Topology and its Applications 312 (2022) 108082

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Centric linking systems of locally finite groups

Rémi Molinier

Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France

ARTICLE INFO

Article history:
Received 24 June 2020
Received in revised form 3 March 2022
Accepted 14 March 2022
Available online 17 March 2022

MSC:
55R35
20D20
55R40

Keywords:
Locally finite group
Sylow p-subgroup
Classifying space
Linking system

ABSTRACT

These notes are defining the notion of centric linking system for a locally finite group. If a locally finite group G has countable Sylow p-subgroups, we prove that, with a finiteness condition on the set of intersections of Sylow p-subgroups, the p-completion of its classifying space is homotopy equivalent to the p-completion of the nerve of its centric linking system.

1. Introduction

Through all the paper, p will denote a fixed prime number. Bousfield and Kan introduced in the 70's a notion of *p-completion* for spaces [1]. It consists in a functor $(-)^\wedge_p$ from spaces to spaces together with a natural transformation $\lambda\colon \mathrm{Id} \to (-)^\wedge_p$ and its main property is that a map f induces an isomorphism in mod p cohomology if and only if $f^\wedge_p$ is an homotopy equivalence. Even more, when the space X is *p-good* (i.e. $\lambda_X\colon X \to X^\wedge_p$ induces an isomorphism in mod p homology) λ_X is a final object among homotopy classes of maps defined on X inducing an isomorphism in mod p cohomology. For example, classifying spaces of finite groups and compact lie groups are p-good.

The notion of centric linking system of a finite group was first introduced by Broto, Levi, and Oliver [2] to study the p-completion of classifying spaces of finite groups. It was an important tool in the proof by Oliver of the Martino-Priddy Conjecture [3,4] which roughly states that, for two finite groups G and H,

E-mail address: remi.molinier@univ-grenoble-alpes.fr.
URL: https://www-fourier.ujf-grenoble.fr/~molinier/.

https://doi.org/10.1016/j.topol.2022.108082
0166-8641/

$BG_p^\wedge \simeq BH_p^\wedge$ if and only if they have the same p-local structure. Later, Broto, Levi, and Oliver defined the notion of centric linking system associated to a saturated fusion system over a finite p-group [5] or a discrete p-toral group [6] to construct classifying spaces for fusion systems and develop the homotopy theory of fusion systems. Moreover, in [6], they generalized centric linking systems of finite groups to centric linking systems of locally finite groups with discrete p-toral Sylow p-subgroups. They also proved the following nice result ([6, Theorem 8.7]): given a locally finite group G with discrete p-toral Sylow p-subgroups and satisfying some technical stabilization condition on centralizers, the p-completion of the nerve of its centric linking system has the homotopy type of the p-completion of the classifying space of G.

On the other hand, Chermak and Gonzalez [7], using the language of localities, are considering fusion systems over countable p-groups. This allows to consider fusion systems of a much more larger class of groups which contains in particular algebraic groups over the algebraic closure of $\mathbb{F}_p$. The groups they are considering are countable locally finite groups with a finite dimensionality condition on a certain poset of p-subgroups. This condition guarantees in particular the existence of Sylow p-subgroups and allows a study of the p-local structure of these groups.

Here we generalize the notion of centric linking system to any locally finite group. We are in particular interested in the case of locally finite groups with countable Sylow p-subgroups. We prove in Theorem 1 that, for a locally finite group G with countable Sylow p-subgroups, with a finiteness condition on the number of conjugacy classes in the poset of intersections of Sylow p-subgroups of G, the p-completion of the nerve of the centric linking system has the homotopy type of the classifying space of G. This generalizes the previous result of Broto, Levi, and Oliver and can be a starting point of an homotopy theory of discrete localities developed in [7]. The surprising part of this result is that we get some information on these p-completions even if we do not know that the spaces we are considering are p-good.

Notation. We will write $H \leq G$ to express that H is a subgroup of G. Moreover, for G a group and H a subgroup of G we denote by H^g the subgroup $g^{-1}Hg$. Also, for $\mathcal{C}$ a small category, we denote by $|\mathcal{C}|$ the geometric realisation of the nerve of $\mathcal{C}$.

Acknowledgement. The author is grateful to Andy Chermak for suggesting the problem. The author also deeply thank the referee for his careful reading and spotting some mistakes in the first place.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

2. Sylow p-subgroups

In this paper, a group is said to be *locally finite* if every finitely generated subgroup of G is finite. A *p-group* is a locally finite group where every element of P has finite order a power of p and a *p-subgroup* is a subgroup which is a p-group.

