 0$, and so $\alpha = \beta\gamma$ with $\beta \in P_r^\psi$ and $\gamma \in P_t^\psi$; also, by (27.5), for $w \in \mathfrak{Z}^\omega$ we have

$$ab^{-s}\varPhi\big\{\delta(w)^{m/2-s\mathbf{a}}\delta\big(w,\, s;\, f\|\pi_r(\beta)\big)\|\pi_t(\gamma)\big\} = \varPhi\big\{\varPhi^{l-t}\big(\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^\psi\|\alpha]\big)\big\}$$

$$= \Phi^{l-t+1}\bigl(\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta_f^{\psi}\|\alpha]\bigr) \neq 0$$

with some $a \in \mathbf{C}^\times$ and $0 < b \in \mathbf{R}$. Applying our result in the case $\Phi^{l-t} = \Phi$ to the quantity involving w on the left-hand side, we have $\pi_t(\gamma) = \xi'\eta'$ with $\xi' \in P_r^\omega$ and $\eta' \in P_{t-1}^\omega$. Now we can easily verify that

$$P_t^\psi \cap P_r^\psi = \{\alpha \in P_t^\psi \mid \pi_t(\alpha) \in P_r^\omega\}, \tag{27.6a}$$

$$\pi_t(P_t^\psi \cap P_r^\psi) = P_r^\omega, \tag{27.6b}$$

and hence we can put $\xi' = \pi_t(\xi)$ and $\eta' = \pi_t(\eta)$ with $\xi \in P_t^\psi \cap P_r^\psi$ and $\eta \in P_t^\psi \cap P_{t-1}^\psi$. Then $\alpha = \beta\xi\xi^{-1}\gamma$, $\xi^{-1}\gamma \in P_t^\psi$, $\beta\xi \in P_r^\psi$, and $\pi_t(\xi^{-1}\gamma) = \eta' \in P_{t-1}^\omega$, and hence $\xi^{-1}\gamma \in P_{t-1}^\psi$ by (27.6a). Therefore induction proves our lemma.

27.7. Lemma. *Put $\omega = \varphi_t$, $P = P_t^\psi$, and $Q = P_r^\psi$, $r \le t < l$. For each ξ in the set X of Lemma 27.3 let Z_ξ be a complete set of representatives for $(P \cap Q)\backslash Q/(\xi\Gamma\xi^{-1} \cap Q)$. For every $\zeta \in Z_\xi$ such that $\zeta\xi\Gamma \cap P \neq \varnothing$ choose and fix an element $\eta \in \zeta\xi\Gamma \cap P$. Let Y denote the set of all such η's. Then the following assertios hold:*

(1) *Y is a finite set, $P = \bigsqcup_{\eta\in Y}(P\cap Q)\eta(\Gamma\cap P)$, and $G^\omega = \bigsqcup_{\eta\in Y} P_t^\omega \pi_t\bigl(\eta(\Gamma\cap P)\bigr)$.*

(2) *If R_η is a complete set of representatives for $(\eta\Gamma\eta^{-1} \cap Q \cap P)\backslash(\eta\Gamma\cap P)$, then $\bigsqcup_{\eta\in Y}\zeta^{-1}R_\eta$ gives $\bigsqcup_{\xi\in X}(\xi\Gamma\xi^{-1} \cap Q)\backslash(\xi\Gamma \cap QP)$, where ζ is taken for each η so that $\zeta \in Z_\xi$ and $\eta \in \zeta\xi\Gamma$.*

(3) *π_t gives a bijection of R_η onto $(\eta_t\Delta\eta_t^{-1} \cap P_r^\omega)\backslash\eta_t\Delta$, where $\Delta = \pi_t(\Gamma \cap P)$ and $\eta_t = \pi_t(\eta)$.*

PROOF. Clearly $Q\xi\Gamma = \bigsqcup_{\zeta\in Z_\xi}(P\cap Q)\zeta\xi\Gamma$ for every $\xi \in X$. Thus $G^\psi = \bigsqcup_{\xi\in X} Q\xi\Gamma = \bigsqcup_{\xi,\zeta}(P\cap Q)\zeta\xi\Gamma$, and hence $P = \bigsqcup_{\xi,\zeta}(P\cap Q)(\zeta\xi\Gamma\cap P) = \bigsqcup_{\eta\in Y}(P\cap Q)\eta(\Gamma\cap P)$. Applying π_t to this equality, we obtain $G^\omega = \bigcup_{\eta\in Y} P_r^\omega\pi_t\bigl(\eta(\Gamma\cap P)\bigr)$ by (27.6b). To see that this union is disjoint, suppose that $\pi_t(\eta) = \alpha\pi_t(\eta'\gamma)$ with $\alpha \in P_r^\omega$, $\gamma \in \Gamma \cap P$ and $\eta, \eta' \in Y$. By (27.6b) we can put $\alpha = \pi_t(\beta)$ with $\beta \in P \cap Q$. Then $\pi_t(\beta\eta'\gamma\eta^{-1}) = 1$. Since $\beta\eta'\gamma\eta^{-1} \in P$, (27.6a) shows that $\beta\eta'\gamma\eta^{-1} \in P\cap Q$, and hence $\eta' \in (P\cap Q)\eta(\Gamma\cap P)$. Thus $\eta = \eta'$, which proves the expected disjointness. Since $P_r^\omega\backslash G^\omega/\Gamma'$ is finite for any congruence subgroup Γ' of G^ω, Y must be finite. Now $Q = Q^{-1} = \bigsqcup_{\zeta\in Z_\xi}(\xi\Gamma\xi^{-1}\cap Q)\zeta^{-1}(P\cap Q)$, and hence $QP = \bigsqcup_{\zeta\in Z_\xi}(\xi\Gamma\xi^{-1}\cap Q)\zeta^{-1}P$. Therefore $(\xi\Gamma\xi^{-1}\cap Q)\backslash(\xi\Gamma\cap QP)$ is represented by the disjoint union of $(\xi\Gamma\xi^{-1}\cap Q)\backslash\bigl[\xi\Gamma\cap(\xi\Gamma\xi^{-1}\cap Q)\zeta^{-1}P\bigr]$, which is represented by $(\xi\Gamma\xi^{-1}\cap Q\cap \zeta^{-1}P\zeta)\backslash(\xi\Gamma\cap\zeta^{-1}P)$, ad can easily be verified. With $\eta \in \zeta\xi\Gamma\cap P$, this is clearly represented by $\zeta^{-1}R_\eta$. Finally we have $\pi_t(\eta\Gamma\cap P) = \eta_t\Delta$, and our last assertion can easily be verified by means of (27.6a, b).

27.8. Lemma. *The notation being as in Lemma 27.7, suppose that $\lambda_t(\gamma)^k = 1$ for every $\gamma \in \Gamma \cap P$; let $p_\xi \in \mathcal{S}_k^\varphi\bigl(\xi\Gamma\xi^{-1}, Q\bigr)$ with $\varphi = \varphi_r$ for each $\xi \in X$. Then*

$$\Phi^{l-t}\Bigl\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}}\sum_{\xi\in X} E_k^{\psi,\varphi}(\mathfrak{z},\, s;\, p_\xi,\, \xi\Gamma\xi^{-1})\|_k\xi\Bigr\} = \delta(w)^{m/2-s\mathbf{a}}\sum_{\eta\in Y}|c_\eta|^{m-2s\mathbf{a}}E_k^{\omega,\varphi}\bigl(w,\, s;\, q_\eta,\, \eta_t\Delta\eta_t^{-1}\bigr)\|_k\,\eta_t \tag{27.7}$$

at least for $\mathrm{Re}(s) > (n+r+1)/2$ in Case SP and $\mathrm{Re}(s) > l + r + \dim(Z)$ in Cases UT and UB. Here $c_\eta = \lambda_r(\zeta^{-1})\lambda_t(\eta)$ and $q_\eta = c_\eta^{-k}p_\xi\|_k\pi_r(\zeta^{-1})$ with $\zeta \in Z_\xi$ such that $\eta \in \zeta\xi\Gamma \cap P$.

REMARK. If Γ is sufficiently small, we have $0 \ll \lambda_t(\gamma) \in \mathfrak{g}^\times$ for every $\gamma \in \Gamma \cap P$ (even if $K \neq F$). Then $\lambda_t(\gamma)^k = \lambda_t(\gamma)^{\mu\mathbf{a}} = 1$. Thus the condition that $\lambda_t(\gamma)^k = 1$ for every $\gamma \in \Gamma \cap P$ is satisfied for a sufficiently small Γ.

PROOF. Let us first show (27.7) by formally applying Φ^{l-t} termwise. Each term is of the form $\Phi^{l-t}\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta(\mathfrak{z}, s; p_\xi)\|\alpha]\}$ with $\alpha \in (\xi\Gamma\xi^{-1} \cap Q)\backslash\xi\Gamma$. By Lemma 27.6 this is nonzero only when $\alpha \in QP$. Thus putting $T_\xi = (\xi\Gamma\xi^{-1} \cap Q)\backslash(\xi\Gamma \cap QP)$, we see that the left-hand side of (27.7) equals (formally)

$$\sum_{\xi \in X} \sum_{\alpha \in T_\xi} \Phi^{l-t}\left\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[\delta(\mathfrak{z}, s; p_\xi)\|\alpha]\right\}. \tag{27.8}$$

By Lemma 27.7 (2), $\bigsqcup_{\xi \in X} T_\xi$ can be replaced by $\bigsqcup_{\eta \in Y} \zeta^{-1} R_\eta$. Let $\gamma \in R_\eta$ and $\varepsilon = \gamma\eta^{-1}$. Then $\eta^{-1}\varepsilon\eta \in \Gamma \cap P$, and hence $|\lambda_t(\varepsilon)|^{\mathbf{a}} = 1$ and $\lambda_t(\varepsilon)^k = 1$; thus $\lambda_t(\gamma)^k = \lambda_t(\eta)^k$ and $|\lambda_t(\gamma)|^{\mathbf{a}} = |\lambda_t(\eta)|^{\mathbf{a}}$. Therefore, taking (ζ^{-1}, γ) as (β, γ) in Lemma 27.6, we see that (27.8) equals

$$\delta(w)^{m/2-s\mathbf{a}} \sum_{\eta \in Y} |c_\eta|^{m-2s\mathbf{a}} \sum_{\gamma \in R_\eta} \delta(w, s; q_\eta)\|\pi_t(\gamma) \tag{27.9}$$

with c_η and q_η as stated in our lemma. Since $\lambda_r(\beta) = \lambda_t(\beta)\lambda_r(\pi_t(\beta))$ for every $\beta \in P \cap Q$, we easily see that $q_\eta \in \mathcal{S}_k^\varphi(\eta_t \Delta \eta_t^{-1}, P_r^\omega)$. By Lemma 27.7 (3), $\pi_t(R_\eta)$ gives $(\eta_t\Delta\eta_t^{-1} \cap P_r^\omega)\backslash\eta_t\Delta$, and hence the last sum over γ in (27.9) is $E^{\omega,\varphi}(w, s; q_\eta, \eta_t\Delta\eta_t^{-1})\|\eta_t$, which proves (27.7) at least in the formal sense. Now the condition $\mathrm{Re}(s) > l + r + \dim(Z)$ in Case UB guarantees the local uniform convergence of the series of (26.28) on $\mathfrak{Z}^\psi$, as well as its uniform convergence in a suitable domain X, as proven in [S97, Proposition A3.7]; see [S97, (A3.5.3)] for the explicit description of X. Now viewing the sum in question as an integral over a discrete set, we can apply the Lebesgue convergence theorem to justify our formal calculation. The necessary condition for the theorem is given by [S97, Lemma A3.6, (A3.6.4), and line 11 from the bottom on page 226]. Cases SP and UT can be handled in a similar way.

27.9. Lemma. (1) *Let* $\alpha \in G^\psi$ *and* $f \in \mathcal{S}_k^\varphi(\Gamma, P_r^\psi)$, $0 \le r < l$; *suppose* $\mathrm{Re}(s) > (n + r + 1)/2$ *in Case SP and* $\mathrm{Re}(s) > l + r + \dim(Z)$ *in Cases UT and UB. Then*

$$\Phi^{l-r}\left\{\delta(\mathfrak{z})^{m/2-s\mathbf{a}}[E_k^{\psi,\varphi}(\mathfrak{z}, s; f, \Gamma)\|_k\alpha]\right\} = \begin{cases} 0 & \text{if } \alpha \notin \Gamma P_r^\psi, \\ \delta(w)^{m/2-s\mathbf{a}} f & \text{if } \alpha = 1. \end{cases}$$

(2) If $\mathfrak{F}(\mathfrak{z}) = \delta(\mathfrak{z})^{m/2-s\mathbf{a}} \sum_{\xi \in X} E_k^{\psi,\varphi}(\mathfrak{z}, s; p_\xi, \xi\Gamma\xi^{-1})\|_k\xi$ with X and p_ξ of Lemma 27.8, then $\Phi^{l-r}(\mathfrak{F}\|_k\eta^{-1}) = \delta(w)^{m/2-s\mathbf{a}} p_\eta$ for every $\eta \in X$ and s as in (1).

PROOF. Though these are essentially special cases of Lemma 27.8, it is easier to derive them directly from Lemma 27.6. In fact, the quantity of (1) is the sum of $\Phi^{l-r}\{\delta^{m/2-s\mathbf{a}}[\delta_f^\psi\|\gamma\alpha]\}$ for $\gamma \in (\Gamma \cap P_r^\psi)\backslash\Gamma$. By Lemma 27.6 the nonvanishshing can occur only if $\gamma\alpha \in P_r^\psi$, that is, only if $\alpha \in \Gamma P_r^\psi$. This proves (1) for $\alpha \notin \Gamma P_r^\psi$. If $\alpha = 1$, nonvanishshing can appear only from $(\Gamma \cap P_r^\psi)\backslash(\Gamma \cap P_r^\psi)$. which is represented by 1. Therefore, taking $\beta = \gamma = 1$ in Lemma 27.6, we obtain (1). This formal proof can be justified for the same reason as in the proof of Lemma 27.8. Assertion (2) follows immediately from (1), since $\xi\eta^{-1} \in \xi\Gamma\xi^{-1}P_r^\psi$ only if $\xi = \eta$.

27.10. Lemma. *Let* $0 \le r < l$. *Put* $N = n + r$ *in Cases SP and UT and* $N = l + r + \dim(Z)$ *in Case UB; put also* $\lambda(a) = (a+1)/2$ *in Case SP and* $\lambda(a) = a$ *in Cases UT and UB. Suppose that* $\mu > \lambda(N)$ *if* $F \neq \mathbf{Q}$ *and* $\mu > \lambda(N)+1$ *if* $F = \mathbf{Q}$.

Suppose also that $\mu \in \Lambda(r, k)$ if $r > 0$ in Cases SP and UT, and $\mu > 4r+2\dim(Z)$ in Case UB, where $\Lambda(*, *)$ is defined by (23.30). Then equality (27.7) and the equalities of Lemma 27.9 are valid for $s = \mu/2$.

PROOF. Our task is to derive $\Phi^{l-t}\{\mathcal{A}(\mathfrak{z}, \mu/2)\} = \mathcal{A}'(w, \mu/2)$ from the type of relation $\Phi^{l-t}\{\mathcal{A}(\mathfrak{z}, s)\} = \mathcal{A}'(w, s)$ established in Lemmas 27.8 and 27.9. Clearly it is sufficient to prove the case $l - t = 1$.

We first consider the assertion of our lemma concerning (27.7) for $t = n - 1$ and $r = 0$ in Cases SP and SU. This means that the desired conclusion is valid for $\mathcal{A}(z, s) = \delta(z)^{m/2-s\mathbf{a}}\big[E_k^{n,0}(z, s; 1, \Gamma)\|\alpha\big]$, $\alpha \in G^n$. The proof given in [S95a, pp.579-580] is quite technical and requires rather involved preliminaries [S95a, pp.549-551], and so we refer the reader to those pages for the detailed proof. The fact was proven only in Case SP, but the proof is applicable to Case SU with obvious modifications. (We can easily give the analogues of Lemmas 2.3, 2.4, and 2.6 of [S95a] in Case SU. In Case UT we may assume that Γ is contained in $SU(\eta_n)$, as explained in the proof of Lemma 17.13; the condition $\mathrm{Re}(2s) > n + 1$ should be changed for $\mathrm{Re}(s) > n$. In the proof we need Proposition 6.16, as well as the holomorphy of $E_{\mu\mathbf{a}}^{n,0}(z, \mu/2; \mu\mathbf{a}, \Gamma)$ and $E_{\mu\mathbf{a}}^{n-1,0}(z, \mu/2; \mu\mathbf{a}, \Gamma)$ which is guaranteed by Theorem 17.7 (i). Also we have to assume that $\mu \neq \lambda(n)$ (see [S95a, p580, line 11]), which is why we have to assume $\mu > \lambda(n)$ instead of $\mu \geq \lambda(n)$ even if $F \neq \mathbf{Q}$.)

Let us now consider the action of Φ on $E_k^{\psi,\varphi}$ in the general case. Let $\mathcal{A}(\mathfrak{z}, s)$ denote the function inside the brackets of (27.7). We first take Case UB. By [S97, Proposition 20.10] $\mathcal{A}(\mathfrak{z}, s)$ is a finite linear combination of functions of the form $\delta(\mathfrak{z})^{m/2-s\mathbf{a}}\big[E_1(\mathfrak{z}, s; \mathbf{f}, \chi, D^{\psi})\|_k\alpha\big]$ with $\alpha \in G^{\psi}$. We may assume that each $\mathbf{f}$ is an eigenform of Hecke operators, since $\mathcal{S}_k^{\varphi}(D^{\varphi})$ with D^{φ} of (26.32) is spanned by such eigenforms (see [S97, Proposition 20.4 (1)]). Taking $p = 1$ in (26.42) and applying $\|\alpha$, we obtain, employing [S97, (22.3.3)],

$$\begin{aligned}&\mu c_m(\mathbf{s})C'(s)\mathcal{Z}(s, \mathbf{f}, \chi)E_1(\mathfrak{z}, s; \mathbf{f}, \chi, D^{\psi})\|_k\alpha\\&\quad= \Lambda_{\mathfrak{c}}^n(s, \chi)\sum_{a\in\mathcal{B}}\chi_{\mathbf{h}}\big(\det(a)\big)\int_{\mathfrak{D}_a}\big(H_{1,a}\|_k[\alpha, 1]_S\big)^{\circ}(\mathfrak{z}, w; s)f_a(w)\delta(w)^m\mathbf{d}w,\end{aligned}$$

where $n = \dim(V_r)$. Recall that $\mathcal{F}^{\circ}(\mathfrak{z}, w) = \delta\big(w, \wp_r(\mathfrak{z})\big)^{-m}\mathcal{F}\big(\iota_U(\mathfrak{z}, w)\big)$ for a function $\mathcal{F}$ on $\mathcal{H}^N$, where $N = l + r + \dim(Z)$, and $H_{1,a}(z)$ for $z \in \mathcal{H}^N$ corresponds to $E_{q_0}(z)$ with some $q_0 \in G_{\mathbf{h}}^N$ in the sense of (17.23a). Put $\mathfrak{E}_a = H_{1,a}\|_k[\alpha, 1]_S$ and $\mathfrak{F}_a(z, s) = \delta(z)^{m/2-s\mathbf{a}}\mathfrak{E}_a(z, s)$ for $z \in \mathcal{H}^N$. By Lemma 17.13, we can put

$$(*) \qquad \mathfrak{E}_a(z, s) = \sum_{i\in I} b_i c_i^s E(z, s; k, \Gamma_i)\|_k\alpha_i$$

with a finite set I and b_i, c_i, Γ_i, α_i as described there. The map $\iota_U : \mathfrak{Z}^{\psi}\times\mathfrak{Z}^{\varphi} \to \mathcal{H}^N$ is defined by [S97, (22.2.1)]. Put $\omega = \varphi_{l-1}$ nd define $\iota_{U_1} : \mathfrak{Z}^{\omega} \times \mathfrak{Z}^{\varphi} \to \mathcal{H}^{N-1}$ in the same way, where U_1 denotes the element of $U(\eta_{N-1})_{\mathbf{a}}$ corresponding to U. Then we can easily verify that $\iota_U\big(\mathrm{diag}[\rho\mathbf{i}_1, \mathfrak{z}_1], w\big) = \mathrm{diag}\big[\rho\mathbf{i}_1, \iota_{U_1}(\mathfrak{z}_1, w)\big]$ for $\mathfrak{z}_1 \in \mathfrak{Z}^{\omega}$ and $w \in \mathfrak{Z}^{\varphi}$. (Relevant formulas are [S97, (6.10.2), (22.1.2), (22.1.3), (22.1.4), (22.1.6), and (22.2.8)]. From these we can easily derive that $U = 1_2 \times U_1$ in the sense of (23.5).) Also $\wp_r^{\psi}\big(\mathrm{diag}[\rho\mathbf{i}_1, \mathfrak{z}_1]\big) = \wp_r^{\omega}(\mathfrak{z}_1)$.

Put $\Phi\big(\mathfrak{F}_a(z, s)\big) = \mathfrak{F}'_a(z_1, s)$, where $z_1 \in \mathcal{H}^{N-1}$. Then $\mathfrak{F}'_a(z_1, s) = \delta(z_1)^{m/2-s\mathbf{a}} \cdot\mathfrak{E}'_a(z_1, s)$ for $\mathrm{Re}(s) > \lambda(N)$ with a function $\mathfrak{E}'_a$ of type (*) defined for $z_1 \in \mathcal{H}^{N-1}$. This is a special case of Lemma 27.8. Moreover, $\Phi\big(\mathfrak{F}_a(z, \mu/2)\big) = \mathfrak{F}'_a(z_1, \mu/2)$ for

$\mu > \lambda(N)$ if $F \neq \mathbf{Q}$ and $\mu > \lambda(N)+1$ if $F = \mathbf{Q}$, as we said in the second paragraph of this proof. Now

$$\mathfrak{F}_a^\circ(\mathrm{diag}[\rho\mathbf{i}_1,\, \mathfrak{z}_1],\, w;\, s) = \delta(w,\, \wp_r(\mathfrak{z}_1))^{-m}\mathfrak{F}_a(\mathrm{diag}[\rho\mathbf{i}_1,\, \iota_{U_1}(\mathfrak{z}_1,\, w)],\, s),$$

so that $(\varPhi(\mathfrak{F}_a)^\circ)(\mathfrak{z}_1,\, w;\, s) = (\mathfrak{F}_a')^\circ(\mathfrak{z}_1,\, w;\, s)$, which holds for $\mathrm{Re}(s) > N$ and also for $s = \mu/2$ as above. By [S97, (22.2.7)] we have

$$\delta(\iota_U(\mathfrak{z},\, w)) = h|\delta(w,\, \wp_r(\mathfrak{z}))|^{-2}\delta(\mathfrak{z})\delta(w)$$

with a positive costant h, and the formula holds for $(\mathfrak{z}_1,\, U_1)$ in place of $(\mathfrak{z},\, U)$ with the same h. Put $\mathfrak{A}(\mathfrak{z},\, s) = \delta(\mathfrak{z})^{m/2-s\mathbf{a}}E_1(\mathfrak{z},\, s;\, \mathbf{f},\, \chi,\, D^\psi)\|_k\alpha$. Then

$$(**) \qquad \mathcal{D}(s)\mathfrak{A}(\mathfrak{z},\, s)$$
$$= h^{s\mathbf{a}-m/2}\sum_{a\in\mathcal{B}} e_a\int_{\mathfrak{D}_a}(\mathfrak{F}_a)^\circ(\mathfrak{z},\, w;\, s)|\delta(w,\, \wp_r(\mathfrak{z}))|^{m-2s\mathbf{a}}f_a(w)\delta(w)^{m/2+s\mathbf{a}}\mathbf{d}w$$

with $e_a \in \mathbf{C}$ and $\mathcal{D}(s) = c_m(\mathbf{s})C'(s)\mathcal{Z}(s,\, \mathbf{f},\, \chi)\Lambda_{\mathfrak{c}}^n(s,\, \chi)^{-1}$.

To see the behavior of the integrals under $\varPhi$, take $\mathfrak{z} = \mathrm{diag}[\rho\mathbf{i}_1,\, \mathfrak{z}_1]$ with $\rho > 1$ and $\mathfrak{z}_1$ in a compact subset of $\mathfrak{Z}^\omega$. Since $\iota(\mathfrak{z},\, w) = \mathrm{diag}[\rho\mathbf{i}_1,\, \iota(\mathfrak{z}_1,\, w)]$, we see that $\iota(\mathfrak{z},\, w)$ belongs to the subset $\mathfrak{X}$ of $\mathcal{H}^N$ described in [S97, p.189, line 7 from the bottom] if w belongs to the Siegel set $\mathfrak{S}'$ employed there. Therefore the argument of [S97, §22.12] proves the absolute and uniform convergence of the integrals of (**) for such $\mathfrak{z}$ and s belonging to a compact subset of $\mathbf{C}$. (In [S97, §22.12], $\mathfrak{z}$ was taken in a compact set, but the argument there is valid for any $\mathfrak{z}$ such that $\iota(\mathfrak{z},\, \mathfrak{S}') \subset \mathfrak{X}$.) Thus we can apply $\varPhi$ to (**) and obtain

$$(***) \qquad \mathcal{D}(s)(\varPhi\mathfrak{A})(\mathfrak{z}_1,\, s)$$
$$= h^{s\mathbf{a}-m/2}\sum_{a\in\mathcal{B}} e_a\int_{\mathfrak{D}_a}(\mathfrak{F}_a')^\circ(\mathfrak{z}_1,\, w;\, s)|\delta(w,\, \wp_r(\mathfrak{z}_1))|^{m-2s\mathbf{a}}f_a(w)\delta(w)^{m/2+s\mathbf{a}}\mathbf{d}w$$
$$= \delta(\mathfrak{z}_1)^{m/2-s\mathbf{a}}\sum_{a\in\mathcal{B}} e_a\int_{\mathfrak{D}_a}(\mathfrak{E}_a')^\circ(\mathfrak{z}_1,\, w;\, s)f_a(w)\delta(w)^m\mathbf{d}w.$$

The last integrals define meromorphic functions of s on the whole $\mathbf{C}$; moreover they are meaningful at $s = \mu/2$. Now $\mathcal{D}$ is finite and nonzero at $s = \mu/2$, as already seen in the proof of Theorerm 23.11 given in §25.7 in Cases SP and UT and in the proof of Theorem 26.13 in Case UB. Thus $\varPhi(\mathfrak{A}(*,\, \mu/2))$ is meaningful and equals $\mathcal{D}(\mu/2)^{-1}$ times the last line of (***) at $s = \mu/2$.

Returning to the question at the beginning, let $\mathcal{A}'(\mathfrak{z}_1,\, s)$ denote the right-hand side of (27.7) with $l - t = 1$. Then $(\varPhi\mathcal{A})(\mathfrak{z}_1,\, s) = \mathcal{A}'(\mathfrak{z}_1,\, s)$ for $\mathrm{Re}(s) > \lambda(N)$. Since $\mathcal{A}$ is a finite linear combination of functions of type $\mathfrak{A}$, we see that $\mathcal{A}'(\mathfrak{z}_1,\, s)$ is a inear combination of some functions that can be given by the last line of (***) times $\mathcal{D}(s)^{-1}$. Therefore $\varPhi(\mathcal{A}(*,\, \mu/2))$ is meaningful and must coincide with $\mathcal{A}'(\mathfrak{z}_1,\, \mu/2)$. This proves the assertion of our lemma concerning (27.7) in Case UB for $l - t = 1$, which proves, by induction, the general case. Cases SP and UT are similar. As for the assertion concerning the equalities of Lemma 27.9, the left-hand sides are special cases of that of (27.7); the only point is that the right-hand sides can be given explicitly as stated in that lemma. Therefore the above proof is applicable to them. This completes the proof.

27.11. With k, m, and μ as in §27.1, for $0 \le r \le l$ we put, in Case UB,

$$(27.10) \qquad E_k^{\psi,r}(z;\, f,\, \Gamma) = E_k^{\psi,\varphi_r}(z,\, \mu/2;\, f,\, \Gamma) \qquad (z \in \mathfrak{Z}^\psi).$$

for $f \in \mathcal{S}_k^{\varphi_r}(\Gamma, P_r^\psi)$, whenever the right-hand side is finite. (Hereafter we use z instead of $\mathfrak{z}$ for the variable on $\mathfrak{Z}^\psi$.) In general this may not be holomorphic in z. We denote by $\tilde{\mathcal{E}}_k^{\psi,r}$ the vector space spanned over $\mathbf{C}$ by $E_k^{\psi,r}(z;\, f,\, \Gamma)\|_k\alpha$ for all $\alpha \in G^\psi$, all congruence subgroups Γ of G^ψ, and all such f; we then put $\mathcal{E}_k^{\psi,r} = \tilde{\mathcal{E}}_k^{\psi,r} \cap \mathcal{M}_k^\psi$. Clearly these spaces are stable under the operator $\|_k\xi$ for every $\xi \in G^\psi$. In Cases SP and UT we take $E_k^{n,r}(z,\, s;\, f,\, \Gamma)$ as $E_k^{\psi,r}(z,\, s;\, f,\, \Gamma)$. Therefore the symbols $\tilde{\mathcal{E}}_k^{n,r}$ and $\mathcal{E}_k^{n,r}$ are more natural in those two cases, but we state our results in all three cases by using $\tilde{\mathcal{E}}_k^{\psi,r}$ and $\mathcal{E}_k^{\psi,r}$, and give the proof mainly in Case UB, although we indicate necessary modifications in the other two cases. The reader is reminded of the remark we made in §27.1; for example, $l = n$ and $\psi = \eta_n$ in Cases SP and UT; the case of half-integral weight can be included.

If $r = l$, then $E_k^{\psi,\psi}(z,\, s;\, f,\, \Gamma) = f(z)$ for every $f \in \mathcal{S}_k^\psi(\Gamma)$ by (23.14) and the remark at the end of §26.9, so that

$$\tilde{\mathcal{E}}_k^{\psi,l} = \mathcal{E}_k^{\psi,l} = \mathcal{S}_k^\psi. \tag{27.11}$$

We have $k = m = \mu\mathbf{a}$ in Case SP. Now, in Cases UT and UB we can restrict Γ to the congruence subgroups of $SU(\psi)$, by virtue of (23.13a) and (26.28a). Since $j_\gamma^k = c_\gamma j_\gamma^{\mu\mathbf{a}}$ for $\gamma \in G^\psi$ with $c_\gamma \in \overline{\mathbf{Q}}^\times$ and $c_\gamma = 1$ for $\gamma \in SU(\psi)$, the spaces $\tilde{\mathcal{E}}_k^{\psi,r}$ and $\mathcal{E}_k^{\psi,r}$ depend only on μ. We have to be more careful, however, when we speak of the rationality of modular forms over a number field.

27.12. Lemma. *Let $0 \le r \le l$. Impose the condition of Lemma 27.10 on μ if $r < l$. Then the function of (27.10) is meaningful and the following assertions hold:*

(1) $\mathcal{E}_k^{\psi,r} = \tilde{\mathcal{E}}_k^{\psi,r}$.

(2) $\Phi^{l-t}(\mathcal{E}_k^{\psi,r}) \subset \mathcal{E}_k^{\varphi_t,r}$ *for $r \le t \le l$ and in particular* $\Phi^{l-r}(\mathcal{E}_k^{\psi,r}) = \mathcal{S}_k^{\varphi_r}$.

(3) $\Phi^s(\mathcal{E}_k^{\psi,r}) = 0$ *if $s > l - r$.*

(4) *If $g \in \mathcal{E}_k^{\psi,r}$ and $\Phi^{l-r}(g\|_k\alpha) = 0$ for every $\alpha \in G^\psi$, then $g = 0$.*

PROOF. All the assertions are trivial if $r = l$, and so we assume $r < l$; notice that (3) for $r = l$ follows from Lemma 27.2 (2). Taking $m = \mu\mathbf{a}$ in Theorems 17.7 (i), 23.11 (I), 26.13 (1), we see that $E_k^{\psi,r}(z;\, f,\, \Gamma)$ is meaningful and holomorphic. Thus we obtain (1). Assertion (2) follows from Lemmas 27.8, 27.9, and 27.10. Then $\Phi^{l-r+1}(\mathcal{E}_k^{\psi,r}) = \Phi\big(\Phi^{l-r}(\mathcal{E}_k^{\psi,r})\big) = \Phi(\mathcal{S}_k^{\varphi_r}) = 0$ by Lemma 27.2 (2), which gives (3). To prove (4), let $g = \sum_{i\in I} E_k^{\psi,r}(z;\, f_i,\, \Gamma_i)\|\alpha_i$ with a finite set of indices I, $\alpha_i \in G^\psi$, congruence subgroups Γ_i, and $f_i \in \mathcal{S}_k^{\varphi_r}(\Gamma_i, P_r^\psi)$. Take a congruence subgroup Γ so that $\Gamma \subset \bigcap_{i\in I}\alpha_i^{-1}\Gamma_i\alpha_i$ and $\lambda_t(\Gamma \cap \mathcal{P}_t^\psi)^k = 1$ for every $t \ge r$. For each i we can find a finite set B_i such that $\Gamma_i\alpha_i = \bigsqcup_{\beta\in B_i}\beta\Gamma$. Then $\beta\Gamma\beta^{-1} \subset \Gamma_i$ for every $\beta \in B_i$, and $\Gamma_i = \bigsqcup_{\beta\in B_i}\beta\Gamma\beta^{-1}\beta\alpha_i^{-1}$. Therefore, taking Γ_i, $\beta\Gamma\beta^{-1}$, and $\beta\alpha_i^{-1}$ as Γ, Γ', and α in (23.13a) or (26.28a), we find that

$$g = \sum_{i\in I}\sum_{\beta\in B_i} E_k^{\psi,r}(z;\, c_{i,\beta}f_i,\, \beta\Gamma\beta^{-1})\|\beta$$

with some constants $c_{i,\beta}$. Thus we may assume at the beginning that $\Gamma_i = \alpha_i\Gamma\alpha_i^{-1}$ for every $i \in I$ with some Γ. Now if $\beta \in \sigma\xi\Gamma$ with $\sigma \in P_r^\psi$ and $\xi \in G^\psi$, then $(\beta\Gamma\beta^{-1} \cap P_r^\psi)\backslash\beta\Gamma$ can be given by σT with $T = (\xi\Gamma\xi^{-1} \cap P_r^\psi)\backslash\xi\Gamma$. Also, for $\tau \in T$ we have

$$\delta_{s,f}\|\sigma\tau = \lambda_r(\sigma)^{-k}|\lambda_r(\sigma)|^{m-2s\mathbf{a}}\delta\big(z,\, s;\, f\|\pi_r(\sigma)\big)\|\tau$$

by (23.12) or [S97, (12.3.5)], and hence

$$E_k^{\psi,r}(z;\, f,\, \beta\Gamma\beta^{-1})\|\beta = \lambda_r(\sigma)^{-k} E_k^{\psi,r}\big(z;\, f\|\pi_r(\sigma),\, \xi\Gamma\xi^{-1}\big)\|\xi.$$

Therefore, with $X = P_r^\psi\backslash G^\psi/\Gamma$ we can put $g = \sum_{\xi\in X} E_k^{\psi,r}(z;\, h_\xi,\, \xi\Gamma\xi^{-1})\|\xi$. Suppose that $\Phi^{l-r}(g\|\alpha) = 0$ for every $\alpha \in G^\psi$. Then by Lemmas 27.9 (2) and 27.10 we have $h_\eta = \Phi^{l-r}(g\|\eta^{-1}) = 0$ for every $\eta \in X$, so that $g = 0$, which proves (4).

We are now ready to state our main theorems on the structure of the spaces of holomorphic Eisenstein series.

27.13. Theorem. *Suppose $n > 1$ in Cases SP and UT, and $2l + \dim(Z) > 2$ in Case UB; put $\mathcal{E}_k^{\psi,r}(\Gamma) = \mathcal{E}_k^{\psi,r} \cap \mathcal{M}_k^\psi(\Gamma)$. Then we have*

$$\mathcal{M}_k^\psi = \textstyle\bigoplus_{r=0}^l \mathcal{E}_k^{\psi,r} \quad \textit{and} \quad \mathcal{M}_k^\psi(\Gamma) = \textstyle\bigoplus_{r=0}^l \mathcal{E}_k^{\psi,r}(\Gamma)$$

for every congruence subgroup Γ of G^ψ provided the following condition is satisfied:

(27.12) Case SP: $\mu \geq 3n/2$ *if* $n > 2$; $\mu > 3$ *if* $n = 2$ *and* $F = \mathbf{Q}$; $\mu > 2$ *if* $n = 2$ *and* $F \neq \mathbf{Q}$;
Case UT: $\mu > 4$ *if* $F = \mathbf{Q}$ *and* $n = 2$; $\mu \geq 3n - 2$ *otherwise;*
Case UB: $\mu > 3$ *if* $F = \mathbf{Q}$ *and* $l = \dim(Z) = 1$; $\mu \geq 2\dim(W) - 3$ *otherwise.*

PROOF. We need the condition on μ of Lemma 27.10 for every $r < l$, which is why (27.12) is required. Suppose that $\sum_{r=0}^l p_r = 0$ with $p_r \in \mathcal{E}_k^{\psi,r}$. By Lemma 27.12 (3) we have $\Phi^l(p_0\|\alpha) = -\sum_{r=1}^l \Phi^l(p_r\|\alpha) = 0$ for every $\alpha \in G^\psi$, and hence $p_0 = 0$ by Lemma 27.12 (4). Similarly we find that $\Phi^{l-1}(p_1\|\alpha) = 0$ for every $\alpha \in G^\psi$, which means that $p_1 = 0$ for the same reason. Repeating this process, we obtain $p_r = 0$ for every r, which proves that the $\mathcal{E}_k^{\psi,r}$ for $0 \leq r \leq l$ form a direct sum. Now given $g \in \mathcal{M}_k^\psi$, take Γ so that $g \in \mathcal{M}_k^\psi(\Gamma)$ and $\lambda_r(\Gamma \cap P_r^\psi)^k = 1$ for every r; take also $X_r = P_r^\psi\backslash G^\psi/\Gamma$. Put $h_\xi = \Phi^l(g\|\xi^{-1})$ for each $\xi \in X_0$ and $f_0 = \sum_{\xi\in X_0} E_k^{\psi,0}(z;\, h_\xi,\, \xi\Gamma\xi^{-1})\|\xi$. Then by Lemma 27.9 (2) and Lemma 27.10, $\Phi^l\big((g - f_0)\|\xi^{-1}\big) = 0$ for every $\xi \in X_0$. By Lemma 27.3 we have $\Phi^{l-1}\big((g - f_0)\|\alpha\big) \in \mathcal{S}_k^{\varphi_1}$ for every $\alpha \in G^\psi$. Put $p_\eta = \Phi^{l-1}\big((g - f_0)\|\eta^{-1}\big)$ for $\eta \in X_1$, and $f_1 = \sum_{\eta\in X_1} E_k^{\psi,1}\big(z;\, p_\eta,\, \eta\Gamma\eta^{-1}\big)\|\eta$. Then $\Phi^{l-1}\big((g - f_0 - f_1)\|\eta^{-1}\big) = 0$ for every $\eta \in X_1$, and hence $\Phi^{l-2}\big((g - f_0 - f_1)\|\alpha\big) \in \mathcal{S}_k^{\varphi_2}$ for every $\alpha \in G^\psi$ by Lemma 27.3. Continuing in this fashion, we find some elements $f_r \in \mathcal{E}_k^{\psi,r}$ for $r \leq l-1$ so that if we put $f_l = g - \sum_{r=0}^{l-1} f_r$, then $\Phi(f_l\|\alpha) = 0$ for every $\alpha \in G^\psi$, which means that $f_l \in \mathcal{S}_k^\psi = \mathcal{E}_k^{\psi,l}$. This proves the first equality. If $g \in \mathcal{M}_k^\psi(\Gamma)$, then $\sum_{r=0}^l f_r\|\gamma = \sum_{r=0}^l f_r$ for every $\gamma \in \Gamma$, so that $f_r\|\gamma = f_r$, that is, $f_r \in \mathcal{M}_k^\psi(\Gamma) \cap \mathcal{E}_k^{\psi,r}$, which completes the proof.

27.14. Theorem. *Let n and μ be as in Theorem 27.13; put $\mathcal{E}_k^\psi = \sum_{r=0}^{l-1} \mathcal{E}_k^{\psi,r}$ and $\mathcal{E}_k^\psi(\Gamma) = \mathcal{E}_k^\psi \cap \mathcal{M}_k^\psi(\Gamma)$. Then*

$$\mathcal{M}_k^\psi = \mathcal{S}_k^\psi \oplus \mathcal{E}_k^\psi, \qquad \mathcal{E}_k^\psi(\Gamma) = \textstyle\bigoplus_{r=0}^{l-1} \mathcal{E}_k^{\psi,r}(\Gamma),$$

$$\mathcal{E}_k^\psi = \big\{\, f \in \mathcal{M}_k^\psi \bigm| \langle f,\, g\rangle = 0 \ \textit{for every}\ g \in \mathcal{S}_k^\psi \,\big\},$$

$$\mathcal{E}_k^{\psi,r} = \big\{\, f \in \mathcal{E}_k^\psi \bigm| \Phi(f\|_k\alpha) \in \mathcal{E}_k^{\varphi_{l-1},r} \ \textit{for every}\ \alpha \in G^\psi \big\} \quad \textit{if} \quad r < l.$$

Moreover, if $\mathfrak{q} : \mathcal{M}_k^\psi \to \mathcal{S}_k^\psi$ *denotes the projection map determined by the decomposition* $\mathcal{M}_k^\psi = \mathcal{S}_k^\psi \oplus \mathcal{E}_k^\psi$, *then* $\mathfrak{q}(f)^\sigma = \mathfrak{q}(f^\sigma)$ *for every* $\sigma \in \mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$.

PROOF. The first and second equalities follow immediately from Theorem 27.13. If $f \in \mathcal{E}_k^{\psi,r}$, $r < l$, and $g \in \mathcal{S}_k^\psi$, then $\langle f, g \rangle = 0$ by [S97, Lemma A3.8]. (Take the present $\delta_{s,f}$, P_r^ψ, 1 to be g, P, τ in that lemma.) This together with the first equality gives the third equality. Put $\omega = \varphi_{l-1}$; suppose $r < l$; then by Lemma 27.12 (2), $\Phi(f\|\alpha) \in \mathcal{E}_k^{\omega,r}$ for every $f \in \mathcal{E}_k^{\psi,r}$ and every $\alpha \in G^\psi$. Conversely suppose that $f = \sum_{s=0}^{l-1} g_s$ with $g_s \in \mathcal{E}_k^{\psi,s}$ and that $\Phi(f\|\alpha) \in \mathcal{E}_k^{\omega,r}$ for every $\alpha \in G^\psi$. Since $\Phi(g_s\|\alpha) \in \mathcal{E}_k^{\omega,s}$ by Lemma 27.12 (2) and the $\mathcal{E}_k^{\omega,s}$ for $0 \le s < l$ form a direct sum, we obtain $\Phi(g_s\|\alpha) = 0$ for $s \neq r$. By Lemma 27.2 (2), $g_s \in \mathcal{E}_k^{\psi,s} \cap \mathcal{S}_k^\psi = \{0\}$ for $s \neq r$, and hence $f = g_r \in \mathcal{E}_k^{\psi,r}$, which proves the third equality. The last assertion concerning $\mathfrak{q}$ will be proven at the end of the proof of Theorem 27.16.

27.15. Theorem. *Let* $0 \le r < l$. *Put* $N = n + r$ *in Cases SP and UT and* $N = l + r + \dim(Z)$ *in Case UB; put also* $\lambda(a) = (a+1)/2$ *in Case SP and* $\lambda(a) = a$ *in Cases UT and UB. Suppose that* $\mu > \lambda(N)$ *if* $F \neq \mathbf{Q}$ *and* $\mu > \lambda(N)+1$ *if* $F = \mathbf{Q}$. *Suppose also that* $\mu \in \Lambda(r, k)$ *if* $r > 0$ *in Cases SP and UT, and* $\mu > 4r + 2\dim(Z)$ *in Case UB, where* $\Lambda(*, *)$ *is defined by (23.30). Given a congruence subgroup* Γ *of* G^ψ, *let* $\mathcal{E}_k^{\psi,r}(\Gamma) = \mathcal{M}_k(\Gamma) \cap \mathcal{E}_k^{\psi,r}$ *and let* $X = P_r^\psi\backslash G^\psi/\Gamma$ *with a fixed* $r < l$. *Then* $f \mapsto \bigl(\Phi^{l-r}(f\|\xi^{-1})\bigr)_{\xi\in X}$ *gives a* $\mathbf{C}$*-linear isomorphism of* $\mathcal{E}_k^{\psi,r}(\Gamma)$ *onto* $\prod_{\xi\in X} \mathcal{S}_k^{\varphi_r}\bigl(\xi\Gamma\xi^{-1}, P_r^\psi\bigr)$. *Moreover, if* $f \in \mathcal{E}_k^{\psi,r}(\Gamma)$ *and* $p_\xi = \Phi^{l-r}(f\|\xi^{-1})$, *then* $f = \sum_{\xi\in X} E_k^{\psi,r}(z; p_\xi, \xi\Gamma\xi^{-1})\|_k\xi$.

PROOF. If $f \in \mathcal{E}_k^{\psi,r}(\Gamma)$, then $\Phi^{l-r}(f\|\xi^{-1}) \in \mathcal{S}_k^{\varphi_r}\bigl(\xi\Gamma\xi^{-1}, P_r^\psi\bigr)$ by Lemma 27.2 (4) and Lemma 27.12 (2), and so our map is meaningful. The injectivity of the map follows from Lemma 27.3 and Lemma 27.12 (4). Now, given $p_\xi \in \mathcal{S}_k^{\varphi_r}\bigl(\xi\Gamma\xi^{-1}, P_r^\psi\bigr)$ for each $\xi \in X$, put $g = \sum_{\xi\in X} E_k^{\psi,r}(z; p_\xi, \xi\Gamma\xi^{-1})\|_k\xi$. Then $g \in \mathcal{E}_k^{\psi,r}(\Gamma)$ and $\Phi^{l-r}(g\|\xi^{-1}) = p_\xi$ for every $\xi \in X$ by Lemma 27.10, which proves the surjectivity.

27.16. Theorem. *Let* μ *and* r *be as in Theorem 27.15. Put* $\mathcal{E}_k^{\psi,r}(\Gamma, \overline{\mathbf{Q}}) = \mathcal{E}_k^{\psi,r} \cap \mathcal{M}_k^\psi(\Gamma, \overline{\mathbf{Q}})$ *and* $\mathcal{E}_k^\psi(\Gamma, \overline{\mathbf{Q}}) = \mathcal{E}_k^\psi \cap \mathcal{M}_k^\psi(\Gamma, \overline{\mathbf{Q}})$. *Then*

(1) $E_k^{\psi,r}(z; f, \Gamma)$ *is* $\overline{\mathbf{Q}}$*-rational if* f *is* $\overline{\mathbf{Q}}$*-rational.*

(2) $\mathcal{E}_k^{\psi,r}(\Gamma) = \mathcal{E}_k^{\psi,r}(\Gamma, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$.

(3) $\mathcal{E}_k^\psi(\Gamma) = \mathcal{E}_k^\psi(\Gamma, \overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$ *if (27.12) is satisfied.*

PROOF. Assertion (1) will be proven in §§28.12 and 29.9. Now every element g of $\mathcal{E}_k^{\psi,r}(\Gamma)$ can be written $g = \sum_{\xi\in X} E_k^{\psi,r}(z; q_\xi, \xi\Gamma\xi^{-1})\|\xi$ with $q_\xi \in \mathcal{S}_k^{\varphi_r}(\xi\Gamma\xi^{-1}, P_r^\psi)$ by Theorem 27.15. By Lemma 23.13 and (26.25), q_ξ is a $\mathbf{C}$-linear combination of elements of $\mathcal{S}_k^{\varphi_r}(\xi\Gamma\xi^{-1}, P_r^\psi, \overline{\mathbf{Q}})$. Therefore (2) follows from (1). Assertion (3) is immediate from (2) and the second equality of Theorem 27.14.

To prove the last assertion of Theorem 27.14, take $\sigma \in \mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$. Then $(\mathcal{E}_k^\psi)^\sigma = \mathcal{E}_k^\psi$ by (3); also $(\mathcal{S}_k^\psi)^\sigma = \mathcal{S}_k^\psi$ by Theorem 10.8 (1) and Proposition 26.8 (2). The desired fact now follows immediately from these equalities.

27.17. Remark. (A) We excluded the case $n = 1$ in Theorems 27.13 and 27.14. If $n = 1$ in Case UT, we have $SU(\eta_1) = SL_2(F)$. Thus the case $n = 1$ is about the Eisenstein series for $SL_2(F)$. In this case the equalities of Theorem 27.14 are true for $k = \mu\mathbf{a}$ with $1 \le \mu \in 2^{-1}\mathbf{Z}$; if $\mu = 1/2$, however, we need the residues of $E^{n,0}_{\mu\mathbf{a}}(z, s)$ at $s = 3/4$. For details, the reader is referred to [S85b]. Theorems 27.15 and 27.16 include the case in which $n = 1$ and $r = 0$; we have to assume $\mu \ge 3/2$ if $F \ne \mathbf{Q}$ and $\mu \ge 5/2$ if $F = \mathbf{Q}$. In fact, we can show that the orthogonal complement of $\mathcal{S}^1_{\mu\mathbf{a}}$ in $\mathcal{M}^1_{\mu\mathbf{a}}$ is spanned by $\mathbf{Q}$-rational elements for every $\mu \ge 1/2$. The proof is given in [S87b, Theorem 9.1]; the case in which $\mu = 3/2$ and $F = \mathbf{Q}$ is complicated; we have to invoke the results of Pei [P] and Miyake [M].

(B) The case $\mu = \lambda(N)$ is not included in Theorem 27.16. In fact, in Corollaries 28.12 and 29.8 below we shall prove that $E^{n,r}_k(z, f, \Gamma)$ is $\overline{\mathbf{Q}}$-rational for every $\overline{\mathbf{Q}}$-rational f, if $m = \mu\mathbf{a}$, $\mu = \lambda(N)$, $\mu > (3r/2) + 1$ in Case SP, $\mu > 3r$ in Case UT, and $\mu > 2n$ in Case UB.

(C) The series defining $E^{\psi,\varphi_r}_k(z, s; f, \Gamma)$ is convergent for $\mathrm{Re}(s) > (n + r + 1)/2$ in Case SP, for $\mathrm{Re}(s) > n + r$ in Case UT, and for $\mathrm{Re}(s) > l + r + \dim(Z)$ in Case UB. In these cases the value $E^{\psi,\varphi_r}_k(z, \mu/2; f, \Gamma)$ is clearly holomorphic, and we do not need Lemma 27.10 for the proof of our lemmas and theorems after that lemma. Though the results even in such convergent cases are by no means trivial, it may be emphasized that one of the main points of our treatment in this section is in the fact that the above theorems are valid beyond the range of convergence.

28. Main theorems on arithmeticity in Cases SP and UT

28.1. Throughout this section we put $d = [F : \mathbf{Q}]$ and denote by Φ the Galois closure of K in $\mathbf{C}$ over $\mathbf{Q}$; also we fix a weight $k \in \mathbf{Z}^{\mathbf{b}}$ and put $m = k$ in Case SP and $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ in Cases UT and UB; k may be half-integral in Case SP. Let $G = Sp(n, F)$ in Case SP, $G = U(\eta_n)$ in Case UT, and $G = G^\varphi$ with (V, φ) as in Section 26 in Case UB. Given a congruence subgroup Γ of G and $f, g \in C^\infty(\mathcal{H}, \mathbf{C})$ such that $f\|_k\gamma = f$ and $g\|_k\gamma = g$ for every $\gamma \in \Gamma$, we define the inner product $\langle g, f\rangle$ by

$$(28.1)\qquad \langle g, f\rangle = \mathrm{vol}(\mathfrak{D})^{-1}\int_{\mathfrak{D}} \overline{g(z)}f(z)\delta(z)^m\mathbf{d}z, \quad \mathrm{vol}(\mathfrak{D}) = \int_{\mathfrak{D}}\mathbf{d}z, \quad \mathfrak{D} = \Gamma\backslash\mathcal{H},$$

whenever the integral is convergent, where $\mathbf{d}z = \prod_{v\in\mathbf{a}}\mathbf{d}z_v$ with $\mathbf{d}z_v$ of Lemma 3.4 in Cases SP and UT; $\mathbf{d}z$ on $\mathfrak{Z}^\varphi$ is given by [S97, (10.9.1)]. This is essentially the same as that of (12.35a). Here, as well as in (28.2) below, we assume that $G_\mathbf{a}$ is not compact. For compact $G_\mathbf{a}$, inner products are defined in [S97, (10.9.5)], but we shall not employ them in the present book.

Let C be an open subgroup of $G_\mathbf{A}$ such that $C \cap G_\mathbf{h}$ is compact. Take a finite subset $\mathcal{B}$ of $G_\mathbf{h}$ so that $G_\mathbf{A} = \bigsqcup_{p\in\mathcal{B}} GpC$. Let W be a subfield of $\mathbf{C}$ such that $\Phi\mathbf{Q}_{\mathrm{ab}} \subset W$ in Cases SP and UT and $\overline{\mathbf{Q}} \subset W$ in Case UB. Let $(g_p)_{p\in\mathcal{B}} \leftrightarrow \mathbf{g} \in \mathcal{M}_k(C)$ in the sense of §20.1 or §26.10 (or [S97, §10.7]). We call $\mathbf{g}$ W-rational if $g_p \in \mathcal{S}_k(W)$ for every $p \in \mathcal{B}$. If $q \in G_\mathbf{h} \cap GpC$, then $q = \alpha pu$ with $\alpha \in G$ and $u \in C$, so that $g_q = g_p\|_k\alpha^{-1}$. Therefore $g_q \in \mathcal{M}_k(W)$ by Theorems 9.13 (3), 10.7 (6), and Proposition 11.13. Thus the W-rationality is independent of the choice of $\mathcal{B}$. (In Case SP we can always take $\mathcal{B} = \{1\}$, and hence we can speak of the rationality of g_1 over any field, but we shall not employ it in this section.) Let $\mathcal{S}_k(C, W)$ denote the set of all W-rational elements of $\mathcal{S}_k(C)$. By Theorem 10.8 and Proposition 26.8 (2) we have $\mathcal{S}_k(C) = \mathcal{S}_k(C, W) \otimes_W \mathbf{C}$. In Cases SP and UT we easily see that $\mathbf{g} \in \mathcal{S}_k(C, W)$ if and only if $c_\mathbf{g}(\tau, q)$ of Proposition 20.2 belongs to W for every (τ, q).

Now given also $(f_b)_{b\in\mathcal{B}} \leftrightarrow \mathbf{f} \in \mathcal{S}_k(C)$, we put

$$(28.2)\qquad \langle\, \mathbf{g},\, \mathbf{f}\,\rangle = \#(\mathcal{B})^{-1}\sum_{b\in\mathcal{B}}\langle\, g_b,\, f_b\,\rangle \quad \text{if } G_{\mathbf{a}} \text{ is not compact.}$$

This is well-defined independently of the choice of C and $\mathcal{B}$; see [S97, §10.9].

In the following lemma we assume that φ is isotropic in Case UB, since if φ is anisotropic, then Proposition 15.7 guarantees the map $\mathfrak{p}$ with no condition on k and p.

28.2. Lemma. *Let k be a weight and let $0 \le p \in \mathbf{Z}^{\mathbf{a}}$. Suppose that for every $v \in \mathbf{a}$ we have $k_v > n + p_v$ in Case SP, $m_v \ge 2n + p_v$ in Case UT, and $m_v \ge \dim(V) + p_v$ in Case UB; suppose also that (27.12) is satisfied if $m = \mu\mathbf{a}$ with $\mu \in 2^{-1}\mathbf{Z}$. Let $D = \Phi\mathbf{Q}_{\mathrm{ab}}$ if $m \notin 2^{-1}\mathbf{Z}^{\mathbf{a}}$ in Cases SP and UT; let $D = \overline{\mathbf{Q}}$ otherwise. Then there exists a $\mathbf{C}$-linear map $\mathfrak{p} : \mathcal{N}_k^p \to \mathcal{S}_k$ with the following properties:*

(1) $\langle\, g,\, f\,\rangle = \langle\, \mathfrak{p}(g),\, f\,\rangle$ for every $f \in \mathcal{S}_k$ and every $g \in \mathcal{N}_k^p$.

(2) $\mathfrak{p}(g)^\sigma = \mathfrak{p}(g^\sigma)$ for every $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$ and every $g \in \mathcal{N}_k^p$.

PROOF. Take the operator $\mathfrak{A}$ of Proposition 15.3. Given $g \in \mathcal{N}_k^p$, put $h = \mathfrak{A}g$. We have $h \in \mathcal{M}_k$ by that proposition, and $\langle\, g,\, f\,\rangle = \langle\, \mathfrak{A}g,\, f\,\rangle$ for every $f \in \mathcal{S}_k$ by Corollary 15.4. Since $\mathfrak{A}$ is a $\mathbf{Q}$-rational polynomial of the operators $L_{\omega,v}$ for $v \in \mathbf{a}$ with $\omega(x) = \det(x)^k$, we see that $\mathfrak{A}(g^\sigma) = (\mathfrak{A}g)^\sigma$ for every $\sigma \in \mathrm{Aut}(\mathbf{C}/D)$ by Theorems 14.9 and 14.12. If $m \notin 2^{-1}\mathbf{Z}^{\mathbf{a}}$ then $\mathcal{S}_k = \mathcal{M}_k$, and so we have properties (1) and (2) with $\mathfrak{A}$ as $\mathfrak{p}$. If $m = \mu\mathbf{a}$ with $\mu \in 2^{-1}\mathbf{Z}$, then take $\mathfrak{q}$ of Theorem 27.14 and put $\mathfrak{p} = \mathfrak{q}\mathfrak{A}$. Then we obtain (1) and (2) from that theorem.

