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1 author:

Kazım Ilhan Ikeda
Bogazici University

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Kâzım İlhan İkeda

–Dedicated to Robert Langlands for his 75th birthday –

Abstract. Let K be a global field. The aim of this speculative paper is to discuss the possibility of constructing the non-abelian version of global class field theory of K by "glueing" the non-abelian local class field theories of K_v in the sense of Koch, for each $v \in \mathbb{h}_K$, following Chevalley's philosophy of idèles, and further discuss the relationship of this theory with the global reciprocity principle of Langlands.

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1. Non-abelian local class field theory in the sense of Koch

For details on non-abelian local class field theory (in the sense of Koch), we refer the reader to the papers [21, 23, 24, 50] as well as Laubie's work [37]. For the basic theory of local fields and for the standard notation that we shall follow, we refer the reader to [3, 39].

Let K be a local field. That is, K is a complete discrete valuation field with finite residue class field κ_K of $q = p^f$ elements. We shall furthermore fix an extension φ_K of the Frobenius automorphism Fr_K of K^{nr} to K^{sep}. Namely, we fix a *Lubin-Tate splitting* $\varphi_K = \varphi$ over K.

The non-abelian local class field theory for K establishes an algebraic and topological isomorphism

$$\mathbf{\Phi}_K^{(\varphi)} : G_K \xrightarrow{\sim} \nabla_K^{(\varphi)}$$

between the absolute Galois group G_K of the local field K and a certain topological group $\nabla_K^{(\varphi)}$ which depends on K and on the choice of the Lubin-Tate splitting φ over

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K. In this paper, we shall denote the inverse $\mathbf{\Phi}_K^{(\varphi)^{-1}}$ of the isomorphism $\mathbf{\Phi}_K^{(\varphi)}$ by

$$\{\cdot, K\}_\varphi : \nabla_K^{(\varphi)} \xrightarrow{\sim} G_K.$$

The construction of the topological group $\nabla_K^{(\varphi)}$ involves the theory of *APF*-extensions of K and the fields of norms construction of Fontaine and Wintenberger. Moreover, the isomorphism $\mathbf{\Phi}_K^{(\varphi)}$, which is called the *non-abelian local reciprocity law of* K, is "natural" in the sense that properties such as "existence", "functoriality" and a certain "ramification theoretic" property are all satisfied. The isomorphism $\{\cdot, K\}_\varphi$ is called the *non-abelian local norm-residue symbol of* K.

Remark 1.1. We should point out that the non-abelian local class field theory for K works under a technical assumption on the local field K. That is, the inclusion $\boldsymbol{\mu}_p(K^{sep}) \subset K$ should be satisfied. Under this assumption, the image of the non-abelian local reciprocity map $\mathbf{\Phi}_K^{(\varphi)}$ can be described explicitly. But, this restriction on the local field K can be dropped without any effect on the general theory. Namely, for any local field K, we glue the non-abelian local class field theory for the local field $K(\mu)$ and the abelian local class field theory for the local cyclotomic extension $K(\mu)/K$, where μ is any primitive p^{th} root of unity (for example, look at Section 8 of [23]).

So, "non-abelianization" of local class field theory in the sense proposed first by Koch, and developed further by de Shalit, Fesenko, Gurevich, Laubie and others, is now a complete and solid theory. Thus, it is then a natural attempt to construct the non-abelian version of global class field theory of a global field by "glueing" the non-abelian local class field theories of respective completions of this global field following Chevalley's philosophy of idèles.

2. Non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of a global field K

From now on K denotes a global field; that is, K is a finite extension of $\mathbb{Q}$ or a finite extension of $\mathbb{F}_q(T)$ (that is, the field of rational functions of a curve defined over a finite field $\mathbb{F}_q$). For details about global fields and the abelian global class field theory, we refer the reader to [39, 46]. Let $\mathbb{a}_K$ denote the set of all archimedean primes of K (so in case K is a function field, then $\mathbb{a}_K = \varnothing$). For each $v \in \mathbb{h}_K$, where $\mathbb{h}_K$ denotes the set of all henselian (=non-archimedean) primes of K, let K_v denote the completion of K with respect to the v-adic absolute value. Fixing a Lubin-Tate splitting φ_{K_v} over K_v, the non-abelian local reciprocity law

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : G_{K_v} \xrightarrow{\sim} \nabla_{K_v}^{(\varphi_{K_v})}$$

or equivalently the "Weil form" of the non-abelian local reciprocity law

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$$

of the local field K_v is defined. Here, ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ is a certain dense subgroup of the topological group $\nabla_{K_v}^{(\varphi_{K_v})}$ (for details, see [23]). Moreover, following [50], or Section

8 of [21] together with [24] for detailed account, for each $v \in \mathbb{h}_K$, there exists the subgroup ${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}$ of ${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ satisfying the equality

$$\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}\left(G_{K_v}^{\underline{0}}\right) = {}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}. \tag{2.1}$$

The following two tables summarize the abelian and the non-abelian local class field theories of K_v, where $v \in \mathbb{h}_K$:

Abelian local class field theory	
$G_{K_v}^{ab}$	$\widehat{K_v^\times}$
$W_{K_v}^{ab}$	$K_v^\times$
$W_{K_v}^{ab^0}$	U_{K_v}
$W_{K_v}^{ab^\delta}$, $\delta \in (i-1, i]$	$U_{K_v}^i$

and

Non-abelian local C.F.T. (φ_K fixed)	
G_{K_v}	$\nabla_{K_v}^{(\varphi_{K_v})}$
W_{K_v}	${}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$
$W_{K_v}^0$	${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}$
$W_{K_v}^\delta$, $\delta \in (i-1, i]$	${}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{i}}$

Recall that, the passage from abelian local class field theory to abelian global class field theory follows via the idèle group $\mathbb{J}_K$ of the global field K, where the topological group $\mathbb{J}_K$ is defined by the "restricted direct product"

$$\mathbb{J}_K := \prod_{v \in \mathbb{h}_K \cup \mathbb{a}_K}' \left(K_v^\times : U_{K_v}\right)$$

of the collection $\{K_v^\times\}_{v \in \mathbb{h}_K \cup \mathbb{a}_K}$ with respect to the collection $\{U_{K_v}\}_{v \in \mathbb{h}_K}$, equipped with the restricted direct product topology. Namely, abelian global class field theory for K establishes an algebraic and topological isomorphism

$$(\cdot, K) = \mathsf{Art}_K^{-1} : \widehat{K^\times \backslash \mathbb{J}_K} \xrightarrow{\sim} \mathrm{Gal}(K^{ab}/K)$$

or equivalently the "Weil form" of abelian global class field theory for K establishes an algebraic and topological isomorphism

$$(\cdot, K) = \mathsf{Art}_K^{-1} : K^\times \backslash \mathbb{J}_K \xrightarrow{\sim} W_K^{ab}$$

satisfying certain "naturality" conditions. The noncompact group $K^\times \backslash \mathbb{J}_K$ is called the idèle class group of K and $\widehat{K^\times \backslash \mathbb{J}_K}$ denotes the profinite completion of $K^\times \backslash \mathbb{J}_K$.

The construction of the global norm residue symbol $(\cdot, K)$ or the Artin reciprocity law Art_K of the global field K can roughly be sketched as follows : First, for each finite abelian extension L/K, consider the well-defined homomorphism

$$\mathsf{R}_{L/K} : \mathbb{J}_K \to \mathrm{Gal}(L/K)$$

defined by

$$\mathsf{R}_{L/K} : (x_v)_v \mapsto \prod_v \prod_{v | \mu} (x_v, L_\mu / K_v)$$

for every $(x_v)_v \in \mathbb{J}_K$. Almost all v are unramified in the extension L/K which yields the well-definedness of the map $\mathsf{R}_{L/K}$. Next, pass to the projective limit

$$\varprojlim_{L/K} \mathsf{R}_{L/K} : \mathbb{J}_K \to \varprojlim_{L/K} \mathrm{Gal}(L/K) = \mathrm{Gal}(K^{ab}/K)$$

over all possible such L/K. Finally, check that the morphism $\varprojlim_{L/K} \mathsf{R}_{L/K}$ induces a homomorphism

$$K^{\times}\backslash \mathbb{J}_K \to \; G_K^{ab}$$

which factors as

$$K^{\times}\backslash \mathbb{J}_K \xrightarrow{\sim} \; W_K^{ab} \to G_K^{ab}.$$

Thus, it is natural to push this idea to the extreme, and introduce the non-abelian idèle group $\mathscr{J}_K$ of K, following the analogy between the tables above and taking into account the philosophy of Miyake [40, 41] (also look at Iwasawa [26]), as follows.

Restricted free products. Let $\{G_i\}_{i\in I}$ be a collection of locally compact topological groups and for all but finitely many $i \in I$ let O_i be a compact open subgroup of G_i. Denote the finite subset of I consisting of all $i \in I$ for which O_i is not defined by I_∞. For every finite subset S of I satisfying $I_\infty \subseteq S$, define the topological group

$$G_S := \mathop{*}_{i\notin S} O_i * \left(\mathop{*}_{i\in S} G_i \right)$$

as the free product of the topological groups O_i, for $i \in I - S$, and G_i, for $i \in S$, which exists in the category of topological groups (cf. Morris [43]). Then, the *restricted free product* of the collection $\{G_i\}_{i\in I}$ with respect to the collection $\{O_i\}_{i\in I-I_\infty}$, which is denoted by $*'_{i\in I}(G_i : O_i)$, is defined by the injective limit

$$\mathop{*}_{i\in I}{}'(G_i : O_i) := \varinjlim_{S} G_S$$

defined over all possible such S, where the connecting morphism

$$\tau_S^T : G_S \to G_T$$

for $S \subseteq T$ is defined naturally by the *"universal mapping property of free products"* [1] (cf. Hilton-Wu [18] and Morris [43]). The topology on $*'_{i\in I}(G_i : O_i)$ is defined by declaring $X \subseteq *'_{i\in I}(G_i : O_i)$ to be open if $X \cap G_S$ is open in G_S for every S. So, endowed with this topology, $*'_{i\in I}(G_i : O_i)$ is a topological group.

Proposition 2.1. *Let $\{G_i\}_{i\in I}$ be a collection of locally compact topological groups and for all but finitely many $i \in I$ let O_i be a compact open subgroup of G_i. Denote the finite subset of I consisting of all $i \in I$ for which O_i is not defined by I_∞. Assume that, for each $i \in I$, a continuous homomorphism*

$$\phi_i : G_i \to H$$

is given. Then, there exists a unique continuous homomorphism

$$\phi_S : G_S \to H$$

[1] If $\{G_i\}_{i\in I}$ is a collection of topological groups and $*_{i\in I} G_i$ is the free product of this collection together with the canonical embeddings $\iota_{i_o} : G_{i_o} \hookrightarrow *_{i\in I} G_i$, for each $i_o \in I$, then the universal mapping property of free products states that, if for each $i_o \in I$, $\phi_{i_o} : G_{i_o} \to H$ is a continuous homomorphism, then there exists a unique continuous homomorphism $\phi : *_{i\in I} G_i \to H$, such that $\phi \circ \iota_{i_o} = \phi_{i_o}$, for every $i_o \in I$.

defined for each finite subset S of I satisfying $I_\infty \subseteq S$, and a unique continuous homomorphism

$$\phi = \varinjlim_{S} \phi_S : \ast'_{i\in I}(G_i : O_i) \to H$$

satisfying

$$\phi_S = \phi \circ c_S : G_S \xrightarrow{c_S} \ast'_{i\in I}(G_i : O_i) \xrightarrow{\phi} H,$$

where $c_S : G_S \to \ast'_{i\in I}(G_i : O_i)$ is the canonical homomorphism, for every S.

Proof. The collection of continuous homomorphisms $\{\phi_i : G_i \to H\}_{i\in I}$ determines a unique continuous homomorphism $\phi_S : G_S \to H$, for each S, by the universal mapping property of free products. Moreover, for $S \subseteq T$, the diagram

$$\begin{tikzcd}
G_S \arrow[dd, "\tau_S^T"'] \arrow[dr, "c_S"] \arrow[drr, bend left, "\phi_S"] & & \\
 & G \arrow[r, dotted, "\exists!\,\phi"] & H \\
G_T \arrow[ur, "c_T"'] \arrow[urr, bend right, "\phi_T"'] & &
\end{tikzcd}$$

is commutative, where $G = \ast'_{i\in I}(G_i : O_i)$, by the universality of direct limits. □

Notation 2.2. As a notation, for a topological group G, the n-fold free product $\overbrace{G \ast \cdots \ast G}^{n\text{-copies}}$ of G is denoted by $G^{\ast n}$.

Non-abelian idèle group $\mathscr{J}_K^{\underline{\varphi}}$ of the global field K. Now, the following definition introduces the major object that we intend to study in this work.

Definition 2.3. For each $v \in \mathbb{h}_K$ fix a Lubin-Tate splitting φ_{K_v} and let $\underline{\varphi} = \{\varphi_{K_v}\}_{v\in\mathbb{h}_K}$. The topological group $\mathscr{J}_K^{\underline{\varphi}}$ defined by the "restricted free product"

$$\mathscr{J}_K^{\underline{\varphi}} := \mathop{\ast'}_{v\in\mathbb{h}_K\cup\mathbb{a}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_{1}\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \right)$$

is called the *non-abelian idèle group of the global field K*. In case K is a number field,

$$\mathscr{J}_K^{\underline{\varphi}} = \mathop{\ast'}_{v\in\mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_{1}\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2},$$

where the finite (=henselian) part $\mathscr{J}_{K,\mathbb{h}}^{\underline{\varphi}}$ of $\mathscr{J}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{J}_{K,\mathbb{h}}^{\underline{\varphi}} := \mathop{\ast'}_{v\in\mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} : {}_{1}\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \right),$$

and the infinite (=archimedean) part $\mathscr{J}_{K,\mathbb{a}}^{\underline{\varphi}}$ of $\mathscr{J}_K^{\underline{\varphi}}$ by

$$\mathscr{J}_{K,\mathbb{a}}^{\underline{\varphi}} := W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}.$$

Here, as usual r_1 and r_2 denote the number of real and the number of pairs of complex-complex conjugate embeddings of the global field K in $\mathbb{C}$.

The non-abelian idèle group $\mathscr{I}_K^{\varphi}$ of K is an "extremely large" topological group, whose definition depends *only on the global field K*.

Remark 2.4. The non-abelian local class field theory in the sense of Koch has evolved in two directions (look at [23] and [37]). The non-abelian idèle groups introduced in Definition 2.3 are defined *à la Fesenko*; namely, the construction is based on the work [23]. However, it is also possible to start with *Laubie class field theory*, and define the non-abelian idèle groups accordingly. Taking into account the work [21], the latter construction produces nothing new. More precisely, if $\mathscr{I}_K^{\varphi \text{Laubie}}$ denotes the non-abelian idèle group of K constructed in terms of Laubie class field theory, then there exists a topological isomorphism $\mathscr{I}_K^{\varphi \text{Laubie}} \xrightarrow{\sim} \mathscr{I}_K^{\varphi}$ by [21].

The following theorem describes the abelianization $\mathscr{I}_K^{\varphi ab}$ of the topological group $\mathscr{I}_K^{\varphi}$.