Definition 1. Let G be a group and $S \leq G$ be a p-subgroup. We say that S is a *Sylow p-subgroup* of G if

(i) S is maximal in the poset of p-subgroups of G, and
(ii) every p-subgroup of G is conjugate to a subgroup of S.

We denote by $\mathrm{Syl}_p(G)$ the set of all Sylow p-subgroups of G.

Lemma 1. *Let G be a group with $Syl_p(G) \neq \emptyset$.*

(a) Any two elements in $Syl_p(G)$ are conjugate.

(b) Let S be a p-subgroup of G maximal in the poset of p-subgroups of G, then $S \in Syl_p(G)$.

Proof. Let S be a p-subgroup of G maximal in poset of p-subgroups of G and $S' \in \mathrm{Syl}_p(G)$. Since S' is a Sylow p-subgroup of G, there is $g \in G$ such that $S^g \leq S'$. Assume that $S^g < S'$. Then $(S')^{g^{-1}}$ is a p-subgroup of G which contains strictly S and this contradicts the maximality of S. Thus $S^g = S'$ and this proves (a) and (b). □

For G a group such that $\mathrm{Syl}_p(G)$ is non-empty, we denote by $\Omega_p(G)$ the poset, ordered by inclusion, of all subgroups of G which are intersections of Sylow p-subgroups of G. Since $\mathrm{Syl}_p(G)$ is closed by conjugation in G, $\Omega_p(G)$ is also closed by conjugation in G. If $S \in \mathrm{Syl}_p(G)$ we will also define $\Omega_S(G) = \mathrm{Sub}(S) \cap \Omega_p(G)$ the poset of subgroups of S which are in $\Omega_p(G)$.

Definition 2. Let G be a group with $\mathrm{Syl}_p(G) \neq \emptyset$. For P a p-subgroup of G we define $P^\circ \in \Omega_p(G)$ as the intersection of all Sylow p-subgroups of G containing P.

For G a group and P, Q two subgroups of G we denote by $N_G(P, Q)$ the set of elements $g \in G$ such that $P^g \leq Q$. The next proposition gathers the main properties of the map $P \mapsto P^\circ$.

Proposition 1. *Let G be a group with $Syl_p(G) \neq \emptyset$ and P, Q be two p-subgroups of G.*

(a) $P \leq P^\circ$ and, if $P \leq Q$ then $P^\circ \leq Q^\circ$.
(b) If $P \in \Omega_p(G)$, $P^\circ = P$.
(c) $N_G(P, Q) \subseteq N_G(P^\circ, Q^\circ)$.
(d) If $Q \in \Omega_p(G)$ then $N_G(P^\circ, Q) = N_G(P, Q)$.

Proof. (a) follows from the definition of $(-)^\circ$. (b) is a direct consequence of the definition of $\Omega_p(G)$. To prove (c) let $g \in N_G(P, Q)$. By a direct calculation we have $(P^\circ)^g = (P^g)^\circ$. Thus, by (a),

$$(P^\circ)^g = (P^g)^\circ \leq Q^\circ$$

and $g \in N_G(P^\circ, Q^\circ)$. Finally, (d) follows from (a) and (c). □

In the main Theorem, we will need to filter $\Omega_p(G)$ in a proper way. The following Lemma shows that the existence of such a filtration is equivalent to require that one works with finitely many conjugacy classes.

Lemma 2. *Let G be a group with $Syl_p(G) \neq \emptyset$. Let $\Omega_0 \subseteq \Omega_p(G)$ be a collection of subgroups which is stable by conjugation in G and such that for any $P, Q \in \Omega_p(G)$ with $P \leq Q$, if $P \in \Omega_0$ then $Q \in \Omega_0$.*

The following are equivalent.

(i) $\Omega_p(G) \setminus \Omega_0$ contains finitely many G-conjugacy classes.
(ii) There exists an at most countable filtration $\Omega_0 \subseteq \Omega_1 \subseteq \cdots \subset \Omega_p(G)$ such that

(a) $\bigcup_{r \geq 0} \Omega_r = \Omega_p(G)$,
(b) for all $r \geq 0$, $\Omega_{r+1} \setminus \Omega_r$ consists in a single G-conjugacy class, and
(c) for all $r \geq 0$ and $P, Q \in \Omega_r$ such that $P \leq Q$, if $P \in \Omega_r$ then $Q \in \Omega_r$.

