AUTOMORPHIC L-FUNCTIONS

A. BOREL

This paper is mainly devoted to the $L$-functions attached by Langlands [35] to an irreducible admissible automorphic representation $\pi$ of a reductive group $G$ over a global field $k$ and to local and global problems pertaining to them. In the context of this Institute, it is meant to be complementary to various seminars, in particular to the $GL_2$-seminars, and to stress the general case. We shall therefore start directly with the latter, and refer for background and motivation to other seminars, or to some expository articles on this topic in general [3] or on some aspects of it [7], [14], [15], [23].

The representation $\pi$ is a tensor product $\pi = \bigotimes_v \pi_v$ over the places of $k$, where $\pi_v$ is an irreducible admissible representation of $G(k_v)$ [11]. Accordingly the $L$-functions associated to $\pi$ will be Euler products of local factors associated to the $\pi_v$'s. The definition of those uses the notion of the $L$-group $^LG$ of, or associated to, $G$. This is the subject matter of Chapter I, whose presentation has been much influenced by a letter of Deligne to the author. The $L$-function will then be an Euler product $L(s, \pi, r)$ assigned to $\pi$ and to a finite dimensional representation $r$ of $^LG$.

(If $G = GL_n$ then the $L$-group is essentially $GL_n(\mathbb{C})$, and we may tacitly take for $r$ the standard representation $r_n$ of $GL_n(\mathbb{C})$, so that the discussion of $GL_n$ can be carried out without any explicit mention of the $L$-group, as is done in the first six sections of [3].) The local $L$- and $\varepsilon$-factors are defined at all places where $G$ and $\pi$ are “unramified” in a suitable sense, a condition which excludes at most finitely many places. Chapter II is devoted to this case. The main point is to express the Satake isomorphism in terms of certain semisimple conjugacy classes in $^LG$ (7.1). At this time, the definition of the local factors at the ramified places is not known in general. For $GL_n$ and $r_n$, however, there is a direct definition [19], [25]. In the general case, the most ambitious scheme is to associate canonically to an irreducible admissible representation of a reductive group $H$ over a local field $E$ a representation of the Weil-Deligne group $W'_E$ of $E$ into $^lH$, and then use $L$- and $\varepsilon$-factors associated to representations of $W'_E$ [60]. This problem is the main topic of Chapter III.

The $L$-function $L(s, \pi, r)$ associated to $\pi$ and $r$ as above is introduced in §13. In fact, it is defined in general as a product of local factors indexed by almost all places of $k$. It converges absolutely in some right half-plane (13.3; 14.2). Some of the main

conjectural analytic properties (meromorphic continuation, functional equation), and the evidence known so far, are discussed in §14.

From the point of view of [35], a great many problems on automorphic representations and their $L$-functions are special cases of one, the so-called lifting problem or problem of functoriality with respect to $L$-groups. It is discussed in Chapter V. It is closely connected with Artin's conjecture (see §17 and the base-change seminar [17]). In §18 brief mention is made of some known or conjectured relations between automorphic $L$-functions and the Hasse-Weil zeta-function of certain varieties, to be discussed in more detail in the seminars on Shimura varieties [8], [40].

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CHAPTER I. $L$-GROUPS.

$k$ is a field, $\bar{k}$ an algebraic closure of $k$, $k_0$ the separable closure of $k$ in $\bar{k}$, and $\Gamma_k$ the Galois group of $k_0$ over $k$. $G$ is a connected reductive group, over $\bar{k}$ in 1.1, 1.2, 2.1, 2.2, over $k$ otherwise.

§§1, 2 will be used throughout, §3 from Chapter III on. The reader willing to take on faith various statements about restriction of scalars need not read §§4, 5.
1. Classification. We recall first some facts discussed in [58].

1.1. There is a canonical bijection between isomorphism classes of connected reductive \( k \)-groups and isomorphism classes of root systems. It is defined by associating to \( G \) the root datum \( \phi(G) = (X^*(T), \varphi, X_a(T), \varphi^a) \) where \( T \) is a maximal torus of \( G \), \( X^*(T) \) (\( X_a(T) \)) the group of characters (1-parameter subgroups) of \( T \) and \( \Phi (\Phi^a) \) the set of roots (coroots) of \( G \) with respect to \( T \).

1.2. The choice of a Borel subgroup \( B \supseteq T \) is equivalent to that of a basis \( \Delta \) of \( \Phi(G, T) \). The previous bijection yields one between isomorphism classes of triples \( (G, B, T) \) and isomorphism classes of based root data \( \phi_0(G) = (X^*(T), \Delta, X_a(T), \Delta^a) \). There is a split exact sequence

\[
1 \longrightarrow \text{Int} G \longrightarrow \text{Aut} G \longrightarrow \text{Aut} \phi_0(G) \longrightarrow 1.
\]

To get a splitting, we may choose \( x_\alpha \in G_a \) (\( \alpha \in \Delta \)) and then have a canonical bijection

\[
\text{Aut} \phi_0(G) \to \text{Aut} (G, B, T, \{x_\alpha\}_{\alpha \in \Delta}).
\]

Two such splittings differ by an inner automorphism \( \text{Int} \).

1.3. Given \( \gamma \in \Gamma_k \) there is \( g \in G(k) \) such that \( g^{-1} T g = T \), \( g^{-1} B g = B \), whence an automorphism of \( \phi_0(G) \), which depends only on \( \gamma \). We let \( \mu_G : \Gamma_k \to \text{Aut} \phi_0(G) \) be the homomorphism so defined. If \( G' \) is a \( k \)-group which is isomorphic to \( G \) over \( k \) (hence over \( k \)), then \( \mu_G = \mu_G' \iff G, G' \) are inner forms of each other.

1.4. Let \( f : G \to G' \) be an automorphism, whose image is a normal subgroup. Then \( f \) induces a map \( \phi(f) : \phi(G) \to \phi(G') \) (contravariant (resp. covariant) in the first (last) two arguments). Given \( B, T \subseteq G \) as above, there exists a Borel subgroup \( B' \) (resp. a maximal torus \( T' \)) of \( G' \) such that \( f(B) \subseteq B', f(T) \subseteq T' \), whence also a map \( \phi_0(f) : \phi_0(G) \to \phi_0(G') \).

2. Definition of the \( L \)-group.

2.1. The inverse system \( \Psi^\gamma \) to the based root datum \( \Psi_0 = (M, \Delta, M^*, \Delta^a) \) is \( \Psi_0^\gamma = (M^*, \Delta^a, M, \Delta) \). To the \( k \)-group \( G \) we first associate the group \( LG_0 \) over \( C \) such that \( \phi_0(LG_0) = \phi_0(G)^\vee \). We let \( LG^\vee = LG_0 \) be the maximal torus and Borel subgroup defined by \( \phi_0^\vee \), and say they define the canonical splitting of \( LG_0 \).

Let \( f \) be as in 1.4. Then \( f \) also induces a map \( \phi_0^\vee(f) : \phi_0(G)^\vee \to \phi_0(G')^\vee \). An algebraic group morphism of \( LG^\vee \) into \( LG_0 \) associated to it will be denoted \( LF^\vee \). Given one, any other is of the form \( \text{Int} t^{-1} f \text{Int} t' \) (\( t, t' \in L^\vee \)), and maps \( L^\vee \) (resp. \( L^\vee B \)) into \( L^\vee \) (resp. \( L^\vee B^\vee \)).

2.2. Examples. (1) Let \( G = \text{GL}_n \). Then \( LG^\vee = \text{GL}_n \). In fact, let \( M = \mathbb{Z}^n \) with \( \{x_i\} \) its canonical basis. Let \( \{e_i\} \) be the dual basis of \( M^* = \mathbb{Z}^n \). Then \( \Psi_0(\text{GL}_n) = (M, \Delta, M^*, \Delta^a) \) with \( \Delta = \{x_i - x_{i+1} \}, 1 \leq i < n \}, \Delta^a = \{e_i - e_{i+1} \}, 1 \leq i < n \}, \) hence \( \phi_0 = \phi_0^\vee \).

(2) Let \( G \) be semisimple and \( \Psi_0(G) = (M, \Phi, M^*, \Phi^a) \). As usual, let \( P(\Phi) \subseteq M \otimes \mathbb{Q} \) be the lattice of weights of \( \Phi \) and \( Q(\Phi) \) the group generated by \( \Phi \) in \( M \). Define \( P(\Phi^a) \) and \( Q(\Phi^a) \) similarly.

As is known \( G \) is simply connected (resp. of adjoint type) if and only if \( P(\Phi) = M \) (resp. \( Q(\Phi) = M \)). Moreover

\[
P(\Phi) = \{ \lambda \in M \otimes \mathbb{Q} | \langle \lambda, \Phi \rangle \in \mathbb{Z} \}, \quad P(\Phi^a) = \{ \lambda \in M^* \otimes \mathbb{Q} | \langle \lambda, \Phi \rangle \in \mathbb{Z} \}.
\]
Therefore:

$G$ simply connected $\iff L^G^\circ$ of adjoint type;

$G$ of adjoint type $\iff L^G^\circ$ simply connected.

(3) Let $G$ be simple. Up to central isogeny, it is characterized by one of the types $A_n, \ldots, G_2$ of the Killing-Cartan classification. It is well known that the map $\hat{\phi}_0(G) \to \hat{\phi}_0(G^\nu)$ permutes $B_n$ and $C_n$ and leaves all other types stable. Thus if $G = \text{Sp}_{2n}$ (resp. $G = \text{PSp}_{2n}$), then $L^G^\circ = SO_{2n+1}$ (resp. $L^G^\circ = \text{Spin}_{2n+1}$). In all other cases, $G \mapsto L^G^\circ$ preserves the type (but goes from simply connected group to adjoint group, and vice versa).

(4) Let again $G$ be reductive and let $f : G \to G'$ be a central isogeny. Let

$$N = \text{coker} \ f^* : X^\ast(T) \longrightarrow X^\ast(T') \quad (T' = f(T)),$$

$$N' = \text{coker} \ f^* : X^\ast(T') \longrightarrow X^\ast(T).$$

Then $N$ and $N'$ are isomorphic and $\ker L^f^\circ = \text{Hom}(N, C^\ast) \cong N$. In particular, $L^f^\circ$ is an isomorphism if and only if $f$ is one.

(5) Let $f : G \to G'$ be a central surjective morphism, $Q = \ker f$, and $Q^\circ$ the identity component of $Q$. Then $\ker L^f^\circ \cong Q/Q^\circ$.

If $Q$ is connected, then $T'' = T \cap Q$ is a maximal torus of $Q$, and the injectivity of $L^f^\circ$ follows from the fact that the exact sequence $1 \to T'' \to T \to T' \to 1$ necessarily splits. If $Q$ is not connected, then $r : H = G/Q^\circ \to G'$ is a nontrivial separable isogeny, with kernel $Q/Q^\circ$. $L^f^\circ$ factors through $L^r^\circ$ and, by the first part and (4), $\ker L^f^\circ = \ker L^r^\circ \cong Q/Q^\circ$. In particular, if we apply this to the case where $G' = G_\text{ad}$ is the adjoint group of $G$, and use (2), we see that the derived group of $L^G^\circ$ is simply connected if and only if the center of $G$ is connected. As an example, let $G = \text{GSp}_{2n}$ be the group of symplectic similitudes on a $2n$-dimensional space. Then the derived group of $L^G^\circ$ is isomorphic to $\text{Spin}_{2n+1}$. In fact, we have $L^G^\circ = (\text{GL}_1 \times \text{Spin}_{2n+1})/A$ where $A = \{1, a\}$ and $a = (a_1, a_2)$, with $a_1$ of order two in $\text{GL}_1$ and $a_2$ the nontrivial central element of $\text{Spin}_{2n+1}$. If $n = 2$, then $\text{Spin}_{2n+1} = \text{Sp}_{2n}$. It follows that if $G = \text{GSp}_2$, then $L^G^\circ = \text{GSp}_2(C)$.

2.3. We have canonically $\text{Aut} \hat{\phi}_0 = \text{Aut} \hat{\phi}_0^\circ$. Therefore we may view $\mu_G$ as a homomorphism of $\Gamma_k$ into $\text{Aut} \hat{\phi}_0^\circ$. Choose a monomorphism

$$\text{Aut} \hat{\phi}_0^\circ \longrightarrow \text{Aut} (L^G^\circ, L^B^\circ, L^T^\circ)$$

as in 1.2(2). We have then a homomorphism

$$\mu_G : \Gamma_k \longrightarrow \text{Aut} (L^G^\circ, L^B^\circ, L^T^\circ).$$

The associated group to, or $L$-group of, $G$ is then by definition the semidirect product

$$L(G/k) = L^G = L^G^\circ \rtimes \Gamma_k,$$

with respect to $\mu_G$. We note that $\mu_G$ is well defined up to an inner automorphism by an element of $L^T^\circ$. The group $L^G$ is viewed as a topological group in the obvious way. The canonical splitting of $L^G^\circ$ (2.1) is stable under $\Gamma_k$.

We have a canonical projection $L^G \to \Gamma_k$ with kernel $L^G^\circ$. The splittings of the exact sequence

$$1 \longrightarrow L^G^\circ \longrightarrow L^G \longrightarrow \Gamma_k \longrightarrow 1$$
defined as in 1.2 via an isomorphism Aut $\mathcal{W}_G \cong \text{Aut}(L^G, L^B, L^T, \{x_a\})$ are called admissible. They differ by inner automorphisms $\text{Int} t(t \in L^T)$. Note that if $G$ splits over $k$, then $\Gamma_k$ acts trivially on $L^G$ and $L^G$ is simply the direct product of $L^G$ and $\Gamma_k$.

2.4. REMARKS. (1) So far, we can in this definition take $L^G$ over any field. We have chosen $C$ since this is the most important case at present, but it is occasionally useful to use other local fields.

(2) There are various variants of this notion, which may be more convenient in certain contexts. For instance we can divide $\Gamma_k$ by a closed normal subgroup which acts trivially on $w_v$, hence on $L^G_0$, e.g., by $\Gamma_{k'}$ if $k'$ is a Galois extension of $k$ over which $G$ splits. Then $\Gamma_k$ is replaced by $\text{Gal}(k'/k)$, and $L^G$ is a complex reductive Lie group.

We can also define a semidirect product $L^G_0 \rtimes \Sigma$, for any group $\Sigma$ endowed with a homomorphism into $\Gamma_k$, e.g., the Weil group of $k$, if $k$ is a local or global field. In that case, we get the “Weil form” of $L^G$.

(3) Let $G'$ be a $k$-group which is isomorphic to $G$ over $k$. Then $G$ and $G'$ are inner forms of each other if and only if $L^G$ is isomorphic to $L^{G'}$ over $\Gamma_k$. In fact, the first condition is equivalent to $\mu_G = \mu_{G'}$, and the latter is easily seen to be equivalent to the second condition. In particular, since two quasi-split groups over $k$ which are inner forms of each other are isomorphic over $k$, it follows that if $G, G'$ are quasi-split and $L^G \cong L^{G'}$ over $\Gamma_k$, then $G$ and $G'$ are $k$-isomorphic.

2.5. Functoriality. Let $f: G \to G'$ be a $k$-morphism whose image is a normal subgroup. Then $f_\phi: \phi_0(G) \to \phi_0(G')$ clearly commutes with $\Gamma_k$, hence so does $f_\phi^\mathrm{v}: \phi_0(G')^v \to \phi_0(G)^v$ and $f^\phi: L^{G'} \to L^G$. We get therefore a continuous homomorphism $Lf: L^G \to L^G$ such that

$$ L^G \xrightarrow{Lf} L^G \xrightarrow{\mu_G} L^G \xrightarrow{\phi_0} \phi_0(G) $$

is commutative, which extends $Lf$.

2.6. Representations. For brevity, by representation of $L^G$ we shall mean a continuous homomorphism $r: L^G \to \text{GL}_m(C)$ whose restriction to $L^G_0$ is a morphism of complex Lie groups.

