Homology, Homotopy and Applications, vol. 19(2), 2017, pp.61–82

COHOMOLOGY OF LINKING SYSTEMS WITH TWISTED COEFFICIENTS BY A p-SOLVABLE ACTION

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(*communicated by Nathalie Wahl*)

Abstract

In this paper, we study the cohomology of the geometric realization of linking systems with twisted coefficients. More precisely, given a prime p and a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, we compare the cohomology of $\mathcal{L}$ with twisted coefficients with the submodule of $\mathcal{F}^c$-stable elements in the cohomology of S. We start with the particular case of constrained fusion systems. Then, we study the case of p-solvable actions on the coefficients.

1. Introduction

The notion of saturated fusion system was introduced by Puig in the 1990s in a context of modular representation theory. Since their introduction, topologists use them to study classifying spaces of finite groups or, more precisely, their p-completions. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ a saturated fusion system over S and $\mathcal{L}$ an associated linking system. For a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, $|\mathcal{L}|^\wedge_p$ is called its *classifying space*. The theory of p-local finite groups have been studied in details by Broto, Levi, Oliver and others (see [**BLO2, OV1, 5a1, 5a2**]). The linking system and its geometric realization, even without p-completion, play here a fundamental and central role. In fact, for a given saturated fusion systems, the existence and uniqueness of an associated linking system were shown more recently by Chermak [**Ch**] (using the theory of partial groups). The proof of this important conjecture highlights that linking systems and their geometric realizations form a deep link between fusion system theory and homotopy theory (we refer to Aschbacher, Kessar and Oliver [**AKO**] for more details about fusion systems in general).

An old and well-known result due to Cartan and Eilenberg (see [**CE**, Theorem XII.10.1]) expresses the cohomology of a finite group G in a $\mathbb{Z}_{(p)}[G]$-module as the submodule of "stable" elements in the cohomology of a Sylow p-subgroup of G. This submodule of stable elements corresponds to the inverse limit over the "fusion" of the group cohomology functor. One important result in the theory of p-local finite groups is an analog of this theorem for p-local finite groups which tells us that the cohomology of the geometric realization of a linking system can be computed by $\mathcal{F}$-stable elements. More precisely, there is a natural inclusion of BS into $|\mathcal{L}|$ and

Received February 15, 2016, revised November 3, 2016; published on August 9, 2017.
2010 Mathematics Subject Classification: 55R40, 55N25, 55R35, 20J06, 20D20, 20J15.
Key words and phrases: fusion system, p-local finite group, cohomology with twisted coefficients, group cohomology.
Article available at http://dx.doi.org/10.4310/HHA.2017.v19.n2.a4

it induces the following isomorphism. Here, $\mathcal{F}^c$ is the full subcategory of $\mathcal{F}$ consisting of $\mathcal{F}$-centric subgroups of S and, for A a finite $\mathbb{Z}_{(p)}$-module, $H^*(\mathcal{F}^c, A) \subseteq H^*(S, A)$ is the submodule of $\mathcal{F}$-stable elements.

Theorem 1.1. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and A be a finite $\mathbb{Z}_{(p)}$-module. The inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|^\wedge_p, A) \cong H^*(|\mathcal{L}|, A) \xrightarrow{\cong} H^*(\mathcal{F}^c, A).$$

Proof. The case $A = \mathbb{F}_p$ is [**BLO2**, Theorem B] and the general case is proven in [**5a2**, Lemma 6.12]. □

One question asked by Oliver in his book with Aschbacher and Kessar [**AKO**] is the understanding of the cohomology of $|\mathcal{L}|$ with twisted coefficients. Indeed, this cohomology appears for example in the study of extensions of p-local finite groups or, more directly, can give more information about the link between the fusion system and the spaces $|\mathcal{L}|$ or $|\mathcal{L}|^\wedge_p$. Recall that, if a space X has a universal covering space $\widetilde{X}$, the cohomology of X with twisted coefficients in a $\mathbb{Z}[\pi_1(X)]$-module M is the cohomology of the chain complex

$$C^*(X; M) = \mathrm{Hom}_{\mathbb{Z}[\pi_1(X)]}(S_*(\widetilde{X}), M),$$

where $S_*(\widetilde{X})$ is the usual singular chain complex of $\widetilde{X}$.

Levi and Ragnarsson [**LR**] already give some tools along these lines. In an other paper [**Mo1**], the author extends Theorem 1.1 to the case of nilpotent actions on the coefficients. The main ingredient is to construct, as in the trivial coefficient case, an idempotent of $H^*(S, M)$ with image $H^*(\mathcal{F}^c, M)$.

In this paper, we also want to extend Theorem 1.1 to twisted coefficients but when the action factors through a p-solvable group. The methods used here are completely different from the ones used in [**Mo1**] and also more direct. We first have a look at constrained fusion systems. In that case we are able to prove that, with any coefficient module, the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

Theorem A (see Corollary 3.5)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. If $\mathcal{F}$ is constrained and M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module, then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Next we focus on p-solvable actions. The main ingredients here are p-local finite subgroups of index a power of p or prime to p and their homotopy properties. We start by looking at p-local subgroups of index prime to p (see Definition 2.6(b)).

Theorem B (see Theorem 4.3)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and denote by $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p. If M is a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module and if the inclusion of BS in $|O^{p'}(\mathcal{L})|$ induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then the inclusion of BS in $|\mathcal{L}|$ induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

This theorem allows us to prove that if the action on the coefficients factor through a p'-group or, even better, a p-nilpotent group, then the cohomology of $|\mathcal{L}|$ can be computed by stable elements.

It is much more complicated to work with p-local finite groups of index a power of p, especially on the level of stable elements. Indeed, for $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module, it is difficult to compare $H^*(\mathcal{F}^c, M)$ and $H^*(\mathcal{F}_0^c, M)$. The difficulty mostly comes from the fact that we are working on different p-groups: S and S_0. But when we work with a p-local finite group realizable by a finite group G, and if G acts "consistently" on the coefficients it is possible to get some positive results (see Section 5).

Theorem C (see Corollary 5.5)**.** *Let G be a finite group, S a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module and assume that G acts* consistently *on M. If both actions factor through a given p-solvable Γ and all the M-essential subgroups (see Definition 5.3) of S are p-centric, then we have natural isomorphisms,*

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

All of these results lead us to the following conjecture.

Conjecture A (see Conjecture 5.6)**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M a $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$-module. If the action of $\pi_1(|\mathcal{L}|)$ on M is p-solvable, then the inclusion of BS in $|\mathcal{L}|$ induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

We finish this paper with an example for Conjecture 5.6 which does not follow from the other results.

Organization

In Section 2, we give some background on p-local finite groups and stable elements. Section 3 is dedicated to the case of constrained fusion systems, Section 4 to coprime actions and Section 5 to p-solvable actions for a realizable p-local finite group. Finally, we give in Section 6 an example for Conjecture 5.6.

Acknowledgments

I would like to thank Bob Oliver, my PhD adviser, for his help and support all along this work. I am also grateful to Jesper Grodal for his hospitality at the Center for Symmetry and Deformation and many fruitful conversations. Finally, I would like to thank the referee for his very careful reading of the paper and all his useful and accurate comments.

2. Background

We give here a very short introduction to p-local finite groups. The notion of fusion system was first introduced by Puig for modular representation theory purpose. Later, Broto, Levi and Oliver developed the notion of linking systems and p-local finite groups to study p-completed classifying spaces of finite groups and spaces which have similar homotopy properties. We refer the reader interested in more details to Aschbacher, Kessar and Oliver [**AKO**].

2.1. Fusion systems and linking systems

A fusion system over a p-group S is a way to abstract the action of a finite group G with $S \in \mathrm{Syl}_p(G)$ on the subgroups of S by conjugation. For G a finite group and $g \in G$, we will denote by c_g the homomorphism $x \in G \mapsto gxg^{-1} \in G$ and for H, K two subgroups of G, $\mathrm{Hom}_G(H, K)$ will denote the set of all group homomorphism c_g for $g \in G$ such that $c_g(H) \leqslant K$.

Definition 2.1. Let S be a finite p-group. A *fusion system* over S is a small category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S and which satisfies the following two properties for all $P, Q \leqslant S$:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$;

(b) each $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

A fusion system is *saturated* if it satisfy two more technical axioms called the saturation axioms (we refer the reader to [**AKO**, Definition I.2.1] for a proper definition).

The composition in a fusion system is given by composition of homomorphisms. We usually write $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Mor}_{\mathcal{F}}(P, Q)$ to emphasize that the morphisms in $\mathcal{F}$ are homomorphisms. For $P, Q \leqslant S$, we say that P is $\mathcal{F}$*-conjugate* to Q if there is an $\mathcal{F}$-isomorphism between P and Q. We denote by $P^{\mathcal{F}}$ the set of all subgroups of S which are $\mathcal{F}$ conjugate to P.

The typical example of a saturated fusion system is the fusion system $\mathcal{F}_S(G)$ of a finite group G over $S \in \mathrm{Syl}_p(G)$.

For the purpose of this paper, we need to distinguish some collections of subgroups of S.

Definition 2.2. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S.

(a) A subgroup $P \leqslant S$ is $\mathcal{F}$*-centric* if for every $Q \in P^{\mathcal{F}}$, $C_S(Q) = Z(Q)$.