28.3. We now consider only Cases SP and UT, and take our setting to be that of Section 22; Case UB will be treated in Section 29. We take C in the form (19.1) with $\mathfrak{e} = \mathfrak{c}$ and a Hecke eigenform $\mathbf{f} \in \mathcal{S}_k(C)$ as in §20.6 with trivial ψ; we also take $\kappa = 0$ in (22.3b). Then the Euler $\mathfrak{p}$-factor of $\mathcal{Z}(s,\, \mathbf{f},\, \chi)$ is 1 for $\mathfrak{p}|\mathfrak{c}$. We note that the square of $\mathrm{vol}(X)$ of (22.6) is a rational number by [S97, (18.9.3) and (18.9.4)]. Therefore from (22.9) and (22.18b) we obtain

$$(28.3)\qquad \Gamma((s))D(us + s_0,\, \mathbf{f},\, \chi) = \mathrm{vol}(\mathfrak{D})\det(\tau)^{h+s\mathbf{a}}\sum_{p\in\mathcal{A}} a_p b_p^{us}\,\langle\, g_p E(\overline{s} + \lambda_n),\, f_p\,\rangle$$

with $a_p \in \overline{\mathbf{Q}}$, $0 < b_p \in \mathbf{Q}$, $E(s) = E(z,\, s;\, m - m',\, 0,\, \Gamma)$, and a certain finite subset $\mathcal{A}$ of $G_{\mathbf{h}}$; $a_p \in \mathbf{Q}_{\mathrm{ab}}$ in Case SP; $u = 2$, $\lambda_n = (n+1)/2$, and $s_0 = (3n/2) + 1$ in Case SP; $u = 1$, $\lambda_n = n$, and $s_0 = 3n/2$ in Case UT; h is defined by (22.4a) with $\kappa = 0$.

Now $\mathrm{vol}(\mathfrak{D})$ is the Euler-Poincaré characteristic of $\Gamma\backslash\mathcal{H}$ times a constant that depends only on the choice of the measure of $\mathcal{H}$. (This is a well-known principle valid for any arithmetic quotient of a hermitian symmetric space. If the quotient is compact, it follows from the classical generalization of the Gauss-Bonnet formula; the noncompact case was proved in [Ha].) In the present case, the constant is π^{d_0} times a rational number, where d_0 is the complex dimension of $\mathcal{H}$. (The constant depends on the type of G_v. In the symplectic case, the rationality follows, for example, from [Si, II, p.279, Theorem 11], which gives $\mathrm{vol}(\mathcal{D})$ when $\Gamma = Sp(n,\, \mathbf{Z})$. In the unitary case it follows from the formula for $\mathrm{vol}(\mathcal{D})$ in [S97, Proposition 24.9], in which we can take $\det(\theta_v) = 1$ for every $v \in \mathbf{a}$.) Therefore, using the notation of (22.19) and putting $\sigma = us + s_0$, we obtain

$$(28.4)\qquad c_{\mathbf{f}}(\tau,\, r)\Gamma((s))\mathcal{Z}(\sigma,\, \mathbf{f},\, \chi)$$

$$= \det(\tau)^{h+s\mathbf{a}} P(\sigma)\Lambda(\sigma)\pi^{d_0} \sum_{p\in\mathcal{A}} a'_p b_p^{us} \langle g_p E(\overline{s}+\lambda_n),\, f_p \rangle$$

with $a'_p \in \overline{\mathbf{Q}}$; $a'_p \in \mathbf{Q}_{\mathrm{ab}}$ in Case SP. Here Λ, P, a'_p, and b_p depend only on (τ, r). Strictly speaking, (28.4) holds for a Hecke eigenform $\mathbf{f}$ with a special property relative to the pair (τ, r) as explained in Theorem 20.9; also, given $\mathbf{f}$, we can always find (τ, r) with which (28.4) holds. Now, for an arbitrary eigenform $\mathbf{f}$ and arbitrary (τ, r) we have

$$(28.5)\quad \Gamma((s))\mathcal{Z}(\sigma, \mathbf{f}, \chi) \sum_{L<M\in\mathcal{L}_\tau} \mu(M/L)\chi^*\bigl(\det(q^*\widehat{y})\mathfrak{r}\bigr)N\bigl(\det(q^*\widehat{y})\mathfrak{r}\bigr)^{-\sigma} c_{\mathbf{f}}(\tau, y)$$
$$= \det(\tau)^{h+s\mathbf{a}} P(\sigma)\Lambda(\sigma)\pi^{d_0} \sum_{p\in\mathcal{A}} a'_p b_p^{us} \langle g_p E(\overline{s}+\lambda_n),\, f_p \rangle$$

with the same Λ, P, a'_p, and b_p, where the sum on the left-hand side is as in Theorem 20.7. Indeed, we obtained (22.19) from (20.19), and (20.19) is a special case of the equality of Theorem 20.7. Therefore (28.5) can be obtained by employing that equality instead of (20.19). The function Λ is obtained from $\mathfrak{L}_0 \prod_{v\nmid\mathfrak{c}} h_v$ of Theorem 20.9 as explained in §22.9; $\mathfrak{L}_0$ and h_v for $k \notin \mathbf{Z}^{\mathbf{a}}$ are given in Theorem 21.4. Thus we obtain

$$\Lambda(\sigma) = L_{\mathfrak{c}}\bigl(\sigma - \tfrac{n}{2},\, \chi\rho_\tau\bigr) \prod_{i=1}^{n/2} L_{\mathfrak{c}}(2\sigma - 2n - 2 + 2i,\, \chi^2) \quad (\text{Case SP},\ k\in\mathbf{Z}^{\mathbf{a}},\ n\in 2\mathbf{Z}),$$

$$\Lambda(\sigma) = \prod_{i=1}^{(n+1)/2} L_{\mathfrak{c}}(2\sigma - 2n - 2 + 2i,\, \chi^2) \quad (\text{Case SP},\ k\in\mathbf{Z}^{\mathbf{a}},\ n\notin 2\mathbf{Z}),$$

$$\Lambda(\sigma) = \prod_{i=1}^{n/2} L_{\mathfrak{c}}(2\sigma - 2n - 1 + 2i,\, \chi^2) \quad (\text{Case SP},\ k\notin\mathbf{Z}^{\mathbf{a}},\ n\in 2\mathbf{Z}),$$

$$\Lambda(\sigma) = L_{\mathfrak{c}}\bigl(\sigma - \tfrac{n}{2},\, \chi\rho_\tau\bigr) \prod_{i=1}^{(n-1)/2} L_{\mathfrak{c}}(2\sigma - 2n - 1 + 2i,\, \chi^2) \quad (\text{Case SP},\ k\notin\mathbf{Z}^{\mathbf{a}},\ n\notin 2\mathbf{Z}),$$

$$\Lambda(\sigma) = \prod_{i=1}^{n} L_{\mathfrak{c}}(2\sigma - n + 1 - i,\, \chi_1\theta^{n+i-1}) \qquad (\text{Case UT}).$$

Here ρ_τ is the Hecke character of F given in Lemma 20.5 and Theorem 21.4; χ_1 is the restriction of χ to F.

28.4. Lemma. *The notation being as above, let h_p be an element of $\mathcal{S}_k(D)$ given for each $p \in \mathcal{A}$, where D is a subfield of $\mathbf{C}$ containing $\Phi\mathbf{Q}_{\mathrm{ab}}$. Then there exists an element $\mathbf{q}$ of $\mathcal{S}_k(C, D)$ independent of $\mathbf{f}$ such that $\sum_{p\in\mathcal{A}} \langle h_p, f_p \rangle = \langle \mathbf{q}, \mathbf{f} \rangle$.*

PROOF. Recall that $f_p \in \mathcal{S}_k(\Gamma^p)$ with $\Gamma^p = G \cap pCp^{-1}$. Take $\Gamma \subset \bigcap_{p\in\mathcal{A}} \Gamma^p$ so that $h_p \in \mathcal{S}_k(\Gamma)$ for every $p\in\mathcal{A}$; let $\Gamma^p = \bigsqcup_{\xi\in X_p} \Gamma\xi$. Put then $h'_p = \#(X_p)^{-1} \sum_{\xi\in X_p} h_p\|\xi$. Then $h'_p \in \mathcal{S}_k(\Gamma^p, D)$ and $\langle h_p, f_p \rangle = \langle h'_p, f_p \rangle$. For each $b\in\mathcal{B}$ let $\mathcal{A}_b = \{p\in\mathcal{A} \mid p \in GbC\}$ and $q_b = \#(\mathcal{B}) \sum_{p\in\mathcal{A}_b} h'_p\|\alpha_p$, where $\alpha_p \in G$ is chosen so that $p \in \alpha_p bC$. Define $\mathbf{q} \in \mathcal{S}_k(C, D)$ by $\mathbf{q} \leftrightarrow (q_b)_{b\in\mathcal{B}}$. Then $\langle q_b, f_b \rangle = \#(\mathcal{B}) \sum_{p\in\mathcal{A}_b} \langle h'_p\|\alpha_p, f_b \rangle = \#(\mathcal{B}) \sum_{p\in\mathcal{A}_b} \langle h'_p, f_p \rangle$, and hence we obtain the desired conclusion.

28.5. Theorem (Cases SP and UT). *Let $\{\lambda(\mathfrak{a})\}$ be a system of eigenvalues on $\mathcal{S}_k(C)$ in the sense that $\mathbf{f}_0|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}_0$ for every $\mathfrak{a}$ with some $\mathbf{f}_0 \in \mathcal{S}_k(C), \neq 0$ (see §20.6). Let Ψ be the field generated by the $\lambda(\mathfrak{a})$ over $\Phi\mathbf{Q}_{\mathrm{ab}}$; put*

(28.6) $\mathcal{V} = \{\, \mathbf{f} \in \mathcal{S}_k(C) \mid \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f} \quad \textit{for every } \ \mathfrak{a} \,\}$,

(28.6a) $\mathcal{V}(\Psi) = \mathcal{V} \cap \mathcal{S}_k(C, \Psi), \quad \mathcal{V}(\overline{\mathbf{Q}}) = \mathcal{V} \cap \mathcal{S}_k(C, \overline{\mathbf{Q}})$.

Then Ψ is stable under complex conjugation and $\mathcal{V} = \mathcal{V}(\Psi) \otimes_\Psi \mathbf{C}$. Moreover, put $m_0 = \mathrm{Min}_{v \in \mathbf{a}} m_v$ and assume that the following condition is satisfied:

(28.7) *$m_0 > (3n/2) + 1$ in Case SP and $m_0 > 3n$ in Case UT.*

Then

$$\langle \mathbf{g}, \mathbf{g}' \rangle / \langle \mathbf{f}, \mathbf{f} \rangle \in \overline{\mathbf{Q}} \quad \textit{if} \quad \mathbf{g}, \mathbf{g}' \in \mathcal{V}(\overline{\mathbf{Q}}) \quad \textit{and} \quad 0 \neq \mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}}).$$

PROOF. That Ψ is stable under complex conjugation follows from Lemma 23.15. By Lemma 20.12 (3), $\mathcal{S}_k(C)$ is spanned by eigenforms of the operators $T(\mathfrak{a})$, which are normal. From [S97, (11.9.1)], Theorem 9.13 (3), (21.4), and Theorem 10.7 (6) we see that the $T(\mathfrak{a})$ map $\mathcal{S}_k(C, \Psi)$ into itself, and so they generate a ring of semisimple Ψ-linear transformations on $\mathcal{S}_k(C, \Psi)$. Therefore we have $\mathcal{V} = \mathcal{V}(\Psi) \otimes_\Psi \mathbf{C}$ and $\mathcal{S}_k(C, \Psi) = \mathcal{V}(\Psi) \oplus \mathcal{U}$ with a vector subspace $\mathcal{U}$ over Ψ stable under the $T(\mathfrak{a})$. Each eigenform in $\mathcal{U} \otimes_\Psi \mathbf{C}$, being not contained in $\mathcal{V}$, must be orthogonal to $\mathcal{V}$. Thus $\mathcal{U}$ is orthogonal to $\mathcal{V}$.

To prove the main part of our theorem, put $\mathcal{Z}(s) = \mathcal{Z}(s, \mathbf{f}, \chi)$. This depends only on the $\lambda(\mathfrak{a})$. We first consider Case SP. Define $\mu \in \mathbf{Z}^{\mathbf{a}}$ so that $0 \leq \mu_v \leq 1$ and $m_0 - k_v + \mu_v \in 2\mathbf{Z}$. Put $l = \mu + (n/2)\mathbf{a}$, $t' = \mu - [k]$, $\nu = m_0 - (n/2)$, and $\sigma_0 = m_0$. By [S97, Lemma 11.14 (3)] we can find a Hecke character χ of F such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'} |x_{\mathbf{a}}|^{t'}$. We consider (28.4) by employing $\mathbf{g}$ defined as in §A5.5 with such χ and μ; then $g_p \in \mathcal{M}_k(\Phi \mathbf{Q}_{\mathrm{ab}})$. Observe that $n+1 < \nu \leq k_v - l_v$ and $k_v - l_v - \nu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Evaluate (28.4) at $s = (\nu - n - 1)/2$ (which means that $\sigma = \sigma_0$). Putting $Q = \Gamma\bigl(((\nu - n - 1)/2)\bigr)^{-1} \Lambda(\sigma_0)$, we thus obtain

(28.8) $$c_{\mathbf{f}}(\tau, r)\mathcal{Z}(\sigma_0) = \pi^{d_0} \det(\tau)^{h + s\mathbf{a}} P_{\tau,r}(\sigma_0) Q \sum_{p \in \mathcal{A}} \langle\, e_p g_p E(\nu/2),\, f_p \,\rangle$$

with $e_p \in \mathbf{Q}_{\mathrm{ab}}$; we write $P_{\tau,r}$ for P in order to emphasize its dependence on (τ, r). From (22.4a) we see that $s + h_v - m_0 + (n+1)/2 \in \mathbf{Z}$ for every $v \in \mathbf{a}$, so that $\det(\tau)^{h+s\mathbf{a}} \in \Phi \mathbf{Q}_{\mathrm{ab}}$; also we can easily verify that $Q \neq 0$. By (17.21) and Theorem 17.7 (i) we can put $E(\nu/2) = \Delta_{\nu\mathbf{a}}^q y$ with $y \in \mathcal{M}_{\nu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ and $q = (k - l - \nu\mathbf{a})/2$. Therefore, by Lemma 15.8, there exists an element h'_p of $\mathcal{M}_k(\Phi \mathbf{Q}_{\mathrm{ab}})$ such that $\langle\, e_p g_p E(\nu/2),\, f_p \,\rangle = \pi^{n|q|} \langle\, h'_p,\, f_p \,\rangle$. Put $h_p = h'_p$ if $k \notin 2^{-1}\mathbf{Z}^{\mathbf{a}}$; otherwise put $h_p = \mathfrak{q}(h'_p)$ with the map $\mathfrak{q}$ of Theorem 27.14. (Recall that $\mathcal{M}_k = \mathcal{S}_k$ if $k \notin 2^{-1}\mathbf{Z}^{\mathbf{a}}$ by [S97, Proposition 10.6 (3)].) Then $h_p \in \mathcal{S}_k(\overline{\mathbf{Q}})$ and $\langle\, h'_p,\, f_p \,\rangle = \langle\, h_p,\, f_p \,\rangle$ since $\langle \mathcal{S}_k^\psi, \mathcal{E}_k^\psi \rangle = 0$ if $k \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$. By Lemma 28.4 we find an element $\mathbf{q}$ of $\mathcal{S}_k(C, \overline{\mathbf{Q}})$ independent of $\mathbf{f}$ such that $\sum_{p \in \mathcal{A}} \langle\, h_p,\, f_p \,\rangle = \langle\, \mathbf{q}, \mathbf{f} \,\rangle$. Clearly $\mathcal{S}_k(C, \overline{\mathbf{Q}}) = \mathcal{V}(\overline{\mathbf{Q}}) \oplus \bigl(\mathcal{U} \otimes_\Psi \overline{\mathbf{Q}}\bigr)$, and $\mathcal{V}(\overline{\mathbf{Q}})$ is orthogonal to $\mathcal{U} \otimes_\Psi \overline{\mathbf{Q}}$. Let $\mathbf{q}_{\tau,r}$ be the projection of $\mathbf{q}$ to $\mathcal{V}(\overline{\mathbf{Q}})$ with respect to this direct sum decomposition of $\mathcal{S}_k(C, \overline{\mathbf{Q}})$. Once μ, χ, and (τ, r) are fixed, then g_p, $P_{\tau,r}$, and E are independent of $\mathbf{f}$. Thus given $0 \neq \mathbf{f} \in \mathcal{V}$, there exists (τ, r) such that $c_{\mathbf{f}}(\tau, r) \neq 0$ and

(28.9) $$c_{\mathbf{f}}(\tau, r)\mathcal{Z}(\sigma_0) = \pi^{d_0 + n|q|} \det(\tau)^{h+s\mathbf{a}} P_{\tau,r}(\sigma_0) Q \langle\, \mathbf{q}_{\tau,r},\, \mathbf{f} \,\rangle.$$

Let $\mathfrak{S}$ denote the set of all pairs (τ, r) for which such an $\mathbf{f}$ exists. Since $\sigma_0 > 3n/2 + 1$, $\mathcal{Z}(\sigma_0) \neq 0$ by Theorem 20.13. Therefore, given $\mathbf{f} \in \mathcal{V}$, we have $\langle\, \mathbf{q}_{\tau,r},\, \mathbf{f} \,\rangle \neq 0$ for some $(\tau, r) \in \mathfrak{S}$, which implies that the $\mathbf{q}_{\tau,r}$ for all $(\tau, r) \in \mathfrak{S}$ generate $\mathcal{V}$ over $\mathbf{C}$, and hence they generate $\mathcal{V}(\overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$. Also we see that $P_{\tau,r}(\sigma_0) \neq 0$ for every $(\tau, r) \in \mathfrak{S}$. Now, given $\mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$ and an arbitrary $(\tau, r) \in \mathfrak{S}$, we may not be able to use (28.8), but evaluating (28.5) at $s = (\nu - n - 1)/2$, we find that $\langle \mathbf{q}_{\tau,r}, \mathbf{f} \rangle \in \pi^{-d_0-n|q|}Q^{-1}\mathcal{Z}(\sigma_0)\overline{\mathbf{Q}}$. Therefore $\langle \mathbf{g}, \mathbf{f} \rangle \in \pi^{-d_0-n|q|}Q^{-1}\mathcal{Z}(\sigma_0)\overline{\mathbf{Q}}$ for every $\mathbf{g}, \mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$. The main assertion of our theorem immediately follows from this fact.

Next let us treat Case UT. Define $\mu \in \mathbf{Z}^{\mathbf{b}}$ as follows: $\mu_{v\rho} = 0$ and $\mu_v = m_v - m_0$ for every $v \in \mathbf{a}$; put $t' = (\mu_v - k_v + k_{v\rho})_{v\in\mathbf{a}}$. Then (22.15a, b) are satisfied. By [S97, Lemma 11.14 (3)] we can find a Hecke character χ of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{-t'}|x_{\mathbf{a}}|^{t'}$. Put $\nu = m_0 - n$ and $\sigma_0 = m_0/2$. Define $\mathbf{g}$ as in §A5.5 and m' as in Proposition 22,2; then $m'_v = \mu_v + n$ for $v \in \mathbf{a}$, and hence $2n < \nu = m_v - m'_v$ for every $v \in \mathbf{a}$; $g_p \in \mathcal{M}_{\mu+n\mathbf{a}}(\overline{\mathbf{Q}})$, since χ^* is algebraic-valued by Lemma 17.11. Evaluate (28.4) at $s = (m_0 - 3n)/2$. Then σ and $E(\overline{s} + \lambda_n)$ become σ_0 and $E(\nu/2)$, and $m - m' = \nu\mathbf{a}$; $\det(\tau)^{h+s\mathbf{a}} \in \Phi$. By Theorem 17.7 (i) we find that $E(\nu/2) \in \mathcal{M}_{\nu\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ so that $e_p g_p E(\nu/2) \in \mathcal{M}_k(\overline{\mathbf{Q}})$. Then we can repeat our argument in Case SP, by putting $h_p = \mathfrak{q}\big(e_p g_p E(\nu/2)\big)$ without employing $\Delta^p_{\nu\mathbf{a}}$.

28.6. Corollary. *The notation being the same as in (28.2) and Theorem 28.5, let* $0 \neq \mathbf{f} \leftrightarrow (f_b)_{b\in\mathcal{B}} \in \mathcal{V}(\overline{\mathbf{Q}})$. *Then* $\langle g, f_a \rangle \in \langle \mathbf{f}, \mathbf{f} \rangle\overline{\mathbf{Q}}$ *for every* $g \in \mathcal{M}_k(\overline{\mathbf{Q}})$ *and every* $a \in \mathcal{B}$.

PROOF. We may assume that $g \in \mathcal{S}_k(\overline{\mathbf{Q}})$. Indeed, $\mathcal{M}_k \neq \mathcal{S}_k$ only if $m = \mu\mathbf{a}$ with $\mu \in 2^{-1}\mathbf{Z}$, in which case we put $g' = \mathfrak{q}(g)$ with the map $\mathfrak{q}$ of Theorem 27.14. Then $g' \in \mathcal{S}_k(\overline{\mathbf{Q}})$ and $\langle g, f \rangle = \langle g', f \rangle$ for every $f \in \mathcal{S}_k$, so that it is sufficient to treat the case $g \in \mathcal{S}_k(\overline{\mathbf{Q}})$. Given $g \in \mathcal{S}_k(\overline{\mathbf{Q}})$, we may assume, changing C for its suitable subgroup, that $g \in \mathcal{S}_k(\Gamma^a)$, where $\Gamma^a = G \cap aCa^{-1}$. Fixing $a \in \mathcal{B}$, define $\mathbf{g} \leftrightarrow (g_b)_{b\in\mathcal{B}} \in \mathcal{S}_k(C, \overline{\mathbf{Q}})$ so that $g_a = g$ and $g_b = 0$ for $a \neq b \in \mathcal{B}$. We have $\mathcal{S}_k(C, \overline{\mathbf{Q}}) = \mathcal{V}(\overline{\mathbf{Q}}) \oplus (\mathcal{U} \otimes_{\Psi} \overline{\mathbf{Q}})$ with $\mathcal{U}$ as in the proof of Theorem 28.5. Let $\mathbf{g}'$ be the projection of $\mathbf{g}$ to $\mathcal{V}(\overline{\mathbf{Q}})$ with respect to that decomposition. Then $\#(\mathcal{B})^{-1}\langle g, f_a \rangle = \langle \mathbf{g}, \mathbf{f} \rangle = \langle \mathbf{g}', \mathbf{f} \rangle$, which belongs to $\langle \mathbf{f}, \mathbf{f} \rangle\overline{\mathbf{Q}}$ by Theorem 28.5.

28.7. Lemma. *In the setting of Sections 24 and 25, let* $R \in \mathcal{M}^{n+r}_{\nu\mathbf{a}}(\Xi)$ *with a subfield* Ξ *of* $\mathbf{C}$ *containing* $\mathbf{Q}_{\mathrm{ab}}$ *and the Galois closure of* K *over* $\mathbf{Q}$, *and let* $S_0 = D_{e,e'}\Delta^p_{\nu\mathbf{a}}R$ *and* $S(z, w) = S_0\big(\mathrm{diag}[z, w]\big)$ *for* $(z, w) \in \mathcal{H}^n \times \mathcal{H}^r$, *where* $D_{e,e'}$, *which we ignore if* $n \neq r$, *is as in §25.2, and* $\Delta^p_{\nu\mathbf{a}}$ *with* $0 \leq p \in \mathbf{Z}^{\mathbf{a}}$ *is as in Lemma 15.8 and the proof of Theorem 17.9. Let* L_v *be the operator of (15.3) defined on* $\mathcal{H}^r$ *with* $\omega(a, b) = \det(b)^m$, $m = \nu\mathbf{a} + 2p + e - e'$. *Then there exists an element* $T(z, w)$ *of* $\sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\Xi L_v^i S$ *which is holomorphic in* w *and such that* $\langle S(z, w), f(w) \rangle = \langle T(z, w), f(w) \rangle$ *for every* $f \in \mathcal{S}^r_m$, *provided* $\nu \geq (n + r)/2$ *in Case SP and* $\nu \geq n + r$ *in Case UT.*

This will be proven in §A8.12.

28.8. Theorem (Cases SP and UT). *The notation being as in Theorem 28.5, let* $0 \neq \mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$; *let* χ *be a Hecke character of* K *such that* $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ *with* $\ell \in \mathbf{Z}^{\mathbf{a}}$, *and let* $\sigma_0 \in 2^{-1}\mathbf{Z}$. *In addition to (28.7), assume the following condition:*

Case SP: $2n+1-k_v+\mu_v \leq \sigma_0 \leq k_v - \mu_v$ *where* $\mu_v = 0$ *if* $[k_v] - \ell_v \in 2\mathbf{Z}$ *and* $\mu_v = 1$ *if* $[k_v] - \ell_v \notin 2\mathbf{Z}$; $\sigma_0 - k_v + \mu_v \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$ *if* $\sigma_0 > n$ *and* $\sigma_0 - 1 + k_v - \mu_v \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$ *if* $\sigma_0 \leq n$.

Case UT: $4n - (2k_{v\rho} + \ell_v) \leq 2\sigma_0 \leq m_v - |k_v - k_{v\rho} - \ell_v|$ *and* $2\sigma_0 - \ell_v \in 2\mathbf{Z}$ *for every* $v \in \mathbf{a}$.

Further exclude the following cases:

(A) Case SP: $\sigma_0 = n+1$, $F = \mathbf{Q}$, *and* $\chi^2 = 1$;
(B) Case SP: $\sigma_0 = n+(3/2)$, $F = \mathbf{Q}$, $\chi^2 = 1$ *and* $[k]-\ell \in 2\mathbf{Z}$;
(C) Case SP: $\sigma_0 = 0$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi = 1$;
(D) Case SP: $0 < \sigma_0 \le n$, $\mathfrak{c} = \mathfrak{g}$, $\chi^2 = 1$, *and the conductor of* χ *is* $\mathfrak{g}$;
(E) Case UT: $2\sigma_0 = 2n+1$, $F = \mathbf{Q}$, $\chi_1 = \theta$, *and* $k_v - k_{v\rho} = \ell_v$;
(F) Case UT: $0 \le 2\sigma_0 < 2n$, $\mathfrak{c} = \mathfrak{g}$, $\chi_1 = \theta^{2\sigma_0}$, *and the conductor of* χ *is* $\mathfrak{r}$.

Here χ_1 *is the restriction of* χ *to* $F_{\mathbf{A}}^\times$ *and* θ *is the Hecke character of* F *corresponding to* K/F. *Then*

$$\mathcal{Z}(\sigma_0, \mathbf{f}, \chi)/\langle \mathbf{f}, \mathbf{f}\rangle \in \pi^{n|m|+d\varepsilon}\overline{\mathbf{Q}}, \tag{28.10}$$

where $d = [F:\mathbf{Q}]$, $|m| = \sum_{v\in\mathbf{a}} m_v$, and

$$\varepsilon = \begin{cases} (n+1)\sigma_0 - n^2 - n & (\text{Case SP, } k \in \mathbf{Z}^{\mathbf{a}} \ \textit{and} \ \sigma_0 > n), \\ n\sigma_0 - n^2 & (\text{Case SP, } k \notin \mathbf{Z}^{\mathbf{a}} \ \textit{or} \ \sigma_0 \le n), \\ 2n\sigma_0 - 2n^2 + n & (\text{Case UT}). \end{cases}$$

Notice that $n|m| + d\varepsilon \in \mathbf{Z}$ in all cases. If $k \notin \mathbf{Z}^{\mathbf{a}}$, for example, the above condition on σ_0 shows that $\sigma_0 + k_v \in \mathbf{Z}$, which implies that $n|m| + d\varepsilon \in \mathbf{Z}$.

PROOF. There are two ways to prove this: the first one applies to the whole critical strip, and the second one only to the right half of the strip. However, the latter can cover certain cases to which the former does not apply. Let us begin with the first method, using the notation of §§25.4 and 25.5, in which the weight of $\mathbf{f}$ was written h instead of k; at the end of the proof we shall reinstate k as the weight of $\mathbf{f}$, and obtain the condition on σ_0 in terms of k as stated above.

Cast UT (1st method). Define d_v, e, e', and k as in §25.4; put $m = (k_v + k_{v\rho})_{v\in\mathbf{a}}$ and $m' = (h_v + h_{v\rho})_{v\in\mathbf{a}}$. (At the end we must change m' into m.) We evaluate (25.8a) at $s = \sigma_0$; we have to change $\mathfrak{c}$, as we did in §25.4, so that the conductor of χ divides $\mathfrak{c}$. Put $\nu = 2\sigma_0$ and $H'_{b,a}(\mathfrak{z}, s) = \Lambda_{\mathfrak{c}}^{2n}(s, \chi)H_{b,a}(\mathfrak{z}, s)$ for $\mathfrak{z} \in \mathcal{H}^{2n}$. We first treat the case $2\sigma_0 < 2n$. For the reason explained at the beginning of §25.5, $H'_{b,a}$ is a function of type (17.24). Therefore, by (17.30), $H'_{b,a}(\mathfrak{z}, \nu/2) = \Delta^p_{\mu\mathbf{a}}R_1$ with $R_1(\mathfrak{z}) = D_r(\mathfrak{z}, \nu/2, k', \chi, \mathfrak{c})$, where $\mu = 4n - \nu$, $p = (m - \mu\mathbf{a})/2$, and k' is such that $(k'_v + k'_{v\rho})_{v\in\mathbf{a}} = \mu\mathbf{a}$. By Theorem 17.12 (iii) we can put $R_1 = \pi^\alpha R$ with $\alpha = dn(2n+1)$ and $R \in \mathcal{M}_{\mu\mathbf{a}}(\overline{\mathbf{Q}})$. (Our H' is a function on $\mathcal{H}^{2n}$, so that we have to take $2n$ in place of n in Theorem 17.7. For example, $\pi^{-2n|p|}\Delta^p_{\mu\mathbf{a}}R$ is $\overline{\mathbf{Q}}$-rational.) We have to assume that $0 \le m_v - \mu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. Since $m_v = h_v + h_{v\rho} - d_v = 2h_{v\rho} + \ell_v$, this means that

$$\text{(i)} \qquad 4n - (2h_{v\rho} + \ell_v) \le \nu < 2n \quad \text{and} \quad \nu - \ell_v \in 2\mathbf{Z} \quad \text{for every } v \in \mathbf{a}.$$

Case (F) of our theorem must be excluded. In (25.8a) we have $c_{m'}(\mathbf{s}') \in \pi^{d_0}\mathbf{Q}^\times$ for the same reason as in (25.11); employing the formulas of Theorem 12.13 and Lemma 25.3, we see that $\Psi(\mathbf{s}) \ne 0$ if in addition to (i) we assume

$$\text{(ii)} \qquad \nu \le m_v + 2d_v \quad \text{whenever} \quad d_v < 0.$$

Define S_0 as in Lemma 28.7 for the present R. Then $\pi^\alpha S = \left[D_{e,e'}H'_{b,a}(\mathfrak{z}, \nu/2)\right]^\circ$ and $\pi^{n|e'-e-2p|}S$ is a $\overline{\mathbf{Q}}$-rational function, to which Lemma 25.8 is applicable. Take $T(z, w)$ as in Lemma 28.7 and put $T_{a,b}(z, w) = T(z, \eta w)j^h_\eta(w)^{-1}$. For the same reason as in Lemma 25.8, we can put $T_{a,b}(z, w) = \pi^{n|2p+e-e'|}\sum_i t_{abi}(z)g_{abi}(w)$ with $t_{abi} \in \mathcal{N}_h^{p'}(\overline{\mathbf{Q}})$ and $g_{abi} \in \mathcal{M}_h(\overline{\mathbf{Q}})$. The integral over $\mathcal{D}_a$ of (25.8a) is a constant times $\langle f'_a, S(z, \eta w)j_\eta^h(w)^{-1}\rangle$, where $f'_a(w) = \overline{f_a(-w^*)}$. By the property of T in Lemma 28.7, the last inner product equals $\langle f'_a, T_{a,b}(z, w)\rangle$, which means that we can replace $\Lambda_\mathfrak{c}^{2n}J'_{b,a}$ at $s = \nu/2$ in (25.8a) by $\pi^\alpha T_{a,b}(z, -w^*)$. Putting $g'_{abi}(w) = \overline{g_{abi}(-w^*)}$ and changing t_{abi} for its suitable scalar multiple, we find that

$$\text{(iii)} \qquad \mathcal{Z}(\sigma_0, \mathbf{f}, \chi)f_b(z) = \pi^\gamma \sum_{a,i}\langle g'_{abi}, f_a\rangle t'_{abi}(z)$$

with $\gamma = \alpha + n|2p + e - e'|$ and some $t'_{abi} \in \mathcal{N}_h^{p'}(\overline{\mathbf{Q}})$. Take the Fourier expansion with respect to z, and compare nonzero Fourier coefficients. Then we find that $\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) = \pi^\gamma\sum_a\langle g_a, f_a\rangle$ with some $g_a \in \mathcal{M}_h(\overline{\mathbf{Q}})$. Applying Corollary 28.6 to the right-hand side, we obtain (28.10). (If ν satisfies (i) and $d_v \geq 0$, then $2n < 2h_{v\rho} + \ell_v = h_v + h_{v\rho} - |d_v|$, so that $2\sigma_0$ satisfies the conditions stated in our theorem. Conversely, those conditions imply (i) and (ii) if $2\sigma_0 < 2n$.)

The case $2\sigma_0 \geq 2n$ is similar; we use Theorem 17.12 (i) or (v) and (17.27); we need (ii) in order to insure that $\Psi(\mathbf{s}) \neq 0$; also, instead of (i) we assume

$$\text{(iv)} \qquad 2n \leq 2\sigma_0 \leq m_v \quad \text{and} \quad 2\sigma_0 - \ell_v \in 2\mathbf{Z} \quad \text{for every } v \in \mathbf{a}.$$

Now $h_v + h_{v\rho} - |d_v| = m_v$ if $d_v \geq 0$, and $h_v + h_{v\rho} - |d_v| = m_v + 2d_v$ if $d_v < 0$. Therefore, changing (h, m') into (k, m), we obtain the condition on σ_0 as stated in our theorem. (If $d_v \geq 0$, then (iv) implies that $2h_{v\rho} + \ell_v \geq 2n$, and hence $4n - (2h_{v\rho} + \ell_v) \leq 2n \leq 2\sigma_0$; if $d_v < 0$, then $-\ell_v < h_{v\rho} - h_v$, and hence $4n - (2h_{v\rho} + \ell_v) < 4n - (h_v + h_{v\rho}) < n \leq 2\sigma_0$, if we assume $h_v + h_{v\rho} > 3n$.) Also we have to exclude the case in which $2\sigma_0 = 2n + 1$, $F = \mathbf{Q}$, and $\chi_1 = \theta$. Even in that case we have (28.10) if $k_v - k_{v\rho} \neq \ell_v$, as will be shown by the second method.

Case SP (1st method). The case of integral k can be proved in the same way as above. Writing again h for the weight, take $e \in \mathbf{Z}^\mathbf{a}$ so that $\chi_\mathbf{a}(x) = x_\mathbf{a}^{h+e}|x_\mathbf{a}|^{-h-e}$ and $0 \leq e_v \leq 1$ for every $v \in \mathbf{a}$; put $k = h + e$. (That is what we did at the end of §25.4.) We evaluate (25.8a) at $s = \sigma_0/2$. Suppose $\sigma_0 \leq n$. We employ (17.22) and Theorem 17.7 (v) with $\nu = 2n + 1 - \sigma_0$. We have to assume that $0 \leq k_v - \nu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$, that is, $2n + 1 - h_v - e_v \leq \sigma_0 \leq n$ and $\sigma_0 - 1 + h_v - e_v \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. In this case $\Psi(\mathbf{s}) \neq 0$ with no extra condition. Writing (k, μ) for (h, e), we obtain "the left half" of our theorem with the condition as stated in our theorem. "The right half" can be obtained from (17.21) and Theorem 17.7 (iii). (If $2n+1-k_v+\mu_v \leq \sigma_0 \leq k_v-\mu_v$, then $k_v-\mu_v \geq n+1/2$ and $2n+1-k_v+\mu_v \leq n+1/2$. Thus $\sigma_0 \leq k_v - \mu_v$ if $\sigma_0 \leq n$, and $\sigma_0 \geq 2n + 1 - k_v + \mu_v$ if $\sigma_0 > n$.)

The case of half-integral weight can be handled by employing [S95b, (8.4)] instead of (25.8a).

Case SP (2nd method). Define $\mu \in \mathbf{Z}^\mathbf{a}$ and take σ_0 as in our theorem; assume that $\sigma_0 \geq n+1/2$. Put $l = \mu+(n/2)\mathbf{a}$ and $\nu = \sigma_0-(n/2)$; then $\nu \geq (n+1)/2$ and $0 \leq k - l - \nu\mathbf{a} \in 2\mathbf{Z}^\mathbf{a}$. Define $\mathbf{g}$ as in §A5.5 with the present μ and χ. Evaluating (28.4) at $s = (\nu-n-1)/2$, we have again (28.8) with $Q = \Gamma\big(((\nu-n-1)/2)\big)^{-1}\Lambda(\sigma_0)$ for the present ν and σ_0. By the same procedure as in the proof of Theorem 28.5 we find an element $\mathbf{q}_\nu$ of $\mathcal{V}(\overline{\mathbf{Q}})$ such that

$$\text{(28.11)} \qquad c_\mathbf{f}(\tau, r)\mathcal{Z}(\sigma_0) = \pi^{d_0+n|q|}\det(\tau)^{h+s\mathbf{a}}P_{\tau,r}(\sigma_0)Q\langle \mathbf{q}_\nu, \mathbf{f}\rangle$$

with $q = (k - l - \nu\mathbf{a})/2$. We have $\det(\tau)^{h+s\mathbf{a}} \in \Phi\mathbf{Q}_{\mathrm{ab}}$ for the same reason as in the proof of Theorem 28.5. Let us now assume that $k \in \mathbf{Z}^\mathbf{a}$; then $\sigma_0 \in \mathbf{Z}$. By (22.4a) we have $q_v + s + h_v = k_v - (n+1)/2$, and so from the explicit form of $\Gamma((s))$ in §22.3 we see that $\pi^{d_0+n|q|}\Gamma((s))^{-1} \in \pi^{n|k|-d\gamma}\mathbf{Q}^\times$, where $\gamma = n^2/4$ if $n \in 2\mathbf{Z}$ and $\gamma = (n^2-1)/4$ if $n \notin 2\mathbf{Z}$. Applying Lemma 17.5 (2) to each factor of Λ, we find that $\Lambda(\sigma_0) \in \pi^{d(n+1)\sigma_0 - d\beta}$, where $\beta = n(3n+4)/4$ if $n \in 2\mathbf{Z}$ and $\beta = (n+1)(3n+4)/4$ if $n \notin 2\mathbf{Z}$. Therefore, dividing (28.11) by $\langle \mathbf{f}, \mathbf{f} \rangle$, we obtain our assertion in Case SP from Theorem 28.5, at least for $\sigma_0 \geq n + 1/2$. In view of Theorem 17.7 (i) we have to exclude the case $\nu = (n+2)/2$ if $F = \mathbf{Q}$. Also, if $\nu = (n+3)/2$, we cannot apply Lemma 15.8 to $E(\nu/2)$. However, if $\mu \neq 0$, then $\mathbf{g}$ is a cusp form, and hence, by Theorem 17.9 (ia), $\pi^{-n|q|}e_p g_p E(\nu/2)$ is a $\Phi\mathbf{Q}_{\mathrm{ab}}$-rational element of $\mathcal{R}_\omega^a$ of (15.4) with suitable ω and a, so that we can take $h_p = \pi^{-n|q|}\mathfrak{p}(e_p g_p E(\nu/2))$ with the map $\mathfrak{p}$ of Proposition 15.6 (3), and eventually obtain $\mathbf{q}_\nu$ from h_p. This is why we have the condition $[k] - \ell \in 2\mathbf{Z}$ in the bad case (B). The case of half-integral weight can be handled in the same way.

Case UT (2nd method). Define $\mu \in \mathbf{Z}^{\mathbf{b}}$ by (22.15a, b) with $t' = -\ell$; put $m' = (\mu_v + \mu_{v\rho} + n)_{v\in\mathbf{a}}$ and $\nu = 2\sigma_0 - n$. Then $n \leq \nu \leq m_v - m'_v$ and $m_v - m'_v - \nu \in 2\mathbf{Z}$ for every $v \in \mathbf{a}$. We have (28.4) with $\mathbf{g}$ defined by using the present μ and χ. We evaluate (28.4) at $s = (\nu/2) - n$; then we have $\det(\tau)^{h+s\mathbf{a}} \in \Phi$ again. We see that $\chi_{\mathbf{a}}(x) = \operatorname{sgn}(x)^{2\sigma_0\mathbf{a}}$ for $x \in F_{\mathbf{a}}^\times$, so that we can apply Lemma 17.5 (2) to $\Lambda(\sigma_0)$. Therefore we obtain our assertion in Case UT, at least for $\sigma_0 \geq n$, in the same manner as in Case SP. There is no problem if $\mathbf{g}$ is a cusp form, and so the conditions in Case (E) are stated as above.

We now consider the arithmeticity of the Eisenstein series of Section 23.

28.9. Theorem (Cases SP and UT). *Let n, r, $\mathbf{f}$, g, χ be as in Theorem 23.11; let m_0 be as in Theorem 28.5. Suppose that $\mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$ and g is $\overline{\mathbf{Q}}$-rational; suppose also that $m_0 > (3r/2) + 1$ in Case SP and $m_0 > 3r$ in Case UT. (Notice that $\mathbf{f} \in \mathcal{S}_k^r(C)$, so that n in Theorem 28.5 is now r.) Let $\lambda_n = (n+1)/2$ in Case SP and $\lambda_n = n$ in Case UT. In the setting of Theorem 23.11 we can state the arithmeticity of each function as follows:*

(I) Let μ be as in Theorem 23.11 (I). Suppose that $\mu > (3r/2) + 1$ in Case SP and $\mu > 3r$ in Case UT. If $F = \mathbf{Q}$, suppose moreover that $\mu \notin \{\lambda_{n+r} + 1/2,\ \lambda_{n+r} + 1\}$ in Case SP and $\mu \neq \lambda_{n+r} + 1$ in Case UT. Then $\pi^{-\alpha}E_k^{n,r}(z, \mu/2;\, g, \Gamma)$ and $\pi^{-\alpha}E_q(z, \mu/2;\, \mathbf{f}, \chi, C)$ are $\overline{\mathbf{Q}}$-rational, where $\alpha = \sum_{v\in\mathbf{a}}(m_v - \mu)(n-r)/2$.

(II) Suppose $F = \mathbf{Q}$ and $\mu = \lambda_{n+r} + 1$. Then $\pi^{-\alpha}E_q(z, \mu/2;\, \mathbf{f}, \chi, C)$ with α as above is $\overline{\mathbf{Q}}$-rational in Case SP if $\chi^2 \neq 1$, and also in Case UT if $\chi_1 \neq \theta^\mu$.

(III) Let μ be as in Theorem 23.11 (III). Exclude Cases (A), (B), (C), (D), and (F) of the same theorem. Then $\pi^{-\beta}\langle \mathbf{f}, \mathbf{f} \rangle^{-1}\mathcal{F}_q(z, \mu/2;\, \mathbf{f}, \chi, C)$ is $\overline{\mathbf{Q}}$-rational, where $\beta = \sum_{v\in\mathbf{a}}(m_v + \mu)(n+r)/2 - de$ with $e = [(n+r)^2/4]$ in Case SP and $e = (n+r)(n+r-1)/2$ in Case UT.

PROOF. We evaluate (24.29) at $s = \mu/2$ as we did in §25.7. By (25.11), $c_m(\mathbf{s}) \in \pi^{d_0}\mathbf{Q}^\times$ at $s = \mu/2$, and by Lemma 17.5 (2), $\Lambda_{\mathfrak{c}}^{2r}(\mu/2, \chi) \in \pi^{dM}\mathbf{Q}_{\mathrm{ab}}^\times$ with an integer M which can be explicitly given. Therefore, by (25.9) and (25.10), we find that

$$\mathcal{Z}(u\mu/2, \mathbf{f}, \chi)E_q(z, \mu/2;\, \mathbf{f}, \chi, C) = \pi^\gamma \sum_{a\in\mathcal{B}} \langle h'_{ai}, f_a \rangle g'_{ai}(z), \tag{28.12}$$

where $\gamma = \sum_{v\in\mathbf{a}}(m_v - \mu)(n+r)/2 + dM$ and $g'_{ai}, h'_{ai} \in \mathcal{N}_k^t(\Psi)$ with some t. Now (25.9) was obtained by applying Lemma 24.11 to $(H_{q.a})^\circ$. Instead, by virtue of Lemma 28.7, we can replace $J_{q.a}$ by a function $T_{q.a}(z, w)$, which is holomorphic in w, and similar to $T_{b,a}$ in the proof of Theorem 28.8. We eventually find (28.12) with $h'_{ai} \in \mathcal{M}_k(\Psi)$. Here we have to verify that $J_{q,a}$ is a function of type $\Delta^p_{\nu\mathbf{a}}R$ of Lemma 28.7, which is so if we exclude the bad cases stated in (I) and (II). Then we find that $\langle h'_{ai}, f_a \rangle \in \langle \mathbf{f}, \mathbf{f} \rangle \overline{\mathbf{Q}}$ by Corollary 28.6. By Theorem 20.13 and (28.10), $\mathcal{Z}(\sigma_0, \mathbf{f}, \chi) \in \pi^{r|m|+d\varepsilon} \langle \mathbf{f}, \mathbf{f} \rangle \overline{\mathbf{Q}}^\times$ for $\sigma_0 = u\mu/2$ with ε given by the formula there in which we have to take r in place of n. An explicit calculation of M shows that $M = r\mu + \varepsilon$. Therefore we obtain our assertions for E_q in (I) and (II). We obtain the assertion for $E_k^{n,r}$ for the same reason as in §25.7.

As for (III), the desired result can be derived from (24.29a), Theorem 17.7 (iii, v), Theorem 17.12 (i, iii), (17.22), and (17.30) in a similar way. If $k \notin \mathbf{Z}^{\mathbf{a}}$, then we use [S95b, (7.22)].

28.10. In the above theorem the case in which $F = \mathbf{Q}$ and $\mu = \lambda_{n+r} + 1$ is excluded, though that case can be handled under a certain condition on χ. It is conjecturable that the arithmeticity as in the above theorem is always true for $\mu = \lambda_{n+r} + 1$ even when $F = \mathbf{Q}$. In fact, we can at least prove the rationality as stated in the above theorem if the following inequality holds:

$$(n+r)(\lambda_{n+r} - 1) > 4\lambda_r - 2 + (n+r-2)m. \tag{28.13}$$

Here we assume $F = \mathbf{Q}$, and so $m = (m_v)_{v\in\mathbf{a}} \in 2^{-1}\mathbf{Z}$. Indeed, if $\mu = \lambda_{n+r} + 1$, then the degree of near holomorphy, written t in the above proof, is given by $t = (n+r)(m - \mu + 2)/2$. Then Lemma 28.2 is applicable to h'_{ai} of (28.12) if $m > 2\lambda_r - 1 + t$, which is equivalent to (28.13). Thus, under (28.13), that lemma allows us to replace h'_{ai} of (28.12) with $\mathfrak{p}(h'_{ai})$, which belongs to $\mathcal{S}_k^r(\Psi)$. Therefore we obtain the desired arithmeticity.

28.11. Corollary. *The notation being as in Theorem 28.9, suppose that $m = \mu\mathbf{a}$ with $\mu \in 2^{-1}\mathbf{Z}$ such that $\mu > (3r/2) + 1$ in Case SP, $\mu > 3r$ in Case UT, and $\mu \geq \lambda_{n+r}$ in both cases; if $F = \mathbf{Q}$, suppose in addition that $\mu \notin \{\lambda_{n+r} + 1/2,\ \lambda_{n+r}+1\}$ in Case SP, and $\mu \neq \lambda_{n+r}+1$ in Case UT. Then $E_k^{n,r}(z, \mu/2; g, \Gamma)$ and $E_q(z, \mu/2; \mathbf{f}, \chi, C)$ are $\overline{\mathbf{Q}}$-rational.*

PROOF. This follows immediately from Theorem 28.9 (I), since $\alpha = 0$ if $m = \mu\mathbf{a}$.

28.12. *Proof of Theorem 27.16 (1) in Cases SP and UT.* Let us first treat Case SP. The desired fact is included in Theorem 28.9, but the proof of the theorem requires the map $\mathfrak{q}$ of Theorem 27.14, which is guaranteed by Theorem 27.16. Therefore, to prove Theorem 27.16, we need the same theorem, but this does not produce any vicious circle, since the proof of Theorem 28.9 on G^n requires Theorem 27.16 on G^r with $r < n$, and we can justify the whole proof by induction on the dimensionality, which is as follows.

First of all, the case $r = 0$ of Theorem 27.16 (1) is guaranteed by Theorem 17.7 (i). Therefore, if $n = 1$, the stability of $\mathcal{E}_k^\psi$ under $\mathrm{Aut}(\mathbf{C}/\overline{\mathbf{Q}})$ is true at least for $\mu > 2$; in fact, it is true even for $\mu = 2$ as noted in Remark 27.17 (A). Thus we have the map $\mathfrak{q}$ of Theorem 27.14 for the forms on G^1; see [S87b, Theorem 9.1 and Proposition 9.4] for the most comprehensive results in the case $n = 1$. Let μ be as in Theorem 27.15 with $1 = r < n$. Notice that $\mu \in \Lambda(1, \mu\mathbf{a})$ if and only if $\mu \geq 2$. Therefore $\mathcal{Z}(s, \mathbf{f}, \chi) \neq 0$ at $s = \mu$ if $\mu \geq 2$, and hence the proof of Theorem 28.5, as well as that of Corollary 28.6, is valid for the forms of weight $\mu\mathbf{a}$ on G^1 with such a μ. Consequently (28.10) is valid for $n = 1$ and $\sigma_0 = \mu$; then our proof of Theorem 28.9 establishes Theorem 27.16 (1) for $r = 1$. This settles Theorem 27.16 for $n = 2$. To prove Theorem 27.16 for an arbitrary $n > 2$, we assume that it is true for $G^{n'}$ for every $n' < n$. Let $1 < r < n$ and let μ be as in Theorem 27.15. Our induction assumption guarantees the map $\mathfrak{q}$ for the forms on G^r. Since $\mu > (3r/2) + 1$, Theorem 28.5 and Corollary 28.6 are valid for the forms of weight $\mu\mathbf{a}$ on G^r, and again the proof of Theorem 28.9 establishes Theorem 27.16 (1) for those r and μ. This completes the proof in Case SP. Case UT can be handled similarly.

28.13. Remark. (A) For $m = \mu\mathbf{a}$ we already stated in Theorem 27.16 the arithmeticity of $E_k^{n,r}(z, \mu/2; g, \Gamma)$ under the condition that $\mu > \lambda_{n+r}$ if $F \neq \mathbf{Q}$ and $\mu > \lambda_{n+r} + 1$ if $F = \mathbf{Q}$. Thus the result for $\mu = \lambda_{n+r}$ given in Corollary 28.11 is not included in that theorem.

(B) In [S76] and several subsequent papers the author obtained various results concerning the critical values of the zeta functions on GL_2 and $GL_2 \times GL_2$, as well as some other related zeta functions. The most comprehensive results in the case with a totally real number field as the basic field were given in [S88] and [S91], in which references to the papers in the intermediate period can be found. In [S76] we formulated the result of type (28.10) in the form, for example,

$$\left[\pi^{-k}\langle f, f\rangle^{-1}D(m, f, g)\right]^{\sigma} = \pi^{-k}\langle f^{\sigma}, f^{\sigma}\rangle^{-1}D(m, f^{\sigma}, g^{\sigma}) \tag{28.14}$$

for every $\sigma \in \mathrm{Aut}(\mathbf{C})$, where f is a Hecke eigenform of weight k and g is another modular form with respect to congruence subgroups of $SL_2(\mathbf{Z})$; $D(s, f, g)$ is of type (22.4). Though we can in fact state (28.10) in such a form, in the present book we content ourselves with a weaker statement, since the proof of the results in the form (28.14) would make our exposition longer and more tedious. At any rate we believe that what was done in Section 10 combined with careful examinations of the behavior of the Eisenstein series of Sections 16 and 17 under $\mathrm{Aut}(\mathbf{C})$ can give the desired formulas with no extra new ideas. Also, we can prove, employing (22.9), the algebraicity of the critical values of the functions of (22.4) in the form similar to (28.10), but the task of giving precise statements for these can be left to the reader as easy exercises.

There are a few more technical points. In [S76] we relied on the existence of a "primitive" Hecke eigenform, which is not guaranteed in general, in the case of $Sp(n, F)$, for example. However, in [S81a] we introduced a method by which this difficulty can be avoided, and mentioned that the higher-dimensional symplectic case could be handled by the same method. Indeed, in this section we proved the expected results by the basic idea of [S76] combined with the technique of [S81a]. The latter requires the nonvanishing of $\mathcal{Z}$ at a critical point, which we proved in Theorem 20.13, and the result is best possible in the sense of Theorem 22.11. However, given a point σ smaller than the limit of Theorem 20.13, it seems possible to find a suitable χ such that $\mathcal{Z}(\sigma, \mathbf{f}, \chi) \neq 0$. A result of this nature for the zeta function on GL_2 was proved by Rohrlich in [R]. A similar result in the higher-dimensional case will certainly allow us to state our theorems in improved forms.

(C) As we already noted at the end of Section 15, if $G = SL_2(F)$, the map $\mathfrak{p}$ of Lemma 28.2 can be established for every $k \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$ with no condition on the degree of near holomorphy, and the nonvanishing of $\mathcal{Z}$ can be given in a better form, as noted in Theorem 22.13. Therefore Theorems 28.5 and 28.8 can be stated in stronger forms. For example, if $k \in \mathbf{Z}^{\mathbf{a}}$, these theorems are true if we assume $m_0 \geq 2$ instead of (28.7). For details, see [S91] and the papers cited there.

Also, take $r = 1$ in the setting of Theorem 28.9 in Case SP and assume $m_0 \geq 2$. Then the conclusions of (I) and (II) are true for every μ as in Theorem 23.11 (I) and (III), excluding the cases specified there, as already noted in §28.12.

29. Main theorems on arithmeticity in Case UB

29.1. We now consider Case UB with (V, φ), G^φ, and $\mathfrak{Z}^\varphi$ as in §26.1; we use the symbols r_v, t_v, and $\mathbf{i}^\varphi$ defined by (26.1) and (26.5). Also, we put $d = [F : \mathbf{Q}]$, $n = \dim(V)$, and

$$\mathbf{a}' = \{\, v \in \mathbf{a} \,|\, r_v > 0 \,\}. \tag{29.1}$$

This is the same as $\mathbf{a}'$ of (12.5). By (3.23) and (3.24a, b) we have

$$j_\alpha^k(z) = \prod_{v\in\mathbf{a}} \det(\alpha_v)^{-k_{v\rho}} \prod_{v\in\mathbf{a}'} j_v(\alpha, z)^{k_v + k_{v\rho}} \quad (k \in \mathbf{Z}^{\mathbf{b}},\ \alpha \in G_{\mathbf{A}}^\varphi). \tag{29.2}$$

Thus we can ignore the k_v for $v \in \mathbf{a}'$ in the definition of $\mathcal{M}_k^\varphi$, but we need them in the definition of Eisenstein series of type $E_k^{\psi,\varphi}$.

In our proof it will become necessary to study the nature of $(A_m^k f)(z, w)$ defined by [S97, (23.6.4)]. To recall the definition of $A_m^k f$, we take $\psi = \varphi$ in the setting of §26.11. Then both ι and ι_U are maps of $\mathfrak{Z}^\varphi \times \mathfrak{Z}^\varphi$ into $\mathcal{H}_n^{\mathbf{a}}$. Let $k \in \mathbf{Z}^{\mathbf{b}}$ and $q \in \mathbf{Z}^{\mathbf{a}'}$. (This q is not q in §26.11; q there is now 0; also we use q in place of m in [S97, (23.6.4)], so that we shall speak of A_q^k.) We consider $S_p(T)$ of §13.13 for Type A with $p \in \mathbf{Z}^{\mathbf{a}'}$ and $T = \prod_{v\in\mathbf{a}'} T_v, T_v = \mathbf{C}_n^n$ for every $v \in \mathbf{a}'$. Then, for $u \in T$ and $(z, w) \in \mathfrak{Z}^\varphi \times \mathfrak{Z}^\varphi$ we put

$$\xi(u) = \prod_{\substack{v\in\mathbf{a}\\ q_v>0}} \det\left[\ell_v(u_v)\right]^{q_v}, \quad \zeta_{z,w}(u) = \prod_{\substack{v\in\mathbf{a}\\ q_v<0}} \det\left[\ell'_v\left({}^tM(w_v)u_vN(z_v)\right)\right]^{|q_v|}. \tag{29.3}$$

Here $\ell_v(Y)$ (resp. $\ell'_v(Y)$) denote the lower left (resp. the lower right) $(r_v \times r_v)$-block of Y; $M(w_v)$ and $N(z_v)$ are defined by [S97, (6.11.4)]. Then $\xi \in S_{p'}(T)$ and $\zeta_{z,w} \in S_{p''}(T)$ with $p' = \left(\mathrm{Max}\{r_v q_v,\, 0\}\right)_{v\in\mathbf{a}'}$ and $p'' = \left(\mathrm{Max}\{-r_v q_v,\, 0\}\right)_{v\in\mathbf{a}'}$; there is an irreducible subspace W (resp. Z) of $S_{p'}(T)$ (resp. $S_{p''}(T)$) containing ξ (resp $\zeta_{z,w}$); see [S97, §23.6]. Now for a function f on $\mathcal{H}_n^{\mathbf{a}}$ we define $A_q^k f$ to be a function on $\mathfrak{Z}^\varphi \times \mathfrak{Z}^\varphi$ given by

$$(A_q^k f)(z, w) = B_{z,w}(f\|_k U^{-1})\bigl(\iota(z, w)\bigr), \quad B_{z,w} g = (E^Z D_\rho^W g)(\xi, \zeta_{z,w}), \tag{29.4}$$

where U is as in [S97, (22.1.6)], g is a function on $\mathcal{H}_n^{\mathbf{a}}$, $\rho(x) = \det(x)^k$, and $E^Z D_\rho^W g$, as defined in §13.13, is a function on $\mathcal{H}_n^{\mathbf{a}}$ with values in $S_1\bigl(W, S_1(Z, \mathbf{C})\bigr)$. In [S97, Lemma 23.9] we showed

$$\prod_{v\in\mathbf{a}} \det(\gamma)^{k_v - k_{v\rho}} A_q^k\bigl(f\|_k[\beta, \gamma]_S\bigr)(z, w) = j_\beta^{k+q}(z)^{-1}\overline{j_\gamma^{k+q}(w)}^{-1}(A_q^k f)(\beta z, \gamma w) \tag{29.5}$$

for $(\beta, \gamma) \in G^\varphi \times G^\varphi$, where $[\beta, \gamma]_S$ is defined by (26.36). (The reader is reminded of the difference between the present notation about the factor of automorphy and that of [S97]; see §5.4.)

