3. WEYL TRICK AND SCHUR'S LEMMA

1. Complete reducibility

1.1. **Unitary representations.** In this section we assume that (π, V) is a unitary representation of G. This means that there exists a Hilbert structure on V which is preserved by the action of G, in that

$$< \pi(g)v, \pi(g)w >=< v, w >, \quad \forall g \in G,\ v, w \in V$$

Assume π is not irreducible. Then V has a proper G-invariant subspace W. But further reducing W if necessary, we may assume that W itself is irreducible. Since π is unitary, it follows that $W^\perp$ is G-ivnariant, for if $u \in W^\perp$ then (applying g^{-1})

$$< \pi(g)u, w >=< u, \pi(g^{-1})w >= 0, \quad \forall w \in W$$

showing that $\pi(g)u \in W^\perp$.

Hence we can write: $V = W \oplus W^\perp$, with both W and $W^\perp$ G-invariant. If $W^\perp$ has proper G-invariant subspaces, we can further decompose $W^\perp$ as a direct sum of (orthogonal) G-invariant subspaces. Eventually this process has to stop (the dimension is lowering) and we end up with a decomposition

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$$

where W_j, $1 \le j \le k$ are mutually orthogonal, G-invariant, irreducible subspaces. We have thus the following

1.2. **Proposition.** Unitary representations are completely decomposable.

1.3. **Example.** S_3 acts unitary on $\mathbb{C}^3$ (check!) and $L_0 = \mathbb{C} \cdot (1,1,1)$ is an invariant subspace. Then $L_0^\perp = U$ is G-invariant and $\mathbb{C}^3 = L_0 \oplus U$.

1.4. **Example.** In this example consider the following representation of $G = \mathbb{R}$ on $\mathbb{C}^2$:

$$\pi : \mathbb{R} \to GL(\mathbb{C}^2), \quad \pi(x) = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$$

The one-dimensional subspace $S = \{(t, 0) : t \in \mathbb{C}\}$ is $\mathbb{R}$-invariant, yet the representation π is not completely reducible [homework].

2. Weyl trick

2.1. **Question.** Given a (finite) group G, which representations of G are unitary?

2.2. **Theorem.** Assume (π, V) is a representation of the finite group G. Then V admits a Hilbert structure that is G-invariant.

2.2.1. *Proof.* Assume $<,>_0$ is a Hibert structure on V. Define:

$$< v, w >= \frac{1}{G} \sum_{g \in G} < \pi(g)v, \pi(g)w >_0$$

Then it is easy to see that $<,>$ is a G-invariant inner product, and it is positive definite since $< v, v >= \frac{1}{|G|} \sum_{g \in G} \|\pi(g)v\|_0^2 \ge 0$.

3. Complete reducibility rev.

3.1. **Corollary.** (G finite group): given an arbitrary (finite dimensional) representation of a finite group G, there exist integer numbers $m_a(\pi) \ge 0$ (possibly zero) such that

$$\pi = \bigoplus_{a \in \widehat{G}} m_a(\pi) a$$

The decomposition is unique [homework].

3.1.1. *Equivalent formulation.* Assume $a = (\pi_a, V_a) \in \widehat{G}$, in other words V_a is the space on which the representation a occurs. Then there exists a intertwining isomorphism:

$$T : \prod_{a \in \widehat{G}} V_a^{m_a} \to V$$

Although the map T is not unique, the following things are unique (depending on π only):

- the indices m_a, and implicitly the irreps $a \in \widehat{G}$ that actually occur in the decomposition (such that $m_a(\pi) > 0$)
- the a-isotypic component $V(a) = T(V_a^{m_a})$ of V. This is the direct sum of G-invariant irreducible subspaces in the class of a.

3.2. **Trace.** In particular we have $\chi_\pi = \sum_{a \in \widehat{G}} m_a(\pi)\chi_a$.

4. Schur's Lemma

4.1. **Theorem.** Assume (π, V) and (σ, W) are two irreducible representations of G, and $T : V \to W$ a G- intertwining operator. Then $T = 0$, if π and σ are inequivalent G-representations, and a multiple of the identity map, otherwise.

4.1.1. *Proof.* The proof is an immediate consequence of the observation that both the kernel and the image of an intertwining map are G-invariant subspaces.