Proof. Assume (i) and let $\mathcal{X} = \{C_1, C_2, \ldots, C_n\}$ be the set of G-conjugacy classes of $\Omega_p(G) \setminus \Omega_p^c(G)$. We can endow $\mathcal{X}$ with a partial order $\preceq$ defined by $C_i \preceq C_j$ if and only if there exists $P_i \in C_i$ and $P_j \in C_j$ such that $P_i \leq P_j$. Up to a permutation of the elements of $\mathcal{X}$ we can assume that $C_i \preceq C_j$ implies that $i \geq j$ (in particular, C_0, resp. C_n, is a minimal, resp. maximal, element of $\mathcal{X}$). We then define, for $r \in \{0, 1, 2, \dots, n\}$,

$$\Omega_r = \Omega_0 \cup \bigcup_{i=1}^{r} C_i.$$

One can then see that $\Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_n$ satisfies properties (a), (b) and (c).

Now assume (ii) and let $(\Omega_r)_r$ be such a filtration. Let $O_p(G)$ be the intersection of all the Sylow p-subgroups of G. By (a), there exists r_0 such that $O_p(G) \in \Omega_{r_0}$. In particular, thanks to (c), $\Omega_{r_0} = \Omega_p(G)$. Therefore, since at each step of the filtration one is only adding one conjugacy class by (b), $\Omega_p(G) \smallsetminus \Omega_0$ contains exactly r_0 G-conjugacy classes. □

We do not discuss here the existence of Sylow p-subgroups in locally finite groups. The reader interested in that matter can find some answers in [8].

3. Centric linking systems

In this section, we will mostly work with locally finite groups even though some definitions make sense for any groups or at least torsion groups. Most of the materials in this section are already available in [2] and [6].

For G a locally finite group, we denote by $\mathcal{T}_p(G)$ the *transporter system* of G, this is the category with set of objects the collection of p-subgroups of G and for morphisms

$$\mathrm{Mor}_{\mathcal{T}_p(G)}(P, Q) = N_G(P, Q) := \{g \in G \mid P^g \leq Q\}.$$

One important collection of p-subgroups is the collection of p-centric subgroups.

Definition 3. Let G be a locally finite group. A p-subgroup $P \leq G$ is *p-centric* if $C_G(P)/Z(P)$ has no element of order p. We denote by $\Omega_p^c(G) \subseteq \Omega_p(G)$ the subposet of G consisting of all subgroups in $\Omega_p(G)$ which are p-centric. Also $\mathcal{T}_p^c(G) \subseteq \mathcal{T}_p(G)$ will denote the full subcategory of $\mathcal{T}_p(G)$ with set of objects the collection of p-centric subgroups of G.

Though the last definition is the most natural to define p-centric subgroups, the next lemma gives a more convenient characterization.

For G a locally finite group, we define $O^p(G) \trianglelefteq G$ the subgroup of G generated by all elements of order prime to p.

Lemma 3. *Let G be a locally finite group and P a p-subgroup of G. The following are equivalent.*

(i) P is p-centric.
(ii) $C_G(P) = Z(P) \times O^p(C_G(P))$ and every element of $O^p(C_G(P))$ has order prime to p.

Proof. The proof is the same as in [6, Proposition 8.5]. □

We can now define the notion of centric linking system.

Definition 4. Let G be a locally finite group. The *centric linking system* of G is the category $\mathcal{L}_p^c(G)$ whose set of objects is the collection of all the p-centric subgroups of G, and where

$$\mathrm{Mor}_{\mathcal{L}_p^c(G)}(P,Q) = N_G(P,Q)/O^p(C_G(P)).$$

If $S \in \mathrm{Syl}_p(G)$, the equivalent full subcategory $\mathcal{L}_S^c(G) \subseteq \mathcal{L}_p^c(G)$ with objects the subgroups of S which are p-centrics is called the *centric linking system of G over S*.

The following lemma allows us to compare the geometric realisation of the centric part of the transporter category and the centric linking system.

Lemma 4. *Let $\Psi\colon \mathcal{C} \to \mathcal{C}'$ be a functor between small categories. Assume the following:*

(i) Ψ is bijective on isomorphism classes of objects and is surjective on morphism sets,
(ii) for each object $c \in \mathcal{C}$, the subgroup

$$K(c) = Ker\,[Aut_{\mathcal{C}}(c) \to Aut_{\mathcal{C}'}(\Psi(c))]$$

is a locally finite group with all elements of order prime to p, and
(iii) for each pair of objects c and d, and each $f, g\colon c \to d$ in $\mathcal{C}$, $\Psi(f) = \Psi(g)$ if and only if there is some $\sigma \in K(c)$ such that $g = f \circ \sigma$ (i.e. $Mor_{\mathcal{C}'}(\Psi(c), \Psi(d)) \cong Mor_{\mathcal{C}}(c,d)/K(c)$).