Clearly, $\ker r$ always contains an open subgroup of $\Gamma_k$, hence $r$ factors through $L^G_0 \rtimes \Gamma_{k'/k}$, where $k'$ is a finite Galois extension of $k$ over which $G$ splits. The group $L^G_0 \rtimes \Gamma_{k'/k}$ is canonically a complex algebraic group and $r$ is a morphism of complex algebraic groups.

3. Parabolic subgroups.

3.1. Notation. We let $\mathcal{P}(G/k)$ denote the set of parabolic $k$-subgroups of $G$, and write $\mathcal{P}(G)$ for $\mathcal{P}(G/k)$. Let $p(G/k)$ be the set of conjugacy classes (with respect to $G(k)$ or $G(\bar{k})$, it is the same) of parabolic $k$-subgroups, and $p(G) = p(G/k)$. Let $p(G)_k$ be the set of conjugacy classes of parabolic subgroups which are defined over $k$ (i.e., if $P \in \sigma \in p(G)_k$, then $\gamma P \sigma$ for all $\gamma \in \Gamma_k$). In particular $p(G/k) \subseteq p(G)_k$. There is equality if $G$ is quasi-split/k.

3.2. We recall there is a canonical bijection between $p(G)$ and the subsets of $\mathcal{A}$. 
Then \( p(G)_k \) corresponds to the \( I'_{k^*} \)-stable subsets of \( \Delta \) and \( p(G/k) \) to those \( I'_{k^*} \)-stable subsets which contain the set \( \Delta_0 \) of simple roots of a Levi subgroup of a minimal parabolic \( k \)-group. In particular we have \( p(G/k) = p(G)_k \) if \( G \) is quasi-split over \( k \). Given \( P \in \mathcal{P}(G) \), we let \( J(P) \) be the subset of \( \Delta \) assigned to the class of \( P \).

Since two conjugate parabolic subgroups whose intersection is a parabolic subgroup are identical, we see in particular that if \( P \) is defined over \( k \), \( P' \sim P \), and the class of \( P' \) is defined over \( k \), then \( P' \) is defined over \( k \).

3.3. Parabolic subgroups of \( \mathcal{L} G \). A closed subgroup \( P \) of \( \mathcal{L} G \) is parabolic if \( \gamma_0(P) = I'_{k^*} \) and \( P^o = \mathcal{L} G^o \cap P \) is a parabolic subgroup of \( \mathcal{L} G^o \). Then \( P = N_{\mathcal{L} G}(P^o) \). In other words, a parabolic subgroup is the normalizer of a parabolic subgroup \( P^o \) of \( \mathcal{L} G^o \), provided the normalizer meets every class modulo \( \mathcal{L} G^o \). We say \( P \) is standard if it contains \( \mathcal{L} B \). The standard parabolic subgroups are the subgroups

\[
\mathcal{L} P^o \cong I'_{k^*},
\]

where \( \mathcal{L} P^o \) runs through the standard parabolic subgroup of \( \mathcal{L} G^o \) such that \( J(\mathcal{L} P^o) \subset \Delta^\vee \) is stable under \( I'_{k^*} \).

Every parabolic subgroup of \( \mathcal{L} G \) is conjugate (under \( \mathcal{L} G \) or, equivalently, \( \mathcal{L} G^o \)) to one and only one standard parabolic subgroup.

We let \( \mathcal{P}(\mathcal{L} G) \) be the set of parabolic subgroups of \( \mathcal{L} G \) and \( \mathcal{P}(\mathcal{L} G^o) \) the set of their conjugacy classes.

The given bijection \( \Delta \leftrightarrow \Delta^\vee \) yields then, in view of 3.2, a bijection

\[
\mathcal{P}(\mathcal{L} G) \leftrightarrow \mathcal{P}(\mathcal{L} G^o).
\]

We shall say that a parabolic subgroup of \( \mathcal{L} G \) is relevant if its class corresponds to one of \( p(G/k) \) under this map. We let \( \mathcal{P}(\mathcal{L} G) \) be the set of relevant parabolic subgroups and \( \mathcal{P}(\mathcal{L} G^o) \) the set of their conjugacy classes, the relevant conjugacy classes of parabolic subgroups. Thus, by definition

\[
p(G/k) \leftrightarrow \mathcal{L} p(\mathcal{L} G^o).
\]

Thus, if \( G \) and \( G' \) are inner forms of each other, \( \mathcal{L} p(\mathcal{L} G) \) and \( \mathcal{L} p(\mathcal{L} G') \) are the same, but \( \mathcal{L} p(\mathcal{L} G) \) and \( \mathcal{L} p(\mathcal{L} G') \) are not. If \( G' \) is quasi-split, then \( \mathcal{L} p(\mathcal{L} G') \cong \mathcal{L} p(\mathcal{L} G) \); hence we have an injection

\[
\mathcal{L} p(\mathcal{L} G) \subset \mathcal{L} p(\mathcal{L} G') = p(\mathcal{L} G').
\]

If \( \mathcal{O} G \) is anisotropic over \( k \), then \( \mathcal{L} p(\mathcal{L} G) \) consists of \( G \) alone.

3.4. Levi subgroups. Let \( P \) be a parabolic subgroup of \( \mathcal{L} G \). The unipotent radical \( N \) of \( P^o \) is normal in \( P \) and will also be called the unipotent radical of \( P \). Then \( P/N \cong P^o/N \cong I'_{k^*} \). In fact, it follows from (1) that \( P \) is a split extension of \( N \), and is the semidirect product of \( N \) by the normalizer in \( P \) of any Levi subgroup \( M^o \) of \( P^o \). Those normalizers will be called the Levi subgroups of \( P \).

Let \( P \in \mathcal{P}(\mathcal{L} G/k) \), \( M \) a Levi \( k \)-subgroup of \( P \). Let \( \mathcal{L} P \) be the standard parabolic subgroup in the class associated to that of \( P \) (see (3)). Then \( \mathcal{L} M \) may be identified to a Levi subgroup of \( \mathcal{L} P \). In fact if \( M \) corresponds to \( (X^*(T), J, X_*(T), J^o) \), then \( \mathcal{L} M^o \) corresponds to \( (X_*(T), J^o, X^*(T), J) \) and \( \mathcal{L} M^o \cong I'_{k^*} \) is equal to \( \mathcal{L} M \) by definition and is a Levi subgroup of \( \mathcal{L} P \), as defined above.

A Levi subgroup of a parabolic subgroup \( P \) of \( \mathcal{L} G \) is relevant if \( P \) is.
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For the sake of brevity, we shall sometimes say "Levi subgroup in \( G \)" for "Levi subgroup of a parabolic subgroup of \( G \)." Similarly for \( \mathcal{L}G \).

3.5. LEMMA. The proper Levi subgroups in \( \mathcal{L}G \) are the centralizers in \( \mathcal{L}G \) of tori in \( \mathcal{D}(\mathcal{L}G^o) \), which project onto \( \Gamma_k \).

Let \( M \) be a proper Levi subgroup in \( \mathcal{L}G \). It is conjugate to a subgroup \( \mathcal{L}'(S)^o \rtimes \Gamma_k \), where \( S \subseteq \mathcal{L}T^o \) is the identity component of the kernel of a subset \( J \subseteq \mathcal{A}' \) stable under \( \Gamma_k \). Let then \( S' \) be the one-dimensional subtorus of \( S \cap \mathcal{D}(\mathcal{L}G^o) \) on which the remaining simple roots are all equal. It is clear that \( \mathcal{L}'(S')^o = \mathcal{L}'(S) \), and that \( S' \) is pointwise fixed under \( \Gamma_k \). We have then \( M = \mathcal{L}'(S') \).

Let now \( S \) be a nontrivial torus in \( \mathcal{D}(\mathcal{L}G^o) \) such that \( \mathcal{L}'(S) \) meets every connected component of \( \mathcal{L}G \). Fix an ordering on \( X^*(S) \). There is a proper parabolic subgroup \( P^o \) of \( \mathcal{L}G^o \) of the form \( \mathcal{L}'(S)^o \cdot \text{U} \), such that the weights of \( S \) in the unipotent radical \( \text{U} \) of \( P^o \) are the roots of \( \mathcal{L}G^o \) with respect to \( S \) which are positive for this ordering. \( \text{U} \) is normalized by \( \mathcal{L}'(S) \); hence \( \mathcal{L}'(S) \cdot \text{U} \) is a proper parabolic subgroup \( P \) of \( \mathcal{L}G \), and then \( \mathcal{L}'(S) \) is a Levi subgroup of \( P \).

3.6. PROPOSITION. Let \( H \) be a subgroup of \( \mathcal{L}G \) whose projection on \( \Gamma_k \) is dense in \( \Gamma_k \). Then the Levi subgroups in \( \mathcal{L}G \) which contain \( H \) minimally form one conjugacy class with respect to the centralizer of \( H \) in \( \mathcal{L}G^o \).

Let \( C \) be the identity component of the centralizer of \( H \) in \( \mathcal{D}(\mathcal{L}G^o) \), and \( D \) a maximal torus of \( H \). If \( D = \{1\} \), then, by 3.5, \( H \) is not contained in any proper Levi subgroup in \( \mathcal{L}G \), and there is nothing to prove. So assume \( D \neq \{1\} \).

Let \( \Gamma'' \) be a normal open subgroup of \( \Gamma_k \) which acts trivially on \( \mathcal{L}G^o \). It is then normal in \( \mathcal{L}G \), and \( H \cdot \Gamma'' \) projects onto \( \Gamma_k \). Since \( \mathcal{D}(D) \) contains \( H \cdot \Gamma'' \), it projects onto \( \Gamma_k \), hence is a proper Levi subgroup by 3.5. Let \( M \) be a Levi subgroup containing \( H \). By 3.5, \( M = Z(S) \), where \( S \) is a torus in \( \mathcal{D}(\mathcal{L}G^o) \). Then \( S \subseteq C \), there exists \( c \in C \) such that \( c \cdot S \cdot c^{-1} \subseteq D \), hence \( c \cdot M \cdot c^{-1} = \mathcal{L}'(S') \supseteq \mathcal{L}'(D) \).

4. Remarks on induced groups. (To be used mainly to discuss restriction of scalars in \S 5 and 6.4.)

4.1. Let \( A \) be a group, \( A' \) a subgroup of finite index of \( A \) and \( E \) a group on which \( A' \) operates by automorphisms. Then we let

\[
(1) \quad \text{Ind}_{A'}^{A}(E) = I_{A'}^{A}(E) = \{ f: A \rightarrow E | f(a'a) = a' \cdot f(a) \ (a \in A; \ a' \in A') \}.
\]

It is a group (composition being defined by taking products of values). It is viewed as an \( A \)-group by right translations:

\[
(2) \quad r_{a}f(x) = f(xa) \quad (x, a \in A).
\]

For \( s \in A' \setminus A \), let

\[
(3) \quad E_s = \{ f \in I_{A'}^{A}(E) | f(a) = 0 \text{ if } a \notin s \}.
\]

Then \( E_s \) is a subgroup, \( I_{A'}^{A}(E) \) is the direct product of the \( E_s \)'s \( (s \in A' \setminus A) \), and these subgroups are permuted by \( A \). The subgroup \( E_{e} \) is stable under \( A' \) and is isomorphic to \( E \) as an \( A' \) module under the map \( f \mapsto f(e) \). The product of the \( E_s \)'s \( (s \in A' \setminus A, s \neq e) \) is also stable under \( A' \). We have therefore canonical homomorphisms.
4.2. Let \( B \) be a group, \( \mu: B \to A \) a homomorphism. Let \( B' = \mu^{-1}(A') \) and assume that \( \mu \) induces a bijection: \( B' \setminus B \simeq A' \setminus A \). Let \( E \) be a group on which \( A' \) operates by automorphisms, also viewed as a \( B' \)-group via \( \mu \). Then \( f \mapsto \mu \circ f \) induces an isomorphism

\[
\mu': I^A(E) \cong I^B(E),
\]

whose inverse is \( \mu \)-equivariant.

This follows immediately from the definitions.

4.3. Let \( A, E \) be as before, \( C \) a group and \( \nu: C \to A \) a homomorphism. Let \( \varphi: C \to E \times A \) be a homomorphism over \( A \). The map \( \psi: C \to E \) such that \( \varphi(c) = (\varphi(c), \nu(c)) \) is a 1-cocycle of \( C \) in \( E \) and \( \varphi \mapsto \psi \) induces a bijection

\[
H^1(C; E) \cong \varphi_A(C, E),
\]

where, by definition, \( \varphi_A(C, E) \) denotes the set of homomorphisms \( \varphi: C \to E \times A \) over \( A \), modulo inner automorphisms by elements of \( E \).

4.4. Let \( A, A', B, B' \) and \( E \) be as in 4.2. We have a commutative diagram with exact rows

\[
1 \longrightarrow I^A(E) \longrightarrow I^A(E) \times A \longrightarrow A \longrightarrow 1
\]

\[
1 \longrightarrow E \longrightarrow E \times A' \longrightarrow A' \longrightarrow 1
\]

where the vertical maps are natural inclusions (4.1).

Let \( \varphi: B \to I^A(E) \times A \) be a homomorphism over \( A \). Using 4.1(4), we get by restriction a homomorphism \( \bar{\varphi}: B' \to E \times A' \) over \( A' \).

4.5. Lemma. The map \( \varphi \mapsto \bar{\varphi} \) of 4.4 induces a bijection \( \varphi_A(B, I^A(E)) \cong \varphi_A(B', E) \).

We have, using 4.2, 4.3:

\[
\varphi_A(B, I^A(E)) = H^1(B; I^A(E)) = H^1(B; I^B(E)),
\]

\[
\Phi_A(B', E) = H^1(B'; E).
\]

By a variant of Shapiro's lemma, contained, e.g., in [4, 1.29]:

\[
H^1(B; I^B(E)) \cong H^1(B'; E),
\]

and it is clear that the isomorphisms (1), (2) carry this isomorphism over to \( \varphi \mapsto \bar{\varphi} \).

5. Restriction of scalars. In this section, \( k' \) is a finite extension of \( k \) in \( k' \), \( G' \) is a connected \( k' \)-group, and \( G = R_{k'/k} G' \).

5.1. The Galois group \( \Gamma_{k'} \) of \( k' \) over \( k' \) is an open subgroup (of finite index) of \( \Gamma_k \) and \( \Sigma_{k', k} = \Gamma_{k'} \setminus \Gamma_k \) may be identified with the set of \( k \)-monomorphisms of \( k' \) into \( k \). We have, in the notation of 4.1 (with \( A = \Gamma_k, A' = \Gamma_{k'} \))

\[
G(k) = I^A_k(G'(k)) = \prod_{\sigma \in \Gamma_k \setminus \Gamma_{k'}} {\sigma}G'(k).
\]
Assume $G'$ to be reductive. Then we see easily that $\phi(G) = (M, \varphi, M^*, \varphi')$ is related to $\phi(G') = (M', \varphi', M'^*, \varphi'^*)$ by

$$M = I_k^G (M'), \quad \varphi = \bigcup_{a \in A \setminus A'} \varphi' \cdot a.$$  

Similarly, if $A'$ is a basis of $\varphi'$, then

$$A = \bigcup_a A' \cdot a$$

is one for $\varphi$.

From this it follows that we have a natural isomorphism

$$L G' \overset{\sim}{\longrightarrow} I_k^G (L G'^*)$$

We have then a commutative diagram

$$1 \longrightarrow L G'^* \longrightarrow L G' = L G'^* \rtimes \Gamma' \longrightarrow \Gamma' \longrightarrow 1$$

$$1 \longrightarrow L G'^* = I_k^G (L G'^*) \longrightarrow L G = L G^* \rtimes \Gamma_k \longrightarrow \Gamma_k \longrightarrow 1$$

5.2. The map $P' \mapsto R_{k'/k} P'$ induces a bijection between $\mathcal{P}(G'/k')$ and $\mathcal{P}(G/k)$. Moreover $P'$ is a Borel subgroup of $G'$ if and only if $R_{k'/k} P'$ is one of $G$. Hence $G'$ is quasi-split/k' if and only if $G$ is quasi-split over $k$ (see [5, §6]).

Since $G(k) \simeq G'(k')$, we also get a bijection $p(G'/k') \simeq p(G/k)$.