Theorem 2.5. *The abelianization $\mathscr{I}_K^{\varphi ab}$ of the topological group $\mathscr{I}_K^{\varphi}$ is indeed $\mathbb{J}_K$.*

Proof. The proof follows by first noting that the direct limit functor is exact and then by abelianizing free products of groups. □

3. Non-abelian global reciprocity law : A proposal

For $v \in \mathbb{h}_K \cup \mathbb{a}_K$, choose an embedding

$$e_v : K^{sep} \hookrightarrow K_v^{sep}.$$

This embedding determines a continuous homomorphism [2] (look at [51] for details)

$$e_v^{\text{Weil}} : W_{K_v} \longrightarrow W_K,$$

and therefore, for each $v \in \mathbb{h}_K$, a continuous homomorphism

$$\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\text{Weil}}} : {}_{\mathbb{Z}}\nabla_{K_v}^{\varphi_{K_v}} \xrightarrow[\sim]{\{\cdot, K_v\}_{\varphi_{K_v}}} W_{K_v} \xrightarrow{e_v^{\text{Weil}}} W_K.$$

Theorem 3.1 (Global non-abelian norm-residue symbol "Weak form"). *There exists a well-defined continuous homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Weil}}} : \mathscr{I}_K^{\varphi} \to W_K, \tag{3.1}$$

which satisfies

$$(\mathsf{NR}_K^{\varphi^{\text{Weil}}})_S = \mathsf{NR}_K^{\varphi^{\text{Weil}}} \circ c_S : (\mathscr{I}_K^{\varphi})_S \xrightarrow{c_S} \mathscr{I}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi^{\text{Weil}}}} W_K,$$

where, following Proposition 2.1, $c_S : (\mathscr{I}_K^{\varphi})_S \to \mathscr{I}_K^{\varphi}$ is the canonical homomorphism defined for every finite subset S of $\mathbb{h}_K \cup \mathbb{a}_K$ containing $\mathbb{a}_K$.

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[2] which is unique if K is a function field and unique up to composition with an inner automorphism of W_K defined by an element of the connected component W_K^o of W_K if K is a number field.

Proof. Proposition 2.1 applied to the collection of continuous homomorphisms

$$\left\{\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\mathrm{Weil}}} : {}_{\mathbb{Z}}\nabla_{K_v}^{\varphi_{K_v}} \to W_K\right\}_{v\in\mathbb{h}_K} \bigcup \left\{e_v^{\mathrm{Weil}} : W_{K_v} \to W_K\right\}_{v\in\mathbb{a}_K}$$

completes the proof. $\square$

We conjecture that the continuous homomorphism $\mathsf{NR}_K^{\underline{\varphi}^{\mathrm{Weil}}} : \mathscr{J}_K^{\underline{\varphi}} \to W_K$ should be considered as the *global non-abelian norm-residue symbol of* K. More precisely,

Conjecture 3.2 (Global non-abelian norm-residue symbol "Strong form"). *The homomorphism*

$$\mathsf{NR}_K^{\underline{\varphi}^{\mathrm{Weil}}} : \mathscr{J}_K^{\underline{\varphi}} \to W_K$$

is open, continuous and surjective.

Regarding this conjecture, the following remark is in order.

Remark 3.3. The conjecture 3.2 seems to be related with :

(1) the well-known fact that the absolute Galois group $G_{\mathbb{Q}}$ of the global field $\mathbb{Q}$ is topologically generated by the inertia subgroups;
(2) the "Riemann's existence theorem" for global fields (for details, see [44]).

4. Local-global compatibility of the non-abelian norm residue symbols

Clearly, for each $v \in \mathbb{h}_K \cup \mathbb{a}_K$, there exists a natural homomorphism

$$q_v : (\mathscr{J}_K^{\underline{\varphi}})_v := \left\{\begin{matrix} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}, & v \in \mathbb{h}_K \\ W_{\mathbb{R}}, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}, & v \in \mathbb{a}_{K,\mathbb{C}} \end{matrix}\right\} \to \mathscr{J}_K^{\underline{\varphi}},$$

which is defined explicitly via the commutative triangle

$$\begin{tikzcd} & (\mathscr{J}_K^{\underline{\varphi}})_S \arrow[dd, "c_S"] \\ (\mathscr{J}_K^{\underline{\varphi}})_v \arrow[ur, "\iota_v^{(S)}"] \arrow[dr, "q_v"'] & \\ & \mathscr{J}_K^{\underline{\varphi}} \end{tikzcd}$$

where S is a finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subseteq S$ and $v \in S$. Note that, the definition of the continuous homomorphism $q_v : (\mathscr{J}_K^{\underline{\varphi}})_v \to \mathscr{J}_K^{\underline{\varphi}}$ does not depend on the choice of S. In fact, if T is another finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subseteq T$

and $v \in T$, then by the universal mapping property of free products and by the definition of the connecting morphism $\tau^{S}_{S\cap T} : (\mathscr{J}^{\varphi}_{\underline{K}})_{S\cap T} \to (\mathscr{J}^{\varphi}_{\underline{K}})_{S}$, the diagram

$$\begin{tikzcd}
 & (\mathscr{J}^{\varphi}_{\underline{K}})_{S\cap T} \arrow[dd, "\tau^{S}_{S\cap T}"] \\
(\mathscr{J}^{\varphi}_{\underline{K}})_{v} \arrow[ur, "\iota_v^{(S\cap T)}"] \arrow[dr, "\iota_v^{(S)}"'] & \\
 & (\mathscr{J}^{\varphi}_{\underline{K}})_{S}
\end{tikzcd}$$

commutes. Thus,

$$c_{S\cap T} \circ \iota_v^{(S\cap T)} = c_S \circ \tau^{S}_{S\cap T} \circ \iota_v^{(S\cap T)} = c_S \circ \iota_v^{(S)} = q_v,$$

which also proves that

$$c_T \circ \iota_v^{(T)} = c_S \circ \iota_v^{(S)}.$$

The next Theorem is the *"local-global compatibility"* of $\{.,K_v\}_{\varphi_{K_v}}$ and $\mathsf{NR}^{\varphi}_{\underline{K}}$ for $v \in \mathbb{h}_K$.

Theorem 4.1. *For each $v \in \mathbb{h}_K$, the following square*

$$\begin{tikzcd}
{}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \arrow[r, "q_v"] \arrow[d, "\{.,K_v\}_{\varphi_{K_v}}"'] & \mathscr{J}^{\varphi}_{\underline{K}} \arrow[d, "\mathsf{NR}^{\varphi\ \mathrm{Weil}}_{\underline{K}}"] \\
W_{K_v} \arrow[r, "e_v^{\mathrm{Weil}}"'] & W_K
\end{tikzcd}$$

is commutative.

Proof. Note that,

$$e_v^{\mathrm{Weil}} \circ \{.,K_v\}_{\varphi_{K_v}} = \mathsf{NR}_{K_v}^{(\varphi_{K_v})\ \mathrm{Weil}}.$$

Now, to prove that

$$\mathsf{NR}^{\varphi\ \mathrm{Weil}}_{\underline{K}} \circ q_v = \mathsf{NR}_{K_v}^{(\varphi_{K_v})\ \mathrm{Weil}},$$

first recall that $q_v = c_S \circ \iota_v^{(S)}$, where $v \in S$ a finite subset of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $\mathbb{a}_K \subset S$. Then,

$$\begin{aligned}
\mathsf{NR}^{\varphi\ \mathrm{Weil}}_{\underline{K}} \circ q_v &= \mathsf{NR}^{\varphi\ \mathrm{Weil}}_{\underline{K}} \circ (c_S \circ \iota_v^{(S)}) \\
&= (\mathsf{NR}^{\varphi\ \mathrm{Weil}}_{\underline{K}})_S \circ \iota_v^{(S)} \\
&= \mathsf{NR}_{K_v}^{(\varphi_{K_v})\ \mathrm{Weil}},
\end{aligned}$$

which completes the commutativity of the square. $\square$

5. ℓ-adic representations of the non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of K

By Theorem 3.1, and also by Conjecture 3.2 as well, it is evident that any representation $\sigma : W_K \to \mathrm{GL}(V)$ of the absolute Weil group of the global field K on an F- vector space V defines naturally a representation $\sigma_*^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$ of the non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of K on the F-linear space V via the composition $\sigma_*^{\varphi} : \mathscr{J}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}} W_K \xrightarrow{\sigma} \mathrm{GL}(V)$. Here, F denotes any field. Moreover, if F is assumed to be a topological field and the F-representation V of W_K is a continuous representation, then the continuity of the non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} : \mathscr{J}_K^{\varphi} \to W_K$ of K yields the continuity of the corresponding F-representation V of $\mathscr{J}_K^{\varphi}$. Thus, the (continuous) F-representation theory of the topological group $\mathscr{J}_K^{\varphi}$[3] is closely related to the (continuous) F-representation theory of the absolute Weil group W_K of the global field K.

In this work, we shall focus on the *continuous ℓ-adic representations of the non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of K* and their basic *"analytic invariants"*, namely the *L-functions* and the *ε-factors* attached to these representations, whose definitions will be made precise below. By the preceding paragraph, such representations are deeply connected with the ℓ-adic representations of the absolute Weil group W_K of the global field K, or equivalently[4] to the ℓ-adic representations of the absolute Galois group G_K of K.

Remark 5.1. In fact, instead of considering the ℓ-adic representations of the non-abelian idèle group $\mathscr{J}_K^{\varphi}$ of the global field K, we can, and we should, more generally consider the *"ℓ-adic Hecke-Langlands parameters" $\mathscr{J}_K^{\varphi} \to {}^L H(\overline{\mathbb{Q}}_\ell)$ of H*, where H is any connected reductive group over K with the dual group $\widehat{H}$, which is a connected

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[3]In fact, we have not seen any work on the representation theory of the restricted free product of topological groups. The closest work in the mathematical literature in this direction are the 1995 paper of Młotkowski [42] and the 2010 paper of Hebisch and Młotkowski [16].

[4]We reproduce the following observation of Brian Conrad : Let K be a global field and W_K be the absolute Weil group of K equipped with the natural continuous homomorphism $\phi : W_K \to G_K$ with dense image (look at [51]). Let H be a finite-type $\overline{\mathbb{Q}}_\ell$-group. Then any continuous homomorphism $\xi : W_K \to H(\overline{\mathbb{Q}}_\ell)$ factors continuously through G_K as

$$\begin{tikzcd} & G_K \arrow[dd, "\xi_o"] \\ W_K \arrow[ur, "\phi"] \arrow[dr, "\xi"'] & \\ & H(\overline{\mathbb{Q}}_\ell). \end{tikzcd}$$

In fact, the target $H(\overline{\mathbb{Q}}_\ell)$ is totally-disconnected. Therefore, such an arrow $\xi : W_K \to H(\overline{\mathbb{Q}}_\ell)$ should kill the identity component W_K^o of W_K. In the number field case, $\phi : W_K \to G_K$ is a surjective topological quotient map with $\ker(\phi) = W_K^o$. In the function field case, $\phi : W_K \to G_K$ is the inclusion and W_K is a dense subgroup of G_K. Thus, in both cases, $\xi : W_K \to H(\overline{\mathbb{Q}}_\ell)$ uniquely defines $\xi_o : G_K \to H(\overline{\mathbb{Q}}_\ell)$.

reductive group over $\mathbb{C}$, and ${}^{L}H(\overline{\mathbb{Q}}_\ell)$ is the $\overline{\mathbb{Q}}_\ell$-rational points[5] of the L-group ${}^{L}H := \widehat{H} \rtimes W_K$ of H. These continuous homomorphisms are closely related with the *"ℓ-adic Langlands parameters" $W_K \to {}^{L}H(\overline{\mathbb{Q}}_\ell)$ of H*, which are the *"ℓ-adic avatars"* of the *Langlands parameters $L_K \to {}^{L}H$ of H*, where L_K denotes the global Langlands group of K, which is a conjectural group in case K is a number field. For details, look at [1, 5, 14, 25, 49]. We shall return to this discussion in Section 8.

Continuous ℓ-adic representations of $\mathscr{J}_K^{\varphi}$. Let

$$\rho : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$$

be a continuous representation of the topological group $\mathscr{J}_K^{\varphi}$ on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. Such a representation (ρ, V) of $\mathscr{J}_K^{\varphi}$ is called a *continuous ℓ-adic representation* of $\mathscr{J}_K^{\varphi}$. Then, the natural homomorphism $q_v : (\mathscr{J}_K^{\varphi})_v \to \mathscr{J}_K^{\varphi}$ defines a local continuous representation

$$\rho_v = \rho \circ q_v : (\mathscr{J}_K^{\varphi})_v := \left\{ \begin{matrix} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}, & v \in \mathbb{h}_K \\ W_{\mathbb{R}}, & v \in \mathbb{a}_{K,\mathbb{R}} \\ W_{\mathbb{C}}, & v \in \mathbb{a}_{K,\mathbb{C}} \end{matrix} \right\} \xrightarrow{q_v} \mathscr{J}_K^{\varphi} \xrightarrow{\rho} \mathrm{GL}(V)$$

of the local group $(\mathscr{J}_K^{\varphi})_v$ on the vector space V over $\overline{\mathbb{Q}}_\ell$, for each $v \in \mathbb{a}_K \cup \mathbb{h}_K$. Moreover, we have the following important theorem, which is essentially the "representation theoretic" incarnation of Proposition 2.1.

Theorem 5.2. *For each $v \in \mathbb{a}_K \cup \mathbb{h}_K$, let*

$$\rho_v : (\mathscr{J}_K^{\varphi})_v \to \mathrm{GL}(V)$$

be a continuous representation of the local group $(\mathscr{J}_K^{\varphi})_v$ on the vector space V over $\overline{\mathbb{Q}}_\ell$. Then the collection $\{\rho_v\}_{v \in \mathbb{a}_K \cup \mathbb{h}_K}$ defines a unique continuous representation

$$\rho : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$$

on the vector space V over $\overline{\mathbb{Q}}_\ell$, such that

$$\rho_v : (\mathscr{J}_K^{\varphi})_v \xrightarrow{q_v} \mathscr{J}_K^{\varphi} \xrightarrow{\rho} \mathrm{GL}(V),$$

for every $v \in \mathbb{a}_K \cup \mathbb{h}_K$.

Proof. By Proposition 2.1, for the collection $\{\rho_v\}_{v \in \mathbb{a}_K \cup \mathbb{h}_K}$, there exists a unique continuous representation ρ_S of $(\mathscr{J}_K^{\varphi})_S$ on the linear space V over $\overline{\mathbb{Q}}_\ell$ defined for each finite subset S of $\mathbb{a}_K \cup \mathbb{h}_K$ satisfying $\mathbb{a}_K \subseteq S$, and a unique continuous representation ρ of $\mathscr{J}_K^{\varphi}$ on the linear space V over $\overline{\mathbb{Q}}_\ell$ satisfying

$$\rho_S = \rho \circ c_S : (\mathscr{J}_K^{\varphi})_S \xrightarrow{c_S} \mathscr{J}_K^{\varphi} \xrightarrow{\rho} \mathrm{GL}(V)$$

for every such set S, which further satisfies

$$\rho_v = \rho_S \circ \iota_v^{(S)} = \rho \circ c_S \circ \iota_v^{(S)} = \rho \circ q_v,$$

for every $v \in \mathbb{a}_K \cup \mathbb{h}_K$. $\square$

[5] The complex reductive group $\widehat{H}$ is defined over $\mathbb{Z}$. Thus, we can consider the the group of $\overline{\mathbb{Q}}_\ell$-rational points of the L-group ${}^{L}H$.

This theorem immediately yields the following corollary, whose proof is straightforward.