29.2. Lemma. *Let $f \in \mathcal{N}_k^e(\overline{\mathbf{Q}})$ with $e \in \mathbf{Z}^{\mathbf{a}}$; put $h = k + q$, $\alpha = \sum_{v\in\mathbf{a}} r_v q_v$, and $e' = (e'_v)_{v\in\mathbf{a}}$, $e'_v = \mathrm{Max}\{e_v + r_v q_v, 0\}$. Then*

$$\pi^{-\alpha}\mathfrak{q}^{-1}(A_q^k f)(z, w) = \sum_{i\in I} g_i(z)\overline{h_i(w)} \tag{29.6}$$

with a finite set of indices I and $g_i, h_i \in \mathcal{N}_h^{e'}(\overline{\mathbf{Q}})$, where $\mathfrak{q} = p_K\bigl(\sum_{v\in\mathbf{a}}(k_v - k_{v\rho})\tau_v, \sum_{v\in\mathbf{a}} t_v\tau_v\bigr)$.

PROOF. By (29.5), $A_q^k f$ has at least the desired automorphy property of the right-hand side of (29.6). To prove its near holomorphy, we can reduce the problem to the nature of $(A_q^k f)(z, \mathbf{i}^\varphi)$ or $(A_q^k f)(\mathbf{i}^\varphi, w)$ by the same argument as in the proof of Lemma 26.12. Put $g = f\|U^{-1}$. Then $g \in \mathcal{N}^e$, and so, by Proposition 13.15 (1), the components of $E^Z D_\rho^W g$ belong to $\mathcal{N}^{e'}$. Since $\zeta_{z,w}$ and $\iota(z, w)$ are holomorphic in $(z, \overline{w})$, we see that $(A_q^k f)(z, w)$ is a polynomial in $\bigl(\iota(z, w) - \iota(z, w)^*\bigr)^{-1}$ whose coefficients are holomorphic functions in $(z, \overline{w})$. Therefore the argument in the proof of Lemma 26.12 shows the desired near holomorphy.

To prove the $\overline{\mathbf{Q}}$-rationality, take the symbols z_0, w_0, $\mathfrak{z}_0$, and $\mathfrak{z}_1$ as in §26.11. Put $\lambda = \lambda(U^{-1}, \mathfrak{z}_1)$, $\mu = \mu(U^{-1}, \mathfrak{z}_1)$, and $j = j(U^{-1}, \mathfrak{z}_1)$ for simplicity. From the definition of U in [S97, (22.1.6)] we see that U_v has algebraic entries for every $v \in \mathbf{a}$; also by Lemma 4.13, z_0, w_0, $\mathfrak{z}_1$, and $\mathfrak{z}_0$ have algebraic coordinates. Therefore ξ, ζ_{z_0,w_0}, $M(w_0)$, $N(z_0)$, λ, μ, and j are all $\overline{\mathbf{Q}}$-rational. Define ξ' and ζ' by $\xi'(u) = \xi(\lambda^{-1}u\cdot {}^t\mu^{-1})$ and $\zeta'(u) = \zeta_{z_0,w_0}({}^t\lambda u\mu)$. Then $(B_{z_0,w_0}g)(\mathfrak{z}_1) = (E^Z D_\rho^W g)(\xi, \zeta_{z_0,w_0})(\mathfrak{z}_1) = j^{-k}(E^Z D_\rho^W f)(\xi', \zeta')(\mathfrak{z}_0)$ by the generalization of (12.21) and (12.24a, b) mentioned in §13.13. Put $\xi_1(u) = \xi'\bigl(\mathfrak{p}_{v\rho}(\mathfrak{z}_0)^{-1}u\cdot {}^t\mathfrak{p}_v(\mathfrak{z}_0)^{-1}\bigr)$ and $\zeta_1(u) = \zeta'\bigl({}^t\mathfrak{p}_{v\rho}(\mathfrak{z}_0)u\mathfrak{p}_v(\mathfrak{z}_0)\bigr)$. Fixing our attention on one $v \in \mathbf{a}$, we suppress the subscript v. By (26.41a, b)) we have

$$\begin{aligned}\zeta_1(u) &= \det\bigl[\ell'\bigl({}^tM(w_0)\cdot{}^t\lambda\cdot{}^t\mathfrak{p}_{v\rho}(\mathfrak{z}_0)u\mathfrak{p}_v(\mathfrak{z}_0)\mu N(z_0)\bigr)\bigr]^{-q}\\ &= \det\bigl[\ell'\bigl(\mathrm{diag}\bigl[{}^t\mathfrak{p}_\rho(z_0), {}^t\mathfrak{p}(w_0)\bigr]\cdot{}^tM(w_0)\cdot{}^t\lambda u\mu N(z_0)\mathrm{diag}\bigl[\mathfrak{p}_\rho(w_0), \mathfrak{p}(z_0)\bigr]\bigr)\bigr]^{-q}\\ &= \det\bigl(\mathfrak{p}(w_0)\mathfrak{p}(z_0)\bigr)^{-q}\zeta'(u).\end{aligned}$$

From the explicit forms of $M(w)$ and $N(z)$ in [S97, (6.11.4)] we see that $\ell\bigl(M(w_0)u\cdot{}^tN(z_0)\bigr) = \ell'(u)$. Therefore we obtain $\xi_1 = \det\bigl(\mathfrak{p}(w_0)\mathfrak{p}(z_0)\bigr)^{-q}\xi'$ by the same type of calculation as for ζ_1 and ζ'. Now, in the proof of Lemma 26.12 we have shown that $\mathfrak{P}_k(\mathfrak{z}_0) = \mathfrak{P}_k(z_0)\mathfrak{P}_k(w_0)\mathfrak{q}$. Therefore

$$\begin{aligned}\bigl(\mathfrak{P}_{\rho\otimes\tau^W\otimes\sigma^Z}(\mathfrak{z}_0)^{-1}E^Z D_\rho^W f\bigr)(\xi', \zeta')(\mathfrak{z}_0) &= \bigl[\mathfrak{P}_k(z_0)\mathfrak{P}_k(w_0)\mathfrak{q}\bigr]^{-1}(E^Z D_\rho^W f)(\xi_1, \zeta_1)(\mathfrak{z}_0)\\ &= \bigl[\mathfrak{P}_h(z_0)\mathfrak{P}_h(w_0)\mathfrak{q}\bigr]^{-1}(E^Z D_\rho^W f)(\xi', \zeta')(\mathfrak{z}_0)\\ &= j^k\bigl[\mathfrak{P}_h(z_0)\mathfrak{P}_h(w_0)\mathfrak{q}\bigr]^{-1}(B_{z_0,w_0}g)(\mathfrak{z}_1).\end{aligned}$$

By Theorem 14.9 (2), $\pi^{-\alpha}E^Z D_\rho^W f \in \mathcal{N}^{e'}_{\rho\otimes\tau^W\otimes\sigma^Z}(\overline{\mathbf{Q}})$. Since ξ' and ζ' are $\overline{\mathbf{Q}}$-rational, $\pi^{-\alpha}$ times the first quantity of the above series of equalities is algebraic, and hence $\pi^{-\alpha}\bigl[\mathfrak{P}_h(z_0)\mathfrak{P}_h(w_0)\bigr]^{-1}\mathfrak{q}^{-1}(A_q^k f)(z_0, w_0) \in \overline{\mathbf{Q}}$. Now we can repeat the proof of Lemma 26.12 with $A_q^k f$ in place of f° there, since the necessary properties of the function are guaranteed by what we proved in the above. Thus we obtain (29.6).

29.3. Lemma. *In the setting of §26.11, let* $R \in \mathcal{M}^{\eta}_{\nu\mathbf{a}}(\overline{\mathbf{Q}})$, $S_0 = B\big((\Delta^p_{\nu\mathbf{a}}R)\|_k U^{-1}\big)$, *and* $S(z, w) = S_0\big(\iota(z, w)\big)$ *for* $(z, w) \in \mathfrak{Z}^{\psi} \times \mathfrak{Z}^{\varphi}$, *where* $\nu \in \mathbf{Z}$, $0 \le p \in \mathbf{Z}^{\mathbf{a}}$, *$k$ is an element of* $\mathbf{Z}^{\mathbf{b}}$ *such that* $(k_v + k_{v\rho})_{v\in\mathbf{a}} = 2p + \nu\mathbf{a}$, $\Delta^p_{\nu\mathbf{a}}$ *is as in Lemma 15.8 and the proof of Theorem 17.9 in the unitary case, and B denotes the operator* $B_{z,w}$ *of (29.4) with* $q_v \ge 0$ *for every v, which we consider only if* $\psi = \varphi$. *(Thus* $Bg = (D^W_\rho g)(\xi)$, *which does not involve the parameters* (z, w), *and* $S_0 = \Delta^p_{\nu\mathbf{a}}(R\|_{\nu\mathbf{a}}U^{-1})$ *if* $\psi \ne \varphi$.*) Let* $h = k + e - e'$ *and* $m = (h_v + h_{v\rho})_{v\in\mathbf{a}}$; *let* L_v *be the operator of (15.3) defined on* $\mathfrak{Z}^{\varphi}$ *with* $\omega(a, b) = \det(b)^m$. *Then there exists an element* $T(z, w)$ *of* $\sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\overline{\mathbf{Q}}L_v^i\overline{S}$ *which is holomorphic in w and such that* $\langle\,\overline{S(z, w)},\, f(w)\,\rangle = \langle\, T(z, w),\, f(w)\,\rangle$ *for every* $f \in \mathcal{S}^{\varphi}_h$, *provided* $\nu \ge \mathrm{Max}_{v\in\mathbf{a}}r_v$.

This will be proven in §A8.13. Once this is established, we have an expression

$$\pi^{-N|p|-\alpha}\mathfrak{q}^{-1}T(z, w) = \sum_{i\in I}\overline{g_i(z)}h_i(w), \tag{29.7}$$

where α and $\mathfrak{q}$ are as in (29.6), I is a finite set of indices, $N = 2^{-1}\big(\dim(V) + \dim(W)\big)$, $g_i \in \mathcal{N}^{e'}_h(\overline{\mathbf{Q}})$, and $h_i \in \mathcal{M}_h(\overline{\mathbf{Q}})$. Indeed, put $f = \pi^{-N|p|}\Delta^p_{\nu\mathbf{a}}R$. Then f is $\overline{\mathbf{Q}}$-rational by Theorem 14.12, and $S = \pi^{N|p|}A^k_q f$, to which Lemma 29.2 is applicable. Thus S has an expression of the form (29.6), so that $\overline{S}$ has an expression of the form (29.7) with $\overline{\mathbf{Q}}$-rational nearly holomorphic g_i and h_i. By Theorem 14.9 (2), the same is true for every element of $\sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\overline{\mathbf{Q}}L_v^i\overline{S}$, and hence, for T in particular. Now T is holomorphic in w, and so the proof of Lemma 26.12, modified in an obvious way, gives the desired expression (29.7).

29.4. The notation being as in §26.10, let $\{\lambda(\mathfrak{a})\}$ be a system of eigenvalues on $\mathcal{S}^{\varphi}_h(D^{\varphi})$ in the sense that $\mathbf{f}_0|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}_0$ for every $\mathfrak{a}$ with some $\mathbf{f}_0 \in \mathcal{S}^{\varphi}_h(D^{\varphi})$, $\ne 0$, where $h \in \mathbf{Z}^{\mathbf{b}}$. Put

$$\mathcal{V} = \big\{\,\mathbf{f} \in \mathcal{S}^{\varphi}_h(D^{\varphi}) \;\big|\; \mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f} \quad \textit{for every } \mathfrak{a}\,\big\}, \tag{29.8a}$$

$$\mathcal{V}(\overline{\mathbf{Q}}) = \mathcal{V} \cap \mathcal{S}^{\varphi}_h(D^{\varphi}, \overline{\mathbf{Q}}) \qquad \text{(see §28.1)}. \tag{29.8b}$$

By Lemma 26.14, the $\lambda(\mathfrak{a})$ are algebraic. Therefore we have $\mathcal{V} = \mathcal{V}(\overline{\mathbf{Q}}) \otimes_{\overline{\mathbf{Q}}} \mathbf{C}$. For the same reason as in the proof of Theorem 28.5 we have $\mathcal{S}^{\varphi}_h(D^{\varphi}, \overline{\mathbf{Q}}) = \mathcal{V}(\overline{\mathbf{Q}}) \oplus \mathcal{U}$ with a vector space $\mathcal{U}$ over $\overline{\mathbf{Q}}$ that is orthogonal to $\mathcal{V}(\overline{\mathbf{Q}})$. We are going to state our main theorems on the arithmeticity, in which we need also a Hecke character χ of K such that $\chi_{\mathbf{a}}(x) = x_{\mathbf{a}}^{\ell}|x_{\mathbf{a}}|^{-\ell}$ with $\ell \in \mathbf{Z}^{\mathbf{a}}$. Here ℓ is basically arbitrary; also we denote by h, instead of k, the weight of a Hecke eigenform. However, whenever we consider an Eisenstein series $E_p(z, \nu/2;\, \mathbf{f},\, \chi,\, D^{\psi})$, we denote the weight by k and assume that $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$, as we did in §26.10.

29.5. Theorem. *The notation being as above, put* $m_0 = \mathrm{Min}_{v\in\mathbf{a}'}\{h_v + h_{v\rho}\}$ *and assume that* $m_0 > 2n$ *and* $G^{\varphi}_{\mathbf{a}}$ *is not compact. (See Theorem 29.7 below for the result in the compact case.) Then*

$$\langle\, \mathbf{g},\, \mathbf{g}'\,\rangle/\langle\, \mathbf{f},\, \mathbf{f}\,\rangle \in \overline{\mathbf{Q}} \quad \textit{if} \quad \mathbf{g},\, \mathbf{g}' \in \mathcal{V}(\overline{\mathbf{Q}}) \quad \textit{and} \quad 0 \ne \mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}}). \tag{29.9}$$

Moreover, let $0 \ne \mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$ and let σ_0 be an element of $2^{-1}\mathbf{Z}$ such that

$$2n - (2h_{v\rho} + \ell_v) \le 2\sigma_0 \le \begin{cases} 2h_v - \ell_v & \textit{if} \;\; r_v(h_v - h_{v\rho} - \ell_v) < 0, \\ 2h_{v\rho} + \ell_v & \textit{otherwise}, \end{cases} \tag{29.10a}$$

$$2\sigma_0 - \ell_v \in 2\mathbf{Z} \;\text{ for every } v \in \mathbf{a}, \tag{29.10b}$$

$$\sigma_0 < 0 \;\; \textit{or} \;\; 2\sigma_0 \ge n \;\; \textit{if} \;\; \mathfrak{c} = \mathfrak{g},\;\; \chi_1 = \theta^{2\sigma_0}, \textit{ and the conductor of } \chi \textit{ is } \mathfrak{r}, \tag{29.10c}$$

where χ_1 is the restriction of χ to $F_{\mathbf{A}}^\times$ and θ is the Hecke character of K corresponding to K/F. If φ is isotropic, suppose also that

(29.11a) $\qquad h_v - h_{v\rho} \geq \ell_v \quad$ *for every* $v \in \mathbf{a}$,

(29.11b) $\qquad 2\sigma_0 \neq n+1 \;$ *if* $F = \mathbf{Q}$ *and* $\chi_1 = \theta^{n+1}$.

Then

$$\mathcal{Z}(\sigma_0,\, \mathbf{f},\, \chi) \in \pi^\gamma \mathfrak{q}\,\langle \mathbf{f},\, \mathbf{f}\rangle\overline{\mathbf{Q}}, \tag{29.12}$$

where $\gamma = (dn/2)(2\sigma_0 - n + 1) + \sum_{v\in\mathbf{a}}\{r_v(h_v + h_{v\rho}) + (t_v/2)(2h_{v\rho} + \ell_v)\}$ *and* $\mathfrak{q} = p_K\big(\sum_{v\in\mathbf{a}} \ell_v\tau_v,\, \sum_{v\in\mathbf{a}} t_v\tau_v\big)$ *with the CM-type* $\tau = \sum_{v\in\mathbf{a}}\tau_v$ *of* K *fixed in §3.5 and the period symbol* p_K *of §11.3.*

Notice that $\gamma \in \mathbf{Z}$. Indeed, for every $v \in \mathbf{a}$ we have $n\sigma_0 + (t_v/2)\ell_v = 2r_v\sigma_0 + (t_v/2)(2\sigma_0 + \ell_v) \in \mathbf{Z}$ by (29.10b), from which we can easily derive that $\gamma \in \mathbf{Z}$.

PROOF. Put $q = (h_v - h_{v\rho} - \ell_v)_{v\in\mathbf{a}}$ and $k = h - q$; then $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$. We needed the ideals $\mathfrak{b}$ and $\mathfrak{c}$ for the definition of D^φ. Changing $\mathfrak{c}$ for its suitable multiple, we may assume, without changing $\mathcal{Z}(s,\, \mathbf{f},\, \chi)$, that the conductor of χ divides $\mathfrak{c}$. Define $E_{\mathbf{A}}$ by (16.27) in Case UT with these k, χ, $\mathfrak{b}$, $\mathfrak{c}$. Then (16.24a, b) are satisfied with $\kappa = 0$. In [S97, (23.11.3)] we proved

$$\begin{aligned}\varepsilon(s)c_h(\mathbf{s}')\Psi(\mathbf{s})\mathcal{Z}(s,\, \mathbf{f},\, \chi)f_b(z) \\ = \sum_{a\in\mathcal{B}} \chi_{\mathbf{h}}\big(\det(a)\big)\int_{\mathcal{D}_a} \Lambda_{\mathfrak{c}}^n(s,\, \chi)(A_q^k H_{b,a})(z,\, w)f_a(w)\delta(w)^m\mathbf{d}w.\end{aligned} \tag{29.13}$$

Here $m = (h_v + h_{v\rho})_{v\in\mathbf{a}}$; the symbols $\mathcal{B}$, $(f_b)_{b\in\mathcal{B}}$, $\Lambda_{\mathfrak{c}}^n$, $H_{b,a}$, and c_h are essentially the same as those in the proof of Theorem 26.13, though we have $\psi = \varphi$ here; they are explained in [S97, §§23.10 and 23.11]; Ψ is given in [S97, Lemma 23.8]; $\varepsilon(s)$ is the factor written $e \cdot d^s$ in [S97, (23.11.3)], which equals $e(k,\, m,\, \mathbf{s})c(\chi)$ of [S97, (23.10.3)]. (In fact, k and m there are $(k_v + k_{v\rho})_{v\in\mathbf{a}}$ and q here. Also we should add corrections: The symbol $\zeta\big(\mathfrak{y}\kappa_\alpha^* \cdot {}^t\mu_\alpha^{-1}\big)$ of [S97, p.192, line 3 and line 9 from the bottom] should be $\zeta\big(-\mathfrak{y}\kappa_\alpha^* \cdot {}^t\mu_\alpha^{-1}\big)$; also, the right-hand side of the equality in [S97, p.194, line 5 from the bottom] needs an extra factor $(-1)^t$, where t is the sum of $r_v m_v$ for all $v \in \mathbf{a}$ such that $m_v > 0$; the quantity of [S97, p.195, line 5] needs an extra factor $(-1)^{rm}$.) Putting $m_v^* = k_v + k_{v\rho}$ and $\mu = 2\sigma_0$, we evaluate (29.13) at $s = \mu/2$ with μ under the condition

$$2n - m_v^* \leq \mu \leq m_v^* \text{ and } \mu - m_v^* \in 2\mathbf{Z} \text{ for every } v \in \mathbf{a}. \tag{*}$$

Observe that $m_v^* = 2h_{v\rho} + \ell_v$ and (*) follows from (29.10a, b), since $m_v^* = 2h_v - \ell_v - 2q_v$. Now from [S97, (A2.9.2)] we easily see that $c_h(\mathbf{s}') \in \pi^{d_0}\overline{\mathbf{Q}}^\times$ at $s = \sigma_0$, where d_0 is the complex dimension of $\mathfrak{Z}^\varphi$. As for $\Psi(\mathbf{s})$, by [S97, (23.2.1) and Lemma 23.8], its v-factor in the obvious sense equals

$$\textstyle\prod_{i=1}^{r_v}\prod_{j=1}^{|q_v|}\big(-s + (m_v^*/2) + i - j\big) \tag{**}$$

if $r_v > 0$ and $q_v < 0$. For such a v, we easily see that (**) is nonzero at $s = \sigma_0$ under (29.10a). Examining the case $r_v q_v \geq 0$ in a similar way, we find that $\Psi(\mathbf{s}) \in \mathbf{Q}^\times$ at $s = \sigma_0$ under (29.10a). Clearly $\varepsilon(\sigma_0) \in \overline{\mathbf{Q}}^\times$. Now $H_{b,a}$ is a function of type E_r of (17.23a) obtained from $E_{\mathbf{A}}$, and hence $\Lambda_{\mathfrak{c}}^n(s,\, \chi)H_{b,a}$ is a function of type D_r of (17.24). Therefore, by Theorem 17.12 (v), its value at $s = \mu/2$ for μ as in (*) is an element of $\pi^\beta\mathcal{N}_k^e(\overline{\mathbf{Q}})$, with β and e given there. (We write e instead of t employed there.) This result combined with (29.6) gives

$$\pi^{d_0-\gamma}\mathfrak{q}^{-1}\mathcal{Z}(\sigma_0,\,\mathbf{f},\,\chi)f_b(z) = \sum_{a\in\mathcal{B}} \mathrm{vol}(\mathcal{D}_a)\sum_{i\in I_a}\langle\, h_{a,b,i},\, f_a\,\rangle\, g_{a,b,i}(z) \tag{29.14}$$

with $h_{a,b,i},\, g_{a,b,i} \in \mathcal{N}_h^{e'}(\overline{\mathbf{Q}})$, where $\gamma = \beta + \sum_{v\in\mathbf{a}} r_v q_v$ and e' is as in Lemma 29.2. For the reason explained in §28.2 we have $\mathrm{vol}(\mathcal{D}_a) = \tau_a\pi^{d_0}$ with $0 < \tau_a \in \overline{\mathbf{Q}}$. Let us now assume that φ is anisotropic. Then $\mathcal{D}_a$ is compact, and so, by Proposition 15.7 (3), we can find $h'_{a,b,i} \in \mathcal{S}_h^{\varphi}(\overline{\mathbf{Q}})$ such that $\tau_a h_{a,b,i} - h'_{a,b,i}$ belongs to the set $\mathcal{T}_\omega^p$ in that proposition. Then the right-hand side of (29.14) is $\pi^{d_0}\sum_{a,i}\langle\, h'_{a,b,i},\, f_a\,\rangle\, g_{a,b,i}(z)$.

Let w be a CM-point of $\mathfrak{Z}^{\varphi}$; let $\mathfrak{P}_h(w)$ denote a fixed element of $\mathbf{C}^\times$ that represents the coset $\mathfrak{P}_h(w)$ defined by (11.17a). Then $g_{a,b,i}(w) \in \mathfrak{P}_h(w)\overline{\mathbf{Q}}$ by our definition of $\mathcal{N}_h^{e'}(\overline{\mathbf{Q}})$. Therefore the last sum $\sum_{a,i}$ at $z = w$ can be written $\mathfrak{P}_h(w)\sum_a\langle\, h''_{a,i},\, f_a\,\rangle$ with some $h''_{a,i} \in \mathcal{S}_h^{\varphi}(\overline{\mathbf{Q}})$. By means of the same technique as in Lemma 28.4 we can find an element $\mathbf{j}$ of $\mathcal{S}_h^{\varphi}(D^{\varphi},\,\overline{\mathbf{Q}})$ such that $\sum_a\langle\, h''_{a,i},\, f_a\,\rangle = \langle\, \mathbf{j},\, \mathbf{f}\,\rangle$. Let $\mathbf{g}$ be the projection of $\mathbf{j}$ to $\mathcal{V}(\overline{\mathbf{Q}})$ with respect to the decomposition of $\mathcal{S}_h^{\varphi}(D^{\varphi},\,\overline{\mathbf{Q}})$ mentioned in §29.4. Observe that $\mathbf{g}$ depends on b and w, but it is independent of $\mathbf{f}$. Writing $\mathbf{g}_{b,w}$ for $\mathbf{g}$, we thus have

$$\mathcal{Z}(\sigma_0,\,\mathbf{f},\,\chi)\mathfrak{P}_h(w)^{-1}f_b(w) = \pi^{\gamma}\mathfrak{q}\,\langle\, \mathbf{g}_{b,w},\, \mathbf{f}\,\rangle \tag{29.15}$$

for every $\mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$. Now suppose $m_0 > 2n$. Put $\varepsilon_v = 0$ if $h_v + h_{v\rho} - m_0 \in 2\mathbf{Z}$ and $\varepsilon_v = 1$ if $h_v + h_{v\rho} - m_0 \notin 2\mathbf{Z}$. Define $\ell \in \mathbf{Z}^{\mathbf{a}}$ as follows: $\ell_v = h_v - h_{v\rho} - \varepsilon_v$ if $r_v > 0$ and $\ell_v = m_0 - 2h_{v\rho}$ if $r_v = 0$. By [S97, Lemma 11.14 (3)] we can find χ as in our theorem with this ℓ. We can easily verify that (29.10a, b) are satified with $\sigma_0 = m_0/2$, and so we can consider (29.15) with this χ and $\mu = m_0$. By [S97, Proposition 20.4 (3)], $\mathcal{Z}(m_0/2,\,\mathbf{f},\,\chi) \neq 0$, since $m_0 > 2n$. Therefore (29.15) shows that given $\mathbf{f}$, we can find $(b,\,w)$ so that $\langle\, \mathbf{g}_{b,w},\, \mathbf{f}\,\rangle \neq 0$. Consequently the $\mathbf{g}_{b,w}$ for all $(b,\,w)$ span $\mathcal{V}$ over $\mathbf{C}$, and hence $\mathcal{V}(\overline{\mathbf{Q}})$ over $\overline{\mathbf{Q}}$. Now $\mathfrak{P}_h(w)^{-1}f_b(w) \in \overline{\mathbf{Q}}$, and hence from (29.15) we see that $\langle\, \mathbf{f}',\, \mathbf{f}\,\rangle \in \pi^{-\gamma}\mathfrak{q}^{-1}\mathcal{Z}(m_0/2,\,\mathbf{f},\,\chi)\overline{\mathbf{Q}}$ for every $\mathbf{f},\,\mathbf{f}' \in \mathcal{V}(\overline{\mathbf{Q}})$, from which (29.9) follows immediately.

Returning to an arbitrary σ_0 satisfying (29.10a, b, c), choose $(b,\,w)$ so that $f_b(w) \neq 0$. Dividing (29.15) by $\langle\, \mathbf{f},\, \mathbf{f}\,\rangle$ and using the formula for β in Theorem 17.12 (v), we obtain (29.12) when φ is anisotropic.

Let us next assume that φ is isotropic. Then we cannot use Proposition 15.7. Instead we use Lemma 29.3, which requires that $q_v \geq 0$ for every $v \in \mathbf{a}$. Now $\Lambda_{\mathfrak{c}}^n H_{b,a}$ is D_r of type (17.24) as we already noted, and its value at $s = \mu/2$ is, by (17.27) or (17.30), of the form $\Delta_{\nu\mathbf{a}}^p R$ with $R \in \mathcal{M}_{\nu\mathbf{a}}$. Here $p = (m^* - \nu\mathbf{a})/2$, $\nu = \mu$ if $\mu \geq n$ and $\nu = 2n - \mu$ if $\mu < n$; R is of the form $D_r(\mathfrak{z},\,\nu/2;\,k',\,\chi,\,\mathfrak{c})$ with k' such that $(k'_v + k'_{v\mathbf{a}})_{v\in\mathbf{a}} = \nu\mathbf{a}$. Taking $(k',\,\nu)$ as $(k,\,\mu)$ of Theorem 17.12 (v), we find that $R \in \pi^c\mathcal{M}_{\nu\mathbf{a}}(\overline{\mathbf{Q}})$ with some $c \in \mathbf{Z}$, if we assume (29.10c) and (29.11b). By Lemma 29.3 we can replace $\Lambda_{\mathfrak{c}}^n(\mu/2,\,\chi)(A_q^k H_{b,a})(z,\,w)$ by $\overline{T(z,\,w)}$ with T of the form (29.7). Therefore we can repeat what we did in the anisotropic case, and obtain (29.14) under (29.11a), with $h_{a,b,i} \in \mathcal{M}_h^{\varphi}(\overline{\mathbf{Q}})$. Replacing these by elements of $\mathcal{S}_h^{\varphi}(\overline{\mathbf{Q}})$ by virtue of Theorem 27.14 (which is necessary only when $m \in \mathbf{Za}$), we eventually obtain (29.15). Then we make a special choice $\ell_v = h_v - h_{v\rho} - \varepsilon_v$ with $\varepsilon_v = 0$ or 1. Since (29.11a) is satisfied by such an ℓ, our proof of (29.9) is valid in the present case. Then (29.12) follows immediately from (29.15).

29.6. Theorem. *Let the notation be as in Theorem 26.13 (see also [S97, Theorem 20.7]); let q be as in §26.9. Let $\mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$ and $f \in \mathcal{S}_k^{\varphi}(\overline{\mathbf{Q}})$, where $\mathcal{V}(\overline{\mathbf{Q}})$*

is defined within $\mathcal{S}_k^{\varphi}(D^{\varphi})$ *for a fixed system of eigenvalues* $\{\lambda(\mathfrak{a})\}$ *as in §29.3; put* $m_0 = \mathrm{Min}_{v\in\mathbf{a}'}\{k_v + k_{v\rho}\}$; *suppose that* $m_0 > 2n$ *and* $G_{\mathbf{a}}^{\varphi}$ *is not compact.*

(I) *For* ν *as in Theorem 26.13 (i), the functions* $\pi^{-\beta}E_p(z,\, \nu/2;\, \mathbf{f},\, \chi,\, D^{\psi})$ *is* $\overline{\mathbf{Q}}$*-rational, where* $\beta = \sum_{v\in\mathbf{a}}(m_v - \nu)q/2$, *except when* φ *is isotropic,* $F = \mathbf{Q}$, $\nu = q+n+1$, *and* $\chi_1 = \theta^{\nu}$; *moreover,* $\pi^{-\beta}E_k^{\psi,\varphi}(z,\, \nu/2;\, f,\, \Gamma)$ *with the same* β *is* $\overline{\mathbf{Q}}$*-rational, except when* φ *is isotropic,* $F = \mathbf{Q}$, *and* $\nu = q+n+1$.

(II) *For* ν *as in Theorem 26.13 (ii), the function* $\pi^{-\alpha}\mathfrak{q}^{-1}\langle \mathbf{f},\, \mathbf{f}\rangle^{-1}\mathcal{F}_p(z,\, \nu/2;\, \mathbf{f},\, \chi,\, D^{\psi})$ *is* $\overline{\mathbf{Q}}$*-rational, where* $\alpha = \sum_{v\in\mathbf{a}}(m_v + \nu - n - q + 1)(n+q)/2$ *and* $\mathfrak{q} = p_K\bigl(\sum_{v\in\mathbf{a}} \ell_v\tau_v,\, \sum_{v\in\mathbf{a}} t_v\tau_v\bigr)$, *provided the following two cases are excluded: (i)* $0 \le \nu < q+n$, $\mathfrak{c} = \mathfrak{g}$, *and* $\chi_1 = \theta^{\nu}$; *(ii)* φ *is isotropic,* $F = \mathbf{Q}$, $\nu = q+n+1$, *and* $\chi_1 = \theta^{\nu}$.

PROOF. To prove (I), we evaluate (26.42) or its consequence (26.44) at $s = \nu/2$. First suppose φ is anisotropic. Then, by Proposition 15.7 (3), we can replace h_{ai} of (26.44) by an element of $\mathcal{S}_k^{\varphi}(\overline{\mathbf{Q}})$. If φ is isotropic, we express $H_{p,a}$ in the form $H_{p,a} = \Delta_{\nu\mathbf{a}}^{e}R$ with some e and $R \in \mathcal{M}_{\nu\mathbf{a}}(\overline{\mathbf{Q}})$. This is possible by Theorem 17.12 (i) and (17.27). (Here we have to exclude Case (ii) of (II).) Then we apply Lemma 29.3 and (29.7) to $H_{p,a}$ and eventually find an expression of type (26.44) with $h_{ai} \in \mathcal{M}_k^{\varphi}(\overline{\mathbf{Q}})$. The whole procedure is similar to what was done in the proof of Theorem 29.5. Since $\mathrm{vol}(\mathcal{D}_a) \in \pi^{d_0}\overline{\mathbf{Q}}$, we eventually find, in both isotropic and anisotropic cases, that

$$\mathcal{Z}(\nu/2,\, \mathbf{f},\, \chi)E_p(z,\, \nu/2) = \pi^{\alpha+\gamma}\mathfrak{q}\sum_{a,\, i}\langle h_{ai},\, f_a\rangle\, g_{ai}(z) \tag{29.16}$$

with some $h_{ai} \in \mathcal{M}_k^{\varphi}(\overline{\mathbf{Q}})$ and $g_{ai} \in \mathcal{N}_k^{\psi,t}(\overline{\mathbf{Q}})$, where $\mathfrak{q}$ is as in Lemma 29.2, $\alpha = (n+q)\sum_{v\in\mathbf{a}}(m_v - \nu)/2$, and $\gamma = dn\nu - dn(n-1)/2$. By the same technique as in the proof of Corollary 28.6 we can derive from (29.9) that $\langle h,\, f_a\rangle \in \langle \mathbf{f},\, \mathbf{f}\rangle\overline{\mathbf{Q}}$ for every $h \in \mathcal{M}_k^{\varphi}(\overline{\mathbf{Q}})$ and every $a \in \mathcal{B}$. Also, $\mathcal{Z}(\nu/2,\, \mathbf{f},\, \chi) \neq 0$ for $\nu > 2n$ by [S97, Theorem 20.4 (3)]. Therefore, dividing (29.16) by $\langle \mathbf{f},\, \mathbf{f}\rangle$ and employing (29.12) with $h = k$, we obtain our assertion of (I) on $\pi^{-\beta}E_p(z,\, \ldots,\, D^{\varphi})$, which combined with [S97, Proposition 20.10] proves the result about $\pi^{-\beta}E_k^{\psi,\varphi}(z,\, \ldots,\, \Gamma)$. Assertion (II) follows from (26.43) combined with Theorem 17.12 (v) in a similar way.

29.7. Theorem. *The notation being as in §29.4, suppose that* φ *is totally definite; let* $\mathbf{f} \in \mathcal{V}(\overline{\mathbf{Q}})$.

(I) *Let* σ_0 *be an element of* $2^{-1}\mathbf{Z}$ *such that*

$$2n - (2h_{v\rho} + \ell_v) \le 2\sigma_0 \le 2h_{v\rho} + \ell_v \quad \text{and} \quad 2\sigma_0 - \ell_v \in 2\mathbf{Z} \text{ for every } v \in \mathbf{a}. \tag{29.17}$$

Then, under (29.10c), $\mathcal{Z}(\sigma_0,\, \mathbf{f},\, \chi) \in \pi^{\beta}p_K\bigl(\sum_{v\in\mathbf{a}}(2h_{v\rho} + \ell_v)\tau_v,\, n\tau\bigr)\overline{\mathbf{Q}}$, *where* $\beta = dn\sigma_0 - dn(n-1)/2 + (n/2)\sum_{v\in\mathbf{a}}(2h_{v\rho} + \ell_v)$.

(II) *Assertions (I) and (II) of Theorem 29.6 are true in the present case, if we replace* $\langle \mathbf{f},\, \mathbf{f}\rangle$ *there by* $p_K\bigl(\sum_{v\in\mathbf{a}} 2k_{v\rho}\tau_v,\, n\tau\bigr)$ *and put* $t_v = n$ *for every* $v \in \mathbf{a}$.

PROOF. Define $k \in \mathbf{Z}^{\mathbf{b}}$ by $k_{v\rho} = h_{v\rho}$ and $k_v = h_{v\rho} + \ell_v$ for $v \in \mathbf{a}$. Then $\ell = (k_v - k_{v\rho})_{v\in\mathbf{a}}$. Moreover, by (29.2), $\mathcal{S}_k^{\varphi}(D^{\varphi}) = \mathcal{S}_h^{\varphi}(D^{\varphi})$, and the Hecke operators on $\mathcal{S}_h^{\varphi}(D^{\varphi})$ stay the same by the change of h for k; also, $\mathcal{M}_k(\overline{\mathbf{Q}}) = \mathcal{M}_h(\overline{\mathbf{Q}})$ by (26.41c). Therefore our calculations of [S97, §22.4 through §22.11] are valid with the present k. In particular, by [S97, (22.11.3)] we have

$$C'(s)\mathcal{Z}(s,\, \mathbf{f},\, \chi)f_b = \sum_{a\in\mathcal{B}} c_a f_a \Lambda_{\mathfrak{c}}^{n}(s,\, \chi)(H_{b.a})^{\circ}(\mathbf{i},\, \mathbf{i};\, s), \tag{*}$$

where $c_a \in \overline{\mathbf{Q}}$, C' is as in [S97, (22.6.5)], and $\Lambda_{\mathfrak{c}}^n$ is the function of (20.20) in Case UT. (See [S97, (20.3.3)]; also we may assume, without changing $\mathcal{Z}$, that the conductor of χ divides $\mathfrak{c}$ as noted in §26.10.) Put $\mu = 2\sigma_0$ and $M(\mathfrak{z}, s) = \Lambda_{\mathfrak{c}}^n(s, \chi)H_{b,a}(\mathfrak{z}, s)$; notice that $k_v + k_{v\rho} = 2h_{v\rho} + \ell_v$. For the same reason as in the proof of Theorem 26.13, we can apply Theorem 17.12 (v) to $M(\mathfrak{z}, \mu/2)$ to find that under (29.17) and (29.10c), $M(\mathfrak{z}, \mu/2)$ belongs to $\pi^\beta \mathcal{N}_k^p(\overline{\mathbf{Q}})$ with $\beta = (n/2)\sum_{v\in\mathbf{a}}(m_v + \mu) - dn(n-1)/2$ and some p. By [S97, (22.6.5)], $C'(\mu/2) \in \overline{\mathbf{Q}}^\times$. By Lemma 26.12, $M^\circ(\mathbf{i}, \mathbf{i}; \mu/2) \in \pi^\beta p_K\bigl(\sum_{v\in\mathbf{a}}(2h_{v\rho}+\ell_v)\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$. Take b so that $f_b \neq 0$; then $f_a/f_b \in \overline{\mathbf{Q}}$ by (26.41c). Therefore we obtain (I) from (*). Assertion (II) can be obtained from [S97, (22.11.2)] and Lemma 26.12 in a similar way.

29.8. Corollary. *The notation being as in Theorems 29.6 and 29.7, suppose that* $\mathbf{f}$ *and* f *are* $\overline{\mathbf{Q}}$*-rational, and* $m = \nu\mathbf{a}$ *with an integer* $\nu \geq \mathrm{Max}\{2n+1, n+q\}$; *suppose also that* $\nu \neq n+q+1$ *if* $F = \mathbf{Q}$ *and* φ *is isotropic. Then* $E_k^{\psi,\varphi}(z, \nu/2; f, \Gamma)$ *and* $E_p(z, \nu/2; \mathbf{f}, \chi, D^\varphi)$ *are* $\overline{\mathbf{Q}}$*-rational.*

This is an immediate consequence of those two theorems.

29.9. *Proof of Theorem 27.16 (1) in Case UB.* Our reasoning is similar to that of §28.12. We have $(W, \psi) = (V, \varphi) \oplus (H_q, \eta_q')$ and $(V, \varphi) = (Z, \zeta) \oplus (H_r, \eta_r')$ with $r = l - q$ as in §26.4 and Theorem 26.13; thus $\dim(W) = n + 2q$ with $n = \dim(V) = 2r + \dim(Z)$, and the lowest dimensional case is $\dim(W) = 3$. We may assume that $\dim(Z) > 0$, since if $Z = \{0\}$, then our group is reduced to Case UT. Also, there is no problem if $W = Z$, and so we assume that $l > 0$. Since ζ is anisotropic, we have Theorems 29.5, 29.6, and 29.7 (without employing the map $\mathfrak{q}$ of Theorem 27.14) for the forms on G^ζ if $\mu > 2\dim(Z)$. Then we obtain Corollary 29.8 for $E_k^{\psi,\zeta}$, that is, Theorem 27.16 (1) for $r = 0$. This establishes Theorem 27.16 in Case UB when $\dim(W) = 3$. Now we make the induction assumption that the theorem is true for G^φ with $\dim(V) < \dim(W)$. Let $0 < r < l$ and let μ be as in Theorem 27.15. Our asumption guarantees the map $\mathfrak{q}$ for the forms on G^φ, since $\mu > 2n$. Then Theorems 29.5 and 29.6 are valid for the forms on G^φ with such a μ. Consequently we obtain Corollary 29.8 for $E_k^{\psi,\varphi}$, that is, Theorem 27.16 (1) for such r and μ. This completes the proof.

29.10. Remark. (A) In the above we stated the case of totally definite φ separately, but we can state the results for anisotropic φ uniformly, and view the totally definite case as a special case. The only point we must remember in that case is that $\mathcal{M}_k(\overline{\mathbf{Q}}) = p_K\bigl(\sum_{v\in\mathbf{a}} k_{v\rho}\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$ as stated in (26.41c), and consequently $\langle \mathbf{f}, \mathbf{f}\rangle \in p_K\bigl(\sum_{v\in\mathbf{a}} 2k_{v\rho}\tau_v, n\tau\bigr)\overline{\mathbf{Q}}$. Also, the condition $m_0 > 2n$ does not apply to the totally definite case.

(B) In Theorem 27.16 we stated a result of the type given in the above corollary, but the case $\nu = n + q \geq 2n+1$ proved in the corollary is not included in that theorem. Also, the case in which $\nu = n + q + 1 \geq 2n + 1$, $F = \mathbf{Q}$, and φ is anisotropic is included in the above corollary, but not in Theorem 27.16.

(C) We assumed (29.11a) when φ is isotropic, but probably this is unnecessary, since Lemma 29.3 is likely to be true without the condition that $q_v \geq 0$ for every v. At least the case $G^\varphi = U(\eta)$ is covered by Theorem 28.8. We can also handle some cases by means of Lemma 28.2 without assuming (29.11a). This method restricts σ_0 to a certain range, which is not large, but often nonempty. We leave the precise statement to the reader, as it is an easy exercise.

APPENDIX

A1. The series associated to a symmetric matrix and Gauss sums

A1.1. In this section we give some results necessary for the explicit calculations of the factors of automorphy of half-integral weight, and also prove the part of Theorem 16.2 concerning A_ζ^1 and f_ζ^1.

We consider F_v with a fixed v in $\mathbf{h}$ prime to 2. For simplicity we drop the subscript v. Thus F denotes a finite algebraic extension of $\mathbf{Q}_p$ with an odd prime number p, $\mathfrak{g}$ the integral closure of $\mathbf{Z}_p$ in F, and $\mathfrak{p}$ the maximal ideal of $\mathfrak{g}$. We denote by $\mathfrak{d}$ the different of F relative to $\mathbf{Q}_p$ and by π an unspecified prime element of F; also we put $q = [\mathfrak{g} : \mathfrak{p}]$. Let $\mathbf{e}_v$ be the $\mathbf{T}$-valued character of the additive group F defined in §1.6. Here, to avoid a possible confusion, we use $\mathbf{e}_v$ without dropping the subscript v. Recall that $\mathfrak{d}^{-1} = \left\{ a \in F \,\middle|\, \mathbf{e}_v(a\mathfrak{g}) = 1 \right\}$. We define the quadratic residue symbol $\left(\dfrac{c}{\mathfrak{p}}\right)$ for $c \in \mathfrak{g}$ as usual by the property that

(A1.1) $$1 + \left(\frac{c}{\mathfrak{p}}\right) = \text{the number of } x \pmod{\mathfrak{p}} \text{ such that } x^2 - c \in \mathfrak{p}.$$

We put then $\left(\dfrac{c}{\mathfrak{p}^k}\right) = \left(\dfrac{c}{\mathfrak{p}}\right)^k$ for every $k \in \mathbf{Z}$.

Given $a \in F^\times$ such that $a\mathfrak{d} = \mathfrak{p}^{-m}$ with $0 \le m \in \mathbf{Z}$, we put

(A1.2) $$\tau(a) = \sum_{x \in \mathfrak{g}/\mathfrak{p}^m} \mathbf{e}_v(ax^2).$$

This is well-defined.

A1.2. Lemma.
$$\tau(a) = \begin{cases} q^{m/2} & \text{if } \ m \in 2\mathbf{Z}, \\ q^{(m-1)/2} \displaystyle\sum_{y \in \mathfrak{g}/\mathfrak{p}} \left(\frac{y}{\mathfrak{p}}\right) \mathbf{e}_v(\pi^{m-1}ay) & \text{if } \ m \notin 2\mathbf{Z}. \end{cases}$$

PROOF. Assuming $m \ge 2$, we have
$$\begin{aligned}\tau(a) &= \sum_{y \in \mathfrak{g}/\mathfrak{p}^{m-1}} \sum_{z \in \mathfrak{g}/\mathfrak{p}} \mathbf{e}_v\big(a(y + \pi^{m-1}z)^2\big) \\ &= \sum_{y \in \mathfrak{g}/\mathfrak{p}^{m-1}} \mathbf{e}_v(ay^2) \sum_{z \in \mathfrak{g}/\mathfrak{p}} \mathbf{e}_v(2a\pi^{m-1}yz),\end{aligned}$$
since $a\pi^{2m-2} \in \mathfrak{d}^{-1}$. The last sum over z is nonzero only if $y \in \mathfrak{p}$, in which case the sum is q. Thus $\tau(a) = q\tau(\pi^2 a)$ if $m \ge 2$. Repeating this procedure, we

obtain $\tau(a) = q^{m/2}$ if m is even, since $\tau(a) = 1$ if $m = 0$. If m is odd, we have $\tau(a) = q^{(m-1)/2}\tau(\pi^{m-1}a)$. Putting $b = \pi^{m-1}a$ and employing (A1.1), we see that

$$\tau(b) = \sum_{y \in \mathfrak{g}/\mathfrak{p}} \left\{1 + \left(\frac{y}{\mathfrak{p}}\right)\right\} \mathbf{e}_v(by) = \sum_{y \in \mathfrak{g}/\mathfrak{p}} \left(\frac{y}{\mathfrak{p}}\right) \mathbf{e}_v(by).$$

This completes the proof.

A1.3. Lemma. *If* $a\mathfrak{d} = \mathfrak{p}^{-m}$ *as above, we have:* (1) $\tau(a)^2 = q^m \left(\frac{-1}{\mathfrak{p}}\right)^m$; (2) $\tau(ca) = \tau(a)\left(\frac{c}{\mathfrak{p}}\right)^m$ *if* $c \in \mathfrak{g}^\times$.

PROOF. The first equality for even m and the second one for an arbitrary m follow easily from Lemma A1.2. Put $\psi(y) = \left(\frac{y}{\mathfrak{p}}\right)$. Assuming $m = 1$, we have

$$\begin{aligned}\tau(a)\overline{\tau(a)} &= \sum_{y \in (\mathfrak{g}/\mathfrak{p})^\times} \psi(y)\mathbf{e}_v(ay) \sum_{z \in (\mathfrak{g}/\mathfrak{p})^\times} \psi(z)^{-1}\mathbf{e}_v(-az) \\ &= \sum_{y,z \in (\mathfrak{g}/\mathfrak{p})^\times} \psi(yz^{-1})\mathbf{e}_v\big(a(y-z)\big) = \sum_{x \in (\mathfrak{g}/\mathfrak{p})^\times} \psi(x) \sum_{z \in (\mathfrak{g}/\mathfrak{p})^\times} \mathbf{e}_v\big(az(x-1)\big).\end{aligned}$$

The last sum over z is $q-1$ or -1 according as $x \equiv 1$ or $x \not\equiv 1 \pmod{\mathfrak{p}}$. Thus $|\tau(a)|^2 = q - \sum_{x \in (\mathfrak{g}/\mathfrak{p})^\times} \psi(x) = q$. Clearly $\overline{\tau(a)} = \psi(-1)\tau(a)$, and hence $\tau(a)^2 = \psi(-1)q$ if $m = 1$. This together with Lemma A1.2 proves (1) for odd m.

A1.4. We now define symbols S, L, Λ, $\gamma(s)$, and $\omega(s)$ as follows:

(A1.3) $$S = S^n = \left\{ x \in F_n^n \,\middle|\, {}^t x = x \right\},$$

(A1.4) $$L = \mathfrak{g}_n^1, \qquad \Lambda = \Lambda^n = S^n \cap \mathfrak{g}_n^n,$$

(A1.5) $$\gamma(s) = \int_L \mathbf{e}_v(xs \cdot {}^t x/2)dx \qquad (s \in S),$$

(A1.6) $$\omega(s) = \nu(\delta s)^{1/2}\gamma(s) \qquad (s \in S).$$

Here dx is the Haar measure of F_n^1 such that $\int_L dx = 1$, and δ is an element of $\mathfrak{g}$ such that $\mathfrak{d} = \delta\mathfrak{g}$; $\nu(\ )$ is defined by (1.16). (In the next section we use a different measure.) In Lemma A1.6 (4) below we shall show that $\omega(s) = \gamma(s)/|\gamma(s)|$, which is consistent with (16.6). If $n = 1$, $s \in F^\times$, and $s\mathfrak{d} = \mathfrak{p}^{-m}$ with $0 \le m \in \mathbf{Z}$, then

(A1.7) $$\tau(s/2) = q^m \int_{\mathfrak{g}} \mathbf{e}_v(sx^2/2)dx = q^m\gamma(s).$$

A1.5. Lemma. (1) *Given* $\sigma \in S^n$, *there exists an element* α *of* $GL_n(\mathfrak{g})$ *such that* $\alpha\sigma \cdot {}^t\alpha$ *is diagonal.*

(2) *If* $\sigma \in S^n \cap GL_n(\mathfrak{g})$ *and* $n > 1$, *then* $x\sigma \cdot {}^t x = 1$ *for some* $x \in \mathfrak{g}_n^1$.

(3) *If* $\sigma, \tau \in S^n \cap GL_n(\mathfrak{g})$ *and* $\det(\sigma)/\det(\tau)$ *is a square in* $\mathfrak{g}/\mathfrak{p}$, *then* $\sigma = {}^t\alpha\tau\alpha$ *with some* $\alpha \in GL_n(\mathfrak{g})$.

PROOF. Since $2x\sigma \cdot {}^t y = (x+y)\sigma \cdot {}^t(x+y) - x\sigma \cdot {}^t x - y\sigma \cdot {}^t y$ and $2 \notin \mathfrak{p}$, the ideal generated by $x\sigma \cdot {}^t y$ for all $x, y \in L$ coincides with the ideal generated by $z\sigma \cdot {}^t z$ for all $z \in L$. Thus, to prove (1), excluding the trivial case $\sigma = 0$ and multiplying σ by an element of $F^\times$, we may assume that this ideal is $\mathfrak{g}$. Take $z \in L$ so that $z\sigma \cdot {}^t z \in \mathfrak{g}^\times$ and put $M = \left\{ y \in L \,\middle|\, y\sigma \cdot {}^t z = 0 \right\}$. Given $x \in L$, put $y = x - (x\sigma \cdot {}^t z)(z\sigma \cdot {}^t z)^{-1}z$. Then $y \in M$ and thus $L = \mathfrak{g}z \oplus M$. This means that for some $\alpha \in GL_n(\mathfrak{g})$ we have $\alpha\sigma \cdot {}^t\alpha = \mathrm{diag}[z\sigma \cdot {}^tz,\ \tau]$ with $\tau \in \Lambda^{n-1}$. Applying induction to τ, we obtain (1).

Clearly it is sufficient to prove (2) for diagonal σ of size 2. Thus put $\sigma = \mathrm{diag}[a,\ d]$ with $a,\ d \in \mathfrak{g}^\times$. Suppose $d/a = -c^2$ with $c \in \mathfrak{g}^\times$. Then for $r,\ s \in \mathfrak{g}$ we have $ar^2 + ds^2 = a(r - cs)(r + cs)$. Since $2 \notin \mathfrak{p}$ we can find $r,\ s \in \mathfrak{g}$ so that $r+cs = 1$ and $r-cs = a^{-1}$. Then $ar^2+ds^2 = 1$ as desired. If $-d/a$ is not a square, let $K = F(\xi)$ with $\xi^2 = -d/a$. Then K is an unramified quadratic extension of F, and the integral closure of $\mathfrak{g}$ in K is $\mathfrak{g}[\xi]$; moreover $a \cdot N_{K/F}(r + s\xi) = ar^2 + ds^2$. It is well- known that $N_{K/F}\big(\mathfrak{g}[\xi]^\times\big) = \mathfrak{g}^\times$, and hence $N_{K/F}(r+s\xi) = a^{-1}$ for some $r,\ s \in \mathfrak{g}$. This completes the proof of (2).

To prove (3), we first note that an element of $\mathfrak{g}^\times$ is a square if and only if it is a square modulo $\mathfrak{p}$, since $2 \notin \mathfrak{p}$. Let σ and τ be as in (3). Our assertion is trivial if $n = 1$; thus we assume $n > 1$. By (2) we have $x\sigma \cdot {}^tx = 1$ for some $x \in L$. Then the above proof of (1) shows that $\alpha\sigma \cdot {}^t\alpha = \mathrm{diag}[1,\ \sigma']$ with some $\alpha \in GL_n(\mathfrak{g})$ and some $\sigma' \in S^{n-1}$. Clearly $\sigma' \in GL_{n-1}(\mathfrak{g})$. For the same reason we may assume that $\tau = \mathrm{diag}[1,\ \tau']$ with $\tau' \in S^{n-1} \cap GL_{n-1}(\mathfrak{g})$. Then $\det(\sigma')/\det(\tau')$ is a square. Applying our induction to σ' and τ', we obtain (3).

A1.6. Lemma. (1) $\gamma(cs) = \gamma(s)\left(\dfrac{c}{\nu_0(\delta s)}\right)$ *if* $c \in \mathfrak{g}^\times$, *where* ν_0 *is defined in* §*1.7;* (2) $\omega(s)^2 = \left(\dfrac{-1}{\nu_0(\delta s)}\right)$; (3) $|\gamma(s)| = \nu(\delta s)^{-1/2}$; (4) $\omega(s) = \gamma(s)/|\gamma(s)|$; (5) $\gamma(s) = \gamma(s + b)$ *and* $\omega(s) = \omega(s + b)$ *if* $\delta b \in \Lambda$; (6) $\gamma(-s) = \overline{\gamma(s)}$ *and* $\omega(-s) = \omega(s)^{-1}$.

PROOF. By Lemma A1.5 (1) we may assume that $s = \mathrm{diag}[s_1,\ \dots,\ s_n]$ with $s_i \in F$; we may also assume that $s_i \in \mathfrak{d}^{-1}$ if and only if $i > r$ with some $r \le n$. Put $s_i\mathfrak{d} = \mathfrak{p}^{-m_i}$ with $0 < m_i \in \mathbf{Z}$ for $i \le r$ and $\lambda = \sum_{i=1}^r m_i$. Then $\nu_0(\delta s) = \mathfrak{p}^\lambda$, and

$$\gamma(s) = \prod_{i=1}^r \int_{\mathfrak{g}} \mathbf{e}_v(s_ix^2/2)dx = \prod_{i=1}^r q^{-m_i}\tau(s_i/2) = \nu(\delta s)^{-1}\prod_{i=1}^r \tau(s_i/2)$$

by (A1.7). (If $r = 0$, then $\nu_0(\delta s) = \mathfrak{g}$ and $\gamma(s) = 1$.) Therefore, from Lemma A1.3 (2) we obtain (1). Also, by Lemma A1.3 (1) we have $\gamma(s)^2 = \nu(\delta s)^{-1}\left(\dfrac{-1}{\nu_0(\delta s)}\right)$, which proves (2) and (3). Then $|\omega(s)| = 1$, which implies (4). Clearly $\gamma(s) = \gamma(s + b)$ and $\nu(\delta s) = \nu\big(\delta(s + b)\big)$ if $\delta b \in \Lambda$. Therefore we obtain (5); the formulas of (6) are obvious.

A1.7. Given $\zeta \in \Lambda^n$, we define two infinite series $\alpha^0(\zeta,\ s)$ and $\alpha^1(\zeta,\ s)$ by

$$\alpha^0_\zeta(s) = \alpha^0(\zeta,\ s) = \sum_{\sigma \in S/\Lambda} \mathbf{e}^n_v(-\delta^{-1}\zeta\sigma)\nu(\sigma)^{-s}, \tag{A1.8}$$

$$\alpha^1_\zeta(s) = \alpha^1(\zeta,\ s) = \sum_{\sigma \in S/\Lambda} \omega(\delta^{-1}\sigma)\mathbf{e}^n_v(-\delta^{-1}\zeta\sigma)\nu(\sigma)^{-s}. \tag{A1.9}$$

The sums are formally meaningful in view of Lemma A1.6 (5). These are the same as α^i_v of (16.7a, b). The series α^0_ζ was investigated in [S97, §§13-15]. (In this section we assume $v \nmid 2$, and hence T^n of [S97, (13.1.4)] coincides with the present Λ^n.) The purpose of the remaining part of this section is to determine α^1_ζ as a rational function of q^{-s} as stated in Theorem 16.2.

For positive integers m and n we put

$$\Lambda_m(n) = \mathfrak{p}^{-m}\Lambda/\Lambda. \tag{A1.10}$$

Define formal power series $A_\zeta^i(t)$ in an indeterminate t as in Theorem 16.2, and using the symbol $e(\sigma)$ introduced there, define also $A_\zeta^{1,m}(t)$ by

$$A_\zeta^{1,m}(t) = \sum_{\sigma\in\Lambda_m(n)} \omega(\delta^{-1}\sigma)\mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)t^{e(\sigma)} \qquad (0 < m \in \mathbf{Z}). \tag{A1.11}$$

Clearly $A_\zeta^{1,m}(t)$ is a polynomial in t and $\lim_{m\to\infty} A_\zeta^{1,m}(t) = A_\zeta^1(t)$ if A_ζ^1 is convergent at t. In fact, α_ζ^0 is convergent for sufficiently large $\mathrm{Re}(s)$ as observed in [S97, p.104, lines 1-3], and hence the same is true for $\alpha_\zeta^1(s)$, since $|\omega(\delta^{-1}\sigma)| = 1$.