If $J' < A'$ is stable under $\Gamma'_k$ then $J = \bigcup_{a \in A' \setminus A'} (J')$ is stable under $\Gamma'_k$. This map is easily seen to yield a bijection between $\Gamma'_k$-stable subsets of $A'$ and $\Gamma_k$-stable subsets of $A$, whence also canonical bijections

$$p(L G') \overset{\sim}{\longrightarrow} p(L G), \quad \bigl\langle p(L G') \bigr\rangle \overset{\sim}{\longrightarrow} \bigl\langle p(L G) \bigr\rangle.$$  

CHAPTER II. QUASI-SPLIT GROUPS. THE UNRAMIFIED CASE.

In this chapter, $G$ is a connected reductive quasi-split $k$-group. From 6.2 on, $G$ is assumed to split over a cyclic extension $k'$ of $k$, and $\sigma$ denotes a generator of $\text{Gal}(k'/k)$.


6.1. Assume $B$ and $T$ to be defined over $k$. Then the action of $\Gamma_k$ on $X^*(T)$ or $X_k(T)$ is determined by its restriction to $T_d$ of $T$. It follows that $W$ may be identified with the subgroup of the elements of $W$ which leave $T_d$ stable or, equivalently, with the fixed-point set of $\Gamma'_k$ in $W$. If we go over to the $L$-group and identify canonically $W$ with $W(L G^*, L T^*)$, then $W$ is also the fixed-point set of $\Gamma'_k$ in $W$, and it operates on the greatest subtorus $S$ of $L T^*$ which is pointwise fixed under $\Gamma'_k$. The group $S$ always contains regular elements; hence any element of $W$ is determined by its restriction to $S$. We let $\mathfrak{N}$ be the inverse image of $W$ in the normalizer $N$ of $L T^*$ in $L G^*$.

6.2. Lemma. Every element $w \in \mathfrak{N}$ has a representative in $\mathfrak{N}$ which is fixed under $\sigma$. 

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Write \( A^\circ = D_1 \cup \cdots \cup D_m \), where the \( D_i \)'s are the distinct orbits of \( I'(k'/k) \) in \( A^\circ \).

Let \( \delta_i \) be the common restriction to \( S \) of the elements of \( D_i \) and \( S_i \) the identity component of the kernel of \( \delta_i \). Then, \( S \)'s, viewed as a group of automorphisms of \( S \), is generated by the reflections \( s_i \) to the \( S_i (1 \leq i \leq m) \), and it suffices to prove the lemma for \( w = s_i (1 \leq i \leq m) \). [The "reflection" \( s_i \) is the unique element \( \neq 1 \) of \( W \) which leaves \( S \) stable, fixes \( S_i \) pointwise and is of order two.]

We let \( \text{Lie}(M) \) denote the Lie algebra of the complex Lie group \( M \). For \( \alpha \in A^\circ \), let, as usual

\[
g_\alpha = \{ X \in \text{Lie}(tG) | \text{Ad} t(X) = \alpha(t) \cdot X \ (t \in L T^\circ) \}.
\]

It is one-dimensional. Fix \( i \) between 1 and \( m \). By construction of \( L G \), we can find nonzero elements \( e_\alpha \in g_\alpha \) (resp. \( e_{-\alpha} \in g_{-\alpha} (\alpha \in D_i) \)) which are permuted by \( \sigma \). We have then

\[
[e_\alpha, e_{-\alpha}] = c \cdot \alpha,
\]

where \( c \) is \( \neq 0 \), and independent of \( \alpha \in D_i \) since \( I'(k'/k) \) is transitive on \( D_i \). Here \( X_\alpha(L T^\circ) \otimes C \) is identified with \( \text{Lie}(L T^\circ) \), and \( \alpha \) with \( \alpha \otimes 1 \). The element

\[
f_{\pm i} = \sum_{\delta \in D_i} e_{\pm \delta},
\]

is fixed under \( \sigma \). Moreover, since the difference of two simple roots is not a root, we have

\[
h_i = [f_i, f_{-i}] = \sum_{\delta \in D_i} [e_\delta, e_{-\delta}] = c \cdot \sum_{\delta \in D_i} \alpha.
\]

Using (3) and (4), we get

\[
[h_i, f_{\pm i}] = c \cdot \sum_{\delta, \beta \in D_i} \langle \alpha, \beta \rangle e_{\pm \beta}.
\]

By the transitivity of \( \text{Gal}(k'/k) \) on \( D_i \), the number

\[
d = \sum_{\delta \in D_i} \langle \alpha, \beta \rangle
\]

is also independent of \( \beta \in D_i \); therefore

\[
[h_i, f_{\pm i}] = c \cdot d \cdot f_{\pm i}.
\]

We claim that \( d \neq 0 \), in fact that \( d = 1, 2 \). The irreducible components of \( D_i \) are permuted transitively by \( \text{Gal}(k'/k) \) and have a transitive group of automorphisms. Therefore they are of type \( A_1 \) or \( A_2 \). Then, accordingly, \( d = 2 \) or \( d = 1 \). It follows that \( h_i, f_i \) and \( f_{-i} \) span a three-dimensional simple algebra pointwise fixed under \( \sigma \). Then so is the corresponding analytic subgroup \( G_i \) of \( L G \). The group \( G_i \) centralizes \( S_i \) and \( S \cap G_i \) is a maximal torus of \( G_i \), with Lie algebra spanned by \( h_i \). Then the nontrivial element of \( W (G_i, S \cap G_i) \) is the required element.

Remark. An equivalent statement is proved, in a different manner, in [35, pp. 19–22].

6.3. We let \( Y = L(T_d)^\circ \). The group \( X_\alpha(T_d) \) may be identified to the fixed-point set of \( I_h \) in \( X_\alpha(T) \). The inclusion of \( X_\alpha(T_d) = X^*(Y) \) into \( X_\alpha(T) = X^*(L T^\circ) \) induces a surjective morphism \( L T^\circ \rightarrow Y \), to be denoted \( \nu \).

The map \( A : t \mapsto t^{-1} \cdot \sigma t \) is an endomorphism of \( L T^\circ \), whose differential \( dA \) at 1 is \( (d\sigma - \text{Id}) \). Let
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(1) \[ U = (\ker A)^0, \quad V = \text{im} \, A. \]

Then $U$ is pointwise fixed under $\sigma$, the Lie algebra of $U$ (resp. $V$) is the kernel (resp. image) of $dA$. Since $dA$ is semisimple, they are transversal to each other; hence

(2) \[ L^T^0 = U \cdot V, \quad \text{and } U \cap V \text{ is finite}. \]

Moreover,

(3) \[ V = \ker \nu, \quad \nu(U) = Y. \]

In the rest of this chapter, we let $L^G$ stand for the “finite Galois form” $L^G \cong \text{Gal}(k'/k)$ of the $L$-group. We now want to discuss the semisimple conjugacy classes in $L^G \times \sigma$ with respect to $L^G$. We have

(4) \[ g^{-1} \cdot (h \times \sigma) \cdot g = g^{-1} \cdot h \cdot \sigma \times \sigma \quad (g, h \in L^G); \]

therefore $L^G$-conjugacy in $L^G \times \sigma$ is equivalent to $\sigma$-conjugacy in $L^G$.

6.4. Lemma. Let $\nu' : L^T^0 \times \sigma \to Y$ be defined by $\nu'(t \times \sigma) = \nu(t)$ $(t \in L^T^0)$. Then $\nu'$ induces a bijection

(1) \[ \tilde{\nu} : (L^T^0 \times \sigma)/\text{Int}_kN \longrightarrow Y/\text{kW}. \]

Let $n \in _kkN$. By 6.2, we may write $n = w \cdot s$ with $w = \sigma w$ and $s \in L^T^0$. Then the $L^T^0$-component of $n^{-1}(t \times \sigma) \cdot n$ is

\[ s^{-1} \cdot w^{-1} \cdot t \cdot w \sigma s = s^{-1} \cdot \sigma s \cdot (w^{-1} \cdot t \cdot w) \in V \cdot w^{-1} \cdot t \cdot w; \]

hence

\[ \nu'(n^{-1}(t \times \sigma) \cdot n) = \nu(w^{-1} \cdot t \cdot w) = w^{-1} \cdot \nu(t) \cdot w = w^{-1} \cdot \nu'(t \times \sigma) \cdot w. \]

Thus $\nu'$ is equivariant with respect to the projection $_kkN \to _kkW$ and therefore induces a map of the left-hand side of (1) into the right-hand side of (1), which is obviously surjective. Let $t, t' \in L^T^0$ and assume that $\nu'(t \times \sigma) = \nu(t') \cdot w$ for some $w \in _kkW$. Then we have $\nu(t) = \nu(w^{-1} \cdot t' \cdot w)$, where $w$ is a representative of $w$ fixed under $\sigma$, whence $t = v \cdot (w^{-1} \cdot t' \cdot w)$, with $v \in V$. We can write $v = s^{-1} \sigma s$ for some $s \in L^T^0$, and get $t \times \sigma = n^{-1}(t' \times \sigma)n$, with $n = ws$.

6.5. Lemma. Let $(L^G \times \sigma)_s$ be the set of semisimple elements in $L^G \times \sigma$. Then the map

\[ \tilde{\mu} : (L^T^0 \times \sigma)/\text{Int}_kN \longrightarrow (L^G \times \sigma)_s/\text{Int} \, L^G, \]

induced by inclusion is a bijection.

By results of F. Gantmacher [12, Theorem 14], $\tilde{\mu}$ is surjective. Let now $s, t \in L^T^0$ and $g \in L^G$ be such that $g^{-1} \cdot (s \times \sigma) \cdot g = t \times \sigma$, i.e., such that $g^{-1} \cdot \sigma s \cdot g = t$. Using the Bruhat decomposition of $L^G$ with respect to $L^B$, we can write uniquely $g = u \cdot n \cdot v$, with $u, v$ in the unipotent radical of $L^B$ and $n$ in the normalizer $N$ of $L^T^0$. These groups are stable under $\sigma$, and normalized by $L^T$. We have then

\[ s \cdot \sigma u \cdot \sigma n \cdot \sigma v = u \cdot n \cdot v \cdot t, \quad (s \cdot \sigma n \cdot s^{-1}) \cdot s \cdot \sigma n \cdot \sigma v = u \cdot n \cdot t \cdot (t^{-1} \cdot v \cdot t); \]

hence $s \sigma n \cdot s^{-1} = n \cdot t$. Therefore the connected component of $n$ in $N$ is stable under
σ, i.e., n represents an element of k W; hence n ∈ k N, and (t ∗ σ) and (s ∗ σ) are conjugate under k N.

REMARK. This proof was suggested to me by T. Springer.

6.6. If M is a complex affine variety, we let C[M] denote its coordinate algebra. The algebra C[Y] may be identified with the group algebra of X*(Y) = X_a(T_d). The quotient Y/k W is also an affine variety (in fact isomorphic to an affine space) and C[Y/k W] = C[Y^W].

Let Rep(G) ⊂ C[G] be the subalgebra generated by the characters of finite dimensional holomorphic representations. Its elements are constant on conjugacy classes. In particular, they define by restriction functions on (LG ∗ σ)_{alg}/Int LG^o.

6.7. PROPOSITION. The map

\[ \alpha = \bar{\mu} \circ \bar{\nu}^{-1} : Y/k W \longrightarrow (LG^o ∗ \sigma)_{alg}/Int LG^o \]

is a bijection, which induces an isomorphism of C[Y/k W] onto the algebra A of restrictions of elements of Rep(G).

REMARK. We shall use 6.7 only when k is a nonarchimedean local field. In that case 6.7 is proved in [35, pp. 18–24].

PROOF. That α is bijective follows from 6.4, 6.5. We prove the second assertion as in [35]. Let ρ be a finite dimensional holomorphic representation of LG and \( f_\rho \) the function on T^o defined by \( f_\rho(t) = tr(\rho(t) ∗ \sigma) \). It can be written as a finite linear combination \( f = \sum c_\lambda \lambda \) of characters \( \lambda \in X^*(T^o) \). Since \( tr(\rho) \) is a class function on LG, we have \( f_\rho(s^{-1} ∗ t ∗ s) = f_\rho(t) \) for all s, t ∈ T^o. By the linear independence of characters, it follows that if \( c_\lambda \neq 0 \), then \( \lambda \) is trivial on V (cf. 6.3(1)), hence is fixed under σ, i.e., may be identified to an element of X*(Y). Thus we may view \( f_\rho \) as an element of C[Y]. But invariance by conjugation and 6.4 imply that \( f \in C[Y/k W] \), whence a map \( \beta : A \rightarrow C[Y/k W] \), which is obviously induced by α. There remains to see that β is surjective. Note that C[Y/k W] is spanned, as a vector space, by the functions

\[ \varphi_\lambda = \sum_{w \in k W} w ∗ \lambda, \]

where \( \lambda \) runs through a fundamental domain C of k W on X_a(T_d). But it is standard that we may take for C the intersection of X_a(T_d) with the Weyl chamber of W in X_a(T) defined by B. Therefore every \( \lambda \in C \) is a dominant weight for \( T^o \) with respect to T^o. It is then the highest weight of an irreducible representation \( \pi_\lambda \) of \( LG^o \). Since it is fixed under σ, the representation \( ^o \pi_\lambda : g \mapsto \pi_\lambda(\sigma g) \) is equivalent to \( \pi_\lambda \). From this it is elementary that \( \pi_\lambda \) extends to an irreducible representation \( \tilde{\pi}_\lambda \) of LG of the same degree as \( \pi_\lambda \). The highest weight space is one-dimensional, stable under σ. Let c be the eigenvalue of σ on it. Then the trace gives rise to a function equal to c ∗ \( \varphi_\lambda \) modulo a linear combination of functions \( \varphi_\mu \) with \( \mu < \lambda \), in the usual ordering. That im β contains \( \varphi_\lambda (\lambda \in C) \) is then proved by induction on the ordering.

7. The Satake isomorphism and the L-group. Local factors.

7.1. We keep the previous notation and conventions. We assume moreover k to be a nonarchimedean local field, k' to be unramified over k, and σ to be the image of a Frobenius element \( Fr \) in \( I'_k \).

Let Q be a special maximal compact subgroup of G(k) [61]. We assume \( Q \cap T \) is the greatest compact subgroup of T(k) and Q contains representatives of k W. Let
$H(G(k), Q)$ be the Hecke algebra of locally constant, $Q$-bi-invariant, and compactly supported complex valued functions on $G(k)$. The Satake isomorphism provides a canonical identification $H \cong C[Y_i, W]$, hence also one of $Y_i W$ with the characters of $H$ [6].

By 6.7, we have now a canonical isomorphism of $H$ with the algebra $A$ of restrictions of characters of finite dimensional representations of $^L G$ to semisimple $^L G^0 \rtimes \sigma$, hence also a canonical bijection between characters of $H(G(k), Q)$ and semisimple classes in $^L G^0 \rtimes \sigma$. Furthermore, each such class can be represented by an element of the form $(t, \sigma)$, with $t \in ^L T^0$ fixed under $\sigma$ (and is determined modulo the finite group $U \cap V$, in the notation of 6.3).

7.2. Local factors. Assume now that $U$ is hyperspecial [61]. Let $\psi$ be an additive character of $k$. Let $(\pi, U_\pi)$ be an irreducible admissible representation of $G(k)$ of class 1 for $Q$ and $r$ a representation of $^L G$. Then the space of fixed vectors of $Q$ in $U_\pi$ is one-dimensional, acted upon by $H$ via a character $\chi_\pi$. To the latter is assigned by 7.1 a semisimple class $S_{\chi}$ in $^L G^0 \rtimes \sigma$. We then put

$$L(s, \pi, r) = \det(1 - r((g \rtimes \sigma)), q^{-s})^{-1}, \quad \varepsilon(s, \pi, r, \psi) = 1,$$

where $q$ is the order of the residue field, and $(g, \sigma)$ any element of $S_{\chi}$.

CHAPTER III. WEIL GROUPS AND REPRESENTATIONS. LOCAL FACTORS.

