ON THE GYROKINETIC LIMIT FOR THE TWO-DIMENSIONAL VLASOV-POISSON SYSTEM

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ABSTRACT. We investigate the gyrokinetic limit for the two-dimensional Vlasov-Poisson system in the regime studied by Golse and Saint-Raymond [12] and by Saint-Raymond [26]. We present another proof of the convergence towards the Euler equation under several assumptions on the energy and on the L^∞ norms of the initial data.

1. Introduction and main results

The purpose of this paper is to investigate an asymptotic regime for the following Vlasov-Poisson system as ε tends to zero:

(1.1)

$$\begin{cases} \partial_t f_\varepsilon + \dfrac{v}{\varepsilon}\cdot\nabla_x f_\varepsilon + \left(\dfrac{E_\varepsilon}{\varepsilon} + \dfrac{v^\perp}{\varepsilon^2}\right)\cdot\nabla_v f_\varepsilon = 0, \quad (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2 \\ E_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\rho_\varepsilon(t,y)\,dy, \quad \rho_\varepsilon(t,x) = \int_{\mathbb{R}^2} f_\varepsilon(t,x,v)\,dv, \\ f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v). \end{cases}$$

Here, $f_\varepsilon = f_\varepsilon(t,x,v) : \mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}_+$ stands for the density of a two-dimensional distribution of electric particles, called a plasma. The evolution of the plasma in the plane is submitted to the self-consistent electric field $E_\varepsilon(t,x)$ and to a large external and constant magnetic field, orthogonal to the plane, which is represented by the term $v^\perp = (v_1,v_2)^\perp = (-v_2,v_1)$. The limit $\varepsilon\to 0$ corresponds to the situation where the strength of the magnetic field tends to infinity. In the periodic setting, namely $(x,v)\in\mathbb{T}\times\mathbb{R}^2$, the gyrokinetic limit was studied by Golse and Saint-Raymond [12], then by Saint-Raymond [26], and also by Brenier [6] in a different regime. In particular, Golse and Saint-Raymond proved that under suitable bounds on the initial data[1], the sequence of spatial densities $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in[2] $L^\infty(\mathbb{R}_+,\mathcal{M}^+(\mathbb{T}\times\mathbb{R}^2))$ weakly $*$ and that any accumulation point ρ is a measure-valued solution[3] to the 2D Euler equation for the vorticity:

$$\begin{cases} \partial_t\rho + E^\perp\cdot\nabla\rho = 0 \\ E^\perp = 2\pi\nabla^\perp\Delta^{-1}\rho. \end{cases} \tag{1.2}$$

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[1] See (1.3), (1.4) and (1.6) below.

[2] Here, $\mathcal{M}^+(\mathbb{R}^2)$ denotes the space of bounded, positive Radon measures on $\mathbb{R}^2$.

[3] In a sense that will be specified in Definition 2.1 below.

The main result of this paper will concern initial densities f_ε^0 satisfying the following assumptions:

$$f_\varepsilon^0 \in L^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2), \quad f_\varepsilon^0 \geq 0, \quad f_\varepsilon^0 \text{ is compactly supported.} \tag{1.3}$$

Moreover, defining for $f\in L^1$ and $\rho=\int f\,dv$ the energy

$$\mathcal{H}(f)=\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2 f(x,v)\,dx\,dv-\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln|x-y|\rho(x)\rho(y)\,dx\,dy,$$

we will assume that

$$\begin{aligned}&\sup_{\varepsilon>0}\left(\|f_\varepsilon^0\|_{L^1}+\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon^0(x)\,dx\right)<+\infty,\\ &\sup_{\varepsilon>0}\mathcal{H}(f_\varepsilon^0)<+\infty.\end{aligned} \tag{1.4}$$

Finally,

$$\varepsilon^2\Theta\left(\|f_\varepsilon^0\|_{L^\infty}\right)\to 0, \quad \text{as } \varepsilon\to 0, \tag{1.5}$$

where $\Theta(\tau)=\tau\ln(\tau+2)$.

Adapting the classical Cauchy theory for the Vlasov-Poisson equation [2, 20, 24] for any $\varepsilon>0$, one obtains a unique global weak solution f_ε to (1.1) belonging to $L^\infty(\mathbb{R}_+,L^1\cap L^\infty(\mathbb{R}^2))$, compactly supported, such that $f_\varepsilon(0)=f_\varepsilon^0$. In particular, the associated spatial density ρ_ε belongs to $L^\infty_{\text{loc}}(\mathbb{R}_+,L^\infty(\mathbb{R}^2))$. Finally, the energy and the L^p norms of the solution are non-increasing in time.

Our main result on the asymptotics of (1.1) can be then stated as follows:

Theorem 1.1. *Let f_ε^0 satisfy* (1.3), (1.4) *and* (1.5). *Let f_ε be the corresponding global weak solution to* (1.1). *There exists a subsequence $\varepsilon_n\to 0$ as $n\to+\infty$ such that $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to ρ in*[4] $C(\mathbb{R}_+,\mathcal{M}^+(\mathbb{R}^2)-w^*)$. *Moreover, ρ belongs to $L^\infty(\mathbb{R}_+,H^{-1}(\mathbb{R}^2))$ and it is a global solution of the 2D Euler equation*(1.2) *in the sense of Definition 2.1.*

Theorem 1.1 is a slight improvement of the convergence result in [26], which handles initial densities satisfying (1.3), (1.4) and (1.6):

$$\varepsilon\|f_\varepsilon^0\|_{L^\infty}\to 0, \quad \text{as } \varepsilon\to 0. \tag{1.6}$$

Typically, the assumption (1.5) allows for initial data such that for some $\beta>1$

$$\sup_{\varepsilon>0}\varepsilon^2|\ln\varepsilon|^\beta\|f_\varepsilon^0\|_{L^\infty}<+\infty.$$

Thus, Theorem 1.1 includes initial data that converge to monokinetic data:

$$f_\varepsilon^0(x,v)=\rho_0(x)\frac{1}{\eta_\varepsilon^2}F\left(\frac{v-u_\varepsilon(x)}{\eta_\varepsilon}\right),$$

where for instance $u_\varepsilon\in L^2(\mathbb{R}^2)$, $\rho_0\in L^\infty(\mathbb{R}^2)$, $F\in L^1\cap L^\infty(\mathbb{R}^2)$, and $\varepsilon^2\Theta(\eta_\varepsilon^{-2})$ vanishes as $\varepsilon\to 0$.