Corollary 5.3. *There exists a bijective correspondence*

$$\left\{\{\rho_v\}_{v\in \Bbbk_K\cup \Bbbk_K} : (\mathscr{I}_K^{\varphi})_v \xrightarrow[\text{cont.}]{\rho_v} \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell),\ {}^\forall v\right\} \rightleftarrows \mathsf{Rep}^{\text{cont}-n}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}_K^{\varphi})$$

between the set of all collections $\{\rho_v\}_{v\in \Bbbk_K\cup \Bbbk_K}$ *consisting of n-dimensional continuous* ℓ*-adic representations* $\rho_v : (\mathscr{I}_K^{\varphi})_v \to \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell)$ *for each* $v\in \Bbbk_K\cup \Bbbk_K$ *and the set* $\mathsf{Rep}^{\text{cont}-n}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}_K^{\varphi})$ *of all n-dimensional continuous* ℓ*-adic representations of* $\mathscr{I}_K^{\varphi}$ *defined by Theorem 5.2.*

Hecke-Weil L**-functions.** To simplify the discussion, from now on, fix an isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ using the axiom of choice.

Our aim now is to define the "Hecke-Weil" L-function $L^{Hecke-Weil}(s,\rho)$ at $s\in\mathbb{C}$ attached the continuous ℓ-adic representation ρ of $\mathscr{I}_K^{\varphi}$ on the n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$.

Definition 5.4. Let

$$\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$$

be a continuous representation of $\mathscr{I}_K^{\varphi}$ on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. The *"Hecke-Weil" L-function* $L^{Hecke-Weil}(s,\rho)$ at $s\in\mathbb{C}$ attached to the representation $\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$ is defined (formally) by the Euler product

$$L^{Hecke-Weil}(s,\rho) := \prod_{\substack{v\in \Bbbk_K\\ \ell\nmid q_v}} L^{Hecke-Weil}(s,\rho_v) \prod_{\substack{v\in \Bbbk_K\\ \ell\mid q_v}} L^{Hecke-Weil}(s,\rho_v) \times \prod_{v\in \Bbbk_K} L^{Hecke-Weil}(s,\rho_v), \qquad s\in\mathbb{C},$$

where for each $v\in \Bbbk_K$, the cardinality of the residue class field κ_{K_v} of K_v is denoted by q_v. The *local "Hecke-Weil" L-factor* $L^{Hecke-Weil}(s,\rho_v)$ at $s\in\mathbb{C}$ is defined

- for $v\in \Bbbk_K$ with $\ell\nmid q_v$, by

$$L^{Hecke-Weil}(s,\rho_v) := L^{Artin-Weil}(s,\rho_v\circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}), \qquad s\in\mathbb{C},$$

  where the right-hand side is defined as the usual local Artin-Weil L-factor of the local representation $\rho_v\circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \to \mathrm{GL}(V)$ of the local absolute Weil group W_{K_v} of K_v on the vector space V over $\overline{\mathbb{Q}}_\ell$ at $s\in\mathbb{C}$;
- for $v\in \Bbbk_K$ with $\ell\mid q_v$, again by

$$L^{Hecke-Weil}(s,\rho_v) := L^{Artin-Weil}(s,\rho_v\circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}), \qquad s\in\mathbb{C},$$

  where this time, the right-hand side is defined in terms of *"Fontaine's* $\mathbb{D}_{\text{pst}}$*-functor"* via the *"p-adic Hodge theory"*, which we shall not discuss in the text, and refer the reader to [10, 11, 52];
- for $v\in \Bbbk_K$, to be the *"gamma-factor"* $\Gamma(s,\rho_v)$ defined explicitly for example in [30], which we shall not discuss in the text, and refer the reader to [30].

Remark 5.5. Deninger introduced a unified construction of local L-factors of motives (pure as well as mixed motives) as a certain power series on infinite-dimensional cohomologies (for details, look at [8]). It would be very interesting to study Artin-Weil and Hecke-Weil L-functions in the framework of Deninger's theory.

Hecke-Weil ε-factors. Next, we shall define the *ε-factor* $\varepsilon^{Hecke-Weil}(s,\rho)$ at $s\in\mathbb{C}$ attached the continuous ℓ-adic representation ρ of $\mathscr{J}_K^{\varphi}$ on the n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. In order to do so, let us briefly review the construction of the global ε-factors of continuous ℓ-adic representations of the absolute Weil group W_K of the global field K introduced by Deligne, Dwork, and Langlands (look at [51] for details). Let

$$\sigma : W_K \to \mathrm{GL}(V)$$

be a continuous representation of W_K on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. Let $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ be a non-trivial additive unitary character of $\mathbb{A}_K$ which is trivial on K, namely, ψ is a non-trivial *global additive character of* K. For each $v \in \mathbb{a}_K \cup \mathbb{h}_k$, the local component $\psi_v : (K_v)_+ \to \mathbb{C}^\times$ is a non-trivial additive character of the local field K_v. Let $d^+\mu$ be the *Haar measure* on $\mathbb{A}_K$ normalized by $\int_{K\backslash\mathbb{A}_K} d^+\mu(x) = 1$. Fix a decomposition $d^+\mu = \prod_v d^+\mu_v$ of $d^+\mu$, where for almost all v the local measure $d^+\mu_v$ is a "normalized" additive Haar measure on K_v, normalized in the sense that $\mathrm{vol}(O_{K_v}) = 1$. Now, the *global ε-factor* $\varepsilon^{Artin-Weil}(s,\sigma)$ of the ℓ-adic representation σ of W_K on V at $s\in\mathbb{C}$ is defined by the product

$$\varepsilon^{Artin-Weil}(s,\sigma) = \prod_{v\in\mathbb{a}_K\cup\mathbb{h}_K} \varepsilon^{Artin-Weil}(s,\sigma_v,\psi_v,d^+\mu_v)$$

of local factors $\varepsilon^{Artin-Weil}(s,\sigma_v,\psi_v,d^+\mu_v)$.

Recall that, the local factor $\varepsilon^{Artin-Weil}(\sigma_v,\psi_v,d^+\mu_v)$, for $v\in\mathbb{h}_K$, is defined[6] explicitly for the case $\dim_{\overline{\mathbb{Q}}_\ell}(V) = n = 1$ by

$$\varepsilon^{Artin-Weil}(\sigma_v,\psi_v,d^+\mu_v) = \sigma_v(\mathsf{Art}_{K_v}(c)) \frac{\int\limits_{U_K} \sigma_v(\mathsf{Art}_{K_v}(x))^{-1}\psi_v(x/c)d^+\mu_v(x)}{\left| \int\limits_{U_K} \sigma_v(\mathsf{Art}_{K_v}(x))^{-1}\psi_v(x/c)d^+\mu_v(x) \right|},$$

where the arrow $\mathsf{Art}_{K_v} : W_{K_v}^{ab} \xrightarrow{\sim} K_v^\times$ is the Artin reciprocity law of K_v, and $c\in K_v^\times$ satisfies $v(c) = a(\sigma_v) + n(\psi_v)$. Recall that, the number $n(\psi_v)$ is the conductor of the additive character $\psi_v : (K_v)_+ \to \mathbb{C}^\times$ and $a(\sigma_v)$ is the Artin conductor of the quasi-character $\sigma_v : W_{K_v} \to \mathbb{C}^\times$. On the other hand, there is *no* explicit formula for the local ε-factor $\varepsilon^{Artin-Weil}(\sigma_v,\psi_v,d^+\mu_v)$ in case $\dim_{\overline{\mathbb{Q}}_\ell}(V) = n > 1$. The best we have is the *"existence and uniqueness theorem of Deligne, Dwork and Langlands for local ε-factors"*.

Remark 5.6. It seems possible to give an explicit formula for the local ε-factor $\varepsilon^{Artin-Weil}(\sigma_v,\psi_v,d^+\mu_v)$ for the general case $\dim_{\overline{\mathbb{Q}}_\ell}(V) = n \geq 1$ in terms of the

[6] Therefore, if $\sigma_v : W_{K_v} \to \mathrm{GL}(V)$ is a 1-dimensional representations of W_{K_v} on the $\mathbb{Q}_\ell$-linear space V, then there exists an explicit expression of the local ε-factor $\varepsilon^{Artin-Weil}(\sigma_v,\psi_v,d^+\mu_v)$ in terms of the local Artin reciprocity law of K_v.

non-abelian local reciprocity law $\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \xrightarrow{\sim} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})}$ of the local field K_v (look at [22])[7]. Also, for a naive attempt in this direction, look at [19].

Definition 5.7. Let

$$\rho : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$$

be a continuous representation of $\mathscr{J}_K^{\varphi}$ on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. Let $\psi : \mathbb{A}_K \to \mathbb{C}^\times$ be a non-trivial global additive character of K. For each $v \in \mathbb{a}_K \cup \mathbb{h}_k$, the local component of ψ is denoted by $\psi_v : (K_v)_+ \to \mathbb{C}^\times$. Let $d^+\mu$ be the Haar measure on $\mathbb{A}_K$ normalized by $\int_{K\backslash \mathbb{A}_K} d^+\mu(x) = 1$. Fix a decomposition $d^+\mu = \prod_v d^+\mu_v$ of $d^+\mu$, where for almost all v the local measure $d^+\mu_v$ is a normalized additive Haar measure on K_v, normalized in the sense that $\mathrm{vol}(O_{K_v}) = 1$. The *global "Hecke-Weil" ε-factor* $\varepsilon^{Hecke-Weil}(s,\rho)$ of the representation $\rho : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$ at $s \in \mathbb{C}$ is defined (formally) by the product

$$\varepsilon^{Hecke-Weil}(s,\rho) := \prod_{v \in \mathbb{a}_K \cup \mathbb{h}_K} \varepsilon^{Hecke-Weil}(s,\rho_v,\psi_v,d^+\mu_v), \qquad s \in \mathbb{C}.$$

For each $v \in \mathbb{a}_K \cup \mathbb{h}_K$, the *local "Hecke-Weil" ε-factor* $\varepsilon^{Hecke-Weil}(s,\rho_v,\psi_v,d^+\mu_v)$ at $s \in \mathbb{C}$ is defined by

$$\varepsilon^{Hecke-Weil}(s,\rho_v,\psi_v,d^+\mu_v) := \varepsilon^{Artin-Weil}(s,\rho_v \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})},\psi_v,d^+\mu_v), \qquad s \in \mathbb{C}.$$

Remark 5.8. Note that, Definition 5.7 is provisional. In fact, it seems that, to define the global Hecke-Weil ε-factors from the local factors properly in the light of Remark 5.6, we have to construct the *"non-commutative Tamagawa measure"* on $\mathscr{J}_K^{\varphi}$ following the lines sketched in [28, 53].

The assignement $\sigma \rightsquigarrow \sigma_*^{(\varphi)}$. Now, let

$$\sigma : W_K \to \mathrm{GL}(V)$$

be a continuous representation of W_K on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. Then we have the following theorem.

Theorem 5.9. *The ℓ-adic continuous representation $\sigma : W_K \to \mathrm{GL}(V)$ defines naturally a continuous representation*

$$\sigma_*^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$$

of $\mathscr{J}_K^{\varphi}$ on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$ via the composition

$$\sigma_*^{\varphi} : \mathscr{J}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}} W_K \xrightarrow{\sigma} \mathrm{GL}(V),$$

[7] As Tate points out in [51],

> "..., we have no way to define $\varepsilon(\chi,\psi,dx)$ without using the reciprocity law isomorphism $F^* \approx W_F^{ab}$. In fact it was his [Langlands'] idea about "nonabelian reciprocity laws" relating representations of degree n of W_F to irreducible representations π of $\mathrm{GL}(n,F)$, and the possibility of defining $\varepsilon(\pi,\psi,dx)$ for the latter, which led Langlands to conjecture and prove ..."

which satisfies

$$\sigma_{*\ v}^{\underline{\varphi}} \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} = \sigma_v = \sigma \circ e_v^{\mathrm{Weil}},$$

for every $v \in \mathbb{h}_K$. *Moreover, the following identities hold :*

$$L^{Artin-Weil}(s,\sigma) = L^{Hecke-Weil}(s,\sigma_*^{\underline{\varphi}}).$$

and

$$\varepsilon^{Artin-Weil}(s,\sigma) = \varepsilon^{Hecke-Weil}(s,\sigma_*^{\underline{\varphi}}).$$

Proof. By Theorem 4.1 on the local-global compatibility of non-abelian norm residue symbols, note that

$$\begin{aligned}\sigma_{*\ v}^{\underline{\varphi}} &= \sigma_*^{\underline{\varphi}} \circ q_v \\ &= \sigma \circ \mathsf{NR}_{\underline{K}}^{\varphi\,\mathrm{Weil}} \circ q_v \\ &= \sigma \circ e_v^{\mathrm{Weil}} \circ \{.,K_v\}_{\varphi_{K_v}}.\end{aligned}$$

That is,

$$\sigma_{*\ v}^{\underline{\varphi}} \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} = \sigma \circ e_v^{\mathrm{Weil}} = \sigma_v.$$

Now the proof follows directly from Theorem 3.1 and Definitions 5.4 and 5.7. □

Notation 5.10. Let $\mathscr{I}_{\underline{K}}^{\varphi}$ denote the image $\mathrm{im}(\mathsf{NR}_{\underline{K}}^{\varphi\,\mathrm{Weil}})$ of the global non-abelian norm-residue symbol $\mathsf{NR}_{\underline{K}}^{\varphi\,\mathrm{Weil}} : \mathscr{I}_{\underline{K}}^{\varphi} \to W_K$ of K.

Remark 5.11. If σ_1 and σ_2 are n-dimensional continuous ℓ-adic representations of W_K, then clearly by Theorem 5.9,

$$(\sigma_1)_*^{\underline{\varphi}} = (\sigma_2)_*^{\underline{\varphi}} \Leftrightarrow \sigma_1 \mid_{\mathscr{I}_{\underline{K}}^{\varphi}} = \sigma_2 \mid_{\mathscr{I}_{\underline{K}}^{\varphi}}.$$

Moreover, *if Conjecture 3.2 holds*; namely, if $\mathscr{I}_{\underline{K}}^{\varphi} = W_K$, then the assignement

$$\sigma \rightsquigarrow \sigma_*^{\underline{\varphi}}$$

introduced in Theorem 5.9, where σ is the continuous representation of W_K on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$ is an *injection*.

Remark 5.12. *If Conjecture 3.2 holds*, then the assignement

$$\sigma \rightsquigarrow \sigma_*^{\underline{\varphi}}$$

introduced in Theorem 5.9, where σ is the continuous representation of W_K on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$ clearly *preserves irreducibility and semi-simplicity*. That is,

- σ is irreducible $\Rightarrow \sigma_*^{\underline{\varphi}}$ is irreducible;
- σ is semi-simple $\Rightarrow \sigma_*^{\underline{\varphi}}$ is semi-simple.

Note that, 1-dimensional ℓ-adic representations of the absolute Weil group W_K of K can be identified with the *ℓ-adic Hecke characters* of K via abelian global class field theory. Thus, an immediate consequence of Theorem 5.9 is the following corollary, whose proof is straightforward.

