A UNIQUENESS CRITERION FOR UNBOUNDED SOLUTIONS TO THE VLASOV-POISSON SYSTEM

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ABSTRACT. We prove uniqueness for the Vlasov-Poisson system in two and three dimensions under the condition that the L^p norms of the macroscopic density growth at most linearly with respect to p. This allows for solutions with logarithmic singularities. We provide explicit examples of initial data that fulfill the uniqueness condition and that exhibit a logarithmic blow-up. In the gravitational two-dimensional case, such states are intimately related to radially symmetric steady solutions of the system. Our method relies on the Lagrangian formulation for the solutions, exploiting the second-order structure of the corresponding ODE.

1. INTRODUCTION

The purpose of this article is to establish a uniqueness result for the Vlasov-Poisson system in dimension $n = 2$ or $n = 3$

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \\ E(t, x) = \gamma \displaystyle\int_{\mathbb{R}^n} \frac{x - y}{|x - y|^n} \rho(t, y)\, dy \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^n} f(t, x, v)\, dv, \end{cases} \tag{1.1}$$

where $\gamma = \pm 1$. The system (1.1) is a physical model for the evolution of a system of particles interacting via a self-induced force field E. The interaction is gravitational if $\gamma = -1$ or Coulombian if $\gamma = 1$. The unknown $f = f(t, x, v) \geq 0$ denotes the microscopic density of the particles at time t, position x and velocity v, and $\rho = \rho(t, x) \geq 0$ denotes their macroscopic density.

A wide literature has been devoted to the Cauchy theory for the Vlasov-Poisson system. Ukai and Okabe [15] established global existence and uniqueness of smooth solutions in two dimensions. In any dimension, global existence of weak solutions with finite energy is a result due to Arsenev [2], where the energy of f, defined by $\mathcal{E}(f) = \frac{1}{2} \iint |v|^2 f\, dx\, dv + \frac{\gamma}{2} \int |E|^2\, dx$, is formally preserved by the flow of (1.1). In three dimensions, global existence and uniqueness of compactly supported classical solutions where obtained by Pfaffelmoser [18] by Lagrangian techniques. Simultaneously, Lions and

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Perthame [11] constructed global weak solutions with finite velocity moments. More precisely, they proved that if

$$f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f_0 < \infty \quad \text{for some } m > 3,$$

then there exists a corresponding solution $f \in L^\infty(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ such that

$$\forall T > 0, \quad \sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t, x, v)\, dx\, dv < \infty.$$

If $m > 6$ such a solution generates a uniformly bounded force field. We also refer to the works by Gasser, Jabin and Perthame [8], Salort [20] and Pallard [16, 17] for further results concerning global existence and propagation of the moments. Another issue in the setting of weak solutions consists in determining sufficient conditions for uniqueness. Robert [19] established uniqueness among weak solutions that are compactly supported. This result was extended by Loeper [12], who proved uniqueness on $[0, T]$ in the class of weak measure-valued solutions with bounded macroscopic density, namely

$$f \in C([0, T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*) \quad \text{and } \rho \in L^\infty([0, T], L^\infty(\mathbb{R}^n)), \tag{1.2}$$

where $\mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n)$ denotes the space of bounded positive measures. The main result of this paper generalizes Loeper's uniqueness condition (1.2) as follows:

Theorem 1.1. *Let $f_0 \in \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n)$ be a nonnegative bounded measure. Let $T > 0$. There exists at most one weak solution $f \in C([0, T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*)$ of the Vlasov-Poisson system on $[0, T]$ with $f(0) = f_0$ such that*

$$\sup_{[0,T]} \sup_{p\geq 1} \frac{\|\rho(t)\|_{L^p}}{p} < +\infty. \tag{1.3}$$

We refer to [12, Definition 1.1] for a precise definition of weak measure-valued solutions. Note in particular that the assumption $\rho \in L^\infty([0, T], L^p(\mathbb{R}^n))$ for any $p > 1$ ensures that such a definition makes sense.

Our next task is to determine sufficient conditions on the initial data for which any corresponding weak solution satisfies the uniqueness criterion of Theorem 1.1. We observe that (1.3) is fulfilled if for example[1]

$$\forall t \in [0, T], \quad \rho(t, x) \leq C(1 + \ln_- |x - \xi(t)|) \tag{1.4}$$

for some $\xi(t) \in \mathbb{R}^n$ (see (4.3)). Such densities where constructed by Caprino, Marchioro, Miot and Pulvirenti [4] as solutions of a related equation to (1.1). On the other hand, there exist solutions of (1.1) that satisfy (1.4) initially, as will be shown in Theorems 1.3 and 4.2. However, in general, it is not clear whether a logarithmic divergence like (1.4) persists at positive times. In fact, in order to propagate a control on the L^p norms of the macroscopic density we also need a description of the initial data at the microscopic level.

[1]Here and in the sequel, we set $\ln_- |x| = \max(0, -\ln |x|)$.