In the case where (1.5) is replaced by the uniform bound

$$\sup_{\varepsilon>0}\|f_\varepsilon^0\|_{L^\infty}<+\infty, \tag{1.7}$$

[4] Here, $\rho\in C(\mathbb{R}_+,\mathcal{M}^+(\mathbb{R}^2)-w^*)$ if and only if $\rho(t)\in\mathcal{M}^+(\mathbb{R}^2)$ for all $t\in\mathbb{R}_+$ and moreover, $t\mapsto\int\phi(x)\,d\rho(t,x)$ is continous, for all $\varphi\in C_c(\mathbb{R}^2)$.

any accumulation point is a true solution of the 2D Euler equation:

Theorem 1.2. *Let f_ε^0 satisfy* (1.3), (1.4) *and* (1.7). *Let f_ε be the corresponding global weak solution to* (1.1). *There exists a subsequence $\varepsilon_n \to 0$ as $n \to +\infty$ such that $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to ρ in $C(\mathbb{R}_+, L^2(\mathbb{R}^2)-w)$. Moreover,*

1. *$\rho \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^2))$;*
2. *$(E_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to some E in $C(\mathbb{R}_+, L^2_{loc}(\mathbb{R}^2))$;*
3. *For all $t \in \mathbb{R}_+$, $E(t) = (x/|x|^2) * \rho(t)$;*
4. *ρ is a global weak solution of the 2D Euler equation* (1.2) *in the sense of distributions.*

Besides the already mentioned articles by Golse and Saint-Raymond [12] and Saint-Raymond [26], a wide literature has been devoted to the mathematical analysis of the Vlasov equation in the limit of large magnetic or electric field. Brenier [6] derived the Euler equation in a different scaling, for smooth and well-prepared data, by means of a different method based on the modulated energy. Various asymptotic regimes for linear or non linear Vlasov equations were investigated by Frénod and Sonnendrücker [9, 10, 11], Golse and Saint-Raymond [13, 25], Han-Kwan [15], Ghendrih, Hauray and Nouri [17], Hauray and Nouri [16], and more recently by Bostan, Finot and Hauray [5] and by Barré, Chiron, Goudon and Masmoudi [3]. The convergence results in [12, 26] rely on the derivation of an equation for the spatial density with a good control of the large velocities. Here, the main ingredient of proof is based on a different weak formulation for the spatial density, following from the ODE satisfied by a suitable combination of the characteristics along which the density is essentially constant, see Proposition 2.7. This approach actually amounts to focusing on the equation satisfied by the shifted density $f_\varepsilon(t, x - \varepsilon v^\perp, v)$, see Proposition 2.11. These so-called gyrocoordinates $(x - v^\perp, v)$ were used in [17] (see also [16]) for the derivation of a gyrokinetic model from a linear Vlasov equation. We also mention that a similar change of variable in the space variable, in addition to a transformation by rotation in the velocity variable, has been considered in [11] and in the recent work [5].

2. Proof of Theorem 1.1

2.1. **Vortex sheet solution of the Euler equation.** We first define the notion of weak solution to the Euler equation (1.2), called vortex sheet solution, which is invoked in Theorem 1.1.

Definition 2.1 (According to [7, 27])**.** Let $\rho_0 \in \mathcal{M}^+(\mathbb{R}^2)\cap H^{-1}(\mathbb{R}^2)$ be compactly supported. We say that $\rho \in C(\mathbb{R}_+, \mathcal{M}^+(\mathbb{R}^2)-w^*)\cap L^\infty(\mathbb{R}_+, H^{-1}(\mathbb{R}^2))$ is a global weak solution of the Euler equation with initial datum ρ_0 if we have for all $\Phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \Phi(t,x)\,d\rho(t,x) = \int_{\mathbb{R}^2} \Phi(0,x)\,d\rho_0(x) + \int_0^t \int_{\mathbb{R}^2} \partial_s\Phi(s,x)\,d\rho(s,x)\,ds + \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} H_\Phi(x,y)\,d\rho(s,x)\,d\rho(s,y)\,ds,$$

where

$$H_\Phi(x,y)=\frac{1}{2}\frac{(x-y)^\perp}{|x-y|^2}\cdot\left(\nabla\Phi(x)-\nabla\Phi(y)\right).$$

For any compactly supported ρ_0 in $\mathcal{M}^+(\mathbb{R}^2)\cap H^{-1}(\mathbb{R}^2)$, global existence of a corresponding vortex sheet solution (satisfying a slightly different formulation than the one above) was established by Delort [7]. The formulation of Definition 2.1, which has been introduced later by Schochet [27], is motivated by the observation that when ρ is a bounded and integrable map,

$$\langle \operatorname{div}(E^\perp\rho),\Phi\rangle_{\mathcal{D}',\mathcal{D}}=-\iint H_\Phi(x,y)\rho(x)\rho(y)\,dx\,dy. \tag{2.1}$$

Moreover, H_Φ is defined and continuous off the diagonal $\Delta=\{(x,x)|x\in\mathbb{R}^2\}$ and bounded on $\mathbb{R}^2\times\mathbb{R}^2$, since $\|H_\Phi\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2))}\leq\|\Phi\|_{W^{2,\infty}}$. Hence the expression (2.1) makes sense for ρ as in Definition 2.1, since the atomic support a positive measure in H^{-1} is empty [7].

2.2. **Uniform a priori estimates.** In all the remainder of this section, f_ε denotes the global weak solution of (1.1) with initial data f_ε^0 satisfying (1.3), (1.4) and (1.5). Replacing $\|f_\varepsilon^0\|_{L^\infty}$ by $\max(1,\|f_\varepsilon^0\|_{L^\infty})$ if necessary, we will always assume that

$$\|f_\varepsilon^0\|_{L^\infty}\geq 1.$$

The purpose of this paragraph is to collect a priori estimates and basic properties for f_ε for later use. The notation C will stand for a constant independent of ε, changing possibly from a line to another.