Corollary 5.13. *An ℓ-adic Hecke character*

$$\chi : K^{\times}\backslash\mathbb{J}_K \to \overline{\mathbb{Q}}_{\ell}^{\times}$$

of K defines naturally a continuous 1-dimensional ℓ-adic representation

$$X_*^{\underline{\varphi}} : \mathscr{J}_K^{\underline{\varphi}} \to \overline{\mathbb{Q}}_{\ell}^{\times}$$

of the non-abelian idèle group $\mathscr{J}_K^{\underline{(\varphi)}}$ of K via the composition

$$\begin{tikzcd}
X_*^{\underline{\varphi}} : \mathscr{J}_K^{\underline{\varphi}} \arrow[r, "\mathsf{NR}_K^{\underline{\varphi}\,\mathrm{Weil}}"] & W_K \arrow[r, "can."] \arrow[rrr, bend left, "X"] & W_K^{ab} \arrow[r, "\mathsf{Art}_K"] & K^{\times}\backslash\mathbb{J}_K \arrow[r, "\chi"] & \overline{\mathbb{Q}}_{\ell}^{\times},
\end{tikzcd}$$

which satisfies

$$X_{*\,v}^{\underline{\varphi}} \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} = X_v = X \circ e_v^{\mathrm{Weil}},$$

for every v. Moreover, the following identities hold :

$$L^{Artin-Weil}(s,X) = L^{\mathrm{Hecke}}(s,\chi) = L^{\mathrm{Hecke-Weil}}(s,X_*^{\underline{\varphi}}).$$

and

$$\varepsilon^{Artin-Weil}(s,X) = \varepsilon^{\mathrm{Hecke}}(s,\chi) = \varepsilon^{\mathrm{Hecke-Weil}}(s,X_*^{\underline{\varphi}}).$$

Thus, Hecke and Artin-Weil L-functions are special cases of Hecke-Weil L-functions via Theorem 5.9 and its Corollary 5.13 as described in the following diagram :

$$\begin{tikzcd}
 & \left\{\begin{array}{c}\text{Hecke-Weil}\\ L\text{-functions}\end{array}\right\} & \\
\left\{\begin{array}{c}\text{Hecke}\\ L\text{-functions}\end{array}\right\} \arrow[ur, "\text{Corollary 5.13}"] \arrow[rr, "\text{abelian global CFT}"'] & & \left\{\begin{array}{c}\text{Artin-Weil}\\ L\text{-functions}\end{array}\right\} \arrow[ul, "\substack{\text{non-abelian global CFT}\\ \text{Theorem 5.9}}"']
\end{tikzcd}.$$

Also, Hecke and Artin-Weil ε-factors are special cases of Hecke-Weil ε-factors via Theorem 5.9 and its Corollary 5.13 as described in the following diagram :

$$\begin{tikzcd}
 & \left\{\begin{array}{c}\text{Hecke-Weil}\\ \varepsilon\text{-factors}\end{array}\right\} & \\
\left\{\begin{array}{c}\text{Hecke}\\ \varepsilon\text{-factors}\end{array}\right\} \arrow[ur, "\text{Corollary 5.13}"] \arrow[rr, "\text{abelian global CFT}"'] & & \left\{\begin{array}{c}\text{Artin-Weil}\\ \varepsilon\text{-factors}\end{array}\right\} \arrow[ul, "\substack{\text{non-abelian global CFT}\\ \text{Theorem 5.9}}"']
\end{tikzcd}.$$

Analytic properties of Hecke-Weil L-functions. Note that, the assignement

$$\sigma \rightsquigarrow \sigma_*^{\varphi}$$

introduced in Theorem 5.9, where σ is the continuous representation of W_K on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$ that is under consideration, also sheds light on the basic analytic properties of the Hecke-Weil L-function $L^{Hecke-Weil}(s, \sigma_*^{\varphi})$ attached to the corresponding continuous representation σ_*^{φ} of $\mathscr{I}_K^{\varphi}$ on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$.

In fact, it is well-known that, $L^{Artin-Weil}(s, \sigma)$ is a convergent product for $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) \gg 0$. Moreover, by *Brauer's induction theorem*, the product $L^{Artin-Weil}(s, \sigma)$ defines a meromorphic function in the whole complex plane, and satisfies the functional equation

$$L^{Artin-Weil}(s, \sigma) = \varepsilon^{Artin-Weil}(s, \sigma) L^{Artin-Weil}(1-s, \sigma^{\vee}),$$

where $\sigma^{\vee}$ is the *contragardient*[8] of σ.

Now, we shall single out the continuous ℓ-adic representations of the non-abelian idèle group $\mathscr{I}_K^{\varphi}$ of K that have nice "analytic invariants".

Definition 5.14. An n-dimensional continuous ℓ-adic representation ρ of $\mathscr{I}_K^{\varphi}$ is called *Galois type*, if it factors through the non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}$: $\mathscr{I}_K^{\varphi} \to W_K$ of K as

$$\begin{tikzcd} \mathscr{I}_K^{\varphi} \arrow[r, "\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}"] \arrow[rr, bend left, "\rho"] & W_K \arrow[r, "\rho^{\gamma}"] & \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell) \end{tikzcd}$$

for some n-dimensional continuous ℓ-adic representation $\rho^{\gamma} : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K.

Remark 5.15. Certainly, if σ is a continuous representation of W_K on an n-dimensional space V over $\overline{\mathbb{Q}}_\ell$, then the continuous ℓ-adic representation σ_*^{φ} of $\mathscr{I}_K^{\varphi}$ on V assigned by Theorem 5.9 is clearly of Galois type, and the following equality holds

$$(\sigma_*^{\varphi})^{\gamma} \mid_{\mathscr{I}_K^{\varphi}} = \sigma \mid_{\mathscr{I}_K^{\varphi}}$$

by Remark 5.11. Moreover, *if Conjecture 3.2 holds,* then the assignement

$$\sigma \rightsquigarrow \sigma_*^{\varphi}$$

introduced in Theorem 5.9, where σ is the continuous representation of W_K on the n-dimensional space V over $\overline{\mathbb{Q}}_\ell$ induces a bijective correspondence

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K) \rightleftarrows \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n;\mathrm{Gal}}(\mathscr{I}_K^{\varphi})$$

[8] Recall that, the contragradient of a continuous n-dimensional ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K is defined to be the representation $\sigma^{\vee} : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K defined by $\sigma^{\vee}(w) = {}^t\sigma(w^{-1})$ for every $w \in W_K$.

between the set $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)$ of all n-dimensional continuous ℓ-adic representations of W_K and the set $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n;\mathrm{Gal}}(\mathscr{I}_K^{\varphi})$ of all Galois type n-dimensional continuous ℓ-adic representations of $\mathscr{I}_K^{\varphi}$, which is a subset of $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(\mathscr{I}_K^{\varphi})$. Note that, by Remark 5.12, this bijective correspondence preserves irreducibility and semi-simplicity. We shall return to this discussion within the framework of Tannakian categories in Section 6.

Theorem 5.16. *Assume that ρ is an n-dimensional continuous ℓ-adic Galois type representation of $\mathscr{I}_K^{\varphi}$. Then the attached Hecke-Weil L-function $L^{Hecke-Weil}(s,\rho)$ is a convergent product for $s \in \mathbb{C}$ satisfying* $\mathrm{Re}(s) \gg 0$. *Moreover, the product $L^{Hecke-Weil}(s,\rho)$ defines a meromorphic function in the whole complex plane, and satisfies the functional equation*

$$L^{Hecke-Weil}(s,\rho) = \varepsilon^{Hecke-Weil}(s,\rho)L^{Hecke-Weil}(1-s,\rho^{\vee}),$$

where $\rho^{\vee}$ is the contragradient of the representation ρ of $\mathscr{I}_K^{\varphi}$ on the $\overline{\mathbb{Q}}_\ell$-linear space V.

Proof. As the n-dimensional continuous ℓ-adic representation ρ of $\mathscr{I}_K^{\varphi}$ is of Galois type,

$$(\rho^{\gamma})_*^{\varphi} = \rho.$$

Now, by Theorem 5.9, the identity

$$L^{Artin-Weil}(s,\rho^{\gamma}) = L^{Hecke-Weil}(s,\rho)$$

follows. Thus, the Hecke-Weil L-function $L^{Hecke-Weil}(s,\rho)$ is a convergent product for $s \in \mathbb{C}$ satisfying $\mathrm{Re}(s) \gg 0$. Moreover, the product $L^{Hecke-Weil}(s,\rho)$ defines a meromorphic function in the whole complex plane, and satisfies a functional equation

$$L^{Hecke-Weil}(s,\rho) = \varepsilon^{Hecke-Weil}(s,\rho)L^{Artin-Weil}(1-s,(\rho^{\gamma})^{\vee}),$$

where $(\rho^{\gamma})^{\vee}$ is the contragradient of the representation ρ^{γ} of W_K on the $\overline{\mathbb{Q}}_\ell$-linear space V. Therefore, it suffices to prove that, for the representation ρ of $\mathscr{I}_K^{\varphi}$ on the $\overline{\mathbb{Q}}_\ell$-linear space V, the identity

$$\rho^{\vee} = ((\rho^{\gamma})^{\vee})_*^{\varphi}$$

is satisfied. Then, by Theorem 5.9, the equality

$$L^{Artin-Weil}(s,(\rho^{\gamma})^{\vee}) = L^{Hecke-Weil}(s,\rho^{\vee})$$

yields the functional equation

$$L^{Hecke-Weil}(s,\rho) = \varepsilon^{Hecke-Weil}(s,\rho)L^{Hecke-Weil}(1-s,\rho^{\vee}).$$

Now, for any $x \in \mathscr{I}_K^{\varphi}$,

$$\begin{aligned}((\rho^{\gamma})^{\vee})^{\varphi}_{*}(x) &= (\rho^{\gamma})^{\vee}(\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}(x)) \\ &= {}^{t}[(\rho^{\gamma})(\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}(x)^{-1})] \\ &= {}^{t}[(\rho^{\gamma})(\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}(x))^{-1}] \\ &= {}^{t}[\rho(x)^{-1}] \\ &= {}^{t}\rho(x^{-1}) \\ &= \rho^{\vee}(x),\end{aligned}$$

which completes the proof. □

"Fine-tuning" Theorem 5.9. In the remaining of this section, we shall "fine-tune" Theorem 5.9 by specializing the continuous representation $\sigma : W_K \to \mathrm{GL}(V)$ of the global Weil group W_K of K on the n-dimensional linear space V over $\overline{\mathbb{Q}}_\ell$ to the following cases : The n-dimensional ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K

- is assumed to be semi-simple;
- has a model over some finite extension E over $\mathbb{Q}_\ell$;
- is unramified at almost all places of K;
- in case K is a number field, is "$\mathbb{B}$-admissible at v in the sense of Fontaine" (look at [10, 11]) for each henselian place v of K satisfying $\ell \mid q_v$. That is, for $v \mid \ell$ (so for such a place v of K, the extension $K_v/\mathbb{Q}_\ell$ is finite), the local representation $\sigma_v : W_{K_v} \xrightarrow{e_v^{\mathrm{Weil}}} W_K \xrightarrow{\sigma} \mathrm{GL}(V)$ is "$\mathbb{B}$-admissible in the sense of Fontaine" (look at [10, 11]); and finally,
- is pure of weight $w \in \mathbb{Z}$.

Note that, except for the first two assumptions on $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$, all the remaining ones on the ℓ-adic representation are defined "locally".

Remark 5.17. We make these assumptions, because the global Langlands reciprocity principle for $\mathrm{GL}(n)$ over the global field K asserts the existence of a unique natural bijective correspondence between the collection of all equivalence classes of irreducible n-dimensional ℓ-adic representations of the Weil group W_K of K satisfying these assumptions, which are called, in case K is assumed to be a number field, *"geometric Galois representations"* by Fontaine and Mazur (for details, on ℓ-adic representations of W_K, look at [13] for the function field case, and [4, 10, 11, 12, 54] for the number field case), and the collection of all equivalence classes of cuspidal automorphic representations of the adèle group $\mathrm{GL}(n, \mathbb{A}_K)$, which are furthermore *"algebraic"* in the sense of Clozel in case K is a number field[9] [10] (for details on cuspidal automorphic representations of $\mathrm{GL}(n, \mathbb{A}_K)$, look at [13] for the function field

[9] For example Maass forms on $\mathrm{GL}(2, \mathbb{Q})$ do not correspond to Galois representations.

[10] As Ngô points out in [45],

> "In the number fields case, only a part of ℓ-adic representations of W_K coming from motives should correspond to a part of automorphic representations."

case, and [6] for the number field case). Moreover, this correspondence satisfies certain "naturality" conditions meaning that the bijection should recover abelian global class field theory when $n = 1$, should respect the corresponding L-functions and ε-factors, should well-behave with respect to certain *linear algebraic operations*, and should satisfy *local-global compatibility*.

Now, let $v \in \mathbb{h}_K$ such that $\ell \nmid q_v$. A continuous ℓ-adic representation $\sigma_v : W_{K_v} \to \mathrm{GL}(V)$ of W_{K_v} on an ℓ-adic space V is said to be *Frobenius semi-simple*, if the $\overline{\mathbb{Q}}_\ell$-linear operator $\sigma_v(\varphi_{K_v}|_{W_{K_v}})$ acts semi-simply on V, for some fixed Frobenius $\varphi_{K_v}|_{W_{K_v}}$ in W_{K_v}. Recall that, there exits an equivalence of the following abelian categories :

$$\left\{\begin{matrix}\text{Weil-Deligne representations}\\ \text{of } K \text{ with coefficients in } \overline{\mathbb{Q}}_\ell\end{matrix}\right\} \approx \left\{\begin{matrix}\text{Continuous Frobenius semi-simple rep-}\\ \text{resentations of } W_K \text{ on } \overline{\mathbb{Q}}_\ell\text{-linear spaces}\end{matrix}\right\}.$$

Definition 5.18. Let $v \in \mathbb{h}_K$ such that $\ell \nmid q_v$. A continuous representation $\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\varphi}$ on an ℓ-adic space V is called *Frobenius semi-simple at v*, if

$$\rho_v \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \xrightarrow[\sim]{\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\rho_v} \mathrm{GL}(V)$$

is a Frobenius semi-simple representation of W_{K_v} on V. That is, the linear operator $\rho_v \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}(\varphi_{K_v}|_{W_{K_v}})$ acts semi-simply on the ℓ-adic linear space V.

Let $\left\{\xi_v : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \to \mathrm{GL}(V)\right\}_{v \in \mathbb{h}_K} \cup \{\xi_v : W_{K_v} \to \mathrm{GL}(V)\}_{v \in \mathbb{a}_K}$ be a collection of continuous ℓ-adic representations on V. If $\xi_v : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \to \mathrm{GL}(V)$ is a continuous Frobenius semi-simple representation in the sense that $\xi_v \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} : W_{K_v} \xrightarrow[\sim]{\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}} {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \xrightarrow{\xi_v} \mathrm{GL}(V)$ is a Frobenius semi-simple representation of W_{K_v} on V for each $v \in \mathbb{h}_K$ satisfying $\ell \nmid q_v$, then Theorem 5.2 yields the existence of a unique continuous representation $\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\varphi}$ on the ℓ-adic space V such that $\rho_v = \xi_v$ for $v \in \mathbb{h}_K$ satisfying $\ell \nmid q_v$. Thus, we make the following definition.

Definition 5.19. The continuous representation $\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\varphi}$ on the n-dimensional ℓ-adic space V is called *Frobenius-admissible*, if ρ is Frobenius semi-simple at v for all $v \in \mathbb{h}_K$ satisfying $\ell \nmid q_v$.

Assume that the continuous ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(V)$ is furthermore semi-simple. Then the local representation $\sigma_v : W_{K_v} \xrightarrow{e_v^{\mathrm{Weil}}} W_K \xrightarrow{\sigma} \mathrm{GL}(V)$ is Frobenius semi-simple for each $v \in \mathbb{h}_K$ satisfying $\ell \nmid q_v$ by the *local-global compatibility of the Langlands correspondence for* $\mathrm{GL}(n)$ (look at [15] for details). Thus, the following corollary follows now from Theorem 5.9, and Definitions 5.18 and 5.19.