In this section, $k$ is a local field, $W_k$ (resp. $W_k^*$) the absolute Weil group (resp. Weil-Deligne group) of $k$. If $H$ is a reductive $k$-group, then $\Pi(H(k))$ is the set of infinitesimal equivalence classes of irreducible admissible representations of $H(k)$.

$G$ denotes a connected reductive $k$-group.

The main local problem is to define a partition of $\Pi(G(k))$ into finite sets $\Pi_{\phi, G}$ or $\Pi_{\phi}$ indexed by the set $\phi(G)$ of admissible homomorphisms of $W_k'$ into $^L G$, modulo inner automorphisms (see §8 for $\phi(G)$), and satisfying a certain number of conditions. So far, this has been carried out for any $G$ if $k = R, C$ [37], for tori over any $k$ [34] and (essentially) for $G = GL_2$ (cf. 12.2). §9 recalls the results for tori; §10 describes some of the conditions to be imposed on this parametrization; §11 summarizes the construction over $R$ or $C$. Such a parametrization would allow one to assign canonically local $L$- and $\varepsilon$-factors to any $\pi \in \Pi(H(k))$ and any complex representation of $^L G$. Two elements $\pi, \pi'$ in the same set $\Pi_{\phi}$ would always have the same local factors, and are hence called $L$-indistinguishable. In the case of $GL_n$ however, local factors have been defined in an a priori quite different way, so that the parametrization problem becomes subordinated to one concerning $L$- and $\varepsilon$-factors. This is discussed in §12.

8. Definition of $\phi(G)$.

8.1. Jordan decomposition in $W_k'$. If $k = R, C$, then $W_k' = W_k$ and, by definition, every element of $W_k'$ is semisimple.

Let $k$ be nonarchimedean. Then $x \in W_k'$ is said to be unipotent if and only if it belongs to $G_\sigma$; the element $x$ is semisimple if either $\varepsilon(x) \neq 0$ or $x$ is in the inertia group. Here $\varepsilon: W_k' \to Z$ is the canonical homomorphism $W_k' \to W_k \to k^* \to Z$. Every element $x \in W_k'$ admits a unique Jordan decomposition $x = x_s \cdot x_u$ with $x_s$ semisimple, $x_u$ unipotent and $x_s x_u = x_u x_s$ [60].
8.2. The set $\Phi(G)$. We consider homomorphisms $\alpha : W_k' \to LG$ over $\Gamma_k$, i.e., such that the diagram

$$
\begin{array}{c}
W_k' \\
\downarrow \\
\Gamma_k
\end{array}
\xrightarrow{\alpha} LG
$$

is commutative, and which satisfy moreover the following conditions:

(i) $\alpha$ is continuous, $\alpha(G_\alpha)$ is unipotent, in $LG^0$, and $\alpha$ maps semisimple elements into semisimple elements (in $LG$: $x = (u, \gamma)$ is said to be semisimple if its image under any representation (2.6) is so).

(ii) If $\alpha(W_k')$ is contained in a Levi subgroup of a parabolic subgroup $P$ of $LG$, then $P$ is relevant (3.3).

Such $\alpha$'s are called admissible. We let $\Phi(G)$ be the set of their equivalence classes modulo inner automorphisms by elements of $LG_0$.

If we write $\alpha(w) = (a(w), v(w))$ with $a(w) \in LG^0$ then $w \mapsto a(w)$ is a 1-cocycle of $W_k'$ (acting on $LG^0$ via $W_k' \to \Gamma_k$) in $LG^0$. It follows that

$$
\Phi(G) \subset H^1(W_k'; LG^0).
$$

Let $H$ be a subgroup of $W_k'$. Then $\alpha: W_k' \to LG$ is said to be trivial on $H$ if $v(H)$ acts trivially on $LG^0$ and $\alpha(H) = \{1\}$. Note that if $v(H)$ acts trivially on $LG^0$, then $\alpha|_H$ is a homomorphism.

8.3. Assume $G'$ is an inner quasi-split form of $G$. Then

1. $\Phi(G) \subset \Phi(G').$

In fact $LG \cong LG'$ and $lp(LG') \Rightarrow lp(LG)$; therefore $\alpha \in \Phi(G) \Rightarrow \alpha \in \Phi(G')$.

8.4. PROPOSITION. Let $k'$ be a finite separable extension of $k$; let $G'$ be a connected reductive $k'$-group and $G = R_{k/k'} G'$. Then there is a canonical bijection $\Phi(G) \cong \Phi(G')$.

We consider the situation of 5.2, 5.4 with $A = \Gamma_k, A' = \Gamma_{k'}, B = W_k', B' = W_{k'}, E = LG^0$. We have the injections (8.2):

$$
\Phi(G) \subset H^1(W_k'; LG^0), \quad \Phi(G') \subset H^1(W_{k'}; LG'^0).
$$

Moreover $LG^0 = \Gamma_k^0(LG'^0)$ (see 5.1); whence, by Shapiro's lemma and 5.2:

$$
H^1(W_k'; LG^0) \xrightarrow{\cong} H^1(W_{k'}; LG'^0).
$$

But it is clear that this isomorphism maps $\Phi(G)$ onto $\Phi(G')$.

8.5. Let $Z_L = C(LG^0)$. If $a: W_k \to Z_L$ and $b: W_k' \to LG^0$ are 1-cocycles, then $ab: w \mapsto a(w)b(w)$ is again a 1-cocycle of $W_k'$ in $LG^0$. If $a$ is continuous and $b$ corresponds to $\varphi \in \Phi(G)$, then $ab$ corresponds to an element of $\Phi(G)$. We get therefore maps

1. $H^1(W_k'; Z_L) \times H^1(W_k'; LG^0) \longrightarrow H^1(W_k'; LG^0),$

2. $H^1(W_k; Z_L) \times \Phi(G) \longrightarrow \Phi(G),$

which define actions of the group $H^1(W_k'; Z_L)$ on the sets $H^1(W_k'; LG^0)$ and $\Phi(G)$. 
8.6. Proposition. Let $\varphi : W_k \to LG$ be an admissible homomorphism. Then the Levi subgroups in $LG$ which contain $\varphi(W_k)$ minimally form one conjugacy class with respect to the centralizer of $\varphi(W_k)$ in $LG$.

Since $\varphi(W_k)$ projects onto a dense subgroup of $\Gamma$ by definition, this follows from 3.6.

Remark. Formally, this also applies to the archimedean case, but the proof in that case is simpler [37, pp. 78–79]. In fact, the argument there applies in all cases to admissible homomorphisms of the Weil (rather than Weil-Deligne) group because $\varphi(W_k)$ is always fully reducible. In this case, the Levi subgroups which contain $\varphi(W_k)$ minimally are those of the parabolic subgroups which contain $\varphi(W_k)$ minimally. Those parabolic subgroups form therefore one class of associated groups.

9. The correspondence for tori.

9.1. Let $T$ be a complex torus. A continuous homomorphism $\varphi : T \to C^*$ is described by a pair of elements $\lambda, \mu \in X^*(T) \otimes C$ such that $\lambda - \mu \in X^*(T)$, by the rule $\varphi(t) = e^{t\lambda}$. Similarly, a continuous homomorphism $\varphi : C^* \to T$ is given by $\mu, \nu \in X_*(T) \otimes C$ such that $\mu - \nu \in X_*(T)$; we have $\varphi(z) = z^{\mu/\nu}$, meaning that, for any $\lambda \in X^*(T)$,

$$
\lambda(\varphi(z)) = z^{\langle \lambda, \mu \rangle}.
$$

This can also be interpreted in the following way: identify $X_*(T) \otimes C$ with the Lie algebra $\text{Lie}(T(C))$. Then the exponential map yields an isomorphism $(X_*(T) \otimes C)/2\pi iX_*(T) = T(C)$. Then $\mu, \nu \in \text{Lie}(T(C))$ are such that $\varphi(e^h) = e^{h\mu + \nu h}$ ($h \in C$).

9.2. Let $G = T$ be a $k$-torus, and $l = \dim T$.

Any $\varphi \in \Phi(G)$ is trivial on $G_{\varphi}$; hence

$$
(1) \quad \Phi(G) = H^1(\Gamma; L^T) = H^1(\Gamma; X^*(T) \otimes C^*),
$$

where $H^1$ refers to continuous cocycles. On the other hand

$$
(2) \quad H^1(G) = \text{Hom}((X_*(T) \otimes k^*_T)^{\Gamma_{k}}, C^*).
$$

We have canonically [34, Theorem 1]

$$
(3) \quad H^1(G) = \Phi(G).
$$

In fact, $L^T$ and $W_k$ are replaced in [34] by a finite Galois form $L^T \otimes \Gamma_{k'/k}$ and a relative Weil group $W_{k'/k}$, where $k'$ is a finite Galois extension of $k$ whose Galois group acts trivially on $L^T$; this is easily seen to not change $\Phi(G)$. The proof then consists in showing first that the transfer from $W_{k'/k}$ to $k'^*$ yields an isomorphism

$$
(4) \quad H_1(W_{k'/k} ; X_*(T)) \longrightarrow H_1(k'^*; X_*(T))^{\Gamma_{k'/k}} = (k'^* \otimes X_*(T))^{\Gamma_{k'/k}},
$$

and second that the pairing

$$
(5) \quad H^1(L^T \otimes \Gamma_{k'/k} ; X_*(T)) \times H_1(W_{k'/k} ; X_*(T)) \longrightarrow C^*;
$$

associated to the evaluation map $(t, \lambda) \mapsto \lambda(t)$ ($t \in L^T$; $\lambda \in X_*(T)$) yields an
isomorphism of the first group onto the group of characters of the second group, which is then (3) by definition.

For illustrations, we discuss some simple cases.

9.3. \(k = C\). Then \(W_R = C^*\) and \(\Phi(G) = \text{Hom}(C^*, \mathbb{L}T^o)\). The correspondence follows from 9.1 since both \(\text{Hom}(C^*, \mathbb{L}T^o)\) and \(\text{Hom}(T, C^*)\) are canonically identified with \(\{(\lambda, \mu) \mid \lambda, \mu \in X^*(T) \otimes C, \lambda - \mu \in X^*(T)\}\).

9.4. \(k = R\). We have

\[
W_R = C^* \times \{\tau\} \quad \text{with} \quad \tau^2 = -1, \quad \tau \cdot z \cdot \tau^{-1} = \bar{z} (z \in C^*).
\]

Put \(C^* = S \times R^+\), with \(S = \{z \in C^*, z \cdot \bar{z} = 1\}\). Then \(\text{Int} \tau\) is the identity on \(R^+\), the inversion on \(S\).

Write \(\varphi(z) = (a, \sigma)\), where \(a\) is determined modulo \(\sigma\)-conjugacy, hence may be assumed to be fixed under \(\sigma\) (6.3(3)). We have \(\varphi(-1) = a^2\). Let \(\mu, \nu\) be the elements of \(X^*(T) \otimes C\) such that

\[
\varphi(z) = z^\mu \cdot z^\nu \quad (z \in C^*), \quad \mu - \nu \in X^*(T),
\]

(see 9.1). We have \(\varphi(z) = \sigma(\varphi(z))(z \in C^*)\); hence \(\nu = \sigma(\mu)\). Fix \(h \in X^*(T) \otimes C\) such that \(a = \exp 2\pi i h\). Then the character \(\pi\) associated to \(\varphi\) is given by

\[
\pi(e^t) = \exp(\langle a, x - \sigma \cdot \bar{x} \rangle) \cdot \exp(\langle \mu, x + \sigma \cdot x \rangle)/2 \quad (x \in X_s(T) \otimes C)
\]

[37, p. 27]. Here 
 denotes the complex conjugation of \(\text{Lie}(T(C)) = X_s(T) \otimes C\) with respect to \(X_s(T) \otimes R\); hence \(x \mapsto \sigma \cdot \bar{x}\) is the complex conjugation with respect to \(\text{Lie}(T(R))\). It follows that \(e^t \in T(R)\) if and only if \(x - \sigma \cdot \bar{x} \in 2\pi i \cdot X_s(T)\).

**Examples.** (a) Let \(T\) be anisotropic over \(R\). Then \(\sigma = -1\) and \(a = 1, h = 0\). We have \(e^t \in T(R)\) if \(x\) is purely imaginary and then (3) yields \(\pi = \mu\).

The fact that \(\varphi(-1) = 1\) shows that \(\mu \in X^*(T)\), confirming that \(\Pi(T(R)) = X^*(T)\).

(b) Let \(T\) be split over \(R\). Then \(\sigma = 1, \mu = \nu\), \(\varphi(z) = (z \cdot \bar{z})^a, a^2 = 1\) and \(h \in X^*(T)/2\). We have \(e^t \in T(R)\) if and only if \(x - \bar{x} \in 2\pi i \cdot X^*(T)\). It is then easily checked that \(\pi\) is given by \(\mu\) on the connected identity component of \(T(R)\), while its restriction to the torsion subgroup of \(T(R)\) is the character naturally defined by \(h\).

9.5. *The unramified case.* Let \(k\) be nonarchimedean, and assume \(T\) to split over an unramified extension \(k'\) of \(k\). A character \(\chi\) of \(T(k)\) is said to be unramified if it is trivial on the greatest compact subgroup \(\mathfrak{g}T(k)\) of \(T(k)\). On the other hand, \(\varphi \in \Phi(T)\) is unramified if it is trivial (see 6.2) on the inertia group. The bijection \(\Phi(T) \sim \Pi(T)\) induces a bijection between the sets \(\Phi_{\text{unr}}(T)\) and \(\Pi_{\text{unr}}(T)\) of unramified elements [34]. In view of its importance, we describe it in more detail.

Given \(t \in T(k')\), let \(\nu(t) \in \text{Hom}(X^*(T), Z)\) be defined by \(\nu(t)(m) = \text{ord } m(t)\) \((m \in X^*(T))\). It is well known, and easily deduced from Hilbert's Theorem 90, that \(H^1(T_{k'/k}, \mathfrak{g}T(k')) = 0\), where \(\mathfrak{g}T(k')\) is the group of units in the ring \(\mathfrak{g}k'\) of integers of \(k'\). Since \(T\) splits over \(k'\), it follows that \(H^1(T_{k'/k}, \mathfrak{g}T(k')) = 0\). By Galois cohomology, this implies that \((T(k')/\mathfrak{g}T(k'))^{r/k} = T(k)/\mathfrak{g}T(k)\), therefore \(t \mapsto \nu(t)\) yields a bijection

\[
T(k)/\mathfrak{g}T(k) \sim \text{Hom}(X^*(T), Z)^{r/k} = X_s(T)^{r/k} = X_s(T_d),
\]

where \(T_d\) is the greatest \(k\)-split torus of \(T\) (this can also be expressed by saying that
the inclusion $T_d \subset T$ induces an isomorphism $T_d(k)/0T_d(k) \sim T(k)/0T(k)$, cf. [6].
We have then
\[(2) \quad H_{\text{uni}}(T(k)) = \text{Hom}(X_k(T)^{*}, C^{*}) = Y.\]

The group $\Gamma'_{k}$ operates on $L T^\circ$ via the cyclic group $\Gamma_{k'/k}$ which is generated by the image $\sigma$ of a Frobenius element Fr. An unramified $\varphi$ is completely determined by $\varphi(\text{Fr})$, which can be written $\varphi(\text{Fr}) = (t, \text{Fr})$, where $t \in L T^\circ$ is determined up to conjugacy by $L T^\circ$. Thus $\varphi_{\text{uni}}(T) = (L T^\circ \rtimes \sigma)/\text{Int} L T^\circ$; and elementary special case of 6.4 provides a canonical isomorphism of the latter set onto $Y$, whence the desired isomorphism.

10. Desiderata. In order to formulate them, we need two preliminary constructions.

10.1. The character $\chi_{\varphi}$ of $C(G)$ associated to $\varphi \in \Phi(G)$ (cf. [37, pp. 20–34]). We want to associate to $\varphi \in \Phi(G)$ a character of the center $C(G)$ of $G$. Let $\mathfrak{G}_{\text{red}}$ be the greatest centrally torus of $\mathfrak{G}$. Then $G_{\text{red}} \to G$ yields a surjective homomorphism $L G \to L G_{\text{red}}$, whence a map $\Phi(G) \to \Phi(G_{\text{red}})$. In view of 7.2, this allows us to associate to $\varphi \in \Phi(G)$ a character $\chi_{\varphi}$ of $G_{\text{red}}$. Thus, if $C(G(k)) = G_{\text{red}}(k)$, our problem is solved.