(b) A subgroup $P \leqslant S$ is $\mathcal{F}$*-radical* if $O_p(\mathrm{Aut}_{\mathcal{F}}(P)) = \mathrm{Inn}(P)$.

(c) A subgroup $P \leqslant S$ is $\mathcal{F}$*-quasicentric* if for each $Q \leqslant PC_S(P)$ containing P, and each $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\alpha|_P = \mathrm{Id}$, α has a p-power order.

We let $\mathcal{F}^{cr} \subseteq \mathcal{F}^c \subseteq \mathcal{F}^q \subseteq \mathcal{F}$ denote the full subcategories of $\mathcal{F}$ with objects the $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups, the $\mathcal{F}$-centric subgroups and the $\mathcal{F}$-quasicentric subgroups, respectively.

If $\mathcal{F} = \mathcal{F}_S(G)$, a subgroup $P \leqslant S$ is

(a) $\mathcal{F}$*-centric* if and only if it is p-centric (i.e. $Z(P) \in \mathrm{Syl}_p(C_G(P))$),

(b) $\mathcal{F}$*-radical* if $P/Z(P) = O_p(N_G(P)/C_G(P))$.

(c) $\mathcal{F}$*-quasicentric* if and only if $O^p(C_G(P))$ has order prime to p.

The notion of linking system has been introduced by Broto, Levi and Oliver [**BLO2**] and generalized by Broto, Castellana, Grodal and Oliver in [**5a1**]. We refer the reader to theses papers, or [**AKO**, Part III], for a proper definition. We recall here some basic facts about linking systems which will be needed here.

For G a finite group, $S \in \mathrm{Syl}_p(G)$ and $\mathcal{H}$ a collection of subgroups of S, the *transporter category* of G over S with set of objects $\mathcal{H}$ is the category $\mathcal{T}_H^{\mathcal{H}}(G)$ with objects $\mathcal{H}$ and for $P, Q \in \mathcal{H}$, $\mathrm{Mor}_{\mathcal{L}}(P,Q) = T_G(P,Q) = \{g \in G \mid P^g \leqslant Q\}$. For $\mathcal{F}$ a saturated fusion system over a p-group S, a *linking system* associated to $\mathcal{F}$ is a certain finite category with objects a collection $\mathcal{H}$ of subgroups of S together with two functors

$$\mathcal{T}_S^{\mathcal{H}}(S) \xrightarrow{\delta} \mathcal{L} \xrightarrow{\pi} \mathcal{F}.$$

δ is the identity on objects and injective on morphisms and π is injective on objects and surjective on morphisms. The collection $\mathcal{H}$ has to be stable by overgroups and $\mathcal{F}$-conjugation and the following proposition tell you which collection you can have.

Proposition 2.3. *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Let $\mathcal{L}$ be a linking system associated to $\mathcal{F}$.*

(a) $Ob(\mathcal{F}^{cr}) \subseteq Ob(\mathcal{L}) \subseteq Ob(\mathcal{F}^q)$, and there exists a linking system $\mathcal{L}^q$ associated to $\mathcal{F}$ such that $Ob(\mathcal{L}^q) = Ob(\mathcal{F}^q)$, and $\mathcal{L}$ is a full subcategory of $\mathcal{L}^q$.

(b) For every subset $Ob(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq Ob(\mathcal{F}^q)$ stable by $\mathcal{F}$-conjugacy and overgroups, the full subcategory $\mathcal{L}^{\mathcal{H}}$ of $\mathcal{L}^q$ with set of objects $\mathcal{H}$ is also a linking system associated to $\mathcal{F}$.

Proof. The first point of (a) can be found for example in [**O4**, Proposition 4(g)]. For the second statement of (a), you can find a proof in [**AKO**, Proposition III.4.8]. Finally, (b) is a consequence of the definition of linking systems. □

If $\mathcal{H} = \mathrm{Ob}(\mathcal{F}^q)$, $\mathcal{L}$ is called a *quasicentric linking system* and if $\mathcal{H} = \mathrm{Ob}(\mathcal{F}^c)$, $\mathcal{L}$ is called a *centric linking system*.

Definition 2.4. A p-local finite group is a triple $(S, \mathcal{F}, \mathcal{L})$ where $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is an associated linking system. If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is an other p-local finite group, we will say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p-local subgroup* of $(S, \mathcal{F}, \mathcal{L})$ if $S_0 \leqslant S$ and $\mathcal{F}_0 \subseteq \mathcal{F}$ is a subsystem of $\mathcal{F}$. Notice that we do not require that $\mathcal{L}_0$ is a subcategory of $\mathcal{L}$.

The typical example you should have in mind is the following. For G a finite group and $S \in \mathrm{Syl}_p(G)$ let $\mathcal{L}_S^q(G)$ be the category with objects the $\mathcal{F}_S(G)$-quasicentric subgroups of G and, for $P, Q \in \mathrm{Ob}(\mathcal{L})$,

$$\mathrm{Mor}_{\mathcal{L}}(P,Q) = T_G(P,Q)/O^p(C_G(P)).$$

Then $(S, \mathcal{F}_S(G), \mathcal{L}_S^q(G))$ defines a p-local finite group where $\mathcal{L}_S^q(G)$ is a quasicentric linking system. We also denote by $\mathcal{L}_S^c(G)$ the full subcategory of $\mathcal{L}_S^q(G)$ with objects the p-centric subgroups of S and it is a centric linking system.

We finish with some basic homotopy properties about linking systems which will be needed in this paper. We refer the reader interested in more details to [**AKO**, Part III]. For $(S, \mathcal{F}, \mathcal{L})$ a p-local finite group, we write $|\mathcal{L}|$ for the geometric realization of $\mathcal{L}$ and $\pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|)$ for its fundamental group. The following theorem will allow us to change the set of objects of $\mathcal{L}$ without changing the homotopy type of $|\mathcal{L}|$.

Theorem 2.5 ([**5a1**, Theorem 3.5])**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be two linking systems associated to $\mathcal{F}$ with a different set of objects. Then the inclusion induces a homotopy equivalence of space $|\mathcal{L}_0| \simeq |\mathcal{L}|$.*

2.2. p-local finite subgroups of index a power of p or prime to p

The notions p-local subgroups of index a power of p or prime to p have been introduced and studied by Broto, Castellana, Grodal, Levi and Oliver [**5a2**]. Here we just give the definitions what we need about these p-local subgroups and we refer the reader to [**5a2**] for more details.

Definition 2.6. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$. Set $\mathfrak{hyp}(\mathcal{F}) = \langle g^{-1}\alpha(g) \mid g \in P \leqslant S, \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P))\rangle \trianglelefteq S$.

(a) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p-local subgroup of index a power of p* if $S_0 \geqslant \mathfrak{hyp}(\mathcal{F})$ and, for every $P \leqslant S_0$, $O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \leqslant \mathrm{Aut}_{\mathcal{F}_0}(P)$.

(b) We say that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is a *p-local subgroup of index prime to p* if $S_0 = S$ and, for every $P \leqslant S$, $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)) \leqslant \mathrm{Aut}_{\mathcal{F}_0}(P)$.

Notice that $\mathfrak{hyp}(\mathcal{F})$ is denoted $O^p_{\mathcal{F}}(S)$ in [**5a2**, Definition 2.1]. These particular p-local subgroups satisfy the following properties.

Proposition 2.7 ([**5a2**, Proposition 3.8])**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$.*

(a) If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index a power of p, then $P \leqslant S_0$ is $\mathcal{F}_0$-quasicentric if, and only if, P is $\mathcal{F}$-quasicentric.

(b) If $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ is of index prime to p, then $P \leqslant S$ is $\mathcal{F}_0$-centric if, and only if, P is $\mathcal{F}$-centric.

For an infinite group G, we denote by $O^{p'}(G)$ the intersection of all normal subgroups in G of finite index prime to p. For $\mathcal{F}$ a fusion system over a p-group S, let $O^{p'}_*(\mathcal{F})$ be the fusion system generated by $O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P))$ for all $P \leqslant S$ and define

$$\mathrm{Out}^0_{\mathcal{F}}(S) = \langle \alpha \in \mathrm{Out}_{\mathcal{F}}(S) \mid \alpha|_P \in \mathrm{Hom}_{O^{p'}_*(\mathcal{F})}(P, S), \text{ for some } P \leqslant S\rangle.$$

Since $\mathrm{Aut}_{\mathcal{F}}(S)$ normalizes $O^{p'}_*(\mathcal{F})$, $\mathrm{Out}^0_{\mathcal{F}}(S) \trianglelefteq \mathrm{Out}_{\mathcal{F}}(S)$.

Proposition 2.8. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

(a) $\mathcal{F} = \langle Aut_{\mathcal{F}}(S), O^{p'}_*(\mathcal{F})\rangle$.

(b) π and the inclusion of $BAut_{\mathcal{F}}(S)$ in $|\mathcal{F}^c|$ induce isomorphisms,

$$\theta\colon \pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \xrightarrow{\cong} \pi_1(|\mathcal{F}^c|) \xrightarrow{\cong} Out_{\mathcal{F}}(S)/Out^0_{\mathcal{F}}(S).$$

Proof. The point (a) is proved in [**5a2**, Lemma 3.4]. For (b), the second isomorphism is given in [**5a2**, Proposition 5.2] and the first one in [**5a2**, Theorem 5.5] and the comment which follows. □

According to Proposition 2.7, when dealing with p-local subgroups of index prime to p, we will work with centric linking systems.