Proposition 2.2. *We have*

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\left(\|f_\varepsilon(t)\|_{L^1}+\mathcal{H}(f_\varepsilon(t))\right)<+\infty,$$

and

$$\sup_{t\in\mathbb{R}_+}\varepsilon^2\Theta(\|f_\varepsilon(t)\|_{L^\infty})\leq\varepsilon^2\Theta(\|f_\varepsilon^0\|)\to 0,\quad \textit{as } \varepsilon\to 0.$$

Proof. This is an immediate consequence of the fact that for (1.1), the energy and the norms of f_ε satisfy

$$\forall t\in\mathbb{R}_+,\quad \mathcal{H}(f_\varepsilon(t))\leq\mathcal{H}_\varepsilon(0),\quad \|f_\varepsilon(t)\|_{L^p}\leq\|f_\varepsilon^0\|_{L^p}.$$

□

Proposition 2.3. *We have for all $t\in\mathbb{R}_+$ and for all $0<\varepsilon<1$*

$$\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(t,x)\,dx\leq C\left(1+\varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2f_\varepsilon^0(x,v)\,dx\,dv\right)$$
$$+C\varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|^2f_\varepsilon(t,x,v)\,dx\,dv.$$

Proof. Let $T>0$ and $R_\varepsilon>0$ such that $\operatorname{supp}(f_\varepsilon(t))$ is included in $\overline{B}(0,R_\varepsilon)\times\overline{B}(0,R_\varepsilon)$ on $[0,T]$. We set $\varphi(x,v)=\left(|x|^2+2\varepsilon x\cdot v^\perp\right)\chi(|x|/R_\varepsilon)\chi(|v|/R_\varepsilon)$, where χ is a smooth cut-off function such that $\chi=1$ on $B(0,1)$ and $\chi=0$

on $B(0,2)^c$. For $t \in [0,T)$, we compute using the weak formulation of (1.1) for the test function φ,

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(|x+\varepsilon v^\perp|^2 - \varepsilon^2|v|^2\right) f_\varepsilon(t,x,v)\,dx\,dv \\
&= \frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \varphi(x,v) f_\varepsilon(t,x,v)\,dx\,dv \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(t,x,v)\left(\frac{v}{\varepsilon}\cdot\nabla_x\varphi + \frac{E_\varepsilon}{\varepsilon}\cdot\nabla_v\varphi + \frac{v^\perp}{\varepsilon^2}\cdot\nabla_v\varphi\right)dx\,dv \\
&= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(t,x,v)\left(\frac{v}{\varepsilon}\cdot(2x+2\varepsilon v^\perp) - 2E_\varepsilon\cdot x^\perp - 2\frac{v^\perp}{\varepsilon}\cdot x^\perp\right)dx\,dv \\
&= -2\int_{\mathbb{R}^2} \rho_\varepsilon(t,x)E_\varepsilon(t,x)\cdot x^\perp\,dx.
\end{aligned}$$

On the other hand, in view of the definition of E_ε, we obtain by a classical symmetrization argument

$$\begin{aligned}
\int_{\mathbb{R}^2} \rho_\varepsilon(t,x)E_\varepsilon(t,x)\cdot x^\perp\,dx &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\frac{x-y}{|x-y|^2}\cdot x^\perp\,dx\,dy \\
&= \frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\frac{x-y}{|x-y|^2}\cdot(x^\perp - y^\perp)\,dx\,dy = 0.
\end{aligned}$$

Since $|x|^2 \le 2(|x+\varepsilon v^\perp|^2 - \varepsilon^2|v|^2) + 4\varepsilon^2|v|^2$, it follows that

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f_\varepsilon(t,x,v)\,dx\,dv \\
&\le 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} \left(|x+\varepsilon v^\perp|^2 - \varepsilon^2|v|^2\right) f^0_\varepsilon(x,v)\,dx\,dv + 4\varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv \\
&\le C\left(\int_{\mathbb{R}^2} |x|^2\rho^0_\varepsilon(x)\,dx + \varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f^0_\varepsilon(x,v)\,dx\,dv\right) \\
&+ C\varepsilon^2\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv.
\end{aligned}$$

□

Proposition 2.4. *We have*

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\left(\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f_\varepsilon(t,x,v)\,dx\,dv + \int_{\mathbb{R}^2} |x|^2\rho_\varepsilon(t,x)\,dx\right) < +\infty,$$

and

$$\sup_{t\in\mathbb{R}_+} \|\rho_\varepsilon(t)\|_{L^2} \le C\|f^0_\varepsilon\|^{1/2}_{L^\infty}.$$

Finally, setting

$$J_\varepsilon(t,x) = \int_{\mathbb{R}^2} |v| f_\varepsilon(t,x,v)\,dv,$$

we have

$$\sup_{t\in\mathbb{R}_+} \|J_\varepsilon(t)\|_{L^{4/3}} \le C\|f^0_\varepsilon\|^{1/4}_{L^\infty}$$

and

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0} \|J_\varepsilon(t)\|_{L^1} < +\infty.$$

Proof. The proof is classical, but we provide some details for sake of completeness. We omit the dependence on t for simplicity. Setting

$$K_\varepsilon = \iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^2 f_\varepsilon(x,v)\,dx\,dv,$$

we have the interpolation inequality (see e.g. [12, Lemma 3.1] or [26, Lemma 2.4])

$$\|\rho_\varepsilon\|_{L^2} \leq C\|f_\varepsilon\|_{L^\infty}^{1/2} K_\varepsilon^{1/2} \leq C\|f_\varepsilon^0\|_{L^\infty}^{1/2} K_\varepsilon^{1/2}.$$

On the other hand, Cauchy-Schwarz inequality and Proposition 2.3 yield

$$\begin{aligned} K_\varepsilon &\leq 2\mathcal{H}(f_\varepsilon) + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln_+|x-y|\rho_\varepsilon(x)\rho_\varepsilon(y)\,dx\,dy \\ &\leq 2\mathcal{H}(f_\varepsilon^0) + \iint_{\mathbb{R}^2\times\mathbb{R}^2} (|x|+|y|)\,\rho_\varepsilon(x)\rho_\varepsilon(y)\,dx\,dy \\ &\leq C + 2\|\rho_\varepsilon\|_{L^1}^{3/2}\left(\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(x)\,dx\right)^{1/2} \\ &\leq C + C\left(1+\varepsilon^2 K_\varepsilon(0)+\varepsilon^2 K_\varepsilon\right)^{1/2}. \end{aligned}$$

For the same reasons, we have

$$\begin{aligned} K_\varepsilon(0) &\leq 2\mathcal{H}(f_\varepsilon^0) + \iint_{\mathbb{R}^2\times\mathbb{R}^2} \ln_+|x-y|\rho_\varepsilon^0(x)\rho_\varepsilon^0(y)\,dx\,dy \\ &\leq C + C\|\rho_\varepsilon^0\|_{L^1}^{3/2}\left(\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon^0(x)\,dx\right)^{1/2} \leq C \end{aligned}$$

in view of (1.5). So we conclude that $K_\varepsilon \leq C$, and by Proposition 2.3, it also follows that $\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(t,x)\,dx \leq C$.