In the general case, $G$ is enlarged to a bigger connected reductive $G_1$ generated by $G$ and a central torus, whose center is a torus. One shows that $\Phi(G_1) \to \Phi(G)$ is surjective. Using the previous step, we get a character of $C(G_1)$, hence one of $C(G)$ by restriction. It is shown to be independent of the choice of $G_1$ (loc. cit.), and is $\chi_{\varphi}$ by definition.

The map $\varphi \mapsto \chi_{\varphi}$ is compatible with restriction of scalars [37, 2.11].

10.2. The character $\pi_{\alpha}$ associated to $\alpha \in H^1(W_k'; \mathbb{Z}_L)$ [37, pp. 34–36]. We recall that $Z_L$ denotes the center of $L G^\circ$ (8.5). We can always find a $k$-torus $D$ such that $H^1(\Gamma_{k'}; D) = 0$, and a $k$-group $\tilde{G}$ isogeneous to $G \times D$ such that there is an exact sequence
\[(1) \quad 1 \longrightarrow D \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1.\]

Since $H^1(\Gamma_{k'}; D) = 0$, the map $\mu: \tilde{G}(k) \to G(k)$ is surjective. Let $G_{\text{sc}}$ be the universal covering of the derived group $\mathfrak{D}G$ of $G$. We have a commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & G_{\text{sc}} \\
\downarrow & & \downarrow \\
1 & \longrightarrow & G \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \tilde{G}/G_{\text{sc}} & \longrightarrow & 1 \\
\end{array}
\]
Going over to $L$-groups, we get

\[ \begin{array}{c}
L^D \\
\downarrow \\
L^G \\
\downarrow \\
L_{G_{sc}} \\
\downarrow \\
L(D) \\
\downarrow \\
1
\end{array} \]

Since $L_{G_{sc}}$ is of adjoint type (2.2), we see that $Z_L = \ker L\alpha^\circ$. Moreover, it is easily seen that

\[ (2) \quad Z_L \cong \ker L\beta^\circ. \]

This yields a map

\[ (3) \quad H^1(W_k^\prime; Z_L) \rightarrow \ker\{ H^1(W_k^\prime; L(G/G_{sc})^\circ) \rightarrow H^1(W_k^\prime; L^D) \}. \]

This allows us to associate to $\alpha \in H^1(W_k^\prime; Z_L)$ a character $\sigma_\alpha$ of $(G/G_{sc})(k)$ which is trivial on $D(k)$, hence a character $\pi_\alpha$ of $G(k) = \tilde{G}(k)/D(k)$. It can be shown to be independent of the choice of $D$. The map $\alpha \mapsto \pi_\alpha$ is compatible with restriction of scalars \cite[2.12] and satisfies:

\[ (4) \quad \chi_{\alpha\varphi} = \pi_\alpha \cdot \chi_\varphi \quad (\alpha \in H^1(W_k^\prime; Z_L), \varphi \in \Phi(G)). \]

10.3. Conditions on the sets $\Pi_\varphi$.

(1) If $\pi \in \Pi_\varphi$, then $\pi(z) = \chi_\varphi(z) \cdot \Id (z \in C(G))$.

(2) If $\varphi' = \alpha \cdot \varphi$, $\varphi' \in \Phi(G)$, $\alpha \in H^1(W_k^\prime; Z_L)$ (see 6.5), then $\Pi_{\varphi}' = \{ \pi_\alpha \otimes \pi | \pi \in \Pi_\varphi \}$.

(3) The following conditions on a set $\Pi_\varphi$ are equivalent:

(i) One element of $\Pi_\varphi$ is square-integrable modulo $C(G)$.

(ii) All elements of $\Pi_\varphi$ are square-integrable modulo $C(G)$.

(iii) $\varphi(W_k^\prime)$ is not contained in any proper Levi subgroup in $L^G$.

(4) Assume $\varphi(G_a) = \{ 1 \}$. The following conditions on a set $\Pi_\varphi$ are equivalent:

(i) One element of $\Pi_\varphi$ is tempered.

(ii) All elements of $\Pi_\varphi$ are tempered.

(iii) $\varphi(W_k^\prime)$ is bounded.

(5) Let $H$ be a connected reductive $k$-group and $\eta: H \rightarrow G$ a $k$-morphism with commutative kernel and cokernel. Let $\varphi \in \Phi(G)$ and $\varphi' = L\eta \circ \varphi$. Then any $\pi \in \Pi_\varphi$, viewed as an $H(k)$-module, is the direct sum of finitely many irreducible admissible representations belonging to $\Pi_{\varphi'}$.

10.4. The unramified case.

We say that $\varphi \in \Phi(G)$ is unramified if it is trivial, in the sense of 6.2, on $G_a$ and on the inertia group $I$. If so, $\text{Im } \varphi$ may be assumed to be in $LT$. Therefore, if $\Phi(G)$ contains an unramified element, then $G$ is quasi-split (see 8.2 (ii)).

Assume now $G$ to be quasi-split, to split over an unramified Galois extension
$k'$ of $k$, and let $\varphi \in \Phi(G)$ be unramified. There exists $t \in (L^\infty)^{\Gamma_k}$ such that

$$\varphi(\text{Fr}) = (t, \text{Fr}),$$

(1) and we have

$$\varphi(w) = (t, \text{Fr})^{\varepsilon(w)} \quad (w \in W_k),$$

(2) where $\varepsilon: W_k \rightarrow \mathbb{Z}$ is the canonical homomorphism. The element $t$ defines an unramified character $\chi$ of a maximal $k$-torus $T$ of a Borel $k$-subgroup $B$ of $G$ (9.5). It is then required that $\Pi_\varphi$ consists of the constituents of the unramified normalized principal series $PS(\chi)$ which have a nonzero vector fixed under some hyperspecial maximal compact subgroup. Conversely let $(\pi, \nu)$ be an irreducible admissible representation with a nonzero vector fixed under some hyperspecial maximal compact subgroup. There exists then an unramified character $\chi$ of $T$ such that $(\pi, \nu)$ is a constituent of $PS(\chi)$ (and $\chi$ is determined modulo the relative Weyl group). We have then $(\pi, \nu) \in \Pi_\varphi$, for the unramified $\varphi$ which maps $\text{Fr}$ to $(t, \text{Fr})$, where $t$ represents $\chi$ (9.5). Note that if $U$ is a special maximal compact subgroup of $G(k)$, then $G(k) = B(k) \cdot U$; hence the fixed-point set of $U$ in $PS(\chi)$ is at most one-dimensional. It follows that $PS(\chi)$ has at most one irreducible constituent with nonzero fixed vectors under $U$.

This assignment is consistent with 7.2. Namely, if $\pi \in \Pi_\varphi$, then the semisimple class $S_\pi^\infty$ in $L^G \times \sigma$ corresponding to the character of the Hecke algebra defined by $\pi$ is indeed represented by $t \times \sigma$. This follows from [6].

Remark. Originally, it was thought that $\Pi_\varphi$ should consist of those constituents of $PS(\chi)$ which had a nonzero fixed vector under some special maximal compact subgroup. However it was pointed out during the Institute by I. Macdonald that such representations may belong to the discrete series. If so, this condition would contradict 10.3(3). Upon a suggestion of J. Tits, this has led to the restriction to hyperspecial maximal compact subgroups made above. Those cannot belong to the discrete series, so that 10.3(3) and 10.4 are consistent.

10.5. Example. Assume that $k = \mathbb{R}$ and that $G$ is semisimple, possesses a Cartan subgroup $T$ which is anisotropic over $\mathbb{R}$, and is an inner form of a split group. Then $L^G$ is the direct product of $L^{G^0}$ and $\Gamma_k$, the Weyl group $W$ contains $-\text{Id}$ and $G(\mathbb{R})$ has a discrete series. We want to describe the parametrization of the latter in terms of $\Phi(G)$. As the notation implies, we shall view $L^T$ as the $L$-group of $T$. Let $\varphi \in \Phi(G)$. It is given by a continuous homomorphism $\varphi': \mathbb{R}_w \rightarrow L^G$. We may assume that $\text{Im } \varphi'$ is contained in the normalizer of $L^{T^0}$. Let $n = \varphi(z)$ and let $w \in W$ be the element of $W$ represented by $n$. Then $w^2 = 1$. Let $\mu, \nu \in X^*(T) \otimes \mathbb{C}$ be such that

$$\varphi(z) = z^\mu \cdot \bar{z}^\nu \quad (z \in C^*), \quad \mu - \nu \in X^*(T)$$

(see 9.1). We have

$$\varphi(z) = n \cdot \varphi(z) \cdot n^{-1} = z^{w \cdot \mu} \cdot \bar{z}^{w \cdot \nu};$$

hence $\mu = w \cdot \nu, \nu = w \cdot \mu$. Assume now that $\text{Im } \varphi$ is not contained in any proper Levi subgroup in $L^G$, or, equivalently, that $\text{Im } \varphi'$ is not contained in any proper Levi subgroup in $L^{G^0}$. Then $w = -\text{Id}$ and $\mu$ is regular: in fact, the proper Levi subgroups in $L^{G^0}$ are the centralizers of nontrivial tori. This implies first that $w$ does
not fix pointwise any nontrivial torus in $L^\infty T$, hence $w = -\text{Id}$; if now $\mu$ were singular, then the centralizer of $\mu(C^\times)$ would contain a semisimple subgroup $H \neq \{1\}$ stable under $\text{Int} n$, the latter would leave pointwise fixed a torus $S \neq \{1\}$ of $H$, and $\text{Im } \phi'$ would be contained in the centralizer $Z(S)$ of $S$, a contradiction. Since $\nu = w \cdot \mu = -\mu$, we have

$$\lambda(\phi(-1)) = (-1)^{(2\mu, \delta)}, \text{ for all } \lambda \in X_{\mu}(T).$$

Let $\delta$ be half the sum of the roots $\alpha$ of $G$ with respect to $T$ such that $\langle \mu, \alpha \rangle > 0$. Then, Lemma 3.2 of [37] implies in particular

$$\lambda(\phi'(-1)) = (-1)^{(2\delta, \lambda)}, \text{ for all } \lambda \in X_{\mu}(T).$$

It follows that

$$\mu \in \delta + X^*(T).$$

Therefore $\mu$ is among the elements of $X^*(T) \otimes \mathbb{Q}$ which parametrize the discrete series in Harish-Chandra's theorem. We then let $\Pi_{\phi}$ be the set of discrete series representations of $G(\mathbb{R})$ with infinitesimal character $\chi_{\mu}$. If $G(\mathbb{R})$ is compact, then $\Pi_{\phi}$ consists of the irreducible finite dimensional representation with dominant weight $\mu - \delta$. In that case, no proper parabolic subgroup of $L^\infty G$ is relevant; hence $\Phi(G)$ consists of the $\phi$ considered here.

10.6. Let $G = GL_m$ nonarchimedean. Let $\phi$ be an admissible representation of $W_k$. If it is irreducible, then $\phi(G_a) = 1$. If it is indecomposable, then it is a tensor product $\rho \otimes \text{sp}(m)$, where $m$ divides $n$, $\rho$ is irreducible of degree $n/m$, and $\text{sp}(m)$ is $m$-dimensional, trivial on 1, maps a generator of the Lie algebra of $G_a$ onto the nilpotent matrix with ones above the diagonal, zero elsewhere, and $\phi \in W_k$ onto the diagonal matrix with entries $a(w)$ (0 ≤ $i < n$) [9, 3.1.3]. If $\chi$ is a character of $W_k$ (hence of $k^\times$), and $\phi = \chi \otimes \text{sp}(n)$, then $\Pi_{\phi}$ consists of the special representation with central character determined by $\chi$. In fact, the Weil-Deligne group came up for the first time precisely to fit the special representations of $GL_2$ into the general scheme (see [9]).

11. Outline of the construction over $R$, $C$. We sketch here the various steps which yield the sets $\Pi_{\phi}$ when $k = R$. For the proofs see [37].

We note first that we may always assume $\phi(W_a) \subset N(L^\infty T)$, and we can write (9.1)

$$\phi(z) = z^\mu z^\nu \quad (z \in C^\times; \mu, \nu \in X^*(T) \otimes C, \mu - \nu \in X^*(T)).$$

11.1. Lemma. Let $\phi \in \Phi(G)$. Assume $\phi(W_R)$ is not contained in any proper Levi subgroup in $L^\infty G$. Then

(i) $G$ has a Cartan $k$-subgroup $C$ such that $(\otimes C \cap S)(R)$ is compact [28, 3.1].

(ii) $\mu$ is regular; $\phi(C^\times)$ contains regular elements [37, 3.3].

The group $L^\infty C$ may be viewed as a maximal torus of $L^\infty G$; hence there is an isomorphism $L^\infty C \rightarrow L^\infty T$ defined modulo an element of $W$. Therefore $\phi$ defines an orbit of $W$ in $\Phi(C)$, hence, by 9.2, an orbit $X_{\phi}$ of $W$ in $X(C(R))$. [Note that $W$, which is defined in $G(C)$, operates on $C(R)$, since $C(R) \cap \otimes G$ is compact, hence on $X(C(R))].$

11.2. Let $G_0 = C(R)((\otimes G)(R))$. Let $A_0$ be the set of representations of $G_0$ which
are square-integrable modulo the center, and have infinitesimal character
\( \chi_2(\lambda \in X_P) \). The induced representations \( \pi = I_G(D_\phi) (\pi_0) \) (\( \pi_0 \in A_p \)) are irreducible [37, p. 50]. By definition, \( II_\phi \) is the set of equivalence classes of these representations [37, p. 54].

11.3. Let \( \phi \in \Phi(G) \). Let \( L^M \) be a minimal relevant Levi subgroup containing \( \text{Im} \phi \). It is essentially unique (8.6). We assume \( L^M \neq L^G \); we may view \( \phi \) as an element of \( \Phi(M) \). By 11.2, there is associated to it a finite set of \( II_{\phi,M} \) of discrete series representations of \( M \).

We may assume \( L^M \) to be a Levi subgroup of a relevant parabolic subgroup \( L^P \) corresponding to \( P \in \mathcal{P}(G/k) \). Then \( U = X^*(T) \otimes R = X_*(T^0) \otimes R \). Let \( V \) be the subspace of elements of \( U \) which are orthogonal to roots of \( L^M \), and fixed under \( I_\phi \). It may be identified with the dual \( \sigma' \) of the Lie algebra of a split component \( A \) of \( P \).

Let \( \zeta = \text{character of } C(M) \) defined by the elements of \( II_{\phi,M} \). We may assume that \( |\zeta| \in \text{Cl}(\alpha_P^+) \). Let \( P_1, P \) be the smallest parabolic \( k \)-subgroup containing \( P \) such that \( |\zeta| \), when restricted to \( \alpha_{P_1} \), is an element of the Weyl chamber \( \alpha_P^+ \). Let \( M_1 = \pi(\alpha_{P_1}) \) and \( P' = P \cap M_1 \). Then \( P' \) is a parabolic subgroup of \( M_1 \). Moreover the restriction of \( |\zeta| \) to the split component \( M_1 \cap A_P \) of \( P' \) is one; therefore, for each \( \rho \in II_{\phi,M} \), the induced representation \( \text{Ind}_{M_1}^{M_1}(\rho) \) is tempered. Let \( II_\phi' \) be the set of all constituents of such representations. Then by definition, \( II_\phi' \) is the set of Langlands quotients \( J(P_1, \sigma) \) with \( \sigma \in II_\phi' \) (cf. [37, p. 82]).