Theorem 2.9 ([**5a2**, Theorem 5.5])**.** *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group with $\mathcal{L}$ a centric linking system. For each subgroup $H \leqslant Out_{\mathcal{F}}(S)$ containing $Out^0_{\mathcal{F}}(S)$, there is a unique p-local finite subgroup $(S,\mathcal{F}_H,\mathcal{L}_H)$ of index prime to p such that $Out_{\mathcal{F}_H}(S) = H$ and $\mathcal{L}_H = \pi^{-1}(\mathcal{F}^c_H)$.*

Moreover, $|\mathcal{L}_H|$ is homotopy equivalent, via its inclusion in $|\mathcal{L}|$, to the covering space of $|\mathcal{L}|$ with fundamental group $\widetilde{H} \geqslant O^{p'}(\pi_{\mathcal{L}})$ such that $\theta(\widetilde{H}/O^{p'}(\pi_{\mathcal{L}})) = H/Out^0_{\mathcal{F}}(S)$ (where θ is the isomorphism given in Proposition 2.8(b)).

Thus, for a p-local finite group $(S,\mathcal{F},\mathcal{L})$, with $\mathcal{L}$ a centric linking system, we can define the *minimal p-local subgroup of index prime to p*, $(S,O^{p'}(\mathcal{F}),O^{p'}(\mathcal{L}))$ corresponding to $(S,\mathcal{F}_H,\mathcal{L}_H)$ with $H = \mathrm{Out}^0_{\mathcal{F}}(S)$ in Theorem 2.9.

2.3. Cohomology and stable elements

The first result about stable elements is due to Cartan and Eilenberg [**CE**, Chap. XII, Theorem 10.1]. It also served as a guideline in the establishment of Theorem 1.1 by Broto, Levi and Oliver. Here we recall the definition of $\mathcal{F}^c$-stable elements in a context of twisted coefficients. We refer the reader to [**Mo1**] for more details. As in [**Mo1**], we will denote by $\omega\colon \mathcal{L} \to \pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|,S)$ the functor which maps each object to the unique object in the target and sends each morphism $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ to the class of the loop $\iota_Q.\varphi.\overline{\iota_P}$ where $\iota_P = \delta(1) \in \mathrm{Mor}_{\mathcal{L}}(P,S)$, $\iota_Q = \delta(1) \in \mathrm{Mor}_{\mathcal{L}}(Q,S)$ and $\overline{\iota_P}$ is the edge ι_P followed in the opposite direction.

Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. Recall first that $\delta\colon \mathcal{T}_S^{\mathrm{Ob}(\mathcal{L})}(S) \to \mathcal{L}$ induces an inclusion $\delta_S\colon BS \to |\mathcal{L}|$. In particular, it induces a natural map $S \to \pi_{\mathcal{L}}$ and thus, for every $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module M, we have a natural action of S, or any subgroup of S, on M. Now, let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, the group cohomology bifunctor $H^*(-,-)$ induces a functor

$$H^*(-,M)\colon \mathcal{F}^c \longrightarrow \mathbb{Z}_{(p)}\text{-Mod}$$

(a priori, $H^*(g,M)$ is defined for $g \in \mathrm{Mor}(\mathcal{L})$ but [**Mo1**, Proposition 2.2] proves that $H^*(-,M)$ is well defined on $\mathcal{F}^c$).

Definition 2.10. Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. An element $x \in H^*(S,M)$ is called *$\mathcal{F}$-centric stable*, or *$\mathcal{F}^c$-stable*, if for all $P \in \mathrm{Ob}(\mathcal{F}^c)$ and all $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$,

$$\varphi^*(x) = \mathrm{Res}^S_P(x).$$

We denote by $H^*(\mathcal{F}^c,M) \subseteq H^*(S,M)$ the submodule of all $\mathcal{F}^c$-stable elements.

Notice that

$$H^*(\mathcal{F}^c,M) = \varprojlim_{\mathcal{F}^c} H^*(-,M) = \varprojlim_{\mathcal{L}} H^*(-,M),$$

where the last equality holds if $\mathcal{L}$ is a centric linking system.

3. Constrained fusion systems

Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. Here, we assume that $\mathcal{F}$ is a constrained fusion system.

Definition 3.1. Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $Q \leqslant S$ is *normal in* $\mathcal{F}$ if

(i) $Q \trianglelefteq S$, and
(ii) for all $P, R \leqslant S$ and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, φ extends to a morphism $\overline{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ, RQ)$ such that $\overline{\varphi}(Q) = Q$.

We write $O_p(\mathcal{F})$ for the maximal subgroup of S which is normal in $\mathcal{F}$. We say that $\mathcal{F}$ is *constrained* if $O_p(\mathcal{F})$ is $\mathcal{F}$-centric.

An important and classical result about constrained fusion systems is the following.

Proposition 3.2 ([**5a1**, Proposition 4.3])**.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $\mathcal{L}$ a centric linking system. If $\mathcal{F}$ is constrained, there exists a finite group G such that*

(a) S is a Sylow p-subgroup of G,
(b) $C_G(O_p(G)) \leqslant O_p(G)$,
(c) $\mathcal{F}_S(G) = \mathcal{F}$.

Moreover, $G \cong Aut_{\mathcal{L}}(O_p(\mathcal{F}))$ and $\mathcal{L} \cong \mathcal{L}_S^c(G)$.

This group G is called a *model* of $\mathcal{F}$ and it is unique in a precise way (see [**AKO**, Theorem III.5.10]). This model can also be recovered from the homotopy type of the geometric realization of a linking system associated to $\mathcal{F}$.

Lemma 3.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group with $\mathcal{L}$ a centric linking system. If $\mathcal{F}$ is constrained, then $|\mathcal{L}|$ is a classifying space of a model G of $\mathcal{F}$.*

Proof. By Proposition 3.2, we can assume that $\mathcal{L} = \mathcal{L}_S^c(G)$. Set

$$\mathcal{H} = \{P \in \mathrm{Ob}(\mathcal{L}) \mid P \geqslant O_p(G)\}$$

and let $\mathcal{L}^{\mathcal{H}}$ be the full subcategory of $\mathcal{L}$ with set of objects $\mathcal{H}$. By [**5a1**, Proposition 1.6], $\mathcal{H}$ contains all $\mathcal{F}$-centric and $\mathcal{F}$-radical subgroups. Thus, by Proposition 2.3, $\mathcal{L}^{\mathcal{H}}$ is a linking system associated to $\mathcal{F}$ and, by Theorem 2.5, $|\mathcal{L}^{\mathcal{H}}| \cong |\mathcal{L}|$.

It remains to prove that $|\mathcal{L}^{\mathcal{H}}| \cong BG$. For that purpose, consider the following functor:

$$\begin{array}{rrcl} F\colon & \mathcal{L}^{\mathcal{H}} & \longrightarrow & \mathcal{L}^{\{O_p(G)\}}, \\ & P \in \mathcal{L}^{\mathcal{H}} & \longmapsto & O_p(G), \\ & g \in T_G(P,Q) & \longmapsto & g \in N_G(O_p(G)) = G. \end{array}$$

It gives us a retraction by deformation of $|\mathcal{L}^{\mathcal{H}}|$ on the geometric realization of the full subcategory of $\mathcal{L}$ with unique object $O_p(G) \leqslant S$. As $\mathrm{Aut}_{\mathcal{L}}(O_p(G)) = N_G(O_p(G)) = G$, this last category is $\mathcal{B}(G)$. In particular, its geometric realization is a classifying space of G. □

Proposition 3.4. *Let G be a finite group and S a Sylow p-subgroup of G. If we have $C_G(O_p(G)) \leqslant O_p(G)$, then, for every $\mathbb{Z}_{(p)}[G]$-module M, the inclusion of S in G induces a natural isomorphism*

$$H^*(G, M) \cong H^*(\mathcal{F}_S^c(G), M).$$

Proof. Let $(S,\mathcal{F},\mathcal{L}) = (S,\mathcal{F}_S(G),\mathcal{L}^c_S(G))$. By assumption, $\mathcal{F}_S(G)$ is constrained and G is a model of $\mathcal{F}_S(G)$. From Cartan-Eilenberg Theorem, we know that

$$Res^G_S \colon H^*(G,M) \longrightarrow H^*(S,M)$$

is injective and that $\operatorname{Im}(\operatorname{Res}^G_S) = \varprojlim_{\mathcal{T}_S(G)} H^*(-,M)$. Moreover,

$$H^*(\mathcal{F}^c,M) = \varprojlim_{\mathcal{F}^c} H^*(-,M) = \varprojlim_{\mathcal{L}} H^*(-,M) = \varprojlim_{\mathcal{T}^c_S(G)} H^*(-,M) \geqslant \varprojlim_{\mathcal{T}_S(G)} H^*(-,M).$$

Thus, it remains to prove that $\varprojlim_{\mathcal{T}^c_S(G)} H^*(-,M) \leqslant \varprojlim_{\mathcal{T}_S(G)} H^*(-,M)$.