To determine α_ζ^1, we need the number $N_m(\psi,\,\varphi)$ defined for $\varphi \in \Lambda^n$ and $\psi \in \Lambda^h$ defined by

$$N_m(\psi,\,\varphi) = \#\left\{\,\rho_m(x)\,\middle|\, x \in \mathfrak{g}_n^h,\ {}^t x\psi x - \varphi \prec \mathfrak{p}^m\,\right\}, \tag{A1.12}$$

where we write $X \prec \mathfrak{p}^m$ if a matrix X has entries in $\mathfrak{p}^m$ (as we did in §1.8) and ρ_m is the natural map of $\mathfrak{g}_n^h$ to $\mathfrak{g}_n^h/(\mathfrak{p}^m)_n^h$ with any h and n.

A1.8. Lemma. *$N_m(1_h, \mathrm{diag}[1,\,\sigma]) = N_m(1_h,\,1)N_m(1_{h-1},\,\sigma)$ for every $\sigma \in \Lambda^n$.*

PROOF. Suppose ${}^t xx - \mathrm{diag}[1,\,\sigma] \prec \mathfrak{p}^m$ with $x \in \mathfrak{g}_{n+1}^h$; put $x = [u\quad y]$ with $u \in \mathfrak{g}_1^h$ and $y \in \mathfrak{g}_n^h$. Then

$$(*) \qquad {}^t uu - 1 \in \mathfrak{p}^m, \quad {}^t uy \prec \mathfrak{p}^m, \quad {}^t yy - \sigma \prec \mathfrak{p}^m.$$

The number of $\rho_m(u)$ for u satisfying the first relation is $N_m(1_h,\,1)$. For each fixed u we are going to show that the number of $\rho_m(y)$ for all y satisfying the last two relations of (*) is $N_m(1_{h-1},\,\sigma)$. Then we obtain our lemma. To do this, fix u, and take $a \in GL_h(\mathfrak{g})$ so that ${}^t ua = [b\quad 0_{h-1}^1]$ with $b \in \mathfrak{g}$. Let c be the upper left entry of $a^{-1}\cdot{}^t a^{-1}$ and d the lower right submatrix of ${}^t aa$ of size $h-1$; let $w = a^{-1}y$ and let z (resp. z') be the lower $h-1$ rows (resp. the top row) of w. Then $b^2c - 1 \in \mathfrak{p}^m$, and y satisfies the last two relations of (*) if and only if $z' \prec \mathfrak{p}^m$ and ${}^t zdz - \sigma \prec \mathfrak{p}^m$. Thus the number of $\rho_m(y)$ equals $N_m(d,\,\sigma)$. Now $\det(d) = c\det(a)^2$ by Lemma 1.3 (1). Since $b^2c - 1 \in \mathfrak{p}^m$, this means that $\det(d)$ is a square of a unit, and hence $d = {}^t ee$ with some $e \in GL_{h-1}(\mathfrak{g})$ by Lemma A1.5 (3). Thus $N_m(d,\,\sigma) = N_m(1_{h-1},\,\sigma)$, which completes the proof.

A1.9. *Proof of the part of Theorem 16.2 concerning A_ζ^1 and f_ζ^1.* Let $0 < k \in \mathbf{Z}$ and $\sigma \in \mathfrak{p}^{-m}\Lambda$. Taking the k-th power of (A1.5) and (A1.6), we obtain

$$\nu(\sigma)^{-k/2}\omega(\delta^{-1}\sigma)^k = q^{-mnk} \sum_{x\in(\mathfrak{g}/\mathfrak{p}^m)_n^k} \mathbf{e}_v^k(\delta^{-1}x\sigma\cdot{}^t x/2),$$

where $\mathbf{e}_v^k(y) = \mathbf{e}_v(\mathrm{tr}(y))$ for a matrix y of size k. Therefore, for $\zeta \in \Lambda$ we have

$$\begin{aligned}
&\sum_{\sigma\in\Lambda_m(n)} \mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)\omega(\delta^{-1}\sigma)^k\nu(\sigma)^{-k/2}\\
&\qquad = q^{-mnk} \sum_{\sigma\in\Lambda_m(n)}\ \sum_{x\in(\mathfrak{g}/\mathfrak{p}^m)_n^k} \mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)\mathbf{e}_v^k(\delta^{-1}x\sigma\cdot{}^t x/2)\\
&\qquad = q^{-mnk} \sum_{x\in(\mathfrak{g}/\mathfrak{p}^m)_n^k}\ \sum_{\sigma\in\Lambda_m(n)} \mathbf{e}_v^n\big((2\delta)^{-1}({}^t xx - 2\zeta)\sigma\big)\\
&\qquad = q^{mn(n+1)/2-mnk}N_m(1_k,\,2\zeta).
\end{aligned}$$

Let $A_\zeta^{0,m}(t)$ be the right-hand side of (A1.11) without the factor $\omega(\delta^{-1}\sigma)$. Put $\theta = \left(\dfrac{-1}{\mathfrak{p}}\right)$. By Lemma A1.6 (2), $\theta^{he(\sigma)} = \omega(\delta^{-1}\sigma)^{2h}$, and hence

$$\begin{aligned} A_\zeta^{0,m}(\theta^h q^{-h}) &= \sum_{\sigma\in\Lambda_m(n)} \mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)\theta^{he(\sigma)}q^{-he(\sigma)} \\ &= \sum_{\sigma\in\Lambda_m(n)} \mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)\omega(\delta^{-1}\sigma)^{2h}\nu(\sigma)^{-h}. \end{aligned}$$

Combining this with the above result, we obtain, for sufficiently large h,

$$\begin{aligned} A_\zeta^0(\theta^h q^{-h}) &= \lim_{m\to\infty} A_\zeta^{0,m}(\theta^h q^{-h}) \\ &= \lim_{m\to\infty} q^{mn(n+1)/2-2mnh} N_m(1_{2h},\, 2\zeta). \end{aligned}$$

Similarly, taking $k = 2h+1$, we obtain

$$A_\zeta^{1,m}(\theta^h q^{-h-1/2}) = \sum_{\sigma\in\Lambda_m(n)} \mathbf{e}_v^n(-\delta^{-1}\zeta\sigma)\omega(\delta^{-1}\sigma)^{2h+1}\nu(\sigma)^{-h-1/2},$$

so that

$$A_\zeta^1(\theta^h q^{-h-1/2}) = \lim_{m\to\infty} q^{mn(n+1)/2-mn(2h+1)} N_m(1_{2h+1},\, 2\zeta).$$

Putting $\varepsilon = 1/2$ and $\tau = \mathrm{diag}[\varepsilon,\, \zeta]$, by Lemma A1.8 we have

$$\begin{aligned} A_\zeta^1(\theta^h q^{-h-1/2}) &= \lim_{m\to\infty} \frac{q^{m(n+1)(n+2)/2-m(n+1)(2h+2)} N_m(1_{2h+2},\, 2\tau)}{q^{m-m(2h+2)} N_m(1_{2h+2},\, 1)} \\ &= A_\tau^0(\theta^{h+1}q^{-h-1})/A_\varepsilon^0(\theta^{h+1}q^{-h-1}). \end{aligned}$$

Since this holds for infinitely many h, we have $A_\zeta^1(t) = A_\tau^0(\theta q^{-1/2}t)/A_\varepsilon^0(\theta q^{-1/2}t)$. Let $\zeta = \mathrm{diag}[\xi,\, 0]$ with $\xi \in \widetilde{S}_v^r \cap GL_r(F)$. By [S97, Theorem 13.6], $A_\varepsilon^0(t) = 1-t$ and $A_\tau^0 = f_\tau g_\tau$ with a polynomial g_τ with coefficients in $\mathbf{Z}$ whose constant term is 1 and a rational function f_τ given as follows:

$$f_\tau(t) = \frac{(1-t)\prod_{i=1}^{[(n+1)/2]}(1-q^{2i}t^2)}{\left(1-\lambda q^{(2n+1-r)/2}t\right)\prod_{i=1}^{[(n-r)/2]}(1-q^{2n+2-r-2i}t^2)} \quad \textit{if } r \textit{ is odd},$$

$$f_\tau(t) = \frac{(1-t)\prod_{i=1}^{[(n+1)/2]}(1-q^{2i}t^2)}{\prod_{i=1}^{[(n-r+1)/2]}(1-q^{2n+3-r-2i}t^2)} \quad \textit{if } r \textit{ is even},$$

where $\lambda = \lambda(\mathrm{diag}[\varepsilon,\, \xi])$ with the symbol $\lambda(\ )$ defined in §16.1. If r is odd, we see that $\lambda = \theta\lambda(\xi)$; if further $\xi \in GL_r(\mathfrak{g})$, then $\mathrm{diag}[\varepsilon,\, \xi] \in GL_{r+1}(\mathfrak{g})$, and so $g_\tau = 1$ as noted in [S97, Theorem 13.6]. If $\zeta = 0$, then $\tau = \mathrm{diag}[\varepsilon,\, 0]$, and so $g_\tau = 1$ by the same theorem. Therefore substituting $\theta q^{-1/2}t$ for t in the formulas for f_τ, we obtain our assertions concerning A_ζ^1 and f_ζ^1 of Theorem 16.2.

A2. Metaplectic groups and factors of automorphy

A2.1. Our basic field is a totally real algebraic number field F and we employ the symbols $\mathbf{a}$, $\mathbf{h}$, $\mathbf{v}$, and $\mathfrak{g}$ introduced in §1.4; we denote by $\mathfrak{d}$ the different of F relative to $\mathbf{Q}$. For $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R_{2n}^{2n}$ with a, b, c, and d in R_n^n, we put $a = a_x$, $b = b_x$, $c = c_x$, and $d = d_x$ if there is no fear of confusion.

With a positive integer n we now put

$$X = F_n^1, \qquad L = \mathfrak{g}_n^1, \qquad L^* = \mathfrak{d}^{-1}L, \qquad \eta = \eta_n = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix},$$
$$G = Sp(n, F), \qquad P = \{\, \alpha \in G \mid c_\alpha = 0 \,\},$$
$$\Omega_{\mathbf{A}} = \{\, x \in G_{\mathbf{A}} \mid \det(c_x) \in F_{\mathbf{A}}^\times \,\}, \qquad \Omega_v = \{\, x \in G_v \mid \det(c_x) \neq 0 \,\}.$$

(The present G is G of (3.26) in Case SP.) We let G act on $X \times X = F_{2n}^1$ by right multiplication. Then we can define *the metaplectic groups* $Mp(X_{\mathbf{A}})$ and $Mp(X_v)$ for each $v \in \mathbf{v}$ in the sense of [W64] with respect to the alternating form $(x, y) \mapsto x\eta_n \cdot {}^t y$ on $F_{2n}^1 \times F_{2n}^1$. Recall that these groups, written $M_{\mathbf{A}}$ and M_v for simplicity, are groups of unitary transformations on $L^2(X_{\mathbf{A}})$ and on $L^2(X_v)$; there are exact sequences

(A2.1a) $$1 \longrightarrow \mathbf{T} \longrightarrow M_{\mathbf{A}} \longrightarrow G_{\mathbf{A}} \longrightarrow 1,$$

(A2.1b) $$1 \longrightarrow \mathbf{T} \longrightarrow M_v \longrightarrow G_v \longrightarrow 1 \qquad (v \in \mathbf{v}).$$

We denote by pr the projection maps of $M_{\mathbf{A}}$ and M_v to $G_{\mathbf{A}}$ and G_v. There is a natural lift $r : G \to M_{\mathbf{A}}$ by which we can consider G a subgroup of $M_{\mathbf{A}}$. There are also two types of lifts

(A2.2) $$r_P : P_{\mathbf{A}} \to M_{\mathbf{A}}, \qquad r_\Omega : \Omega_{\mathbf{A}} \to M_{\mathbf{A}},$$

which satisfy the formulas

(A2.3a) $$[r_P(\alpha)f](x) = |\det(a_\alpha)|_{\mathbf{A}}^{1/2}\mathbf{e}_{\mathbf{A}}(xa_\alpha \cdot {}^t b_\alpha \cdot {}^t x/2)f(xa_\alpha) \quad \textit{if} \quad \alpha \in P_{\mathbf{A}},$$

(A2.3b) $$r_\Omega(\alpha\beta\gamma) = r_P(\alpha)r_\Omega(\beta)r_P(\gamma) \quad \textit{if} \quad \alpha, \gamma \in P_{\mathbf{A}} \quad \textit{and} \quad \beta \in \Omega_{\mathbf{A}},$$

(A2.3c) $$[r_\Omega(\beta)f](x) = |\det(c_\beta)|_{\mathbf{A}}^{1/2} \int_{X_{\mathbf{A}}} f(xa_\beta + yc_\beta)\mathbf{e}_{\mathbf{A}}\big(q_\beta(x, y)\big)dy \ \textit{ if } \ \beta \in \Omega_{\mathbf{A}},$$
$$q_\beta(x, y) = (1/2)xa_\beta \cdot {}^t b_\beta \cdot {}^t x + (1/2)yc_\beta \cdot {}^t d_\beta \cdot {}^t y + xb_\beta \cdot {}^t c_\beta \cdot {}^t y.$$

Moreover $r_P = r$ on P and $r_\Omega = r$ on $G \cap \Omega_{\mathbf{A}}$. There are also similar lifts of P_v and Ω_v into M_v given by the same formulas with the subscript $\mathbf{A}$ replaced by v. We denote these lifts also by r_P and r_Ω, since the distinction will be clear from the context. Here the measure on $X_{\mathbf{A}}$ is the n-fold product of the measure $\prod_{v \in \mathbf{a} \cup \mathbf{h}} d_v x$ on $F_{\mathbf{A}}$, $\int_{\mathfrak{g}_v} d_v x = N(\mathfrak{d}_v)^{-1/2}$ for $v \in \mathbf{h}$, $\int_0^1 d_v x = 1$ for $v \in \mathbf{a}$. We can let $Sp(n, \mathbf{R})$ act on $\mathfrak{H}_n$ and define factors of automorphy $\mu(\alpha, z)$ for $\alpha \in Sp(n, \mathbf{R})$ as in §3.3. We define a space $\mathcal{H}$ and a vector space $\mathcal{U}$ by

(A2.4) $$\mathcal{H} = \mathfrak{H}_n^{\mathbf{a}}, \qquad \mathcal{U} = (\mathbf{C}^n)^{\mathbf{a}}.$$

For $\sigma \in M_{\mathbf{A}}$, $\alpha = \mathrm{pr}(\sigma) \in G_{\mathbf{A}}$, and $z \in \mathcal{H}$ we put

(A2.5a) $$\mu(\sigma, z) = \mu(\alpha, z) = \big(\mu(\alpha_v, z_v)\big)_{v \in \mathbf{a}},$$

(A2.5b) $$j_\sigma(z) = j_\alpha(z) = \big(j(\alpha_v, z_v)\big)_{v \in \mathbf{a}}, \quad j(\alpha_v, z_v) = \det\big(\mu(\alpha_v, z_v)\big),$$

(A2.5c) $$j_\sigma(z)^{\mathbf{a}} = j_\alpha(z)^{\mathbf{a}} = \prod_{v \in \mathbf{a}} j(\alpha_v, z_v).$$

We let $G_v = Sp(n, \mathbf{R})$ act on $\mathbf{C}^n \times \mathfrak{H}_n$ by

(A2.6) $$\alpha(u, z) = \big({}^t\mu(\alpha, z)^{-1}u, \ \alpha z\big)$$

for $\alpha \in G_v$, and let $G_{\mathbf{A}}$ act on $\mathcal{H}$ and $\mathcal{U} \times \mathcal{H}$ by

(A2.7) $$\alpha(z) = (\alpha_v z_v)_{v \in \mathbf{a}}, \qquad \alpha(u, z) = \big({}^t\mu(\alpha_v, z_v)^{-1}u_v, \ \alpha_v z_v\big)_{v \in \mathbf{a}}$$

for $z \in \mathcal{H}$ and $\alpha \in G_{\mathbf{A}}$, ignoring $\alpha_{\mathbf{h}}$. We define the action of an element σ of $M_{\mathbf{A}}$ (resp. M_v) on $\mathcal{H}$ and $\mathcal{U} \times \mathcal{H}$ (resp. $\mathfrak{H}_n$ and $\mathbf{C}^n \times \mathfrak{H}_n$) to be the same as that of $\mathrm{pr}(\sigma)$.

We now define a $\mathbf{C}$-valued function $\varphi(x;\, u,\, z)$ for $x \in X_{\mathbf{a}}$, $z \in \mathcal{H}$, and $u \in \mathcal{U}$ by

(A2.8a) $$\varphi(x;\, u,\, z) = \prod_{v \in \mathbf{a}} \varphi_v(x_v;\, u_v,\, z_v),$$

(A2.8b) $$\varphi_v(x_v;\, u_v,\, z_v) = \mathbf{e}\big((1/2)^t u(z - \overline{z})^{-1} u + (1/2) x z \cdot {}^t x + x u\big),$$

where the subscript v is suppressed on the right-hand side of the last formula.

A2.2. Lemma. *Let M' be the group formed by all the couples $(\alpha,\, g)$ with $\alpha \in Sp(n, \mathbf{R})$ and a holomorphic function g on $\mathfrak{H}_n$ such that $g(z)^2 = t \cdot j_\alpha(z)$ with $t \in \mathbf{T}$, the law of composition being $(\alpha,\, g)(\alpha',\, g') = \big(\alpha\alpha',\, g(\alpha'(z))g'(z)\big)$. Then for each $v \in \mathbf{a}$, M_v is isomorphic to M' via the map $\xi \mapsto \big(\mathrm{pr}(\xi),\, g_\xi\big) \in M'$ for $\xi \in M_v$ with g_ξ determined by*

(A2.9) $$(\xi\varphi_v)(x;\, u,\, z) = g_\xi(z)^{-1} \varphi_v\big(x;\, \xi(u,\, z)\big) \qquad (\xi \in M_v).$$

In particular $g_\xi(z) = \det(-iz)^{1/2}$ if $\xi = r_\Omega(\eta_n)$.

PROOF. Let $\tau = r_P(\alpha)$ with $\alpha \in P$. Then from (A2.3a) we can easily derive that $(\tau\varphi_v)(x;\, u,\, z) = g_\tau^{-1}\varphi_v(x;\, \tau(u,\, z))$ with $g_\tau = |\det(d_\alpha)|_v^{1/2}$. Also. if $\sigma = r_\Omega(\eta_n)$, formula (A2.3c) together with an easy calculation shows that

(A2.10) $$\begin{aligned}(\sigma\varphi_v)(x;\, u,\, z) &= \int_{\mathbf{R}_n^1} \varphi_v(y;\, u,\, z)\mathbf{e}(-x \cdot {}^t y)dy \\ &= \det(-iz)^{-1/2}\varphi_v\big(x;\, \sigma(u,\, z)\big).\end{aligned}$$

Now from (A2.1b) and Lemma 7.5 we see that M_v is generated by $r_P(P_v)$, $r_\Omega(\eta_n)$, and $\mathbf{T}$. Therefore we have (A2.9) with a certain g_ξ such that $g_\xi(z)^2/j_\alpha(z) \in \mathbf{T}$. Then $\xi \mapsto \big(\mathrm{pr}(\xi),\, g_\xi\big)$ defines a homomorphism of M_v into M'. In particular, an element t of $\mathbf{T}$ is mapped to $(1,\, t^{-1})$. Therefore we see that the map is injective. Given $(\alpha,\, g) \in M'$, take $\xi \in M_v$ so that $\mathrm{pr}(\xi) = \alpha$. Then $g_\xi^2/g^2 \in \mathbf{T}$, so that $g_\xi = tg$ with $t \in \mathbf{T}$, and so $(\alpha,\, g)$ is the image of $t\xi$. Thus our map is surjective. The last assertion of our lemma follows from (A2.10).

Thus, writing $g(\xi,\, z)$ for $g_\xi(z)$, we have $g(\xi\xi',\, z) = g(\xi,\, \xi' z)g(\xi',\, z)$ for $\xi,\, \xi' \in M_v$. Moreover, we have shown that

(A2.11) $$g\big(r_P(\alpha),\, z\big) = |\det(d_\alpha)|_v^{1/2} \quad \text{if} \ \ \alpha \in P_v,\, v \in \mathbf{a}.$$

A2.3. For two fractional ideals $\mathfrak{x}$ and $\mathfrak{y}$ in F such that $\mathfrak{x}\mathfrak{y} \subset \mathfrak{g}$ we define a subgroup $D[\mathfrak{x},\, \mathfrak{y}]$ of $G_{\mathbf{A}}$ by (16.20a) and put $D_v[\mathfrak{x},\, \mathfrak{y}] = G_v \cap D[\mathfrak{x},\, \mathfrak{y}]$; we define also a subgroup C^θ of $G_{\mathbf{A}}$, which may be called the "theta-subgroup," by

(A2.12a) $$C^\theta = G_{\mathbf{a}} \textstyle\prod_{v \in \mathbf{h}} C_v^\theta,$$

(A2.12b) $$C_v^\theta = \big\{\, \xi \in D_v[\mathfrak{d}^{-1},\, \mathfrak{d}] \;\big|\; \chi_v\big((x,\, y)\xi\big) = \chi_v(x,\, y) \ \ \textit{for every}\ x \in L_v \ \textit{and}\ y \in L_v^* \,\big\} \quad \textit{if} \ \ v \in \mathbf{h},$$

where $\chi_v(x,\, y) = \mathbf{e}_v(x \cdot {}^t y/2)$ for $x,\, y \in (F_v)_n^1$. Then it can be shown that

(A2.13) $$C_v^\theta = \big\{\, \xi \in D_v[\mathfrak{d}^{-1}, \mathfrak{d}] \;\big|\; (a_\xi \cdot {}^t b_\xi)_{ii} \in 2\mathfrak{d}_v^{-1} \ \ \textit{and} \ \ (c_\xi \cdot {}^t d_\xi)_{ii} \in 2\mathfrak{d}_v \ \ \textit{for}\ 1 \leqq i \leqq n \big\},$$

where α_{ii} denotes the (i, i)-entry of α. We see from (A2.12b) that C_v^θ is indeed a group. Also we easily see that

(A2.14) $$C_v^\theta \supset D_v[2\mathfrak{d}^{-1}, 2\mathfrak{d}] \cup D_v[2\mathfrak{d}^{-1}, 2\mathfrak{d}]\varepsilon_v$$

with $\varepsilon \in G_{\mathbf{A}}$ given by

(A2.15) $$\varepsilon_{\mathbf{a}} = 1_{2n}, \qquad \varepsilon_v = \begin{bmatrix} 0 & -\delta_v^{-1}1_n \\ \delta_v 1_n & 0 \end{bmatrix} \quad \textit{for} \quad v \in \mathbf{h},$$

where δ is an arbitrarily fixed element of $F_{\mathbf{h}}^\times$ such that $\mathfrak{d} = \delta\mathfrak{g}$. Since C_v^θ coincides with $Ps(X_v, L_v)$ of [W64, n°36], we obtain a lift

(A2.16) $$r_v : C_v^\theta \to M_v$$

which is written $\mathbf{r}'_L$ there.

We are going to define a factor of automorphy, $h(\sigma, z)$, of weight $\mathbf{a}/2$ for $z \in \mathcal{H}$ and σ in the set

(A2.17) $$\mathfrak{M} = \{ \sigma \in M_{\mathbf{A}} \mid \mathrm{pr}(\sigma) \in P_{\mathbf{A}}C^\theta \}.$$

Clearly $r_P(P_{\mathbf{A}}) \subset \mathfrak{M}$. Notice that $\eta \in G \cap \mathfrak{M}$ since $\eta_{\mathbf{h}} = \mathrm{diag}[\delta 1_n, \delta^{-1}1_n]\varepsilon \in P_{\mathbf{A}}C^\theta$.

We denote by $\mathcal{S}(X_{\mathbf{h}})$ and $\mathcal{S}(X_{\mathbf{A}})$ the Schwartz-Bruhat spaces of $X_{\mathbf{h}}$ and $X_{\mathbf{A}}$. We shall often view an element ℓ of $\mathcal{S}(X_{\mathbf{h}})$ as a function on X by restricting ℓ to the image of X in $X_{\mathbf{h}}$ (see §1.6). Given $\ell \in \mathcal{S}(X_{\mathbf{h}})$, we put

(A2.18) $$\ell_{\mathbf{A}}(x; u, z) = \ell(x_{\mathbf{h}})\varphi(x_{\mathbf{a}}; u, z) \quad \textit{for } x \in X_{\mathbf{A}},\ z \in \mathcal{H},\ u \in \mathcal{U}.$$

For fixed z and u we view $\ell_{\mathbf{A}}$ as an element of $\mathcal{S}(X_{\mathbf{A}})$, so that $\sigma\ell_{\mathbf{A}}$ for $\sigma \in M_{\mathbf{A}}$ is meaningful. Now there is another action of $\mathfrak{M}$ on $\mathcal{S}(X_{\mathbf{h}})$ as follows:

A2.4. Theorem. *For every $\sigma \in \mathfrak{M}$ we can define its action on $\mathcal{S}(X_{\mathbf{h}})$ which is a $\mathbf{C}$-linear automorphism, written $\ell \mapsto {}^\sigma\ell$ for $\ell \in \mathcal{S}(X_{\mathbf{h}})$, and also a holomorphic function $h(\sigma, z)$ of $z \in \mathcal{H}$ by the formula*

(1) $$(\sigma\ell_{\mathbf{A}})(x; u, z) = h(\sigma, z)^{-1}({}^\sigma\ell)_{\mathbf{A}}(x; \sigma(u, z)).$$

Moreover, this action and h have the following properties:

(2) $h(\sigma, z)^2 = \zeta j_\sigma(z)^{\mathbf{a}}$ *with* $\zeta \in \mathbf{T}$.

(3) $h(t \cdot r_P(\gamma), z) = t^{-1}|\det(d_\gamma)_{\mathbf{a}}|_{\mathbf{A}}^{1/2}$ *if* $t \in \mathbf{T}$ *and* $\gamma \in P_{\mathbf{A}}$.

(4) $h(\rho\sigma\tau, z) = h(\rho, z)h(\sigma, \tau z)h(\tau, z)$ *and* ${}^{(\rho\sigma\tau)}\ell = {}^\rho({}^\sigma({}^\tau\ell))$ *if* $\mathrm{pr}(\rho) \in P_{\mathbf{A}}$ *and* $\mathrm{pr}(\tau) \in C^\theta$; *in particular,* $h(-\sigma, z) = h(\sigma, z)$.

(5) ${}^\sigma\ell$ *depends only on* ℓ *and* $\mathrm{pr}(\sigma)_{\mathbf{h}}$.

(6) $\{\sigma \in \mathfrak{M} \mid {}^\sigma\ell = \ell\}$ *contains an open subgroup of* $M_{\mathbf{A}}$ *for every* $\ell \in \mathcal{S}(X_{\mathbf{h}})$. *In particular, if* ℓ *is the characteristic function of* $\prod_{v\in\mathbf{h}} L_v$, *then* ${}^\sigma\ell = \ell$ *for every* σ *such that* $\mathrm{pr}(\sigma) \in C^\theta$.

(7) *If* $\ell(x) = \prod_{v\in\mathbf{h}} \ell_v(x_v)$ *with* $\ell_v \in \mathcal{S}(X_v)$, ℓ_v *is the characteristic function of* L_v *for almost all* v, *and* $\sigma = r_\Omega(\tau)$, $\tau \in P_{\mathbf{A}}C^*$ *with* $C^* = \{\alpha \in C^\theta \mid L_v^*(c_\alpha)_v = L_v$ *for every* $v \in \mathbf{h}\}$, *then* ${}^\sigma\ell(x) = \prod_{v\in\mathbf{h}}[r_\Omega(\tau_v)\ell_v](x_v)$.

(8) *If* ℓ *is as in (7) and* $\mathrm{pr}(\sigma) = \beta\alpha$ *with* $\beta \in P_{\mathbf{A}}$ *and* $\alpha \in C^\theta$, *then* ${}^\sigma\ell(x) = \prod_{v\in\mathbf{h}}[r_P(\beta_v)r_v(\alpha_v)\ell_v](x_v)$, *where* r_v *is the lift of (A2.16).*

Proof. Let $\tau \in M_{\mathbf{A}}$ and $\alpha = \mathrm{pr}(\tau)$; suppose $\alpha \in C^\theta$. Take $\xi_v \in M_v$ for each $v \in \mathbf{a}$ so that $\mathrm{pr}(\xi_v) = \alpha_v$. Then we can define an element γ of $M_{\mathbf{A}}$ such that $\mathrm{pr}(\gamma) = \alpha$ and

$$(\text{A2.19}) \qquad (\gamma\ell_{\mathbf{A}})(x;\, u,\, z) = \prod_{v\in\mathbf{h}}[r_v(\alpha_v)\ell_v](x_v)\prod_{v\in\mathbf{a}}(\xi_v\varphi_v)(x_v;\, u_v,\, z_v)$$

for $\ell = \prod_{v\in\mathbf{h}}\ell_v$ as in (7) (see [W64, n°38]). Then $\tau = \zeta\gamma$ with $\zeta \in \mathbf{T}$. Now every element σ of $\mathfrak{M}$ can be written $\sigma = r_P(\beta)\tau$ with such a τ and $\beta \in P_{\mathbf{A}}$. Applying $r_P(\beta)$ to (A2.19), we obtain

$$(\text{A2.20}) \qquad (\sigma\ell_{\mathbf{A}})(x;\, u,\, z) = \zeta\ell'(x_{\mathbf{h}})\prod_{v\in\mathbf{a}}[r_P(\beta_v)\xi_v\varphi_v](x_v;\, u_v,\, z_v),$$

where $\ell' = \prod_{v\in\mathbf{h}} r_P(\beta_v)r_v(\alpha_v)\ell_v$. By (A2.9) we can write (A2.20) in the form

$$(\text{A2.21}) \qquad (\sigma\ell_{\mathbf{A}})(x;\, u,\, z) = h(z)^{-1}\ell'_{\mathbf{A}}\big(x;\, \sigma(u,\, z)\big)$$

with $h(z) = \zeta^{-1}\prod_{v\in\mathbf{a}} g\big(r_P(\beta_v)\xi_v,\, z_v\big)$. We have assumed that $\ell = \prod_{v\in\mathbf{h}}\ell_v$, but clearly $\ell \mapsto \ell'$ can be extended to a $\mathbf{C}$-linear automorphism of $\mathcal{S}(X_{\mathbf{h}})$, which we write again $\ell \mapsto \ell'$. Then (A2.21) holds for every $\ell \in \mathcal{S}(X_{\mathbf{h}})$ with the same $h(z)$. We put $h(\sigma,\, z) = h(z)$ and ${}^{\sigma}\ell = \ell'$. To show that these are independent of the choice of β and τ, take ℓ to be the characteristic function λ of $\prod_{v\in\mathbf{h}} L_v$, and recall that $r_v(C_v^{\theta})\lambda_v = \lambda_v$ (see [W64, n°21]). Now (A2.3a) shows that $(r_P(\beta_v)\lambda_v)(0) = |\det(d_\beta)|_v^{-1/2}$. Therefore, putting $x = 0$ and $u = 0$ in (A2.21), we obtain

$$(\text{A2.22}) \qquad (\sigma\lambda_{\mathbf{A}})(0;\, 0,\, z) = |\det(d_\beta)_{\mathbf{h}}|_{\mathbf{A}}^{-1/2}\cdot h(z)^{-1}.$$

Since $\mathrm{pr}(\sigma) \in \beta D[\mathfrak{d}^{-1},\, \mathfrak{d}]$, from (1.19) we obtain $|\det(d_\beta)_{\mathbf{h}}|_{\mathbf{A}}^{-1} = N\big(\mathrm{il}_{\mathfrak{d}}(\mathrm{pr}(\sigma))\big)$, which depends only on $\mathrm{pr}(\sigma)$. Thus $h(z)$ is determined by σ, and consequently formula (1) is established with ${}^{\sigma}\ell$ well-defined. Clearly (2), (4), (5), and (8) follow easily from our definition of $h(\sigma,\, z)$ and ${}^{\sigma}\ell$; (3) follows from (A2.11) if we take $\xi_v = 1$ and $\alpha = 1$; the first part of (6) can be derived from the fact that $\{\gamma \in C_v^{\theta} \mid r_v(\gamma)\ell_v = \ell_v\}$ is open for every $\ell_v \in \mathcal{S}(X_v)$ (see [W64, n°21, n°36]). Finally let $\sigma = r_\Omega(\tau)$ with $\tau = \beta\alpha$ with $\beta \in P_{\mathbf{A}}$ and $\alpha \in C^*$. Then $r_\Omega(\tau_v) = r_P(\beta_v)r_\Omega(\alpha_v)$; moreover $r_\Omega(\alpha_v) = r_v(\alpha_v)$ by [W64, p.168, last line]. Therefore (7) follows from (8). The second half of (6) follows also from (8).

Given $\ell \in \mathcal{S}(X_{\mathbf{h}})$, we define a theta function $\theta(u,\, z;\, \ell)$ for $(u,\, z) \in \mathcal{U}\times\mathcal{H}$ by

$$(\text{A2.23}) \qquad \theta(u,\, z;\, \ell) = \sum_{\xi\in X}\ell_{\mathbf{A}}(\xi;\, u,\, z).$$

A2.5. Proposition. *For every $\alpha \in G\cap\mathfrak{M}$ we have*

$$\theta\big(\alpha(u,\, z);\, {}^{\alpha}\ell\big) = h(\alpha,\, z)\theta(u,\, z;\, \ell).$$

Moreover, for every $\beta \in G$ and $\xi \in \prod_{v\in\mathbf{a}} M_v$ such that $\mathrm{pr}(\xi) = \beta_{\mathbf{a}}$, there is a $\mathbf{C}$-linear automorphism $\ell \mapsto \ell'$ of $\mathcal{S}(X_{\mathbf{h}})$ such that

$$\theta\big(\beta(u,\, z);\, \ell'\big) = \theta(u,\, z;\, \ell)\prod_{v\in\mathbf{a}} g(\xi_v,\, z_v).$$

PROOF. By virtue of [W64, Theorem 4 or 6] we have $\sum_{\xi\in X}(\alpha\ell_{\mathbf{A}})(\xi;\, u,\, z) = \sum_{\xi\in X}\ell_{\mathbf{A}}(\xi;\, u,\, z)$ for every $\alpha \in G$, which combined with (1) of Theorem A2.4 proves the first assertion. Now, by Lemma 7.5, G is generated by $G\cap\mathfrak{M}$, since $G\cap\mathfrak{M}$ contains η and P. Therefore the second assertion follows from the first one.

To state our formulas on $h(\sigma,\, z)$, we need the symbols $\gamma(s)$ and $\omega(s)$ of (16.5) and (16.6), as well as $\mathrm{il}_{\mathfrak{d}}$ of (1.19).

A2.6. Lemma. *Let $\sigma \in \mathfrak{M}$; if σ or $\sigma\eta^{-1}$ belongs to $r_\Omega(\Omega_{\mathbf{A}} \cap P_{\mathbf{A}}C^\theta)$, then $h(\sigma, z)$ is completely determined by Theorem A2.4 (2) and the following formulas:*

$$\lim_{r\to\infty} h(\sigma, r\mathbf{i})/|h(\sigma, r\mathbf{i})| = \omega(-c_\alpha^{-1}d_\alpha) \quad \text{if } \sigma = r_\Omega(\alpha) \text{ with } \alpha \in \Omega_{\mathbf{A}} \cap P_{\mathbf{A}}C^\theta,$$

$$\lim_{r\to 0} h(\sigma, r\mathbf{i})/|h(\sigma, r\mathbf{i})| = \omega(\delta^{-2}d_\alpha^{-1}c_\alpha) \quad \text{if } \sigma = r_\Omega(\alpha\eta^{-1})\eta \text{ with } \alpha \in \Omega_{\mathbf{A}}\eta \cap P_{\mathbf{A}}C^\theta,$$

where $\mathbf{i}$ is the origin of $\mathcal{H}$ defined by (16.21). In particular

$$h(\eta, z) = \prod_{v\in\mathbf{a}} \det(-iz_v)^{1/2}, \tag{A2.24}$$

where $\det(-iz_v)^{1/2}$ is chosen so that it is positive when $\mathrm{Re}(z_v) = 0$.

In order to speak of $\omega(s)$, we need to know $\gamma(s) \neq 0$. That $\gamma(-c_\alpha^{-1}d_\alpha) \neq 0$ and $\gamma(\delta^{-2}d_\alpha^{-1}c_\alpha) \neq 0$ will be shown in the following proof.

PROOF. Write a, b, c, d for a_α, b_α, c_α, d_α; let $\sigma = r_\Omega(\alpha)$ with $\alpha \in \Omega_{\mathbf{A}} \cap P_{\mathbf{A}}C^\theta$; let λ be the characteristic function of $\prod_{v\in\mathbf{h}} L_v$. By (A2.22) and (A2.3c) we have

$$\begin{aligned}
N\big(\mathrm{il}(\alpha)\big)^{1/2}h(\sigma, z)^{-1} &= (\sigma\lambda_{\mathbf{A}})(0;\, 0,\, z)\\
&= |\det(c)|_{\mathbf{A}}^{1/2}\int_{X_{\mathbf{A}}} \lambda_{\mathbf{A}}(yc;\, 0,\, z)\mathbf{e}_{\mathbf{A}}(yc\cdot {}^t d\cdot {}^t y/2)dy\\
&= |\det(c)|_{\mathbf{A}}^{-1/2}\int_{X_{\mathbf{A}}} \lambda_{\mathbf{A}}(x;\, 0,\, z)\mathbf{e}_{\mathbf{A}}(xc^{-1}d\cdot {}^t x/2)dx\\
&= |\det(c)|_{\mathbf{A}}^{-1/2}\prod_{v\in\mathbf{a}}\int_{X_v} \varphi_v(x;\, 0,\, (z+c^{-1}d)_v)d_vx\prod_{v\in\mathbf{h}}\int_{L_v}\mathbf{e}_v(xc^{-1}d\cdot {}^t x/2)d_vx.
\end{aligned}$$

From (A2.10) we see that the integral over X_v is equal to $\det(-i(z+c^{-1}d)_v)^{-1/2}$. The integral over L_v is $N(\mathfrak{d}_v)^{-n/2}\gamma_v(c^{-1}d)$. Our equality shows that $\gamma_v(c^{-1}d) \neq 0$. Since $\overline{\gamma_v(s)} = \gamma_v(-s)$, we obtain the first formula. We can take η to be α, since $\eta \in P_{\mathbf{A}}C^\theta$ as we noted it immediately after (A2.17). Then we find our assertion concerning $h(\eta, z)$. Now if $\rho = \mathrm{diag}[\delta 1_n,\, \delta^{-1}1_n]$, then (8) of Theorem A2.4 together with (A2.3a) shows that ${}^\eta\lambda = \prod_{v\in\mathbf{h}} r_P(\rho_v)\lambda_v = N(\mathfrak{d})^{-n/2}\lambda'$, where $\lambda'(x) = \lambda(\delta x)$. Let $\sigma = r_\Omega(\alpha\eta^{-1})\eta$ with $\alpha \in \Omega_{\mathbf{A}}\iota \cap P_{\mathbf{A}}C^\theta$. By (1) of Theorem A2.4 we have

$$\begin{aligned}
(\sigma\lambda_{\mathbf{A}})(x;\, 0,\, z) &= \big(r_\Omega(\alpha\eta^{-1})\eta\lambda_{\mathbf{A}}\big)(x;\, 0,\, z)\\
&= N(\mathfrak{d})^{-n/2}h(\eta, z)^{-1}\big(r_\Omega(\alpha\eta^{-1})\lambda'_{\mathbf{A}}\big)\big(x;\, 0,\, \eta(z)\big).
\end{aligned}$$

Since $\alpha\eta^{-1} = \begin{bmatrix} -b & a \\ -d & c \end{bmatrix}$, a calculation similar to the above one shows that

$$\begin{aligned}
\big(r_\Omega(\alpha\eta^{-1})\lambda'_{\mathbf{A}}\big)\big(0;\, 0,\, \eta(z)\big) &= |\det(d)|_{\mathbf{A}}^{-1/2}N(\mathfrak{d})^{n/2}\\
&\quad\cdot\gamma(-\delta^{-2}d^{-1}c)\prod_{v\in\mathbf{a}}\det\big(i(z^{-1}+d^{-1}c)_v\big)^{-1/2}.
\end{aligned}$$

Since (A2.22) is nonzero, we see that $\gamma(\delta^{-2}d_\alpha^{-1}c_\alpha) \neq 0$. Putting $z = r\mathbf{i}$ and taking the limit as $r \to 0$, we obtain the second formula.

A2.7. Proposition. *Let ψ^* be the quadratic ideal character of F corresponding to the extension $F(\sqrt{-1})/F$. Suppose $\alpha \in G \cap P_{\mathbf{A}}D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]$. Then d_α is invertible, $\det(d_\alpha)\mathrm{il}_{\mathfrak{d}}(\alpha)^{-1}$ is prime to 2, and*

$$h(\alpha, z)^2 = \mathrm{sgn}\big(N_{F/\mathbf{Q}}(\det(d_\alpha)\big)\psi^*\big(\det(d_\alpha)\mathrm{il}_\mathfrak{d}(\alpha)^{-1}\big)j_\alpha(z)^\mathbf{a}.$$

PROOF. From Lemma 1.11 (2), (3) we obtain $\nu_0(\delta^{-1}d_\alpha^{-1}c_\alpha) = \det(d_\alpha)\mathrm{il}_\mathfrak{d}(\alpha)^{-1}$ and also the first two assertions of our proposition. Since $\alpha = \alpha\eta^{-1}\eta = r_\Omega(\alpha\eta^{-1})\cdot r_\Omega(\eta)$, Lemma A2.6 shows that

$$\lim_{z\to 0} h(\alpha, z)^2/|h(\alpha, z)|^2 = \omega(\delta^{-2}d_\alpha^{-1}c_\alpha)^2.$$

By Lemma A1.6 (2), $\prod_{v\nmid 2}\omega_v(\delta^{-2}d_\alpha^{-1}c_\alpha)^2 = \psi^*\big(\nu_0(\delta^{-1}d_\alpha^{-1}c_\alpha)\big)$. This combined with Theorem 2.4 (2) proves our last assertion, since $\lim_{z\to 0} j_\alpha(z)^\mathbf{a} = N_{F/\mathbf{Q}}\big(\det(d_\alpha)\big)$.

A2.8. Proposition. (1) *If* $\alpha \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$, *then* $\gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha) = 1$ *for* $v|2$, $\det(d_\alpha)\mathfrak{g}$ *is prime to 2, and*

$$(*)\qquad \lim_{z\to 0} h(\alpha, z) = |N_{F/\mathbf{Q}}\big(\det(d_\alpha)\big)|\gamma(\delta^{-2}d_\alpha^{-1}c_\alpha) = \sum_{x\in L/Ld_\alpha} \mathbf{e}_\mathbf{h}(\delta^{-2}xd_\alpha^{-1}c_\alpha\cdot {}^tx/2).$$

(2) *Let* $\beta = \xi\alpha\xi^{-1}$ *with* α *as above and* $\xi = \mathrm{diag}[1_n, s1_n]$, $0 \ll s \in \mathfrak{g}$; *suppose* $\beta \in \Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$. *Let* ψ_s^* *be the ideal character of* F *corresponding to the extension* $F(\sqrt{s})/F$. *Then* $\det(d_\alpha)\mathfrak{g}$ *is prime to* $2s$ *and*

$$h(\beta, z) = h(\alpha, sz)\psi_s^*\big(\det(d_\alpha)\mathfrak{g}\big).$$

PROOF. Let $\alpha\in\Gamma[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$; then $\det(d_\alpha)\mathfrak{g}$ is clearly prime to 2 and $\mathrm{il}_\mathfrak{d}(\alpha) = \mathfrak{g}$, so that $\nu_0(\delta^{-1}d_\alpha^{-1}c_\alpha) = \det(d_\alpha)\mathfrak{g}$ by Lemma 1.11 (2). Since $d_\alpha^{-1}c_\alpha \prec 2\mathfrak{d}_v$ for $v|2$, we see from (16.5) that $\gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha) = 1$ for $v|2$. By Lemma A1.6 (3), $|\gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha)| = \nu\big((\delta^{-1}d_\alpha^{-1}c_\alpha)_v\big)^{-1/2}$ for $v\nmid 2$, and hence $|\gamma(\delta^{-2}d_\alpha^{-1}c_\alpha)| = N\big(\det(d_\alpha)\mathfrak{g}\big)^{-1/2} = \lim_{z\to 0}|h(\alpha, z)|^{-1}$. This combined with Lemma A2.6 proves our first equality of (*). By (16.5) we can easily express the quantity in question as a sum over L/Ld_α as stated. Next, if β is as in (2), then $c_\beta = sc_\alpha$ and $d_\beta = d_\alpha$, so that $h(\beta, z)/h(\alpha, sz)$ is a constant, which equals $\gamma(\delta^{-2}d_\beta^{-1}c_\beta)/\gamma(\delta^{-2}d_\alpha^{-1}c_\alpha)$ by (1). If $v\nmid 2\det(d_\alpha)$, then d_α^{-1} is v-integral, so that $\gamma_v(\delta^{-2}d_\beta^{-1}c_\beta) = \gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha) = 1$. Now $\det(d_\alpha)\mathfrak{g}$ is prime to $2s$ since $b_\beta \prec 2\mathfrak{d}^{-1}$ and $c_\beta \prec 2s\mathfrak{d}$. If $v|\det(d_\alpha)$ and $v\nmid 2$, then by Lemma A1.6 (1) we have

$$\gamma_v(\delta^{-2}d_\beta^{-1}c_\beta) = \gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha)\left(\frac{s}{\nu_0(\delta^{-1}d_\alpha^{-1}c_\alpha)_v}\right) = \gamma_v(\delta^{-2}d_\alpha^{-1}c_\alpha)\left(\frac{s}{\det(d_\alpha)\mathfrak{g}_v}\right).$$

Therefore we obtain the formula of (2).

A2.9. *Proof of Theorem 6.8.* Observe that $\theta(u, z; \ell)$ coincides with $\varphi_F(u, z; \lambda)$ if $\lambda(x) = \ell({}^tx)$. Then (1), (2), (3), and (5) of Theorem 6.8 are special cases of Theorem A2.4 (2), (4), (3), and Proposition A2.7, respectively. Given such λ and ℓ, put $D_\lambda = \{\sigma \in C^\theta \mid {}^\sigma\ell = \ell\}$, where we write ${}^\sigma\ell$ for ${}^\tau\ell$ with any $\tau \in M_\mathbf{A}$ such that $\sigma = \mathrm{pr}(\tau)$. This is meaningful by Theorem A2.4 (5), and D_λ is an open subgroup of C^θ by Theorem A2.4 (6); moreover $D_\lambda = C^\theta$ if ℓ is the characteristic function of $\prod_{v\in\mathbf{h}} L_v$. Now let the notation be as in (4) of Theorem 6.8. Then we can put $\alpha^{-1} = \gamma\sigma$ with $\gamma = \mathrm{diag}[{}^td, d^{-1}]$ and $\sigma\in D_\lambda$. Put $\ell_1 = {}^{\alpha^{-1}}\ell$. By Theorem A2.4 (4) we have $\ell = {}^\alpha\ell_1$ and $\ell_1 = {}^{\gamma\sigma}\ell = {}^\gamma\ell$, and so by Proposition A2.5,

$$\theta\big(\alpha(u, z); \ell\big) = h_\alpha(z)\theta(u, z; {}^\gamma\ell).$$

If ℓ is as in Theorem A2.4 (8), then by that assertion and (A2.3a) we have $({}^{\gamma}\ell)(x) = \ell(x \cdot {}^t d)$, so that $({}^{\gamma}\ell)({}^t x) = \lambda(dx)$. Clearly this is valid for an arbitrary $\lambda \in \mathcal{S}(F_{\mathbf{h}}^n)$. Thus we obtain (4) of Theorem 6.8.

Proof of Theorem 6.9. The first part of (2) of Theorem 6.9 follows from Proposition A2.5. To prove (6.33), take $\alpha = \eta$ in Proposition A2.5 and substitute $\eta(u, z)$ for (u, z). Then we obtain $\theta(-u, z; {}^{\eta}\ell) = h_\eta(\eta z)\theta\big(\eta(u, z); \ell\big)$. From (A2.24) we easily see that $h_\eta(\eta z) = h_\eta(z)^{-1}$, and hence $\theta\big(\eta(u, z); \ell\big) = h_\eta(z)\theta(-u, z; {}^{\eta}\ell)$. Now $\theta(-u, z; \ell_1) = \theta(u, z; \ell_2)$ with $\ell_2(x) = \ell_1(-x)$. Therefore $\theta\big(\eta(u, z); \ell\big) = h_\eta(z) \cdot\theta(u, z; \ell')$ with $\ell'(x) = ({}^{\eta}\ell)(-x)$. By Theorem A2.4 (7) and (A2.3c) we obtain

$$ {}^{\eta}\ell(x) = |D_F|^{-n/2}\int_{\mathbf{F}_{\mathbf{h}}^n} \ell({}^t y)\mathbf{e}_{\mathbf{h}}(-{}^t xy)dy, $$

and hence we obtain λ' as given in (6.33).

To prove (1), we first consider the case $\alpha \in G$. Take any nonzero ℓ in $\mathcal{S}(X_{\mathbf{h}})$ and define ℓ' as in Proposition A2.5 with any choice of ξ for a given $\alpha \in G$. (Take α as β there.) We may assume that $r(z) = \prod_{v\in\mathbf{a}} g(\xi_v, z_v)$. By Theorem A2.4(6) we can find a congruence subgroup $\varGamma$ of G such that ${}^{\gamma}\ell = \ell$ and ${}^{\gamma}\ell' = \ell'$ for every $\gamma \in \varGamma$. Let $\gamma \in \alpha^{-1}\varGamma\alpha \cap \varGamma$. By Proposition A2.5 we then have

$$ \begin{aligned} h(\alpha\gamma\alpha^{-1}, \alpha z)\theta\big(\alpha(u, z); \ell'\big) &= \theta\big(\alpha\gamma\alpha^{-1}\alpha(u, z); \ell'\big) = r(\gamma z)\theta\big(\gamma(u, z); \ell\big) \\ &= r(\gamma z)h(\gamma, z)\theta(u, z; \ell) = r(\gamma z)h(\gamma, z)r(z)^{-1}\theta\big(\alpha(u, z); \ell'\big). \end{aligned} $$

Since $\theta(u, z; \ell')$ is a nonzero function, we obtain (1) when $\alpha \in G$. To treat the general case, we observe that every element of $\widetilde{G}_+$ is the product of an element of G and an element of the form $\mathrm{diag}[1_n, s1_n]$ with $s \in F^\times, \gg 0$. Therefore we can reduce our problem to the equality $h(\alpha\gamma\alpha^{-1}, \alpha z) = h_\gamma(z)$ for γ in some congruence subgroup when $\alpha = \mathrm{diag}[1_n, s1_n]$. We easily see that it is sufficient to prove it for $s \in \mathfrak{g}$. Taking α and γ here to be ξ and α of Proposition A2.8 (2), we btain $h(\alpha\gamma\alpha^{-1}, z) = h_\gamma(sz)\psi_s^*\big(\det(d_\gamma)\mathfrak{g}\big)$. Since $s \gg 0$, we have $\psi_s^*\big(\det(d_\gamma)\mathfrak{g}\big) = 1$ if $\det(d_\gamma) - 1 \in 4s\mathfrak{g}$. Thus we can define the desired congruence subgroup by that congruence condition. This completes the proof of Theorem 6.9.

Let us now study the behavior of $h(\sigma, z)$ and the action of $\mathfrak{M}$ on $\mathcal{S}(V_{\mathbf{h}})$ under the reflection $z \mapsto -z^\rho$, where x^ρ is defined for $x \in (\mathbf{C}_m^m)^{\mathbf{a}}$ by $x_v^\rho = \overline{x}_v$ for each $v \in \mathbf{a}$. Observe that this reflection maps $\mathcal{H}$ onto itself. Putting $\alpha^* = E\alpha E^{-1}$ for $\alpha \in G_{\mathbf{A}}$ with $E = \mathrm{diag}[1_n, -1_n]$, we see that $\overline{(C^\theta)^*} = C^\theta$, $\alpha^*(-z^\rho) = -\alpha(z)^\rho$ for $z \in \mathcal{H}$, $\mu_0(\alpha, z)^\rho = \mu_0(\alpha^*, -z^\rho)$, and $\overline{\mu(\alpha, z)} = \mu(\alpha^*, -z^\rho)$.

A2.10. Proposition. *There exists an automorphism of $M_{\mathbf{A}}$ which is written $\sigma \mapsto \sigma^*$, consistent with $\alpha \mapsto E\alpha E^{-1}$ for $\alpha \in \underline{G}$, and determined by the relation $(\sigma f)^* = \sigma^* f^*$ for $f \in L^2(X_{\mathbf{A}})$, where $f^*(x) = \overline{f(-x)}$. Moreover, $\mathrm{pr}(\sigma)^* = \mathrm{pr}(\sigma^*)$, $r_P(\alpha)^* = r_P(\alpha^*)$ for $\alpha \in P_{\mathbf{A}}$, $r_\Omega(\beta)^* = r_\Omega(\beta^*)$ for $\beta \in \Omega_{\mathbf{A}}$, $t^* = t^{-1}$ for $t \in \mathbf{T}$, $\mathfrak{M}^* = \mathfrak{M}$, $\overline{h(\sigma, z)} = h(\sigma^*, -z^\rho)$ and ${}^{\sigma^*}(\ell^*) = ({}^{\sigma}\ell)^*$ for every $\sigma \in \mathfrak{M}$ and $\ell \in \mathcal{S}(X_{\mathbf{h}})$, where ℓ^* is defined by $\ell^*(x) = \overline{\ell(-x)}$.*

PROOF. From (A2.3a, c) we easily see that $[r_P(\alpha)f]^* = r_P(\alpha^*)f^*$ for $\alpha \in P_{\mathbf{A}}$ and $[r_\Omega(\beta)f]^* = r_\Omega(\beta^*)f^*$ for $\beta \in \Omega_{\mathbf{A}}$. From Lemma 7.5 we easily see that $M_{\mathbf{A}}$ is generated by $r_P(P_{\mathbf{A}})$, $r_\Omega(\Omega_{\mathbf{A}})$, and $\mathbf{T}$, and hence these equalities prove our assertions except the last two, which follow from Theorem A2.4 (1) and (A2.22) combined with the relation $(\ell_{\mathbf{A}})^*(x; u, z) = (\ell^*)_{\mathbf{A}}(x; \overline{u}, -z^\rho)$ that can easily be verified.

A2.11. We conclude this section by investigating $h(\alpha, z)$ for α in the subgroup $Sp(r, F) \times Sp(s, F)$ of $Sp(r+s, F)$. To emphasize the dimension, let us denote the symbols G, P, X, $\mathcal{H}$, $\mathcal{U}$, $M_{\mathbf{A}}$, and $\mathfrak{M}$ by $G^{(n)}$, $P^{(n)}$, $X^{(n)}$, $\mathcal{H}^{(n)}$, $\mathcal{U}^{(n)}$, $M_{\mathbf{A}}^{(n)}$, and $\mathfrak{M}^{(n)}$. Let $n = r+s$ with positive integers r and s. For $\beta \in G_{\mathbf{A}}^{(r)}$ and $\gamma \in G_{\mathbf{A}}^{(s)}$ we define an element $\beta \times \gamma$ of $G_{\mathbf{A}}^{(n)}$ by (23.5).

Let us now study how this injection $(\beta, \gamma) \mapsto \beta \times \gamma$ of $G_{\mathbf{A}}^{(r)} \times G_{\mathbf{A}}^{(s)}$ into $G_{\mathbf{A}}^{(n)}$ can be extended to their metaplectic coverings. First, for $f \in L^2(X_{\mathbf{A}}^{(r)})$ and $f' \in L^2(X_{\mathbf{A}}^{(s)})$ we define $f \otimes f' \in L^2(X_{\mathbf{A}}^{(n)})$ by $(f \otimes f')(x, x') = f(x)f'(x')$ for $x \in X_{\mathbf{A}}^{(r)}$ and $x' \in X_{\mathbf{A}}^{(s)}$. Given $\sigma \in M_{\mathbf{A}}^{(r)}$ and $\sigma' \in M_{\mathbf{A}}^{(s)}$, we have a unique unitary operator $\langle \sigma, \sigma' \rangle$ on $L^2(X_{\mathbf{A}}^{(n)})$ such that $\langle \sigma, \sigma' \rangle (f \otimes f') = \sigma f \otimes \sigma' f'$ for all such f and f'. Then, checking the formula of [W64, (15)], we find that $\langle \sigma, \sigma' \rangle \in M_{\mathbf{A}}^{(n)}$ and $\mathrm{pr}(\langle \sigma, \sigma' \rangle) = \mathrm{pr}(\sigma) \times \mathrm{pr}(\sigma')$. (Notice that $(\sigma, \sigma') \mapsto \langle \sigma, \sigma' \rangle$ is not injective.) For $z \in \mathcal{H}^{(r)}$ and $z' \in \mathcal{H}^{(s)}$ define $[z, z'] \in \mathcal{H}^{(n)}$ by $[z, z']_v = \mathrm{diag}[z_v, z'_v]$; similarly for $u \in \mathcal{U}^{(r)}$ and $u' \in \mathcal{U}^{(s)}$ define $[u, u'] \in \mathcal{U}^{(n)}$ by $[u, u']_v = {}^t({}^t u_v, {}^t u'_v)$. Clearly

$$\varphi\bigl(x, x'; [u, u'], [z, z']\bigr) = \varphi(x; u, z)\varphi(x'; u', z'). \tag{A2.25}$$

Observe that $\langle \sigma, \sigma' \rangle \in \mathfrak{M}^{(n)}$ if $\sigma \in \mathfrak{M}^{(r)}$ and $\sigma' \in \mathfrak{M}^{(s)}$.

A2.12. Proposition. *If $\sigma \in \mathfrak{M}^{(r)}$ and $\sigma' \in \mathfrak{M}^{(s)}$, we have*

$$h\bigl(\langle \sigma, \sigma' \rangle, [z, z']\bigr) = h(\sigma, z)h(\sigma', z') \quad \textit{and} \quad {}^{\langle \sigma, \sigma' \rangle}(\ell \otimes \ell') = ({}^{\sigma}\ell) \otimes ({}^{\sigma'}\ell')$$

for $\ell \in \mathcal{S}(X_{\mathbf{h}}^{(r)})$ and $\ell' \in \mathcal{S}(X_{\mathbf{h}}^{(s)})$, where $(\ell \otimes \ell')(x, x') = \ell(x)\ell'(x')$. Moreover, if $\alpha \in G^{(r)}$ and $\alpha' \in G^{(s)}$, then $\alpha \times \alpha'$ as an element of $M_{\mathbf{A}}^{(n)}$ coincides with $\langle \alpha, \alpha' \rangle$, and hence

$$h\bigl(\alpha \times \alpha', [z, z']\bigr) = h(\alpha, z)h(\alpha', z')$$

if $\alpha \in G^{(r)} \cap \mathfrak{M}^{(r)}$ and $\alpha' \in G^{(s)} \cap \mathfrak{M}^{(s)}$.