Again by interpolation, we have

$$\|J_\varepsilon\|_{L^{4/3}} \leq C\|f_\varepsilon\|_{L^\infty}^{1/4} K_\varepsilon^{3/4} \leq C\|f_\varepsilon^0\|_{L^\infty}^{1/4} K_\varepsilon^{3/4},$$

and by Cauchy-Schwarz inequality, we obtain

$$\|J_\varepsilon\|_{L^1} \leq C\|f_\varepsilon\|_{L^1}^{1/2} K_\varepsilon^{1/2} \leq C\|f_\varepsilon^0\|_{L^1}^{1/2} K_\varepsilon^{1/2},$$

so the conclusion follows.

□

To conclude this paragraph, we introduce a smooth, positive function $\tilde{\rho}_\varepsilon$, compactly supported in $B(0,1)$, such that

$$\int_{\mathbb{R}^2}\tilde{\rho}_\varepsilon(x)\,dx = \int_{\mathbb{R}^2}\rho_\varepsilon(x)\,dx, \quad \sup_{\varepsilon>0}\|\tilde{\rho}_\varepsilon\|_{L^\infty} < +\infty \tag{2.2}$$

and we set

$$\tilde{E}_\varepsilon(x) = \int_{\mathbb{R}^2}\frac{x-y}{|x-y|^2}\tilde{\rho}_\varepsilon(y)\,dy.$$

Since $\int(\rho_\varepsilon(t)-\tilde{\rho}_\varepsilon) = 0$ and $\rho_\varepsilon(t)-\tilde{\rho}_\varepsilon$ is compactly supported, it is well-known that $E_\varepsilon(t)-\tilde{E}_\varepsilon$ belongs to $L^2(\mathbb{R}^2)$, see e.g. [23, Proposition 3.3]. In addition,

Proposition 2.5. *We have*

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{L^2}<+\infty.$$

Proof. The computations below are quite standard and we perform them for sake of completeness. We first integrate by parts, using that $E_\varepsilon(t)-\tilde{E}_\varepsilon = 2\pi\nabla G*(\rho_\varepsilon(t)-\tilde{\rho}_\varepsilon)$, with G the fundamental solution of the Laplacian in $\mathbb{R}^2$. Then we expand, which yields

$$\begin{aligned}\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{L^2}^2 &= -2\pi\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln|x-y|\,(\rho_\varepsilon-\tilde{\rho}_\varepsilon)(t,x)\,(\rho_\varepsilon-\tilde{\rho}_\varepsilon)(t,y)\,dx\,dy\\ &\le -2\pi\iint_{\mathbb{R}^2\times\mathbb{R}^2}\ln|x-y|\rho_\varepsilon(t,x)\rho_\varepsilon(t,y)\,dx\,dy\\ &\quad-2\pi\iint_{B(0,1)^2}\ln_-|x-y|\tilde{\rho}_\varepsilon(x)\tilde{\rho}_\varepsilon(y)\,dx\,dy\\ &\quad+4\pi\iint_{\mathbb{R}^2\times B(0,1)}\ln_+|x-y|\rho_\varepsilon(t,x)\tilde{\rho}_\varepsilon(y)\,dx\,dy.\end{aligned}$$

Then we use Proposition 2.4 and (2.2) to infer that

$$\begin{aligned}&\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{L^2}^2\\ &\le C\Big(\mathcal{H}(f_\varepsilon(t))+\|\tilde{\rho}_\varepsilon\|_{L^\infty}^2+\iint_{\mathbb{R}^2\times B(0,1)}(|x|+|y|)\rho_\varepsilon(t,x)\tilde{\rho}_\varepsilon(y)\,dx\,dy\Big)\\ &\le C\Big(\mathcal{H}(f_\varepsilon(t))+\|\tilde{\rho}_\varepsilon\|_{L^\infty}^2+\|\tilde{\rho}_\varepsilon\|_{L^\infty}\|\rho_\varepsilon(t)\|_{L^1}^{1/2}\left(\int_{\mathbb{R}^2}|x|^2\rho_\varepsilon(t,x)\,dx\right)^{1/2}\Big)\\ &+C\|\tilde{\rho}_\varepsilon\|_{L^\infty}\|\rho_\varepsilon(t)\|_{L^1}\\ &\le C.\end{aligned}$$

□

Proposition 2.6. *We have $E_\varepsilon-\tilde{E}_\varepsilon\in L^\infty(\mathbb{R}_+,H^1(\mathbb{R}^2))$ and*

$$\sup_{t\in\mathbb{R}_+}\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{H^1(\mathbb{R}^2)}\le C\|f_\varepsilon^0\|_{L^\infty}^{1/2}.$$

In particular, for all $q\ge 2$ we have

$$\sup_{t\in\mathbb{R}_+}\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{L^q}\le C\sqrt{q}\|f_\varepsilon^0\|_{L^\infty}^{1/2}.$$

Proof. On the one hand, $\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{L^2}\le C$ by virtue of Proposition 2.5. On the other hand, standard elliptic regularity theory yields a constant $C>0$ such that

$$\|\nabla(E_\varepsilon(t)-\tilde{E}_\varepsilon)\|_{L^2}\le C\|\rho_\varepsilon(t)-\tilde{\rho}_\varepsilon\|_{L^2}\le C\|f_\varepsilon^0\|_{L^\infty}^{1/2},$$

where we have used Proposition 2.4 and (2.2) in the last inequality.

Finaly, the second statement follows from the Sobolev embedding theorem $H^1(\mathbb{R}^2)\subset L^q(\mathbb{R}^2)$ for all $q\ge 2$, with the dependence of the constant with respect to q given in, e.g., [19, Paragraph 8.5, p. 206]. □

2.3. **Lagrangian trajectories and weak formulation.** We introduce the field

$$b_\varepsilon(t,x,v)=\left(\frac{v}{\varepsilon},\frac{E_\varepsilon(t,x)}{\varepsilon}+\frac{v^\perp}{\varepsilon^2}\right),$$

which satisfies

$$\frac{b_\varepsilon}{1+|x|+|v|}\in L^1_{\text{loc}}(\mathbb{R}_+,L^1(\mathbb{R}^2\times\mathbb{R}^2))+L^1_{\text{loc}}(\mathbb{R}_+,L^\infty(\mathbb{R}^2\times\mathbb{R}^2)),$$

see e.g. [4, Proposition 6.2]. Moreover, by Proposition 2.6, we have[5]

$$Db_\varepsilon\in L^1_{\text{loc}}(\mathbb{R}_+,L^2_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}^2)).$$