Then for any functor $F\colon \mathcal{C}' \to Top$, the induced map

$$\underset{\mathcal{C}'}{\mathrm{hocolim}}(F) \to \underset{\mathcal{C}}{\mathrm{hocolim}}(F \circ \Psi)$$

is an $\mathbb{F}_p$-homology equivalence, and induces a homotopy equivalence between the p-completions.

Proof. This is [2, Lemma 1.3] except that we are just asking $K(c)$ to be a locally finite group with all elements of order prime to p instead of a finite p'-group. But this suffices to ensure that coinvariants preserve exact sequences of $\mathbb{Z}_{(p)}[K(c)]$-modules, which is the only way the condition on $K(c)$ is used in the proof of [2, Lemma 1.3]. $\square$

In particular, when G is locally finite, the canonical projection functor $\mathcal{T}_p^c(G) \to \mathcal{L}_p^c(G)$ satisfies all of the hypotheses of Lemma 4. Hence, the induced map gives a homotopy equivalence

$$|\mathcal{T}_p^c(G)|_p^\wedge \simeq |\mathcal{L}_p^c(G)|_p^\wedge \tag{1}$$

4. Higher limits over orbit categories

In this section we recollect results about higher limits over orbit categories needed to prove the main result. They are manly adaptations of results from [6,9].

Definition 5. Let G be a group and $\mathcal{H}$ a collection of subgroups of G. The *orbit category* of G over $\mathcal{H}$ is the category $\mathcal{O}_{\mathcal{H}}(G)$ with set of objects $\mathcal{H}$ and morphisms

$$\mathrm{Mor}_{\mathcal{O}_{\mathcal{H}}(G)}(H,H') = H' \diagdown N_G(H,H') \cong \mathrm{Map}_G(G/H, G/H').$$

When $1 \in \mathcal{H}$, for $M \in \mathbb{Z}[G]$-module, we define

$$\Lambda^*_{\mathcal{H}}(G;M) = \varprojlim_{\mathcal{O}_{\mathcal{H}}(G)}{}^{*}\,(F_M),$$

where $F_M\colon \mathcal{O}_{\mathcal{H}}(G) \to \mathcal{A}\mathrm{b}$ is the functor defined by setting $F_M(H) = 0$ if $H \neq 1$ and $F_M(1) = M$.

Finally, when $\mathcal{H} = S_p(G)$ the collection of all p-subgroups of G, we will write $\mathcal{O}_p(G)$ in place of $\mathcal{O}_{S_p(G)}(G)$ and, for M a $\mathbb{Z}[G]$-module, $\Lambda^*_p(G; M)$ in place of $\Lambda^*_{S_p(G)}(G; M)$.

If G is a group with $\mathrm{Syl}_p(G) \neq \emptyset$ then, by Proposition 1, we have a functor $(-)^\circ\colon \mathcal{O}_p(G) \to \mathcal{O}_{\Omega_p(G)}(G)$ and we have the following adjunction.

Lemma 5. *Let G be a group with $Syl_p(G) \neq \emptyset$. The two functors*

$$\begin{tikzcd} \mathcal{O}_{\Omega_p(G)}(G) \arrow[r, bend left, "incl"] & \mathcal{O}_p(G) \arrow[l, bend left, "(-)^\circ"] \end{tikzcd}$$

are adjoints. More precisely, $(-)^\circ$ is left adjoint to the inclusion functor.