PROOF. Formula (A2.25) together with (A2.22) proves the first equality, which together with Theorem A2.4 (1) proves the second one. Let $\alpha \in G^{(r)}$ and $\alpha' \in G^{(s)}$. If $\alpha \in P^{(r)}$ and $\alpha' \in P^{(s)}$, formula (A2.3a) gives $\alpha \times \alpha' = \langle \alpha, \alpha' \rangle$. Since G is generated by P and η, our proof will be complete if we can show $\eta_r \times 1 = \langle \eta_r, 1 \rangle$ and $1 \times \eta_s = \langle 1, \eta_s \rangle$. Now, from Lemma A2.6 we easily see that

$$h\bigl(\alpha \times 1, [z, z']\bigr) = h(\alpha, z) \quad \textit{if} \quad \alpha \in G^{(r)} \quad \textit{and} \quad \det(d_\alpha) \neq 0. \tag{A2.26}$$

Observe that $\eta_r = \beta\gamma$ with $\beta = \begin{bmatrix} 0 & -1_r \\ 1_r & 2 \cdot 1_r \end{bmatrix}$ and $\gamma = \begin{bmatrix} 1_r & -2 \cdot 1_r \\ 0 & 1_r \end{bmatrix}$. Clearly $\det(d_\beta)\det(d_\gamma) \neq 0$, and hence the equality of (A2.26) is true with β and γ in place of α. Since $\gamma \in C^\theta$, Theorem A2.4 (4) shows that

$$h(\eta_r, z) = h(\beta, \gamma z)h(\gamma, z) = h\bigl(\beta \times 1, [\gamma z, z']\bigr)h\bigl(\gamma \times 1, [z, z']\bigr) = h\bigl(\eta_r \times 1, [z, z']\bigr).$$

On the other hand we already know that $h\bigl(\langle \eta_r, 1 \rangle, [z, z']\bigr) = h(\eta_r, z)$. Observe that if $h(\sigma, w) = h(\tau, w)$ for some w and $\mathrm{pr}(\sigma) = \mathrm{pr}(\tau)$, then $\sigma = \tau$. Therefore we have $\langle \eta_r, 1 \rangle = \eta_r \times 1$. Similarly $\langle 1, \eta_s \rangle = 1 \times \eta_s$, which completes the proof.

A2.13. *Proof of Proposition 16.9.* We consider only half-integral k; the case of integral k can be handled by the same methods (see Remark 16.12). We identify any element of $P_{\mathbf{A}}$ with its image under r_P, as we did in §16.5. We start with an obvious equality

$$\text{(i)} \qquad E^*_{\mathbf{A}}(x,\, s) = \chi(\delta)^{-n} \sum_{\alpha \in A} \mu(\alpha x\widetilde{\zeta})\varepsilon(\alpha x\widetilde{\zeta})^{-s}, \qquad A = P\backslash G.$$

Let $\alpha \in G$ and $\mathrm{pr}(x) \in P_{\mathbf{A}}G_{\mathbf{a}}$; suppose $\mu(\alpha x\widetilde{\zeta}) \neq 0$. Then $\alpha x\widetilde{\zeta} \in P_{\mathbf{A}}\widetilde{D}$, so that $\alpha \in P_{\mathbf{A}}D[\mathfrak{b}^{-1},\, \mathfrak{bc}]\zeta^{-1}P_{\mathbf{A}}$. Now (1.18) shows that $\det(c_y)_v \neq 0$ if $y \in D[\mathfrak{b}^{-1},\, \mathfrak{bc}]\zeta^{-1}$ and $v|\mathfrak{c}$. Therefore we see that $\det(c_\alpha) \neq 0$, and so $\alpha \in P\eta P$ by [S97, Lemma 2.12 (2)], where η is η_n of (1.8). Thus we can take $P\backslash P\eta P$ in place of A in (i). By [S97, Lemma 18.8 (2)] we can take ηR as $P\backslash P\eta P$, where R is given by (16.11). Thus

$$\text{(ii)} \qquad E^*(x,\, s) = \chi(\delta)^{-n} \sum_{\alpha \in \eta R} \mu(\alpha x\widetilde{\zeta})\varepsilon(\alpha x\widetilde{\zeta})^{-s} \ \text{ if } \ \mathrm{pr}(x) \in P_{\mathbf{A}}G_{\mathbf{a}}.$$

For $\sigma \in S_{\mathbf{A}}$ put $\tau(\sigma) = \begin{bmatrix} 1_n & \sigma \\ 0 & 1_n \end{bmatrix}$. Clearly $\tau(S) = R$. Putting $\xi = \mathrm{diag}[q,\, \widehat{q}]$, from (16.42) we obtain

$$\text{(iii)} \qquad c(h,\, q,\, s) = \chi(\delta)^{-n} \int_{S_{\mathbf{A}}/S} E^*\bigl(\tau(\sigma)\xi,\, s\bigr)\mathbf{e}^n_{\mathbf{A}}(-h\sigma)d\sigma,$$

where we take the measure of $S_{\mathbf{A}}/S$ to be 1. To simplify our notation, put $g(x) = \mu(x)\varepsilon(x)^{-s}$. Putting $x = \tau(\sigma)\xi$ and $\alpha = \eta\tau(a)$ with $a \in S$ in (ii), we obtain

$$E^*\bigl(\tau(\sigma)\xi,\, s\bigr) = \chi(\delta)^{-n} \sum_{a \in S} g\bigl(\eta\tau(a+\sigma)\xi\widetilde{\zeta}\bigr).$$

Substituting this into (iii), we find that

$$\text{(iv)} \qquad c(h,\, q,\, s) = \chi(\delta)^{-n} \int_{S_{\mathbf{A}}} g\bigl(\eta\tau(\sigma)\xi\widetilde{\zeta}\bigr)\mathbf{e}^n_{\mathbf{A}}(-h\sigma)d\sigma.$$

Since $\tau(b)\xi = \xi\tau(q^{-1}b\widehat{q})$, from (16.41) we see that $E^*\bigl(\tau(\sigma+b)\xi\bigr) = E^*\bigl(\tau(\sigma)\xi\bigr)$ if $b \in S_{\mathbf{h}}$ and $q^{-1}b\widehat{q} \prec \mathfrak{d}^{-2}\mathfrak{bc}$. Then (iii) shows that $c(h,\, q,\, s) \neq 0$ only if $({}^tqhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v\widetilde{S}_v$ for every $v \in \mathbf{h}$, which is the first statement of Proposition 16.9. Consequently $\alpha^\lambda_{\mathfrak{c}}(\varepsilon_b^{-1} \cdot {}^tqhq,\, 2s,\, \chi)$ is meaningful.

Our next task is to determine the value of $g(x\widetilde{\zeta})$ for $x = \eta\tau(\sigma)\xi$. Putting $y = x\zeta$, we have

$$y_{\mathbf{h}} = \begin{bmatrix} -\delta\widehat{q} & 0 \\ \delta\sigma\widehat{q} & -\delta^{-1}q \end{bmatrix}_{\mathbf{h}}, \quad y_{\mathbf{a}} = x_{\mathbf{a}} = \begin{bmatrix} 0 & -\widehat{q} \\ q & \sigma\widehat{q} \end{bmatrix}_{\mathbf{a}}, \quad (d_y^{-1}c_y)_{\mathbf{h}} = -\delta^2(q^{-1}\sigma\widehat{q})_{\mathbf{h}}.$$

We assume $\det(q_v) > 0$ for every $v \in \mathbf{a}$. Since $r_\Omega(\eta) = \eta$ and we are identifying $\tau(\sigma)\xi$ with $r_P\bigl(\tau(\sigma)\xi\bigr)$, we have, by (A2.3b), $x = r_\Omega(\eta)r_P\bigl(\tau(\sigma)\xi\bigr) = r_\Omega\bigl(\eta\tau(\sigma)\xi\bigr) = r_\Omega(x)$. Now $j^k(x\widetilde{\zeta}, \mathbf{i}) = j^k_x(\mathbf{i})$ by (16.30), and $j^k_x(\mathbf{i}) = \lambda\det(qi+\sigma\widehat{q})^k_{\mathbf{a}}$ with $\lambda \in \mathbf{T}$. (The branch of $\det(z)^k$ was chosen in §16.8.) Then from Lemma A2.6 we can easily derive that $\lambda\mathbf{e}\bigl(n[F:\mathbf{Q}]/8\bigr) = \omega(-q^{-1}\sigma\widehat{q})$. Since $\omega(-s) = \omega(s)^{-1}$, we have $j^k_x(\mathbf{i})^{-1} = C\omega(q^{-1}\sigma\widehat{q})\det(qi+\sigma\widehat{q})^{-k}_{\mathbf{a}}$ with $C = \mathbf{e}\bigl(n[F:\mathbf{Q}]/8\bigr)$.

Now $g\bigl(x\widetilde{\zeta}\bigr) \neq 0$ only if $y \in P_{\mathbf{A}}D[\mathfrak{b}^{-1},\, \mathfrak{bc}]$, which is so if and only if $(\delta^2q^{-1}\sigma\widehat{q})_v \prec \mathfrak{b}_v\mathfrak{c}_v$ for every $v|\mathfrak{c}$, by virtue of the characterization of $P_{\mathbf{A}}D$ in Lemma 1.9. Assuming the last condition, let $y_{\mathbf{h}} = pw$ with $p \in P_{\mathbf{h}}$ and $w \in D[\mathfrak{b}^{-1},\, \mathfrak{bc}]$; take $b \in F^\times_{\mathbf{h}}$ so that $b\mathfrak{g} = \mathfrak{b}$. By Lemma 1.11 (2),

$$\det(d_yd_p^{-1})\mathfrak{g} = \det(d_y)\mathrm{il}_{\mathfrak{b}}(y)^{-1} = \nu_0(b^{-1}d_y^{-1}c_y) = \nu_0(b^{-1}\delta^2q^{-1}\sigma\widehat{q}).$$

Since $d_y = d_pd_w$, we obtain

$$\text{(v)} \qquad \begin{aligned} \chi_{\mathbf{h}}\bigl(\det(d_p)\bigr)^{-1}\chi_{\mathfrak{c}}\bigl(\det(d_w)\bigr)^{-1} &= \chi_{\mathbf{h}}\bigl(\det(d_y)\bigr)^{-1}(\chi_{\mathbf{h}}/\chi_{\mathfrak{c}})\bigl(\det(d_p^{-1}d_y)\bigr) \\ &= \chi_{\mathbf{h}}\bigl(\det(-\delta q^{-1})\bigr)\chi^*\bigl(\nu_0(b^{-1}\delta^2q^{-1}\sigma\widehat{q})\bigr). \end{aligned}$$

Here notice that $\nu_0(b^{-1}d_y^{-1}c_y)$ is prime to $\mathfrak{c}$ by Lemma 1.11 (3). By (16.23a), $\varepsilon(y_{\mathbf{h}}) = N(\mathrm{il}_{\mathfrak{b}}(y))^{-2} = |\delta^{-n}\det(q_{\mathbf{h}})|_{\mathbf{A}}^2 \nu(b^{-1}\delta^2 q^{-1}\sigma\widehat{q})^2$ and $\varepsilon(y_{\mathbf{a}}) = \varepsilon(x_{\mathbf{a}}) = |j_x(\mathbf{i})|^{2\mathbf{a}}$. Temporarily denote the quantity of (v) by $\mu(y_{\mathbf{h}})$. Combining all these and taking ε_b as in Proposition 16.9, we obtain

$$\begin{aligned}\chi(\delta)^{-n}g(x\widetilde{\zeta}) &= \chi(\delta)^{-n}\mu(y_{\mathbf{h}})\varepsilon(y_{\mathbf{h}})^{-s}\varepsilon(x_{\mathbf{a}})^{-s}j_x^k(\mathbf{i})^{-1}|j_x(\mathbf{i})|^{k-i\kappa} \\ &= C|\delta^n\det(q_{\mathbf{h}}^{-1})|_{\mathbf{A}}^{2s}\chi_{\mathbf{h}}\big(\det(-q^{-1})\big)\chi^*\big(\nu_0(\delta\varepsilon_b q^{-1}\sigma\widehat{q})\big)\nu(\delta\varepsilon_b q^{-1}\sigma\widehat{q})^{-2s} \\ &\quad\cdot\omega(q^{-1}\sigma\widehat{q})\det(qi+\sigma\widehat{q})_{\mathbf{a}}^{-k}|\det(qi+\sigma\widehat{q})_{\mathbf{a}}|^{k-i\kappa-2s\mathbf{a}}.\end{aligned}$$

For a fixed q the last product can be written in the form

$$C|\delta^n\det(q)|_{\mathbf{A}}^{2s}\chi_{\mathbf{h}}\big(\det(-q^{-1})\big)\prod_{v\in\mathbf{v}} f_v(\sigma_v)$$

with functions f_v on S_v which we choose in the manner obvious from the above expression. Then from (iv) we obtain

$$c(h,\,q,\,s) = C|\delta^n\det(q)|_{\mathbf{A}}^{2s}\chi_{\mathbf{h}}\big(\det(-q^{-1})\big)c(S)\prod_{v\in\mathbf{v}}\int_{S_v} f_v(\sigma_v)\mathbf{e}_v^n(-h\sigma_v)d\sigma_v,$$

where $c(S)$ is the factor determined by $d\sigma = c(S)\prod_{v\in\mathbf{v}} d\sigma_v$, whose value is given in [S97, (18.9.3)]. Since $(\delta^2 q^{-1}\sigma\widehat{q})_v \prec \mathfrak{b}_v\mathfrak{c}_v \subset 2\mathfrak{d}_v$ for every $v|\mathfrak{c}$, we have $\nu_0(\delta\varepsilon_b q^{-1}\sigma\widehat{q})_v = \mathfrak{g}_v$ and $(q^{-1}\sigma\widehat{q})_v \prec 2\mathfrak{d}_v^{-1}$ for $v|\mathfrak{c}$. Therefore, from (16.5) and (16.6) we obtain $\omega\big((q^{-1}\sigma\widehat{q})_v\big) = 1$ for $v|\mathfrak{c}$. Thus, for $v|\mathfrak{c}$, we can take f_v to be the characteristic function of $q_v S(\mathfrak{d}^{-2}\mathfrak{b}\mathfrak{c})_v\cdot{}^tq_v$. Put $\mu_v = N(\mathfrak{b}_v\mathfrak{c}_v)^{-n(n+1)/2}|\det(q_v)|^{n+1}$. Since $({}^tqhq)_v \in (\mathfrak{d}\mathfrak{b}^{-1}\mathfrak{c}^{-1})_v\widetilde{S}_v$, the integral over S_v for $v|\mathfrak{c}$ is the measure of the set $q_v S(\mathfrak{d}^{-2}\mathfrak{b}\mathfrak{c})_v\cdot{}^tq_v$, which equals $\mu_v N(\mathfrak{d}_v)^{n(n+1)}$.

For $v\in\mathbf{h}$, $v\nmid\mathfrak{c}$, taking the variable change $\sigma_v\mapsto b_v q_v\sigma_v\cdot{}^tq_v$, we find that

$$\int_{S_v} f_v(\sigma_v)\mathbf{e}_v^n(-h\sigma_v)d\sigma_v = \mu_v\cdot\int_{S_v}\chi^*\big(\nu_0(\sigma_v)\big)\omega(\delta_v^{-1}\sigma_v)\nu(\sigma_v)^{-2s}\mathbf{e}_v^n(-\tau\sigma_v)d\sigma_v,$$

where $\tau = (\varepsilon_b^{-1}\cdot{}^tqhq)_v$. The last integral is clearly the v-factor of $\alpha_{\mathfrak{c}}^1(\tau,\,2s,\,\chi)$.

As for $v\in\mathbf{a}$, the integral over S_v produces $\xi(y_v,\,h_v;\,\ldots)$. We shall not go into details here, since such a calculation is practically the same as what was done in [S97, §18.11] for integral k. Taking the product of all these factors, we can complete the proof. As we said at the beginning, the case of integral k can be proved in a similar and much simpler way.

A2.14. *Proof of Theorem 17.7 (vi).* Suppose $n=1$ and $\mathfrak{c}=\mathfrak{g}$; then $k\in\mathbf{Z}^{\mathbf{a}}$ and $G=P\cup P\eta P$ by [S97, Lemma 2.12 (1)]. Since $P\backslash P\eta P$ can be given by ηR by [S97, Lemma 18.8 (2)], $P\backslash G$ can be given by $\{1\}\cup\eta R$. Thus, by (16.33), for $\xi\in G_{\mathbf{a}}$ we have

$$\begin{aligned}E_{\mathbf{A}}^*(\xi,\,s) &= \chi(\delta)^{-1}\mu(\xi\zeta)\varepsilon(\xi\zeta)^{-s} + E'(\xi,\,s) \\ \text{with}\quad E'(\xi,\,s) &= \chi(\delta)^{-1}\sum_{\alpha\in\eta R}\mu(\alpha\xi\zeta)\varepsilon(\alpha\xi\zeta)^{-s}.\end{aligned}$$

Given $z = x+iy\in\mathfrak{H}_1^{\mathbf{a}}$, take $\xi = \tau(x)\mathrm{diag}[y^{1/2},\,y^{-1/2}]$. We now apply our computation of §A2.13 (for $k\in\mathbf{Z}^{\mathbf{a}}$, or that of [S97, §18.11]) to $E'(\xi,\,s)$ to obtain

$$E'(\xi,\,s)j_\xi^k(\mathbf{i}) = y^{-k/2}\sum_{h\in F} c(h,\,y^{1/2},\,s)\mathbf{e}_{\mathbf{a}}(hx)$$

with quantities $c(h,\,y^{1/2},\,s)$, to which the formula of Proposition 16.9 is applicable; we simply take $\mathfrak{c}=\mathfrak{g}$ in that formula. Observing that ζ of (16.28) belongs to

$\operatorname{diag}[\varepsilon_b^{-1}, \varepsilon_b]D[\mathfrak{b}^{-1}, \mathfrak{b}]$ with ε_b of Proposition 16.9, we see that $\chi(\delta)^{-1}\mu(\xi\zeta)\varepsilon(\xi\zeta)^{-s} = \chi^*(\mathfrak{b}\mathfrak{d}^{-2})N(\mathfrak{b}^{-1}\mathfrak{d})^{2s}y^{s\mathbf{a}}$. Therefore

$$(*) \qquad L(2s, \chi)E^*(z, s) = \chi^*(\mathfrak{b}\mathfrak{d}^{-2})N(\mathfrak{b}^{-1}\mathfrak{d})^{2s}L(2s, \chi)y^{s\mathbf{a}-k/2} + L(2s, \chi)y^{-k/2}\sum_{h\in F} c(h, y^{1/2}, s)\mathbf{e}_{\mathbf{a}}(hx).$$

Our problem is the nature of $D(z, 0; 2\mathbf{a}, \chi, \mathfrak{g})$ for $n = 1$. If $\chi \neq 1$, then (v) of our theorem says that it belongs to $\pi^d\mathcal{M}_{2\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$. Suppose $\chi = 1$; then $L(2s, \chi) = \zeta_F(2s)$ and the analysis of the Fourier coefficients at $s = 0$ in the setting of (v) is applicable to the second term of (*). Thus (*) with $k = 2\mathbf{a}$ at $s = 0$ gives

$$\zeta_F(0)y^{-\mathbf{a}} + \pi^d\sum_{h\in F} a_h\mathbf{e}_{\mathbf{a}}(hz)$$

with $a_h \in \mathbf{Q}_{\mathrm{ab}}$. By Lemma 17.5 (3), $\zeta_F(0) = 0$ if $F \neq \mathbf{Q}$. It is well-known that $\zeta(0) = -1/2$. Thus, for $k = 2\mathbf{a}$ and $s = 0$, (*) produces an element of $\pi^d\mathcal{M}_{2\mathbf{a}}(\mathbf{Q}_{\mathrm{ab}})$ or $\pi\mathcal{N}_2^1(\mathbf{Q}_{\mathrm{ab}})$ according as $F \neq \mathbf{Q}$ or $F = \mathbf{Q}$. This result is for E^*. Transforming E^* back to E by ζ_0 as in (16.35), we can complete the proof.

A3. Transformation formulas of general theta series

A3.1. For $X = (x_{ij}) \in \mathbf{C}_q^q$ and $Y \in \mathbf{C}_n^n$ we define $X \otimes Y \in \mathbf{C}_{nq}^{nq}$ by

$$X \otimes Y = \begin{bmatrix} x_{11}Y & \cdots & x_{1q}Y \\ \cdots & \cdots & \cdots \\ x_{q1}Y & \cdots & x_{qq}Y \end{bmatrix}.$$

Let V be a q-dimensional vector space over F and $S : V \times V \to F$ a nondegenerate F-bilinear symmetric form. For each $v \in \mathbf{v}$ we have an F_v-bilinear symmetric form $S_v : V \times V \to F_v$. For each $v \in \mathbf{a}$ put $I_v = \operatorname{diag}[1_{r_v}, -1_{s_v}]$ with the signature (r_v, s_v) of S_v; we also take and fix an F_v-linear bijection $A_v : V_v \to (F^q)_v$ so that $S_v(x, y) = {}^t(A_vx)I_v(A_vy)$ and put $T_v(x, y) = {}^t(A_vx)(A_vy)$. For $p = (p_1, \ldots, p_n) \in V_v^n$ with $p_i \in V_v$ we define elements $S_v[p]$ and $T_v[p]$ of $(F_v)_n^n$ by $S_v[p] = (S_v(p_i, p_j))_{i,j=1}^n$, $T_v[p] = (T_v(p_i, p_j))_{i,j=1}^n$. Notice that $S_v[p]$ is meaningful also for $v \in \mathbf{h}$.

A3.2. Let us again emphasize the dimension as we did in Section A2, by using the symbols $G^{(n)}, \mathcal{H}^{(n)}, \mathcal{U}^{(n)}$; in addition we use $D^{(n)}[\mathfrak{x}, \mathfrak{y}]$ and $\theta^{(n)}$ for $D[\mathfrak{x}, \mathfrak{y}]$ and θ. We now define a theta function $g(u, z; \lambda)$ for $z \in \mathcal{H}^{(n)}$, $u \in \mathcal{U}^{(nq)}$, and $\lambda \in \mathcal{S}(V_{\mathbf{h}}^n)$ by

$$\text{(A3.0a)} \qquad g(u, z; \lambda) = \sum_{\xi\in V^n} \lambda(\xi_{\mathbf{h}})\Phi(\xi; u, z),$$

$$\text{(A3.0b)} \qquad \Phi(p; u, z) = \prod_{v\in\mathbf{a}} \Phi_v(p_v; u_v, z_v) \qquad (p \in V_{\mathbf{A}}^n),$$

$$\Phi_v(p; u, x + iy) = \mathbf{e}\Big({}^tu(1_q \otimes 4iy)^{-1}u + 2^{-1}\operatorname{tr}\big(xS_v[p] + iyT_v[p]\big) + \operatorname{tr}\big(u'(A_vp)\big)\Big)$$

$$\text{for } p \in V_v^n,\ x + iy \in \mathfrak{H}_n, \text{ and } u \in \mathbf{C}^{nq},$$

where $A_vp = (A_vp_1\ \ldots\ A_vp_n) \in (F_v)_n^q$ for $p = (p_1, \ldots, p_n) \in V_v^n$, $u' = (u_1\ \ldots\ u_q) \in \mathbf{C}_q^n$ for ${}^tu = ({}^tu_1\ \ldots\ {}^tu_q) \in \mathbf{C}_{nq}^1$ with $u_i \in \mathbf{C}^n$. If $q = 1$, $V = F$, $S(x, x) = x^2$, and $A_v = 1$ for all v, then we see that $g(u, z; \lambda)$ coincides with $\theta^{(n)}(u, z; \lambda)$. In the general case g can be obtained as a "pullback" of $\theta^{(nq)}$ as will be shown below. It should be noted that $g(u, z; \lambda) = 0$ for every (u, z) only if $\lambda = 0$.

We let $G_{\mathbf{A}}$ act on $\mathcal{H}^{(n)} \times \mathcal{U}^{(nq)}$ by $\alpha(u, z) = (w, \alpha z)$ for $\alpha \in G_{\mathbf{A}}$ with

$$w_v = \mathrm{diag}[1_{r_v} \otimes {}^t\mu(\alpha, z)_v^{-1},\ 1_{s_v} \otimes {}^t\overline{\mu(\alpha, z)_v^{-1}}]u_v. \tag{A3.1}$$

We now put $\mathfrak{M}_q = \mathfrak{M}$ if q is odd and $\mathfrak{M}_q = G_{\mathbf{A}}$ if q is even, and define a factor of automorphy $J^S(\alpha, z)$ for $\alpha \in \mathfrak{M}_q$ by

$$J^S(\alpha, z) = \begin{cases} \prod_{v \in \mathbf{a}} j_\alpha(z)_v^{(q/2)-s_v} |j_\alpha(z)_v|^{s_v} & \textit{if } q \textit{ is even,} \\ h(\alpha, z)^q \prod_{v \in \mathbf{a}} j_\alpha(z)_v^{-s_v} |j_\alpha(z)_v|^{s_v} & \textit{if } q \textit{ is odd.} \end{cases} \tag{A3.2a}$$

If q is even, J^S is a factor of automorphy; if q is odd, however, Theorem A2.4 (4) implies the following weaker property:

$$J^S(\alpha\beta\gamma, z) = J^S(\alpha, z)J^S(\beta, \gamma z)J^S(\gamma, z) \quad \textit{if } \ \mathrm{pr}(\alpha) \in P_{\mathbf{A}},\ \beta \in \mathfrak{M}, \ \textit{and } \ \mathrm{pr}(\gamma) \in C^\theta. \tag{A3.2b}$$

A3.3. Theorem. *Let χ be the Hecke character of F corresponding to the extension $F(\det(S)^{1/2})/F$ or $F((-1)^{q/4}\det(S)^{1/2})$ according as q is odd or even; let* pr *denote the identity map of $G_{\mathbf{A}}$ onto itself if q is even. Then every $\sigma \in \mathfrak{M}_q$ gives a* $\mathbf{C}$*-linear automorphism $\lambda \mapsto {}^\sigma\lambda$ of $\mathcal{S}(V_{\mathbf{h}}^n)$ with the following properties:*

(0) $J^S(\alpha, z)^{-1}g(\alpha(u, z); {}^\alpha\lambda) = g(u, z; \lambda)$ if $\alpha \in G \cap \mathfrak{M}_q$.

(1) *The map $\lambda \mapsto {}^\sigma\lambda$ does not depend on $\{A_v\}_{v\in\mathbf{a}}$ (though it depends on S).*

(2) ${}^{(\sigma\tau)}\lambda = {}^\sigma({}^\tau\lambda)$ *for every σ* and *τ* in $G_{\mathbf{A}}$ if q is even; ${}^{(\rho\sigma\tau)}\lambda = {}^\rho({}^\sigma({}^\tau\lambda))$ *whenever* $\mathrm{pr}(\rho) \in P_{\mathbf{A}}$ *and* $\mathrm{pr}(\tau) \in C^\theta$ if q is odd.

(3) *${}^\sigma\lambda$ depends only on λ and $\mathrm{pr}(\sigma)_{\mathbf{h}}$.*

(4) $\mathrm{pr}(\{\sigma \in \mathfrak{M}_q \mid {}^\sigma\lambda = \lambda\})$ *contains an open subgroup of $G_{\mathbf{A}}$ for every $\lambda \in \mathcal{S}(V_{\mathbf{h}}^n)$.*

(5) *For $\mathrm{pr}(\sigma) = \tau \in P_{\mathbf{h}}$ we have*

$$({}^\sigma\lambda)(x) = |\det(a_\tau)_{\mathbf{h}}|_{\mathbf{A}}^{q/2}\chi_{\mathbf{h}}(\det(a_\tau))\mathbf{e}_{\mathbf{h}}\Bigl(\mathrm{tr}\bigl(S[x]a_\tau \cdot {}^tb_\tau\bigr)/2\Bigr)\lambda(xa_\tau),$$

where $S[x] = (S_v[x_v])_{v\in\mathbf{h}}$ and $(x_1, \dots, x_n)a = (\sum_{i=1}^n x_i a_{ij})_{j=1}^n$ for $(x_1, \dots, x_n) \in V_{\mathbf{h}}^n$ and $a \in (F_{\mathbf{h}})_n^n$.

(6)
$$({}^\eta\lambda)(x) = i^r \int_Y \lambda(y)\mathbf{e}_{\mathbf{h}}\left(-\sum_{i=1}^n S(x_i, y_i)\right) d^S y.$$

Here $Y = V_{\mathbf{h}}^n$, $d^S y$ is the Haar measure on Y such that the measure of $(\sum_{i=1}^q \mathfrak{g}_v e_i)^n$ for each $v \in \mathbf{h}$ with a basis $\{e_i\}_{i=1}^q$ of V over F is $N(\mathfrak{d}_v)^{-qn/2}$ $|\det(S(e_i, e_j))|_v^{n/2}$, and $r = (n/2)\sum_{v\in\mathbf{r}}(r_v - s_v)$ or $-n\sum_{v\in\mathbf{r}} s_v$ according as q is even or odd.

(7) $J_S(-\eta, \eta z)J_S(\eta, z) = 1$ *and* ${}^{-\eta}({}^\eta\lambda) = \lambda$.

Since V has no fixed coordinate system, $\det(S)$ means the coset of $\det(S(e_i, e_j))$ modulo $\{a^2 \mid a \in F^\times\}$ with $\{e_i\}_{i=1}^n$ as in (6) above. Clearly χ is well-defined. Thus our theorem is "coordinate-free" (as far as V is concerned). In the proof, however, we use a matrix representation, and so hereafter we assume that $V = F^q$ and $S(x,y) = {}^txSy$ for $x, y \in F_n^q$ with ${}^tS = S \in GL_q(F)$. Then $A_v \in GL_q(F_v)$, $S_v = {}^tA_vI_vA_v$, and $T_v = {}^tA_vA_v$ for each $v \in \mathbf{a}$. In this setting, the formula of (6) can be written

(A3.3) $$({}^{\eta}\lambda)(x) = i^r|N_{F/\mathbf{Q}}(\det(S)|^{-n/2}\int_Y \lambda(y)\mathbf{e}_{\mathbf{h}}\big(-\mathrm{tr}({}^t xSy)\big)dy,$$

where $Y = (F_n^q)_{\mathbf{h}}$, and the measure of $(\mathfrak{g}_v)_n^q$ is $N(\mathfrak{d}_v)^{-nq/2}$ for each $v \in \mathbf{h}$.

The proof of our theorem requires some preliminaries. We first define an embedding $\psi : \mathcal{H}^{(n)} \to \mathcal{H}^{(nq)}$ and an injective homomorphism $\alpha \mapsto \alpha_S$ of $G_{\mathbf{A}}^{(n)}$ into $G_{\mathbf{A}}^{(nq)}$ by $\psi(z) = \big(\psi_v(z_v)\big)_{v\in\mathbf{a}}$,

$$\psi_v(x+iy) = S_v \otimes x + iT_v \otimes y, \qquad \alpha_S = \begin{bmatrix} 1_q \otimes a_\alpha & S \otimes b_\alpha \\ S^{-1} \otimes c_\alpha & 1_q \otimes d_\alpha \end{bmatrix}.$$

It can easily be verified that $\psi\big(\alpha(z)\big) = \alpha_S\big(\psi(z)\big)$ and

$$\mu\big(\alpha_S,\, \psi(z)\big)_v = (A_v \otimes 1_n)^{-1}\mathrm{diag}[1_{r_v} \otimes \mu(\alpha,\, z)_v,\, 1_{s_v} \otimes \overline{\mu(\alpha,\, z)_v}](A_v \otimes 1_n)$$

for each $v \in \mathbf{a}$. From these we obtain immediately

(A3.4) $$j_{\alpha_S}\big(\psi(z)\big)^{\mathbf{a}} = j_\alpha(z)^{q\mathbf{a}}\prod_{v\in\mathbf{a}} j_\alpha(z)_v^{-s_v}\overline{j_\alpha(z)_v}^{s_v}.$$

To find a relationship between $J^S(\alpha,\, z)$ and $h\big(\alpha_S,\, \psi(z)\big)$, we take integral ideals $\mathfrak{b}$ and $\mathfrak{c}$ such that $\alpha_S \in D^{(nq)}[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]$ for every $\alpha \in D^{(n)}[2\mathfrak{b}\mathfrak{d}^{-1},\, 2\mathfrak{c}\mathfrak{d}]$. Then $S \in GL_q(\mathfrak{g}_v)$ for every $v \nmid \mathfrak{b}\mathfrak{c}$.

A3.4. Lemma. *For $\alpha \in G \cap P_{\mathbf{A}}D[2\mathfrak{b}\mathfrak{d}^{-1},\, 2\mathfrak{c}\mathfrak{d}]$ we have*

$$h\big(\alpha_S,\, \psi(z)\big) = \chi_{\mathbf{a}}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}(\alpha)^{-1}\big)J^S(\alpha,\, z),$$

where χ^* is the ideal character associated with χ of Theorem A3.3.

PROOF. From Theorem A2.4 (1), (2), and (A3.4) we easily see that

(A3.5) $$h\big(\alpha_S,\, \psi(z)\big) = tJ^S(\alpha,\, z) \quad \textit{with} \quad t \in \mathbf{T}.$$

By Lemma A2.6 we have

$$\lim_{z\to 0} h(\alpha,\, z)/|h(\alpha,\, z)| = \omega, \quad \lim_{z\to 0} h\big(\alpha_S,\, \psi(z)\big)/\big|h\big(\alpha_S,\, \psi(z)\big)\big| = \omega'$$

with $\omega = \omega(\delta^{-2}d_\alpha^{-1}c_\alpha)$ and $\omega' = \omega(\delta^{-2}S^{-1} \otimes d_\alpha^{-1}c_\alpha)$. Let $\alpha \in P_{\mathbf{A}}\tau$ with $\tau \in D[2\mathfrak{b}\mathfrak{d}^{-1},\, 2\mathfrak{c}\mathfrak{d}]$; write simply c and d for c_τ and d_τ. Clearly $d_\alpha^{-1}c_\alpha = d^{-1}c$. From (A3.2a) and (A3.4) we see that $\omega' = t\chi_{\mathbf{a}}\big(\det(d_\alpha)\big)\xi$, where $\xi = 1$ or $\xi = \omega^q$ according as q is even or odd. If $v \nmid 2\mathfrak{b}\mathfrak{c}$, then $S^{-1} = {}^tT_v \cdot \mathrm{diag}[s_1,\, \ldots,\, s_q]T_v$ with $T_v \in GL_q(\mathfrak{g}_v)$ and $s_i \in \mathfrak{g}_v^\times$. By Lemma A1.6 (1) we see that

$$\gamma_v(\delta^{-2}S^{-1} \otimes d^{-1}c) = \prod_{i=1}^q \gamma_v(\delta^{-2}s_i d^{-1}c) = \gamma_v(\delta^{-2}d^{-1}c)^q\left(\frac{\det(S)}{\nu_0(\delta^{-1}d^{-1}c)_v}\right).$$

By Lemma 1.11 (2) we have $\nu_0(\delta^{-1}d^{-1}c) = \nu_0(\delta^{-1}d_\alpha^{-1}c_\alpha) = \det(d_\alpha)\mathrm{il}_{\mathfrak{d}}(\alpha)^{-1}$. If $v|2\mathfrak{b}\mathfrak{c}$, then $|\det(d)|_v = 1$, and hence both $(2^{-1}\delta^{-1}d^{-1}c)_v$ and $(2^{-1}\delta^{-1}S^{-1}\otimes d^{-1}c)_v$ are v-integral. Therefore (A1.5) shows that $\gamma_v = \gamma_v' = 1$. Notice that $\prod_{v\nmid 2}\omega_v^q = \psi^*\big(\nu_0(\delta^{-1}d^{-1}c)\big)^{q/2}$ by Lemma A1.6 (2), if q is even, where ψ^* is as in Proposition A2.7. Combining all these, we obtain our lemma.

PROOF OF THEOREM A3.3. We are identifying V with F^q and V^n with F_n^q, and hence $S_v[p] = {}^tpS_vp$ and $T_v[p] = {}^tpT_vp$ for $p \in (F_n^q)_v$. Define $\mathcal{A} : \mathcal{U}^{(nq)} \to \mathcal{U}^{(nq)}$ by $\mathcal{A}(u)_v = ({}^tA_v \otimes 1_n)u_v$ and also $\omega : F_{nq}^1 \to F_n^q$ by $\omega(x_1\ \ldots\ x_q) = {}^t({}^tx_1\ \ldots\ {}^tx_q)$ for $x_i \in F_n^1$. Then a straightforward calculation shows that

(A3.6a) $$g(u,\, z;\, \lambda) = \theta^{(nq)}\big(\mathcal{A}(u),\, \psi(z);\, \lambda \circ \omega\big),$$

(A3.6b) $$\alpha_S\big(\mathcal{A}(u),\, \psi(z)\big) = \big(\mathcal{A}(w),\, \psi(\alpha z)\big) \quad \textit{if} \quad \alpha(u,\, z) = (w,\, \alpha z).$$

For $\alpha \in G \cap \mathfrak{M}_q$ we can find an element $\xi \in \prod_{v \in \mathbf{a}} Mp((F^1_{nq})_v)$ such that $\mathrm{pr}(\xi) = (\alpha_S)_{\mathbf{a}}$ and $\prod_{v \in \mathbf{a}} g(\xi_v, \psi(z)_v) = J^S(\alpha, z)$. Then, for $\ell = \lambda \circ \omega$ Proposition A2.5 together with (A3.6a, b) shows that

$$J^S(\alpha, z)g(u, z; \lambda) = \theta^{(nq)}\big(\alpha_S(\mathcal{A}(u), \psi(z)); \ell'\big) = g\big(\alpha(u, z); \lambda'\big)$$

with $\lambda' = \ell' \circ \omega^{-1}$. Putting $\lambda' = {}^{\alpha}\lambda$, we obtain (0). Let $\mathfrak{k}$ be the conductor of χ; let $E = D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{c}\mathfrak{d}]$ and $E' = \{\, \alpha \in E \mid \chi_{\mathfrak{k}}\big(\det(d_\alpha)\big) = 1 \,\}$, where $\chi_{\mathfrak{k}} = \prod_{v|\mathfrak{k}} \chi_v$. If $\alpha \in G \cap P_{\mathbf{A}}E$, then Lemma A3.4 and Proposition A2.5, combined with the above argument, show that

$$(*) \qquad {}^{\alpha}\lambda \circ \omega = \chi_{\mathbf{a}}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}_{\mathfrak{d}}(\alpha)^{-1}\big) \cdot {}^{\beta}(\lambda \circ \omega)$$

with $\beta = \alpha_S$. In particular, ${}^{\alpha}\lambda \circ \omega = {}^{\beta}(\lambda \circ \omega)$ if $\alpha \in G \cap E'$. This together with Theorem A2.4, (6) shows that ${}^{\alpha}\lambda = \lambda$ for every α in a congruence subgroup Γ of G depending on λ. Take an open subgroup D of E' so that $\Gamma \supset G \cap D$. We take $D \subset C^\theta$ if q is odd. By strong approximation, given $\sigma \in \mathfrak{M}_q$, we can take $\alpha \in G$ so that $\mathrm{pr}(\sigma) \in \alpha D$. Then $\alpha \in \mathfrak{M}_q$. Define ${}^{\sigma}\lambda$ to be ${}^{\alpha}\lambda$. It is then easy to verify that this is well-defined and has property (2) for even q, and also properties (3) and (4). To prove (5), let $\mathrm{pr}(\sigma) = \tau \in P_{\mathbf{h}}$. Given $\lambda \in \mathcal{S}((F^q_n)_{\mathbf{h}})$, take $\alpha \in G$ so that $\tau \in \alpha E'$ and that ${}^{\sigma}\lambda = {}^{\alpha}\lambda$. Then we easily see that

$$(**) \qquad \chi_{\mathbf{a}}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}_{\mathfrak{d}}(\alpha)^{-1}\big) = \chi\big(\det(a_\tau)\big).$$

Let $\ell = \lambda \circ \omega$ and $\beta = \alpha_S$. From (*) and (**) we obtain ${}^{\beta}\ell = \chi\big(\det(a_\tau)\big)({}^{\alpha}\lambda) \circ \omega$. Changing E' for a smaller group if necessary, we may assume that τ_S and α_S have the same effect on ℓ. Then we obtain

$$(***) \qquad \chi\big(\det(a_\tau)\big)({}^{\alpha}\lambda) \circ \omega = {}^{\varphi}\ell \quad \textit{with} \quad \mathrm{pr}(\varphi) = \tau_S.$$

Taking φ to be $r_P(\tau_S)$, from Theorem A2.4 (8) and (A2.3a) we obtain

$${}^{\varphi}\ell(y) = |\det(1_q \otimes a_\tau)|_{\mathbf{A}}^{1/2}\, \mathbf{e_h}\big(y(S \otimes a_\tau \cdot {}^t b_\tau) \cdot {}^t y/2\big)\ell\big(y(1_q \otimes a_\tau)\big),$$

which combined with (***) proves (5). To prove (6), or rather (A3.3), we first observe that (A3.5) is valid for $\alpha = \eta$. Now the first formula of Lemma A2.6 shows that both $h(\eta, z)$ and $h\big(\eta_S, \psi(z)\big)$ are positive if $z \in \mathbf{R}\mathbf{i}$ with $\mathbf{i}$ of that lemma, and hence $i^r \cdot h\big(\eta_S, \psi(z)\big) = J^S(\eta, z)$ with r as in (6). Then from (A3.6a, b) we obtain ${}^{\eta}\lambda \circ \omega = i^r \cdot {}^{\gamma}(\lambda \circ \omega)$ with $\gamma = \eta_S$. Observe that γ belongs to the set $P_{\mathbf{A}}C^*$ of Theorem A2.4 (7) (of degree nq). Therefore, by (A2.3c) we have

$${}^{\gamma}\ell(x) = |\det(c_\gamma)_{\mathbf{h}}|_{\mathbf{A}}^{1/2} \int_Y \ell(yc_\gamma)\mathbf{e_h}(xb_\gamma \cdot {}^t c_\gamma \cdot {}^t y)dy \qquad (Y = (F^1_{nq})_{\mathbf{h}}),$$

which can easily be transformed to (A3.3). To prove (2) when q is odd, first let $\sigma \in \mathfrak{M}$ and $\mathrm{pr}(\rho) \in C^\theta$. With an open normal subgroup D of C^θ, take $\alpha \in G \cap \mathrm{pr}(\sigma)D$ and $\beta \in G \cap \mathrm{pr}(\rho)D$. Then $\alpha\beta \in G \cap \mathrm{pr}(\sigma\rho)D$. Take D so small that ${}^{\sigma}({}^{\beta}\lambda) = {}^{\alpha}({}^{\beta}\lambda)$, ${}^{\beta}\lambda = {}^{\rho}\lambda$, and ${}^{(\sigma\rho)}\lambda = {}^{(\alpha\beta)}\lambda$. Since $\beta \in G \cap C^\theta$, (A3.2b) shows that $J^S(\alpha\beta, z) = J^S(\alpha, \beta z)J^S(\beta, z)$, and hence we obtain ${}^{(\alpha\beta)}\lambda = {}^{\alpha}({}^{\beta}\lambda)$ from (0). Thus ${}^{(\sigma\rho)}\lambda = {}^{(\alpha\beta)}\lambda = {}^{\alpha}({}^{\beta}\lambda) = {}^{\sigma}({}^{\beta}\lambda) = {}^{\sigma}({}^{\rho}\lambda)$. Next let $\mathrm{pr}(\tau) \in P_{\mathbf{A}}$ and $\sigma \in \mathfrak{M}$. Then $\sigma = \pi\rho$ with $\mathrm{pr}(\pi) \in P_{\mathbf{A}}$ and $\mathrm{pr}(\rho) \in C^\theta$. From (***) we see that ${}^{(\tau\pi)}\zeta = {}^{\tau}({}^{\pi}\zeta)$ for every $\zeta \in \mathcal{S}((F^q_m)_{\mathbf{h}})$. Therefore ${}^{(\tau\sigma)}\lambda = {}^{(\tau\pi\rho)}\lambda = {}^{(\tau\pi)}({}^{\rho}\lambda) = {}^{\tau}({}^{\pi}({}^{\rho}\lambda)) = {}^{\tau}({}^{(\pi\rho)}\lambda) = {}^{\tau}({}^{\sigma}\lambda)$. This proves (2). To prove (1), it is sufficient to show that ${}^{\alpha}\lambda$ for $\alpha \in G \cap \mathfrak{M}_q$ is independent of $\{\, A_v \,\}$. If q is even, this follows from (5) and (6), since P and η generate G. Suppose q is odd. By (2) and (5), it is sufficient to show that ${}^{\alpha}\lambda$ for $\alpha \in G \cap C^\theta$ is independent of $\{\, A_v \,\}$. Now $\alpha^m \in D[2\mathfrak{b}\mathfrak{d}^{-1}, 2\mathfrak{c}\mathfrak{d}]$ for some positive integer m. Therefore Lemma A3.4 shows that $h(\alpha_S, \psi(z)) = tJ^S(\alpha, z)$ with a root of unity t. Then ${}^{\alpha}\lambda \circ \omega = t^{-1} \cdot {}^{\gamma}(\lambda \circ \omega)$ with $\gamma = \alpha_S$. Now the set of all $\{A_v\}$ is not necessarily connected, but it can easily be shown that the set of all $\{T_v\}$ is connected. Since $h(\alpha_S, \psi(z))$ is continuous in $\{T_v\}$, we see that t does not depend on $\{A_v\}$. This proves (1). Finally, to prove (7), we recall that $h(-\eta, z) = h(\eta, z)$ as stated in Theorem A2.4 (4). Therefore from (A2.24) we easily see that $h(-\eta, z)h(\eta, z) = 1$. This combined with (A3.2a) proves the first half of (7). Then the second half of (7) follows from (0).

A3.5. We are going to introduce a series involving harmonic polynomials on V^n. Given a finite-dimensional complex vector space W and $0 \le a \in \mathbf{Z}$, we denote by $\mathfrak{S}_a(W)$ the vector space of all $\mathbf{C}$-valued homogeneous polynomial functions on W of degree a. We then denote by $\mathcal{P}_a(\mathbf{C}^q_m)$ the vector subspace of $\mathfrak{S}_a(\mathbf{C}^q_m)$ spanned by the functions p satisfying the condition

$$\text{(A3.7a)} \qquad \sum_{i=1}^{q} \partial^2 p/\partial x_{ih}\partial x_{ik} = 0 \quad \textit{for every} \quad h \textit{ and } k,$$

where $x = (x_{ih})$ is a variable on $\mathbf{C}^q_m$. For instance, we can take $p(x) = \varphi({}^t\rho x)$ with $\varphi \in \mathfrak{S}_a(\mathbf{C}^m_m)$ and $\rho \in \mathbf{C}^q_m$ satisfying the condition

$$\text{(A3.7b)} \qquad {}^t\rho_h\rho_k = 0 \quad \textit{whenever} \quad \partial^2\varphi/\partial y_{hi}\partial y_{kj} \neq 0 \textit{ for some } i \textit{ and } j,$$

where ρ_h denotes the h-th column of ρ, and $y = (y_{hi})$ is a variable on $\mathbf{C}^m_m$.

A3.6. Lemma. *Let* $\omega(x) = \exp\left(\sum_{h,k=1}^{m}\sum_{i=1}^{q} c_{hk}x_{ih}x_{ik}\right)$ *for* $x \in \mathbf{R}^q_m$ *with* $c_{hk} = c_{kh} \in \mathbf{C}$. *Then* $[p(D)(\omega\psi)](0) = [p(D)\psi](0)$ *for every* $p \in \mathcal{P}_a(\mathbf{C}^q_m)$ *and every* C^∞ *function* ψ *in* x, *where* D *is the* $(q \times m)$*-matrix whose* (i, h)*-entry is* $\partial/\partial x_{ih}$.

PROOF. We first observe that if α is a polynomial in n variables $y_1, \dots, y_n$ and $\alpha_i = \partial\alpha/\partial y_i$, then

$$(*) \qquad [\alpha(\partial/\partial y_1, \dots, \partial/\partial y_n)(y_i\beta)](0) = [\alpha_i(\partial/\partial y_1, \dots, \partial/\partial y_n)\beta](0)$$

for every i and every C^∞ function β. This is completely elementary. Now our lemma is trivial if $a = 0$. Assume that it is true for degree $< a$ and that $a > 0$. We have $ap(x) = \sum_{i,h} x_{ih}p_{ih}(x)$ with $p_{ih} = \partial p/\partial x_{ih}$, and hence

$$\begin{aligned}
[ap(D)(\omega\psi)](0) &= \sum_{i,h}[p_{ih}(D)\partial/\partial x_{ih}(\omega\psi)](0) \\
&= \sum_{i,h}[p_{ih}(D)(\omega \cdot \partial\psi/\partial x_{ih})](0) + \sum_{i,h,k} 2c_{hk}[p_{ih}(D)(x_{ik}\omega\psi)](0).
\end{aligned}$$

Since $p_{ih} \in \mathcal{P}_{a-1}(\mathbf{C}^q_m)$, by our induction assumption the first sum on the last line is $\sum_{i,h}[p_{ih}(D)\partial\psi/\partial x_{ih}](0)$, which is $[ap(D)\psi](0)$. By (*) the second sum equals $\sum_{i,h,k} 2c_{hk}[(\partial p_{ih}/\partial x_{ik})(D)(\omega\psi)](0)$, which is 0 by (A3.7a). This completes the proof.

A3.7. Coming back to the space V and the form S, for each $v \in \mathbf{a}$ put

$$\text{(A3.8a)} \qquad X_v^+ = \{x \in V_v \mid (A_v x)_i = 0 \quad \textit{for} \quad i > r_v\},$$

$$\text{(A3.8b)} \qquad X_v^- = \{x \in V_v \mid (A_v x)_i = 0 \quad \textit{for} \quad i \le r_v\},$$

where y_i for $y \in \mathbf{R}^q$ means the i-th component of y. Clearly $V_v = X_v^+ \oplus X_v^-$. For $y \in V_v$, we denote by y^+ and y^- the projections of y to X_v^+ and X_v^-.

Given m and m' in $\mathbf{Z}^\mathbf{a}$ whose components are all nonnegative, we denote by $\mathcal{P}_{m,m'}(V^n)$ the vector space over $\mathbf{C}$ spanned by all functions p on $V^n_\mathbf{a} = \prod_{v\in\mathbf{a}} V^n_v$ of the form

$$p(x) = \prod_{v\in\mathbf{a}} p_v(A_v x_v^+) p'_v(A_v x_v^-) \tag{A3.9}$$

with $p_v \in \mathcal{P}_{m_v}(\mathbf{C}_n^{r_v})$, $p'_v \in \mathcal{P}_{m'_v}(\mathbf{C}_n^{s_v})$, where $A_v x_v^\pm = (A_v x_{v1}^\pm, \ldots, A_v x_{vn}^\pm)$ for $x_v = (x_{v1}, \ldots, x_{vn})$ with $x_{vi} \in V_v$. Write p of (A3.9) as $p = (p_v, p'_v)$. We let every element μ of $GL_n(\mathbf{C})^{\mathbf{a}}$ act $\mathbf{C}$-linearly on $\mathcal{P}_{m,m'}(V^n)$ by defining $\mu p = (\mu_v p_v, \overline{\mu}_v p'_v)$ with $(\nu s)(y) = s(y\nu)$ for $s = p_v, p'_v$, and $\nu = \mu_v$ or $\overline{\mu}_v$. Notice that this action is compatible with both (A3.7a) and (A3.7b).

Now for $z \in \mathcal{H}$, $\lambda \in \mathcal{S}(V_{\mathbf{h}}^n)$, and $p \in \mathcal{P}_{m,m'}(V^n)$ we consider a series

$$f(z;\, \lambda,\, p) = \sum_{\xi\in V^n} \lambda(\xi_{\mathbf{h}}) p(\xi_{\mathbf{a}}) \varPhi(\xi;\, 0,\, z). \tag{A3.10}$$

A3.8. Theorem. *The notation* ${}^\alpha\lambda$ *being as in Theorem A3.3, we have*

$$J^S(\alpha, z)^{-1} f\big(\alpha(z);\, {}^\alpha\lambda,\, {}^t\mu(\alpha, z)^{-1} p\big) = f(z;\, \lambda,\, p)$$

for every $\alpha \in G \cap \mathfrak{M}_q$.

PROOF. Write the variable u in the form ${}^t u_v = (u_{v1}^1, \ldots, u_{v1}^n, \ldots, u_{vq}^1, \ldots, u_{vq}^n)$. We can then define a differential operator $B = \prod_{v\in\mathbf{a}} p_v(D_v) p'_v(D'_v)$ on $\mathcal{U}$, where $D_v = (\partial/\partial u_{vi}^j)$ with $1 \le i \le r_v$, $1 \le j \le n$, $D'_v = (\partial/\partial u_{vi}^j)$ with $r_v < i \le q$, $1 \le j \le n$, $E_v = (\partial/\partial a_{vi}^j)$, $E'_v = (\partial/\partial b_{vi}^j)$ with $1 \le i \le q$, $1 \le j \le n$. Employing Lemma A3.6 we can easily verify that $[Bg(u, z;\, \lambda)]_{u=0} = (2\pi i)^N f(z;\, \lambda,\, p)$, where $N = \sum_{v\in\mathbf{a}} (m_v + m'_v)$. Therefore we obtain our assertion by applying B to the equality of Theorem A3.3 (0).

We can associate with the above f a function $f_{\mathbf{A}}(x;\, \lambda,\, p)$ with a variable x on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$, according as q is even or odd, by

$$f_{\mathbf{A}}(\alpha w;\, \lambda,\, p) = J^S(w, \mathbf{i})^{-1} f\big(w(\mathbf{i});\, {}^w\lambda,\, {}^t\mu(w, \mathbf{i})^{-1} p\big) \tag{A3.11}$$

for $\alpha \in G$ and $w \in \mathfrak{M}_q$, where $\mathbf{i}$ is as in Lemma A2.6; we take $\mathrm{pr}(w) \in C^\theta$ if q is odd. This is well-defined by virtue of Theorem A3.8. Now we have

A3.9. Proposition. *For every* $\alpha \in G$ *and* $y \in \mathfrak{M}_q$ *such that* $y(\mathbf{i}) = \mathbf{i}$ *and that* $\mathrm{pr}(y) \in C^\theta$ *if* q *is odd, we have*

$$f_{\mathbf{A}}(\alpha x y;\, \lambda,\, p) = J^S(y, \mathbf{i})^{-1} f_{\mathbf{A}}\big(x;\, {}^y\lambda,\, {}^t\mu(y, \mathbf{i})^{-1} p\big).$$

PROOF. Given x, take $\beta \in G$ and $w \in \mathfrak{M}_q$ so that $x = \beta w$ and $\mathrm{pr}(w) \in C^\theta$. Then

$$\begin{aligned} f_{\mathbf{A}}(\alpha x y,\, \lambda,\, p) &= f_{\mathbf{A}}(wy,\, \lambda,\, p) \\ &= J^S(wy, \mathbf{i})^{-1} f\big(w(\mathbf{i}),\, {}^{(wy)}\lambda,\, {}^t\mu(wy, \mathbf{i})^{-1} p\big) \\ &= J^S(y, \mathbf{i})^{-1} J^S(w, \mathbf{i})^{-1} f\big(w(\mathbf{i}),\, {}^w({}^y\lambda),\, {}^t\mu(w, \mathbf{i})^{-1} \cdot {}^t\mu(y, \mathbf{i})^{-1} p\big) \\ &= J^S(y, \mathbf{i})^{-1} f_{\mathbf{A}}(x,\, {}^y\lambda,\, {}^t\mu(y, \mathbf{i})^{-1} p). \end{aligned}$$

A3.10. Proposition. *The notation being as in Proposition A2.10, we have* $J^S(\sigma^*,\, -z^\rho) = \overline{J^S(\sigma, z)}$ *and* ${}^{\sigma^*}(\lambda^*) = ({}^\sigma\lambda)^*$ *for every* $\sigma \in \mathfrak{M}_q$ *and* $\lambda \in \mathcal{S}(V_{\mathbf{h}}^n)$, *where* λ^* *is defined by* $\lambda^*(x) = \overline{\lambda(-x)}$.

PROOF. The first equality follows from Proposition A2.10 and (A3.2a) immediately. By virtue of strong approximation and (4) of Theorem A3.3, it is sufficient to prove the second assertion when $\sigma \in G \cap \mathfrak{M}_q$, in which case the desired fact follows from (0) of Theorem A3.3, since $g(\overline{u},\, -z^\rho;\, \lambda^*) = \overline{g(u,\, z;\, \lambda)}$.

A3.11. Remark. (I) Define an algebraic group $O(S)$ by

$$O(S) = \{\, \alpha \in GL(V) \mid S(\alpha x,\, \alpha x) = S(x,\, x) \,\}.$$

Fixing $(A_v)_{v\in\mathbf{a}}$ as above, put $A_v^\alpha = A_v\alpha_v$ for every $\alpha \in O(S)_\mathbf{a}$. Then we can define our series g and f with A_v^α in place of A_v. Thus g and f are essentially parametrized by $O(S)_\mathbf{a}$.

(II) If $r \le \mathrm{Min}(q, m)$, we easily see that the subdeterminants of $x \in \mathbf{C}_m^q$ of degree r define elements of $\mathcal{P}_r(\mathbf{C}_m^q)$. In particular, if $r_v = q = n$, we can take $p_v(A_v x_v^+) = \det(A_v x_v)$ in (A3.9). In this case $\mu_v p_v = \det(\mu_v)p_v$.