Corollary 5.20. *Let $\sigma : W_K \to \mathrm{GL}(V)$ be a continuous ℓ-adic semi-simple representation of W_K on the n-dimensional linear space V over $\overline{\mathbb{Q}}_\ell$. Then, the naturally* *defined continuous representation*

$$\sigma_*^{\varphi} : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(V)$$

of $\mathscr{I}_K^{\varphi}$ on V is Frobenius-admissible.

Recall that, a continuous n-dimensional ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ has a *model over some finite extension E over* $\mathbb{Q}_\ell$, if the corresponding ℓ-adic Galois representation $\sigma_o : G_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ has a *model over some finite extension E over* $\mathbb{Q}_\ell$. That is, if the continuous representation $\sigma_o : G_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ factors through a continuous homomorphism $\sigma_\bullet : G_K \to \mathrm{GL}(n, E)$ as

$$\sigma_o : G_K \xrightarrow{\sigma_\bullet} \mathrm{GL}(n, E) \hookrightarrow \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell),$$

where the continuity of the arrow $\sigma_\bullet$ is defined with respect to the Krull topology on G_K and the ℓ-adic topology on $\mathrm{GL}(n, E)$. Thus, it is natural to make the following definition.

Definition 5.21. A continuous representation $\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ is said to have a finite *model over some finite extension E over* $\mathbb{Q}_\ell$, if it factors through a continuous homomorphism $\rho_\bullet : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(n, E)$ as

$$\rho : \mathscr{I}_K^{\varphi} \xrightarrow{\rho_\bullet} \mathrm{GL}(n, E) \hookrightarrow \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell),$$

where the continuity of the arrow $\rho_\bullet$ is defined with respect to the restricted free product topology on $\mathscr{I}_K^{\varphi}$ and the ℓ-adic topology on $\mathrm{GL}(n, E)$.

By *Baire category theorem*, it follows that any continuous representation $G_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ has a model over some finite extension E over $\mathbb{Q}_\ell$ (look at [25] and [29] for details). Therefore, the following corollary immediately follows from Theorem 5.9 and from Definition 5.21.

Corollary 5.22. *Let $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ be a continuous ℓ-adic representation of W_K. The naturally defined continuous ℓ-adic representation*

$$\sigma_*^{\varphi} : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$$

of $\mathscr{I}_K^{\varphi}$ has a model over some finite extension E over $\mathbb{Q}_\ell$.

More generally, the following proposition follows from Definition 5.14 and from Definition 5.21 directly.

Proposition 5.23. *Assume that ρ is an n-dimensional continuous ℓ-adic Galois type representation of $\mathscr{I}_K^{\varphi}$. Then ρ has a model over some finite extension E over $\mathbb{Q}_\ell$.*

Recall that, a continuous n-dimensional ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K is said to be *unramified at $v \in \mathbb{h}_K$*, if $\sigma_v(I_{K_v}) = 1$. Thus, we introduce the following type of ℓ-adic representations of the non-abelian idèle group $\mathscr{I}_K^{\varphi}$ of the global field K.

Definition 5.24. Let $v \in \mathbb{h}_K$. A continuous representation $\rho : \mathscr{I}_K^{\underline{\varphi}} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\underline{\varphi}}$ on an ℓ-adic space V is called *unramified at* v, if

$$\rho_v\left(\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}(I_{K_v})\right) \overset{(2.1)}{=} \rho_v\left({}_1\nabla_{K_v}^{(\varphi_{K_v})\underline{0}}\right) = 1;$$

that is, if the local representation $\rho_v : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \to \mathrm{GL}(V)$ is unramified. Otherwise, the representation is called *ramified at* v.

The continuous n-dimensional ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of W_K is said to be *unramified at almost all places* of K, if $\sigma_v(I_{K_v}) = 1$ for every $v \notin S$ for some finite subset $S := S(\sigma)$ of $\mathbb{h}_K$. Thus, we introduce the following type of ℓ-adic representations of the non-abelian idèle group $\mathscr{I}_K^{\underline{\varphi}}$ of the global field K.

Definition 5.25. The continuous representation $\rho : \mathscr{I}_K^{\underline{\varphi}} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\underline{\varphi}}$ on the ℓ-adic space V is said to be *unramified at almost all places* of K, if if there exists a finite subset $S(\rho) =: S$ of $\mathbb{h}_K$ such that, for each finite prime $v \notin S$, the representation ρ of $\mathscr{I}_K^{\underline{\varphi}}$ is unramified at v.

Now, by Theorem 5.9, and by Definitions 5.24 and 5.25, the following result follows immediately.

Corollary 5.26. *Let $\sigma : W_K \to \mathrm{GL}(V)$ be a continuous representation of W_K on the ℓ-adic space V, which is unramified at almost all places of K. Then, the naturally defined continuous representation*

$$\sigma_*^{\underline{\varphi}} : \mathscr{I}_K^{\underline{\varphi}} \to \mathrm{GL}(V)$$

of $\mathscr{I}_K^{\underline{\varphi}}$ on the ℓ-adic space V is unramified at almost all places of K.

Recall that, a continuous ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(V)$ of W_K on the space V is *pure of weight* $w \in \mathbb{Z}$, if

- there exists a finite subset S of $\mathbb{h}_K$ such that, for each finite prime $v \notin S$, the local representation $\sigma_v : W_{K_v} \to \mathrm{GL}(V)$ is *unramified*; namely, $\sigma_v(I_{K_v})$ is trivial; and
- the eigenvalues of $\sigma_v(\varphi_{K_v}|_{W_{K_v}})$ are algebraic integers whose complex conjugates have complex absolute value $q_v^{w/2}$.

Thus, we make the following definition.

Definition 5.27. A continuous representation $\rho : \mathscr{I}_K^{\underline{\varphi}} \to \mathrm{GL}(V)$ of $\mathscr{I}_K^{\underline{\varphi}}$ on a $\overline{\mathbb{Q}}_\ell$-linear space V is called *pure of weight* $w \in \mathbb{Z}$, if

- there exists a finite subset $S(\rho) =: S$ of $\mathbb{h}_K$ such that, for each finite prime $v \notin S$, the local representation $\rho_v : {}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \to \mathrm{GL}(V)$ is unramified; that is, ρ is unramified at v, and
- the eigenvalues of $\rho_v\left(\mathbf{\Phi}_{K_v}^{(\varphi_{K_v})}\left(\varphi_{K_v}|_{W_{K_v}}\right)\right)$ are algebraic integers whose complex conjugates have complex absolute value $q_v^{w/2}$.

Corollary 5.28. *Assume that the continuous ℓ-adic representation $\sigma : W_K \to \mathrm{GL}(V)$ is pure of weight w. Then, the naturally defined representation*

$$\sigma_*^{\varphi} : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$$

of $\mathscr{J}_K^{\varphi}$ on the n-dimensional linear space V over $\overline{\mathbb{Q}}_\ell$ is pure of weight w.

Proof. Follows from Theorem 5.9, Definitions 5.24, 5.25 and Corollary 5.26, and Definition 5.27. □

Finally, in case K is a number field, a continuous ℓ-adic semi-simple representation $\sigma : W_K \to \mathrm{GL}(V)$ of W_K on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$ is called $\mathbb{B}$*-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$*, where $\mathbb{B}$ denotes a topological $\mathbb{Q}_\ell$-algebra endowed with a continuous linear action of $G_{\mathbb{Q}_\ell}$, if *a corresponding model $\sigma_\bullet : G_K \to \mathrm{GL}(V_\bullet)$ on an n-dimensional linear space $V_\bullet$ over some finite extension E over $\mathbb{Q}_\ell$ of degree $[E:\mathbb{Q}_\ell] = n_o$ is $\mathbb{B}$-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$*. That is, *the local Galois representation $(\sigma_\bullet)_v : G_{K_v} \to \mathrm{GL}(V_\bullet)$ on the linear space $V_\bullet$ over E is $\mathbb{B}$-admissible for each finite place v satisfying $\ell \mid q_v$*. Recall that, following closely [4, 10, 11], *for each finite place v satisfying $\ell \mid q_v$, $(\sigma_\bullet)_v : G_{K_v} \to \mathrm{GL}(V_\bullet)$ is called $\mathbb{B}$-admissible*, if the following equality

$$\dim_{\mathbb{B}^{G_{K_v}}}(D_{\mathbb{B}}(V_\bullet)) = \dim_{\mathbb{Q}_\ell}(V_\bullet)$$

holds, where

$$D_{\mathbb{B}}(V_\bullet) := (\mathbb{B} \otimes_{\mathbb{Q}_\ell} V_\bullet)^{G_{K_v}}$$

is a $\mathbb{B}^{G_{K_v}}$-linear space (note that $\mathbb{B}^{G_{K_v}}$ is a field).

Remark 5.29. Note that, the definition of $\mathbb{B}$-admissibility of $\sigma : W_K \to \mathrm{GL}(V)$ at v for each henselian place v of K satisfying $\ell \mid q_v$ does *not* depend on the choice of a model $\sigma_\bullet : G_K \to \mathrm{GL}(V_\bullet)$ on an n-dimensional linear space $V_\bullet$ over some finite extension E over $\mathbb{Q}_\ell$.

Thus, we make the following definition.

Definition 5.30. Let $\mathbb{B}$ denote a topological $\mathbb{Q}_\ell$-algebra endowed with a continuous linear action of $G_{\mathbb{Q}_\ell}$. A continuous *Galois type* representation $\rho : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(V)$ of $\mathscr{J}_K^{\varphi}$ on a $\overline{\mathbb{Q}}_\ell$-linear space V is called $\mathbb{B}$*-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$*, if there exists a representation $\rho^\gamma : W_K \to \mathrm{GL}(V)$ on the $\overline{\mathbb{Q}}_\ell$-linear space V satisfying

$$\begin{tikzcd} \mathscr{J}_K^{\varphi} \arrow[r, "\mathsf{NR}_K^{\varphi\,\mathrm{Weil}}"] \arrow[rr, bend left, "\rho"] & W_K \arrow[r, "\rho^\gamma"] & \mathrm{GL}(V) \end{tikzcd}$$

and which is $\mathbb{B}$-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$.

Remark 5.31. In fact, following [10, 11], it is possible to define $\mathbb{B}$-admissible representations, at a henselian place v of K satisfying $\ell \mid q_v$, of $\mathscr{J}_K^{\varphi}$ on a $\overline{\mathbb{Q}}_\ell$-linear space V for any regular $(\mathbb{Q}_\ell, (\mathscr{J}_K^{\varphi})_v)$-ring $\mathbb{B}$. However, for this work, it suffices to restrict to the special case introduced in Definition 5.30.

Certainly, by Theorem 5.9 and by Definition 5.30, the following corollary follows directly.

Corollary 5.32. *Let $\sigma : W_K \to \mathrm{GL}(V)$ be a continuous representation of W_K on the ℓ-adic space V, which is $\mathbb{B}$-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$. Then, the naturally defined continuous representation*

$$\sigma_*^\varphi : \mathscr{J}_K^\varphi \to \mathrm{GL}(V)$$

of $\mathscr{J}_K^\varphi$ on the ℓ-adic space V is $\mathbb{B}$-admissible at v for each henselian place v of K satisfying $\ell \mid q_v$.

Remark 5.33. In what follows, we are primarily interested in the case $\mathbb{B} = \mathbb{B}_{dR}$.

That is, we have established the following table :

$\sigma : W_K \xrightarrow{\text{cont.}} \mathrm{GL}(V)$	$\rightsquigarrow$	$\sigma_*^\varphi : \mathscr{J}_K^\varphi \xrightarrow{\text{cont.}} \mathrm{GL}(V)$
semi-simple		Frobenius-admissible
has a model over some finite extension $E/\mathbb{Q}_\ell$		has a model over some finite extension $E/\mathbb{Q}_\ell$
unramified at almost all places of K	$\Longrightarrow$	unramified at almost all places of K
pure of weight $w \in \mathbb{Z}$		pure of weight $w \in \mathbb{Z}$
(if K is a number field) $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ s.t. $\ell \mid q_v$		(if K is a number field) $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ s.t. $\ell \mid q_v$

6. Tannakian categories

Note that, the global non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} : \mathscr{J}_K^\varphi \to W_K$ of K naturally produces a functor

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K) \to \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont;Gal}}(\mathscr{J}_K^\varphi) \tag{6.1}$$

from the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ of ℓ-adic continuous representations of W_K to the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont;Gal}}(\mathscr{J}_K^\varphi)$ of Galois type continuous representations of $\mathscr{J}_K^\varphi$ with coefficients $\overline{\mathbb{Q}}_\ell$ introduced by Definition 5.14, which is defined by pulling back a representation of W_K on a $\overline{\mathbb{Q}}_\ell$-linear space V to $\mathscr{J}_K^\varphi$. Moreover, Theorem 5.9 states that this functor

- preserves the respective representation spaces;
- preserves the respective L-functions and ε-factors;

and Corollaries 5.20, 5.26, and 5.28 state that it furthermore

- sends semi-simple ℓ-adic continuous representations of W_K to the Frobenius-admissible representations of $\mathscr{J}_K^\varphi$;
- sends ℓ-adic continuous representations of W_K which are unramified at almost all places of K to ℓ-adic continuous representations of $\mathscr{J}_K^\varphi$ which are unramified at almost all places of K;

– sends ℓ-adic continuous pure representations of W_K of weight w to ℓ-adic continuous pure representations of $\mathscr{J}_K^{\varphi}$ of weight w.

If K is a number field, Corollary 5.32 state that the functor (6.1) furthermore

– sends ℓ-adic continuous representations of W_K which are $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ such that $\ell \mid q_v$ to ℓ-adic continuous representations of $\mathscr{J}_K^{\varphi}$ which are $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ such that $\ell \mid q_v$.

Notation 6.1. Let $\mathscr{N}_K^{\varphi}$ denote the kernel $\ker(\mathsf{NR}_K^{\varphi\,\mathrm{Weil}})$ of the global non-abelian norm-residue symbol $\mathsf{NR}_K^{\varphi\,\mathrm{Weil}} : \mathscr{J}_K^{\varphi} \to W_K$ of K.

Remark 6.2. Assume that Conjecture 3.2 *holds*. Let

$$\rho : \mathscr{J}_K^{\varphi} \to GL(V)$$

be *any* (that is, not necessarily of Galois type !) continuous representation of $\mathscr{J}_K^{\varphi}$ on a vector space V over $\overline{\mathbb{Q}}_{\ell}$. Then the following conditions are clearly equivalent :

– we have an inclusion $\mathscr{N}_K^{\varphi} \subseteq \ker(\rho)$;
– the representation (ρ, V) of $\mathscr{J}_K^{\varphi}$ factors through

$$\rho : \mathscr{J}_K^{\varphi} \xrightarrow[c_K]{\text{canonical}} \mathscr{N}_K^{\varphi} \backslash \mathscr{J}_K^{\varphi} \xrightarrow{\rho_o} GL(V),$$

where $\mathscr{N}_K^{\varphi} \backslash \mathscr{J}_K^{\varphi} \xrightarrow[\mathsf{NR}_{K_o}^{\varphi}]{\sim} W_K$ under the global non-abelian norm-residue symbol NR_K^{φ} of K.
– the representation (ρ, V) of $\mathscr{J}_K^{\varphi}$ is of Galois type.

Thus, by Remark 6.2, we get an alternative description of the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont;Gal}}(\mathscr{J}_K^{\varphi})$ as follows.