Let then $x \in H^*(\mathcal{F}^c,M) = \varprojlim_{\mathcal{T}^c_S(G)} H^*(-,M)$. For $P \leqslant S$ and $g \in N_G(P,S)$ we have, in $\mathcal{T}_S(G)$, the following commutative diagram:

$$\begin{array}{ccc} PO_p(G) & \xrightarrow{\;g\;} & gPg^{-1}O_p(G) \\ {\scriptstyle e}\big\uparrow & & \big\uparrow{\scriptstyle e} \\ P & \xrightarrow{\;g\;} & gPg^{-1}, \end{array}$$

where e is the trivial element of G. Hence, as the top part of the diagram is in $T^c_S(G)$ and $x \in \varprojlim_{\mathcal{T}^c_S(G)} H^*(-,M)$,

$$\begin{aligned} c^*_g \circ \operatorname{Res}^S_{gPg^{-1}}(x) &= \operatorname{Res}^{PO^p(G)}_P \circ c^*_g \circ \operatorname{Res}^S_{gPg^{-1}O_p(G)}(x) \\ &= \operatorname{Res}^{PO^p(G)}_P \circ \operatorname{Res}^S_{PO_p(G)}(x) \\ &= \operatorname{Res}^S_P(x). \end{aligned}$$

Thus $x \in \varprojlim_{\mathcal{T}_S(G)} H^*(-,M)$ and this complete the proof. □

Corollary 3.5. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. If $\mathcal{F}$ is constrained and M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then δ_S induces a natural isomorphism,*

$$H^*(|\mathcal{L}|,M) \cong H^*(\mathcal{F}^c,M).$$

4. Actions factoring through a p'-group

In this section, for each p-local finite group $(S,\mathcal{F},\mathcal{L})$ we will assume that $\mathcal{L}$ is a **centric linking system**.

Lemma 4.1. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and $(S,O^{p'}(\mathcal{F}),O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then the inclusion $O^{p'}(\mathcal{L}) \subseteq \mathcal{L}$ induces the following isomorphism,*

$$H^*(|\mathcal{L}|,M) \cong H^*(|O^{p'}(\mathcal{L})|,M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Proof. By Theorem 2.9, $|O^{p'}(\mathcal{L})|$ is, up to homotopy, a covering space of $|\mathcal{L}|$ with fundamental group $O^{p'}(\pi_{\mathcal{L}}) \trianglelefteq \pi_{\mathcal{L}}$. It gives us a fibration sequence

$$|O^{p'}(\mathcal{L})| \to |\mathcal{L}| \to B\left(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})\right).$$

Consider then the Serre spectral sequence associated

$$H^{s+t}(|\mathcal{L}|, M) \Leftarrow H^s\left(\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}), H^t(|O^{p'}(\mathcal{L})|, M)\right).$$

M is a $\mathbb{Z}_{(p)}$-module, thus $H^q(|O^{p'}(\mathcal{L})|, M)$ is also a $\mathbb{Z}_{(p)}$-module. As $\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})$ is a p'-group, the E_2-page is concentrated in the first column with terms

$$H^t(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})}.$$

Thus the spectral sequence collapses on the E_2-page and the lemma follows. □

Lemma 4.2. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then*

$$H^*(\mathcal{F}^c, M) = H^*(O^{p'}(\mathcal{F})^c, M)^{Aut_{\mathcal{F}}(S)/Aut_{O^{p'}(\mathcal{F})}(S)}.$$

Proof. Notice first that, by Proposition 2.7, $\mathrm{Ob}(O^{p'}(\mathcal{F})^c) = \mathrm{Ob}(\mathcal{F}^c)$. Hence, we are working with the same underlying set of objects. Thus, by definition, $H^*(\mathcal{F}^c, M) \subseteq H^*(O^{p'}(\mathcal{F})^c, M)^{\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)}$. On the other hand, by Proposition 2.8, we have $\mathcal{F} = \langle O^{p'}(\mathcal{F}), \mathrm{Aut}_{\mathcal{F}}(S)\rangle$ which gives the converse inclusion. □

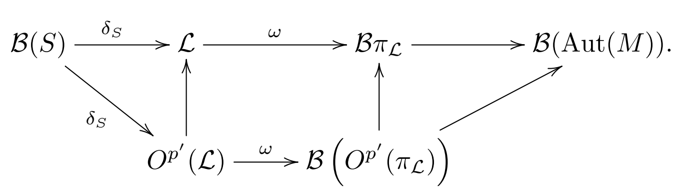
Theorem 4.3. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $(S, O^{p'}(\mathcal{F}), O^{p'}(\mathcal{L}))$ its minimal p-local subgroup of index prime to p. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and if the inclusion δ_S induces an isomorphism*

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M),$$

then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. Recall that, by Theorem 2.9, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$. Then we have the following commutative diagram:



Moreover, by Proposition 2.8 and Theorem 2.9, the projection $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}$ induces an isomorphism

$$\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}}) \cong \pi_1(|\mathcal{F}^c|) \cong \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S).$$

Then, by the two previous lemmas, we obtain

$$\begin{aligned} H^*(|\mathcal{L}|, M) &\cong H^*(|O^{p'}(\mathcal{L})|, M)^{\pi_{\mathcal{L}}/O^{p'}(\pi_{\mathcal{L}})} \\ &\cong \left(\varprojlim_{O^{p'}(\mathcal{F})^c} H^*(-, M) \right)^{\mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_{O^{p'}(\mathcal{F})}(S)} \\ &\cong H^*(\mathcal{F}^c, M). \end{aligned}$$

For the second isomorphism, we have to be careful with respect to the action of $\pi_{\mathcal{L}}$ on the left side of the isomorphism and $\mathrm{Aut}_{\mathcal{F}}(S)$ on the right side. In fact, here, by Definition 2.10 of $\mathcal{F}^c$-stable elements, we can see it on the chain level. The map $\delta_S^*\colon H^*(|O^{p'}(\mathcal{L})|, M) \longrightarrow H^*(S, M)$, induced by $\delta_S\colon BS \longrightarrow |O^{p'}(\mathcal{L})|$, gives on the chain level,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{Z}_{(p)}[S]}\left(C_*\left(\widetilde{|O^{p'}(\mathcal{L})|}\right), M\right) & \longrightarrow & \mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{O^{p'}(\mathcal{L})}]}(C_*(|\mathcal{E}(S)|), M) \\ f & \longmapsto & f|_{C_*(|\mathcal{E}(S)|)}, \end{array}$$

where $\mathcal{E}(S)$ is defined as the category with set of object S and for each $(s, s') \in S$, $\mathrm{Mor}_{\mathcal{E}(S)}(s, s') = \{\varphi_{s,s'}\}$ (in particular, $|\mathcal{E}(S)|$ is a universal covering space of BS). Then, for $\varphi \in \mathrm{Aut}_S(\mathcal{F})$, if we choose a lift $\widetilde{\varphi} \in \mathrm{Aut}_{\mathcal{L}}(S)$, φ acts on the left side by

$$f \longmapsto \omega(\widetilde{\varphi}^{-1}) f \omega(\widetilde{\varphi}) \ ,$$

and on the right side by,

$$f \longmapsto \omega(\widetilde{\varphi})^{-1} f \circ \varphi^* \ .$$

Finally, the action of φ on $\mathcal{E}(S)$ corresponds to the action of $\omega(\widetilde{\varphi})$ on $|\mathcal{E}(S)|$ (indeed, a lift of $\omega(\widetilde{\varphi})$ in $\widetilde{|O^{p'}(\mathcal{L})|}$ joins every vertex $s \in S$ of $|\mathcal{E}(S)|$ to the vertex $\varphi(s)$ and similarly for higher simplices). Hence, the two actions coincide. □

Corollary 4.4. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. If the action of $\pi_{\mathcal{L}}$ on M factors through a p'-group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

Proof. By Theorem 4.3, it is enough to prove that δ_S induces an isomorphism

$$H^*(|O^{p'}(\mathcal{L})|, M) \cong H^*(O^{p'}(\mathcal{F})^c, M).$$

But, as the action on M factor through a p'-group, $\pi_1(|O^{p'}(\mathcal{L})|) = O^{p'}(\pi_{\mathcal{L}})$ acts trivially on M and Theorem 1.1 gives the wanted isomorphism. □

We already know, from a previous article [**Mo1**, Theorem 4.3] that, if M is a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and the action of $\pi_{\mathcal{L}}$ on M factor through a p-group, then δ_S induces an isomorphism

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M)$$

(it is a direct corollary of [**Mo1**, Theorem 4.3] because, any action of a p-group on an abelian p-group is nilpotent). Hence, with the same arguments, we get another corollary of Theorem 4.3.

Corollary 4.5. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and M be a finite $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module. If the action of $\pi_{\mathcal{L}}$ on M factors through an extension of a normal p-group by a p'-group then δ_S induces an isomorphism,*

$$H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).$$

5. Realizable fusion systems and actions factoring through a p-solvable group

Consider here a finite group G, S a Sylow p-subgroup of G and let $(S, \mathcal{F}, \mathcal{L})$ be the associated p-local finite group with $\mathcal{L} = \mathcal{L}_S^c(G)$. Set $\mathcal{T} = \mathcal{T}_S^c(G)$ be the centric transporter category of G, $\mathcal{L}^q = \mathcal{L}_S^q(G)$ be the quasicentric linking system associated to G and $\mathcal{T}^q = \mathcal{T}_S^q(G)$ be the associated quasicentric transporter category. We also write $\pi_{\mathcal{T}} = \pi_1(|\mathcal{T}|)$.