Proof. This is a direct consequence of Proposition 1. $\square$

Lemma 6 *(cf. [6, Lemma 5.10]). Let G be a group and Q be a p-subgroup of G. Let $F\colon \mathcal{O}_p(G)^{op} \to \mathcal{A}b$ be a functor such that $F(P) = 0$ except when P is G-conjugate to Q. Set $F'\colon \mathcal{O}_p(N_G(Q)/Q) \to \mathcal{A}b$ to be the functor $F'(P/Q) = F(P)$. Then*

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*}(F) \cong \varprojlim_{\mathcal{O}_p(N_G(Q)/Q)}{}^{*}(F') \cong \Lambda^*_p(N_G(Q)/Q; F(Q)).$$

Proof. This is a direct application of [6, Proposition 5.3] with $\mathcal{C} = \mathcal{O}_p(G)$, $\Gamma = N_G(Q)/Q$ and $\mathcal{H} = S_p(G)$. $\square$

Lemma 7 *(cf. [6, Proposition 5.12]). Let G be a locally finite group. Assume there is a countable p-subgroup $S \leq G$ such that every p-subgroup of G is conjugate to a subgroup of S. Fix a $\mathbb{Z}[G]$-module M and assume that there exists a finite subgroup $H \leq G$ such that $\Lambda^*_p(K; M) = 0$ for all subgroup $K \leq G$ containing H. Then $\Lambda^*_p(G; M) = 0$. In particular, $\Lambda^*_p(G; M) = 0$ if M is a $\mathbb{Z}_{(p)}[G]$-module and the kernel of the action of G on M contains an element of order p.*

Proof. The proof is exactly the same as the proof of [6, Proposition 5.12]. Indeed, they prove the result for S a discrete p-toral group but the only property of discrete p-toral groups they used is that S is an increasing union of finite groups, which is also true for countable locally finite groups. $\square$

Lemma 8. *Let G be a group. Let $\mathcal{H}' \subseteq \mathcal{H}$ be collections of p-subgroups of G closed by G-conjugation such that for all $P, Q \in \mathcal{H}$, if $P \in \mathcal{H}'$ and $P \leq Q$ then $Q \in \mathcal{H}'$. Let $F\colon \mathcal{O}_{\mathcal{H}}(G)^{op} \to \mathcal{A}b$ be a functor and denote by $F|_{\mathcal{O}_{\mathcal{H}'}(G)}$ the restriction of F to $\mathcal{O}_{\mathcal{H}'}(G)$. If for all $P \in \mathcal{H} \setminus \mathcal{H}'$, $F(P) = 0$, then*

$$\varprojlim_{\mathcal{O}_{\mathcal{H}}(G)}{}^{*}(F) = \varprojlim_{\mathcal{O}_{\mathcal{H}'}(G)}{}^{*}(F|_{\mathcal{O}_{\mathcal{H}'}(G)}).$$

Proof. The proof is exactly the same as the proof of [9, Lemma 1.6(a)]. $\square$

5. p-completion of classifying spaces

This section is devoted to the main Theorem.

Theorem 1. *Let G be a locally finite group with $Syl_p(G) \neq \emptyset$ and $S \in Syl_p(G)$. Assume that S is countable and that $\Omega_S(G) \setminus \Omega_S^c(G)$ contains, up to conjugacy, finitely many subgroups. Then,*

$$|\mathcal{L}_S^c(G)|_p^\wedge \simeq |\mathcal{L}_p^c(G)|_p^\wedge \simeq BG_p^\wedge.$$

Proof. The proof is based on the proof of [6, Theorem 8.7]. We will write $\Omega := \Omega_p(G)$ and $\Omega^c := \Omega_p^c(G)$ for short. The first homotopy equivalence holds since the categories $\mathcal{L}_S^c(G)$ and $\mathcal{L}_p^c(G)$ are equivalent. It remains to prove the last homotopy equivalence.

For $i > 0$, let $F_i \colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$ be the ith mod p cohomology functor defined for $Q \in S_p(G)$ by $F_i(Q) = H^i(BQ, \mathbb{F}_p)$. We also set $\mathcal{I} \colon \mathcal{O}_p(G) \to \mathrm{Top}$ and $\Phi \colon \mathcal{O}_p(G) \to \mathrm{Top}$ be the functors defined for $Q \in S_p(G)$ by

$$\mathcal{I}(Q) = G/Q \qquad \text{and} \qquad \Phi(Q) = EG \times_G \mathcal{I}(Q)$$

where EG is a fixed universal covering space of BG endowed with his natural G-action ($\Phi(Q)$ corresponds to a Borel construction).