4.2. **Corollary.** For $a, b \in \widehat{G}$, $\mathrm{Hom}_G(V_a, V_b) = \delta_{ab}\mathbb{C}\cdot I_a$. Equivalence classes of representations. Notation: $\widehat{G}$ collection of equivalence classes of irreducible representations. Notation: $Hom_G(V_1, V_2)$ intertwining operators.

4.3. **Isotypic vectors revisited.**

4.3.1. *Observation.* If $V = V_1 \oplus V_2$ is a direct sum of G-invariant subspaces, then the projection $P : V \to V_1$ is an intertwining operator.

4.3.2. *Uniqueness of decomposition.* Assume $V = \oplus_{i=1}^m U_i = \oplus_{j=1}^n W_j$ are two different complete decomposition of V into irreducible (not necessarily mutually orthogonal) subspaces. Then the projections $P_{ji} : W_j \hookrightarrow V \to U_i$ are intertwining operators between irreducible subspaces. By Schur's lemma each such P_{ji} is either 0 or an isomorphism. A careful bookkeeping shows that one can relabel the irreducible subspaces such that $m = n$ and $U_i \simeq W_i$, $1 \leq i \leq m$.

4.3.3. A more "invariant" description of $V(a)$ is: the set of vectors that lie in the linear span of images of all possibles maps $T \in Hom_G(V_a, V)$.

4.4. **Example.** $V = \mathbb{C} \times \mathbb{C}^3$, $\sigma \cdot (x_0, y) = (x_0, y_{\sigma^{-1}})$. Then

$$V \simeq \chi_0 \oplus \chi_0 \oplus \sigma = 2 \cdot \chi_0 \oplus \sigma$$

with σ the standard irreducible representation in dimension 2.

4.5. **Abelian groups.** Assume G is abelian. Since $\widehat{G}$ consists of one-dimensional representations, it means that every representation (π, V) of G can be decomposed as $\pi = \oplus_{i=1}^n \chi$, where χ_i are group characters $\chi : G \to \mathbb{C}^\times$. In other words, there exists a basis $\mathcal{B}$ on V such that the action of π with respect to this basis is given by matrices of type

$$\pi(g)_{\mathcal{B}} = \begin{bmatrix} \chi_1(g) & & \\ & \ddots & \\ & & \chi_n(g) \end{bmatrix}, \quad \forall g \in G$$

5. Duality in Hilbert spaces

5.1. **Riesz representation theorem.** V with Hilbert structure. For $w \in V$, let $\lambda_w \in V^*$ given by $\lambda_w(v) =< v, w >$. Then $w \mapsto \lambda_w$ is a $\mathbb{R}$-linear isomorphism $V^* \simeq V$. In particular, it is bijective.

Note that λ is not an isomorphism of complex vector spaces since $\lambda_{cw} = \bar{c}\lambda_w$, for $w \in V$ and $c \in \mathbb{C}$.

5.2. **Adjoint.** Let V, W Hilbert spaces. The adjoint map $* : \mathcal{L}(V, W) \to \mathcal{L}(W, V)$ is defined by given by $< Av, w >=< v, A^*w >$.

Note that the operation is well defined due to the Riesz representation theorem.

5.3. **Skew-bilinear maps.** Let V, W two finite dimensional Hilbert spaces. $B : V \times W \to \mathbb{C}$ with the properties:

$$\begin{cases} B(c_1 v_1 + c_2 v_2, w) = c_1 B(v_1, w) + c_2 B(v_2, w), & \forall c_i \in \mathbb{C}, v_i \in V, w \in W \\ B(v, c_1 w_1 + c_2 w_2) = \overline{c}_1 B(v, w_1) + \overline{c}_2 B(v, w_2), & \forall c_i \in \mathbb{C}, v \in V, w_i \in W \end{cases}$$

Then there exists linear map $A : V \to W$ such that $B(v, w) =< Av, w >$.

5.3.1. *Proof.* For a fixed $w \in W$, the $v \mapsto B(v, w)$ is in V^*, so there exists $T(w) \in V$ such that $B(v, w) =< v, A(w) >$. It is easy to see that the map $w \mapsto T(w)$ is $\mathbb{C}$-linear. Then $B(v, w) =< v, T(w) >=< T^* v, w >$, so $A = T^*$ is the map we're after.