Therefore, the DiPerna and Lions theory [8] applies, providing a unique Lagrangian flow associated to b_ε, which we denote by $(X_\varepsilon,V_\varepsilon)$. We refer to the recent survey [1] or to [4], which handles specifically the Vlasov-Poisson case. In particular, for almost every $(x,v)\in\mathbb{R}^2\times\mathbb{R}^2$, $t\mapsto(X_\varepsilon(t,x,v),V_\varepsilon(t,x,v)$ is an absolutely continuous map which satisfies

$$\begin{cases}X_\varepsilon(t,x,v)=x+\dfrac{1}{\varepsilon}\displaystyle\int_0^t V_\varepsilon(s,x,v)\,ds\\ V_\varepsilon(t,x,v)=v+\dfrac{1}{\varepsilon^2}\displaystyle\int_0^t\Big(V_\varepsilon^\perp(s,x,v)+\varepsilon E_\varepsilon(s,X_\varepsilon(s,x,v))\Big)\,ds.\end{cases}\tag{2.3}$$

Moreover, the solution f_ε is the push-forward[6] of the initial density f_ε^0 by the flow,

$$f_\varepsilon(t)=(X_\varepsilon(t),V_\varepsilon(t))_\# f_\varepsilon^0.\tag{2.4}$$

Recalling that ρ_ε belongs to $L^\infty_{\text{loc}}(\mathbb{R}_+,L^\infty(\mathbb{R}^2))$ for all $0<\varepsilon<1$, we infer that E_ε satisfies

$$\forall T>0,\quad \sup_{t\in[0,T]}\|E_\varepsilon(t)\|_{L^\infty}\le C(\varepsilon,T),$$
$$\sup_{t\in[0,T]}|E_\varepsilon(t,x)-E_\varepsilon(t,y)|\le C(\varepsilon,T)|x-y|(1+|\ln|x-y||)$$

(see e.g. [18, Lemma 4]). Thus it turns out that for all $(x,v)\in\mathbb{R}^2\times\mathbb{R}^2$ the map $t\mapsto(X_\varepsilon(t,x,v),V_\varepsilon(t,x,v))$ belongs to $W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^2\times\mathbb{R}^2)$ and is the unique solution to the ODE (2.3).

We define then the following combination of the characteristics:

$$Z_\varepsilon(t,x,v)=X_\varepsilon(t,x,v)+\varepsilon V_\varepsilon^\perp(t,x,v).$$

Proposition 2.7. *For all $(x,v)\in\mathbb{R}^2\times\mathbb{R}^2$, the map $t\mapsto Z_\varepsilon(t,x,v)$ belongs to $W^{1,\infty}(\mathbb{R}_+,\mathbb{R}^2)$ and it satisfies*

$$\dot Z_\varepsilon(t,x,v)=E_\varepsilon^\perp(t,X_\varepsilon(t,x,v)),\quad \textit{for a.e. } t\in\mathbb{R}_+.$$

Proof. We have for a.e. $t\in\mathbb{R}_+$

$$\dot Z_\varepsilon(t)=\frac{V_\varepsilon(t)}{\varepsilon}+\varepsilon\left(\frac{V_\varepsilon^\perp(t)+\varepsilon E_\varepsilon(t,X_\varepsilon(t))}{\varepsilon^2}\right)^\perp=E_\varepsilon^\perp(t,X_\varepsilon(t)).$$

□

[5] Db denotes the differential matrix of b with respect to x and v.

[6] In view of the support properties of f_ε, this means here that for all $\varphi\in L^1_{\text{loc}}(\mathbb{R}^2\times\mathbb{R}^2)$, we have $\iint f_\varepsilon(t,x,v)\varphi(x,v)\,dx\,dv=\iint f_\varepsilon^0(x,v)\varphi(X_\varepsilon(t,x,v),V_\varepsilon(t,x,v))\,dx\,dv$.

We can now derive a weak formulation for the spatial density.

Proposition 2.8. *Let $\Phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. We have*

$$\int_{\mathbb{R}^2} \rho_\varepsilon(t,x)\Phi(t,x)\,dx - \int_{\mathbb{R}^2} \rho_\varepsilon^0(x)\Phi(0,x)\,dx = \int_0^t \int_{\mathbb{R}^2} \partial_s\Phi(s,x)\rho_\varepsilon(s,x)\,dx\,ds$$
$$+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} H_{\Phi(s,\cdot)}(x,y)\rho_\varepsilon(s,x)\rho_\varepsilon(s,y)\,dx\,dy\,ds + R_\varepsilon(t),$$

where R_ε converges to zero locally uniformly on $\mathbb{R}_+$ as $\varepsilon \to 0$. More precisely,

$$|R_\varepsilon(t)| \le C(1+t)\|\Phi\|_{L^\infty(W^{2,\infty})} \left(\varepsilon^2\Theta(\|f_\varepsilon^0\|_{L^\infty})\right)^{1/2}.$$

Proof. Thanks to (2.4), we may write

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(t,x,v)\Phi(t,x)\,dx\,dv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(t,Z_\varepsilon(t,x,v))\,dx\,dv + R_{\varepsilon,1}(t),$$

where

$$R_{\varepsilon,1}(t) = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\left(\Phi(t,X_\varepsilon(t,x,v)) - \Phi(t,Z_\varepsilon(t,x,v))\right)\,dx\,dv.$$

On the one hand, we have by the mean-value theorem

$$|R_{\varepsilon,1}(t)| \le \|D\Phi(t)\|_{L^\infty}\varepsilon \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)|V_\varepsilon(t,x,v)|\,dx\,dv$$

hence using (2.4) and Proposition 2.2 we get

$$\sup_{t\in\mathbb{R}_+} |R_{\varepsilon,1}(t)| \le C\,\varepsilon\|D\Phi(t)\|_{L^\infty}.$$

On the other hand, Proposition 2.7 implies that for all $(x,v) \in \mathbb{R}^2\times\mathbb{R}^2$, the map $t \mapsto \Phi(t,Z_\varepsilon(t,x,v))$ belongs to $W^{1,\infty}(\mathbb{R}_+)$ therefore

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(t,Z_\varepsilon(t,x,v))\,dx\,dv$$
$$= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(0,Z_\varepsilon(0,x,v))\,dx\,dv$$
$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\int_0^t \frac{d}{ds}\Phi(s,Z_\varepsilon(s,x,v))\,ds\,dx\,dv$$
$$= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(0,x+\varepsilon v^\perp)\,dx\,dv$$
$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\int_0^t \partial_s\Phi(s,Z_\varepsilon(s,x,v))\,ds\,dx\,dv$$
$$+ \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\int_0^t \nabla\Phi(s,Z_\varepsilon(s,x,v))\cdot E_\varepsilon^\perp(s,X_\varepsilon(s,x,v))\,ds\,dx\,dv.$$