We have a functor

$$\rho\colon \mathcal{T}_S(G) \longrightarrow \mathcal{B}(G),$$

which sends each object in the source to the unique object o_G in the target and sends, for every $P, Q \leqslant S$, $g \in T_G(P, Q)$ to $g \in G = \mathrm{Mor}_{\mathcal{B}}(G)(o_G)$. As $|\mathcal{B}(G)| = BG$, this induces a homomorphism

$$\rho_*\colon \pi_{\mathcal{T}} \longrightarrow G.$$

Here for M a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, with action $\varphi\colon \pi_{\mathcal{L}} \to \mathrm{Aut}(M)$ we will suppose that we have the following commutative diagram for some homomorphism $\overline{\varphi}\colon G \to \mathrm{Aut}(M)$:

$$\begin{array}{ccccc} & & \pi_{\mathcal{L}} & & \\ & \overset{\delta_*}{\nearrow} & & \overset{\varphi}{\searrow} & \\ \pi_{\mathcal{T}} & & & & \mathrm{Aut}(M). \\ & \underset{\rho_*}{\searrow} & & \underset{\overline{\varphi}}{\nearrow} & \\ & & G & & \end{array}$$

Then, we can compare the cohomology of $|\mathcal{L}|$ and the cohomology of G when the action factors through a p-solvable group. The main ingredients that we will use are p-local subgroups of index a power of p or prime to p.

The following lemma allows us to compare $H^*(|\mathcal{L}|, M)$ and $H^*(|\mathcal{T}|, M)$.

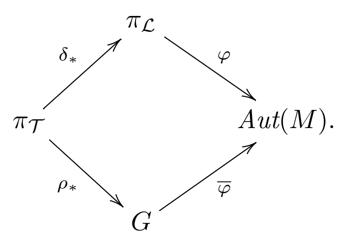
Lemma 5.1. *Let G be a finite group and $(S, \mathcal{F}, \mathcal{L})$ be an associated p-local finite group. Let $\mathcal{T} = \mathcal{T}_S^{Ob(\mathcal{L})}(G) \subseteq \mathcal{T}^q$ be the transporter category associated to G with set of objects $Ob(\mathcal{L})$. If M is a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module, then the canonical functor $\delta\colon \mathcal{T} \to \mathcal{L}$ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(|\mathcal{L}|, M)$.*

Proof. This is a consequence of [**BLO1**, Lemma 1.3] with $\mathcal{C} = \mathcal{T}$, $\mathcal{C}' = \mathcal{L}$ and the functor $T\colon \mathcal{L}^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-Mod which sends each object to M, and each morphism to its action on M. Then δ induces a natural isomorphism $\varprojlim_{\mathcal{T}}{}^*(M) \cong \varprojlim_{\mathcal{L}}{}^*(M)$. Then

$$H^*(|\mathcal{T}|, M) = \varprojlim_{\mathcal{T}}{}^*(M) \cong \varprojlim_{\mathcal{L}}{}^*(M) = H^*(|\mathcal{L}|, M),$$

where the first and last equality is just an interpretation in terms of functor cohomology and can be found in [**LR**, Proposition 3.9]. □

Theorem 5.2. *Let G be a finite group, S a Sylow p-subgroup of G, $\mathcal{L} = \mathcal{L}^c_S(G)$ and $\mathcal{T} = \mathcal{T}^c_S(G)$. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and assume that we have the following commutative diagram:*



If ρ_* is surjective and $\Gamma = Im(\varphi) = Im(\overline{\varphi})$ is p-solvable, then δ and ρ induce natural isomorphisms

$$H^*(|\mathcal{L}|, M) \cong H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

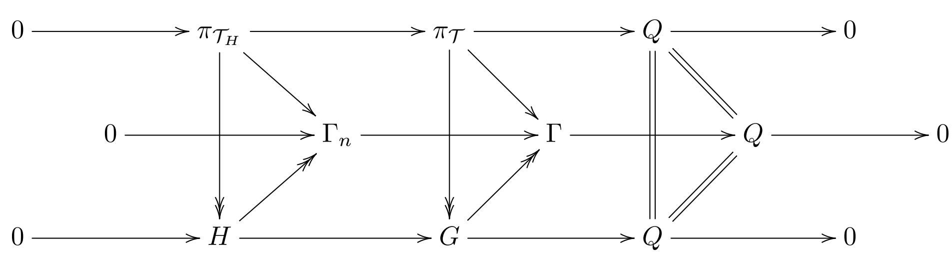
Proof. By Lemma 5.1, we just have to show that ρ induces a natural isomorphism $H^*(|\mathcal{T}|, M) \cong H^*(G, M)$. We prove this by induction on the minimal number n of extensions by p-groups or p'-groups we need to obtain Γ.

If $n = 0$, $\Gamma = 1$ and the action of $\pi_{\mathcal{T}}$ on M is trivial, then it follows from [**OV1**, Proposition 4.5]. Assume that, if Γ is obtained by n extensions, the result is true and suppose that Γ is obtained with $n + 1$ extensions. Consider then the last one

$$0 \to \Gamma_n \to \Gamma \to Q \to 0.$$

Denote $H = \overline{\varphi}_*^{-1}(\Gamma_n)$. Thus $(T, \mathcal{F}_H, \mathcal{L}_H) = (S \cap H, \mathcal{F}_{S\cap H}(H), \mathcal{L}^c_{S\cap H}(H))$ is a p-local subgroup of $(S, \mathcal{F}, \mathcal{L})$ of index a power of p or prime to p.

If Q is a p'-group. In that case, $(T, \mathcal{F}_H, \mathcal{L}_H)$ is a p-local finite subgroup of index prime to p (defined in Definition 2.6). Then $\mathrm{Ob}(\mathcal{F}^c) = \mathrm{Ob}(\mathcal{F}^c_H)$, $\mathcal{T}_H = \mathcal{T}^c_{S\cap H}(H) \subset \mathcal{T}$ and, by [**OV1**, Proposition 4.1(d)], this inclusion of categories induces, up to homotopy, a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. We then have the following commutative diagram with exact rows (here, $\twoheadrightarrow$ means onto):



and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ$$

$$BH \longrightarrow BG \longrightarrow BQ\ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

If Q is a p-group. In that case, we have to be more careful on the collection of subgroups of S we are working with. As in the case when Q is a p'-group we want to apply [**OV1**, Proposition 4.1(d)]. This forces us to use the following collection. Let

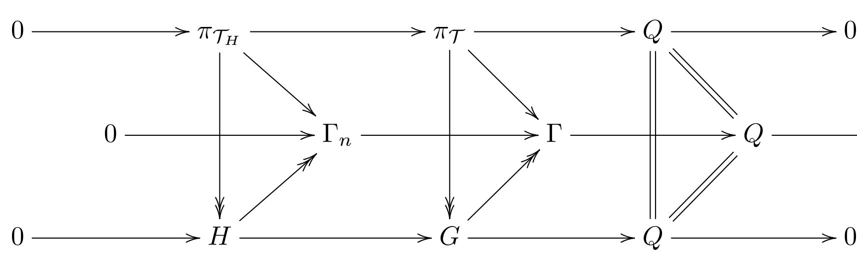
$$\mathcal{H} = \{P \in \mathrm{Ob}(\mathcal{F}^q) \mid P \cap T \in \mathrm{Ob}(\mathcal{F}_H^q)\}\,.$$

Since $H \trianglelefteq G$, no element of $T = S \cap H$ is G-conjugate to any element of $S \smallsetminus T$. Thus, by [**5a2**, Lemma 3.5], for every $P \in \mathrm{Ob}(\mathcal{F}^{cr})$, $P \cap T \in \mathrm{Ob}(\mathcal{F}_H^c) \subseteq \mathrm{Ob}(\mathcal{F}_H^q)$. In particular, $\mathrm{Ob}(\mathcal{F}^{cr}) \subseteq \mathcal{H} \subseteq \mathrm{Ob}(\mathcal{F}^q)$. Hence if $\mathcal{L}^{\mathcal{H}} \subseteq \mathcal{L}^q$ is the full subcategory of $\mathcal{L}^q$ with set of objects $\mathcal{H}$, by Proposition 2.3(b), $\mathcal{L}^{\mathcal{H}}$ defines a linking system associated to $\mathcal{F}$. On the level of transporter systems, the inclusions $\mathcal{T} \subseteq \mathcal{T}^q \supseteq \mathcal{T}^{\mathcal{H}}$ induce natural isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Indeed, we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{T} & \longrightarrow & \mathcal{T}^q & \longleftarrow & \mathcal{T}^{\mathcal{H}} \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\
\mathcal{L} & \longrightarrow & \mathcal{L}^q & \longleftarrow & \mathcal{L}^{\mathcal{H}}.
\end{array}$$

The vertical arrows induce isomorphisms in cohomology by Lemma 5.1 and the lower horizontal one induces an isomorphism since, by Theorem 2.5, the inclusions of categories $\mathcal{L} \subseteq \mathcal{L}^q \supseteq \mathcal{L}^{\mathcal{H}}$ induces $|\mathcal{L}| \simeq |\mathcal{L}^q| \simeq |\mathcal{L}^{\mathcal{H}}|$. Hence the upper arrows induce isomorphisms $H^*(|\mathcal{T}^{\mathcal{H}}|, M) \simeq H^*(|\mathcal{T}^q|, M) \simeq H^*(|\mathcal{T}|, M)$. Finally, by Proposition 2.7, $P \in \mathrm{Ob}(\mathcal{F}_H^q)$ if and only if $P \leqslant T$ and $P \in \mathcal{H}$. In particular, $\mathcal{T}_H^q \subseteq \mathcal{T}^{\mathcal{H}}$. Thus we can assume for this part that $\mathcal{T} = \mathcal{T}^{\mathcal{H}}$ and $\mathcal{T}_H = \mathcal{T}_H^q$.