(III) Take $q \ge n$ and $S = T \in GL_q(F)$ with $r_v = q$ and $s_v = 0$ for every $v \in \mathbf{a}$; take also a subset $\mathbf{a}'$ of $\mathbf{a}$. For $x \in \mathbf{C}_n^q$ and $v \in \mathbf{a}'$ let $p_v(x)$ be the determinant of the first n rows of x; let $p_v = 1$ if $v \notin \mathbf{a}'$. We have clearly

$$f(z;\, \lambda,\, p) = \sum_{\xi\in F_n^q} \lambda(\xi_\mathbf{h})p(\xi_\mathbf{a})\mathbf{e}_\mathbf{a}^n(2^{-1}\cdot {}^t\xi S\xi z).$$

Then Theorem A3.8 shows that this is an element of $\mathcal{M}_k$ with $k = (q/2)\mathbf{a} + \mathbf{a}'$. It is a cusp form if $\mathbf{a}' \ne \varnothing$. Indeed, by Lemma 7.5, G is generated by $G \cap \mathfrak{M}_q$, and hence Theorem A3.8 shows that the tranform of $f(z;\, \lambda,\, p)$ by an element of G (understood in the sense of (10.12) if $k \notin \mathbf{Z}^\mathbf{a}$) is of the form $f(z;\, \lambda',\, p)$ with some λ'. If $\mathbf{a}' \ne \varnothing$, we have clearly $f(z;\, \lambda',\, p) = \sum_{h\in S} c(h)\mathbf{e}_\mathbf{a}^n(hz)$ with $c(h) \ne 0$ only for $\det(h) \ne 0$; thus $f(z;\, \lambda,\, p)$ is a cusp form. To find λ such that $f(z;\, \lambda,\, p) \ne 0$, take $\xi \in \mathfrak{g}_n^q$ so that $p(\xi_\mathbf{a}) \ne 0$ and put $X = \{\, \xi' \in \mathfrak{g}_n^q \mid {}^t\xi' S\xi' = {}^t\xi S\xi \,\}$. Then X is a finite set. Therefore we can easily find $\lambda \in \mathcal{S}\big((F_n^q)_\mathbf{h}\big)$ such that $\lambda(\xi) \ne 0$ and $\lambda(\xi') = 0$ if $\xi \ne \xi' \in X$ or $\xi' \notin \mathfrak{g}_n^q$. Then clearly $f(z;\, \lambda,\, p) \ne 0$.

Let us now derive from Theorem A3.3 explicit formulas for ${}^\alpha\lambda$ in two forms convenient in applications. For simplicity, we put ${}^\alpha\lambda = {}^\beta\lambda$ if $\alpha = \mathrm{pr}(\beta)$ with $\beta \in \mathfrak{M}_q$, q odd. This is meaningful in view of Theorem A3.3 (3). Now our first formula is:

A3.12. Lemma. *Let $\mathfrak{k}$ be the conductor of χ of Theorem A3.3, and χ^* the ideal character associated with χ. For $\lambda \in \mathcal{S}(V_\mathbf{h}^n)$ put*

$$U_\lambda = \{\sigma \in D[2\mathfrak{d}^{-1},\, 2\mathfrak{k}\mathfrak{d}] \mid {}^\sigma\lambda = \lambda \ \textit{and}\ \det(d_\sigma)_v \equiv 1 \pmod{\mathfrak{k}_v} \textit{ for every } v|\mathfrak{k}\}.$$

Then, for every $\alpha \in \mathrm{diag}[p,\, {}^tp^{-1}]U_\lambda$ with $p \in GL_n(F_\mathbf{h})$ we have

$$({}^\alpha\lambda)(x) = |\det(p)_\mathbf{h}|_\mathbf{A}^{q/2}\chi_\mathbf{h}(\det(p))\lambda(xp)$$

where xp is as in Theorem A3.3 (5). In particular, if such an α belongs to G, then

$$({}^\alpha\lambda)(x) = \chi_\mathbf{a}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}_\mathfrak{d}(\alpha)^{-1}\big)N(\mathrm{il}_\mathfrak{d}(\alpha))^{q/2}\lambda(xp).$$

Proof. Let $\alpha = \tau\sigma$ with $\tau = \mathrm{diag}[p,\, {}^tp^{-1}]$ and $\sigma \in U_\lambda$. By Theorem A3.3 (2, 5) we have $({}^\alpha\lambda)(x) = ({}^\tau\lambda)(x) = |\det(p)_\mathbf{h}|_\mathbf{A}^{q/2}\chi_\mathbf{h}(\det(p))\lambda(xp)$. Suppose $\alpha \in G$. Since ${}^tpd_\alpha = d_\sigma$, we have

$$\chi_\mathbf{a}\big(\det(d_\alpha)\big)\chi^*\big(\det(d_\alpha)\mathrm{il}_\mathfrak{d}(\alpha)^{-1}\big) = \chi_\mathbf{h}(\det(p))\chi_\mathfrak{k}\big(\det(d_\sigma^{-1})\big) = \chi_\mathbf{h}(\det(p)),$$

which completes the proof.

A3.13. Proposition. *Given* $\lambda \in \mathcal{S}(V_{\mathbf{h}}^n)$, *let* M *be a* $\mathfrak{g}$*-lattice in* V^n *such that* $\lambda(x+u) = \lambda(x)$ *for every* $u \in M$. *Further let* $\mathfrak{x}$, $\mathfrak{y}$, *and* $\mathfrak{z}$ *be fractional ideals of* F *with the following properties:*

(i) $2(1+\delta_{ij})^{-1}S(x_i, x_j) \in \mathfrak{x}$ *for every* i, j *and every* $x \in V^n$ *such that* $\lambda(x) \neq 0$.

(ii) $2(1+\delta_{ij})^{-1}S(y_i, y_j) \in \mathfrak{y}$ *for every* i, j *and every* $y \in M'$, *where*

$$M' = \left\{ y \in V^n \;\middle|\; \sum_{i=1}^{n} S(x_i, y_i) \in \mathfrak{d}^{-1} \text{ for every } x \in M \right\}.$$

(iii) $\lambda(xa) = \lambda(x)$ *for every* $a \in \prod_{v\in\mathbf{h}} GL_n(\mathfrak{g}_v)$ *such that* $a_v - 1 \in (\mathfrak{z}_v)_n^n$ *for every* $v \in \mathbf{h}$, *where* xa *is as in Theorem A3.3 (5).*

Then with χ *and* $\mathfrak{k}$ *as in Theorem A3.3 and Lemma A3.12 we have*

$$({}^{\gamma}\lambda)(x) = \chi_{\mathfrak{k}}\big(\det(a_\gamma)\big)\lambda\big(x(a_\gamma)_{\mathfrak{z}}\big) \quad \text{for every } \gamma \in B,$$

where $(a_\gamma)_{\mathfrak{z}}$ *is the projection of* a_γ *to* $\prod_{v|\mathfrak{z}} GL_n(F_v)$, $B = D\big[2\mathfrak{d}^{-1}\mathfrak{x}^{-1}, 2\mathfrak{d}^{-1}\mathfrak{y}^{-1} \cap \{2^{-1}\mathfrak{d}\mathfrak{x}(\mathfrak{k}\cap\mathfrak{z})\}\big]$ *if* q *is even, and* $B = D[2\mathfrak{d}^{-1}\mathfrak{a}, 2^{-1}\mathfrak{d}\mathfrak{a}^{-1}\mathfrak{b}]$, $\mathfrak{a} = \mathfrak{x}^{-1}\cap\mathfrak{g}$, $\mathfrak{b} = \mathfrak{k}\cap\mathfrak{z}\cap 4\mathfrak{a}\cap 4\mathfrak{d}^{-2}\mathfrak{a}\mathfrak{y}^{-1}$ *if* q *is odd.*

We first prove:

A3.14. Lemma. *Let* $\mathfrak{b}$ *and* $\mathfrak{c}$ *be fractional ideals in* F *such that* $\mathfrak{b}\mathfrak{c}$ *is integral, and* $\mathfrak{a}$ *an integral ideal such that* $\mathfrak{a} \subset \mathfrak{b}\cap\mathfrak{c}$. *Further let* $E(\mathfrak{b}) = G_{\mathbf{a}}\prod_{v\in\mathbf{h}} E_v(\mathfrak{b})$ *and* $E'(\mathfrak{c}) = G_{\mathbf{a}}\prod_{v\in\mathbf{h}} E'_v(\mathfrak{c})$, *where* $E_v(\mathfrak{b})$ *resp.* $E'_v(\mathfrak{c})$ *denotes the set of all elements of* G_v *of the form* $\begin{bmatrix} 1_n & b \\ 0 & 1_n \end{bmatrix}$ *resp.* $\begin{bmatrix} 1_n & 0 \\ c & 1_n \end{bmatrix}$ *with* $b \in (\mathfrak{b}_v)_n^n$ *resp.* $c \in (\mathfrak{c}_v)_n^n$. *Then* $D[\mathfrak{b}, \mathfrak{c}]$ *is generated by* $E(\mathfrak{b})$, $E'(\mathfrak{c})$, *and* $D[\mathfrak{a}, \mathfrak{a}]$.

PROOF. Since $D_v[\mathfrak{a}, \mathfrak{a}] = D_v[\mathfrak{b}, \mathfrak{c}]$ if $v \nmid \mathfrak{a}$, it is sufficient to show that $D_v[\mathfrak{b}, \mathfrak{c}]$ is generated by $E_v(\mathfrak{b})$, $E'_v(\mathfrak{c})$, and $D_v[\mathfrak{a}, \mathfrak{a}]$. If $v \nmid \mathfrak{b}\mathfrak{c}$, then $D_v[\mathfrak{b}, \mathfrak{c}]$ is conjugate to $D_v[\mathfrak{g}, \mathfrak{g}] = Sp(n, \mathfrak{g}_v)$, and hence our assertion follows from a well-known fact that $Sp(n, \mathfrak{g}_v)$ is generated by $E_v(\mathfrak{g})$, $E'_v(\mathfrak{g})$, and $\mathrm{diag}[a, {}^t a^{-1}]$ with $a \in GL_n(\mathfrak{g}_v)$. If $v|\mathfrak{b}\mathfrak{c}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in D_v[\mathfrak{b}, \mathfrak{c}]$, then ${}^t ad - 1 \prec (\mathfrak{b}\mathfrak{c})_v$, and hence $a \in GL_n(\mathfrak{g}_v)$, and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca^{-1} & 1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{bmatrix}\begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix},$$

which proves our lemma.

PROOF OF PROPOSITION A3.13. We use the matrix representation as in the proof of Theorem A3.3. Let $\alpha \in P_{\mathbf{A}}$ with $a_\alpha = 1_n$. Then Theorem A3.3 (5) shows that ${}^{\alpha}\lambda(x) = \lambda(x)\mathbf{e}_{\mathbf{h}}(\mathrm{tr}({}^t xSxb_\alpha)/2)$. Therefore ${}^{\alpha}\lambda = \lambda$ if $\alpha \in E(2\mathfrak{d}^{-1}\mathfrak{x}^{-1})$. Put $\beta = \eta^{-1}\alpha\eta$ and $\lambda' = {}^{\eta}\lambda$. Substituting $y+z$ for y in (A3.3), we find that $\lambda'(x) = \mathbf{e}_{\mathbf{h}}(\mathrm{tr}({}^t xSz))\lambda'(x)$ for every $z \in M$, and hence $\lambda'(x) \neq 0$ only if $x \in M'$. By (ii) this means that $\lambda'(x) \neq 0$ only if ${}^t xSx$ has entries in $\mathfrak{y}$. Therefore ${}^{\alpha}\lambda' = \lambda'$ if $\alpha \in E(2\mathfrak{d}^{-1}\mathfrak{y}^{-1})$. Suppose $\beta \in E'(2\mathfrak{d}\cap 2\mathfrak{d}^{-1}\mathfrak{y}^{-1})$. Since $\beta \in C^\theta$ and $\alpha \in P_{\mathbf{A}}$, we have ${}^{\eta}({}^{\beta}\lambda) = {}^{(\eta\beta)}\lambda = {}^{(\alpha\eta)}\lambda = {}^{\alpha}({}^{\eta}\lambda) = {}^{\eta}\lambda$, and hence ${}^{\beta}\lambda = \lambda$. By Lemma A3.12 the expected formula for ${}^{\gamma}\lambda$ is true for $\gamma \in D[\mathfrak{e}, \mathfrak{e}]$ with a suitable ideal $\mathfrak{e}$. We have seen that it is also true for $\gamma \in E(2\mathfrak{d}^{-1}\mathfrak{x}^{-1}) \cup E'(2\mathfrak{d}\cap 2\mathfrak{d}^{-1}\mathfrak{y}^{-1})$, which together with Lemma A3.14 proves our proposition for odd q, since ${}^{\delta}({}^{\varepsilon}\lambda) = {}^{\delta\varepsilon}\lambda$ at least for $\delta, \varepsilon \in C^\theta$. If q is even, the associativity is true for all $\delta, \varepsilon \in G_{\mathbf{A}}$, and so ${}^{\beta}\lambda = \lambda$ for $\beta \in E(2\mathfrak{d}^{-1}\mathfrak{y}^{-1})$. Therefore we can take B in the form stated in our proposition.

The following lemma, though unnecessary in the present book, is of independent interest, and so we give here a proof.

A3.15. Lemma. *The notation being as in Lemma A3.14, let* $T(\mathfrak{b}) = G \cap E(\mathfrak{b})$ *and* $T'(\mathfrak{c}) = G \cap E'(\mathfrak{c})$. *Then* $\Gamma[\mathfrak{b}, \mathfrak{c}]$ *is generated by* $T(\mathfrak{b})$, $T'(\mathfrak{c})$, *and* $\Gamma[\mathfrak{a}, \mathfrak{a}]$.

PROOF. Let X be an open normal subgroup of $D[\mathfrak{b}, \mathfrak{c}]$ contained in $D[\mathfrak{a}, \mathfrak{a}]$. Given $\alpha \in \Gamma[\mathfrak{b}, \mathfrak{c}]$, Lemma A3.14 allows us to take $u_1, \ldots, u_m$ in $E(\mathfrak{b}) \cup E'(\mathfrak{c}) \cup D[\mathfrak{a}, \mathfrak{a}]$ so that $\alpha = u_1 \cdots u_m$. By strong approximation, $u_i \in \beta_i X$ with some $\beta_i \in G$. If $u_i \in D[\mathfrak{a}, \mathfrak{a}]$, then $\beta_i \in G \cap D[\mathfrak{a}, \mathfrak{a}] = \Gamma[\mathfrak{a}, \mathfrak{a}]$. If $u_i \in E(\mathfrak{b})$, we can take β_i from $T(\beta)$, and similarly if $u_i \in E'(\mathfrak{c})$, we can take β_i from $T'(\mathfrak{c})$. Then $\alpha = u_1 \cdots u_m \in \beta_1 \cdots \beta_m X$, and hence $\alpha = \beta_1 \cdots \beta_m \gamma$ with $\gamma \in G \cap X \subset \Gamma[\mathfrak{a}, \mathfrak{a}]$, which completes the proof.

A3.16. We now consider the special class of theta series by taking $n = q$. Thus we put $W = F_n^n$, and identify it with V^n. For $z \in \mathfrak{H}^{\mathbf{a}}$, $\lambda \in \mathcal{S}(W_{\mathbf{h}})$, a totally positive symmetric element τ of W, and $\mu \in \mathbf{Z}^{\mathbf{a}}$ such that $0 \le \mu_v \le 1$ for every $v \in \mathbf{a}$, put

$$\theta(z, \lambda) = \sum_{\xi \in W} \lambda(\xi_{\mathbf{h}}) \det(\xi)^{\mu} \mathbf{e}_{\mathbf{a}}^n({}^t\xi\tau\xi z). \tag{A3.12}$$

This is a special case of the function defined by (A3.10). Indeed, let $S(x, y) = 2 \cdot {}^t x\tau y$ for $x, y \in V = F_1^n$; we can take $p(\xi) = \det(\xi)^{\mu}$ for $\xi \in W$ as explained in Remark A3.11 (II). Then θ of (A3.12) can be obtained as $f(z; \lambda, p)$ of (A3.10). We now put $\mathfrak{M}_n = \mathfrak{M}$ with $\mathfrak{M}$ of (A2.17) if n is odd and $\mathfrak{M}_n = G_{\mathbf{A}}$ if n is even. Putting $l = \mu + (n/2)\mathbf{a}$, we define a factor of automorphy $J(\alpha, z)$ for $\alpha \in \mathfrak{M}_n$ by

$$J(\alpha, z) = \begin{cases} j_\alpha^l(z) & \textit{if } n \textit{ is even,} \\ h(\alpha, z)^n j_\alpha^{\mu}(z) & \textit{if } n \textit{ is odd.} \end{cases} \tag{A3.13}$$

From Theorem A3.8 we obtain

$$\theta(\alpha z, {}^{\alpha}\lambda) = J(\alpha, z)\theta(z, \lambda) \quad \textit{for every} \quad \alpha \in G \cap \mathfrak{M}_n. \tag{A3.14}$$

Moreover, for each λ, our function $\theta(z, \lambda)$ is an element of $\mathcal{M}_l$. (See §6.10 for the definition of $\mathcal{M}_l$ if $l \notin \mathbf{Z}^{\mathbf{a}}$.) Now, by the principle of (A3.11) we can associate with the above θ a function $\theta'_{\mathbf{A}}(x, \lambda)$ with a variable x on $G_{\mathbf{A}}$ or $M_{\mathbf{A}}$, according as n is even or odd, by

$$\theta'_{\mathbf{A}}(x, \lambda) = J(w, \mathbf{i})^{-1}\, \theta\big(w(\mathbf{i}), {}^{w}\lambda\big) \tag{A3.15}$$

for $x = \alpha w$ with $\alpha \in G$ and $w \in \mathfrak{M}_n$; we take $\mathrm{pr}(w) \in C^{\theta}$ if n is odd. This is well-defined. It should be noted that $\theta'_{\mathbf{A}}$ is the function associated with θ by the principle of §16.6 only if $J = j^l$, which is not necessarily true if n is odd. In the rest of this section we put

$$S = \{\, x \in W \mid {}^t x = x \,\}.$$

We no longer use S of §3.1 in the rest of Section A3, but reinstate it in Section A4.

As a special case of what we said in Remark A3.11 (III) we have

(A3.16) $\theta(z, \lambda)$ *is a cusp form if* $\mu \neq 0$.

A3.17. Proposition. *Let* χ *be the Hecke character of* F *corresponding to the extension* $F\big(\det(2\tau)^{1/2}\big)/F$ *or* $F\big((-1)^{n/4}\det(\tau)^{1/2}\big)/F$ *according as* n *is odd or even. Then*

$$\begin{aligned}\theta'_{\mathbf{A}}\left(r_P\begin{bmatrix} q & s\widehat{q} \\ 0 & \widehat{q}\end{bmatrix}, \lambda\right) &= \chi\big(\det(q)\big)\det(q)_{\mathbf{a}}^{\mu}|\det(q)|_{\mathbf{A}}^{n/2} \\ &\quad \cdot \textstyle\sum_{\xi \in W} \lambda(\xi_{\mathbf{h}} q)\det(\xi)^{\mu}\mathbf{e}_{\mathbf{a}}^n(\mathbf{i} \cdot {}^t q \cdot {}^t\xi\tau\xi q)\mathbf{e}_{\mathbf{A}}^n({}^t\xi\tau\xi s)\end{aligned} \tag{A3.17}$$

for every $q \in GL_n(F)_{\mathbf{A}}$ *and* $s \in S_{\mathbf{A}}$, *where* r_p *is the identity map of* $G_{\mathbf{A}}$ *onto itself if* n *is even. Moreover if* $\beta = r_P(\mathrm{diag}[r, \widehat{r}])w$ *with* $\beta \in G$, $r \in GL_n(F)_{\mathbf{h}}$, *and* $w \in \mathfrak{M}_n$, $\mathrm{pr}(w) \in D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$, *then*

$$J(\beta, \beta^{-1}z)\theta(\beta^{-1}z, \lambda) = \chi_{\mathbf{h}}\big(\det(r)\big)|\det(r)|_{\mathbf{A}}^{n/2} \cdot \textstyle\sum_{\xi \in W} ({}^{w}\lambda)(\xi_{\mathbf{h}} r)\det(\xi)^{\mu}\mathbf{e}_{\mathbf{a}}^{n}({}^{t}\xi\tau\xi z). \tag{A3.18}$$

PROOF. Given $x \in \mathfrak{M}_n$, take $\alpha \in G$ and w as in (A3.15). Then $\alpha \in \mathfrak{M}_n$. Put $z = w(\mathbf{i})$. By (A3.14) and (A3.15) we have $\theta'_{\mathbf{A}}(x, \lambda) = J(w, \mathbf{i})^{-1}\theta(z, {}^{w}\lambda) = J(w, \mathbf{i})^{-1}J(\alpha, z)^{-1}\theta\big(\alpha z, {}^{\alpha}({}^{w}\lambda)\big)$. By (A3.2b) and Theorem A3.3 (2) we have $J(\alpha w, \mathbf{i}) = J(\alpha, z)J(w, \mathbf{i})$ and ${}^{\alpha}({}^{w}\lambda) = {}^{x}\lambda$ since $\mathrm{pr}(w) \in C^{\theta}$. Therefore

$$\theta'_{\mathbf{A}}(x, \lambda) = J(x, \mathbf{i})^{-1}\theta\big(x(\mathbf{i}), {}^{x}\lambda\big) \quad \textit{if} \quad x \in \mathfrak{M}_n. \tag{A3.19}$$

Take $x = r_P \begin{bmatrix} q & s\widehat{q} \\ 0 & \widehat{q} \end{bmatrix}$. By Theorem A3.3 (5) we have

$$({}^{x}\lambda)(y) = |\det(q)_{\mathbf{h}}|_{\mathbf{A}}^{n/2}\chi_{\mathbf{h}}\big(\det(q)\big)\mathbf{e}_{\mathbf{h}}^{n}({}^{t}y\tau ys)\lambda(yq) \qquad (y \in W_{\mathbf{h}}).$$

This combined with (A3.12) and (A3.19) gives (A3.17). To prove (A3.18), take the element x so that $\mathrm{pr}(x) \in P_{\mathbf{a}}$ and $x(\mathbf{i}) = z$. Since $\beta^{-1}z = w^{-1}x(\mathbf{i})$ with β as in (A3.18) and $G_{\mathbf{a}}$ acts trivially on $\mathcal{S}(W_{\mathbf{h}})$, (A3.15) shows that $J(\beta, \beta^{-1}z)\theta(\beta^{-1}z, \lambda) = J(\beta, \beta^{-1}z)J(w^{-1}x, \mathbf{i})\theta'_{\mathbf{A}}(w^{-1}x, {}^{w}\lambda) = J(\beta w^{-1}x, \mathbf{i})\theta'_{\mathbf{A}}(\beta w^{-1}x, {}^{w}\lambda) = J(g, \mathbf{i}) \cdot\theta'_{\mathbf{A}}(g, {}^{w}\lambda)$, where $g = r_P \begin{bmatrix} rq & rs\widehat{q} \\ 0 & \widehat{rq} \end{bmatrix}$. Thus we obtain (A3.18) from (A3.17).

A3.18. Fixing an integral ideal $\mathfrak{e}$ in F, we put

$$R = \textstyle\prod_{v \in \mathbf{h}}(\mathfrak{g}_v)_n^n \ (\subset W_{\mathbf{h}}), \quad E_v = GL_n(\mathfrak{g}_v), \quad E'_v = \big\{ x \in E_v \,\big|\, x - 1 \prec \mathfrak{e}_v \big\},$$
$$R' = \big\{ x \in R \,\big|\, x_v \in E'_v \text{ for every } v|\mathfrak{e} \big\}, \quad R^* = R' \cdot W_{\mathbf{a}} \ (\subset W_{\mathbf{A}}).$$

We take μ and τ as above and a Hecke character ω of F such that

$$\omega_{\mathbf{a}}(-1)^n = (-1)^{n\|\mu\|}, \quad \|\mu\| = \textstyle\sum_{v \in \mathbf{a}} \mu_v, \quad \textit{if} \ 2 \in \mathfrak{e}, \tag{A3.20}$$

and denote the conductor of ω by $\mathfrak{f}$. Taking an element p of $GL_n(F)_{\mathbf{h}}$, we define a series θ by

$$\theta(z) = \sum_{\xi \in W \cap pR^*} \omega_{\mathbf{a}}\big(\det(\xi)\big)\omega^*\big(\det(p^{-1}\xi)\mathfrak{g}\big)\det(\xi)^{\mu}\mathbf{e}_{\mathbf{a}}^{n}({}^{t}\xi\tau\xi z). \tag{A3.21}$$

Here it is understood that $\omega_{\mathbf{a}}(b)\omega^*(b\mathfrak{a})$ for $b = 0$ and a fractional ideal $\mathfrak{a}$ denotes $\omega^*(\mathfrak{a})$ or 0 according as $\mathfrak{f} = \mathfrak{g}$ or $\mathfrak{f} \neq \mathfrak{g}$. For $\alpha \in \mathfrak{M}_n$ we define $j_{\alpha}^{l}(z)$ by (16.17).

The ideal $\mathfrak{e}$ is needed only for some technical reasons; the series of (A3.21) is most natural when $\mathfrak{e} = \mathfrak{g}$. Notice that θ is identically equal to 0 if condition (A3.20) is not satisfied, which can happen only if n is odd.

A3.19. Proposition. *Let* ρ_{τ} *be the Hecke character of* F *corresponding to the extension* $F(c^{1/2})/F$ *with* $c = (-1)^{[n/2]}\det(2\tau)$; *put* $\mathfrak{f}' = \mathfrak{f} \cap \mathfrak{e}$ *and* $\omega' = \omega\rho_{\tau}$. *Then there exist a fractional ideal* $\mathfrak{b}$ *and an integral ideal* $\mathfrak{c}$ *such that* $\mathfrak{c} \subset \mathfrak{e}$, *the conductor of* ω' *divides* $\mathfrak{c}$, $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \subset D[2\mathfrak{d}^{-1}, 2\mathfrak{d}]$ *if* n *is odd, and*

$$\theta(\gamma z) = \omega'_{\mathfrak{c}}\big(\det(a_{\gamma})\big)j_{\gamma}^{l}(z)\theta(z) \quad \textit{for every} \quad \gamma \in G \cap C, \tag{A3.22}$$

where $C = \big\{ x \in D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}] \,\big|\, a_x - 1 \prec \mathfrak{e} \big\}$. *Moreover, if* $\beta \in G \cap \mathrm{diag}[r, \widehat{r}]C$ *with* $r \in GL_n(F)_{\mathbf{h}}$, *then*

$$\text{(A3.23)}\qquad j_\beta^l(\beta^{-1}z)\theta(\beta^{-1}z) = \omega'\big(\det(r)\big)^{-1}\omega'_{\mathfrak{c}}\big(\det(d_\beta r)\big)|\det(r)|_{\mathbf{A}}^{n/2}$$
$$\sum_{\xi\in W\cap pR^*r^{-1}} \omega_{\mathbf{a}}\big(\det(\xi)\big)\omega^*\big(\det(\xi p^{-1}r)\mathfrak{g}\big)\det(\xi)^\mu \mathbf{e}_{\mathbf{a}}^n({}^t\xi\tau\xi z).$$

In particular, suppose that ${}^tg\cdot 2\tau g \in \mathfrak{x}$ *for every* $g \in pL_0$ *and* ${}^th(2\tau)^{-1}h \in 4\mathfrak{t}^{-1}$ *for every* $h \in \widehat{p}L_0$ *with fractional ideals* $\mathfrak{x}$ *and* $\mathfrak{t}$, *where* $L_0 = \mathfrak{g}_1^n$; *let* $\mathfrak{h}$ *be the conductor of* ρ_τ. *Then we can take* $(\mathfrak{b},\, \mathfrak{c}) = (2^{-1}\mathfrak{d}\mathfrak{x},\, \mathfrak{h}\cap\mathfrak{f}'\cap\mathfrak{x}^{-1}\mathfrak{f}'^2\mathfrak{t})$ *if* n *is even and* $(\mathfrak{b},\, \mathfrak{c}) = (2^{-1}\mathfrak{d}\mathfrak{a}^{-1},\, \mathfrak{h}\cap\mathfrak{f}'\cap 4\mathfrak{a}\cap\mathfrak{a}\mathfrak{f}'^2\mathfrak{t})$ *if* n *is odd, where* $\mathfrak{a} = \mathfrak{x}^{-1}\cap\mathfrak{g}$.

PROOF. For β and r as above, we have $\mathrm{il}_{\mathfrak{d}}(\beta) = \det(r)^{-1}\mathfrak{g}$; thus by Proposition A2.7,

$$\text{(A3.24)}\qquad h(\beta,\, z)^2 = \psi_2\big(\det(d_\beta r)\big)\psi_{\mathbf{h}}\big(\det(r)\big)j_\beta(z)^{\mathbf{a}},$$

where ψ is as in that proposition and $\psi_2 = \prod_{v|2}\psi_v$. It follows that

$$\text{(A3.25)}\qquad J(\beta,\, z) = \big\{\psi_2\big(\det(d_\beta r)\big)\psi_{\mathbf{h}}\big(\det(r)\big)\big\}^{[n/2]}j_\beta^l(z)\quad \text{if}\quad n\notin 2\mathbf{Z}.$$

In particular $J(\alpha,\, z) = \psi_2\big(\det(d_\alpha)\big)^{[n/2]}j_\alpha^l(z)$ if $\alpha\in G\cap D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]$ and $n\notin 2\mathbf{Z}$. Now define $\lambda\in\mathcal{S}(W_{\mathbf{h}})$ by $\lambda(x) = \omega_{\mathbf{h}}\big(\det(p)^{-1}\big)\lambda'(p^{-1}x)$, $\lambda'(x) = \prod_{v\in\mathbf{h}}\lambda'_v(x_v)$, with

$$\lambda'_v(y) = \begin{cases} 1 & \textit{if}\ \ y\in(\mathfrak{g}_v)_n^n,\ \ v\nmid\mathfrak{f}',\\ \omega_v\big(\det(y)^{-1}\big) & \textit{if}\ \ y\in E'_v,\ \ v|\mathfrak{f}',\\ 0 & \textit{otherwise}.\end{cases}$$

Then the function $\theta(z,\, \lambda)$ of (A3.12) coincides with θ of (A3.21). Notice that $\lambda(xa) = \omega_{\mathfrak{f}'}\big(\det(a)\big)^{-1}\lambda(x)$ for every $a\in\prod_{v\in\mathbf{h}}E'_v$. Applying Proposition A3.13 to the present λ, we find a fractional ideal $\mathfrak{b}$ and an integral ideal $\mathfrak{c}$ such that $D[\mathfrak{b}^{-1},\, \mathfrak{b}\mathfrak{c}]\subset D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]$ if n is odd, the conductors of χ and ω divide $\mathfrak{c}$, and that

$$\text{(A3.26)}\qquad ({}^w\lambda)(x) = \chi_{\mathfrak{k}}\big(\det(a_w)\big)\lambda\big(x(a_w)_{\mathfrak{f}'}\big)\quad \textit{for every}\quad w\in D[\mathfrak{b}^{-1},\, \mathfrak{b}\mathfrak{c}],$$

where χ is as in Proposition A3.17, and $\mathfrak{k}$ is its conductor. This combined with (A3.14), (A3.18), and (A3.25) proves our assertions up to formula (A3.23). Let $\mathfrak{h}'$ be the conductor of χ. Then $\mathfrak{h}' = \mathfrak{h}$ if n is even and $\mathfrak{h}\cap 4\mathfrak{g} = \mathfrak{h}'\cap 4\mathfrak{g}$ if n is odd. Therefore Proposition A3.13 gives our assertion on $(\mathfrak{b},\, \mathfrak{c})$ as stated above. (In fact, the ideal $\mathfrak{y}$ of that proposition is $4(\mathfrak{d}^2\mathfrak{f}'^2\mathfrak{t})^{-1}$ in the present case.)

A3.20. Example. Take θ of (A3.21) with trivial ω, $\mu = \mathbf{a}$, $\mathfrak{e} = \mathfrak{g}$, $\tau = 2^{-1}1_n$, $p = 1_n$, and $n\in 4\mathbf{Z}$. Then $l = \big(1 + (n/2)\big)\mathbf{a}$ and ρ_τ is trivial; thus we can take $\mathfrak{x} = \mathfrak{g}$ and $\mathfrak{t} = 4\mathfrak{g}$ in Proposition A3.19, so that $\mathfrak{b} = 2^{-1}\mathfrak{d}$ and $\mathfrak{c} = 4\mathfrak{g}$. Therefore $\theta\in\mathcal{M}_l(\Gamma)$ with $\Gamma = G\cap D[2\mathfrak{d}^{-1},\, 2\mathfrak{d}]$. By (A3.16), θ is a cusp form. If ${}^t\xi\xi = {}^t\xi_1\xi_1$, then $\det(\xi) = \pm\det(\xi_1)$, and so $\det(\xi)^{\mathbf{a}} = \det(\xi_1)^{\mathbf{a}}$ if $[F:\mathbf{Q}]$ is even. If $[F:\mathbf{Q}]$ is odd, then ${}^t(\beta\xi)(\beta\xi) = {}^t\xi\xi$ and $\det(\beta\xi)^{\mathbf{a}} = -\det(\xi)^{\mathbf{a}}$ with $\beta = \mathrm{diag}[-1,\, 1_{n-1}]$. Thus θ is nonzero if and only if $[F:\mathbf{Q}]$ is even.

A4. The constant term of a theta series at each cusp depends only on the genus

A4.1. Our purpose is to prove the fact stated in the title, and to discuss its consequences in connection with Siegel's theory of quadratic forms. Our setting is the same as in §A3.1. For simplicity we use matrix expressions; thus taking ${}^tS = S\in GL_q(F)$, we put $W = F_n^q$,

$$O(S) = \{ \alpha \in GL_q(F) \mid {}^t\alpha S \alpha = S \},$$

and $S[x] = {}^t x S x$ for $x \in W$; we also identify W with V^n. For $\lambda \in \mathcal{S}(W_\mathbf{h})$ we view λ also as a function on $W_\mathbf{A}$ by putting $\lambda(x) = \lambda(x_\mathbf{h})$.

A4.2. Theorem. *Let $\alpha \in G$ and $\lambda \in \mathcal{S}(W_\mathbf{h})$. Define $g(u, z; \lambda)$ by (A3.0a) with W in place of V^n. Let $t(z) = \zeta \prod_{v \in \mathbf{a}} j_\alpha(z)_v^{(q/2)-s_v} |j_\alpha(z)_v|^{s_v}$ with $\zeta \in \mathbf{T}$ and any choice of the branch of $j_\alpha(z)_v^{(q/2)-s_v}$. (If $\alpha \in \mathfrak{M}_q$, we can take $J^S(\alpha, z)$ to be $t(z)$.) Then the following assertions hold:*

(1) *There exists $\lambda_1 \in \mathcal{S}(W_\mathbf{h})$ determined by the equation*

$$(*) \qquad t(z)^{-1} g\big(\alpha(u, z); \lambda\big) = g(u, z; \lambda_1).$$

(2) *Define $\mu \in \mathcal{S}(W_\mathbf{h})$ by $\mu(x) = \lambda(\gamma x)$ with an element $\gamma \in O(S)_\mathbf{h}$; define μ_1 by taking μ in place of λ in (*). Then $\mu_1(x) = \lambda_1(\gamma x)$.*

PROOF. Given λ and $\alpha \in G \cap \mathfrak{M}_q$, take ℓ so that ${}^\alpha\ell = \lambda$ and put $\ell = \lambda_1$. Then from Theorem A3.3 (0) we obtain

$$J^S(\alpha, z)^{-1} g\big(\alpha(u, z); \lambda\big) = g(u, z; \lambda_1).$$

Now $G \cap \mathfrak{M}_q$ contains P and η. By Lemma 7.5 every element of G is a product of finitely many elements in $P \cup \{\eta\}$, since $\eta^{-1} = -\eta$ and $-1 \in P$. Now if the assertion of (1) is true for (α, t) and (α', t'), then we easily see that it is true for $(\alpha\alpha', t'')$, where $t''(z) = \zeta t(\alpha' z) t'(z)$ with any $\zeta \in \mathbf{T}$. Therefore we obtain (1). As for (2), for the same reason it is sufficient to prove it when $\alpha \in P$ or $\alpha = -\eta$ with $t(z) = J^S(\alpha, z)$. If $\alpha \in P$, taking $\beta = \alpha^{-1}$, we have (*) with $\lambda_1 = {}^\beta\lambda$. By Theorem A3.3 (5) we have

$$({}^\beta\lambda)(x) = \varepsilon(\beta)\mathbf{e_h}\Big(\mathrm{tr}\big(S[x]a_\beta \cdot {}^t b_\beta\big)/2\Big)\lambda(x a_\beta)$$

with a constant $\varepsilon(\beta)$ that depends only on β and S. Then clearly $({}^\beta\lambda)(\gamma x) = ({}^\beta\mu)(x)$, which is the desired relation for $\alpha \in P$. Next take $\alpha = -\eta$. Then (*) holds with $\lambda_1 = {}^\eta\lambda$ by Theorem A3.3 (7). Now ${}^\eta\lambda$ is given by (A3.3). For γ and μ as above, we have $d(\gamma y) = dy$, and hence we can easily derive from (A3.3) that $({}^\eta\mu)(x) = ({}^\eta\lambda)(\gamma x)$ as desired. This completes the proof.

A4.3. Theorem. (1) *Suppose that $n = 1$; given $\lambda \in \mathcal{S}(W_\mathbf{h})$ and $\gamma \in O(S)_\mathbf{h}$, define $\mu \in \mathcal{S}(W_\mathbf{h})$ by $\mu(x) = \lambda(\gamma x)$. Then $g(0, z; \lambda) - g(0, z; \mu)$ is rapidly decreasing at the cusps of G.*

(2) *Suppose moreover that S is totally definite. Put*

$$\text{(A4.1)} \qquad f(z, \lambda) = \sum_{g \in W} \lambda(g)\mathbf{e_a}\big(S[g]z/2\big) \qquad (z \in \mathfrak{H}_1^\mathbf{a}).$$

If μ is as above, then $f(z, \lambda) - f(z, \mu)$ is a cusp form.

PROOF. Put $\nu = \lambda - \mu$; then clearly $g(0, z; \lambda) - g(0, z; \mu) = g(0, z; \nu)$, and if α and $t(z)$ are as in Theorem A4.2, then

$$\text{(A4.2)} \qquad t(z)^{-1} g(0, \alpha z; \nu) = g(0, z; \nu_1)$$

with $\nu_1 = \lambda_1 - \mu_1$. By Theorem A4.2 (2), we have $\mu_1(x) = \lambda_1(\gamma x)$, so that $\nu_1(0) = 0$. Since $n = 1$, the definition of the series g shows that the function of (A4.2) must be rapidly decreasing. This proves (1). Assertion (2) is merely a special case of (1).

A4.4. Theorem. *Let L be a $\mathfrak{g}$-lattice in F^q, and let*

$$f(z,\,L) = \sum_{g\in L} \mathbf{e_a}\big(S[g]z/2\big) \qquad (z \in \mathfrak{H}_1^{\mathbf{a}}).$$

Then the following assertions hold:

(1) *$f(z,\,L) - f(z,\,\gamma L)$ is a cusp form for every $\gamma \in O(S)_{\mathbf{h}}$.*

(2) *Let $\{\,L_i\,\}_{i=1}^h$ be a complete set of representatives for the classes of lattices in the genus of L with respect to $O(S)$. Put $e_i = \#\,\{\,\alpha \in O(S) \,|\, \alpha L_i = L_i\,\}$ and*

$$\text{(A4.3)} \qquad p(z) = \left(\sum_{i=1}^h e_i^{-1}\right)^{-1} \sum_{i=1}^h e_i^{-1} f(z,\,L_i).$$

Then $f(z,\,L) - p(z)$ is a cusp form.

PROOF. Clearly $h(z,\,L) = h(z,\,\lambda)$ if we take $\lambda(x) = \prod_{v\in\mathbf{h}} \lambda_v(x_v)$ with the characteristic function of L_v as λ_v. Therefore (1) follows from Theorem A4.3 (2). Assertion (2) follows immediately from (1).

Now p of (A4.3) equals $F(S,\,z)$ with the function F defined by Siegel in [Si, I, p.372, (78); p.542, (129)]. He proved (2) of Theorems A4.4 when $F = \mathbf{Q}$ in [Si, I, p.376]. The number $\sum_{i=1}^h e_i^{-1}$ is *the mass of the genus of L* in his sense. If we put $f(z,\,L) = \sum_{b\in F} r(b,\,L)\mathbf{e_a}(bz/2)$ and $p(z) = \sum_{b\in F} r_0(b)\mathbf{e_a}(bz/2)$, then $r(b,\,L) = \#\{\,g \in L \,|\, S[g] = b\,\}$ and $r_0(b) = \big(\sum_{i=1}^h e_i^{-1}\big)^{-1} \sum_{i=1}^h e_i^{-1} r(b,\,L_i)$. Thus $r_0(b)$ is the weighted average of the numbers of representations of b by S, for which Siegel gave his product formula.

We can also show that p is an Eisenstein series by means of the Siegel-Weil formula. Siegel gave this fact in [Si, I, p.373, Satz 3; p.543, Satz III] for $n > 4$. The case of an arbitrary n is explained in [S99, Section 5].

In Sections A2 through A4 we have assumed that the basic filed F is totally real, but we can actually treat the case of an arbitrary number field. Indeed, such a theory was presented in [S93], which gives at least the generalizations of the results up to Lemma A3.15 without assuming F to be totally real. Also, in [S97, Section A7] we treated theta series of a hermitian form in a similar fashion. In the next section we will give some more results complementary to this theory. Generalizations or analogues of Theorems A4.2, A4.3, and A4.4 can be proved for theta series over an arbitrary number field and also in the hermitian case by the same methods.

A5. Theta series of a hermitian form

This section concerns Case UT. Thus K is a CM-field; see §3.5. For $y \in K_{\mathbf{A}}^\times$ and $\mu \in \mathbf{Z}^{\mathbf{a}}$ we note that $|y|^\mu = \prod_{v\in\mathbf{a}} |y_v|^{\mu_v}$, where $|y_v|$ is the standard absolute value in $\mathbf{C}$, not its square. For $b \in K^\times$, for example, we have $|b_{\mathbf{h}}|_K^{-1} = |b_{\mathbf{a}}|_K = |b|^{2\mathbf{a}}$.

A5.1. Lemma. *Let φ_0 be a Hecke character of F such that $\varphi_0(x) = x^\nu |x|^{-\nu}$ for $x \in F_{\mathbf{a}}^\times$ with $\nu \in \mathbf{Z}^{\mathbf{a}}$. Then there exists a Hecke character φ of K such that $\varphi = \varphi_0$ on $F_{\mathbf{A}}^\times$, $\varphi(y) = y^\nu |y|^{-\nu}$ for $y \in K_{\mathbf{a}}^\times$, and the conductor of φ divides a power of the conductor of φ_0.*

PROOF. Let $\mathfrak{a}$ be the conductor of φ_0. We can find a positive integer m such that if ζ is a root of unity in K and $\zeta - 1 \in \mathfrak{r}\mathfrak{a}^m$, then $\zeta = 1$; see [S97, Lemma 24.3 (1)]. Put $U = \big\{\,b \in K_{\mathbf{a}}^\times \prod_{v\in\mathbf{h}} \mathfrak{r}_v^\times \,\big|\, b - 1 \prec \mathfrak{r}\mathfrak{a}^m\,\big\}$. For $x = abc$ with $a \in K^\times$, $b \in U$, and $c \in F_{\mathbf{A}}^{\times}$, define $\varphi(x) = b_{\mathbf{a}}^{\nu}|b_{\mathbf{a}}|^{-\nu}\varphi_0(c)$. This is a well-defined map of $K^{\times}UF_{\mathbf{A}}^{\times}$ into $\mathbf{T}$. Indeed, suppose $abc = 1$; put $\zeta = a/a^{\rho}$. Then $\zeta = b^{\rho}/b \in U \cap K \subset \mathfrak{r}^{\times}$. Since $|\zeta_v| = 1$ for every $v \in \mathbf{a}$, ζ is a root of unity. By our choice of $\mathfrak{a}^m$, we have $\zeta = 1$, so that $a \in F^{\times}$ and $b \in F_{\mathbf{A}}^{\times} \cap U$. Thus $\varphi_0(c)^{-1} = \varphi_0(ab) = \varphi_0(b) = b_{\mathbf{a}}^{\nu}|b_{\mathbf{a}}|^{-\nu}$. This shows that φ is a well-defined character of $K^{\times}UF_{\mathbf{A}}^{\times}$. Since $K^{\times}UF_{\mathbf{A}}^{\times}$ is a subgroup of $K_{\mathbf{A}}^{\times}$ of finite index, by [S97, Lemma 11.15] we can extend φ to a $\mathbf{T}$-valued character of $K_{\mathbf{A}}^{\times}$, which is clearly a Hecke character with the desired properties.

A5.2. Lemma. *Let* $E^* = \{ a \in GL_n(K)_{\mathbf{a}} \prod_{v \in \mathbf{h}} GL_n(\mathfrak{r}_v) \mid a - 1 \prec \mathfrak{r}\mathfrak{a} \}$ *and* $T = \{ \mathrm{diag}[a, \widehat{a}] \mid a \in E^* \}$ *with an integral* $\mathfrak{g}$*-ideal* $\mathfrak{a}$*, and let* C *be an open subgroup of* $D[\mathfrak{b}^{-1}, \mathfrak{b}\mathfrak{c}]$ *containing* T *and such that* $a_x - 1 \prec \mathfrak{r}\mathfrak{a}$ *for every* $x \in C$*. Suppose* $\mathfrak{a} \supset \mathfrak{c}$ *and* $\mathfrak{a}^{-1}\mathfrak{c}$ *is divisible by the relative discriminant of* K *over* F*. Then* $C = T(C \cap (G_1)_{\mathbf{A}})$.

PROOF. Given $g \in C$, put $u = \det(g)$. Then $uu^{\rho} = 1$ and $u_v \in \mathfrak{r}_v^{\times}$ for every $v \in \mathbf{h}$. Suppose we can find $b \in K_{\mathbf{a}}^{\times}$ such that $u = b/b^{\rho}$, $b - 1 \prec \mathfrak{r}\mathfrak{a}$, and $b_v \in \mathfrak{r}_v^{\times}$ for every $v \in \mathbf{h}$. Then putting $h = \mathrm{diag}[e, \widehat{e}]$ with $e = \mathrm{diag}[b, 1_{n-1}]$, we see that $h \in T$ and $h^{-1}g \in C \cap (G_1)_{\mathbf{A}}$, which proves our lemma. Clearly the problem is to find b_v with the required properties for each $v \in \mathbf{v}$. There is no problem for $v \in \mathbf{a}$. If $v \in \mathbf{h}$ and $v \nmid \mathfrak{c}$, then v is unramified in K, and hence the desired b_v exists by virtue of [S97, Lemma 5.11 (1)]. Suppose $v|\mathfrak{c}$; put $f = \det(a_g)_v$. Since $a_g(d_g)^* - 1 \prec \mathfrak{r}\mathfrak{c}$ and $u - \det(a_g d_g) \prec \mathfrak{r}\mathfrak{c}$, we see that $f \in \mathfrak{r}_v^{\times}$, $f - 1 \in \mathfrak{r}_v\mathfrak{a}_v$, and $u_v - f/f^{\rho} \in \mathfrak{r}_v\mathfrak{c}_v$. Put $w = u_v f^{\rho}/f$. Then $ww^{\rho} = 1$ and $w - 1 \in \mathfrak{r}_v\mathfrak{c}_v$. By our asumption on $\mathfrak{a}^{-1}\mathfrak{c}$ and [S97, Lemma 17.5], there exists an element $c \in \mathfrak{r}_v^{\times}$ such that $c - 1 \in \mathfrak{r}_v\mathfrak{a}_v$ and $w = c/c^{\rho}$. Put $b_v = cf$. Then $u_v = b_v/b_v^{\rho}$ and $b_v - 1 \in \mathfrak{r}_v\mathfrak{a}_v$ as desired. This completes the proof.

A5.3. In [S97, Section A7] we treated theta series of a hermitian form, and proved transformation formulas for them analogous to Theorems A3.3, A3.8 and Propositions A3.13, A3.17, and A3.19. However, the formulas in [S97] were given in terms of $(G_1)_{\mathbf{A}}$ for $G_1 = G \cap SL_{2n}(K)$ with $G = U(\eta_n)$ in Case UT. Let us now show that we can formulate the results in terms of $G_{\mathbf{A}}$. (The group G of [S97, Section A7] is the present G_1.)

Let the notation be as in [S97, Theorem A7.4]; in particular we recall that with $V = K_n^q$ we defined the action of $(G_1)_{\mathbf{A}}$ on $\mathcal{S}(V_{\mathbf{h}})$, that depends on a hermitian element H in K_q^q. We let ε denote (instead of χ we used in [S97, Theorem A7.4]) the quadratic Hecke character of F corresponding to K/F. By Lemma A5.1 there exists a Hecke character φ of K such that $\varphi = \varepsilon$ on $F_{\mathbf{A}}^{\times}$, $\varphi(y) = y^{-\mathbf{a}}|y|^{\mathbf{a}}$ for $y \in K_{\mathbf{a}}^{\times}$, and the conductor of φ divides a power of the conductor of ε. We are going to show that we can extend the action of $(G_1)_{\mathbf{A}}$ on $\mathcal{S}(V_{\mathbf{h}})$ to that of $G_{\mathbf{A}}$. We define $r, s \in \mathbf{Z}^{\mathbf{a}}$ so that H_v has signature (r_v, s_v) for every $v \in \mathbf{a}$ and put

$$J_H(\alpha, z) = \prod_{v \in \mathbf{a}} j_v(\alpha, z)^{r_v} \overline{j_{v\rho}(\alpha, z)}^{s_v} \qquad (\alpha \in G_{\mathbf{A}},\ z \in \mathcal{H}), \tag{A5.1}$$

using the notation of (5.3). We also define $A = (A_v)_{v \in \mathbf{a}}$ and $f(z; u, u'; \lambda)$ as in [S97, (A7.3.2)] for $z \in \mathcal{H}$, $(u, u') \in (C_q^n)^{\mathbf{a}} \times (C_q^n)$ and $\lambda \in \mathcal{S}(V_{\mathbf{h}})$. We defined the action of G_1 on $\mathcal{H} \times (C_q^n)^{\mathbf{a}} \times (C_q^n)$ by [S97, (A7.3.4), (A7.3.5)]. Clearly this action can be extended to the action of G by the same formulas. (The symbols φ, $\mu(\alpha_\tau, z_\tau)$, $\kappa(\alpha_\tau, z_\tau)$ there correspond to $\mathbf{a}$, $\mu(\alpha_v, z_v)$, $\lambda(\alpha_v, z_v)$ here.)

A5.4. Theorem. *Every element σ of $G_{\mathbf{A}}$ gives a $\mathbf{C}$-linear automorphism of $\mathcal{S}(V_{\mathbf{h}})$, written $\lambda \mapsto {}^{\sigma}\lambda$ for $\lambda \in \mathcal{S}(V_{\mathbf{h}})$, with the following properties:*

(0) *If $\sigma \in (G_1)_{\mathbf{A}}$, then this action is the same as that of [S97, Theorem A7.4]. (Notice that G there is G_1 here.)*

(1) $f\big(\alpha(z;\, u,\, u');\, {}^{\alpha}\lambda\big) = J_H(\alpha,\, z) f(z;\, u,\, u';\, \lambda)$ *for every* $\alpha \in G$.

(2) ${}^{(\sigma'\sigma)}\lambda = {}^{\sigma'}({}^{\sigma}\lambda)$.

(3) *The map $\lambda \mapsto {}^{\sigma}\lambda$ depends only on $\sigma_{\mathbf{h}}$, H, and the choice of φ; it does not depend on $\{\, A_v \,\}_{v\in\mathbf{a}}$ or $\sigma_{\mathbf{a}}$.*

(4) *$\{\, \sigma \in G_{\mathbf{A}} \mid {}^{\sigma}\lambda = \lambda \,\}$ is an open subgroup of $G_{\mathbf{A}}$ for every $\lambda \in \mathcal{S}(V_{\mathbf{h}})$.*

(5) *If $\sigma \in P_{\mathbf{h}}$, then*

$$({}^{\sigma}\lambda)(\xi) = |\det(a_\sigma)|_K^{q/2} \varphi_{\mathbf{h}}\big(\det(a_\sigma)\big)^q \mathbf{e_h}\Big(2^{-1}\mathrm{tr}\big(\xi^* H \xi a_\sigma b_\sigma^*\big)\Big)\lambda(\xi a_\sigma),$$

(6) *If* $\eta = \begin{bmatrix} 0 & -1_n \\ 1_n & 0 \end{bmatrix}$, *we have*

$$({}^{\eta}\lambda)(x) = i^p \left|N_{F/\mathbf{Q}}\big(\det(2H^{-1})\big)\right|^n \int_{V_{\mathbf{h}}} \lambda(y)\mathbf{e_h}\Big(-2^{-1}\mathrm{Tr}_{K/F}\big(\mathrm{tr}(y^* H x)\big)\Big)dy.$$

Here $p = n\sum_{v\in\mathbf{a}}(r_v - s_v)$ and dy is the Haar measure on $V_{\mathbf{h}}$ such that the measure of $\prod_{v\in\mathbf{h}}(\mathfrak{r}_v)_n^q$ is $|D_K|^{-nq/2}$, D_K being the discriminant of K.

PROOF. Assertions (1~6) for $\sigma \in (G_1)_{\mathbf{A}}$ are those given in [S97, Theorem A7.4]. In order to distinguish them from the present ones, let us denote those old ones for $\sigma \in (G_1)_{\mathbf{A}}$ by $(1)_1, (2)_1, \ldots, (6)_1$. To define the action of $G_{\mathbf{A}}$ on $\mathcal{S}(V_{\mathbf{h}})$, put $Q = \big\{\, \mathrm{diag}[a,\, \widehat{a}] \,\big|\, a \in GL_n(K) \,\big\}$. For $\lambda \in \mathcal{S}(V_{\mathbf{h}})$ and $p = \mathrm{diag}[a,\, \widehat{a}] \in Q_{\mathbf{A}}$ with $a \in GL_n(K)_{\mathbf{A}}$, we define ${}^{p}\lambda$ by

$$\text{(A5.2)} \qquad ({}^{p}\lambda)(x) = |\det(a_{\mathbf{h}})|_K^{q/2} \varphi_{\mathbf{h}}\big(\det(a)\big)^q \lambda(xa) \qquad (x \in V_{\mathbf{A}}).$$

Clearly ${}^{pp'}\lambda = {}^{p}({}^{p'}\lambda)$ for $p,\, p' \in Q_{\mathbf{A}}$. We have $G_{\mathbf{A}} = Q_{\mathbf{A}}(G_1)_{\mathbf{A}}$ and ${}^{\sigma}\lambda$ is meaningful for $\sigma \in (G_1)_{\mathbf{A}}$. Thus, for $\alpha = p\sigma \in G_{\mathbf{A}}$ with $p \in Q_{\mathbf{A}}$ and $\sigma \in (G_1)_{\mathbf{A}}$ we naturally put ${}^{\alpha}\lambda = {}^{p}({}^{\sigma}\lambda)$. To show that this is well-defined, put $p\sigma = p'\sigma'$ with $p' \in Q_{\mathbf{A}}$ and $\sigma' \in (G_1)_{\mathbf{A}}$; put also $h = p^{-1}p'$. Then $h = \sigma(\sigma')^{-1} \in Q \cap (G_1)_{\mathbf{A}}$. Since the definition of ${}^{p}\lambda$ in (A5.2) is consistent with that of $(5)_1$, we have ${}^{p'}({}^{\sigma'}\lambda) = {}^{p}\big({}^{h}({}^{\sigma'}\lambda)\big) = {}^{p}({}^{h\sigma'}\lambda) = {}^{p}({}^{\sigma}\lambda)$. Thus ${}^{\alpha}\lambda$ is well-defined, and clearly (3) holds.

Before proving (2), we first make the following observation: Given $w \in (G_1)_{\mathbf{A}}$, put $y = p^{-1}wp$ with $p = \mathrm{diag}[a,\, \widehat{a}]$. Then $y \in (G_1)_{\mathbf{A}}$. We have to show that ${}^{p}({}^{y}\lambda) = {}^{w}({}^{p}\lambda)$, which is equivalent to

(*) *If $\lambda'(x) = \lambda(xa)$, then $({}^{w}\lambda')(x) = ({}^{y}\lambda)(xa)$, where $x \in V_{\mathbf{A}}$.*

In view of $(4)_1$, (*) is true for w in a sufficiently small open subgroup D of $(G_1)_{\mathbf{A}}$ depending on λ and a. Now $(G_1)_{\mathbf{A}} = G_1 D$, and G_1 is generated by η_n and $P_1 = \big\{\, g \in G_1 \,\big|\, c_g = 0 \,\big\}$. Therefore, in order to prove (*) for that particular λ, it is sufficient to prove (*) for $w \in P_1$ and $w = \eta_n$, and for an arbitrary λ. If $w \in P_1$, (*) can be derived from $(5)_1$ by a straightforward calculation. If $w = \eta_n$, we obtain the desired fact from $(5)_1$ and $(6)_1$ by observing that $p^{-1}\eta_n p = \eta_n \mathrm{diag}[t,\, \widehat{t}]$ with $t = a^*a$, and $\varepsilon\big(\det(t)\big) = 1$.

Now, to prove (2), let $\alpha = p\sigma$ as above and let $\beta = r\tau$ with $r = \mathrm{diag}[b,\, \widehat{b}] \in Q_{\mathbf{A}}$ and $\tau \in (G_1)_{\mathbf{A}}$. Then $\beta\alpha = rp\xi\sigma$ with $\xi = p^{-1}\tau p$. We have shown that ${}^{p}({}^{\xi}\lambda) = {}^{\tau}({}^{p}\lambda)$, from which we immediately obtain ${}^{(\beta\alpha)}\lambda = {}^{\beta}({}^{\alpha}\lambda)$. As for (1), we have it for $\alpha \in G_1$; if $\alpha \in Q$, we can verify it by a direct calculation. Since $G = QG_1$, we obtain (1) in the general case in view of (2). Similarly (5) follows from $(5)_1$ and (A5.2), since $P = QP_1$. To prove (4), define E^* and T as in Lemma A5.2 with an integral ideal $\mathfrak{a}$ and put $C = \{\alpha \in D[\mathfrak{a}, \mathfrak{a}] \mid \alpha - 1 \prec \mathfrak{ra}\}$. (We are taking $\mathfrak{c}$ in that lemma to be $\mathfrak{a}^2$ here.) Given $\lambda \in \mathcal{S}(V_{\mathbf{h}})$, we can take $\mathfrak{a}$ so that $\lambda(xa) = \lambda(x)$ for every $a \in E^*$ and ${}^{\sigma}\lambda = \lambda$ for every $\sigma \in C \cap (G_1)_{\mathbf{A}}$. We also assume that $\mathfrak{a}$ is divisible by the relative discriminant of K over F and by the conductor of φ. Then ${}^{p}\lambda = \lambda$ for every $p \in T$ by (A5.2). Since $C = T(C \cap (G_1)_{\mathbf{A}})$, we see that ${}^{\alpha}\lambda = \lambda$ for every $\alpha \in C$, which proves (4). This completes the proof, as (6) is $(6)_1$.