Using again (2.4), we obtain

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(t,Z_\varepsilon(t,x,v))\,dx\,dv\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(0,x+\varepsilon v^\perp)\,dx\,dv\\
&+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\partial_s\Phi(s,x+\varepsilon v^\perp)\,ds\,dx\,dv\\
&+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\nabla\Phi(s,x+\varepsilon v^\perp)\cdot E_\varepsilon^\perp(s,x)\,dx\,dv\,ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(t,Z_\varepsilon(t,x,v))\,dx\,dv\\
&=\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Phi(0,x)\,dx\,dv+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\partial_s\Phi(s,x)\,ds\,dx\,dv\\
&+\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\nabla\Phi(s,x)\cdot E_\varepsilon^\perp(s,x)\,dx\,dv\,ds+\sum_{k=2}^5 R_{\varepsilon,k}(t),
\end{aligned}$$

where

$$\begin{aligned}
R_{\varepsilon,2}(t)&=\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon^0(x,v)\Big(\Phi(0,x+\varepsilon v^\perp)-\Phi(0,x)\Big)\,dx\,dv,\\
R_{\varepsilon,3}(t)&=\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\Big(\partial_s\Phi(s,x+\varepsilon v^\perp)-\partial_s\Phi(s,x)\Big)\,dx\,dv\,ds,\\
R_{\varepsilon,4}(t)&=\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\Big(\nabla\Phi(s,x+\varepsilon v^\perp)-\nabla\Phi(s,x)\Big)\cdot(E_\varepsilon^\perp(s,x)-\tilde{E}_\varepsilon^\perp(x))\,dx\,dv\,ds,\\
R_{\varepsilon,5}(t)&=\int_0^t\iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)\Big(\nabla\Phi(s,x+\varepsilon v^\perp)-\nabla\Phi(s,x)\Big)\cdot\tilde{E}_\varepsilon^\perp(x)\,dx\,dv\,ds.
\end{aligned}$$

On the one hand, inserting the definition of E_ε and symmetrizing as in [27], we get

$$\int_{\mathbb{R}^2}\rho_\varepsilon(s,x)\nabla\Phi(s,x)\cdot E_\varepsilon^\perp(s,x)dx=\iint_{\mathbb{R}^2\times\mathbb{R}^2} H_{\Phi(s,\cdot)}(x,y)\rho_\varepsilon(s,x)\rho_\varepsilon(s,y)\,dx\,dy.$$

On the other hand, as before, we obtain

$$|R_{\varepsilon,2}(t)|\le C\varepsilon\|\nabla\Phi(0)\|_{L^\infty}.$$

Besides, Proposition 2.6 yields

$$|R_{\varepsilon,3}(t)|\le C\varepsilon t\|D\partial_s\Phi\|_{L^\infty(L^\infty)}\sup_{s\in\mathbb{R}_+}\iint_{\mathbb{R}^2\times\mathbb{R}^2}|v|f_\varepsilon(s,x,v)\,dx\,dv\le Ct\varepsilon\|D^2\Phi\|_{L^\infty(L^\infty)}.$$

Next, we infer from the mean-value theorem, Hölder inequality and Proposition 2.6 that

$$\begin{aligned}|R_{\varepsilon,4}(t)| &\leq \varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_\varepsilon(s,x,v)|v||E_\varepsilon(s,x) - \tilde{E}_\varepsilon(x)| \, dx \, dv \, ds \\ &\leq Ct\varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \sup_{s\in[0,t]} \left(\|E_\varepsilon(s) - \tilde{E}_\varepsilon\|_{L^q} \|J_\varepsilon(s)\|_{L^{q'}} \right) \\ &\leq Ct\varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \sqrt{q} \|f_\varepsilon^0\|_{L^\infty}^{1/2} \sup_{s\in[0,t]} \|J_\varepsilon(s)\|_{L^{q'}},\end{aligned}$$

where q' is the conjugate exponent of q, and where $q \geq 4$ will be chosen later. Since $q' \in (1, 4/3]$, we have

$$\|J_\varepsilon(s)\|_{L^{q'}} \leq \|J_\varepsilon(s)\|_{L^1}^{1-\frac{4}{q}} \|J_\varepsilon(s)\|_{L^{4/3}}^{\frac{4}{q}},$$

thus Proposition 2.4 yields

$$|R_{\varepsilon,4}(t)| \leq Ct\varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \sqrt{q} \|f_\varepsilon^0\|_{L^\infty}^{\frac{1}{2}+\frac{1}{q}}.$$

Finally, we set

$$q = \max(4, \ln(\|f_\varepsilon^0\|_{L^\infty})),$$

so that

$$|R_{\varepsilon,4}(t)| \leq Ct\varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \Theta \left(\|f_\varepsilon^0\|_{L^\infty} \right)^{1/2}.$$

We turn to the last term. We infer from (2.2) and from classical potential estimates, see e.g. [23], that

$$\sup_{\varepsilon>0} \|\tilde{E}_\varepsilon\|_{L^\infty} \leq C \sup_{\varepsilon>0} \|\tilde{\rho}_\varepsilon\|_{L^1}^{1/2} \|\tilde{\rho}_\varepsilon\|_{L^\infty}^{1/2} \leq C,$$

therefore

$$\begin{aligned}|R_{\varepsilon,5}(t)| &\leq Ct\varepsilon \, \|D^2\Phi\|_{L^\infty(L^\infty)} \|\tilde{E}_\varepsilon\|_{L^\infty} \sup_{s\in[0,t]} \int_{\mathbb{R}^2\times\mathbb{R}^2} |v| f_\varepsilon(s,x,v) \, dx \, dv \\ &\leq Ct\varepsilon \|D^2\Phi\|_{L^\infty(L^\infty)}.\end{aligned}$$

Gathering the previous bounds and recalling that $\Theta(\|f_\varepsilon^0\|_{L^\infty}) \geq 1$, we obtain the desired estimate.

□

2.4. **Passing to the limit.** We establish a property of uniform equicontinuity with respect to time for the spatial densities.