We have $\mathcal{T}_H \subseteq \mathcal{T}$ is a transporter system associated to $\mathcal{F}_H$ and, by definition of $\mathcal{H}$, the hypotheses of [**OV1**, Proposition 4.1(d)], are satisfied. Thus this inclusion induces a covering space of $|\mathcal{T}|$ with covering group $G/H = Q$. Therefore, we have the following diagram with exact rows



and the following fibration sequences

$$|\mathcal{T}_H| \longrightarrow |\mathcal{T}| \longrightarrow BQ\ ,$$
$$BH \longrightarrow BG \longrightarrow BQ\ .$$

Moreover, ρ induces a morphism of fibration sequences between these two.

Hence, in both cases, we have the following Serre spectral sequences

$$H^{s+t}(|\mathcal{T}|, M) \Leftarrow H^s(Q, H^t(|\mathcal{T}_H|, M)),$$

$$H^{s+t}(G, M) \Leftarrow H^s(Q, H^t(H, M)),$$

and ρ induces a morphism ρ^* of spectral sequences between these two. By induction, ρ^* gives an isomorphism on the E_2 page and then induces an isomorphism of spectral sequences. In particular, ρ induces a natural isomorphism

$$H^*(|\mathcal{T}|, M) \cong H^*(G, M).$$

The result follows by induction. □

Assume the hypotheses of Theorem 5.2. It remains to compare $H^*(G, M)$ with the $\mathcal{F}^c$-stable elements. This is also not obvious and they are not isomorphic in all cases. On one hand, by Cartan-Eilenberg Theorem, we have $H^*(G, M) \cong \varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$. On the other hand, we have $H^*(\mathcal{F}^c, M) = \varprojlim_{\mathcal{L}_S^c(G)} H^*(-, M) = \varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. Hence, it remains to compare $\varprojlim_{\mathcal{T}_S(G)} H^*(-, M)$ and $\varprojlim_{\mathcal{T}_S^c(G)} H^*(-, M)$. For that we can use a result of Grodal [**Gr**].

Definition 5.3. let G be a finite group, $S \in \mathrm{Syl}_p(G)$ and M be a $\mathbb{Z}_{(p)}[G]$-module. Let K be the kernel of $G \to \mathrm{Aut}(M)$. A subgroup $P \leqslant S$ is called M*-essential* if

(i) the poset of non-trivial p-subgroup of $N_G(P)/P$ is empty or disconnected,

(ii) $Z(P) \cap K \in \mathrm{Syl}_p(C_G(P) \cap K)$,

(iii) $O_p(N_G(P)/(P(C_G(P) \cap K))) = 1$.

The property (ii) looks like the definition of p-centric and (iii) looks like the definition of $\mathcal{F}$-radical. For the property (i), if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, it is equivalent to $P = S$ or P is $\mathcal{F}$-essential [**AKO**, Definition I.3.2].

Theorem 5.4 ([**Gr**, Corollary 10.4])**.** *Let G be a finite group, S a Sylow p-subgroup of G and M a $\mathbb{Z}_{(p)}[G]$-module.*

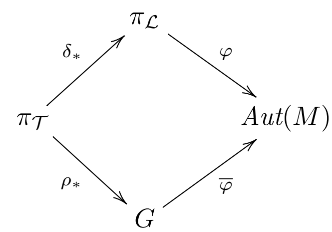
Let $\mathcal{H}$ be a family of subgroup of S containing S and all the subgroups which are M-essential.

Then, the inclusion of S in G induce a natural isomorphism,

$$H^*(G, M) \cong \varprojlim_{\mathcal{T}_S^{\mathcal{H}}(G)} H^*(-, M).$$

From this theorem and Theorem 5.2, we get the following corollary.

Corollary 5.5. *Let G be a finite group, S a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group. Let M be a $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module and assume that we have* *the following commutative diagram:*



that ρ_* *is surjective and that* $\Gamma := Im(\varphi) = Im(\overline{\varphi})$. *If* Γ *is* p*-solvable and all the* M*-essential subgroups of* S *are* p*-centric, then* δ *and* ρ *induce natural isomorphisms,*

$$H^*(|\mathcal{L}|, M) \cong H^*(G, M) \cong H^*(\mathcal{F}^c, M).$$

We also conjecture that it can be generalized to any abstract p-local finite group and any $\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]$-module with a p-solvable action.

Conjecture 5.6. *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a* p*-local finite group and let* M *be a* $\mathbb{Z}_{(p)}[\pi_1(|\mathcal{L}|)]$*-module. If the action of* $\pi_1(|\mathcal{L}|)$ *on* M *is* p*-solvable, then the inclusion of* BS *in* $|\mathcal{L}|$ *induces a natural isomorphism*

$$H^*(|\mathcal{L}|, M) \xrightarrow{\cong} H^*(\mathcal{F}^c, M).$$

Corollary 4.5 and Corollary 5.5 give good evidence for Conjecture 5.6 to be true.

The next section, which is a bit technical, is dedicated to give an example of Conjecture 5.6 where Corollary 5.5 doesn't apply (see Remark 6.7).

6. The p-local structure of wreath products by C_p: an example for Conjecture 5.6

Let G_0 be a finite group, S_0 a Sylow p-subgroup of G_0 and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p-local finite group. We are interested in the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ and the associated p-local finite group $(S, \mathcal{F}, \mathcal{L})$. By [**CL**, Theorem 5.2 and Remark 5.3], we have that $|\mathcal{L}| \simeq |\mathcal{L}_0| \wr BC_p := |\mathcal{L}_0|^p \times_{C_p} EC_p$ and an extension $(\pi_{\mathcal{L}_0})^p \to \pi_{\mathcal{L}} \to C_p$. In addition, we have a section $C_p \to \pi_{\mathcal{L}}$ coming from $* \wr BC_p \to |\mathcal{L}_0| \wr BC_p$ and thus $\pi_{\mathcal{L}} = \pi_{\mathcal{L}_0} \wr C_p$.

We first give a lemma on strongly p-embedded subgroups. For a finite group G, a subgroup $H < G$ is *strongly* p*-embedded*, if $p \mid |H|$ and for each $x \in G \setminus H$, $H \cap xHx^{-1}$ has order prime to p.

Lemma 6.1. *Let* G *be a finite group,* $G_0 \leqslant G$ *a subgroup of index a power of* p*. If* G *contains a strongly* p*-embedded subgroup and* $p \mid |G_0|$*, then* G_0 *contains a strongly* p*-embedded subgroup.*

Proof. Let H be a strongly p-embedded subgroup of G. By [**AKO**, Proposition A.7], H contains a Sylow p-subgroup of G so, up to conjugacy, we can choose H such that H contains a Sylow p-subgroup of G_0. Hence $G_0 \cap H$ contains a Sylow p-subgroup of G_0 and $p \mid |G_0 \cap H|$. We will show that $G_0 \cap H$ is a strongly p-embedded subgroup of G_0.

As $[G:H]$ is prime to p and $[G:G_0]$ is a power of p, $G_0 \cap H$ is a proper subgroup of G_0.

It remains to show that, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p. But $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1} \leqslant H \cap xHx^{-1}$, thus, as H is a strongly p-embedded subgroup of G, this last subgroup has order prime to p for every $x \in G \setminus H$. In particular, for each $x \in G_0 \setminus G_0 \cap H$, $(G_0 \cap H) \cap x(G_0 \cap H)x^{-1}$ has order prime to p and $G_0 \cap H$ is a strongly p-embedded subgroup of G_0. □

We give also a lemma on $\mathcal{F}_1$-essential subgroups for $\mathcal{F}_1 \subseteq \mathcal{F}$ a subsystem of index a power of p. A proper subgroup $P < S$ is $\mathcal{F}$-*essential* if P is $\mathcal{F}$-centric and fully normalized in $\mathcal{F}$, and if $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup.

Lemma 6.2. *Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group and $(S_1,\mathcal{F}_1,\mathcal{L}_1)$ a p-local subgroup of index a power of p. If $P < S_1$ is $\mathcal{F}$-essential, then P is $\mathcal{F}_1$-conjugate to an $\mathcal{F}_1$-essential subgroup and P is $\mathcal{F}_1$-essential if and only if P is fully normalized in $\mathcal{F}_1$.*

Proof. Let $P < S_1$ be an $\mathcal{F}$-essential subgroup. Since $\mathcal{F}_1$ is saturated, P is $\mathcal{F}_1$-conjugate to a subgroup of S_1 fully normalized in $\mathcal{F}_1$. If P is $\mathcal{F}_1$-essential, it is, in particular, fully normalized in $\mathcal{F}_1$. Thus, it remains to prove that if P is fully normalized in $\mathcal{F}_1$, then P is $\mathcal{F}_1$-essential. For the remaining, we assume that P is fully normalized in $\mathcal{F}_1$ and $\mathcal{F}$-essential.

P is $\mathcal{F}_1$-centric: As P is $\mathcal{F}$-centric, $C_S(Q) = Z(Q)$ for all $Q \in P^{\mathcal{F}}$. In particular, for all $Q \in P^{\mathcal{F}_1} \subseteq P^{\mathcal{F}}$, $C_{S_1}(Q) = Z(Q)$ and P is $\mathcal{F}_1$-centric.