A5.5. We now consider the case $q = n$; thus, hereafter $V = K_n^n$. We take $\mu \in \mathbf{Z}^{\mathbf{b}}$ such that $\mu_v \geq 0$ for every $v \in \mathbf{b}$ and $\mu_v \mu_{v\rho} = 0$ for every $v \in \mathbf{a}$, and take also $\tau \in S^+$. We then put $l = \mu + n\mathbf{a}$ and

(A5.3) $$\theta(z, \lambda) = \sum_{\xi \in V} \lambda(\xi) \det(\xi)^{\mu\rho} \mathbf{e}_{\mathbf{a}}^n(\xi^* \tau \xi z) \qquad (\lambda \in \mathcal{S}(V_{\mathbf{h}}),\ z \in \mathcal{H}).$$

Here we understand that $\det(\xi)^{\mu\rho} = 1$ for every ξ if $\mu = 0$. Taking τ to be H in the above theorem and fixing a Hecke character φ, we have an action of $G_{\mathbf{A}}$ on $\mathcal{S}(V_{\mathbf{h}})$. Now we have

(A5.4) $$\theta(\alpha z, {}^{\alpha}\lambda) = j_{\alpha}^{l}(z)\theta(z, \lambda) \ \textit{for every } \alpha \in G.$$

Indeed, $\theta(z, \lambda)$ is the series of [S97, (A7.13.1)], and (A5.4) for $\alpha \in G_1$ was given in [S97, (A7.13.4)]. If $\alpha = \mathrm{diag}[a, \widehat{a}]$ with $a \in GL_n(K)$, then (A5.4) follows immediately from (A5.2), so that (A5.4) holds in general. From (4) of the above theorem we see that $\theta(z, \lambda)$ belongs to $\mathcal{M}_l$. We have, for the same reason as in (A3.16),

(A5.5) *$\theta(z, \lambda)$ is a cusp form if $\mu \neq 0$.*

We now define a function $\theta_{\mathbf{A}}$ on $G_{\mathbf{A}}$ by

(A5.6) $$\theta_{\mathbf{A}}(x, \lambda) = j_x^l(\mathbf{i})^{-1}\theta(x(\mathbf{i}), {}^{x}\lambda) \qquad (x \in G_{\mathbf{A}}).$$

Then from (A5.4) we easily obtain

(A5.7) $$\theta_{\mathbf{A}}(\alpha x w, \lambda) = j_w^l(\mathbf{i})^{-1}\theta_{\mathbf{A}}(x, {}^{w}\lambda) \ \textit{if } \alpha \in G,\ w \in G_{\mathbf{A}},\ \textit{and } w(\mathbf{i}) = \mathbf{i}.$$

Let ω be a Hecke character of K, and $\mathfrak{f}$ the conductor of ω. Taking an integral $\mathfrak{g}$-ideal $\mathfrak{a}$, put

(A5.8a) $$\omega' = \omega\varphi^{-n}, \quad \mathfrak{h} = \mathfrak{f} \cap \mathfrak{a}, \quad R^* = \{w \in V_{\mathbf{A}} \mid w - 1 \prec \mathfrak{ra}\}.$$

We then define $\lambda_0 \in \mathcal{S}(V_{\mathbf{h}})$ as follows: $\lambda_0(x) = \omega_{\mathfrak{h}}(\det(x)^{-1})$ if $x \in R^*$ and $x_v \in GL_n(\mathfrak{r}_v)$ for every $v | \mathfrak{h}$; $\lambda_0(x) = 0$ otherwise. Fixing $r \in GL_n(K)_{\mathbf{h}}$, put $\lambda_1(x) = \omega(\det(r)^{-1})\lambda_0(r^{-1}x)$ for $x \in V_{\mathbf{h}}$. In [S97, Proposition A7.16] and its proof, we found a group

(A5.8b) $$C' = \{ w \in D[\mathfrak{b}^{-1}, \mathfrak{bc}] \mid a_w - 1 \prec \mathfrak{ra} \}$$

with a fractional $\mathfrak{g}$-ideal $\mathfrak{b}$ and an integral $\mathfrak{g}$-ideal $\mathfrak{c}$ divisible by $\mathfrak{h}$ and the conductor of φ such that

(A5.9) $${}^{w}\lambda_1 = \omega'_{\mathfrak{c}}(\det(a_w))^{-1}\lambda_1 \ \text{for every } w \in (G_1)_{\mathbf{A}} \cap C'.$$

(Corrections to [S97, Proposition A7.16]: $D[\mathfrak{b}^{-1}, \mathfrak{bc}]$ there should be replaced by its subgroup defined by $a_w - 1 \prec \mathfrak{a}$; $\omega_0'(\det(a_\gamma))$ in [S97, (A7.16.1)] should be $\omega_\mathfrak{c}'(\det(a_w))$. Here we are taking $\mathfrak{ra}$ as $\mathfrak{a}$ there.) Now (A5.9) is true for every $w \in C'$ if $\mathfrak{c}$ is suitably chosen. Indeed, by Lemma A5.2 we can shoose $\mathfrak{c}$ so that $C' = T\big(C' \cap (G_1)_\mathbf{A}\big)$. Since (A5.9) for $w \in T$ follows from (A5.2), we have (A5.9) for every $w \in C'$.

Put $\mathbf{g}(x) = \theta_\mathbf{A}(x, \lambda_1)$. Then (A5.7) and the last fact show that $\mathbf{g} \in \mathcal{M}_l(C', \omega')$ with the notation of §20.1, since (20.3b) can easily be verified by means of (A5.6). Let $p = \operatorname{diag}[q, \widehat{q}]$ with $q \in GL_n(K)_\mathbf{h}$ and let g_p be the p-component of $\mathbf{g}$ in the sense of (20.3b). We are going to show that

$$\text{(A5.10)} \qquad g_p(z) = \omega'\big(\det(q)\big)^{-1} |\det(q)|_K^{n/2} \sum_{\xi \in V \cap rR^* q^{-1}} \omega_\mathbf{a}\big(\det(\xi)\big)\omega^*\big(\det(r^{-1}\xi q)\mathfrak{r}\big) \det(\xi)^{\mu\rho} \mathbf{e}_\mathbf{a}^n(\xi^*\tau\xi z).$$

Here it is understood that $\omega_\mathbf{a}(b)\omega^*(b\mathfrak{x})$ for $b = 0$ and a fractional ideal $\mathfrak{x}$ denotes $\omega^*(\mathfrak{x})$ or 0 according as $\mathfrak{f} = \mathfrak{r}$ or $\mathfrak{f} \neq \mathfrak{r}$. Indeed, for $z = y(\mathbf{i})$ with $y \in G_\mathbf{a}$ we have $j_y^l(\mathbf{i})^{-1} g_p(z) = \mathbf{g}(py) = j_y^l(\mathbf{i})^{-1}\theta(z, {}^p\lambda_1)$ by (A5.6), so that $g_p(z) = \theta(z, {}^p\lambda_1)$. Therefore, by (A5.2) and (A5.3) we obtain

$$g_p(z) = |\det(q)|_K^{n/2} \varphi\big(\det(q)\big)^n \sum_{\xi \in V} \lambda_1(\xi q) \det(\xi)^{\mu\rho} \mathbf{e}_\mathbf{a}^n(\xi^*\tau\xi z).$$

If $\det(\xi) \neq 0$, then $\lambda_1(\xi q) \neq 0$ only when $\xi \in rR^* q^{-1}$ and $\det(r^{-1}\xi q)\mathfrak{r}$ is prime to $\mathfrak{h}$, in which case

$$\begin{aligned}\omega\big(\det(q)\big)\lambda_1(\xi q) &= \omega_\mathbf{h}\big(\det(r^{-1}\xi q)\big)\omega_\mathbf{a}\big(\det(\xi)\big)\omega_\mathfrak{h}\big(\det(r^{-1}\xi q)\big)^{-1} \\ &= \omega_\mathbf{a}\big(\det(\xi)\big)\omega^*\big(\det(r^{-1}\xi q)\mathfrak{r}\big).\end{aligned}$$

Checking also the case of ξ with zero determinant, we obtain (A5.10). On the other hand, taking $(\mathbf{f}, r)$ of (20.9f) to be $(\mathbf{g}, q)$ here, we find that

$$\text{(A5.11)} \qquad c_\mathbf{g}(\sigma, q) = |\det(q)|_K^{n/2} \omega'\big(\det(q)\big)^{-1} \cdot \sum_\xi \omega_\mathbf{a}\big(\det(\xi)\big)\omega^*\big(\det(r^{-1}\xi q)\mathfrak{r}\big) \det(\xi)^{\mu\rho},$$

where ξ runs over $V \cap rR^* q^{-1}$ under the condition that $\xi^*\tau\xi = \sigma$.

Now in the setting of §22.3, take $(\chi^{-1}, \mathfrak{e})$ there to be $(\omega, \mathfrak{a})$ here. Then we obtain $\mathbf{g}$ in Case UT in that §, since (A5.11) gives exactly (22.15).

In Case SP the matter is simpler. We take θ of (A3.21) with χ^{-1} as ω; we also take (r, q) to be (p, r) in (A3.23). By Proposition A3.19 this corresponds to an element $\mathbf{g}$ of $\mathcal{M}_l(C', \psi')$ with $\psi' = \chi^{-1}\rho_\tau$ and a suitable C'. Combining (A3.23) with (20.9e), we obtain (22.15). Notice that $\det(a_\beta d_\beta) - 1 \in \mathfrak{c}$ for β in (A3.23), since $\operatorname{diag}[r^{-1}, {}^t r]\beta \in C$ with r there.

A6. Estimate of the Fourier coefficients of a modular form

A6.1. Let us first recall a basic fact on reduction theory of symmetric and hermitian matrices. We consider S of (16.1a) and S^+, $S_\mathbf{a}^+$ of (22.1a, b) in Cases SP and UT, and let R denote the group of all upper triangular elements of $GL_n(K)$ whose diagonal elements are all equal to 1. (Here K is as in §3.5, and so $K = F$ in Case SP.) We embed F naturally into $F_\mathbf{a}$ and extend the map $\mathrm{Tr}_{F/\mathbf{Q}} : F \to \mathbf{Q}$

to an $\mathbf{R}$-linear map of $F_{\mathbf{a}} = \mathbf{R}^{\mathbf{a}}$ into $\mathbf{R}$, and denote it by the same symbol $\mathrm{Tr}_{F/\mathbf{Q}}$. Given a positive number $r > 1$, we denote by Δ_r the set of all diagonal matrices $\mathrm{diag}[\delta_1, \ldots, \delta_n]$ with $\delta_i \in F_{\mathbf{a}}^{\times}$ such that

$$\text{(A6.1)} \quad (\delta_i)_v > 0, \quad r^{-1} \le (\delta_i)_v/\mathrm{Tr}_{F/\mathbf{Q}}(\delta_i) \le r, \quad \textit{and} \quad \mathrm{Tr}_{F/\mathbf{Q}}(\delta_i) \le r\mathrm{Tr}_{F/\mathbf{Q}}(\delta_{i+1})$$

for every i and every $v \in \mathbf{a}$. (We of course ignore δ_{n+1}.) For a compact subset C of R we define a *Siegel set* $\mathfrak{S}$ by

$$\text{(A6.2)} \qquad \mathfrak{S} = \mathfrak{S}(r, C) = \{\tau^* d\tau \mid \tau \in C,\ d \in \Delta_r\}.$$

Let U be a subgroup of $GL_n(\mathfrak{r})$ of finite index. Then we can choose r, C, and a finite subset B of $GL_n(K) \cap \mathfrak{r}_n^n$ so that

$$\text{(A6.3)} \qquad S_{\mathbf{a}}^+ = \bigcup_{b \in B} \bigcup_{u \in U} u^* \widehat{b} \mathfrak{S}(r, C) b^{-1} u.$$

This and (A6.5) below are well-known. To state another basic fact on $\Gamma\backslash\mathcal{H}^n$ for any congruence subgroup Γ of G^n, take any nonempty open subset X of $S_{\mathbf{a}}$ and an element y_0 of $S_{\mathbf{a}}^+$, put

$$\text{(A6.4)} \qquad T = \{x + iy \in \mathcal{H}^n \mid x \in X,\ y_0 < y \in S_{\mathbf{a}}^+\},$$

where we write $y_0 < y$ (and $y > y_0$) if $y_v > (y_0)_v$ for every $v \in \mathbf{a}$. Then there exists a finite subset A of G such that

$$\text{(A6.5)} \qquad \mathcal{H}^n = \textstyle\bigcup_{\alpha \in A} \Gamma\alpha T.$$

A6.2. Lemma. *Let L be a $\mathfrak{g}$-lattice in S, and U a subgroup of $GL_n(\mathfrak{r})$ of finite index. Then there exists a positive constant M with the following property: Given $h \in L \cap S^+$, there exists an element u of U such that $\mathrm{tr}\big((u^*hu)_v^{-1}\big) \le M$ for every $v \in \mathbf{a}$.*

PROOF. By (A6.3), given $h \in L \cap S^+$, we have $b^*u^*hub \in \mathfrak{S}(r, C)$ for some $b \in B$ and $u \in U$. Put $b^*u^*hub = \tau^* d\tau$ with $\tau \in C$ and $d = \mathrm{diag}[\delta_1, \ldots, \delta_n] \in \Delta_r$. Then $(u^*hu)^{-1} = b\tau^{-1}d^{-1}\widehat{\tau}b^*$. From (A6.1) we obtain $(\delta_i)_v \le r\mathrm{Tr}_{F/\mathbf{Q}}(\delta_i) \le r^2\mathrm{Tr}_{F/\mathbf{Q}}(\delta_{i+1}) \le r^3(\delta_{i+1})_v$, and hence $(\delta_i)_v^{-1} \le r^3(\delta_1)_v^{-1}$. Since $b\tau^{-1}$ belongs to a compact set independent of h, we have

$$(*) \qquad \mathrm{tr}\big((u^*hu)_v^{-1}\big) \le M'(\delta_1)_v^{-1}$$

with a positive constant M' independent of h. Now τ is upper triangular, so that $\delta_1 = (b^*u^*hub)_{11}$ and this belongs to a fractional ideal $\mathfrak{a}$ depending only on L and U. Thus $\mathrm{Tr}_{F/\mathbf{Q}}(\delta_1) \in \mathrm{Tr}_{F/\mathbf{Q}}(\mathfrak{a})$, and $(\delta_1)_v \ge r^{-1}\mathrm{Tr}_{F/\mathbf{Q}}(\delta_1) \ge M''$ with a positive constant M'' independent of h. Combining this with (*) we obtain our lemma.

A6.3. Lemma. *Let L be a $\mathfrak{g}$-lattice in F^t and $f(x_1, \ldots, x_t)$ be a nonzero polynomial in t indeterminates $x_1, \ldots, x_t$ with complex coefficients of degree d_i with respect to x_i for each i. Then there exists a positive constant M depending only on L and $\{d_i\}$ with the following property: Given $\xi \in F_{\mathbf{a}}^t$, there exists an element b of L such that $f(b) \ne 0$ and $|(\xi_i - b_i)_v| \le M$ for every i and every $v \in \mathbf{a}$.*

PROOF. We may assume that $L = \mathfrak{b}^t$ with a fractional ideal $\mathfrak{b}$. For $0 < r \in \mathbf{R}$ and $a \in F_{\mathbf{a}}$ put $B_r(a) = \{b \in \mathfrak{b} \mid |(a - b)_v| \le r \text{ for every } v \in \mathbf{a}\}$. We can find r such that $\#B_r(a) > \mathrm{Max}(d_1, \ldots, d_t)$ for every $a \in F_{\mathbf{a}}$. Now given $\xi \in F_{\mathbf{a}}^t$, we can find $b_1 \in B_r(\xi_1)$ such that $f(b_1, x_2, \ldots, x_t) \ne 0$; then we find $b_2 \in B_r(\xi_2)$ such that $f(b_1, b_2, x_3, \ldots, x_t) \ne 0$. Eventually we find $b_i \in B_r(\xi_i)$ for $1 \le i \le t$ such that $f(b_1, \ldots, b_t) \ne 0$. This proves our lemma.

A6.4. Proposition. *Let k be a weight and let $f(z) = \sum_{h \in S} c(h)\mathbf{e}_{\mathbf{a}}^n(hz) \in \mathcal{M}_k$ in Cases SP and UT; put $m = k$ in Case SP and $m = (k_v + k_{v\rho})_{v \in \mathbf{a}}$ in Case UT. Then the following assertions hold:*

(1) *If f is a cusp form, then $\delta(z)^{m/2} f(z)$ is bounded on $\mathcal{H}^n$, and $|c(h)| \le M|\det(h)^{m/2}|$ for every $h \in S$ with a constant M depending only on f.*

(2) *If $m_v \ne m_{v'}$ for some v, $v' \in \mathbf{a}$, then f is a cusp form. Consequently, if f is a not a cusp form, then $m = \kappa\mathbf{a}$ with $0 < \kappa \in 2^{-1}\mathbf{Z}$.*

(3) *If $m = \kappa\mathbf{a}$ with $0 < \kappa \in 2^{-1}\mathbf{Z}$, then $|c(h)| \le M \det(h)^{\kappa\mathbf{a}}$ for every $h \in S^+$ with a constant M depending only on f.*

PROOF. The first half of (1) and (2) were stated in [S97, Proposition 10.6] and proved in [S97, §§A4.9 and A4.10] for integral k. The case of half-integral k can be reduced to the case of integral k by considering f^2. To prove the second half of (1), we first take $M > 0$ so that $|\delta(z)^{m/2} f(z)| \le M$ on the whole $\mathcal{H}$. Then $|f(x + iy)| \le M \det(y)^{-m/2}$. Therefore, from (5.24) we obtain $|c(h)| \le M' \det(y)^{-m/2}\mathbf{e}_{\mathbf{a}}^n(-ihy)$ with a positive constant M'. Since f is a cusp form, $c(h) \ne 0$ only if $h \in S^+$. Thus taking $y = h^{-1}$, we find the desired estimate of $c(h)$ as stated in (1).

Before proving (3), we make two elementary observations. Let $g(z) = \sum_{h \in S_+} a(h) \cdot\mathbf{e}_{\mathbf{a}}^n(hz) \in \mathcal{M}_k$. The series is absolutely convergent, and so $|g(z)| \le \sum_h |a(h)\mathbf{e}_{\mathbf{a}}^n(hz)| = \sum_h |a(h)\mathbf{e}_{\mathbf{a}}^n(ihy)|$. If $y_0 < y \in S_{\mathbf{a}}^+$, then $\mathrm{tr}\big(h(y - y_0)\big)_v \ge 0$ for every $v \in \mathbf{a}$, so that $\mathbf{e}_{\mathbf{a}}^n\big(ih(y - y_0)\big) \le 1$. Thus $\mathbf{e}_{\mathbf{a}}^n(ihy) \le \mathbf{e}_{\mathbf{a}}^n(ihy_0)$, and hence $|g(z)| \le \sum_h |a(h)\mathbf{e}_{\mathbf{a}}^n(ihy_0)|$. This means that every element of $\mathcal{M}_k$ is bounded on the set T of (A6.4). Next we observe that

$$\text{(A6.6)} \qquad |\det(x + iy)| \ge \det(y) \qquad (x \in S_v,\ y \in S_v^+).$$

To show this, put $\varepsilon = y^{-1/2}$ and take a unitary matrix u so that $u\varepsilon x\varepsilon u^* = \mathrm{diag}[d_1, \ldots, d_n]$ with $d_i \in \mathbf{R}$. Then $|\det\big((x + iy)y^{-1}\big) = |\det(\varepsilon x\varepsilon + i1_n)| = \prod_{\nu=1}^n |d_\nu + i| \ge 1$, which proves (A6.6).

To prove (3), we first assume that k is integral. Take Γ so that $f \in \mathcal{M}_k(\Gamma)$; put $D = \bigcup_{\alpha \in A} \Gamma\alpha$ with A as in (A6.5) and $f_\beta = f\|_k\beta$ for $\beta \in D$. Since $\{ f_\beta \,|\, \beta \in D \} = \{ f_\alpha \,|\, \alpha \in A \}$, by our observation we have $|f_\beta(z)| \le M_1$ for every $\beta \in D$ and every $z \in T$. Here and in the following we denote by $M_1, M_2, \ldots$ some positive constants depending only on Γ, k, and f. Given $z \in \mathcal{H}^n$, take $\beta \in D$ and $w \in T$ so that $z = \beta w$; put $c = c_\xi$ and $d = d_\xi$ with $\xi = \beta^{-1}$. Take a $\mathfrak{g}$-lattice L in S so that $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \in \Gamma$ for every $s \in L$. By Lemma A6.3 we can find M_2 with the property that for every $x \in S_{\mathbf{a}}$ we have $\det(cs + d) \ne 0$ and $\|(s - x)_v\| \le M_2$ with some $s \in L$, where $\|X\| = \mathrm{Max}_{i,j}\{|X_{ij}|\}$. Putting $z = x + iy$, we choose such an s for this particular x and put $\zeta = \begin{bmatrix} 0 & 1 \\ -1 & s \end{bmatrix}$. Then $\zeta \in G$ and $j_\beta(w)^{-1} = j_\beta(\beta^{-1}z)^{-1} = j_\xi(z) = j(\xi\zeta^{-1}\zeta, z) = j(\xi\zeta^{-1}, \zeta z)j(\zeta, z)$, so that $f(z) = j_\beta^k(w)f_\beta(w) = j^k(\zeta, z)^{-1}j^k(\xi\zeta^{-1}, \zeta z)^{-1}f_\beta(w)$. Put $u = \zeta z$. Then $j(\xi\zeta^{-1}, \zeta z) = (cs + d)u - c = (cs + d)\big(u - (cs + d)^{-1}c\big)$, $j_\zeta(z) = \det(s - z)$, and $u = (s - z)^{-1}$. Since $(cs + d)^{-1}c \in S$, we have , by (A6.6),

$$(*) \qquad \big|\det\big(u - (cs + d)^{-1}c\big)\big| \ge \delta(u) = |j_\zeta(z)|^{-2}\delta(z) = |\det(s - z)|^{-2}\delta(z).$$

Therefore, for $k = \kappa\mathbf{a}$ we have

$$(**) \qquad |f(z)| \le M_1|\det(cs + d)|^{-\kappa\mathbf{a}}|\det(z - s)|^{\kappa\mathbf{a}}\delta(z)^{-\kappa\mathbf{a}}.$$

Since $\xi \in \alpha^{-1}\Gamma$ with some $\alpha \in A$, we have $cs+d \prec \mathfrak{x}$ with a fractional ideal $\mathfrak{x}$ in K depending only on A and Γ. Therefore $|\det(cs+d)|^{\kappa\mathbf{a}} \geq M_3$. Since $\|(x-s)_v\| \leq M_2$, we see that $|\det(x-s+iy)|^{\kappa\mathbf{a}}$ as a polynomial function of y has bounded coefficients. Now $|y_{\mu\nu}|^2 \leq |y_{\mu\mu}y_{\nu\nu}| \leq (y_{\mu\mu}+y_{\nu\nu})^2/4 \leq \mathrm{tr}(y)^2$, and hence $|\det(x-s+iy)|^{\kappa\mathbf{a}} \leq M_4\bigl(1+\mathrm{tr}(y)^n\bigr)^{\kappa\mathbf{a}}$. Thus from (**) we obtain

$$|f(x+iy)| \leq M_5 \det(y)^{-\kappa\mathbf{a}}\bigl(1+\mathrm{tr}(y)^n\bigr)^{\kappa\mathbf{a}} \tag{A6.7}$$

on the whole $\mathcal{H}^n$. We have assumed that k is integral. If k is half-integral, applying (A6.7) to f^2, we find that (A6.7) is also true for half-integral k. Combining (A6.7) with (5.24), we have

$$|c(h)| \leq M_6 \mathbf{e}_{\mathbf{a}}^n(-ihy)\det(y)^{-\kappa\mathbf{a}}\bigl(1+\mathrm{tr}(y)^n\bigr)^{\kappa\mathbf{a}} \tag{***}$$

for every $h \in S$. Suppose $c(h) \neq 0$ and $h \in S^+$. By (5.21), we have $c(u^*hu) = c(h)$ for every u in a subgroup U_1 of $GL_n(\mathfrak{r})$ of finite index that depends only on Γ. We can also find a $\mathfrak{g}$-lattice L_1 in S such that $c(h) \neq 0$ only if $h \in L_1$. By Lemma A6.2 we can find $u \in U_1$ so that $\mathrm{tr}\bigl((u^*hu)_v^{-1}\bigr) \leq M_7$. This means that to prove (3), we may assume that $\mathrm{tr}(h_v^{-1}) \leq M_7$. Take $y = h^{-1}$ in (***). Then $|c(h)| \leq M_8 \det(h)^{\kappa\mathbf{a}}$. This completes the proof of (3).

A6.5. Lemma. *For $\mathfrak{S} = \mathfrak{S}(r, C)$ as in (A6.2) the following assertions hold:*
(1) *For every $x \in \mathfrak{S}$ and every $v \in \mathbf{a}$ we have*

$$\det(x_v) \leq (x_{11}\cdots x_{nn})_v \leq M \cdot \det(x_v)$$

with a positive constant M depending only on $\mathfrak{S}$.

(2) *There exists a positive constant M' depending only on $\mathfrak{S}$ with the property that $(x_{ii})_v/(x_{ii})_{v'} \leq M'$ for every $x \in \mathfrak{S}$, every i, and every $v, v' \in \mathbf{a}$.*

(3) *For every $i \leq n$, the number of elements $e \in \mathfrak{g}^\times$ such that $ex_{ii} = x'_{ii}$ for some $x, x' \in \mathfrak{S}$ is bounded by a constant depending only on $\mathfrak{S}$.*

PROOF. Let $x = \tau^* d\tau$ with $\tau \in C$ and $d = \mathrm{diag}[\delta_1, \ldots, \delta_r] \in \Delta_r$. Focusing our attention on one $v \in \mathbf{a}$, let us drop the subscript v. We easily see that $x_{11} = \delta_1$, and x_{ii} is a linear form of $\delta_1, \ldots, \delta_i$ with coefficients in a compact set. Therefore from (A6.1) we obtain $x_{ii} \leq M\delta_i$ with M depending only on $\mathfrak{S}$. Also we easily see that $x_{ii} - \delta_i \geq 0$. Thus $\det(x) = \delta_1\cdots\delta_n \leq x_{11}\cdots x_{nn} \leq M^n\delta_1\cdots\delta_n \leq M^n\det(x)$, which proves (1). As for (2), we have seen that $(\delta_i)_v \leq (x_{ii})_v \leq M(\delta_i)_v$. From (A6.1) we see that $(\delta_i)_{v'}/(\delta_i)_v \leq r^2$ for every $v, v' \in \mathbf{a}$. Therefore $(x_{ii})_{v'}/(x_{ii})_v \leq M(\delta_i)_{v'}/(\delta_i)_v \leq Mr^2$, which proves (2). To prove (3), suppose $ex_{ii} = x'_{ii}$ for some $x, x' \in \mathfrak{S}$ and $e \in \mathfrak{g}^\times$. Then $(ex_{ii})_v/(ex_{ii})_{v'} \leq Mr^2$, ad hence $e_v/e_{v'} \leq M^2r^4$. Multiplying the inequalities for all $v' \in \mathbf{a}$, we find that $M^{-2}r^{-4} \leq e_v \leq M^2r^4$ for every $v \in \mathbf{a}$. Clearly the number of such $e \in \mathfrak{g}^\times$ is finite. This proves (3).

A6.6. Proposition. *Let U be a subgroup of $GL_n(\mathfrak{r})$ of finite index and let $S(\mathfrak{a})$ be defined by (16.1b) with a fractional ideal $\mathfrak{a}$ in K. Further let $\mathcal{R} = \bigl[S(\mathfrak{a}) \cap S^+\bigr]/U$ (see §22.1). Then $\sum_{\sigma\in\mathcal{R}}\det(\sigma)^{-s\mathbf{a}}$ is convergent for $\mathrm{Re}(s) > \lambda_n$ with λ_n of §22.1.*

PROOF. Take B and $\mathfrak{S} = \mathfrak{S}(r, C)$ as in (A6.3). We first prove that for every $\mathfrak{g}$-lattice Λ in S the series $\sum_{h\in\mathfrak{S}\cap\Lambda}\det(h)^{-s\mathbf{a}}$ is convergent for $\mathrm{Re}(s) > \lambda_n$. For that purpose, take M as in Lemma 6.5 (1). We may assume that $\Lambda \subset S(\mathfrak{a})$ with some $\mathfrak{a}$. Given $h \in \mathfrak{S} \cap \Lambda$, put $h_i = h_{ii}$. Then $\det(h_v)^{-1} \leq M(h_1\cdots h_n)_v^{-1}$ and $|(h_{ij})_v|^2 \leq |(h_ih_j)_v|$. Now for fixed positive constants T, T_0 and $t = (t_v)_{v\in\mathbf{a}} \in \mathbf{R}^{\mathbf{a}}$ such that $t_v > T_0$ and $t_v/t_{v'} \leq T$ for every $v, v' \in \mathbf{a}$ we have

(A6.8) $\qquad \#\{\, a \in \mathfrak{a} \,|\, |a_v| \le t_v \text{ for every } v \in \mathbf{a} \,\} \le M_1 \prod_{v\in\mathbf{a}} t_v^{[K:F]}$

with a positive constant M_1 independent of t. This will be shown at the end of the proof. Now take M' as in Lemma A6.5 (2); put $e = [F : \mathbf{Q}]$. Then $(h_i)_v^e \ge M'^{-e} N_{F/\mathbf{Q}}(h_i) \ge M'^{-e} N(\mathfrak{a}\cap F)$, since $h_i \in \mathfrak{a}\cap F$. Thus we can take $(h_i h_j)_v$ as t_v in (A6.8). Therefore the number of elements $h \in \mathfrak{S}\cap\Lambda$ with given $h_1, \ldots, h_n$ as their diagonal elements are at most $M_1^{n(n-1)/2} \cdot N_{F/\mathbf{Q}}(h_1 \cdots h_n)^{[K:F](n-1)/2}$. Observe that $[K : F](n-1)/2 = \lambda_n - 1$. In view of Lemma 6.5 (3), our series in question converges if $\sum_g |N_{F/\mathbf{Q}}(g)|^{\lambda_n - 1 - s}$ is convergent, where g runs over $\mathfrak{a} \cap F^\times$ modulo multiplication by the elements of $\mathfrak{g}^\times$. Since such a series is convergent for $\mathrm{Re}(s) > \lambda_n$, we obtain the desird result concerning $\sum_{h\in\mathfrak{S}\cap\Lambda} \det(h)^{-s\mathbf{a}}$. Returning to $\sum_{\sigma\in\mathcal{R}} \det(\sigma)^{-s\mathbf{a}}$ as in our proposition, we may assume, by (A6.3), that σ in the sum belongs to $\widehat{b}\mathfrak{S}b^{-1}$ for some $b \in B$. Thus it is sufficient to consider the sum of $\det(\sigma)^{-s\mathbf{a}}$ for all $\sigma \in \widehat{b}\mathfrak{S}b^{-1} \cap S(\mathfrak{a})$, or equivalently, the sum for all $\sigma \in \mathfrak{S}\cap b^* S(\mathfrak{a}) b$, to which the above result is applicable. Therefore we obtain our proposition.

To prove (A6.8), given t, put $\tau = \mathrm{Max}_{v\in\mathbf{a}} t_v$. Then $T_0 \le \tau \le T t_v$, and so the problem can be reduced to the case in which $t_v = \tau$ for every $v \in \mathbf{a}$. Let $\{\alpha_i\}_{i=1}^d$ be a Z-basis of $\mathfrak{a}$, where $d = [K : \mathbf{Q}]$. Suppose $\left|\left(\sum_{i=1}^d m_i \alpha_i\right)_v\right| \le \tau$ with $m_i \in \mathbf{Z}$ for every $v \in \mathbf{a}$. Then $|m_i| \le A\tau$ with a positive constant A depending only on $\{\alpha_i\}_{i=1}^d$. Change A for $\mathrm{Max}(A, T_0^{-1})$. Then $A\tau \ge 1$, and hence the number of such m_i is $\le 3A\tau$. Thus we obtain (A6.8).

A6.7. Let us now investigate the convergence of the series of (22.4). As explained at the beginning of §22.3, $c_{\mathbf{f}}(\sigma, q)$ (resp $c_{\mathbf{g}}(\sigma, q)$) for $\sigma \in S$ are the Fourier coefficients of an element of $\mathcal{S}_k$ (resp $\mathcal{M}_\ell$). By Proposition A6.4 we have $|c_{\mathbf{f}}(\sigma, q)| \le M \det(\sigma)^{m/2}$ and $|c_{\mathbf{g}}(\sigma, q)| \le M' \det(\sigma)^{m'\varepsilon}$ for every $\sigma \in S^+$ with constants M and M' independent of σ, where m and m' are as in Proposition 22.2; $\varepsilon = 1/2$ if $\mathbf{g}$ is a cusp form and $\varepsilon = 1$ otherwise. Therefore

$$|c_{\mathbf{f}}(\sigma, q) c_{\mathbf{g}}(\sigma, q) \det(\sigma)^{-s\mathbf{a}-h}| \le MM' |\det(\sigma)^{(c-s)\mathbf{a}}|,$$

where $c = 0$ if $\mathbf{g}$ is a cusp form, and c is the element of $\mathbf{Q}$ such that $m' = 2c\mathbf{a}$ if $\mathbf{g}$ is not a cusp form. Also, $c_{\mathbf{f}}(\sigma, q) c_{\mathbf{g}}(\sigma, q) \ne 0$ only if σ belongs to a lattice in S depending on q. Therefore, by Proposition A6.6, the series of (22.4) is convergent if $\mathrm{Re}(s)$ is sufficiently large.

A7. The Mellin transforms of Hilbert modular forms

A7.1. This section concerns the case in which $G = SL_2(F)$ and $\mathcal{H} = \mathfrak{H}_1^{\mathbf{a}}$ in the setting of Section 5 and §10.6. We put $\mathfrak{g}_+^\times = \{a \in \mathfrak{g}^\times \,|\, a \gg 0\}$.

Let $f(z) = \sum_{h\in F} a(h) \mathbf{e}_{\mathbf{a}}(hz) \in \mathcal{M}_k$ with an integral or a half-integral weight k. As noted in (5.21) and §6.10, we can find a subgroup U_1 of $\mathfrak{g}_+^\times$ of finite index such that $f(u^2 z) = u^{-k} f(z)$ for every $u \in U_1$, so that $a(u^2 h) = u^k a(h)$ for every $u \in U_1$. We now put

(A7.1) $$D(s, f) = [\mathfrak{g}^\times : U]^{-1} \sum_{h\in F^\times/U} a(h) |h|^{-k/2 - s\mathbf{a}} \qquad (s \in \mathbf{C}^\times),$$

where U is a subgroup of $\mathfrak{g}_+^\times$ of finite index such that $a(uh) = u^{k/2} a(h)$ for every $u \in U$. This is formally well-defined independently of the choice of U. By Proposition A6.4, $|a(h)| \le M |h|^{k/2+\sigma\mathbf{a}}$ for $h \ne 0$ with a constant M independent of h, where

$\sigma = \kappa/2$ if $k = \kappa\mathbf{a}$ with $\kappa \in 2^{-1}\mathbf{Z}$ and $\sigma = 0$ otherwise. Therefore the sum is convergent for $\mathrm{Re}(s) > 1+\sigma$, and defines a holomorphic function of s there.

To find analytic continuation of $D(s, f)$, put

$$f_*(z) = (-iz)^{-k} f(-z^{-1}). \tag{A7.2}$$

Observe that $(f_*)_* = f$, $f_* = i^c f\|_k \eta$, and $f_* \in \mathcal{M}_k$, where $c = \sum_{v\in\mathbf{a}} [k]_v$.

A7.2. Theorem. *The notation being as above, put* $R(s, f) = \Gamma_k(s)D(s, f)$ *with* $\Gamma_k(s) = \prod_{v\in\mathbf{a}} (2\pi)^{-s-k_v/2}\Gamma(s+(k_v/2))$; *put also* $f_*(z) = \sum_{h\in F} a_*(h)\mathbf{e}_{\mathbf{a}}(hz)$. *Then* $R(s, f)$ *can be continued as a meromorphic function of* s *to the whole* $\mathbf{C}$, *and satisfies* $R(-s, f) = R(s, f_*)$. *Moreover,* $R(s, f)$ *is entire except when* $k = \kappa\mathbf{a}$ *with* $\kappa \in 2^{-1}\mathbf{Z}$, *in which case* $R(s, f)$ *is holomorphic on* $\mathbf{C}$ *except for possible simple poles at* $s = -\kappa/2$ *and* $s = \kappa/2$ *with residues* $-a(0)R_F/2$ *and* $a_*(0)R_F/2$, *respectively, where* R_F *denotes the regulator of* F. *In particular,* $R(s, f)$ *is entire if* f *is a cusp form.*

PROOF. Put $g = f - a(0)$, $g_* = f_* - a_*(0)$, $\varphi(z) = \sum_{h\in A} a(h)\mathbf{e}_{\mathbf{a}}(hz)$ with a fixed complete set of representatives A for $F^\times/U$, $Y = \{y \in \mathbf{R}^{\mathbf{a}} \mid y \gg 0\}$, $Y_1 = \{y \in Y \mid y^{\mathbf{a}} \geq 1\}$, $Y_2 = \{y \in Y \mid y^{\mathbf{a}} < 1\}$. Then $g(z) = \sum_{u\in U}\sum_{h\in A} a(uh)\mathbf{e}_{\mathbf{a}}(uhz) = \sum_{u\in U} u^{k/2}\varphi(uz)$, so that $g(iy)y^{k/2+(s-1)\mathbf{a}} = \sum_{u\in U}\varphi(iuy)(uy)^{k/2+(s-1)\mathbf{a}}$. Thus we have, at least formally,

$$\int_{Y/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy = \int_Y \varphi(iy)y^{k/2+(s-1)\mathbf{a}}dy$$
$$= \sum_{h\in A} a(h)\int_Y \mathbf{e}_{\mathbf{a}}(ihy)y^{k/2+(s-1)\mathbf{a}}dy = \sum_{h\in A} a(h)\Gamma_k(s)|h|^{-k/2-s\mathbf{a}} = [\mathfrak{g}^\times : U]R(s, f).$$

Our formal calculation is valid for $\mathrm{Re}(s) > 1+\sigma$, because of the convergence of $\sum_{h\in A} |a(h)h^{-k/2-s\mathbf{a}}|$. Now we have

$$\int_{Y/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy = \int_{Y_1/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy + \int_{Y_2/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy.$$

All three integrals are convergent for $\mathrm{Re}(s) > 1+\sigma$. If $y^{\mathbf{a}} \geq 1$ and $\mathrm{Re}(s) < \mathrm{Re}(s')$, then $|y^{s\mathbf{a}}| \leq |y^{s'\mathbf{a}}|$, and hence the integral over Y_1/U is convergent for every $s \in \mathbf{C}$. To deal with the integral over Y_2/U, take U so that $a_*(uh) = u^{k/2}a_*(h)$ for every $u \in U$. From (A7.2) we obtain $g(iy^{-1}) = y^k g_*(iy) - a(0) + a_*(0)y^k$, and hence

$$\int_{Y_2/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy = \int_{Y_1/U} g(iy^{-1})y^{-k/2-(s+1)\mathbf{a}}dy$$
$$= \int_{Y_1/U} g_*(iy)y^{k/2-(s+1)\mathbf{a}}dy$$
$$- a(0)\int_{Y_1/U} y^{-k/2-(s+1)\mathbf{a}}dy + a_*(0)\int_{Y_1/U} y^{k/2-(s+1)\mathbf{a}}dy,$$

provided $\mathrm{Re}(s) > 1+\sigma$ and the last three integrals are convergent. The integral involving g_* is convergent for every $s \in \mathbf{C}$. We have $a(0) = a_*(0) = 0$ if f is a cusp form. If not, then $k = \kappa\mathbf{a}$ with $\kappa \in 2^{-1}\mathbf{Z}$ by Proposition A6.4 (2). Now we need a formula

$$\int_{Y_1/U} y^{-\alpha\mathbf{a}}dy = (\alpha-1)^{-1}2^{-1}[\mathfrak{g}^\times : U]R_F \quad \textit{if} \ \mathrm{Re}(\alpha) > 1, \tag{A7.3}$$

which will be proven at the end of the proof. Thus, putting $M = [\mathfrak{g}^\times : U]^{-1}$, we obtain

$$\text{(A7.4)}\qquad R(s,f) = M\int_{Y_1/U} g(iy)y^{k/2+(s-1)\mathbf{a}}dy + M\int_{Y_1/U} g_*(iy)y^{k/2-(s+1)\mathbf{a}}dy \\ - a(0)(R_F/2)(s+\kappa/2)^{-1} + a_*(0)(R_F/2)(s-\kappa/2)^{-1}$$

for sufficiently large $\mathrm{Re}(s)$, where the last two terms occur only if $k = \kappa\mathbf{a}$. Since the integrals over Y_1/U are convergent for every s, the right-hand side defines a meromorphic function on the whole $\mathbf{C}$ with poles and residues as described in our theorem. To obtain the functional equation, change f for f_*. Since $(f_*)_* = f$, $R(s, f_*)$ can be obtained by exchanging $\bigl(g, a(0)\bigr)$ for $\bigl(g_*, a_*(0)\bigr)$ in (A7.4). Then we easily see that it coincides with $R(-s, f_*)$. This proves our theorem.

To prove (A7.3), take a set of generators $\{\varepsilon_i\}_{i=1}^r$ of U, where $r = [F : \mathbf{Q}] - 1$. Take $r+1$ real variables $\{t_i\}_{i=0}^r$ and put $y_v = t_0 \exp\bigl(\sum_{i=1}^r t_i \log|\varepsilon_i|_v\bigr)$ for $v \in \mathbf{a}$ with $t_0 > 0$. We easily see that the jacobian of the map $t \mapsto y$ is $\pm(r+1)t_0^r \det\bigl(\log|\varepsilon_i|_v\bigr)_{i,v}$, where v is restricted to the set of arbitrarily chosen r elements of $\mathbf{a}$. This quantity equals $\pm(r+1)t_0^r 2^{-1}[\mathfrak{g}^\times : U]R_F$. Therefore

$$\int_{Y_1/U} y^{-\alpha\mathbf{a}}dy = (r+1)t_0^r 2^{-1}[\mathfrak{g}^\times : U]R_F \int_1^\infty t_0^{r-(r+1)\alpha}dt_0,$$

which gives (A7.3).

A7.3. *Proof of Lemma 18.2.* Given κ and t as in our lemma, put

$$f(z) = \theta(z, \kappa) = \sum_{\xi\in F} \kappa(\xi)\xi^t \mathbf{e}_\mathbf{a}(\xi^2 z/2) = \sum_{h\in F} a(h)\mathbf{e}_\mathbf{a}(hz).$$

This can be obtained by putting $n = 1$ and $\tau = 1/2$ in (A3.12). As explained after (A3.14), $f \in \mathcal{M}_k$ with $k = t + \mathbf{a}/2$; f is a cusp form if $t \neq 0$. Clearly $a(h) = \sum_{\xi^2=2h} \kappa(\xi)\xi^t$. Take U of (18.1) so that $U \subset \mathfrak{g}_+^\times$ and put $U^2 = \{u^2 \mid u \in U\}$. Observing that $\bigsqcup_{h\in F^\times/U^2} \{\xi \in F \mid \xi^2 = 2h\}$ gives $F^\times/U$, we have

$$[\mathfrak{g}^\times : U]D_t(2s, \kappa) = \sum_{h\in F^\times/U^2} \sum_{\xi^2=2h} \kappa(\xi)\xi^t|\xi|^{-t-2s\mathbf{a}}$$
$$= \sum_{h\in F^\times/U^2} a(h)|2h|^{-t/2-s\mathbf{a}} = [\mathfrak{g}^\times : U^2]2^{-t/2-sa}D(s-1/4, f).$$

Consequently

$$\text{(A7.5)}\qquad R_t(s, \kappa) = 2^r R\bigl((s/2)-(1/4), f\bigr),$$

where $r = [F : \mathbf{Q}] - 1$. Taking η as α in (A3.14) and substituting ηz for z, we obtain $\theta(z, {}^\eta\kappa) = J(\eta, \eta z)f(\eta z)$, and hence $f_*(z) = i^{-|t|}\theta(z, {}^\eta\kappa)$. Now ${}^\eta\kappa$ can be obtained from Theorem A3.3 (6), where we take $S(x, y) = xy$ for $x, y \in F$. Putting $\kappa_* = i^{-|t|}\cdot{}^\eta\kappa$, we find κ_* as given in Lemma 18.2. Then all the assertions of Lemma 18.2 can be derived from Theorem A7.2 in view of (A7.5).

A7.4. *Proof of Theorem 18.16 (1).* The notation being as in §18.15, for $0 \leq m \in \mathbf{Z}^\mathbf{a}$ and $\ell \in \mathcal{S}(K_\mathbf{h})$ put

$$f(z) = \theta(z, \ell) = \sum_{\alpha\in K} \ell(\alpha)\alpha^{m\rho}\mathbf{e}_\mathbf{a}(\alpha\alpha^\rho z) = \sum_{h\in F} a(h)\mathbf{e}_\mathbf{a}(hz) \qquad (z \in \mathcal{H}).$$

This can be obtained from (A5.3) by putting $n = 1$, $\tau = 1$, $\mu = m$, and $\lambda = \ell$. Since $n = 1$, we have $SU(\eta) = SL_2(F)$ by Lemma 1.3 (2), and hence $f \in \mathcal{M}_k$

with $k = m + \mathbf{a}$, where $\mathcal{M}_k$ is defined with respect to $G = SL_2(F)$. Clearly $a(h) = \sum_{\alpha\alpha^\rho = h} \ell(\alpha)\alpha^{m\rho}$. Take U of (18.18) so that $U \subset \mathfrak{g}_+^\times$ and put $U^2 = \{u^2 \mid u \in U\}$. Observing that $\bigsqcup_{h \in F^\times/U^2} \{\alpha \in K \mid \alpha\alpha^\rho = h\}$ gives $K^\times/U$, we have

$$[\mathfrak{r}^\times : U] L_m(s, \ell) = \sum_{h \in F^\times/U^2} \sum_{\alpha\alpha^\rho = h} \ell(\alpha)\alpha^{m\rho}|\alpha|^{-m-2s\mathbf{a}}$$

$$= \sum_{h \in F^\times/U^2} a(h)|2h|^{-m/2-s\mathbf{a}} = [\mathfrak{g}^\times : U^2] D_k(s - 1/2, f).$$

Thus

$$L_m(s, \ell) = 2^r[\mathfrak{r}^\times : \mathfrak{g}^\times]^{-1} D_k(s - 1/2, f), \tag{A7.6}$$

and therefore (1) of Theorem 18.16 follows immediately from Theorem A7.2, since f is a cup form if $m \neq 0$.

We can actually include the case $m = 0$ and prove some results for $L_m(s, \ell)$ completely parallel to Lemma 18.2. The precise statement may be left to the reader, as the task is easy and we do not need it in the present book.

A8. Certain unitarizable representation spaces

A8.1. Given a finite-dimensional vector space W over $\mathbf{C}$, we denote by $\mathfrak{S}(W)$ the symmetric algebra over W, and by $\mathfrak{S}_e(W)$, for $0 \leq e \in \mathbf{Z}$, its subspace consisting of all the homogeneous elements of degree e. Then $S_e(W)$ of §12.3 and $\mathfrak{S}_e(W)$ are dual to each other with respect to the pairing

$$\langle \alpha, x_1 \cdots x_e \rangle = \alpha_*(x_1, \ldots, x_e) \qquad \big(x_i \in W, \ \alpha \in S_e(W)\big). \tag{A8.1}$$

We now take our setting to be that of Sections 12 through 14. We denote by $\mathcal{K}$ the maximal compact subgroup of $G_\mathbf{a}$ given by $\mathcal{K} = \{\alpha \in G_\mathbf{a} \mid \alpha(\mathbf{o}) = \mathbf{o}\}$, where $\mathbf{o} = (\mathbf{o}_v)_{v \in \mathbf{a}}$ with $\mathbf{o}_v = 0$ for Types AB and CB and $\mathbf{o}_v = i1_n$ for Types AT and CT. Then we denote by $\mathfrak{g}$ the Lie algebra of $G_\mathbf{a}$, and by $\mathfrak{k}$ the subalgebra of $\mathfrak{g}$ corresponding to $\mathcal{K}$. Since we have a fixed complex structure of $\mathcal{H} = G/\mathcal{K}$, we have a well-known decomposition $\mathfrak{g}_\mathbf{C} = \mathfrak{k}_\mathbf{C} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-$ with the following property:

$$[\mathfrak{k}_\mathbf{C}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \quad [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{k}_\mathbf{C}. \tag{A8.2}$$

We denote by $\mathfrak{U}$ the universal enveloping algebra of $\mathfrak{g}_\mathbf{C}$. We let $\mathfrak{g}$ act on the set $C^\infty(G_\mathbf{a})$ of all C^∞ functions on $G_\mathbf{a}$ by $(Yf)(x) = (d/dt)_{t=0} f\big(x \cdot \exp(tY)\big)$ for $Y \in \mathfrak{g}$ and $x \in G_\mathbf{a}$, and extend the action to that of $\mathfrak{U}$ on $C^\infty(G_\mathbf{a})$ as usual. More generally, we view every $\mathfrak{g}$-module as a $\mathfrak{U}$-module, and vice versa. We say that a $\mathfrak{U}$-module $\mathcal{Y}$ is *unitarizable* if it satisfies the following condition:

(A8.3) *$\mathcal{Y}$ has a positive definite hermitian form $\{\ ,\ \} : \mathcal{Y} \times \mathcal{Y} \to \mathbf{C}$ such that $\{Xg, h\} = -\{g, Xh\}$ for every $g, h \in \mathcal{Y}$ and every $X \in \mathfrak{g}$.*

Let $\mathcal{Y}$ and $\mathcal{Y}'$ be two unitarizable $\mathfrak{U}$-modules. Then we define a $\mathfrak{g}$-module structure on $\mathcal{Y} \otimes_\mathbf{C} \mathcal{Y}'$ as usual by defining $X(g \otimes g') = (Xg) \otimes g' + g \otimes Xg'$ for $g \in \mathcal{Y}$, $g' \in \mathcal{Y}'$, and $X \in \mathfrak{g}$. Defining an inner product on $\mathcal{Y} \otimes_\mathbf{C} \mathcal{Y}'$ by $\{g \otimes g', h \otimes h'\} = \{g, h\}\{h, h'\}$, we easily see that $\mathcal{Y} \otimes_\mathbf{C} \mathcal{Y}'$ is unitarizable.

A8.2. Given a representation $\{\rho, V\}$ of $\mathcal{K}$, we denote by $C^\infty(\rho)$ the set of all elements f of $C^\infty(G_\mathbf{a}, V)$ such that $f(xk^{-1}) = \rho(k)f(x)$ for every $k \in \mathcal{K}$ and $x \in G_\mathbf{a}$. Then $Xf = -d\rho(X)f$ for every $X \in \mathfrak{k}$ and $f \in C^\infty(\rho)$. Let $Ad : G_\mathbf{a} \to \mathrm{Aut}(\mathfrak{g})$ denote the adjoint representation of $G_\mathbf{a}$. For $g \in G_\mathbf{a}$ the action of $Ad(g)$ can be extended to $\mathfrak{U}$, which we denote also by $Ad(g)$. In the following we consider exclusively $Ad(k)$ for $k \in K$. It is well-known, and in fact, can easily be verified, that

$$\text{(A8.4)}\qquad \big[\big(Ad(k)B\big)f\big](x) = \rho(k)(Bf)(xk) \quad \text{if} \quad f \in C^\infty(\rho), B \in \mathfrak{U},\ k \in \mathcal{K},\ x \in G_\mathbf{a}.$$

In view of (A8.2), $\mathfrak{S}(\mathfrak{p}_\pm)$ can be embedded in $\mathfrak{U}$. Then $\mathfrak{S}(\mathfrak{p}_\pm)$ is stable under $Ad(k)$ for $k \in \mathcal{K}$.

Let us now describe $\mathfrak{p}_\pm$ and $\mathcal{K}$ more explicitly, explaining their relation with the symbols T and K^c of §12.1. For the moment, our group is a single factor G_v, not $G_\mathbf{a}$. For each type there exist an injection ι_0 of $\mathcal{K}$ into K^c and $\mathbf{C}$-linear bijections $\iota_\pm$ of T onto $\mathfrak{p}_\pm$ with the following properties:

$$\text{(A8.5a)}\qquad \iota_0(k) = M_k(\mathbf{o}) \quad \text{for} \quad k \in \mathcal{K},$$

$$\text{(A8.5b)}\qquad \mathfrak{p} = \big\{\iota_+(u) + \iota_-(\overline{u}) \,\big|\, u \in T\big\},$$

$$\text{(A8.5c)}\qquad \iota_\pm(u)^* = \iota_\mp(\overline{u}) \quad \text{for} \quad u \in T,$$

$$\text{(A8.5d)}\qquad Ad(k)\iota_+(u) = \iota_+({}^ta^{-1}ub^{-1}) \ \text{and}\ Ad(k)\iota_-(u) = \iota_-(au\cdot{}^tb) \ \text{if}\ \iota_0(k) = (a,\, b).$$

Here $M_\alpha(z)$ is defined by (5.1) and (12.4c); we put $(X + iY)^* = X - iY$ for $X + iY \in \mathfrak{g}_\mathbf{C}$ with $X, Y \in \mathfrak{g}$; the reader is reminded of the convention made in §12.1 that K^c for Type C is identified with the set of all $(a,\, a) \in GL_n(\mathbf{C})^2$.

Now the explicit forms of $\iota_\pm$ are as follows:

$$\text{Types AB and CB:}\quad \iota_+(u) = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix},\quad \iota_-(u) = \begin{bmatrix} 0 & 0 \\ {}^tu & 0 \end{bmatrix} \qquad (u \in T),$$

$$\text{Types AT and CT:}\quad \iota_+(u) = \tfrac{1}{2}\beta \begin{bmatrix} 0 & iu \\ 0 & 0 \end{bmatrix}\beta^{-1},\quad \iota_-(u) = \tfrac{1}{2}\beta \begin{bmatrix} 0 & 0 \\ -i\cdot{}^tu & 0 \end{bmatrix}\beta^{-1}$$

$$\left(\beta = \begin{bmatrix} -i1_n & i1_n \\ 1_n & 1_n \end{bmatrix},\quad u \in T\right).$$

For these, see [S90, Section 5], where classical groups of other types are also treated.

Given a representation $\{\rho, V\}$ of K^c, we have a representation $\rho \circ \iota_0$ of $\mathcal{K}$; for simplicity, we write also ρ for $\rho \circ \iota_0$. In other words, we identify k with $\iota_0(k)$ for $k \in \mathcal{K}$. For such a $\{\rho, V\}$ and $h \in C^\infty(\mathcal{H}, V)$, we define $h^\rho \in C^\infty(G_\mathbf{a}, V)$ by $h^\rho(x) = \rho\big(M_x(\mathbf{o})\big)^{-1} h\big(x(\mathbf{o})\big)$ for $x \in G_\mathbf{a}$. Then $h^\rho \in C^\infty(\rho)$, and moreover, $h \mapsto h^\rho$ is a $\mathbf{C}$-linear bijection of $C^\infty(\mathcal{H}, V)$ onto $C^\infty(\rho)$. Now we have

$$\text{(A8.6a)}\qquad \iota_+(u_1)\cdots\iota_+(u_e)g^\rho = (D_\rho^e g)^{\rho\otimes\tau_v^e}(u_1, \ldots, u_e),$$

$$\text{(A8.6b)}\qquad \iota_-(u_1)\cdots\iota_-(u_e)g^\rho = (E^e g)^{\rho\otimes\sigma_v^e}(u_1, \ldots, u_e)$$

$$\big(g \in C^\infty(\mathcal{H}, V);\ u_1, \ldots, u_e \in T_v\big).$$

For these, see [S90, pp.259-260, Proposition 7.3] and [S94b, Proposition 2.2]. From (A8.6b) we see that g is holomorphic if and only if $Yg^\rho = 0$ for every $Y \in \mathfrak{p}_-$. More generally, g is nearly holomorphic if and only if there exists a positive integer t such that $Y_1 \cdots Y_t g^\rho = 0$ for every $Y_1, \ldots, Y_t \in \mathfrak{p}_-$. Now we denote by $H(\rho)$ the set of all $f \in C^\infty(\rho)$ such that $Yf = 0$ for every $Y \in \mathfrak{p}_-$. Then $H(\rho)$ consists of the functions of the form g^ρ with all holomorphic maps g of $\mathcal{H}$ into V.

In the following treatment, we include the case of half-integral weight. Strictly speaking, in such a case we have to formulate everything in terms of suitable covering groups of $G_{\mathbf{a}}$ and $\mathcal{K}$. Since the formulation is obvious, we do not explicitly employ those coverings, and we merely state the results for both integral and half-integral weights. See [S94b, p.151] for relevant comments on this point.

We insert here an elementary lemma. Let $\mathcal{G}$ be a unimodular Lie group, L its Lie algebra, and Γ a closed unimodular subgroup of $\mathcal{G}$. Then $\Gamma\backslash\mathcal{G}$ has a $\mathcal{G}$-invariant measure μ.

A8.3. Lemma. (1) *Let $X \in L$. If φ is a $\mathbf{C}$-valued Γ-invariant C^1 function on $\mathcal{G}$ such that both φ and $X\varphi$ belong to $L^1(\Gamma\backslash\mathcal{G})$, then $\int_{\Gamma\backslash\mathcal{G}} X\varphi\, d\mu = 0$.*

(2) *Let f and h be $\mathbf{C}$-valued Γ-invariant C^1 functions on $\mathcal{G}$ and let $X \in L$. Then*

$$\int_{\Gamma\backslash\mathcal{G}} Xf \cdot h\, d\mu = -\int_{\Gamma\backslash\mathcal{G}} f \cdot Xh\, d\mu,$$

provided fh, $Xf \cdot h$, and $f \cdot Xh$ all belong to $L^1(\Gamma\backslash\mathcal{G})$.