Lemma 2.9. *There exists $K_0 > 0$ such that for all $s, t \in \mathbb{R}_+$,*

$$\|\rho_\varepsilon(t) - \rho_\varepsilon(s)\|_{W^{-2,1}(\mathbb{R}^2)} \leq K_0 \left(|t-s| + (1+t+s)\varepsilon\Theta(\|f_\varepsilon^0\|_{L^\infty})^{1/2} \right).$$

Proof. This is a simple consequence of Proposition 2.8 and of the estimate $|H_\Phi(x,y)| \leq \|\Phi\|_{W^{2,\infty}}$.

□

We are now in position to complete the proof of Theorem 1.1. A straightforward adaptation of Ascoli's theorem yields:

Lemma 2.10. *Let $T > 0$. Let (F, d) be a complete metric space. Let $(f_n)_{n\in\mathbb{N}}$ be a family of $C([0,T], F)$ such that*

(1) *For all $t \in [0,T]$, $(f_n(t))_{n\in\mathbb{N}}$ is relatively compact in F;*
(2) *There exists $C > 0$ and a sequence $r_n \to 0$ as $n \to +\infty$ such that for all $t, s \in [0,T]$, for all $n \in \mathbb{N}$, $|f_n(t) - f_n(s)| \le C|t-s| + r_n$.*

Then the family $(f_n)_{n\in\mathbb{N}}$ is relatively compact in $C([0,T], F)$.

Using the fact that $(\rho_\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}_+, \mathcal{M}^+(\mathbb{R}^2))$ and recalling Lemma 2.9, we can apply this Lemma to $F = W^{-2,1}(\mathbb{R}^2)$ and we can mimick the proof of Lemma 3.2 in [27, Lemma 3.2] to show that there exists $\varepsilon_n \to 0$ as $n \to +\infty$ such that $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to some ρ in $C(\mathbb{R}_+, \mathcal{M}^+(\mathbb{R}^2) - w^*)$. By Proposition 2.5, $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ is bounded in $L^\infty(\mathbb{R}_+, H^{-1}(\mathbb{R}^2))$. It was proved in [7] (see also [22, 27]) that this implies that the non-linear term

$$\int \iint H_\Phi(x,y)\rho_{\varepsilon_n}(s,x)\rho_{\varepsilon_n}(s,y)\,dx\,dy\,ds$$

converges to

$$\int \iint H_\Phi(x,y)\rho(s,x)\rho(s,y)\,dx\,dy\,ds$$

as $n \to +\infty$ for all test function Φ. On the other hand, all linear terms appearing in the formulation given by Proposition 2.8 pass to the limit. This means that ρ satisfies the conclusion of Theorem 1.1.

2.5. **Alternative proof of Theorem 1.1 without Lagrangian trajectories.** The purpose of this paragraph is to propose another proof of Theorem 1.1, for smooth solutions, that does not rely on the characteristics. Here, we assume that the initial data f_ε^0 satisfy the assumptions of Theorem 1.1 and that moreover

$$f_\varepsilon^0 \in C^{1,\alpha}(\mathbb{R}^2 \times \mathbb{R}^2)$$

for some $\alpha \in (0,1)$. The corresponding solution to (1.1) then belongs to $C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$.

As in [17], we consider the microscopic and macroscopic densities in the gyro-coordinates:

$$\overline{f_\varepsilon}(t,x,v) = f_\varepsilon(t, x - \varepsilon v^\perp, v), \quad \overline{\rho}_\varepsilon(t,x) = \int_{\mathbb{R}^2} \overline{f_\varepsilon}(t,x,v)\,dv.$$

Proposition 2.11. *We have*

$$\partial_t \overline{f_\varepsilon} + E_\varepsilon^\perp(t, x-\varepsilon v^\perp)\cdot\nabla_x \overline{f_\varepsilon} + \left(\frac{v^\perp}{\varepsilon^2} + \frac{E_\varepsilon(t, x-\varepsilon v^\perp)}{\varepsilon}\right)\cdot\nabla_v \overline{f_\varepsilon} = 0,$$

and

$$\partial_t \overline{\rho}_\varepsilon + \nabla_x \cdot \left(\int_{\mathbb{R}^2} E_\varepsilon^\perp(t, x-\varepsilon v^\perp)\overline{f_\varepsilon}\,dv\right) = 0.$$

Proof. We compute

$$\partial_t \overline{f_\varepsilon}(t,x,v) = \partial_t f_\varepsilon(t, x-\varepsilon v^\perp, v), \quad \nabla_x \overline{f_\varepsilon}(t,x,v) = \nabla_x f_\varepsilon(t, x-\varepsilon v^\perp, v),$$
$$\nabla_v \overline{f_\varepsilon}(t,x,v) = (\varepsilon\nabla_x^\perp + \nabla_v) f_\varepsilon(t, x-\varepsilon v^\perp, v),$$

then

$$\left(\frac{v^\perp}{\varepsilon^2} + \frac{E_\varepsilon(t, x - \varepsilon v^\perp)}{\varepsilon}\right) \cdot \nabla_v f_\varepsilon(t, x - \varepsilon v^\perp, v)$$

$$= \left(\frac{v^\perp}{\varepsilon^2} + \frac{E_\varepsilon(t, x - \varepsilon v^\perp)}{\varepsilon}\right) \cdot \left(\nabla_v - \varepsilon \nabla_x^\perp\right) \overline{f_\varepsilon}(t, x, v)$$

$$= -\frac{v}{\varepsilon} \cdot \nabla_x \overline{f_\varepsilon}(t, x, v) + E_\varepsilon^\perp(t, x - \varepsilon v^\perp) \cdot \nabla_x \overline{f_\varepsilon}(t, x, v) + \left(\frac{v^\perp}{\varepsilon^2} + \frac{E_\varepsilon(t, x - \varepsilon v^\perp)}{\varepsilon}\right) \cdot \nabla_v \overline{f_\varepsilon}(t, x, v).$$

Therefore $\overline{f_\varepsilon}$ satisfies the first equation in Proposition 2.11. Next, we integrate with respect to v and we observe that

$$\int_{\mathbb{R}^2} v^\perp \cdot \nabla_v \overline{f_\varepsilon}\, dv = -\int_{\mathbb{R}^2} \nabla_v \cdot v^\perp\, \overline{f_\varepsilon}\, dv = 0,$$

$$\int_{\mathbb{R}^2} E_\varepsilon^\perp(x - \varepsilon v^\perp) \cdot \nabla_v \overline{f_\varepsilon}\, dv = -\int_{\mathbb{R}^2} \nabla_v \cdot [E_\varepsilon(x - \varepsilon v^\perp)]\, \overline{f_\varepsilon}\, dv = \varepsilon \int_{\mathbb{R}^2} \mathrm{curl}(E_\varepsilon)(x - \varepsilon v^\perp)\, \overline{f_\varepsilon}\, dv,$$

where $\mathrm{curl}(G) = \partial_2 G_1 - \partial_1 G_2$. Similarly,

$$\int_{\mathbb{R}^2} E_\varepsilon^\perp(x - \varepsilon v^\perp) \cdot \nabla_x \overline{f_\varepsilon}\, dv = \nabla_x \cdot \left(\int_{\mathbb{R}^2} E_\varepsilon^\perp(x - \varepsilon v^\perp) \overline{f_\varepsilon}\, dv\right) - \int_{\mathbb{R}^2} \mathrm{curl}(E_\varepsilon)(x - \varepsilon v^\perp)\, \overline{f_\varepsilon}\, dv.$$

Now, since E_ε is a gradient, we have $\mathrm{curl}(E_\varepsilon) = 0$, hence the second equation of Proposition 2.11 follows.