To prove the last homotopy equivalence $|\mathcal{L}_p^c(G)|_p^\wedge \simeq BG_p^\wedge$ we will proceed as follows.

$$\begin{aligned}
|\mathcal{L}_p^c(G)|_p^\wedge &\simeq |\mathcal{T}_p^c(G)|_p^\wedge && \text{(see (1) at the end of Section 3)}\\
&\simeq \big(\operatorname*{hocolim}_{\mathcal{O}_{\Omega^c}(G)}(\Phi)\big)_p^\wedge && \text{(see (5) in Step 4)}\\
&\simeq \big(\operatorname*{hocolim}_{\mathcal{O}_{\Omega}(G)}(\Phi)\big)_p^\wedge && \text{(see (4) in Step 3)}\\
&\simeq BG_p^\wedge && \text{(see (2) in Step 2).}
\end{aligned}$$

The remaining of the proof is dedicated to prove these homotopy equivalences and is divided into 4 Steps. Step 1 describes BG in terms of homotopy colimit. Step 2 computes some limits over orbit categories using results from Section 4. These results are then used in Step 3 to compare two spectral sequences to get an homotopy equivalence between homotopy colimits. Finally, Step 4 decomposes $|\mathcal{T}_p^c(G)|$ as an homotopy colimit.

Step 1. From exactly the same arguments as in the Step 1 of the proof of [6, Theorem 8.7], we have

$$\operatorname*{hocolim}_{\mathcal{O}_\Omega(G)}(\Phi) \cong EG \times_G \left(\operatorname*{hocolim}_{\mathcal{O}_\Omega(G)}(\mathcal{I})\right) \simeq BG. \tag{2}$$

Step 2. For $Q \in \Omega \setminus \Omega^c$ and $i \geq 0$, we define the functor $F_i^{[Q]} \colon \mathcal{O}_p(G)^{\mathrm{op}} \to \mathcal{A}\mathrm{b}$ as follows

$$F_i^{[Q]}(P) = \begin{cases} H^i(BP, \mathbb{F}_p) & \text{if } P \text{ is } G\text{-conjugate to } Q \\ 0 & \text{otherwise.} \end{cases}$$

$C_G(Q)Q/Q \leq \mathrm{Aut}_{\mathcal{O}_p(G)}(Q) = N_G(Q)/Q$ acts trivially on $F_i^{[Q]}(Q)$. Moreover, since Q is not p-centric, $C_G(Q)Q/Q \cong C_G(Q)/Z(Q)$ contains an element of order p. Hence, by Lemma 6 and Lemma 7,

$$\varprojlim_{\mathcal{O}_p(G)}{}^{*} (F_i^{[Q]}) \cong \Lambda^*\left(N_G(Q)/Q; F_i^{[Q]}\right) = 0 \quad \text{for all } i.$$

Therefore, by Lemma 5,

$$\varprojlim_{\mathcal{O}_\Omega(G)}{}^{*} (F_i^{[Q]}) \cong \varprojlim_{\mathcal{O}_p(G)}{}^{*} (F_i^{[Q]}) = 0.$$

Step 3. Since S is a Sylow p-subgroup and $\Omega_S(G)\setminus\Omega_S^c(G)$ contains, up to conjugacy, finitely many subgroups, one also have that $\Omega_p(G)\setminus\Omega_p^c(G)$ contains finitely many conjugacy classes. Therefore by Lemma 2, there exists a finite filtration

$$\mathcal{O}_{\Omega^c}(G)=\mathcal{O}_0\subseteq\mathcal{O}_1\subseteq\cdots\subseteq\mathcal{O}_n=\mathcal{O}_\Omega(G)$$

by full subcategories of $\mathcal{O}_\Omega(G)$ such that

(a) for all $r\in\{1,2\dots,n-1\}$, $\mathrm{Ob}(\mathcal{O}_{r+1})\setminus\mathrm{Ob}(\mathcal{O}_r)=\{Q_{r+1}\}^G$ is the G-conjugacy class of a subgroup $Q_{r+1}\in\Omega$, and
(b) for all $r\in\{1,2\dots,n-1\}$ and for all $P\in\mathrm{Ob}(\mathcal{O}_r)$ and $P'\in\Omega$ with $P\le P'\le S$ then $P'\in\mathrm{Ob}(\mathcal{O}_r)$.