11.4. Complex groups. Assume now \( k = C \). Then \( W_k = C^* \), and \( \Phi(G) \) may be identified to the set of homomorphisms of \( C^* \) into \( X^*(T) \otimes C \), modulo the Weyl group \( W \), i.e.,

\[
\{(\lambda, \mu), \text{ where } \lambda, \mu \in X^*(T) \otimes C, \lambda - \mu \in X^*(T)\}
\]

modulo the (diagonal) action of \( W \). In this case \( \text{Im} \phi \) is in the Levi subgroup \( L^T \) of \( L^B \), which is the \( L^P \) of 11.3. The set \( II_{\phi,M} \) consists of one character of \( T \) (cf. 9.1). Choose \( P_1, M_1 \), as in 11.3. Since the unitary principal series of a complex group are irreducible (N. Wallach), the set \( II'_\phi \) consists of one element. Hence so does \( II_\phi' \). Thus each \( II_\phi' \) is a singleton. The classification thus obtained is equivalent to that of Zelovenko.

11.5. Let \( G = \text{GL}_n, k = R \). In this case, it is also true that the tempered representations induced from discrete series are irreducible [22]; therefore each set \( II_\phi' \) (cf. 9.3) consists of only one element, hence so does \( II_\phi \) and we get a bijection between \( \Phi(G) \) and \( ILG(R) \).

Let \( n = 2 \). If \( \phi \) is reducible, then \( \text{Im} \phi \) is commutative; hence \( \phi \) factors through \( (W_B)_{ab} = R^* \) and is described by two characters \( \mu, \nu \) of \( R^* \). Then \( II_\phi \) consists of a principal series representation \( \pi(\mu, \nu) \) (including finite dimensional representations, as usual). In particular there are three \( \phi \)'s with kernel \( C^* \), to which correspond respectively \( \pi(1, 1), \pi(\text{sgn}, \text{sgn}) \) and \( \pi(1, \text{sgn}) \), where \( \text{sgn} \) is the sign character. If \( \phi \) is irreducible, then \( \phi(\tau) \) may be assumed to be equal to \( (s_0, \tau) \), where \( s_0 \) is a fixed element of the normalizer of \( T^0 \) inducing the inversion on it. \( \phi(R^+) \) belongs to the center of \( L^G \), and \( \phi(S) \) is sum of two characters, described by two integers. Then \( II_\phi \) consists of a discrete series representation, twisted by a one-dimensional representation.

11.6. As is clear from these two examples, the main point to get explicit knowl-
edge of the sets $II_{\varphi}$ is the decomposition of representations induced from tempered representations of parabolic subgroups. This last problem has been solved by A. Knapp and G. Zuckerman [29], [30].

11.7. Remark on the nonarchimedean case. Langlands’ classification [37] is also valid over $p$-adic fields [57]. In view of 8.6, it is then clear that the last step (11.3) of the previous construction can also be carried out in the nonarchimedean case. Thus, besides the decomposition of tempered representations, the main unsolved problem in the $p$-adic case is the construction and parametrization of the discrete series.

12. Local factors.

12.1. Let $\pi \in II(G(k))$ and $r$ be a representation of $LG$ (2.6). Assume that $\pi \in II_{\varphi}$ for some $\varphi \in \Phi(G)$. For a nontrivial additive character $\psi$ of $k$, we let

$$L(s, \pi, r) = L(s, r \circ \varphi), \quad \varepsilon(s, \pi, r) = \varepsilon(s, \pi, r, \psi) = \varepsilon(s, r \circ \varphi, \psi),$$

where on the right-hand sides we have the $L$- and $\varepsilon$-factors assigned to the representation $r \circ \varphi$ of $W_{\psi}^{\varphi}$ [60]. In the unramified situation of 10.4, this coincides with the definition given in 7.2.

In view of what has been recalled so far, these local factors are defined if $k$ is archimedean, or if $k$ is nonarchimedean in the unramified case, or if $G$ is a torus.

12.2. Let now $G = GL_n$. In this case there are associated to $\pi \in II(G(k))$ local factors $L(s, \pi)$ and $\varepsilon(s, \pi, \psi)$ defined by a generalization of Tate’s method, in [25] for $n = 2$, in [19] for any $n$, which play a considerable role in the parametrization problem and in the local lifting. A natural question is then whether these factors can be viewed as special cases of 12.1, where $r = r_n$ is the standard representation of $GL_n$, i.e., whether we have equalities

$$L(s, \pi) = L(s, \pi, r_n), \quad \varepsilon(s, \pi, \psi) = \varepsilon(s, \pi, r_n, \psi),$$

with the right-hand side defined by the rule of 12.1.

(a) Let $n = 2$. It has been shown in [25] that the equivalence class of $\pi$ is characterized by the functions $L(s, \pi \otimes \chi), \varepsilon(s, \pi \otimes \chi, \psi)$, where $\chi$ varies through the characters of $k^*$. In this case, the parametrization problem and the proof of (1) are part of the following problem:

(*) Given $\sigma \in \Phi(G)$, find $\pi = \pi(\sigma)$ such that

$$L(s, \sigma \otimes \chi) = L(s, \pi \otimes \chi), \quad \varepsilon(s, \sigma \otimes \chi, \psi) = \varepsilon(s, \pi \otimes \chi, \psi)$$

for all $\chi$'s, and prove that $\sigma \mapsto \pi(\sigma)$ establishes a bijection between $\Phi(G)$ and $II(G(k))$.

This problem was stated and partially solved in [25]. The most recent and most complete results in preprint form are in [62]; they still leave out some cases of even residual characteristic, although some arguments sketched by Deligne might take care of them (see [63] for a survey).

As stated, the problem is local, but, except at infinity, progress was achieved first mostly by global methods: one uses a global field $E$ whose completion at some place $v$ is $k$, a reductive $E$-group $H$ isomorphic to $G$ over $k$, an element $\rho \in \Phi(H/k)$ whose restriction to $\iota(H/k_v) = iG$ is $\sigma$, chosen so that there exists an automorphic representation $\pi(\rho)$ with the $L$-series $L(s, \rho)$ (see §14 for the latter). This construc-
tion relies, among other things, on Artin’s conjecture in some cases, and [38]. In fact, it was already shown in [25] that (\(*\)) for odd residual characteristics follows from Artin’s conjecture, leading to a proof in the equal characteristic case. At present, there are in principle purely local proofs in the odd residue characteristic case [63]. Note also that the injectivity assertion is a statement on two-dimensional admissible representations of $W'$, namely, whether such a representation $\sigma$ is determined, up to equivalence, by the factors $L(s, \sigma \circ \chi)$ and $\varepsilon(s, \sigma \circ \chi, \phi)$. But, so far, the known proofs all use admissible representations of reductive groups [63].

(b) For arbitrary $n$, (1) has been proved in the unramified case, for special representations, and by H. Jacquet for $k = R, C$ [24].

(c) Local $L$- and $\varepsilon$-factors are also introduced for $G = GL_2 \times GL_2$ in [21], at any rate for products $\pi \times \pi'$ of infinite dimensional irreducible representations. Partial extensions of this to $GL_m \times GL_n$ for other values of $m, n$ are known to experts.

(d) For $n = 3$, $G = GL_3 \times SL_2$ in [21], at any rate for products $n \times n'$ of infinite dimensional irreducible representations. Partial extensions of this to $GL_m \times GL_n$ for other values of $m, n$ are known to experts.

12.3. Local factors have also been defined directly for some other classical groups, in particular for $GSp_4$ by F. Rodier [48], extending earlier work of M. E. Novodvorsky and I. Piatetskii-Shapiro, for split orthogonal groups, in an odd number $2n + 1$ of variables by M. E. Novodvorsky [41]. In the latter case $L^G = Sp_{2n}$, and in the unramified case, the local factors coincide (up to a translation in $s$) with those associated by 7.2 to the standard $2n$-dimensional representation of the $L$-group. See also [42].

CHAPTER IV. THE $L$-FUNCTION OF AN AUTOMORPHIC REPRESENTATION.

From now on, $k$ is a global field, $\nu = \nu_k$ the ring of integers of $k$, $A$ or $A$ the ring of adeles of $k$, $V$ (resp. $V_\infty$, resp. $V_f$) the set of places (resp. infinite places, resp. finite places) of $V$. For $v \in V$, $k_v$, $\nu_v$ and $N^v$ have the usual meaning. Unless otherwise stated, $G$ is a connected reductive $k$-group.

13. The $L$-function of an irreducible admissible representation of $G_A$.

13.1. Let $\pi$ be an irreducible admissible representation of $G_A$ and $r$ a representation of $L^G$. There exists a finite Galois extension $k'$ of $k$ over which $G$ splits and such that $r$ factors through $L^G \rtimes \Gamma_{k'/k}$. We want to associate to $\pi$ and $r$ infinite Euler products $L(s, \pi, r)$ and $\varepsilon(s, \pi, r)$, whose factors are defined (at least) for almost all places of $k$.

Let $v \in V$. By restriction, $r$ defines a representation $r_v$ of $L(G|k_v) = L^G \rtimes \Gamma_k$. On the other hand, $\pi = \bigotimes_v \pi_v$, with $\pi_v \in II(G(k_v))$ [11]. Assume the parametrization problem of Chapter III solved. Then there is a unique $\varphi_v \in \Phi(G|k_v)$ such that $\pi_v \in II_{\varphi_v}$. Then we let

\begin{align*}
(1) & & L(s, \pi, r) = II_\pi L(s, \pi_v, r_v), \\
(2) & & \varepsilon(s, \pi, r) = II_\varepsilon \varepsilon(s, \pi_v, r_v, \varphi_v),
\end{align*}

where $\varphi_v$ is an additive character of $k_v$ associated to a given nontrivial additive character of $k$, and the factors on the right are given by 12.1(1).
The local problem is solved for archimedean $v$'s, and for almost all finite $v$'s (see below) so that the factors on the right are defined except for at most finitely many $v \in V_f$. For questions of convergence or meromorphic analytic continuation this does not matter, and we shall also denote such partial products by $L(s, \pi, r)$.

By 10.4, $\varphi_v$ is well defined if the following conditions are fulfilled: $G$ is quasi-split over $k_v$, $G(o_v)$ is a very special maximal compact subgroup of $G(k_v)$, $k'$ is unramified over $k$, and $\pi_v$ is of class one with respect to $G(o_v)$. All but finitely many $v \in V_f$ satisfy those conditions [61].

13.2. Theorem [35]. Let $\pi$ be an irreducible admissible unitarizable representation of $G_A$ and $r$ be a representation of $^L G$ (2.6). Then $L(s, \pi, r)$ converges absolutely for Re $s$ sufficiently large.

We may and do view $r$ as a complex analytic representation of $^L G_0 \cong k' / k$, where $k'$ is a finite Galois extension of $k$ over which $G$ splits (2.7). We let $V_1$ be the set of $v \in V_f$ satisfying the conditions listed at the end of 13.1. We have to show that

$$ L' = \prod_{v \in V_1} L(s, \pi_v, r_v), $$

converges in some right half-plane.

Let $Fr_v$ be the Frobenius element of $I'_k / k_v$, where $v' \in V_k$ lies over $v \in V_1$. We have

$$ \varphi_v(Fr_v) = (t_v, Fr_v), \quad \text{with} \quad t_v \in ^L T^o $$

and

$$ L(s, \pi_v, r_v) = (\det(1 - r((t_v, Fr_v)) N_e^{-s}))^{-1}. $$

To prove the theorem, it suffices therefore to show the existence of a constant $a > 0$ such that

$$ |\mu| \leq (Nv)^a \quad \text{for every} \quad v \in V_1 \quad \text{and eigenvalue} \quad \mu \quad \text{of} \quad r((t_v, Fr_v)). $$

Let $n = [k': k]$. Since we may assume $t_v$ fixed under $I'_k$ (6.3), we have $t_v^n = (t_v, Fr_v)^n$; hence it is equivalent to show (4) for all eigenvalues $\mu$ of $r(t_v)$. These are of the form $t_v^\lambda$, where $\lambda$ runs through the set $P_r$ of weights of $r$, restricted to $^L G^o$.

Thus we have to show the existence of $a > 0$ such that

$$ |t_v|^{\text{Re } \lambda} \leq (Nv)^a \quad \text{for all} \quad v \in V_1 \quad \text{and} \quad \lambda \in P_r. $$

Let $G'$ be a quasi-split inner $k$-form of $G$. Then $^L G = {^L G}'$, and $G$ is isomorphic to $G'$ over $k_v$ for all $v \in V_1$. We may therefore replace $G$ by $G'$; changing the notation slightly, we may (and do) assume $G$ to be quasi-split over $k$. We then fix a Borel $k$-subgroup $B$ of $G$ and view $^L T$ as the $L$-group of a maximal $k$-torus $T$ of $G$.

For a cyclic subgroup $D$ of $I'_k$, let $V_D$ be the set of $v \in V_1$ for which $I'_k$ is equal to the inverse image of $D$ in $I'_k$. The group $U = X_a(T)^o$ is then the group of one-parameter subgroups of a subtorus $S$ of $T$ such that $S / k_s$ is a maximal $k_v$-split torus of $G/k_v$ for all $v \in V_D$. The group

$$ Y = \text{Hom}(U, C^*) = \text{Hom}(X_a(T)^o, C^*) \quad (v \in V_D), $$

is independent of $v$, and is the $Y$ of §6 for $G/k_v$. The root datum $\phi(G/k_v)$, which is determined by the action of $D$, is also independent of $v \in V_D$. 
Given $y \in Y$, let $y_0$ be a "logarithm" of $y$, i.e., an element of $\Hom(X_{a}(T)^D, C)$ such that

\[(7) \quad y(u) = N_{\mathfrak{v}} y_0(u) = N_{\mathfrak{v}} y_0^{u}, \quad \text{for } u \in X_{a}(T)^D.\]

This element is determined modulo a lattice, but its real part $\Re y_0 \in \Hom(U, \mathbb{R})$, defined by

\[(8) \quad y(u) = N_{\mathfrak{v}} \Re y_0^{u}\]

is well defined. If $y$ has values in $\mathbb{R}_{+}$, then we choose $y_0$ to be equal to its real part.

The space $a^*$ is the dual of $a = U \otimes \mathbb{R}$ (the so-called real Lie algebra of $S/k_{a}$), and is acted upon canonically by $sW$ as a reflection group. We let $a^* +$ be the positive Weyl chamber defined by $B$.

Let $\rho_0$ be the unramified character of $T(k_{a})$, given by $t \mapsto |\delta(t)|_{\mathfrak{v}}$, where $| \cdot |_{\mathfrak{v}}$ is the normalized valuation at $\mathfrak{v}$ and $\delta$ half the sum of the positive roots. Then its real logarithm $\rho_0$ is independent of $\mathfrak{v} \in V_{D}$. In fact, it is a positive integral power of $N_{\mathfrak{v}}$ whose exponent is determined by the $k_{\mathfrak{v}}$-roots, their multiplicities, and the indices $q_{0}$ of the Bruhat-Tits theory [61]. But those are determined by the previous data and the action of $I_{k_{a}}$ on the completed Dynkin diagram [61], which is also independent of $\mathfrak{v} \in V_{D}$. We write $\rho_0$ instead of $\rho_{v, 0}$. We have $\rho_0 \in a^* +$.

The representation $\pi_{v}$ is a constituent of an unramified principal series $PS(\chi_{v})$, where $\chi_{v}$ is an unramified character of $T(k_{a})$, or, equivalently, of $S(k_{a})$, determined up to a transformation by an element of $sW$. Thus we may assume $\chi_{v, 0}$ to be contained in the closure $\mathcal{C}(a^{* +})$ of $a^{* +}$. Since $\pi_{v}$ is unitary, the associated spherical function is bounded, and hence $\Re \chi_{v, 0}$ is contained in the convex hull of $sW(\rho_0)$, i.e., we have

\[(9) \quad \langle \rho_0 - \chi_{v, 0}, \lambda \rangle \geq 0, \quad \text{for all } \lambda \in a^{* +}.\]

(See remark following the proof.)