PROOF. Put $c = \int_{\Gamma\backslash\mathcal{G}} X\varphi\, d\mu$ and $F(g, t) = (X\varphi)\big(g \cdot \exp(tX)\big)$ for $g \in \mathcal{G}$ and $t \in \mathbf{R}$. Then $\int_{\Gamma\backslash\mathcal{G}} F(g, t) d\mu(g) = c$ for every t. Similarly $\int_{\Gamma\backslash\mathcal{G}} |F(g, t)| d\mu(g) = \int_{\Gamma\backslash\mathcal{G}} |X\varphi| d\mu$, and hence $F(g, t)$ is integrable on $[0, 1] \times (\Gamma\backslash\mathcal{G})$. Therefore

$$\begin{aligned} c &= \int_0^1 \int_{\Gamma\backslash\mathcal{G}} F(g, t) d\mu(g) dt = \int_{\Gamma\backslash\mathcal{G}} \int_0^1 F(g, t) dt d\mu(g) \\ &= \int_{\Gamma\backslash\mathcal{G}} \big[\varphi(g \cdot \exp(X)) - \varphi(g)\big] d\mu(g) = 0, \end{aligned}$$

which proves (1). Here we employ the fact that $F(g, t) = (d/dt)\varphi\big(g \cdot \exp(tX)\big)$, which holds only under the assumption that φ is C^1. Assertion (2) can be obtained by taking fh to be φ in (1).

A8.4. Theorem. *Let $\rho(a, b) = \det(b)^{\kappa}$ for $(a, b) \in \mathfrak{K}_0$ with $\kappa \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$ in Case SP and $\kappa \in \mathbf{Z}^{\mathbf{a}}$ otherwise; let $\kappa_0 = \mathrm{Min}_{v \in \mathbf{a}} \kappa_v$. Suppose that the following condition is satisfied:*

(A8.7) *$\kappa_0 \geq n/2$ in Case SP, $\kappa_0 \geq n$ in Case UT, and $\kappa_v \geq \mathrm{Min}(m_v, n_v)$ for every $v \in \mathbf{a}$ such that G_v is not compact in Case UB.*

Then the following assertions hold:

(1) *For any nonzero $f \in H(\rho)$ the $\mathfrak{U}$-module structure of $\mathfrak{U}f$ is completely determined by κ independently of the choice of f. Moreover, $w \mapsto wf$ for $w \in \mathfrak{S}(\mathfrak{p}_+)$ gives a bijection of $\mathfrak{S}(\mathfrak{p}_+)$ onto $\mathfrak{U}f$.*

(2) *Such $\mathfrak{U}f$ is irreducible as a $\mathfrak{U}$-module.*

(3) *Such $\mathfrak{U}f$ is unitarizable.*

(4) *If $\{\ ,\ \}$ is an inner product on $\mathfrak{U}f$ as in (A8.3), then $\big\{\mathfrak{S}_p(\mathfrak{p}_+)f, \mathfrak{S}_q(\mathfrak{p}_+)f\big\} = 0$ for $p \neq q$.*

We can show that (1) does not hold without (A8.7); see Proposition A8.9 below for the precise statement.

PROOF. For the present ρ, we see that $d\rho$ is a $\mathbf{C}$-linear map of $\mathfrak{k}$ into $\mathbf{C}$. Therefore, in view of (A8.2), a well-known procedure shows that given $y \in \mathfrak{U}$ there exists an element $w \in \mathfrak{S}(\mathfrak{p}_+)$ such that $yf = wf$ for every $f \in H(\rho)$. Therefore

$\mathfrak{U}f = \mathfrak{S}(\mathfrak{p}_+)f$, and if we can show that $w \mapsto wf$ for $w \in \mathfrak{S}(\mathfrak{p}_+)$ is injective for every such nonzero f, then we obtain (1). For that purpose, for $h \in \mathfrak{U}f$ and $k \in \mathcal{K}$, define ${}^k h$ by ${}^k h(x) = \rho(k)h(xk)$ for $x \in G_{\mathbf{a}}$. Then ${}^k(Bf) = \big(Ad(k)B\big)f$ by (A8.4), and hence $h \mapsto {}^k h$ for $h \in \mathfrak{U}f$ defines a representation of K. We can restrict the action to $\mathfrak{S}_e(\mathfrak{p}_+)f$. In view of (A8.5d) the decomposition of $\mathfrak{S}_e(\mathfrak{p}_+)$ into $\mathcal{K}$-irreducible subspaces follows from the decomposition of (a sum of tensor products of) $\{\tau_r,\, S_r(T_v)\}$ described in Theorem 12.7; in particular, each irreducible subspace has multiplicity 1. Let V be a $\mathcal{K}$-irreducible subspace of $\mathfrak{S}_e(\mathfrak{p}_+)$. Then Vf is $\{0\}$ or isomorphic to V, and $\mathfrak{U}f$ is the direct sum of all such Vf, since no two different V's are isomorphic. Therefore it is sufficient to prove that $Vf \neq \{0\}$ for every V. Let $\mathfrak{g}^v$ be the Lie algebra of G_v and let $\mathfrak{p}_+^v = \mathfrak{p}_+ \cap \mathfrak{g}_{\mathbf{C}}^v$. Then $V = \bigotimes_{v\in\mathbf{a}} W_v$ and $e = \sum_{v\in\mathbf{a}} e_v$ with an irreducible subspace W_v of $\mathfrak{S}_{e_v}(\mathfrak{p}_+^v)$. Therefore it is sufficient to show that $W_v f \neq 0$ for every $v \in \mathbf{a}$. Thus we focus our attention on a single v; in other words, we may assume $G_{\mathbf{a}} = G_v$, $\mathfrak{p}_+ = \mathfrak{p}_+^v$, $T = T_v$, $S_e(T) = S_{e_v}(T_v)$, and $V = W_v$; so hereafter we drop the subscript v. Let us put $\psi^{(e)} = \psi^{\rho\otimes\tau^e}$ for $\psi \in C^\infty\big(\mathcal{H},\, S_e(T)\big)$. By (A8.6a), for $g \in C^\infty(\mathcal{H})$ and $X_i = \iota_+(u_i)$ with $u_1, \ldots, u_e \in T$, we have

$$\text{(A8.8)} \qquad X_1 \cdots X_e g^\rho = (D_\rho^e g)^{(e)}(u_1, \ldots, u_e) = \sum_Z (\varphi_Z D_\rho^e g)^{(e)}(u_1, \ldots, u_e),$$

where Z runs over all the irreducible subspaces of $S_e(T)$. For each Z we take its dual bases $\{\zeta_i\}$ and $\{\omega_i\}$ with respect to (12.8). Then the value of $\varphi_Z D_\rho^e g$, as an element of $S_e(T)$, equals $\sum_i (D_\rho^Z g)(\zeta_i)\cdot\omega_i$ in view of (12.22) and (12.23). (Strictly speaking, we should write $\{\zeta_i^Z\}$ and $\{\omega_i^Z\}$ for these bases, but we suppress the superscript Z, merely remembering that they depend on Z.) Therefore the last quantity of (A8.8) equals $\sum_Z \sum_i \big[(D_\rho^Z g)(\zeta_i)\omega_i\big]^{(e)}(u_1, \ldots, u_e)$. From this we obtain

$$\text{(A8.9)} \qquad z g^\rho = \sum_Z \sum_i \big\langle \big[(D_\rho^Z g)(\zeta_i)\omega_i\big]^{(e)},\, z \big\rangle \qquad \textit{for every} \quad z \in \mathfrak{S}_e(\mathfrak{p}_+).$$

Here we identify $\mathfrak{S}_e(\mathfrak{p}_\pm)$ with $\mathfrak{S}_e(T)$ through $\iota_\pm$, so that (A8.1) takes the form $\langle \alpha,\, x_1 \cdots x_e \rangle = \alpha_*\big(\iota_\pm(x_1), \ldots, \iota_\pm(x_e)\big)$ for $x_i \in \mathfrak{p}_\pm$ and $\alpha \in S_e(T)$. Given V as above and $0 \neq y \in V$, take a $\mathcal{K}$-irreducible subspace Z of $S_e(T)$ so that $\langle Z,\, V \rangle \neq 0$; put $p = \sum_i (D_\rho^Z g)(\zeta_i)\omega_i$ with this particular Z. Then $y g^\rho = \langle p^{(e)},\, y \rangle$, since $\langle Z',\, V \rangle = 0$ for $Z' \neq Z$. Thus our task is to show that $p \neq 0$ if g is nonzero and holomorphic. Now $D_\rho^Z g$ for a holomorphic g is a polynomial function of the function $r(z)$ of (13.4a, b) of degree e with holomorphic coefficients. Its highest homogeneous term is of the form $q(r)g$ with $q \in S_e(T,\, Z)$, as can be seen from (13.25); moreover, q is independent of g as noted there. Therefore the highest homogeneous term of p is $g\sum_i q(r)(\zeta_i)\omega_i$. To find $q(r)(\zeta_i)$, take $s = 0$ and $\alpha = 1$ in Lemma 13.9. Then we find that $q(r)(\zeta_i) = (-1)^e \psi_Z(-\kappa)\zeta_i(r)$, since $\xi^{-1} = ir$ for Types AT and CT and $\xi^{-1}\overline{z} = -r$ for Types AB and CB. From the formula for ψ_Z in Theorem 12.13 we see that the last quantity is nonzero under the condition on κ of our lemma, and hence $p \neq 0$, which completes the proof of (1).

To prove (4), we first observe that $\{Ag,\, h\} = \{g,\, -A^*h\}$ for $g, h \in \mathfrak{U}f$ and $A \in \mathfrak{g}_{\mathbf{C}}$. Therefore (A8.5c), applied to all factors G_v of $G_{\mathbf{a}}$, shows that $\{\mathfrak{p}_+ g,\, h\} = \{g,\, \mathfrak{p}_- h\}$. From (A8.2) we can easily derive that $\mathfrak{S}_p(\mathfrak{p}_-)\mathfrak{S}_q(\mathfrak{p}_+)f = 0$ if $p > q$, and hence $\{\mathfrak{S}_p(\mathfrak{p}_+)f,\, \mathfrak{S}_q(\mathfrak{p}_+)f\} \subset \{f,\, \mathfrak{S}_p(\mathfrak{p}_-)\mathfrak{S}_q(\mathfrak{p}_+)f\} = 0$ if $p > q$. This proves (4).

To prove (2) and (3), it is sufficient to consider again the problem on a single G_v. Indeed, take $f(x) = \prod_{v\in\mathbf{a}} f_v(x_v)$ for $x = (x_v)_{v\in\mathbf{a}} \in G_{\mathbf{a}}$ with $x_v \in G_v$ and

$f_v \in H(\rho_v)$, $\rho_v(a_v, b_v) = \det(b_v)^{\kappa_v}$. Then clearly $\mathfrak{U}f = \bigotimes_{v\in\mathbf{a}} \mathfrak{U}(\mathfrak{g}^v_{\mathbf{C}})f_v$, and hence our problem can be reduced to that on G_v. Therefore we drop the subscript v in the same sense as above. Now, to prove that $\mathfrak{U}f$ is irreducible, it is sufficient to show that for any $z \in \mathfrak{S}(\mathfrak{p}_+)$, $\neq 0$, there exists an element w of $\mathfrak{S}(\mathfrak{p}_-)$ such that $wzf = f$. Since $\mathfrak{p}_-f = 0$, from (A8.2) we easily see that $\mathfrak{S}_e(\mathfrak{p}_-)\mathfrak{S}_i(\mathfrak{p}_+)f = 0$ for $i < e$ and $\mathfrak{S}_e(\mathfrak{p}_-)\mathfrak{S}_e(\mathfrak{p}_+)f \subset \mathbf{C}f$. Therefore it is sufficient to show that for any $z \in \mathfrak{S}_e(\mathfrak{p}_+)$, $\neq 0$, there exists an element w of $\mathfrak{S}_e(\mathfrak{p}_-)$ such that $wzf \neq 0$, even with a special choice of f. Now z can be expressed as a sum of elements y, each y being contained in some V as above. Let $f = g^\rho$ with a holomorphic g. Then we have seen in (A8.9) that $zf = \langle h^{(e)}, z\rangle$ with an element $h \in \mathcal{N}^e_{\rho\otimes\tau^e}$ whose highest term is $\alpha(r)g$ with $\alpha \in S_e(T, S_e(T))$. The above computation of p shows that $\alpha(r) = \sum_Z c_Z \sum_i \zeta_i(r)\omega_i$ with $c_Z \in \mathbf{Q}^\times$. Taking E, $\rho\otimes\tau^e$, and h in place of D_ρ, ρ, and g in (A8.8) and (A8.9), and employing (A8.6b) instead of (A8.6a), we find that

$$wh^{(e)} = \sum_{Z'}\sum_j \langle [(E^{Z'}h)(\zeta'_j)\omega'_j]^{\rho\otimes\tau^e\otimes\sigma^e}, w\rangle \quad \textit{for every} \quad w \in \mathfrak{S}_e(\mathfrak{p}_-),$$

where Z' runs over all the irreducible subspaces of $S_e(T)$, and $\{\zeta'_i\}$ and $\{\omega'_i\}$ are dual bases of Z'; we use Z', since we will have to consider the double sum $\sum_Z\sum_{Z'}$. By (13.14b), for every $\gamma \in Z'$ we have $(-1)^e(E^{Z'}h)(\gamma) = \gamma(\partial/\partial r)h$, which equals $g\cdot\gamma(\partial/\partial r)\alpha(r)$, since $\gamma(\partial/\partial r)$ annihilates the terms of degree $< e$. Taking $\partial/\partial r$ in place of $\mathcal{D}_v$ in (12.28), we obtain $\gamma(\partial/\partial r)\alpha(r) = e![\gamma, \alpha]$. Thus $\sum_{Z'}\sum_j(E^{Z'}h)(\zeta'_j)\omega'_j = (-1)^e e! g\sum_{Z'}\sum_j[\zeta'_j, \alpha]\omega'_j$. Take g to be constant 1 and evaluate our functions at the identity element 1 of $G_{\mathbf{a}}$. By Lemma 12.8 (1), $[Z, Z'] = 0$ for $Z \neq Z'$, and hence we obtain

$$(wzf)(1) = \langle (wh^{(e)})(1), z\rangle = (-1)^e e!\sum_Z c_Z\sum_{i,j}[\zeta_j, \zeta_i]\langle\omega_j, w\rangle\langle\omega_i, z\rangle.$$

Pick a $\mathcal{K}$-irreducible subspace V of $\mathfrak{S}_e(\mathfrak{p}_+)$ so that the V-component of z is nonzero. Let Z be the $\mathcal{K}$-irreducible subspace of $S_e(T)$ such that $\langle Z, V\rangle \neq 0$, and let V' be the image of V under $\iota_-\circ(\iota_+)^{-1}$. Take $\{\omega_i\}$ for this particular Z so that $\langle\omega_1, z\rangle = 1$ and $\langle\omega_i, z\rangle = 0$ for $i > 1$. Put $\beta = \sum_j[\zeta_j, \zeta_1]\omega_j$; then $\beta \neq 0$ by Lemma 12.8 (2). Take $w \in V'$ so that $\langle\beta, w\rangle \neq 0$. Then $(wzf)(1) = (-1)^e e! c_Z\langle\beta, w\rangle \neq 0$, which completes the proof of (2).

To prove (3), we first consider $\mathfrak{U}\varphi$ with $\varphi = f_0^\rho$, where $0 \neq f_0 \in \mathcal{S}_\kappa(\Gamma)$ with a congruence subgroup Γ of G. Then every element of $\mathfrak{U}\varphi$ is left Γ-invariant. We then put

$$\text{(A8.10)} \qquad \{g, h\} = \int_{\Gamma\backslash G_{\mathbf{a}}} \overline{g(x)}h(x)dx$$

for $g, h \in \mathfrak{U}\varphi$. This is convergent, since f_0 is a cusp form. Then by Lemma A8.3 we easily see that $\mathfrak{U}\varphi$ is unitarizable. We employ this fact to prove (3). In view of (1), it is sufficient to prove our assertion for a special choice of f. In Case SP we can take $G_{\mathbf{a}} = G_v = Sp(n, \mathbf{R})$. Put $\kappa = q/2$ with $q \geq n$ in Case SP. Take a real quadratic field F and put $G' = Sp(n, F)$ and $G'_{\mathbf{a}} = G_v \times G_{v'}$ so that $\{v, v'\}$ is the set of archimedean primes of F. By Remark A3.11 (III), we can find a nonzero cusp form φ on G' of weight $(q/2)(v + v') + v'$. Since G' is dense in $G'_{\mathbf{a}}$, we can find an element γ of G' such that $\varphi(\gamma(\mathbf{i})) \neq 0$. Changing φ for $\varphi\|\gamma$, we may assume that $\varphi(\mathbf{i}) \neq 0$. Put $f(x) = \varphi^\sigma(x, 1)$ for $x \in G_v$, where $\sigma(a, a') = \det(a)^{q/2}\det(a')^{1+(q/2)}$ for $(a, a') \in GL_n(\mathbf{C})^2$. We easily see that

$0 \neq f \in H(\rho)$. Let $\mathfrak{g}'$ be the Lie algebra of $G'_{\mathbf{a}}$. Then $\mathfrak{U}$ can be embedded into $\mathfrak{U}(\mathfrak{g}'_{\mathbf{C}})$ and $(\alpha f)(x) = (\alpha\varphi^\sigma)(x, 1)$ for $\alpha \in \mathfrak{U}$. Our proof of (1) shows that the map $\alpha f \mapsto \alpha\varphi^\sigma$ is bijective. Since φ is a cusp form, $\mathfrak{U}(\mathfrak{g}'_{\mathbf{C}})\varphi^\sigma$ is unitarizable as we have shown by means of (A8.10). Therefore, $\mathfrak{U}f$, being $\mathfrak{g}$-isomorphic to a subspace of $\mathfrak{U}(\mathfrak{g}'_{\mathbf{C}})\varphi^\sigma$, must be unitarizable. This proves (3) in Case SP. Case UT can be handled in a similar way, but here we first prove a lemma applicable to both Cases UT and UB.

A8.5. Lemma. *Let $\mathcal{G}=SU(m, n)$, $\mathcal{K}=SU(m, n)\cap\big[U(m)\times U(n)\big]$, and $\rho(a, b) = \det(b)$ for $(a, b) \in \mathcal{K}$. Then there exist a discrete subgroup Γ of $\mathcal{G}$ and a nonzero element f of $H(\rho)$ such that $\Gamma\backslash\mathcal{G}$ is compact and $f(\gamma x) = f(x)$ for every $\gamma \in \Gamma$.*

PROOF. Take a real quadratic field F and a CM-field K such that $[K : F] = 2$; put $\mathbf{a} = \{v, v'\}$. We can then find an element $\mathcal{T}$ of $GL_{m+n}(K)$ such that $\mathcal{T}^* = -\mathcal{T}$, $i\mathcal{T}_v$ has signature (m, n), and $i\mathcal{T}_{v'}$ is positive definite. (The easiest way to find such a $\mathcal{T}$ is to take it to be diagonal.) With this $\mathcal{T}$ we consider the injection of $SU(\mathcal{T})$ into $Sp(d, \mathbf{Q})$ and the embedding $\varepsilon : \mathcal{H} \to \mathfrak{H}_d$ of §11.6, where $d = 2(m+n)$. Put $\psi(z) = \det\big(\kappa(z)\big)$ for $z \in \mathcal{H}$ with κ of (11.7). From (11.11) we obtain

$$j\big(\widetilde{\alpha}, \varepsilon(z)\big) = \psi(\alpha z) j_\alpha(z)^2 \psi(z)^{-1} \quad \textit{for every} \quad \alpha \in SU(\mathcal{T}), \tag{A8.11}$$

since $\det\big[M_\alpha(z)\big] = j_\alpha(z)^2$, which can be seen from (3.23), (3.24a), and (5.1). Now take a nonzero element θ of $\mathcal{M}_{\mathbf{a}/2}$ on $\mathfrak{H}_d$ and a congruence subgroup Γ' of $Sp(d, \mathbf{Q})$ so that $\theta^2 \in \mathcal{M}_{\mathbf{a}}(\Gamma')$; put $g(z) = \psi(z)^{-1/2}\theta\big(\varepsilon(z)\big)$ for $z \in \mathcal{H}$ with any fixed square root $\psi(z)^{-1/2}$ of $\psi(z)^{-1}$, which is meaningful as a holomorphic function on $\mathcal{H}$. Changing θ for $\theta\|_{\mathbf{a}/2}\beta$ with a suitable $\beta \in Sp(d, \mathbf{Q})$ if necessary, we may assume that $g \neq 0$. Take a congruence subgroup Γ_1 of $SU(\mathcal{T})$ so that $\big\{\widetilde{\alpha} \,\big|\, \alpha \in \Gamma_1\big\} \subset \Gamma'$. From (A8.11) we see that $j_\gamma(z)^{-2} g(\gamma z)^2 = g(z)^2$ for every $\gamma \in \Gamma_1$. Then $j_\gamma(z)^{-1} g(\gamma z) = \chi(\gamma) g(z)$ for every $\gamma \in \Gamma_1$ with a character $\chi : \Gamma_1 \to \{\pm 1\}$. We obtain the desired Γ and f by putting $\Gamma = \big\{\gamma \in \Gamma_1 \,\big|\, \chi(\gamma) = 1\big\}$ and $f = g^\rho$. Indeed, $\Gamma\backslash\mathcal{G}$ is compact, since $i\mathcal{T}_{v'}$ is definite.

Returning to the proof of Theorem A8.4 (3) in the unitary cases, take f and Γ as in Lemma A8.5 with $\mathcal{G} = G_v$, and for $g, h \in \mathfrak{U}f^\kappa$ define $\{g, h\}$ by (A8.10), which is meaningful, since $\Gamma\backslash G_v$ is compact. By Lemma A8.3, $\mathfrak{U}f^\kappa$ is unitarizable. This completes the proof of Theorem A8.4.

A8.6. Lemma. *Let $f \in H(\rho)$ and $f' \in H(\rho')$ with $\rho(a, b) = \det(b)^\kappa$ and $\rho'(a, b) = \det(b)^{\kappa'}$ with $\kappa, \kappa' \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$. Suppose that both κ and κ' satisfy condition (A8.7). Then the $\mathbf{C}$-linear span of the products $\alpha f \cdot \beta f'$ for all $\alpha, \beta \in \mathfrak{U}$ is unitarizable.*

PROOF. Let $\mathfrak{U}f \cdot \mathfrak{U}f'$ denote the $\mathbf{C}$-linear span of the products in question. We view $\mathfrak{S}(\mathfrak{p}_+)$ and $\mathfrak{S}_e(\mathfrak{p}_+)$ as representation spaces of $\mathcal{K}$ through Ad. In the proof of Theorem A8.4 we have seen that $g \mapsto {}^k g$ for $g \in \mathfrak{U}f$ and $k \in \mathcal{K}$ defines a representaion of $\mathcal{K}$, which is equivalent to $\big\{Ad, \mathfrak{S}(\mathfrak{p}_+)\big\}$. The same is true for $\mathfrak{U}f'$. Naturally we use ρ' instead of ρ for the action of $\mathcal{K}$ on $\mathfrak{U}f'$; similarly we put $({}^k\psi)(x) = \rho(k)\rho'(k)\psi(xk)$ for $k \in K$ and $\psi \in \mathfrak{U}f \cdot \mathfrak{U}f'$. Define a $\mathbf{C}$-linear map $\sigma : \mathfrak{U}f \otimes \mathfrak{U}f' \to \mathfrak{U}f \cdot \mathfrak{U}f'$ so that $\sigma(g \otimes h) = gh$. We let $\mathcal{K}$ act on $\mathfrak{U}f \otimes \mathfrak{U}f'$ by ${}^k(g \otimes h) = {}^k g \otimes {}^k h$. Then clearly ${}^k\sigma(\varphi) = \sigma({}^k\varphi)$ for $\varphi \in \mathfrak{U}f \otimes \mathfrak{U}f'$, and σ is a $\mathfrak{g}$-homomorphism. Let $\mathfrak{T}_p = \sum_{e+e'=p} \mathfrak{S}_e(\mathfrak{p}_+) f \otimes \mathfrak{S}_{e'}(\mathfrak{p}_+) f'$. Then $\mathfrak{U}f \cdot \mathfrak{U}f' = \sum_{p=0}^\infty \sigma(\mathfrak{T}_p)$. By Theorem A8.4 (3), both $\mathfrak{U}f$ and $\mathfrak{U}f'$ are unitarizable, and hence so is $\mathfrak{U}f \otimes \mathfrak{U}f'$; let $\{\ ,\ \}$ be a hermitian inner product with the property of (A8.3) on that space. Put $M_p = \mathrm{Ker}(\sigma) \cap \mathfrak{T}_p$ and $L_p = \{x \in \mathfrak{T}_p \mid \{x, M_p\} = 0\}$. Then $\mathfrak{T}_p = L_p \oplus M_p$, since $\mathfrak{T}_p$ is finite-dimensional. By Theorem A8.4 (4), $\mathfrak{T}_p$ and $\mathfrak{T}_q$ for $p \neq q$ are orthogonal. Observe also that they have no isomorphic $\mathcal{K}$-irreducible subspaces. Now every element of $\mathrm{Ker}(\sigma)$ is contained in $\sum_{p \in P} \mathfrak{T}_p$ with a finite set P, and hence $\mathrm{Ker}(\sigma)$, being $\mathcal{K}$-stable, is the sum of some $\mathcal{K}$-irreducible subspaces, each of which is contained in $\mathfrak{T}_p$ for some p. Thus $\mathrm{Ker}(\sigma) = \bigoplus_p M_p$, and $\mathfrak{U}f \otimes \mathfrak{U}f' = \mathrm{Ker}(\sigma) \oplus L$ with $L = \bigoplus_p L_p = \{\varphi \in \mathfrak{U}f \otimes \mathfrak{U}f' \mid \{\varphi, \mathrm{Ker}(\sigma)\} = 0\}$, which is a $\mathfrak{g}$-submodule. Now $\mathfrak{U}f \cdot \mathfrak{U}f'$ is $\mathfrak{g}$-isomorphic to L, and hence is unitarizable.

A8.7. Lemma. *Let W be a subfield of $\mathbf{C}$ such that $\Phi\mathbf{Q}_{\mathrm{ab}} \subset W$ in Cases SP and UT and $\overline{\mathbf{Q}} \subset W$ in Case UB, where Φ is the Galois closure of K in $\mathbf{C}$ over $\mathbf{Q}$. Let $\rho(a, b) = \det(b)^{\kappa}$ for $(a, b) \in \mathfrak{K}_0$ with $\kappa \in 2^{-1}\mathbf{Z}^{\mathbf{a}}$ in Case SP and $\kappa \in \mathbf{Z}^{\mathbf{a}}$ otherwise. If f is an element of $\mathcal{N}_{\kappa}^{p}(W)$ such that $\mathfrak{U}f^{\rho}$ is unitarizable, then there exists an element q of $\mathcal{M}_{\kappa}(W)$ such that $\langle f, h\rangle = \langle q, h\rangle$ for every $h \in \mathcal{S}_{\kappa}$.*

Proof. Take dual bases $\{a_\nu\}$ and $\{b_\nu\}$ of T_v with respect to $(u, v) \mapsto \mathrm{tr}({}^t uv)$ as in §12.5 for a fixed $v \in \mathbf{a}$; define an element $\mathcal{L}_v$ of $\mathfrak{U}$ by $\mathcal{L}_v = \sum_{\nu \in N} \iota_+(a_\nu)\iota_-(b_\nu)$, which is clearly independent of the bases. We can take these bases so that $b_\nu = a_\nu = \overline{a}_\nu$ for every ν. Take a hermitian form $\{\ ,\ \}$ on $\mathfrak{U}f^{\rho}$ as in (A8.3). Then $\{Ag, h\} = \{g, -A^*h\}$ for $g, h \in \mathfrak{U}f^{\rho}$ and $A \in \mathfrak{g}_{\mathbf{C}}$, so that by (A8.5c),

$$\{\mathcal{L}_v g,\ h\} = \Big\{\sum_{\nu \in N} \iota_+(a_\nu)\iota_-(a_\nu)g,\ h\Big\} = -\sum_{\nu \in N} \{\iota_-(a_\nu)g,\ \iota_-(a_\nu)h\}$$
$$= \Big\{g,\ \sum_{\nu \in N} \iota_+(a_\nu)\iota_-(a_\nu)h\Big\} = \{g,\ \mathcal{L}_v h\}.$$

Taking in particular $g = h$, we see that if $\mathcal{L}_v g = 0$, then $\sum_{\nu \in N} \{\iota_-(a_\nu)g,\ \iota_-(a_\nu)g\} = 0$, and hence $\iota_-(a_\nu)g = 0$ for every ν. Thus we obtain the direct part of

(A8.12) $\qquad \mathcal{L}_v g = 0$ for every $v \in \mathbf{a} \iff Xg = 0$ for every $X \in \mathfrak{p}_-$.

The converse part is obvious. Let L_v denote the operator $L_{\omega,v}$ of (15.3) with $\omega(x, y) = \det(y)^{\kappa}$. Then, for a function φ on $\mathcal{H}$, we have

$$\mathcal{L}_v \varphi^{\rho} = \sum_{\nu \in N} \iota_+(a_\nu)\iota_-(b_\nu)\varphi^{\rho} = \sum_{\nu \in N} \iota_+(a_\nu)(E_v\varphi)^{\rho \otimes \sigma_v}(b_\nu)$$
$$= \sum_{\nu \in N} \big(D_{\omega \otimes \sigma_v, v} E_v \varphi\big)^{\rho \otimes \sigma_v \otimes \tau_v}(b_\nu, a_\nu) = -(L_v\varphi)^{\rho}$$

by (A8.6a, b) and (12.30a, b). Given f as in our lemma, take Γ so that $f \in \mathcal{N}_{\kappa}^{p}(\Gamma)$ and put $\mathcal{X} = \sum_{v \in \mathbf{a}'} \sum_{i=0}^{\infty} \mathbf{C} L_v^i f$, where $\mathbf{a}' = \{v \in \mathbf{a} \mid G_v \text{ is not compact}\}$. Then $\mathcal{X}$ is finite-dimensional, since it is a subspace of $\mathcal{N}_{\kappa}^{p}(\Gamma)$. Now $\mathcal{X}^{\rho} = \sum_{v \in \mathbf{a}'} \sum_{i=0}^{\infty} \mathbf{C}\mathcal{L}_v^i f^{\rho} \subset \mathfrak{U}f^{\rho}$, and hence we can diagonalize the $\mathcal{L}_v$ on $\mathcal{X}^{\rho}$ simultaneously. Consequently we can diagonalize the L_v on $\mathcal{X}$ simultaneously. Let $h \in \mathcal{S}_{\kappa}$. Then $L_v h = 0$, as h is holomorphic. Suppose $L_v g = \lambda_v g$ with $g \in \mathcal{X}$ and $0 \neq \lambda_v \in \mathbf{C}$ for some v. Then $0 = \langle g, L_v h\rangle = \langle L_v g, h\rangle = \overline{\lambda}_v\langle g, h\rangle$ by Theorem 12.15, and hence $\langle g, h\rangle = 0$. Let $\mathcal{Y} = \{g \in \mathcal{X} \mid L_v g = 0 \text{ for every } v \in \mathbf{a}'\}$, $\mathcal{X}(W) = \mathcal{N}_{\kappa}^{p}(\Gamma, W) \cap \mathcal{X}$, and $\mathcal{Y}(W) = \mathcal{Y} \cap \mathcal{X}(W)$. By Theorems 14.9 (2) and 14.12 (3), L_v maps $\mathcal{X}(W)$ into itself, so that we have a Jordan decomposition $\mathcal{X}(W) = \mathcal{Y}(W) \oplus \mathcal{Z}$ with a subspace $\mathcal{Z}$ over W such that $\mathcal{Z} \otimes_W \mathbf{C}$ is spanned by the eigenfunctions of the L_v not belonging to $\mathcal{Y}$. Given W-rational f, let q be the projection of f to $\mathcal{Y}(W)$

with respect to this decomposition. Then $\langle f, h \rangle = \langle q, h \rangle$ for every $h \in \mathcal{S}_\kappa$. Now $\mathcal{L}_v q^\rho = -(L_v q)^\rho = 0$ for every $v \in \mathbf{a}$, so that $q^\rho \in H(\rho)$ by (A8.12), that is, q is holomorphic. Thus q is the desired element of $\mathcal{M}_\kappa(W)$.

A8.8. *Proof of Lemma 15.8.* The notation being as in the lemma, put $f = g^\rho$ with $\rho(a, b) = \det(b)^l$, $f' = h^{\rho'}$, $s = \Delta_{l'}^p h$, and $\varepsilon(a, b) = \det(b)^{l'+2p}$. Take Z and ψ as in our lemma, and take $\zeta_1 = \psi$ in (A8.9); take also $y \in V$ so that $\langle \omega_1, y \rangle = 1$. (Notice that $\dim(Z) = \dim(V) = 1$.) Then (A8.9) shows that $yf' = s^\varepsilon$. Thus $(gs)^{\rho\varepsilon} = fs^\varepsilon \in \mathfrak{U}f \cdot \mathfrak{U}f'$, so that $\mathfrak{U}\big((gs)^{\rho\varepsilon}\big) \subset \mathfrak{U}f \cdot \mathfrak{U}f'$. By Lemma A8.6, $\mathfrak{U}f \cdot \mathfrak{U}f'$ is unitarizable, so that $\mathfrak{U}\big((gs)^{\rho\varepsilon}\big)$ is unitarizable. Therefore we obtain the desired element q by Lemma A8.7.

A8.9. Proposition. *Let $G_\mathbf{a} = Sp(n, \mathbf{R})$ or $SU(m, n)$; let $\rho(a, b) = \det(b)^\kappa$ for $(a, b) \in \mathfrak{K}_0$ with $\kappa \in 2^{-1}\mathbf{Z}$, $0 \le \kappa \le (n-1)/2$ if $G_\mathbf{a} = Sp(n, \mathbf{R})$ and $\kappa \in \mathbf{Z}$, $0 \le \kappa \le \mathrm{Min}\{m, n\} - 1$ if $G_\mathbf{a} = SU(m, n)$. Then there exist two nonzero elements f_1 and f_2 of $H(\rho)$ such that $\mathfrak{U}f_1$ and $\mathfrak{U}f_2$ are not $\mathfrak{U}$-isomorphic.*

PROOF. Put $\nu = 2\kappa+1$ if $G_\mathbf{a} = Sp(n, \mathbf{R})$ and $\nu = \kappa+1$ if $G_\mathbf{a} = SU(m, n)$; put also $\psi(u) = \det_\nu(u)$ for $u \in T$. Let Z be the irreducible subspace of $S_\nu(T)$ with ψ as its highest weight vector in the sense of Theorem 12.7. By [S94b, (4.9c)], $(D_\rho^Z g)(\zeta) = \zeta(\mathcal{D})g$ with $\mathcal{D}$ of (12.25) for every $\zeta \in Z$ and every $g \in C^\infty(\mathcal{H})$. (The symbols λ and h there correspond to κ and ν here.) Take $V \subset \mathfrak{S}_e(\mathfrak{p}_+)$ corresponding to Z as in the proof of Theorem A8.4. Then, by (A8.9),

$$yg^\rho = \sum_i \big\langle \big[\zeta_i(\mathcal{D})g \cdot \omega_i\big]^{\rho\otimes\tau^e}, \ y \big\rangle \qquad \text{for every} \quad y \in V.$$

Let $f_1 = g_1^\rho$ with a nonzero constant g_1 on $\mathcal{H}$. Then $\zeta_i(\mathcal{D})g_1 = 0$, so that $Vf_1 = 0$. Next take $f_2 = g_2^\rho$ with $g_2(z) = \psi(z)$ for $z \in \mathcal{H}$. Then $\zeta_i(\mathcal{D})g_2 = \nu![\zeta_i, \psi]$ by (12.28), and hence, $(yf_2)(1) = \nu!\big\langle \sum_i [\zeta_i, \psi]\omega_i, y \big\rangle = \nu!\langle \psi, y \rangle$, which is not zero for some $y \in V$. Thus $\mathfrak{U}f_2$ is not isomorphic to $\mathfrak{U}f_1$, which proves our proposition.

A8.10. We have been dealing with a set of objects $\{G, \mathcal{K}, \mathcal{H}, \mathfrak{g}, \mathfrak{U}\}$. Suppose we have two more sets $\{G_i, \mathcal{K}_i, \mathcal{H}_i, \mathfrak{g}_i, \mathfrak{U}_i\}$ of the same type for $i = 1, 2$; suppose also that there exist an injective homomorphism I of $(G_1 \times G_2)_\mathbf{a}$ into $G_\mathbf{a}$ and a holomorphic injection J of $\mathcal{H}_1 \times \mathcal{H}_2$ into $\mathcal{H}$ such that $I(\mathcal{K}_1 \times \mathcal{K}_2) \subset \mathcal{K}$ and $J(\beta z, \gamma w) = I(\beta, \gamma)J(z, w)$ for $(\beta, \gamma) \in (G_1 \times G_2)_\mathbf{a}$ and $(z, w) \in \mathcal{H}_1 \times \mathcal{H}_2$. To avoid possible confusions, we assume our setting to be one of the following types:

Cases SP and UT: $G_1 = G^n$, $G_2 = G^r$, $G = G^{n+r}$, $\mathcal{H}_1 = \mathcal{H}^n$, $\mathcal{H}_2 = \mathcal{H}^r$, $\mathcal{H} = \mathcal{H}^{n+r}$, $J(z, w) = \mathrm{diag}[z, w]$, and $I(\beta, \gamma) = \beta \times \gamma$ with the notation of Sections 24 and 25.

Case UB: $G_1 = G^\psi$, $G_2 = G^\varphi$, $G = G^\eta$, $\mathcal{H}_1 = \mathfrak{Z}^\psi$, $\mathcal{H}_2 = \mathfrak{Z}^\varphi$, and $\mathcal{H} = \mathcal{H}_{q+n}^\mathbf{a}$ with the notation of Section 26 and [S97, Section 21]; I and J will be described later.

In these cases the map extended to the complexification of $\mathcal{K}_1 \times \mathcal{K}_2$ is a collection of several maps each of which is equivalent to the diagonal embedding of $GL_r(\mathbf{C}) \times GL_s(\mathbf{C})$ into $GL_{r+s}(\mathbf{C})$ for some (r, s).

We consider (ρ, κ) for G as in Theorem A8.4, and define similarly ρ_i for G_i with the same κ. Now $\mathfrak{g}_1 \times \mathfrak{g}_2$ can be embedded in $\mathfrak{g}$, so that $\mathfrak{U}_1 \otimes \mathfrak{U}_2$ can be viewed as a subalgebra of $\mathfrak{U}$. For a function h on $G_\mathbf{a}$ we define a function h° on $(G_1 \times G_2)_\mathbf{a}$ by $h^\circ(x, y) = h\big(I(x, y)\big)$ for $(x, y) \in (G_1 \times G_2)_\mathbf{a}$. Clearly $(\alpha \otimes \beta)(h^\circ) =$

$\big((\alpha\otimes\beta)h\big)^\circ$ for $(\alpha,\,\beta)\in\mathfrak{U}_1\times\mathfrak{U}_2$. We define ${}^k h$, as before, by $({}^k h)(z)=\rho(k)h(zk)$ for $k\in\mathcal{K}$ and $z\in G_{\mathbf{a}}$, and similarly put $\big({}^{(k,k')}g\big)(x,\,y)=\rho_1(k)\rho_2(k')g(xk,\,yk')$ for $g\in C^\infty((G_1\times G_2)_{\mathbf{a}})$, $(x,\,y)\in(G_1\times G_2)_{\mathbf{a}}$, and $(k,\,k')\in\mathcal{K}_1\times\mathcal{K}_2$. Clearly $\big({}^{I(k,k')}h\big)^\circ={}^{(k,k')}(h^\circ)$.

A8.11. Lemma. *For every $f\in H(\rho)$, $\neq 0$, the $(\mathfrak{g}_1\times\mathfrak{g}_2)$-module $\{\,h^\circ\,|\,h\in\mathfrak{U}f\,\}$ is unitarizable, provided (A8.7) is satisfied for the group G.*

PROOF. Put $N=\{\,g\in\mathfrak{U}f\,|\,g^\circ=0\,\}$, $T_p=\mathfrak{S}_p(\mathfrak{p}_+)f$, $N_p=N\cap T_p$. By Theorem A8.4 (3), $\mathfrak{U}f$ is unitarizable; let $\{\ ,\ \}$ be a hermitian inner product on $\mathfrak{U}f$ with the property of (A8.3). Put $M_p=\{x\in T_p\,|\,\{x,\,N_p\}=0\,\}$. Then $T_p=L_p\oplus M_p$, since $\mathfrak{T}_p$ is finite-dimensional. By Theorem A8.4 (4), T_p and T_q for $p\neq q$ are orthogonal; also they have no isomorphic $(\mathcal{K}_1\times\mathcal{K}_2)$-irreducible subspaces. Now every element of N is contained in $\sum_{p\in P}T_p$ with a finite set P, and hence N, being $(\mathcal{K}_1\times\mathcal{K}_2)$-stable, is the sum of some $(\mathcal{K}_1\times\mathcal{K}_2)$-irreducible subspaces, each of which is contained in T_p for some p. Thus $N=\bigoplus_p N_p$, and $\mathfrak{U}f=M\oplus N$ with $M=\bigoplus_p M_p=\{g\in\mathfrak{U}f\,|\,\{g,\,N\}=0\}$, which is a $(\mathfrak{g}_1\times\mathfrak{g}_2)$-submodule. Now $(\mathfrak{U}f)^\circ$ is $(\mathfrak{g}_1\times\mathfrak{g}_2)$-isomorphic to M, and hence is unitarizable.

A8.12. *Proof of Lemma 28.7.* Let $\rho_0(a,\,b)=\det(b)^{\nu\mathbf{a}}$, $\rho(a,\,b)=\det(b)^{2p+\nu\mathbf{a}}$, and $\rho'(a,\,b)=\det(b)^m$. For the same reason as in §A8.8, we have $\big(\Delta^p_{\nu\mathbf{a}}R\big)^\rho=\alpha R^{\rho_0}$ with an element α of $\mathfrak{U}$. Assuming that $n=r$, recall that $D_{e,e'}=B_eC_{e'}$, $B_eg=(D^Z_\rho g)(\varphi)$, and $C_{e'}g=(E^Wg)(\varphi')$, where φ is as in (25.3) and φ' is defined by the same formula with e' in place of e. Taking $\Delta^p_{\nu\mathbf{a}}R$ and the present D^Z_ρ as g and D^Z_ρ of (A8.9), put $q=\sum_i(D^Z_\rho g)(\zeta_i)\omega_i$ with $\{\zeta_i\}$ and $\{\omega_i\}$ as in that formula. Then $\beta g^\rho=\langle\, q^{\rho\otimes\tau^Z},\ \beta\,\rangle$ for every β in the $\mathcal{K}$-irreducible subspace V of $\mathfrak{S}_{n|e|}(\mathfrak{p}_+)$ such that $\langle Z,\,V\rangle\neq 0$. Now we view Z as a representation space of the complexification of $\mathcal{K}_1\times\mathcal{K}_2$. Then $\mathbf{C}\varphi$ is stable under the group action, and so $Z=\mathbf{C}\varphi+Z'$ with a subspace Z' stable under the group action. Take $\{\zeta_i\}$ so that $\zeta_1=\varphi$ and $Z'=\sum_{i>1}\mathbf{C}\zeta_i$. From (25.3) we obtain $\varphi\big(\mathrm{diag}[a,\,b]u\cdot\mathrm{diag}[a',\,b']\big)=\det(ba')^e\varphi(u)$, and hence from (12.9) we see that ω_1 has the same property, and $\sum_{i>1}\mathbf{C}\omega_i$ is $(\mathcal{K}_1\times\mathcal{K}_2)$-stable. Take an element β of V such that $\langle\,\omega_1,\,\beta\,\rangle=1$ and $\langle\,\omega_i,\,\beta\,\rangle=0$ for $i>1$. Now (A8.9) evaluated at $\mathfrak{z}\in G^{n+r}_{\mathbf{a}}$ involves $\tau^e\big(M(\mathfrak{z},\,\mathbf{o})\big)$. However, if we take $\mathfrak{z}=I(x,\,y)$, from what we said about the ω_i we can easily derive that $\langle q^{\rho\otimes\tau^Z},\,\beta\rangle$ at $\mathfrak{z}=I(x,\,y)$ equals $(D^Z_\rho g)(\varphi)^{\rho_1}\big(I(x,\,y)\big)$, where $\rho_1(a,\,b)=\rho(a,\,b)\det(b)^e$. Thus $(\beta g^\rho)^\circ=\big[(D^Z_\rho g)(\varphi)^{\rho_1}\big]^\circ$. Taking similarly E^W in place of D^Z_ρ, we can find an element γ of $\mathfrak{U}$ such that $(\gamma\beta g^\rho)^\circ=\big[(D_{e,e'}g)^{\rho'}\big]^\circ$. (This can be justified, since D^Z_ρ is considered on $\prod_{e_v>0}G_v$, and E^Z on $\prod_{e'_v>0}G_v$, and $e_ve'_v=0$ for every v.) Thus $(\gamma\beta\alpha R^{\rho_0})^\circ=(S^{\rho'}_0)^\circ$. We consider $D_{e,e'}$ only when $n=r$, and so if $n\neq r$, what we need is merely $(\alpha R^{\rho_0})^\circ=(S^{\rho'}_0)^\circ$. Now $\pi^{-c}S_0\in\mathcal{N}^{p'}(\Xi)$ with some c and p' by Theorem 14.12 (4), and hence $\pi^{-c}S(z,\,w)$ as a function of w (resp. z) belongs to $\mathcal{N}^{p'}_m(\Gamma)$ (resp. $\mathcal{N}^{p'}_m(\Gamma')$) with a congruence subgroup Γ of G^r (resp. Γ' of G^n). All such nearly holomorphic forms with respect to $\Gamma'\times\Gamma$ form a finite-dimensional vector space with a Ξ-rational basis as the proof of Lemma 24.11 shows.

Now suppose that $\nu\geq(n+r)/2$ in Case SP and $\nu\geq n+r$ in Case UT. Then the $(\mathfrak{g}_1\times\mathfrak{g}_2)$-module $\{\,h^\circ\,|\,h\in\mathfrak{U}R^{\rho_0}\,\}$ is unitarizable by Lemma A8.11. Define L_v on $\mathcal{H}^r$ and $\mathcal{L}_v$ on $G^r_{\mathbf{a}}$ as in the proof of Lemma A8.7 with $\kappa=m$. Then we can repeat the proof of Lemma A8.7. To be explicit, put $\mathcal{X}=\sum_{v\in\mathbf{a}}\sum_{i=0}^\infty\mathbf{C}L^i_vS$ and

$\mathcal{X}(\Xi) = \sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty} \Xi L_v^i S'$, where $S' = \pi^{-c}S$. Then $\mathcal{X}(\Xi)$ is a finite-dimensional vector space over Ξ. For a function $h \in C^\infty(\mathcal{H}^n \times \mathcal{H}^r)$ define $h^{\rho'} \in C^\infty(G_{\mathbf{a}}^n \times G_{\mathbf{a}}^r)$ in an obvious way by restricting ρ' to the complexification of $\mathcal{K}_1 \times \mathcal{K}_2$. Then, as in the proof of Lemma A8.7 we have $\mathcal{L}_v h^{\rho'} = -(L_v h)^{\rho'}$, and so $\mathcal{X}^{\rho'} = \sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty} \mathbf{C}\mathcal{L}_v^i(\gamma\beta\alpha R^{\rho_0})^\circ$, since $(S_0^{\rho'})^\circ = S^{\rho'}$. Thus $\mathcal{X}^{\rho'}$ is contained in a unitarizable space. Therefore, by the same procedure as in the proof of Lemma A8.7 and keeping the variables on $\mathcal{H}^n$ and G^n constant, we can find an element T of $\mathcal{X}(\Xi)$ such that $L_v T = 0$ for every $v \in \mathbf{a}$ and $\langle S'(z, w) - T(z, w), f(w)\rangle = 0$ for every $f \in \mathcal{S}_m^r$. The unitarizability implies (A8.12) in the present case, so that $T(z, w)$ is holomorphic in w. Thus $\pi^c T$ gives the desired element T of Lemma 28.7.

A8.13. *Proof of Lemma 29.3.* We assume that $r_v > 0$ for every $v \in \mathbf{a}$, since we do not need Lemma 29.3 otherwise. The idea of the proof is the same as in §A8.12. However, the map $\iota : \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi$ of [S97, (6.10.2)] is antiholomorphic in the variable w on $\mathfrak{Z}^\varphi$, and so we have to change it for a holomorphic one. Thus put $\widetilde{w} = (\widetilde{w}_v)_{v\in\mathbf{a}}$, where

$$\text{(A8.13)} \qquad \widetilde{z} = \begin{bmatrix} -\overline{x} \\ \overline{y} \end{bmatrix} \quad \textit{if} \quad z = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathfrak{Z}_v^\varphi.$$

Notice that $\widetilde{\mathbf{i}}_v = \mathbf{i}_v$. Also, put $P = (P_v)_{v\in\mathbf{a}}$, $Q = (Q_v)_{v\in\mathbf{a}}$, $R = (R_v)_{v\in\mathbf{a}}$ with $P_v = \mathrm{diag}[-1_{r_v}, 1_{t_v+r_v}]$, $Q_v = \mathrm{diag}[R_v, 1_{r_v}]$, $R_v = \mathrm{diag}[1_{r_v}, -1_{t_v}]$. For $\alpha = (\alpha_v) \in \prod_{v\in\mathbf{a}} U(\varphi_v')$ with φ_v' as in (26.2), put $\widetilde{\alpha} = \alpha^\sim = (P_v\overline{\alpha_v}P_v)_{v\in\mathbf{a}}$. Then we can easily verify that $\widetilde{\alpha} \in \prod_{v\in\mathbf{a}} U(\varphi_v')$, $\widetilde{\alpha w} = \widetilde{\alpha}\widetilde{w}$, $\lambda_v(\widetilde{\alpha}, \widetilde{w}) = R\overline{\lambda_v(\alpha, w)}R_v$, and $\mu_v(\widetilde{\alpha}, \widetilde{w}) = \overline{\mu_v(\alpha, w)}$ for every $v \in \mathbf{a}$.

Now we define $J : \mathfrak{Z}^\psi \times \mathfrak{Z}^\varphi \to \mathcal{H}_{q+n}^{\mathbf{a}}$ and $I : G_{\mathbf{a}}^\psi \times G_{\mathbf{a}}^\varphi \to G_{\mathbf{a}}^\eta$ by $J(z, w) = \iota(z, \widetilde{w})$ and $I(\beta, \gamma) = [\tau_v\beta\tau_v^{-1}, (\sigma_v\gamma_v\sigma_v^{-1})^\sim]_R$ with the symbols of (26.2) and [S97, (6.10.2), (6.10.5), (12.1.4), (22.2.1)].

For $f \in C^\infty(\mathcal{H})$ define $\widetilde{f} = f^\sim \in C^\infty(\mathcal{H})$ by $\widetilde{f}(w) = \overline{f(\widetilde{w})}$. Then $(f\|_\kappa\alpha)^\sim = \widetilde{f}\|_\kappa\widetilde{\alpha}$ for every $\alpha \in G_{\mathbf{a}}^\varphi$ and $\kappa \in \mathbf{Z}^{\mathbf{a}}$. Define L_v and $\mathcal{L}_v$ as in the proof of Lemma A8.7 for a fixed κ, and define a differential operator $\widetilde{L}_v$ on $\mathcal{H}$ by $\widetilde{L}_v f = (L_v\widetilde{f})^\sim$. Then we easily see that $\widetilde{L}_v(f\|_\kappa\alpha) = (\widetilde{L}_v f)\|_\kappa\alpha$ for every $\alpha \in G_{\mathbf{a}}^\varphi$. Now all differential operators D on $\mathcal{H}_v$ such that $D(f\|_{\kappa_v}\alpha) = (Df)\|_{\kappa_v}\alpha$ for every $\alpha \in G_v^\varphi$ form a polynomial ring generated by r_v elements D_i, $1 \le i \le r_v$, such that D_i is of degree $2i$ and $D_1 = L_v$. (See [S90, Theorem 3.6 (3), (4)].) Since $\widetilde{L}_v$ is of degree 2, we have $\widetilde{L}_v = aL_v + b$ with constants a and b. Take a nonzero holomorphic function f on $\mathcal{H}_v$. Then $\widetilde{f}$ is holomorphic, and so $L_v f = L_v\widetilde{f} = 0$, so that $\widetilde{L}_v f = 0$. Therefore b must be 0. Thus $\widetilde{L}_v = aL_v$, and clearly $a \ne 0$. This shows that $L_v g = 0$ if and only if $\widetilde{L}_v g = 0$, that is, if and only if $L_v\widetilde{g} = 0$.

Let $R_1 = R\|U^{-1}$, $\rho_0(a, b) = \det(b)^{\nu\mathbf{a}}$, and $\omega(a, b) = \det(b)^m$. By the same argument as in §A8.12, we find an element ε of $\mathfrak{U}$ such that $(\varepsilon R_1^{\rho_0})^\circ = (S_0^\omega)^\circ$. By Theorem 14.12 (4), $\pi^{-c}\Delta_{\nu\mathbf{a}}^p R$ for some $c \in \mathbf{Z}$ is a $\overline{\mathbf{Q}}$-rational nearly holomorphic function. Define L_v on $\mathfrak{Z}^\varphi$ as in the present lemma and $\mathcal{L}_v$ on $G_{\mathbf{a}}^\varphi$ as in the proof of Lemma A8.7. Let $M(z, w) = \pi^{-c}S_0\bigl(\iota(z, w)\bigr)$. Since $S_0 = B\bigl[(\Delta_{\nu\mathbf{a}}^p R)\|U^{-1}\bigr]$, formula (29.6) is valid with M in place of $A_q^k f$. By (12.32) and Theorem 14.9 (2), $\mathcal{N}_h^{e'}(\Gamma, \overline{\mathbf{Q}})$ is stable under the L_v for any congruence subgroup Γ of G^φ. Thus $\overline{M(z, w)}$ belongs to a finite-dimensional vector space over $\overline{\mathbf{Q}}$ that is stable under the L_v. Put $S_1(z, w) = S(z, \widetilde{w})$. Then $S_1(z, w) = S_0\bigl(J(z, w)\bigr)$ and $\overline{M(z, w)} =$

$\pi^{-c}\overline{S_1(z, \widetilde{w})}$. Let $\mathcal{X} = \sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\mathbf{C}L_v^i\overline{M}$, $\mathcal{X}(\overline{\mathbf{Q}}) = \sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\overline{\mathbf{Q}}L_v^i\overline{M}$, and $\mathcal{X}' = \sum_{v\in\mathbf{a}}\sum_{i=0}^{\infty}\mathbf{C}L_v^iS_1$. Then $\mathcal{X}(\overline{\mathbf{Q}})$ is finite-dimensional over $\overline{\mathbf{Q}}$. For $f \in C^{\infty}(\mathfrak{Z}^{\psi}\times\mathfrak{Z}^{\varphi})$ define $f^{\omega} \in C^{\infty}(G_{\mathbf{a}}^{\psi}\times G_{\mathbf{a}}^{\varphi})$ by $f^{\omega}(x, y) = \omega\big(M_{(x,y)}(\mathbf{o})\big)^{-1}f\big(x(\mathbf{o}), y(\mathbf{o})\big)$. Then $S_1^{\omega} = \big(S_0^{\omega}\big)^{\circ} = \big(\varepsilon R_1^{\rho_0}\big)^{\circ}$, which is contained in the unitarizable space by Lemma A8.11. Therefore the $\mathcal{L}_v$ are diagonalizable on $(\mathcal{X}')^{\omega}$, so that the L_v are diagonalizable on $\mathcal{X}'$. Now $f \mapsto \widetilde{f}$ maps $\mathcal{X}'$ (anti-$\mathbf{C}$-linearly) onto $\mathcal{X}$, and hence the L_v and $\widetilde{L}_v$ are diagonalizable on $\mathcal{X}$. Therefore we have a decomposition $\mathcal{X}(\overline{\mathbf{Q}}) = \mathcal{Y}_0 \oplus \mathcal{Y}$ with $\mathcal{Y}_0 = \big\{h \in \mathcal{X}(\overline{\mathbf{Q}}) \,\big|\, L_vh = 0 \textit{ for every } v \in \mathbf{a}\big\}$ and a vector space $\mathcal{Y}$ spanned by the eigenfunctions g of the L_v such that $L_vg \neq 0$ for some v. Then $\langle g(z, w), f(w)\rangle = 0$ for every $f \in \mathcal{S}_h^{\varphi}$ and every $g \in \mathcal{Y}$ for the same reason as in the proof of Lemma A8.7. Let $T_1(z, w)$ be the projection of $\overline{M(z, w)}$ to $\mathcal{Y}_0$. Then $\big\langle\overline{M(z, w)} - T_1(z, w), f(w)\big\rangle = 0$ for every $f \in \mathcal{S}_h^{\varphi}$ and $L_vT_1 = 0$ for every v. Thus $\widetilde{L}_vT_1 = 0$, and so $L_v\widetilde{T}_1 = (\widetilde{L}_vT_1)^{\sim} = 0$. Since $\widetilde{T}_1 \in \mathcal{X}'$, (A8.12) shows that $\widetilde{T}_1$ is holomorphic in w, so that T_1 is holomorphic in w. Thus π^cT_1 gives the desired T of our lemma.

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Index

Written by one of the leading experts, venerable grandmasters, and most active contributors ... in the arithmetic theory of automorphic forms ... the new material included here is mainly the outcome of his extensive work ... over the last eight years ... a very careful, detailed introduction to the subject ... this monograph is an important, comprehensively written and profound treatise on some recent achievements in the theory.

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The main objects of study in this book are Eisenstein series and zeta functions associated with Hecke eigenforms on symplectic and unitary groups. After preliminaries—including a section, "Notation and Terminology"—the first part of the book deals with automorphic forms on such groups. In particular, their rationality over a number field is defined and discussed in connection with the group action; also the reciprocity law for the values of automorphic functions at CM-points is proved. Next, certain differential operators that raise the weight are investigated in higher dimension. The notion of nearly holomorphic functions is introduced, and their arithmeticity is defined. As applications of these, the arithmeticity of the critical values of zeta functions and Eisenstein series is proved.

Though the arithmeticity is given as the ultimate main result, the book discusses many basic problems that arise in number-theoretical investigations of automorphic forms but that cannot be found in expository forms. Examples of this include the space of automorphic forms spanned by cusp forms and certain Eisenstein series, transformation formulas of theta series, estimate of the Fourier coefficients of modular forms, and modular forms of half-integral weight. All these are treated in higher-dimensional cases. The volume concludes with an Appendix and an Index.

The book will be of interest to graduate students and researchers in the field of zeta functions and modular forms.

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