For $r\in\{1,2\dots,n\}$, we define $F_{i,r}\colon\mathcal{O}_{r+1}^{\mathrm{op}}\to\mathcal{A}\mathrm{b}$ by

$$F_{i,r}(P)=\begin{cases}F_i(P) & \text{if } P\in\mathrm{Ob}(\mathcal{O}_r),\\ 0 & \textit{else}\end{cases}$$

(where we set $\Omega_{n+1}=\Omega_n$). For all $r\in\{1,2\dots,n-1\}$, Property (b) implies

$$\mathrm{Ker}\left[F_{i,r+1}|_{\mathcal{O}_{r+1}}\twoheadrightarrow F_{i,r}\right]=F_i^{[Q_{r+1}]}|_{\mathcal{O}_{r+1}}$$

and, by (2) and Lemma 8 (which can be applied thanks to assumption (b) on the filtration), the higher limits of this functor vanish. Thus

$$\varprojlim_{\mathcal{O}_{r+1}}{}^*(F_{i,r+1}|_{\mathcal{O}_{r+1}})\cong\varprojlim_{\mathcal{O}_{r+1}}{}^*(F_{i,r})\cong\varprojlim_{\mathcal{O}_r}{}^*(F_{i,r}|_{\mathcal{O}_r})\tag{3}$$

where the last isomorphisms follows by Lemma 8. Notice that for all $r\in\{1,2\dots,n\}$, $F_{i,r}|_{\mathcal{O}_r}=F_i|_{\mathcal{O}_r}$ and so, by (3),

$$\varprojlim_{\mathcal{O}_r}{}^*(F_{i,r}|_{\mathcal{O}_r})\cong\varprojlim_{\mathcal{O}_0}{}^*(F_i|_{\mathcal{O}_0})=\varprojlim_{\mathcal{O}_{\Omega^c}(G)}{}^*(F_i|_{\mathcal{O}_{\Omega^c}(G)}).$$

In particular,

$$\varprojlim_{\mathcal{O}_\Omega(G)}{}^*(F_i|_{\mathcal{O}_\Omega(G)})\cong\varprojlim_{\mathcal{O}_{\Omega^c}(G)}{}^*(F_i|_{\mathcal{O}_{\Omega^c}(G)}).$$

The spectral sequence for cohomology of a homotopy colimit ([1, XII.4.5]) now implies that the inclusion $\mathcal{O}_{\Omega^c}(G)\subseteq\mathcal{O}_\Omega(G)$ induces a mod p homology isomorphism of homotopy colimits of Φ and hence a homotopy equivalence

$$(\operatorname*{hocolim}_{\mathcal{O}_{\Omega^c}(G)}(\Phi))_p^\wedge\simeq(\operatorname*{hocolim}_{\mathcal{O}_\Omega(G)}(\Phi))_p^\wedge.\tag{4}$$

Step 4. Now, by exactly the same argument as in [2, Lemma 1.2] we have

$$\operatorname*{hocolim}_{\mathcal{O}_p^c(G)}(\Phi)\cong EG\times_G\left(\operatorname*{hocolim}_{\mathcal{O}_p^c(G)}(\mathcal{I})\right)\simeq|\mathcal{T}_p^c(G)|.$$

Moreover, the adjunction of Lemma 5 restricts to an adjunction between $\mathcal{O}_{\Omega^c}(G)$ and $\mathcal{O}_p^c(G)$, and hence induces a homotopy equivalence

$$(\operatorname*{hocolim}_{\mathcal{O}_{\Omega^c}(G)}(\Phi))^\wedge_p \simeq (\operatorname*{hocolim}_{\mathcal{O}^c_p(G)}(\Phi))^\wedge_p \simeq |\mathcal{T}^c_p(G)|^\wedge_p. \tag{5}$$

This ends the proof of Theorem 1. □

6. Particular cases

Theorem 1 works for a very large class of groups. Here are some examples of groups which satisfy the hypothesis of Theorem 1.

Definition 6. A *discrete p-toral group* is a group P with a normal subgroup $P_0 \trianglelefteq P$ such that

(a) P is isomorphic to a finite product of copies of $\mathbb{Z}/p^\infty := \bigcup_{n\geq 1} \mathbb{Z}/p\mathbb{Z}$; and
(b) P/P_0 is a finite p-group.

One can see that discrete p-toral groups are countable.

Proposition 2. *Let G be a locally finite group such that $Syl_p(G) \neq \emptyset$ and let $S \in Syl_p(G)$. Assume that*

1. *S is a discrete p-toral group, and*
2. *$\Omega_S(G) \setminus \Omega^c_S(G)$ contains, up to conjugacy, finitely many subgroups.*

Then,

$$|\mathcal{L}^c_S(G)|^\wedge_p \simeq |\mathcal{L}^c_p(G)|^\wedge_p \simeq BG^\wedge_p.$$

Proposition 2 gives a small generalization of the second part of [6, Theorem 8.7] where they work with locally finite groups with discrete p-toral Sylow p-subgroups but with a technical condition of stabilization on centralizers. This condition was introduced to ensure the existence of a Sylow p-subgroup but they also proved in [6, Lemma 8.6] that it implies moreover that $\Omega_p(G)$ contains finitely many conjugacy classes.