Let $\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ denote the category of continuous ℓ-adic representations (ρ, V) of $\mathscr{J}_K^{\varphi}$ satisfying the *"congruence relation"*

$$\mathscr{N}_K^{\varphi} \subseteq \ker(\rho).$$

Assuming the validity of Conjecture 3.2, Remark 6.2 yields the equality

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont;Gal}}(\mathscr{J}_K^{\varphi}) = \mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}},$$

which is an alternative description of the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont;Gal}}(\mathscr{J}_K^{\varphi})$. Moreover, combining Remarks 5.15 and 6.2, there exist an equivalence

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(W_K) \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$$

of the categories $\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(W_K)$ and $\mathsf{Rep}_{\overline{\mathbb{Q}}_{\ell}}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$, *"natural"* in the sense that it preserves the respective representation spaces, preserves irreducibility and semi-simplicity, L-functions and ε-factors. More precisely, we have the following theorem.

Theorem 6.3. *Assume that Conjecture 3.2 holds. Then the functor*

$$\mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K) \to \mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)$$

defined by Theorem 5.9 induces an equivalence

$$\mathsf{Rep}^{cont}_{\overline{\mathbb{Q}}_\ell}(W_K) \approx \mathsf{Rep}^{cont}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K} \tag{6.2}$$

between the categories $\mathsf{Rep}^{cont}_{\overline{\mathbb{Q}}_\ell}(W_K)$ *and* $\mathsf{Rep}^{cont}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K}$*, which preserves the respective representation spaces, preserves irreducibility and semi-simplicity, as well as the respective L-functions and the* ε*-factors.*

Proof. Following Remark 6.2, define the functor

$$\mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K} \to \mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K)$$

by

$$(\rho, V) \mapsto (\rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1}, V)$$

for every representation (ρ, V) of $\mathscr{I}^{\varphi}_K$ satisfying the "congruence relation" $\mathscr{N}^{\varphi}_K \subseteq \ker(\rho)$. Then, by Theorem 5.9,

$$(\rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1}, V) \mapsto ((\rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1})_*, V),$$

where

$$\begin{aligned}
\rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1})_* &= \rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1} \circ \mathsf{NR}^{\varphi}_K \\
&= \rho_o \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1} \circ \mathsf{NR}^{\varphi}_{K_o} \circ c_K \\
&= \rho_o \circ c_K \\
&= \rho,
\end{aligned}$$

which shows that the composition

$$\mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K} \to \mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K) \to \mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K}$$

is the identity functor on $\mathsf{Rep}^{\text{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{I}^{\varphi}_K)_{\mathscr{N}^{\varphi}_K}$.

Now, let $\sigma : W_K \to \mathrm{GL}(V)$ be any continuous representation of W_K on an n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$. Then, the corresponding continuous representation

$$\sigma^{\varphi}_* : \mathscr{I}^{\varphi}_K \to \mathrm{GL}(V)$$

of the topological group $\mathscr{I}^{\varphi}_K$ on the n-dimensional vector space V over $\overline{\mathbb{Q}}_\ell$ satisfies the "congruence relation" $\mathscr{N}^{\varphi}_K \subseteq \ker(\sigma^{\varphi}_*)$, as $\alpha \in \mathscr{N}^{\varphi}_K$ yields $\sigma^{\varphi}_*(\alpha) = \sigma \circ \mathsf{NR}^{\varphi}_K(\alpha) = \sigma(1_{W_K}) = 1_V$. Moreover,

$$(\sigma^{\varphi}_*, V) \mapsto (\sigma^{\varphi}_{*_o} \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1}, V),$$

where

$$\sigma^{\varphi}_{*_o} \circ (\mathsf{NR}^{\varphi}_{K_o})^{-1} = \sigma,$$

because for any $w \in W_K$, there exists an $\alpha_w \in \mathscr{J}_K^{\varphi}$ such that $\mathsf{NR}_K^{\varphi}(\alpha_w) = w$. Therefore, $\sigma_{*_o}^{\varphi} \circ (\mathsf{NR}_{K_o}^{\varphi})^{-1}(w) = \sigma_{*_o}^{\varphi}(\mathscr{N}_K^{\varphi}\alpha_w) = \sigma_*^{\varphi}(\alpha_w) = \sigma \circ \mathsf{NR}_K^{\varphi}(\alpha_w) = \sigma(w)$. Hence, the composition

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K) \to \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}} \to \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$$

is the identity functor on $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$.

Thus, there exists an equivalence

$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K) \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$$

between the categories $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ and $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$, which clearly preserves the respective representation spaces, preserves irreducibility and semi-simplicty, as well as the respective L-functions and the ε-factors. □

Remark 6.4. Moreover, the equivalence (6.2) induces the equivalences :

–
$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{ss}} \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{Frob-adm}}$$

between the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{ss}}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ whose objects are the semi-simple objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ and the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{Frob-adm}}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ whose objects are the Frobenius-admissible objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$;

–
$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{nr-ae}} \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{nr-ae}}$$

between the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{nr-ae}}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ whose objects are the objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ which are unramified at almost all places of K and the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{nr-ae}}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ whose objects are the objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ which are unramified at almost all places of K;

–
$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{pure},w} \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{pure},w}$$

between the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathrm{pure},w}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ whose objects are the objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)$ which are pure of weight w and the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathrm{pure},w}$ of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ whose objects are the objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}$ which are pure of weight w.

If K is furthermore assumed to be a number field, then

–
$$\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(W_K)^{\mathbb{B}-\mathrm{adm}} \approx \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}}(\mathscr{J}_K^{\varphi})_{\mathscr{N}_K^{\varphi}}^{\mathbb{B}-\mathrm{adm}}$$

between the full subcategory $\mathsf{Rep}^{\mathrm{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K)^{\mathbb{B}-\mathrm{adm}}$ of $\mathsf{Rep}^{\mathrm{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K)$ whose objects are the objects of $\mathsf{Rep}^{\mathrm{cont}}_{\overline{\mathbb{Q}}_\ell}(W_K)$ which are $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ satisfying $\ell \mid q_v$ and the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}^{\varphi}_{K})^{\mathbb{B}-\mathrm{adm}}_{\mathscr{N}^{\varphi}_{K}}$ of $\mathsf{Rep}^{\mathrm{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}^{\varphi}_{K})_{\mathscr{N}^{\varphi}_{K}}$ whose objects are the objects of $\mathsf{Rep}^{\mathrm{cont}}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}^{\varphi}_{K})_{\mathscr{N}^{\varphi}_{K}}$ which are $\mathbb{B}$-admissible at each $v \in \mathbb{h}_K$ satisfying $\ell \mid q_v$.

7. On the global Langlands reciprocity principle

On the other hand, the proper generalization to the non-abelian setting of the idèle-class character of abelian global class field theory is the notion of *automorphic representation* of $\mathrm{GL}(n,\mathbb{A}_K)$ for $1 \le n$. So, it is natural to ask the relationship between automorphic representations of $\mathrm{GL}(n,\mathbb{A}_K)$ and the n-dimensional ℓ-adic representations of the non-abelian idèle group $\mathscr{J}^{\varphi}_{K}$ of K.

Now, suppose that π is an *irreducible admissible smooth* representation of $\mathrm{GL}(n,\mathbb{A}_K)$ (cf. [6]). Then, following Flath [9], there exists the restricted tensor product decomposition $\pi = \otimes_{v<\infty}{}'\pi_v \otimes \pi_\infty$ of π, where

- π_v is an irreducible admissible representation of the local group $\mathrm{GL}(n,K_v)$, for each $v \in \mathbb{h}_K$;
- π_v is an unramified representation of the local group $\mathrm{GL}(n,K_v)$, for *almost all* $v \in \mathbb{h}_K$.

Fix a rational prime ℓ. We shall define the set S_ℓ as usual by

$$S_\ell := \{v \in \mathbb{h}_K \mid \ \ell \mid q_v\} \cup \mathbb{a}_K.$$

By the (non-archimedean) local Langlands reciprocity principle for $\mathrm{GL}(n)$, which is now a theorem of Laumon-Rapoport-Stuhler [38], Harris-Taylor [15] and Henniart [17], for each $v \in \mathbb{h}_K$ satisfying $\ell \nmid q_v$, that is $v \notin S_\ell$, there is a correspondence

$$\pi_v \longmapsto \mathscr{L}^{(n)}_{K_v}(\pi_v),$$

where

$$\mathscr{L}^{(n)}_{K_v}(\pi_v) : W_{K_v} \to \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell)$$

is an (equivalence class of) n-dimensional Frobenius semi-simple ℓ-adic representation of the Weil group W_{K_v} of K_v such that

$$L^{\mathrm{Langlands}}(s,\pi_v) = L^{\mathrm{Artin-Weil}}(s,\mathscr{L}^{(n)}_{K_v}(\pi_v))$$

and

$$\varepsilon^{\mathrm{Langlands}}(s,\pi_v,\psi_v,d^+\mu_v) = \varepsilon^{\mathrm{Artin-Weil}}(s,\mathscr{L}^{(n)}_{K_v}(\pi_v),\psi_v,d^+\mu_v).$$

Moreover, for almost all $v \in \mathbb{h}_K$, as the local admissible representation π_v of $\mathrm{GL}(n,K_v)$ is unramified, the corresponding ℓ-adic representation $\mathscr{L}^{(n)}_{K_v}(\pi_v)$ of W_{K_v} is unramified as well. On the other hand, if $v \in \mathbb{h}_K$ satisfies $\ell \mid q_v$, then for the time being there is unfortunately *no* known well-formulated conjectural statement neither for the "*p-adic Langlands functoriality*" nor for the "*p-adic reciprocity principle for any reductive group G*", where only for the case $G = \mathrm{GL}(2)$ and $K_v = \mathbb{Q}_p$ some results are known (look at [33] for the theory of p-adic automorphic forms from a very general perspective). Thus, for this case and for the case $v \in \mathbb{o}_K$ as well, we shall "forget" the corresponding irreducible admissible representation π_v. To sum up, we have the following

Proposition 7.1. *If π is an irreducible admissible smooth representation of $\mathrm{GL}(n,\mathbb{A}_K)$, then there exists a unique collection*

$$\left\{\mathscr{L}_{K_v}^{(n)}(\pi_v) : W_{K_v} \to \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell)\right\}_{v\notin S_\ell}$$

consisting of (equivalence classes of) n-dimensional Frobenius semi-simple ℓ-adic representations of W_{K_v} for finite primes v of K satisfying $\ell \nmid q_v$, where almost all of them are unramified.

Now, we can state the theorem.

Theorem 7.2. *For any (equivalence class of) irreducible admissible smooth representation π of $\mathrm{GL}(n,\mathbb{A}_K)$, there corresponds an (equivalence class of) ℓ-adic continuous Frobenius-admissible representation $\lambda_K^{(n)}(\pi) : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell)$ unramified at almost all $v \in \mathbb{h}_K$, satisfying*

$$\lambda_K^{(n)}(\pi)_v \circ \mathbf{\Phi}_{K_v}^{(\varphi_{K_v})} = \mathscr{L}_{K_v}^{(n)}(\pi_v)$$

for every $v \notin S_\ell$, and the equalities

$$L_{S_\ell}^{Artin-Hecke}(s,\lambda_K^{(n)}(\pi)) = L_{S_\ell}^{Langlands}(s,\pi)$$

and

$$\varepsilon_{S_\ell}^{Artin-Hecke}(s,\lambda_K^{(n)}(\pi)) = \varepsilon_{S_\ell}^{Langlands}(s,\pi).$$

Moreover the arrow

$$\lambda_K^{(n)} : \Pi(\mathrm{GL}(n,\mathbb{A}_K)) \to \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(\mathscr{J}_K^{\varphi})^o$$

is bijective, where $\Pi(\mathrm{GL}(n,\mathbb{A}_K))$ denotes the category of irreducible admissible smooth representations of $\mathrm{GL}(n,\mathbb{A}_K)$ and $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(\mathscr{J}_K^{\varphi})^o$ denotes the category of Frobenius-admissible n-dimensional ℓ-adic representations of $\mathscr{J}_K^{\varphi}$ which are unramified at almost all $v \in \mathbb{h}_K$.

Proof. In fact, the injectivity of the arrow follows from the *"strong multiplicity one"* theorem for $\mathrm{GL}(n,\mathbb{A}_K)$ of Piatetski-Shapiro (cf. [47]). For the surjectivity of the arrow, note that, by the local (archimedean and non-archimedean) Langlands reciprocity for $\mathrm{GL}(n)$, any $\rho \in \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(\mathscr{J}_K^{\varphi})^o$ yields a collection $\{\pi_{\rho_v}\}_{v\in\mathbb{h}_K\cup\mathbb{o}_K}$ consisting of irreducible admissible representations π_{ρ_v} of $\mathrm{GL}(n,K_v)$ as v runs over finite and infinite places of K, where for almost all finite v, π_v is unramified. Then, by Flath's theorem, $\pi = \otimes_{v<\infty}{}'\pi_v \otimes \pi_\infty$ is an irreducible admissible smooth representation of $\mathrm{GL}(n,\mathbb{A}_K)$ satisfying $\lambda_K^{(n)}(\pi) = \rho$ and the equalities

$$L_{S_\ell}^{Artin-Hecke}(s,\rho) = L_{S_\ell}^{Langlands}(s,\pi)$$

and

$$\varepsilon_{S_\ell}^{Artin-Hecke}(s,\rho) = \varepsilon_{S_\ell}^{Langlands}(s,\pi),$$

which completes the proof. □

Note that, Theorem 7.2 has a very interesting consequence, which we state now as the following corollary.

Corollary 7.3. *For a finite subset S of $\mathbb{h}_K \cup \mathbb{a}_K$ satisfying $S_\ell \subseteq S$, let $\{\xi_v\}_{v \notin S}$ be a collection consisting of continuous n-dimensional Frobenius semi-simple ℓ-adic representations $\xi_v : (\mathscr{I}_K^{\varphi})_v \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$ of the local group $(\mathscr{I}_K^{\varphi})_v$ for $v \notin S$, where almost all of then are unramified. Then, the collection $\{\xi_v\}_{v \notin S}$ uniquely determines a continuous n-dimensional Frobenius-admissible ℓ-adic representation*

$$\rho : \mathscr{I}_K^{\varphi} \to \mathrm{GL}(n, \overline{\mathbb{Q}}_\ell)$$

of the non-abelian idèle group $\mathscr{I}_K^{\varphi}$ of K, which is unramified at almost all $v \in \mathbb{h}_K$, and which satisfies

$$\rho_v = \xi_v$$

for every $v \notin S_\ell$.