$\mathrm{Out}_{\mathcal{F}_1}(P)$ contains a strongly p-embedded subgroup: Since P is $\mathcal{F}$-essential, the group $\mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup. As $\mathcal{F}_1$ is a subsystem of $\mathcal{F}$ of index a power of p, $\mathrm{Out}_{\mathcal{F}_1}(P)$ is a subgroup of $\mathrm{Out}_{\mathcal{F}}(P)$ of index a power of p. Moreover, as P is a proper subgroup of S_1, $P < N_{S_1}(P)$ and, as P is $\mathcal{F}_1$-centric, every element of $N_{S_1}(P) \smallsetminus Z(P)$ induces a non-trivial element in $\mathrm{Out}_{\mathcal{F}_1}(P)$. Hence $p \mid |\mathrm{Out}_{\mathcal{F}_1}(P)|$ and, by Lemma 6.1, $\mathrm{Out}_{\mathcal{F}_1}(P)$ contains a strongly p-embedded subgroup. □

We can easily describe the essential subgroups of a product of fusion systems.

Lemma 6.3. *Let $(S_1,\mathcal{F}_1,\mathcal{L}_1)$ and $(S_2,\mathcal{F}_2,\mathcal{L}_2)$ be p-local finite groups and set $S = S_1 \times S_2$ and $\mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2$. The $\mathcal{F}$-essential subgroups of S are of the form $Q_1 \times S_2$ with $Q_1 < S_1$ $\mathcal{F}_1$-essential or $S_1 \times Q_2$ with $Q_2 > S_2$ $\mathcal{F}_2$-essential.*

Proof. Let $P \leqslant S$ be an $\mathcal{F}$-essential subgroup. By [**AKO**, Proposition I.3.3], P is $\mathcal{F}$-centric and $\mathcal{F}$-radical. Thus, by [**AOV**, Lemma 3.1], $P = P_1 \times P_2$ with $P_i \leqslant S_i$ and P_i $\mathcal{F}_i$-centric.

Remark also that, if we have two groups G_1 and G_2 such that p divide $|G_1|$ and $|G_2|$ then $G_1 \times G_2$ cannot contain a strongly p-embedded subgroup. To see that let S_i be a Sylow p-subgroup of G_i and set $H = \langle x \in G \mid x(S_1 \times S_2)x^{-1} \cap S_1 \times S_2 \neq 1\rangle$. H contains $G_1 \times \{0\}$ and $\{0\} \times G_2$ so that $H = G$. Thus, by [**AKO**, Proposition A.7], this implies that G has no strongly p-embedded subgroups.

We also have that $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{F}_1}(P_1) \times \mathrm{Out}_{\mathcal{F}_2}(P_2)$. Hence, the only possibility for P to be $\mathcal{F}$-essential is that $P_1 = S_1$ and P_2 is $\mathcal{F}_2$-essential or the contrary. □

Let G_0 be a finite group, S_0 a Sylow p-subgroup of G_0 and $(S_0,\mathcal{F}_0,\mathcal{L}_0)$ be the associated p-local finite group. We consider the wreath product $G = G_0 \wr C_p$, $S = S_0 \wr C_p$

and the associated p-local finite group $(S,\mathcal{F},\mathcal{L})$. Here, for the direct computation, we will take the notation of Alperin and Fong [**AF**]: an element of G will be represented by permutation matrix corresponding to the powers of $(1,2,\dots,p)$ with entries in G_0 and the composition will follow the matrix product with the composition in G_0. Denote by $c\in G$ the element

$$e\otimes P_{(1,2,\dots,p)}=\begin{pmatrix}0 & 0 & \cdots & 0 & e\\ e & 0 & \cdots & 0 & 0\\ \vdots & \ddots & & \vdots & \vdots\\ \vdots & & \ddots & 0 & 0\\ 0 & \cdots & \cdots & e & 0\end{pmatrix},$$

where e is the trivial element of G_0. Here, we are interested in the $\mathcal{F}$-essential subgroups.

Lemma 6.4. *Let $P\leqslant S$ be an $\mathcal{F}$-essential subgroup.*

(E_1) If $P\leqslant S_0^p$, then either $P=S_0^p$ and $N_G(P)=N_{G_0}(S_0)\wr C_p$ or P is $\mathcal{F}_0^p$-essential and $N_G(P)=N_{G_0^p}(P)$.

(E_2) If $P\nleqslant S_0^p$, then $P\cong_{\mathcal{F}}Q\wr C_p$ where Q is $\mathcal{F}_0$-essential and we have $N_G(P)/P\cong N_{G_0}(Q)/Q$ through the diagonal map $G_0\hookrightarrow G_0^p$.

Proof. Let $P\leqslant S$ be an $\mathcal{F}$-essential subgroup.

Assume first that $P\leqslant S_0^p$. If $P=S_0^p$ a direct calculation gives $N_G(P)=N_{G_0}(S_0)\wr C_p$. Else, by Lemma 6.2, we know that P is $\mathcal{F}_0^p$-conjugate to an $\mathcal{F}_0^p$-essential subgroup $Q\leqslant S_0^p$. By Lemma 6.3 we have $N_G(Q)\leqslant G_0^p$ and, in particular, $N_G(Q)=N_{G_0^p}(Q)$. Thus, since P is $\mathcal{F}_0^p$-conjugate to Q, we also have $N_G(P)=N_{G_0^p}(P)$ and, since P is fully normalized in $\mathcal{F}$, it is fully normalized in $\mathcal{F}_0^p$. Hence, by Lemma 6.2, P is $\mathcal{F}_0$-essential.

Secondly, assume that $P\nleqslant S_0^p$. As all choices of a splitting $C_p\to G$ are conjugate in G, we can assume that $P=\langle P_0,x\rangle$ where $P_0=P\cap S_0^p$ and $x=((x_1,x_2,\dots,x_p),c)$ is such that $x^p\in P_0$. Up to conjugation in $S_0\wr C_p$ we can assume that x is of the form $((a,1,1,\dots,1),c)$ where $a\in N_{S_0}(Q)$ where Q is the projection of P_0 on the first factor. If we write $P_0^{(i)}$ the projection of P_0 on its ith factor, as x normalizes P_0, we have that $P_0^{(i)}=P_0^{(j)}$ for all i,j and then $P_0\leqslant(P_0^{(1)})^p=Q^p$.

Notice also that $N_G(P)=\langle N_{G_0^p}(P),x\rangle$. If $g=(g_1,\dots,g_p)\in N_{G_0^p}(P)$, as g normalizes $P\cap G_0^p=P_0$, we have, for all i, $g_i\in N_{G_0}(Q)$. Moreover, if we denote $h=(h_1,\dots,h_p)=gxg^{-1}x^{-1}\in P_0$, we have, for all i, $g_ih_i=g_{i-1}$ (with $g_0=g_p$). Therefore, there is $h'\in Q^p$ such that $g=(g_1,g_1,\dots,g_1).h'\in\langle N_{G_0}(Q)\otimes\mathrm{Id},Q^p\rangle\leqslant N_G(Q^p)$. Hence, every automorphism $c_g\in\mathrm{Aut}_{\mathcal{F}}(P)$ can be extended to an automorphism of $\langle Q^p,x\rangle$. As P is $\mathcal{F}$ essential, by [**AKO**, Proposition I.3.3], $P=\langle Q^p,x\rangle$. Now, $x^p\in Q^p$ implies that $a\in Q$ so $P=\langle Q^p,x\rangle=\langle Q^p,c\rangle=Q\wr C_p$.

Finally, direct computations give that

$$C_G(P)\cong C_{G_0}(Q)\otimes\mathrm{Id}=\left\{\begin{pmatrix}g & 0 & \cdots & 0\\ 0 & g & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & g\end{pmatrix}\ ;\ g\in C_{G_0}(Q)\right\}$$

and

$$N_G(P)/P \cong N_{G_0}(Q)/Q \otimes \mathrm{Id} \cong N_{G_0}(Q)/Q.$$

In particular, as P is p-centric, Q is G_0-centric. Moreover, as $N_G(P)/P = \mathrm{Out}_{\mathcal{F}}(P)$ contains a strongly p-embedded subgroup, $\mathrm{Out}_{\mathcal{F}_0}(Q) = N_{G_0}(Q)/Q$ does as well. Up to conjugacy, we can also assume that Q is fully normalized in $\mathcal{F}_0$ and thus Q is $\mathcal{F}_0$-essential. □

Let us now look at some cohomological results. Recall that for a group G, a subgroup $H \leqslant G$, and M an $\mathbb{F}_p[H]$-module, we define the *induced* and *coinduced* $\mathbb{F}_p[G]$-module by,

$$\mathrm{Ind}_H^G(M) = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p[H]} M, \qquad \mathrm{coInd}_H^G(M) = \mathrm{Hom}_{\mathbb{F}_p[H]}(\mathbb{F}_p[G], M).$$

Recall also that, when the index of H in G is finite, these two $\mathbb{F}_p[G]$-modules are isomorphic (by [**We**, Lemma 6.3.4]).

Lemma 6.5. *Let X be a CW complex and denote by G its fundamental group. If X_0 is a covering space of X with fundamental group $G_0 \trianglelefteq G$ of finite index, then, for every $\mathbb{F}_p[G_0]$-module M, we have a natural isomorphism of $\mathbb{F}_p[G/G_0]$-modules,*

$$H^*(X_0, Ind_{G_0}^G(M)) \cong H^*(X_0, M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0],$$

where, on the right side, G/G_0 is only acting by translation on $\mathbb{F}_p[G/G_0]$.