Finally, Theorem 1 covers also countably locally finite groups which satisfies a condition of "finite dimensionality" which is central in [7]. For G a group and H a subgroup of G we denote by $\widetilde{\Omega}_H(G)$ and $\widetilde{\Omega}^{\mathrm{fin}}_H(G)$ the set of subgroups of H that are intersections, respectively finite intersection, of G-conjugates of H.

Proposition 3. *Let G be locally finite group and S a maximal p-subgroup of G. Assume that*

(a) G is countable,
(b) The supremum of the lengths of chains of proper inclusions in $\Omega^{fin}_S(G)$ exists and is finite.
(c) $\Omega_S(G) \setminus \Omega^c_S(G)$ contains, up to conjugacy, finitely many subgroups.

Then $S \in Syl_p(G)$ and

$$|\mathcal{L}^c_S(G)|^\wedge_p \simeq |\mathcal{L}^c_p(G)|^\wedge_p \simeq BG^\wedge_p.$$

Proof. We first show that $\widetilde{\Omega}_S(G) = \widetilde{\Omega}^{\mathrm{fin}}_S(G)$. Since G is countable so is the set of G-conjugates of S. Let $P \in \widetilde{\Omega}_S(G)$ and assume that P is not a finite intersection of G-conjugates of S. Let $(S_n)_{n\in\mathbb{N}}$ be a collection of G-conjugates of S such that $P = \bigcap_{n\geq 0} S_n$. By (b), the decreasing sequence of subgroups $S_0 \supseteq S_0 \cap S_1 \supseteq \cdots$ stabilizes after a finite stage. So there is a n_0 such that, for all $n \geq 0$, $S_0 \cap S_1 \cap \cdots \cap S_n = S_0 \cap S_1 \cap \cdots \cap S_{n_0}$. Thus $S_0 \cap S_1 \cap \cdots \cap S_{n_0} = \bigcap_{n\geq 0} S_n = P$ which contradicts the assumption on P.

Since $\widetilde{\Omega}_S(G) = \widetilde{\Omega}^{\mathrm{fin}}_S(G)$, one can show that (G, Δ, S) with Δ the collection of all subgroups of S is a locality in the sense of [7, Definition 3.1] (the finite dimensionality condition is a direct consequence of (b)). By [7, Proposition 3.8], we have that S is a Sylow p-subgroup of G. Finally, thanks to (c), one can apply Theorem 1 to get the result. □

Gonzalez and Chermak proved in an unpublished, using the Chevalley commutator formula, that an algebraic group over the algebraic closure of $\mathbb{F}_p$ satisfies the finite dimensionality condition. If moreover you work with connected reductive algebraic groups, thanks to Bruhat decomposition, one have the finiteness condition on the set of conjugacy classes. This gives a nice class of groups in which this results apply.

References

[1] A.K. Bousfield, D.M. Kan, Homotopy Limits, Completions and Localizations, Lecture Notes in Mathematics, vol. 304, Springer-Verlag, Berlin-New York, 1972.
[2] C. Broto, R. Levi, B. Oliver, Homotopy equivalences of p-completed classifying spaces of finite groups, Invent. Math. 151 (3) (2003) 611–664.
[3] B. Oliver, Equivalences of classifying spaces completed at odd primes, Math. Proc. Camb. Philos. Soc. 137 (2) (2004) 321–347.
[4] B. Oliver, Equivalences of classifying spaces completed at the prime two, Mem. Am. Math. Soc. 180 (848) (2006), vi+102.
[5] C. Broto, R. Levi, B. Oliver, The homotopy theory of fusion systems, J. Am. Math. Soc. 16 (4) (2003) 779–856.
[6] C. Broto, R. Levi, B. Oliver, Discrete models for the p-local homotopy theory of compact Lie groups and p-compact groups, Geom. Topol. 11 (2007) 315–427.
[7] A. Chermak, A. Gonzalez, Discrete localities I, arXiv:1702.02595, 2017, http://arxiv.org/abs/1702.02595.
[8] O.H. Kegel, e.A.F. Wehrfritz, Locally Finite Groups, Mathematical Studies, North-Holland Mathematical Library, 1973.
[9] B. Oliver, Existence and uniqueness of linking systems: Chermak's proof via obstruction theory, Acta Math. 211 (1) (2013) 141–175.