For $\lambda \in X^{*}(L^{\circ})$, let $\lambda'$ be the restriction of $\lambda$ to $X_{a}(T)^D$. In view of 10.4 and our conventions, we have then

\[(10) \quad |\lambda(t_{v})| = N_{\mathfrak{v}}(\Re x_{\mathfrak{v}, 0} \circ \lambda').\]

Let $\bar{\lambda} = sW(\lambda') \cap \mathcal{C}(a^{* +})$. Since $\Re \chi_{v, 0} \in \mathcal{C}(a^{* +})$, we have

\[(11) \quad N_{\mathfrak{v}}(\Re x_{\mathfrak{v}, 0} \circ \bar{\lambda}) \leq N_{\mathfrak{v}}(\Re x_{\mathfrak{v}, 0} \circ \bar{\lambda}).\]

Combined with (9), this implies

\[(12) \quad |\bar{\lambda}(t_{v})| \leq N_{\mathfrak{v}}(\rho_0, \bar{\lambda}).\]

If now $\lambda$ runs through $P_{r}$, there are only finitely many possibilities for $\bar{\lambda}$, whence (4), with $a = \sup \langle \rho_0, \lambda \rangle (\lambda \in P_{r})$, for $v \in V_{D}$. Since $V_{1}$ is a finite union of such sets, this proves (4).

**Remark.** The relation (9) is proved in [35, pp. 27-29] for the split case. For a general semisimple simply connected group, see I. Macdonald, *Spherical functions on a group of p-adic type*, Publ. Ramanujan Institute 2, Madras, Theorem 4.7.1, or H. Matsumoto, Lecture Notes in Math., vol. 590, Springer-Verlag, Berlin and New York, Proposition 4.4.11. In fact, we have used it for a general connected reductive
group but the reduction to the case of simply connected semisimple groups is easily carried out by going over to the universal covering of the derived group.

13.3. **Corollary.** Let \( P \) be a parabolic \( k \)-subgroup of \( G \), \( P = M \cdot N \) a Levi decomposition over \( k \) of \( P \). Assume that \( \pi \) is a constituent of a representation \( \text{Ind}_A^G(\sigma) \) induced from a unitarizable irreducible admissible representation \( \sigma \) of \( M_A \), viewed as a representation of \( P_A \) trivial on \( N_A \). Then \( L(s, \pi, r) \) is absolutely convergent in some right half-plane.

We view \( L \) as a subgroup of \( L \). Let \( r' \) be the restriction of \( r \) to \( L \).

Let \( v \in V_f \) be such that the conditions listed at the end of 13.1 are satisfied by \( M, G, \sigma, \) and \( \pi_v \). Then, by the transitivity of induction, it follows that there exists \( \chi_v \) as in the above proof such that \( \sigma_v \) (resp. \( \pi_v \)) is the constituent of class 1 with respect to \( M(o_v)(\sigma) \) of the principal series \( \text{PS}(\chi_v) \) for \( M(k_v) \) (resp. \( G(k_v) \)). Then \( L(s, \pi_v, r) = L(s, \sigma_v, r') \) (7.2, 10.4). This being true for almost all \( v \)'s, we are reduced to 13.2.

14. **The \( L \)-function of an automorphic representation.**

14.1. A smooth representation of \( G_A \) is **automorphic** if it is a subquotient of the regular representation of \( G_A \) in \( G_A \). It is cuspidal if it consists of cusp forms. If so, it is unitary modulo the center. We let \( \mathcal{A}(G/k) \) denote the set of equivalence classes of irreducible admissible automorphic representations of \( G_A \). By Proposition 2 of [39], every \( \pi \in \mathcal{A}(G/k) \) is a constituent of a representation induced from some cuspidal \( \sigma \in \mathcal{A}(M/k) \), where \( M \) is a Levi \( k \)-subgroup of a parabolic \( k \)-subgroup of \( G \). Combined with 13.3 this yields the

14.2. **Theorem (Langlands).** Let \( \pi \in \mathcal{A}(G/k) \) and \( r \) be a representation of \( L \). Then \( L(s, \pi, r) \) is absolutely convergent in some right half-plane.

The \( L \)-function of an irreducible admissible automorphic representation will also be called an automorphic \( L \)-function.

14.3. There are several conjectures on the analytic character of \( L(s, \pi, r) \) for automorphic \( \pi \), all checked in some special cases, going back to the work of Hecke on \( L \)-series attached to Grössencharaktere and to modular forms.

(a) If \( \pi \in \mathcal{A}(G/k) \), then \( L(s, \pi, r) \) admits a meromorphic continuation to the whole complex plane.

(b) Assume that \( \pi \) and \( G \) are such that the local solution to the local problem yields factors \( L \) and \( \varepsilon \) at all places. It is then conjectured that there is a functional equation \( L(s, \pi, r) = \varepsilon(s, \pi, r) \cdot L(1 - s, \bar{\pi}, r) \), where \( \bar{\pi} \) is the contragredient representation to \( \pi \).

(c) In a number of cases, it has been shown that:

(*) If \( \pi \) is cuspidal, \( r \) irreducible nontrivial, then \( L(s, \pi, r) \) is entire.

Here and there, conjectures to the effect that this should be a general phenomenon have been stated. However, there are counterexamples. Heuristically, one sees this is likely to happen if \( \pi \) is lifted from a cuspidal representation of a reductive group \( H \) (in the sense of \( V \) below) and the restriction of \( r \) to \( lH \) contains the trivial representation.

14.4. (a) Let \( G = \text{GL}_n \) and \( r = r_n \) be the standard representation of \( \text{GL}_n(C) \). Then 14.3(b), (c) are proved in [25] for \( n = 2 \), in [19] for \( n \geq 2 \), if \( L \) and \( \varepsilon \) are de-
fined to be the products of the $L$- and $\varepsilon$-factors mentioned in 12.4. As recalled in 12.4, these are the same as those considered here at almost all places, and for $n = 2$, at all places.

(b) If $G = \text{GL}_2 \times \text{GL}_2$ and $r = r_2 \otimes r_2$, similar results are established by Jacquet in [21].

(c) Let $G = \text{GL}_2$. If $r: \text{GL}_2(C) \to \text{GL}_2(C)$ is the adjoint representation, then 14.3(b), (c) are announced in [16]. This extends results of Shimura [54]. If $r = \text{Sym}^2(r_2)$, then 14.3(b) is stated in [15], in the context of the global lifting (see V); for $\text{Sym}^4(r_2)$, it is also proved in [51], in the framework of 14.5 below.

(d) Let $k$ be a function field, $G = \text{GL}_m \times \text{GL}_n$ and $r = r_m \otimes r_n$. Let $\pi$ (resp. $\pi'$) be a cuspidal automorphic representation of the first (resp. second) factor. By the methods of [19], [26], [27], one can define $L$ and $\varepsilon$, and (Jacquet dixit) show 14.3(b), and also the holomorphy, except when $m = n$ and $\pi$ is contragredient to $\pi'$. These methods also yield further examples for other groups and for other representations. It is expected that similar results hold over number fields.

(e) 14.3(a) has also been checked when $G = \text{PSp}(4)$ in some cases in [1], and, in general, in [42]. A functional equation is also established. 14.3(a), (b) are announced in [41] for orthogonal groups in an odd number of variables over functional fields, for the local factors mentioned in 12.3. For a survey and earlier references, see [43]. See also [44].

14.5. We describe some cases in which 14.3(a) has been verified in [33] (see also [18] for a survey). Let $C$ be a split $k$-group, of adjoint type, endowed with its canonical $\sigma$-structure. Fix a Borel subgroup $B$ of $C$ and a maximal torus $T$ of $B$ defined over $\mathfrak{o}$. Let $P$ be a maximal proper standard parabolic subgroup and $P = M \cdot N$ its standard Levi decomposition. Since $C$ is adjoint, it is easily seen that $C(M)$ is a torus. The group $M/C(M)$ is semisimple, split over $k$, of adjoint type, of rank equal to $\text{rk}(C) - 1$. We let $G = M/C(M)$. The group $L^G$ is simply connected (2.2(2)). We have a natural inclusion $L^G \to L^M$, and $L^M$ is the Levi subgroup of a standard parabolic subgroup $LP = L^M \cdot U$ with unipotent radical $U$ (3.3). Let $A$ be the split component of $P$ in $T$, and $L^A^C$ the split component of $L^P$ in $L^T$. The group $L^A^C$ acts on the Lie algebra $\mathfrak{u}$ of $U$ and its eigenspaces are irreducible $L^G^C$-modules. We let $F_p$ denote the set of contragredient representations to these $L^G^C$-modules. The $L$-functions considered in [33] are of the form $L(s, \pi, r)$ with $r \in F_p$ and $\pi$ an irreducible cuspidal automorphic representation of $G$. A number of examples are given in which $L(s, \pi, r)$ admits a meromorphic continuation. This is deduced from the results of [32]: let $m$ be the length of a composition series of $\mathfrak{u}$ with respect to $M$. Then, for suitable numbering of the elements of $F_p$ and strictly positive integers $a_i$, there is a relation

$$M(s) = \prod_{1 \leq i \leq m} L(a_i s, \pi, r_i) \cdot L(s a_i + 1, \pi, r_i)^{-1},$$

where $M(s)$ is the intertwining operator occurring in the theory of Eisenstein series with respect to $P$, and is known to have a meromorphic continuation to the complex plane [32]. If $r = 1$, this and 13.2 yield the meromorphic continuation. In general, if we have the analytic continuation for all $r_i$'s except one, (1) gives it for the remaining one.

14.6. The converse problem is to what extent automorphic representations can be characterized by analytic properties of their $L$-functions, or to give analytic
conditions on a given $L$-function which will insure that it is automorphic. The first main result was Hecke's characterization of the Mellin transform of a parabolic modular form. Then came Weil's extension of this theorem to congruence subgroups [64], [65], its generalization in the context of representations in [25], and the extension to $GL_3$ [46], [27]. In those results, conditions are imposed on the $L$-functions of $\pi$ and of the twists $\pi \otimes \chi$ of $\pi$ by characters. However, the analogous statement is false from $n = 4$ on [46]. It may remain true if one imposes conditions on the twist $\pi \otimes \rho$ of $\pi$ by representations of $GL_{n-1}$ or only of $GL_{n-2}$. For results in that direction, over function fields, see [45].

Note however that in the general problem outlined here, one wishes rather to turn things around and deduce the analytical properties of some given $L$-series by showing directly that it is automorphic (see the seminars on base change and on zeta-functions of Shimura varieties [17], [8], [40]).

14.7. Other problems. (1) One "representation theoretic" form of "Ramanujan's conjecture" is the following: if $\pi = \otimes \pi_v$ is an irreducible nontrivial admissible cuspidal automorphic representation (and $G$ is simple), then each $\pi_v$ is tempered. It is now well known to be false for certain orthogonal or unitary groups, and even for one split group [20].

(2) Let $\pi$ be a unitary irreducible representation of $G_A$. If $G = GL_2$, then its multiplicity in the space of cusp forms $L_s^0(G(k)\backslash G(A))$ is at most one, "multiplicity one theorem" [25]. In fact there is even a "strong multiplicity one theorem" [38]: given $\pi_v$ for almost all $v$'s, there is at most one constituent $\pi$ of the space of cusp-forms with those local factors.

The multiplicity one theorem has been proved for $GL_n$ [52] and the strong form for $GL_3$ [28]. It is unknown whether it is true for $SL_2$. On the other hand, there are counterexamples for some inner forms of $SL_2$ [31].

Chapter V. Lifting Problems.

Although the problems on automorphic $L$-functions discussed in §14 are only partially solved, the solutions provide practically all cases in which an $L$-series (automorphic or not) has been proved to have meromorphic or holomorphic analytic continuation with functional equation. This suggests trying, given an $L$-series and a reductive group $G$, to see whether $G$ has an automorphic representation with the given $L$-series. Many instances of such questions can be viewed more precisely as special cases of the "lifting problem" or of the "problem of functoriality with respect to morphisms of $L$-groups." There is also a local version. For the sake of exposition, we shall start with the latter, but it should be borne in mind that the motivation and requirements stem from the global one, and that local and global are at present inextricably linked in many proofs. These questions were raised by Langlands in [35].

15. $L$-homomorphisms of $L$-groups.

15.1. Let $E$ be a field and $H$, $G$ connected reductive $E$-groups. A homomorphism $u: L^H \rightarrow L^G$ over $\Gamma_h$ is said to be an $L$-homomorphism if it is continuous and if its restriction to $L^{H_0}$ is a complex analytic homomorphism of $L^{H_0}$ into $L^{G_0}$. Let $E$ be local and $G$ quasi-split. If $\varphi \in \Phi(H)$, then $u \circ \varphi \in \Phi(G)$. In fact, condition 8.2(i) is clearly satisfied, by $u \circ \varphi$, and so is 8.2(ii) because every parabolic subgroup of $L^G$
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is relevant, $G$ being assumed to be quasi-split. Therefore $\varphi \mapsto u \circ \varphi$ defines a map $\Phi(H) \to \Phi(G)$, to be denoted $\Phi(u)$.

15.2. Let $E = k$ be a global field. For $v \in V$, the Galois group $\Gamma_k$ is a subgroup of $\Gamma_F$; hence the $L$-group of $G$ viewed as a $k_v$-group, to be denoted $L(G/k_v)$, is a subgroup of $L(G/k)$. Thus, in particular, the $L$-homomorphism $u$ of 15.1 defines by restriction an $L$-homomorphism $u_v: L(H/k_v) \to L(G/k_v)$, hence also a map $\Phi(u_v): \Phi(H/k_v) \to \Phi(G/k_v)$ ($v \in V$).

The “lifting problem” is, roughly speaking, whether such maps are mirrored by maps of representations in the local case, or of automorphic representations in the global case.

15.3. EXAMPLE: BASE CHANGE. Let $H$ be a split over $E$, $F$ a finite Galois extension of $E$, and $G = R_{F/E}H$. Then $L(G)$ is a product of copies of $L(H)$, indexed and permuted by $\Gamma_{F/E}$ ($5.1$). There is then a natural $L$-homomorphism $u$ which is the identity on $\Gamma_E$ and the diagonal map on $L(H)$. If $E$ is a local field, then $W'_F$ is an open normal subgroup of $W_{F'}$, and the map $\Phi(u)$ may be viewed as given by the restriction to $W'_F$.

16. Local lifting.

16.1. Let $k = E$ be a local field, $G$ quasi-split over $E$, $H$ a connected reductive $E$-group and $u: L(H) \to L(G)$ an $L$-homomorphism. The problem of local lifting is, roughly, to establish a correspondence $\Pi(u): \Pi(H(k)) \to \Pi(G(k))$ which preserves $L$- and $\varepsilon$-factors. If the local parametrization problem of III is solved, then $\Pi(u)$ is the map between indistinguishable classes which assigns $\Pi_{\varphi \circ \varepsilon, G}$ to $\Pi_{\varphi, H}$ ($\varphi \in \Phi(H)$). The element $\Pi \in \Pi(G(k))$ is said to be a lift of $\pi \in \Pi(H(k))$ if $\Pi \in \Pi_{\varphi \circ \varepsilon, G}$, where $\varphi \in \Phi(H)$ is such that $\pi \in \Pi_{\varphi, H}$. We have then

$$L(s, \Pi, r) = L(s, \pi, r \circ \varphi), \quad \varepsilon(s, \Pi, r, \varphi) = \varepsilon(s, \pi, r \circ u, \varphi).$$

for every representation $r$ of $L(G)$.

16.2. The local lifting is thus viewed as a map between classes of $L$-indistinguishable representations rather than one between representations. However it is possible to single out one lifting under assumptions which, in the global case, are satisfied almost everywhere: assume $H$, $G$ to be quasi-split, split over an unramified extension $F$ of $E$, endowed with an $o_E$-structure such that $H(o_F)$ and $G(o_F)$ are very special maximal compact subgroups, and $\pi$ of class one with respect to $H(o_E)$. Then $\varphi$ such that $\pi \in \Pi_{\varphi, H}$, and the set $\Pi_{\varphi \circ \varepsilon, G}$ are well defined. Moreover, $\Pi_{\varphi \circ \varepsilon, G}$ contains exactly one element of class one (with respect to $G(o_E)$), to be called the natural lift of $\pi$.