Proof. This can be easily seen on the chain level. Let $\tilde{X}$ be the universal covering space of X. As $\mathbb{F}_p[G/G_0]$-modules, we have the following

$$\mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \mathrm{Ind}_{G_0}^G(M)) = \bigoplus_{g \in [G/G_0]} \mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), g.M),$$

where the action of G/G_0 is permuting the terms in the sum. But, each terms in the sum is isomorphic, as (trivial) $\mathbb{F}_p[G/G_0]$-modules, to $\mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M)$. Thus

$$\mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), \mathrm{Ind}_{G_0}^G(M)) \cong \mathrm{Hom}_{\mathbb{F}_p[G_0]}(C_*(\tilde{X}), M) \otimes_{\mathbb{F}_p} \mathbb{F}_p[G/G_0].$$

This induces the wanted isomorphism in cohomology. □

Proposition 6.6. *Let G_0 be a finite group and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ be the associated p-local finite group. Consider $G = G_0 \wr C_p$, $S = S_0 \wr C_p$ a Sylow p-subgroup of G and $(S, \mathcal{F}, \mathcal{L})$ the associated p-local finite group. Let M be an $\mathbb{F}_p[\pi_{\mathcal{L}_0}]$-module.*

If δ_{S_0} induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M),$$

and

$$H^*(|\mathcal{L}_0|^p, coInd_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}}(M^{\otimes p})) \cong H^*((\mathcal{F}_0^p)^c, coInd_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}}(M^{\otimes p})),$$

then δ_S induces a natural isomorphism

$$H^*(|\mathcal{L}|, coInd_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}}(M^{\otimes p})) \cong H^*(\mathcal{F}^c, coInd_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}}(M^{\otimes p})).$$

Proof. Write $N = \mathrm{coInd}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}}(M^{\otimes p})$ and, for $i \in \{1,2\}$, denote by $H^*(\mathcal{F}^{E_i}, N)$ the stable elements of $H^*(S,N)$ under the full subcategory of $\mathcal{F}$ with objects S and all the subgroups of S of type (E_i) defined in Lemma 6.4.

By the Mackey Formula,

$$\mathrm{Res}_{Q\wr C_p}^{\pi_{\mathcal{L}}}\mathrm{Ind}_{\pi_{\mathcal{L}_0}^p}^{\pi_{\mathcal{L}}} = \mathrm{Ind}_{Q^p}^{Q\wr C_p}\mathrm{Res}_{Q^p}^{\pi_{\mathcal{L}_0}^p}.$$

Thus by Shapiro's Lemma (see for example [**Ev**, Proposition 4.1.3]) and the Kunneth Formula, for every $P = Q \wr C_p$ of type (E_2), we have a natural isomorphism $H^*(Q \wr C_p, N) \cong H^*(Q^p, M^{\otimes p}) \cong H^*(Q,M)^{\otimes p}$ and, by the computation of normalizers in Lemma 6.4,

$$H^*(Q \wr C_p, N)^{\mathrm{Aut}_{\mathcal{F}}(Q\wr C_p)} \cong (H^*(Q,M)^{\mathrm{Aut}_{\mathcal{F}_0}(Q)})^{\otimes p}.$$

Hence, applying this to all the subgroups of type (E_2) and, by naturality of the Shapiro isomorphisms, we have that,

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p}.$$

On the other hand, by [**CL**, Theorem 5.2 and Remark 5.3], $|\mathcal{L}_0|^p$ has the homotopy type of a covering space of $|\mathcal{L}|$ with covering group C_p. Then, if we denote by X the universal covering space of $|\mathcal{L}|$ (which is also the universal covering space of $|\mathcal{L}_0|^p$), we have the following isomorphism on the chain level (because Res and coInd are adjoint functors)

$$\mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}_0}^p]}(C_*(X), M^{\otimes p}) \cong \mathrm{Hom}_{\mathbb{Z}_{(p)}[\pi_{\mathcal{L}}]}(C_*(X), N),$$

which is analogue to the Shapiro isomorphism (see [**Ev**, Proposition 4.1.3]). By the Kunneth Formula, it gives us the following isomorphism on cohomology

$$H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N)$$

and give the following commutative diagram:

$$\begin{array}{ccc} H^*(S_0,M)^{\otimes p} & \xrightarrow{\cong} & H^*(S,N) \\ \downarrow{\scriptstyle (\delta_{S_0})^*} & & \downarrow{\scriptstyle \delta_S^*} \\ H^*(|\mathcal{L}_0|,M)^{\otimes p} & \xrightarrow[\cong]{} & H^*(|\mathcal{L}|,N). \end{array}$$

Thus δ_S induces an isomorphism

$$H^*(\mathcal{F}^{E_2}, N) \cong H^*(\mathcal{F}_0^c, M)^{\otimes p} \cong H^*(|\mathcal{L}_0|, M)^{\otimes p} \cong H^*(|\mathcal{L}|, N).$$

Secondly, by factoring the Shapiro isomorphism (see [**Ev**, Proposition 4.1.3]), the inclusion of S_0^p in S induces an injection $H^*(S,N) \hookrightarrow H^*(S_0^p, N)$. Hence

$$H^*(\mathcal{F}^{E_1}, N) \cong H^*((\mathcal{F}_0^p)^c, N)^{C_p} \leqslant H^*(S_0^p, N).$$

By assumption, $\delta_{S_0^p}$ induces an isomorphism

$$H^*((\mathcal{F}_0^p)^c, N) \cong H^*(|\mathcal{L}_0|^p, N).$$

Moreover, by Lemma 6.5, this last term is isomorphic to $H^*(|\mathcal{L}_0|^p, M^{\otimes p}) \otimes \mathbb{F}_p[C_p]$ and, in particular, it is a projective $\mathbb{F}_p[C_p]$-module.

Consider now the Serre spectral sequence associated to the fibration sequence

$$|\mathcal{L}_0|^p \longrightarrow |\mathcal{L}| \longrightarrow BC_p,$$

with coefficients in N. The E_2 page is the following,

$$E_2^{s,t} = H^s(C_p, H^t(|\mathcal{L}_0|^p, N))$$

and, by projectivity of $H^t(|\mathcal{L}_0|^p, N)$, the E_2 page is concentrated in the 0th column. Hence, we have that, $H^*(|\mathcal{L}_0|^p, N)^{C_p} = E_2^{0,*} \cong H^*(|\mathcal{L}|, N)$.

In conclusion,

$$H^*(\mathcal{F}^c, N) = H^*(\mathcal{F}^{E_1}, N) \cap H^*(\mathcal{F}^{E_2}, N) \cong H^*(|\mathcal{L}|, N)$$

and the theorem follows. □

This proposition is a bit technical but we will use it in a specific case. Consider $p = 5$, the group $G_0 = GL_{20}(F_2)$, the wreath product $G = G_0 \wr C_5$ and $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ and $(S, \mathcal{F}, \mathcal{L})$ the associated 5-local finite groups. By [**Ru**, Theorem 6.3], we know that $(S_0, \mathcal{F}_0, \mathcal{L}_0)$ admits a 5-local subgroup of index 4 which is exotic $(S_e, \mathcal{F}_e, \mathcal{L}_e)$ and that we have a fibration sequence

$$|\mathcal{L}_e| \longrightarrow |\mathcal{L}_0| \longrightarrow BC_4.$$

In particular, we have $\pi_{\mathcal{L}}/\pi_{\mathcal{L}_e}^5 = C_4 \wr C_5$ and we can be interested in comparing $H^*(|\mathcal{L}|, N)$ and $H^*(\mathcal{F}^c, N)$ for

$$N = \mathbb{F}_5[C_4 \wr C_5] = \mathrm{Ind}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}}(M^{\otimes 5}) \cong \mathrm{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}}(M^{\otimes 5})$$

(the action factors through a finite group) with $M = \mathbb{F}_5[C_4]$.

By Corollary 4.4, we have that δ_{S_0} and $\delta_{S_0^p}$ induce natural isomorphisms

$$H^*(|\mathcal{L}_0|, M) \cong H^*((\mathcal{F}_0)^c, M) \text{ and}$$

$$H^*(|\mathcal{L}_0|^5, \mathrm{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}}(M^{\otimes 5})) \cong H^*((\mathcal{F}_0^5)^c, \mathrm{coInd}_{\pi_{\mathcal{L}_0}^5}^{\pi_{\mathcal{L}}}(M^{\otimes 5})),$$

(for the second isomorphism, notice that $|\mathcal{L}_0|^5$ has the homotopy type of a linking system associated to $\mathcal{F}_0^5$ by [**CL**, Proposition 2.17]). Hence, all the hypothesis of Proposition 6.6 are satisfied and

$$H^*(|\mathcal{L}|, N) \cong H^*(\mathcal{F}^c, N).$$

Remark 6.7. This gives us an example of isomorphism between the cohomology of $|\mathcal{L}|$ and the stable elements when the action factors through a p-solvable group which cannot be recovered by a previous result. Notice that, even if the fusion system $\mathcal{F}$ is realizable, as $\mathcal{F}_e$ is exotic, we cannot find a group G with $S \in \mathrm{Syl}_p(G)$ such that G acts on M in the same way as asked in Section 5. This example gives us some additional evidence for Conjecture 5.6.

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