□

We now establish Theorem 1.1. The same arguments as the ones of Subsection 2.4 yield a subsequence such that $(\rho_{\varepsilon_n})_{n \in \mathbb{N}}$ converges to ρ in $C(\mathbb{R}_+, \mathcal{M}^+(\mathbb{R}^2) - w^*)$ as $n \to +\infty$. Let $\Phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. Using Proposition 2.11 and the fact that the Jacobian of $x \mapsto x + \varepsilon v^\perp$ is one for any fixed v, we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^2} \overline{\rho}_{\varepsilon_n}(t, x) \Phi(t, x)\, dx$$

$$= \int_{\mathbb{R}^2} \overline{\rho}_{\varepsilon_n}(t, x) \partial_t \Phi(t, x)\, dx + \int_{\mathbb{R}^2} \nabla \Phi(t, x) \cdot \left(\int_{\mathbb{R}^2} E_{\varepsilon_n}^\perp(t, x - \varepsilon_n v^\perp) \overline{f}_{\varepsilon_n}(t, x, v)\, dv\right) dx$$

$$= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f_{\varepsilon_n}(t, x - \varepsilon_n v^\perp, v) \left[\partial_t \Phi(t, x) + \nabla \Phi(t, x) \cdot E_{\varepsilon_n}^\perp(t, x - \varepsilon_n v^\perp)\right] dx\right) dv$$

$$= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f_{\varepsilon_n}(t, x, v) \left[\partial_t \Phi(t, x + \varepsilon_n v^\perp) + \nabla \Phi(t, x + \varepsilon_n v^\perp) \cdot E_{\varepsilon_n}^\perp(t, x)\right] dx\right) dv.$$

Writing finally

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} f_{\varepsilon_n}(t, x, v) \left[\partial_t \Phi(t, x + \varepsilon_n v^\perp) + \nabla \Phi(t, x + \varepsilon_n v^\perp) \cdot E_{\varepsilon_n}^\perp(t, x)\right] dx\right) dv$$

$$= \int_{\mathbb{R}^2} \rho_{\varepsilon_n}(t, x) \left[\partial_t \Phi(t, x) + \nabla \Phi(t, x) \cdot E_{\varepsilon_n}^\perp(t, x)\right] dx$$

$$+ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_{\varepsilon_n}(t, x, v) \left[\partial_t \Phi(t, x + \varepsilon_n v^\perp) - \partial_t \Phi(t, x)\right] dx\, dv$$

$$+ \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f_{\varepsilon_n}(t, x, v) \left[\nabla \Phi(t, x + \varepsilon_n v^\perp) - \nabla \Phi(t, x)\right] \cdot E_{\varepsilon_n}^\perp(t, x)\, dx\, dv,$$

we conclude as in the previous section.

3. Proof of Theorem 1.2

In this section we adapt the proof of Theorem 1.1 to the case of initial data satisfying the assumptions of Theorem 1.2. We have

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\|f_\varepsilon(t)\|_{L^\infty}<\infty, \tag{3.1}$$

hence it follows from Propositions 2.4 and 2.6 that

$$\sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\|\rho_\varepsilon(t)\|_{L^2(\mathbb{R}^2)}<\infty,\quad \sup_{t\in\mathbb{R}_+}\sup_{\varepsilon>0}\|E_\varepsilon(t)-\tilde{E}_\varepsilon\|_{H^1(\mathbb{R}^2)}<\infty. \tag{3.2}$$

Exactly as in Subsection 2.4, the family $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C(\mathbb{R}_+,\mathcal{M}^+(\mathbb{R}^2)-w^*)$. Moreover, $(\rho_\varepsilon(t))_{\varepsilon>0}$ is weakly relatively compact in $L^2(\mathbb{R}^2)$ for all $t\geq 0$. It follows that for some subsequence $\varepsilon_n\to 0$, $(\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to some ρ in $C(\mathbb{R}_+,L^2(\mathbb{R}^2)-w)$ and in $C(\mathbb{R}_+,\mathcal{M}^+(\mathbb{R}^2)-w^*)$ as $n\to+\infty$. Let $E=(x/|x|^2)*\rho$, so that E belongs to $L^\infty_{\text{loc}}(\mathbb{R}_+,L^1+L^2(\mathbb{R}^2))$. Decomposing

$$\frac{x}{|x|^2}=\frac{x}{|x|^2}\chi_\delta+\frac{x}{|x|^2}(1-\chi_\delta),$$

with χ_δ a cut-off function supported in $B(0,2\delta)$ with value 1 on $B(0,\delta)$, we see immediately that

$$\frac{x}{|x|^2}(1-\chi_\delta)*\rho_{\varepsilon_n}\to\frac{x}{|x|^2}(1-\chi_\delta)*\rho\quad\text{locally uniformly on }\mathbb{R}_+\times\mathbb{R}^2\text{ as }n\to+\infty,$$

while

$$\left\|\left(\frac{x}{|x|^2}\chi_\delta\right)*\rho_{\varepsilon_n}(t)\right\|_{L^2}\leq C\delta\|\rho_{\varepsilon_n}(t)\|_{L^2}\leq C\delta.$$

So we conclude that $(E_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to E in $C(\mathbb{R}_+,L^2_{\text{loc}}(\mathbb{R}^2))$. This implies that $(E^\perp_{\varepsilon_n}\rho_{\varepsilon_n})_{n\in\mathbb{N}}$ converges to $E^\perp\rho$ in the sense of distributions on $\mathbb{R}_+\times\mathbb{R}^2$. Therefore, all terms pass to the limit in Proposition 2.8, and ρ satisfies (1.2) in the sense of distributions. This concludes the proof.

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