Clearly, Theorem 7.2 is closely related with the *global Langlands reciprocity principle for* $\mathrm{GL}(n)$ (for details [6, 7, 13, 14, 20, 36]). Let $\Pi_{\mathrm{aut}}(\mathrm{GL}(n))$ denote the category of equivalence classes of automorphic representations of $\mathrm{GL}(n, \mathbb{A}_K)$, and let $\Pi_{\mathrm{isob}}(\mathrm{GL}(n))$ and $\Pi_{\mathrm{cusp}}(\mathrm{GL}(n))$ denote the full subcategories of $\Pi_{\mathrm{aut}}(\mathrm{GL}(n))$ whose objects are the equivalence classes of *isobaric* and the equivalence classes of *cuspidal* automorphic representations of $\mathrm{GL}(n, \mathbb{A}_K)$ respectively. Following closely [27, 35], an automorphic representation π of $\mathrm{GL}(n, \mathbb{A}_K)$ is called *isobaric*, if $\pi \simeq \pi_1 \boxplus \cdots \boxplus \pi_m$ with each π_i a *cuspidal* automorphic representation of $\mathrm{GL}(n_i, \mathbb{A}_K)$, and $n_1 + n_2 + \cdots + n_m = n$. The automorphic representation $\pi_1 \boxplus \cdots \boxplus \pi_m$ of $\mathrm{GL}(n, \mathbb{A}_K)$, called the *Langlands sum* of $\pi_1, \cdots, \pi_m$, is defined to be the automorphic representation of $\mathrm{GL}(n, \mathbb{A}_K)$, which is *unique up to equivalence*, satisfying

$$L_S(s, \boxplus_{i=1}^m \pi_i) = \prod_{i=1}^m L_S(s, \pi_i).$$

The existence of the Langlands sum operation $\boxplus : \Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \times \Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \to \Pi_{\mathrm{isob}}(\mathrm{GL}(n))$ follows from the theory of Eisenstein series. The global reciprocity principle of Langlands for $\mathrm{GL}(n)$ predicts a *unique* and *"natural"* bijective correspondence

$$\begin{tikzcd} \Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \arrow[r, leftrightarrow, dotted] & \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}} \\ \Pi_{\mathrm{cusp}}(\mathrm{GL}(n)) \arrow[r, leftrightarrow, dotted] \arrow[u, hook] & \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}}_{\mathrm{irr}} \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \arrow[r, leftrightarrow, dotted] & \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o} \\ \Pi_{\mathrm{cusp}}(\mathrm{GL}(n)) \arrow[r, leftrightarrow, dotted] \arrow[u, hook] & \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o}_{\mathrm{irr}} \arrow[u, hook] \end{tikzcd}$$

K : number field $\qquad$ K : function field

between the category $\Pi_{\mathrm{isob}}(\mathrm{GL}(n))$ of equivalence classes of isobaric representations of $\mathrm{GL}(n, \mathbb{A}_K)$ and the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^o$ (resp. the category $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}}$) of equivalence classes of n-dimensional continuous semi-simple ℓ-adic (resp. n-dimensional continuous semi-simple ℓ-adic and de Rham) representations of W_K unramified at almost all $v \in \mathbb{h}_K$ in case K is a function field (resp. in case K is a number field), inducing a bijective correspondence between the full subcategory

$\Pi_{\mathrm{cusp}}(\mathrm{GL}(n))$ of $\Pi_{\mathrm{isob}}(\mathrm{GL}(n))$ consisting of the cuspidal objects of $\Pi_{\mathrm{isob}}(\mathrm{GL}(n))$ and the full subcategory $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o}_{\mathrm{irr}}$ (resp. $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}}_{\mathrm{irr}}$) of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o}$ (resp. $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}}$) consisting of the irreducible objects of $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o}$ (resp. $\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(W_K)^{o,\mathrm{dR}}$) which recovers abelian global class field theory when $n=1$, preserves the corresponding L-functions and ε-factors, which is well-behaved with respect to certain linear algebraic operations, and which is compatible with the local Langlands reciprocity principle for $\mathrm{GL}(n)$. Altough the global reciprocity principle for $\mathrm{GL}(n)$ is still conjectural if K is a number field, in case K is a function field the global correspondence for $\mathrm{GL}(n)$ is now a theorem of L. Lafforgue [31]. Thus, it would be very interesting to understand the following diagram :

$$
\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K)) \arrow[r, shift left, "\lambda_K^{(n)}"] & \mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}^{\mathrm{cont}-n}(\mathscr{J}_K^{\varphi})^{o} \arrow[l, shift left, "\lambda_K^{(n)-1}"] \\
\Pi_{\mathrm{aut}}(\mathrm{GL}(n)) \arrow[u, hook] \arrow[r, dotted, leftrightarrow] & \boxed{?} \arrow[u, hook] \\
\Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \arrow[u, hook] \arrow[r, dotted, leftrightarrow] & \boxed{?} \arrow[u, hook] \\
\Pi_{\mathrm{cusp}}(\mathrm{GL}(n)) \arrow[u, hook] \arrow[r, dotted, leftrightarrow] & \boxed{?} \arrow[u, hook]
\end{tikzcd}
$$

Definition 7.4. An n-dimensional continuous ℓ-adic Frobenius-admissible representation ρ of $\mathscr{J}_K^{\varphi}$ which is unramified at almost all $v \in \mathbb{h}_K$ is called *automorphic type* (resp. *isobaric automorphic type*, *cuspidal automorphic type*), if $\rho = \lambda_K^{(n)}(\pi)$ for some automorphic (resp. isobaric automorphic, cuspidal automorphic) representation π of $\mathrm{GL}(n,\mathbb{A}_K)$.

Now, assume that Conjecture 3.2 *holds*. Let π be an irreducible admissible smooth representation of $\mathrm{GL}(n,\mathbb{A}_K)$ such that the corresponding continuous ℓ-adic Frobenius-admissible representation $\lambda_K^{(n)}(\pi) : \mathscr{J}_K^{\varphi} \to \mathrm{GL}(n,\overline{\mathbb{Q}}_\ell)$ which is unramified at almost all $v \in \mathbb{h}_K$ satisfies the *"congruence relation"*[11]

$$\mathscr{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi));$$

[11] As Arthur points out in [2],

> "However, the condition that π be automorphic is very rigid. It imposes deep and interesting relationships among the components $\{\pi_v\}$ of π."

Also, look at the Takagi Lectures of M. Harris [14] for a detailed account on the non-abelian generalization of the "congruence conditions" appearing in abelian global class field theory.

namely, $\lambda_K^{(n)}(\pi)$ is an object in the category $\mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})^{o}_{\mathscr{N}_K^{\varphi}}$ of n-dimensional continuous ℓ-adic Frobenius-admissible representations ρ of $\mathscr{J}_K^{\varphi}$ unramified at almost all $v \in \mathbb{h}_K$ and satisfying the congruence relation

$$\mathscr{N}_K^{\varphi} \subseteq \ker(\rho).$$

Then, as $\mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})^{o}_{\mathscr{N}_K^{\varphi}}$ is equivalent to the category $\mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^{o}$ of n-dimensional continuous ℓ-adic semi-simple representations of W_K unramified at almost all $v \in \mathbb{h}_K$ by Theorem 6.3 and Remark 6.4, there exists an n-dimensional continuous semi-simple ℓ-adic representation $\mathscr{L}_K^{(n)}(\pi)$ of W_K which is unramified at almost all $v \in \mathbb{h}_K$ corresponding to $\lambda_K^{(n)}(\pi)$ and satisfying the equalities

$$L_{S_\ell}^{Artin-Hecke}(s, \lambda_K^{(n)}(\pi)) = L_{S_\ell}^{Artin-Weil}(s, \mathscr{L}_K^{(n)}(\pi))$$

and

$$\varepsilon_{S_\ell}^{Artin-Hecke}(s, \lambda_K^{(n)}(\pi)) = \varepsilon_{S_\ell}^{Artin-Weil}(s, \mathscr{L}_K^{(n)}(\pi)).$$

Thus, Theorem 7.2 has the following important consequence.

Corollary 7.5. *Assume that Conjecture 3.2 holds. Then, there exists a bijective arrow*

$$\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathscr{N}_K^{\varphi}} \arrow[r, shift left, "\lambda_K^{(n)}"] \arrow[rr, bend left, "\mathscr{L}_K^{(n)}"] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(\mathscr{J}_K^{\varphi})^{o}_{\mathscr{N}_K^{\varphi}} \arrow[l, shift left, "\lambda_K^{(n)^{-1}}"] \arrow[r, shift left, "\text{equiv.(6.2)}"] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^{o} \arrow[l, shift left, "\text{equiv.(6.2)}^{-1}"]
\end{tikzcd}$$

defined by

$$\mathscr{L}_K^{(n)} : \pi \rightsquigarrow \lambda_K^{(n)}(\pi) \rightsquigarrow \mathscr{L}_K^{(n)}(\pi),$$

and satisfying the equalities

$$L_{S_\ell}^{Langlands}(s, \pi) = L_{S_\ell}^{Artin-Weil}(s, \mathscr{L}_K^{(n)}(\pi))$$

and

$$\varepsilon_{S_\ell}^{Langlands}(s, \pi) = \varepsilon_{S_\ell}^{Artin-Weil}(s, \mathscr{L}_K^{(n)}(\pi)),$$

for each π from the category $\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathscr{N}_K^{\varphi}}$ of all irreducible admissible smooth representations π of $\mathrm{GL}(n,\mathbb{A}_K)$ satisfying the congruence relation

$$\mathscr{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi)).$$

To proceed our discussion, *first assume that K is a function field*. As the global reciprocity principle for $\mathrm{GL}(n)$ over a function field K is now a theorem of L. Lafforgue (cf. [31]), assuming that Conjecture 3.2 *holds*, the diagram

$$\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathscr{N}_K^{\varphi}} \arrow[dr, leftrightarrow, "\text{Corollary}\,7.5"] & \\
& \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^o \\
\Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \arrow[ur, leftrightarrow, "\text{Lafforgue}"'] &
\end{tikzcd}$$

and, taking into account Theorem 6.3 and Corollary 7.5, the diagram

$$\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K))^1_{\mathscr{N}_K^{\varphi}} \arrow[dr, leftrightarrow, "\text{Corollary}\,7.5"] & \\
& \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^o_{\mathrm{irr}} \\
\Pi_{\mathrm{cusp}}(\mathrm{GL}(n)) \arrow[ur, leftrightarrow, "\text{Lafforgue}"'] &
\end{tikzcd}$$

where $\Pi(\mathrm{GL}(n,\mathbb{A}_K))^1_{\mathscr{N}_K^{\varphi}}$ denotes the full subcategory of $\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathscr{N}_K^{\varphi}}$ consisting of the objects π of $\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathscr{N}_K^{\varphi}}$ for which $\lambda_K^{(n)}(\pi)$ is irreducible, yield the following equivalences on a given irreducible admissible smooth representation π of $\mathrm{GL}(n,\mathbb{A}_K)$:

Proposition 7.6. *Let K be a function field. Assume that π is an irreducible admissible smooth representation π of $\mathrm{GL}(n,\mathbb{A}_K)$. Then,*

$$\pi \textit{ is an isobaric representation of } \mathrm{GL}(n,\mathbb{A}_K) \Longleftrightarrow \mathscr{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi)),$$

and

$$\pi \textit{ is a cuspidal representation of } \mathrm{GL}(n,\mathbb{A}_K) \Longleftrightarrow \begin{cases} \mathscr{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi)), \\ \lambda_K^{(n)}(\pi) \textit{ is irreducible} \end{cases}$$

provided that Conjecture 3.2 holds.

Next, *assume that K is a number field*. Assuming that Conjecture 3.2 *holds*, we then expect the following diagrams

$$\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K))_{\mathcal{N}_K^{\varphi}} \arrow[r, leftrightarrow, "\text{Corollary 7.5}"] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^o \\
\Pi_{\mathrm{isob}}(\mathrm{GL}(n)) \arrow[u, hook] \arrow[r, leftrightarrow, "\text{Global Langlands}", "\text{Reciprocity for } \mathrm{GL}(n)"'] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^{o,\mathrm{dR}} \arrow[u, hook]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Pi(\mathrm{GL}(n,\mathbb{A}_K))^1_{\mathcal{N}_K^{\varphi}} \arrow[r, leftrightarrow, "\text{Corollary 7.5}"] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^o_{\mathrm{irr}} \\
\Pi_{\mathrm{cusp}}(\mathrm{GL}(n)) \arrow[u, hook] \arrow[r, leftrightarrow, "\text{Global Langlands}", "\text{Reciprocity for } \mathrm{GL}(n)"'] & \mathsf{Rep}^{\mathrm{cont}-n}_{\overline{\mathbb{Q}}_\ell}(W_K)^{o,\mathrm{dR}}_{\mathrm{irr}} \arrow[u, hook]
\end{tikzcd}$$

where $\Pi(\mathrm{GL}(n,\mathbb{A}_K))^1_{\mathcal{N}_K^{\varphi}}$ is defined as in the function field case. It is then natural to pose the following conjecture.

Conjecture 7.7. *Let K be a number field. Assume that Conjecture 3.2 holds. Let π be an admissible smooth representation of* $\mathrm{GL}(n,\mathbb{A}_K)$. *Then,*

(1) *The following statements are equivalent :*
- *π is an isobaric representation of* $\mathrm{GL}(n,\mathbb{A}_K)$*;*
- *the Frobenius-admissible continuous n-dimensional ℓ-adic representation $\lambda_K^{(n)}(\pi)$ of $\mathscr{J}_K^{\varphi}$ unramified at almost all $v \in \mathbb{h}_K$ satisfies*

$$\mathcal{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi))$$

  and is $\mathbb{B}_{\mathrm{dR}}$-admissible at each $v \in \mathbb{h}_K$ satisfying $\ell \mid q_v$.

(2) *The following statements are equivalent :*
- *π is a cuspidal representation of* $\mathrm{GL}(n,\mathbb{A}_K)$*;*
- *the Frobenius-admissible continuous n-dimensional ℓ-adic representation $\lambda_K^{(n)}(\pi)$ of $\mathscr{J}_K^{\varphi}$ unramified at almost all $v \in \mathbb{h}_K$ is irreducible and satisfies*

$$\mathcal{N}_K^{\varphi} \subseteq \ker(\lambda_K^{(n)}(\pi))$$

  and is $\mathbb{B}_{\mathrm{dR}}$-admissible at each $v \in \mathbb{h}_K$ satisfying $\ell \mid q_v$.

Certainly, Conjectures 3.2 and 7.7 imply the global Langlands reciprocity for $\mathrm{GL}(n)$ over number fields.

8. Langlands group L_K of a number field K

In the remaining of the text, we shall closely follow the work of Arthur [1].

It seems possible to apply the ideas developed in this work to the construction of the "hypothetical" Langlands group L_K of a given global field K, especially to the construction of the Langlands group L_K in case K is a number field, whose existence is one of the major problems in the framework of the global Langlands reciprocity and the global functoriality principles. In case K is a function field, then $L_K = W_K$ and we have already covered this case in the previous sections of this work.

Therefore, we shall from now on assume that K is a number field. Then, for each $v \in \mathbb{h}_K \cup \mathbb{a}_K$, the local Langlands group L_{K_v} is defined by

$$L_{K_v} = \begin{cases} WA_{K_v} = W_{K_v} \times \mathrm{SL}(2,\mathbb{C}), & \text{if } v \in \mathbb{h}_K; \\ W_{K_v}, & \text{if } v \in \mathbb{a}_K. \end{cases}$$

Note that, if $v \in \mathbb{h}_K$, instead of the traditional Weil-Deligne group WD_{K_v} of the non-archimedean local field K_v, we shall use the topological group $W_{K_v} \times \mathrm{SL}(2,\mathbb{C})$, which is denoted by WA_{K_v} and called the *Weil-Arthur group* of K_v (cf. Langlands [32]).

Now, introduce the "thickened" version $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ of the non-abelian idèle group $\mathscr{J}_K^{\underline{\varphi}}$ of a number field K as follows.

Definition 8.1. For each $v \in \mathbb{h}_K$ fix a Lubin-Tate splitting φ_{K_v} and let $\underline{\varphi} = \{\varphi_{K_v}\}_{v \in \mathbb{h}_K}$. The topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ defined by the "restricted free product"

$$\mathscr{W}\mathscr{A}_K^{\underline{\varphi}} := \mathop{\ast'}_{v \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SL}(2,\mathbb{C}) : {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \times \mathrm{SL}(2,\mathbb{C}) \right) \ast W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}$$

is called the *Weil-Arthur idèle group of the number field K*. The finite (=henselian) part $\mathscr{W}\mathscr{A}_{K,\mathbb{h}}^{\underline{\varphi}}$ of $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{W}\mathscr{A}_{K,\mathbb{h}}^{\underline{\varphi}} := \mathop{\ast'}_{v \in \mathbb{h}_K} \left({}_{\mathbb{Z}}\nabla_{K_v}^{(\varphi_{K_v})} \times \mathrm{SL}(2,\mathbb{C}) : {}_1\nabla_{K_v}^{(\varphi_{K_v})^{\underline{0}}} \times \mathrm{SL}(2,\mathbb{C}) \right),$$

and the infinite (=archimedean) part $\mathscr{W}\mathscr{A}_{K,\mathbb{a}}^{\underline{\varphi}}$ of $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is defined by

$$\mathscr{W}\mathscr{A}_{K,\mathbb{a}}^{\underline{\varphi}} := W_{\mathbb{R}}^{\ast r_1} \ast W_{\mathbb{C}}^{\ast r_2}.$$

Here, as usual r_1 and r_2 denote the number of real and the number of pairs of complex-complex conjugate embeddings of the global field K in $\mathbb{C}$.

The topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is an "extremely big" group, whose definition *depends only on K*.

The following theorem describes the abelianization $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}\,ab}$ of the topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$.

Theorem 8.2. *The abelianization $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}\,ab}$ of the topological group $\mathscr{W}\mathscr{A}_K^{\underline{\varphi}}$ is indeed $\mathbb{J}_K$.*

Proof. The proof follows by first noting that the direct limit functor is exact and then by abelianizing free products of groups. □

Remark 8.3. From now on, unless otherwise stated, we shall assume the existence of the global Langlands group L_K of a number field K.

For $v \in \mathbb{h}_K \cup \mathbb{a}_K$, as before, choose an embedding

$$e_v : K^{sep} \hookrightarrow K_v^{sep}.$$

This embedding determines a continuous homomorphism[12]

$$e_v^{\text{Langlands}} : L_{K_v} \to L_K,$$

and therefore, for each $v \in \mathbb{h}_K$, a continuous homomorphism

$$\mathsf{NR}_{K_v}^{(\varphi_{K_v})^{\text{Langlands}}} : {}_{\mathbb{Z}}\nabla_{K_v}^{\varphi_{K_v}} \times \mathrm{SL}(2,\mathbb{C}) \xrightarrow[\sim]{\{\cdot,K_v\}_{\varphi_{K_v}} \times \mathrm{id}_{\mathrm{SL}(2,\mathbb{C})}} WA_{K_v} \xrightarrow{e_v^{\text{Langlands}}} L_K.$$

Theorem 8.4 (Ultimate global non-abelian norm-residue symbol "Weak form"). *There exists a well-defined continuous homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K, \tag{8.1}$$

which satisfies

$$(\mathsf{NR}_K^{\varphi^{\text{Langlands}}})_S = \mathsf{NR}_K^{\varphi^{\text{Langlands}}} \circ c_S : (\mathscr{W}\mathscr{A}_K^{\varphi})_S \xrightarrow{c_S} \mathscr{W}\mathscr{A}_K^{\varphi} \xrightarrow{\mathsf{NR}_K^{\varphi^{\text{Langlands}}}} L_K,$$

where, following Proposition 2.1, $c_S : (\mathscr{W}\mathscr{A}_K^{\varphi})_S \to \mathscr{W}\mathscr{A}_K^{\varphi}$ is the canonical homomorphism defined for every finite subset S of $\mathbb{h}_K \cup \mathbb{a}_K$ containing $\mathbb{a}_K$.

We conjecture that the continuous homomorphism $\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$ should be considered as the *"ultimate form" of the global non-abelian norm-residue symbol of K*. More precisely, we pose the following conjecture (to be precise, the following meta-conjecture).

Conjecture 8.5 (Ultimate global non-abelian norm-residue symbol "Strong form"). *The homomorphism*

$$\mathsf{NR}_K^{\varphi^{\text{Langlands}}} : \mathscr{W}\mathscr{A}_K^{\varphi} \to L_K$$

is open, continuous and surjective.

Finally, we have the following remark.

Remark 8.6. (1) In fact, via the same lines of reasoning of this work, we can study the relationship between the automorphic representations π of a general reductive group G over the number field K with the global Langlands parameters $\phi : L_K \to {}^L G$ of G, which is the content of the *global reciprocity principle of Langlands for a general reductive group G*.

(2) It would be interesting to compare Arthur's construction of L_K, which uses the classification of automorphic representations of Langlands in the sense of "beyond endoscopy" [34], with the topological group $\mathscr{W}\mathscr{A}_K^{\varphi}$ constructed in this paper. This comparison may reveal an *unconditional* definition of the global Langlands group L_K of the number field K.

[12] which is unique up to conjugacy.

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References

[1] J. Arthur, *A note on the automorphic Langlands group*, Canad. Math. Bull. **45**(4), 2002, 466–482.

[2] J. Arthur, *The principle of functoriality*, Bulletin (New Series) of the A.M.S **40**(1), 2002, 39–53.

[3] S. Bedikyan, *Abelian local class field theory*, Workshop on Algebraic Number Theory and L-functions, August 29-September 04 2010, Akdeniz Üniversitesi Adrasan Eğitim ve Uygulama Merkezi, Adrasan, Antalya, Turkey.

[4] L. Berger, *An introduction to the theory of p-adic representations*, Geometric Aspects of Dwork Theory (A. Adolphson, F. Baldassari, P. Berthelot, N. Katz, F. Loeser eds.) Vol. I, De Gruyter Proc. in Math., Walter de Gruyter, Berlin and New York, 2004, 255–292.

[5] K. Buzzard and T. Gee, *The conjectural connections between automorphic representations and Galois representations*, `arXiv:1009.0785`.

[6] L. Clozel, *Motifs et formes automorphes : Applications du principe de fonctorialité*, Automorphic Forms, Shimura Varieties, and L-functions (L. Clozel and J. S. Milne, eds.) Vol. I, Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6-16, 1988, Perspectives in Math. **11**, Academic Press, Boston, 1990, 77–160.

[7] J. W. Cogdell, *L-functions and functoriality*, CIMPA Research School on Automorphic Forms and L-functions, Shandong University, August 1-14 2010, 45 pages. `<http://www.math.osu.edu/~cogdell/cimpa-www.pdf>`, acces date: March 21, 2011 at 13:37.

[8] Ch. Deninger, *Motivic L-functions and regularized determinants*, Motives (U. Jannsen, S. Kleiman and J.-P. Serre eds.), Proc. Sympos. Pure Math. **55** Part 1, Amer.Math. Soc., Providence R.I., 1994, 707–744.

[9] D. Flath, *Decomposition of representations into tensor products*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman eds.), Proc. Sympos. Pure Math. **33** Part 1, Amer. Math. Soc., Providence , R.I., 1979, 179–183.

[10] J.-M. Fontaine, *Representations p-adiques semi-stables*, Periodes p-adiques, Bures-sur-Yvette 1988, Asterisque **223**, 1994, 113–184.

[11] J.-M. Fontaine, *Representations ℓ-adiques potentiellement semi-stables*, Periodes p-adiques, Bures-sur-Yvette 1988, Asterisque **223**, 1994, 321–347.

[12] J.-M. Fontaine and B. Mazur, *Geometric Galois representations*, Elliptic Curves, Modular Forms & Fermat’s Last Theorem, 2^{nd} Edition (J. H. Coates and S. T. Yau eds.),

Proc. of a Conference on Elliptic Curves and Modular Forms, Hong Kong, December 18–21, 1993, Ser. Number Theory **I**, International Press, Cambridge, M.A., 1995, 190–227.

[13] E. Frenkel, *Recent advances in the Langlands program*, Bulletin (New Series) of the A.M.S. **41**(2), 2004, 151–184.

[14] M. Harris, *Arithmetic applications of the Langlands program*, Japanese J. Math. (3rd series) **5**, 2010, 1–71.

[15] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Math. Studies **151**, Princeton University Press, Princeton N.J., 2001.

[16] W. Hebisch and W. Młotkowski, *Irreducible representations of the free products of groups*, Indiana Univ. Math. Journal **59** (1), 2010, 131–182.

[17] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps* p*-adique*, Invent. Math. **139**, 2000, 439–455.

[18] P. Hilton and Y.-C. Wu, *A Course in Modern Algebra*, Wiley-Interscience Publ., New York, 1989.

[19] K. I. Ikeda, *On the local Artin conductor* $\mathfrak{f}_{\mathrm{Artin}}(\chi)$ *of a character* χ *of* $\mathrm{Gal}(E/K)$*– II: Main results for the metabelian case*, Proc. Indian Acad. Sci. (Math. Sci.) **113** (2), 2003, 99–137.

[20] K. I. Ikeda, *Langlands functoriality principle*, Workshop on Algebraic Number Theory and *L*-functions, August 29-September 04 2010, Akdeniz Üniversitesi Adrasan Eğitim ve Uygulama Merkezi, Adrasan, Antalya, Turkey.

[21] K. I. Ikeda, S. Kazancıoğlu, and E. Serbest, *On the relationship between the generalized Fesenko and the Laubie reciprocity maps*, submitted.

[22] K. I. Ikeda and Ş. Özden, *On the local* ε*-factors of Weil-Deligne representations*, in preparation.

[23] K. I. Ikeda and E. Serbest, *Non-abelian local reciprocity law*, manuscripta math. **132**, 2010, 19–49.

[24] K. I. Ikeda and E. Serbest, *Ramification theory in non-abelian local class field theory*, Acta Arithmetica **144**, 2010, 373–393.

[25] T. Ito, *On motives and* ℓ*-adic Galois representations associated to automorphic representations of* $\mathrm{GSp}(4)$, The 9^{th} Autumn Workshop on Number Theory, November 6–10, 2006, Hakuba Highmount Hotel, Hakuba, Nagano, Japan.

[26] K. Iwasawa, *On solvable extensions of algebraic number fields*, Annals of Math. **58**, 1953, 548–572.

[27] H. Jacquet and J. Shalika, *Euler products and the classification of automorphic forms II*, Amer. J. of Math. **103**(4), 1981, 777–815.

[28] M. M. Kapranov, *Noncommutative geometry and path integrals*, Algebra, Arithmetic, and Geometry: In Honour of Yu. I. Manin Volume II (Yu. Zarkhin, Yu. Tschinkel eds.), Progress in Math. **270**, Birkhäuser, Boston, 2009, 49–88.

[29] C. Khare, M. Larsen, and R. Ramakrishna, *Transcendental* ℓ*-adic Galois representations*, Math. Res. Lett. **12**, 2005, 685–700.

[30] A. W. Knapp, *Local Langlands correspondence : The archimedean case*, Motives (U. Jannsen, S. Kleiman and J.-P. Serre eds.), Proc. Sympos. Pure Math. **55** Part 2, Amer.Math. Soc., Providence R.I., 1994, 393–410.

[31] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147**(1), 2002, 1–242.

[32] R. P. Langlands, *Functoriality and Reciprocity*, Two Lectures at the Institute for Advanced Studies, Seminar Notes, March 2011.

[33] R. P. Langlands, *A review of Haruzo Hida's p-adic automorphic forms on Shimura varieties*, Bulletin (New Series) of the A.M.S. **44**(2), 2007, 291–308.

[34] R. P. Langlands, *Beyond endoscopy*, Contributions to Automorphic Forms, Geometry, and Number Theory : A Volume in Honor of Joseph Shalika (H. Hida, D. Ramakrishnan, and F. Shahidi eds), The Johns Hopkins University Press, Baltimore MD, 2004, 611–698.

[35] R. P. Langlands, *Automorphic representations, Shimura varieties, and motives. Ein Märchen*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman eds.), Proc. Sympos. Pure Math. **33** Part 2, Amer. Math. Soc., Providence , R.I., 1979, 205–246.

[36] R. P. Langlands, *L-functions and automorphic representations*, Proceedings of the International Congress of Mathematicians, Helsinki, 1978, 165–175.

[37] F. Laubie, *Une théorie non abélienne du corps de classes local*, Compositio Math. **143**, 2007, 339–362.

[38] G. Laumon, M. Rapoport, U. Stuhler, *$\mathscr{D}$-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113**, 1993, 217–338.

[39] Yu. I. Manin and A. A. Panchishkin, *Introduction to Modern Number Theory : Fundamental Problems, Ideas and Theories (2^{nd} ed.)*, Encyclopedia of Mathematical Sciences **49**, Springer-Verlag, Berlin and Heidelberg, 2005.

[40] K. Miyake, *The arithmetic structure of the Galois group of the maximal nilpotent extension of an algebraic number field*, Proc. Japan Acad. **67** Ser. A, 1991, 55–59.

[41] K. Miyake, *Galois theoretic local-global relations in nilpotent extensions of algebraic number fields*, Séminaire de théorie des nombres, Paris 1989–90 (Sinnou David ed.), Progress in Math. **102**, Birkhäuser, Boston, 1992, 191–207.

[42] W. Młotkowski, *Irreducible representations of free products of infinite groups*, Colloquium Math. **LXIX** (2), 1995, 193–211.

[43] S. A. Morris, *Free products of topological groups*, Bull. Austral. Math. Soc. **4**, 1971, 17–29.

[44] J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields* (2nd ed.), Grundlehren der mathematischen Wissenschaften **323**, Springer-Verlag, Berlin and Heidelberg, 2008.

[45] B. C. Ngô, *Endoscopy theory of automorphic forms*, Proceedings of the International Congress of Mathematicians, Hyderabad, 2010, 210–238.

[46] Ş. Özden, *Abelian global class field theory*, Workshop on Algebraic Number Theory and L-functions, August 29-September 04 2010, Akdeniz Üniversitesi Adrasan Eğitim ve Uygulama Merkezi, Adrasan, Antalya, Turkey.

[47] I. Piatetski-Shapiro, *Multiplicity one theorems*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman eds.), Proc. Sympos. Pure Math. **33** Part 1, Amer. Math. Soc., Providence , R.I., 1979, 209–212.

[48] J. D. Rogawski, *Functoriality and the Artin conjecture*, Representation Theory and Automorphic Forms : An Instructional Conference at the International Centre for Mathematical Sciences, Edinburgh, 1996 (T. Bailey and A. Knapp eds.), Proc. Sympos. Pure Math. **61**, A.M.S., Providence R.I., 1997, 331–353.

[49] P. Scholze and S. W. Shin, *On the cohomology of compact unitary group Shimura varieties at ramified split places*, preprint, Bonn, 2011.

[50] E. Serbest, *Non-abelian local class field theory in the sense of Koch*, Workshop on Algebraic Number Theory and L-functions, August 29-September 04 2010, Akdeniz Üniversitesi Adrasan Eğitim ve Uygulama Merkezi, Adrasan, Antalya, Turkey.

[51] J. Tate, *Number theoretic background*, Automorphic Forms, Representations, and L-functions (A. Borel and W. Casselman eds.), Proc. Sympos. Pure Math. **33** Part 2, Amer. Math. Soc., Providence, R.I., 1979, 3–26.

[52] R. Taylor, *Galois representations*, Annales de la Faculté des Sciences de Toulouse **13**, 2004, 73–119.

[53] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables : A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups*, CRM Monograph Series **1**, Amer. Math. Soc., Providence, R.I., 1992.

[54] J.-P. Wintenberger, *On p-adic geometric representations of $G_{\mathbb{Q}}$*, Doc. Math., Extra Vol.: John H. Coates' Sixtieth Birthday (I. Fesenko, S. Lichtenbaum, B. Perrin-Riou, P. Schneider eds.), 2006, 819–827.

Kâzım İlhan İkeda
Yeditepe University
Department of Mathematics
İnönü Mah., Kayışdağı Cad.
26 Ağustos Yerleşimi
34755 Ataşehir, Istanbul
Turkey
e-mail: `ilhan.ikeda@yeditepe.edu.tr`