16.3. A full solution of the local parametrization problem does not seem to be in sight, and it is conceivable that it may require proving at the same time global results such as Artin’s conjecture. Meanwhile, one wants to settle some approximations to it, notably to be able to prove some cases of Artin’s conjecture. Note that if $G = \text{GL}_n$, then the sets $\Pi_{\varphi, G}$ are either known or conjectured to consist of one element ($12.2, 12.3$). Such a lifting problem can then be stated as one of constructing a map $u_\varphi: \Pi(H(k)) = \Pi(G(k))$ satisfying certain conditions. So far, there are two examples:

(a) Base change (cf. 15.3) when $H = \text{GL}_2$ and $F$ is cyclic of prime degree over $E$ [17], [38], [49], [56]. Besides some naturality conditions and 16.3, the main require-
ments relate the characters of \( \pi \) and of the hypothetical \( u_\ast(\pi) \). The results also describe the fibres and the image of \( u_\ast \). [Note that the results of \( [38] \) on this problem are used in \( [62] \), so that we cannot invoke the solution of the local parametrization problem (12.4) for \( GL_2 \) just to use the map \( II(u) \) of 16.1. If we could, then the local questions \( [38] \) would be mainly to relate the characters of \( \pi \) and \( II(u)(\pi) \).]

(b) \( H = GL_2, \ G = GL_3, \) and

(1) \[ u: L^H = GL_2(\mathbb{C}) \rightarrow L^G = GL_3(\mathbb{C}) \]
is given by the adjoint representation of \( L^H \) (see \( [16] \)).

In this case, \( II = u_\ast(\pi) \) must be trivial on the center of \( L^G \) and be such that the \( L \)- and \( \varepsilon \)-factors of \( u_\ast(\pi) \otimes \chi \) (\( \chi \) character of \( E^* \)) are certain given functions. There is at most one such \( II \)(12.4(d)). In \( [16] \), \( II \) is stated to exist, except possibly if \( E \) has even residual characteristic and \( n \) is "extraordinary."

16.4. In 16.3(a), the lifting problem was connected with the existence of relations between characters. This is a direct connection between \( II(H) \) and \( II(G) \), which is of great importance for the use of the trace formula in proving or using the local or global lifting. We now mention two other examples of such relations. Assume that \( G \) is a quasi-split inner form of \( H \). There is then an isomorphism \( u: L^H \simeq L^G \) and an embedding \( \Phi(u): \Phi(H) \subset \Phi(G) \). If \( f: H \rightarrow G \) is a \( k \)-isomorphism such that \( f^{-1} \cdot T \) is an inner automorphism of \( G \) for every \( \gamma \in \Gamma_k \), then \( f \) establishes a bijection between conjugacy classes which are stable under \( \Gamma_k \). Using results of Steinberg \( [59] \), one then sees easily that maximal \( k \)-tori in \( H \) are isomorphic over \( k \) to maximal \( k \)-tori in \( G \). This allows one in some cases to assign regular semisimple classes in \( G(k) \) to such classes in \( H(k) \), so that it makes sense to compare values of characters of \( H(k) \) and of \( G(k) \) on such classes.

(a) Let \( k \) be either \( R \) or nonarchimedean with odd residual characteristic. Let \( G = GL_2 \) and \( H \) be the group of invertible elements in the quaternion algebra over \( k \). The sets \( II_\varphi \) are singletons, \( \Phi(u) \) assigns to a (finite dimensional) irreducible representation \( \pi \) of \( H(k) \) a discrete series representation \( \pi' \) of \( G(k) \). In this case, the semisimple classes of \( H(k) \) correspond to the elliptic classes in \( G(k) \). It is proved in \( [25] \) that the characters of \( \pi \) and \( \pi' \) differ only by a sign on those classes.

(b) Let \( k = R \). For \( \varphi \in \Phi(H), \ \Phi(G), \) let \( \chi_\varphi \) be the sum of the characters of the elements in \( II_\varphi \). Choose \( \varphi \in \Phi(H) \) such that \( II_\varphi \) consists of tempered representations. Then \( \chi_\varphi \) and \( \chi_{u\varphi} \) are equal on the regular semisimple classes of \( H(k) \), up to a sign depending only on \( H \) and \( G \) \( [53, 6.3] \).

16.5. We could also take the Weil forms of the \( L \)-groups. In that case an \( L \)-homomorphism, restricted to \( W_E \), is assumed to satisfy the obvious analogue of 8.2(i). Take in particular the case where \( H = \{1\} \). Then \( u \) is just an element of \( \Phi(G) \). The lifting problem in this case is part of the local problem of III.

17. Global lifting.

17.1. Assume \( G \) to be quasi-split. Let \( H \) be a reductive \( k \)-group and \( u: L^H \rightarrow L^G \) an \( L \)-homomorphism. Let \( u_\varphi: L^H(k_\varphi) \rightarrow L^G(k_\varphi) \) and \( \Phi(u_\varphi): \Phi(H)(k_\varphi) \rightarrow \Phi(G)(k_\varphi) \) be the associated maps (\( \nu \in V \) ) (see 15.1).

Let \( \pi = \bigotimes \varphi \pi_\varphi \) (resp. \( II = \bigotimes \varphi II_\varphi \)) be an irreducible admissible representation of \( H_\varphi \) (resp. \( G_\varphi \)). Then \( II \) is said to be a lift of \( \pi \) if \( II_\varphi \) is one of \( \pi_\varphi \) for every \( \nu \in V \) (16.1).

If that is the case, then, for every representation \( r \) of \( L^G \), we have

(1) \[ L(s, II, r) = L(s, \pi, r \circ u), \quad \varepsilon(s, II, r) = \varepsilon(s, \pi, r \circ u). \]
It is also usually requested that \( \Pi_v \) be the natural lift (16.2) of \( \pi_v \) for almost all \( \nu \)'s. The question is then whether every automorphic \( \pi \) has a lift, which is automorphic, or, somewhat more ambitiously, whether there is a map \( u_*: \mathfrak{a}(H/k) \to \mathfrak{a}(G/k) \) with reasonable properties, which sends \( \pi \in \mathfrak{a}(H/k) \) onto a lift of \( \pi \). One also wants to describe the fibres and the image of \( u_* \).

In that degree of generality, the problem appears to be inaccessible at present. However, there are many results, old and recent, which are very striking illustrations of this principle, some of which will be extensively discussed in various seminars. Here, for orientation, and to give an idea of the scope of the problem, I shall list briefly some special cases, referring to the literature or to other seminars for more details.

**Remark.** Let \( r \) be a representation of \( L \) of degree \( n \). Then it defines an \( \mathbb{L} \)-homomorphism \( u: L \to \mathbb{L}G_n = \mathbb{L}GL_n(C) \times I'k \) in the obvious way. A positive answer to the lifting problem would imply in particular that if \( \pi \) is an automorphic representation of \( H \), then \( L(s, \pi, r) = L(s, \mathfrak{a}, \pi) \), where \( \mathfrak{a} \) is an automorphic representation of \( GL_n \) and \( r \) the standard representation. This would therefore to a large extent reduce the study of automorphic \( \mathbb{L} \)-functions to those of \( GL_n \), with respect to the standard representation.

17.2. Let \( H = \{1\} \), \( G = \mathbb{L}GL_n \). Then an \( \mathbb{L} \)-homomorphism \( u \) is just a continuous complex \( n \)-dimensional representation of \( I'k \). The question is then whether the Artin \( \mathbb{L} \)-series \( L(s, u) \) is an automorphic \( \mathbb{L} \)-series of \( GL_n \) (with respect to the standard representation of \( GL_n(C) \)), which should be cuspidal if \( u \) is irreducible. In view of known results on \( GL_n \) (cf. 14.4) this would imply Artin’s conjecture.

For \( n = 1 \), a positive answer is given by class-field theory. For \( n = 2, 3 \), a positive answer is equivalent to Artin’s conjecture, since there are converses to Hecke theory \([25],[65],[27],[46]\). For \( n = 2 \), it has been proved for dihedral or tetrahedral representations of \( I'k \), and for some others over \( \mathbb{Q} \) (see \([38],[17],[15]\)).

17.3. Let \( k' \) be a Galois extension of \( k \), \( n \) the degree of \( k' \) over \( k \). Take \( H = R_{k'/k}GL_n \), \( G = \mathbb{L}GL_n \). There is a natural homomorphism \( f: L^\infty H \to LGL_n(C) \times I'k \) into the normalizer of a maximal torus \( L^\infty \) of \( LGL_n(C) \). Since the former group is a quotient of \( L^\infty H \), and \( L^\infty G = GL_n(C) \times I'k \), we can define an \( L \)-homomorphism \( u: L^\infty H \to L^\infty G \) by \( u(h, \gamma) = (f(h), \gamma) \) \((h \in L^\infty H, \gamma \in I'k)\). An automorphic representation of \( H \) is a Grössencharakter \( \chi \) of \( k' \). The problem is then whether the Artin \( L \)-series \( L(s, \chi) \) is the \( L \)-series of an automorphic representation of \( G \).

If \( n = 2 \), \( k = \mathbb{Q} \), and \( k' \) is imaginary, this was proved by Hecke; \( \pi \) is associated to a cuspidal holomorphic automorphic form. If \( n = 2 \), \( k = \mathbb{Q} \), and \( k' \) is real quadratic, this was established by H. Maass. \( \pi \) is then associated to a nonholomorphic automorphic form.

For \( n = 3 \), this is proved in \([26],[27]\).

17.4. Base change. This is the global counterpart to 16.3(a). Let \( k' \) be a finite Galois extension of \( k \). Assume \( H \) to be \( k \)-split and \( G = R_{k'/k}H \). There is again an \( L \)-homomorphism \( u: L^\infty H \to L^\infty G \) whose restriction to \( L^\infty H \) is a diagonal map. In this case \( G(A) \) and \( G(k) \) are canonically isomorphic to \( H(A_{k'}) \) and \( H(k') \); therefore the problem is to associate an automorphic representation of \( H(A_{k'}) \) to an automorphic representation of \( H(A_k) \). Again, it should be a counterpart to the restriction to \( W_k \) of homomorphisms \( W_k \to L^\infty H \).

If \( H = \mathbb{L}GL_2 \) and \( k' \) is cyclic of prime degree, the lifting map \( u_* \) for representations is constructed in \([38]\), which also gives a description of its image and fibres.
This extends work of Doi-Naganuma, Jacquet [21] (on the quadratic case) and of Saito [49], Shintani [55], [56] (cf. [17]).

17.5. Let $G$ be quasi-split, and $H$ an inner form of $G$. Then $L^H = L^G$ and $\phi(H/k_v) \subset \phi(G/k_v)$ for all $v$'s (8.3). Moreover, for almost all $v$'s, $H$ and $G$ are isomorphic over $k_v$; hence $\phi(H/k_v) = \phi(G/k_v)$ and $I(H(k_v)) = I(G(k_v))$. The question is then, given $\pi = \otimes_v \pi_v$, is there an automorphic representation $\Pi = \otimes_v \Pi_v$ of $G$ such that $\Pi_v = \pi_v$ for almost all $v$'s?

If $G = \text{GL}_2$ and $H$ is the group of invertible elements of a quaternion algebra $D$ over $k$, a positive answer is given by Jacquet-Langlands [25]. Note that, in that case, because of the “strong multiplicity one theorem,” at most one $\Pi$ may be associated to a given $\pi$ in this way. The possible $\Pi$'s are in fact the cuspidal automorphic representations for which $\Pi_v$ belongs to the discrete series for all $v$'s over which $D$ does not split (loc. cit.).

17.6. If $G = \text{GL}_2$, $G = \text{GL}_3$ and $u$ is given by the adjoint representation, as in 13.4, the global lifting problem has been solved by Gelbart-Jacquet [16], the “local lifting” being the one of 16.2(b).

17.7. Let $M$ be a Levi $k$-subgroup of a parabolic $k$-subgroup $P$ of $G$. Then $L^M$ imbeds naturally into $L^G$ (3.3), whence an $L$-homomorphism $u: L^M \to L^G$. If $\pi$ is cuspidal, then the analytic continuation and residues of Eisenstein series [32] are known to yield a unitary $u^*(\pi)$ in many cases, and, conjecturally, in general.

18. Relations with other types of $L$-functions.

18.1. In 17.2, the lifting problem amounts to identifying an Artin $L$-function with an automorphic $L$-function on $\text{GL}_n$. One can also include in this problem more general representations of Weil groups if one passes to the Weil form of the $L$-groups. For simplicity, let us limit ourselves to relative Weil groups $W_{k'/k}$, where $k'$ is a finite Galois extension of $k$ over which $H$ and $G$ split. An $L$-homomorphism $u: L^H \times W_{k'/k} \to L^G \times W_{k'/k}$ is then a continuous homomorphism compatible with the projections on $W_{k'/k}$ whose restriction to $L^G$ is a complex analytic homomorphism into $L^G$, and such that, for $w \in W_{k'/k}$, $u(w) = (u'(w), w)$ with $u(w)$ semisimple (cf. 8.2(i)).

If $H = \{1\}$, an $L$-homomorphism is said to be an admissible homomorphism of $W_{k'/k}$ into $L^G$. In analogy with the definition of $\Phi(G)$ in the local case, we can consider the set $\Phi_{k'/k}(G)$ of equivalence classes of such homomorphisms, modulo inner automorphisms of $L^G$, and then pass to a suitable limit $\Phi(G)$ over $k'$.

The lifting problem asks in this case to associate to any $\varphi \in \Phi(G)$ an automorphic representation $\pi$, such that, for any representation $r$ of $L^G$, $L(s, \pi, r)$ is equal to the Artin-Hecke $L$-series of $r \circ u$. In particular, is every Artin-Hecke $L$-series that of an automorphic representation of $\text{GL}_n$ with respect to the standard representation?

If $G$ is a torus, then [34] provides a positive answer. In fact, in this case the irreducible admissible automorphic representations of $G$ are the characters of $G(k')G(A)$, and [34] gives a homomorphism with finite kernel of $\Phi_{k'/k}(G)$ onto the set of such characters.

18.2. In the same vein, it is natural to ask whether Hasse-Weil zeta-functions (or even $L$-functions of compatible systems of $l$-adic representations of Galois groups) can be expressed in terms of automorphic $L$-functions. For elliptic curves over function fields, it is a theorem. That it should be the case for elliptic curves over
\( Q \) is the Taniyama-Weil conjecture; it has been checked in a number of special cases (see [2], [14] for surveys from the classical and representation theoretic points of view respectively). Apart from that, this problem has been pursued mostly for Shimura curves and certain Shimura varieties; we refer to the corresponding seminars for a description of the present state of affairs.

Finally, one may ask whether it is possible to characterize a priori those automorphic representations whose \( L \)-series have an arithmetic or algebraico-geometric significance. A necessary condition if \( k \) is a number field is that for an infinite place \( v \), \( \pi_v \) should be associated to a representation \( \sigma_v \) of \( W_k \) whose restriction to \( C^* \) is rational, \( C^* \) being viewed as real algebraic group, i.e., be of type \( A_0 \) in [3, 6.5]. If the \( L \)-series of \( \pi \) is to be an Artin \( L \)-series, then \( \pi \) should even be of type \( A_{00} \) (loc. cit.), i.e., \( \sigma_v \) should be trivial on \( C^* \). Let \( k = Q \). Then there are three possibilities for \( \pi_v \) (11.5). If \( \pi_v = \pi(1, \text{ sgn}) \), then \( \pi \) corresponds to 2-dimensional representations of \( \Gamma_Q \) with odd determinant by the theorem of Deligne-Serre [10], [50]. Modulo the Artin conjecture for such representations, the correspondence is bijective. However, I am not aware of any result for the other two possible values of \( \pi_v \). A positive answer would involve nonholomorphic automorphic forms. In [36], it is shown in many cases for \( \text{GL}_2 \) over \( Q \) that the \( L \)-series of a representation of type \( A_0 \) is that of a compatible system of \( l \)-adic representations of \( \Gamma_Q \). Over a function field, there is no condition such as \( A_0 \). In fact, for \( \text{GL}_2 \), Drinfeld has shown that all irreducible admissible automorphic representations are associated to \( l \)-adic representations (see the lectures on his work by G. Harder and D. Kazhdan).

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