In the above-mentioned previous works [11, 16, 17, 20], the condition (1.2) is met by assuming that the initial data satisfy

$$\forall R > 0, \quad \forall T > 0, \sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \sup_{|y-x|\le RT, |v-w|\le RT^2} f_0(y+vt, w)\, dv < +\infty.$$

In the present paper we shall require instead a suitable control on the velocity moments, having in mind the well-known property that velocity moments control the norms of the density, see (3.1):

Theorem 1.2. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be nonnegative and such that*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |v|^m f_0(x,v)\, dx\, dv < +\infty$$

for some $m > n^2 - n$. Let $T > 0$ and let $f \in C([0,T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*) \cap L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ be a weak solution provided by [11, Theo. 1][2]*. If f_0 satisfies*

$$\forall k \ge 1, \quad \iint_{\mathbb{R}^n\times\mathbb{R}^n} |v|^k f_0(x,v)\, dx\, dv \le (C_0 k)^{\frac{k}{n}},$$

for some constant C_0, then f satisfies the uniqueness condition (1.3)*.*

Typically, Theorem 1.2 allows to consider initial densities with compact support in velocity as well as Maxwell-Boltzmann distributions of the type

$$f_0(x,v) = e^{-|v|^n} |v|^p h_0(x,v), \quad p \ge 0, \quad h_0 \in L^1 \cap L^\infty \cap L^\infty_v(L^1_x).$$

Indeed, denoting by $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt$ the Gamma function we have

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} |v|^k f_0(x,v)\, dx\, dv \le C\|h_0\|_{L^\infty_v(L^1_x)}\, \Gamma\left(\frac{k+p}{n} + 1\right) \le (C_0 k)^{\frac{k}{n}}$$

(see also (4.4)-(4.6) below). Theorem 1.2 also does include some initial data with unbounded macroscopic density:

Theorem 1.3. *There exists a nonnegative $f_0 \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying the assumptions of Theorem 1.2 and such that*[3]

$$\rho_0(x) = \omega_n \ln_- |x|, \quad \forall x \in \mathbb{R}^n.$$

Let us next explain the main idea for proving Theorem 1.1. The argument of Loeper [12] in the context of uniformly bounded macroscopic densities (see also [13, Theo. 3.1, Chapter 2]) uses loglipschitz regularity for the force field

$$|E(t,x) - E(t,y)| \le (\|\rho(t)\|_{L^1} + \|\rho(t)\|_{L^\infty})\, |x-y|(1 + |\ln|x-y||),$$

which enables to perform a Gronwall estimate involving the distance between the Lagrangian flows associated to the solutions.

The loglipschitz regularity fails in the setting of unbounded densities. However, for L^p solutions, Sobolev embeddings imply that E is Hölder continuous with exponent and semi-norm estimated explicitely in terms of p and $\|\rho(t)\|_{L^p}$, see Lemma 2.2 below. This estimate turns out to be sufficient

[2]The result of [11] is stated for $n = 3$. The case $n = 2$ can be obtained by a straightforward adaptation.

[3]Here ω_n denotes the volume of the unit ball of $\mathbb{R}^n$.

to close the Gronwall estimate as $p \to +\infty$ provided the L^p norms satisfy the condition in Theorem 1.1.

The Vlasov-Poisson system presents lots of analogies with the Euler equations for two-dimensional incompressible fluids

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{on } \mathbb{R} \times \mathbb{R}^2, \\ \omega = \operatorname{curl} u, \ \operatorname{div} u = 0, \end{cases} \tag{1.5}$$

where $u : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity and $\omega = \operatorname{curl} u$ is the vorticity. Because of their analogous transport structure, both equations (1.1) and (1.5) are often handled similarly, especially for the uniqueness issue. In [12], Loeper extends his uniqueness proof for (1.1) to (1.5). Also the proof of uniqueness in [19] applies to both equations. We emphasize that this is not the case in the present paper, as is explained in Remarks 2.4 and 2.5. This is due to the fact that for the Vlasov-Poisson system the Lagrangian trajectories satisfy a second-order ODE, while for the Euler equations they satisfy a first-order ODE. This crucial observation was already exploited in [4], where it was proved that a logarithmic divergence on the macroscopic density still yields enough regularity for the force field to get well-posedness for the corresponding ODE.

The paper is organized as follows. In the next Section 2 we recall a Hölder estimate for a field that controls the force field. As a consequence we derive a second-order Gronwall estimate on a distance between the Lagrangian flows of two solutions, which leads to the proof of Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Finally in Section 4 we prove Theorem 1.3 and we display in Proposition 4.1 a large class of initial densities for which uniqueness holds. We conclude by commenting on the link with radially symmetric steady states in the two-dimensional gravitational case.

Notation. In the remainder of the paper, the notation C will denote a constant that can change from one line to another, depending only on T, n, $\|f\|_{L^\infty([0,T],L^1\cap L^\infty(\mathbb{R}^n\times\mathbb{R}^n))}$, and $\iint |v|^m f_0$ (this latter quantity only for the proof of Theorem 1.2) but independent on p and k as $p, k \to +\infty$.Finally, for a function F, we set will $F_+ = \max(F, 0)$ and $F_- = \max(-F, 0)$.

2. Proof of Theorem 1.1

2.1. **Lagrangian formulation for weak solutions.** We consider a weak solution $f \in C([0,T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*)$ of (1.1) on $[0,T]$. We assume that, moreover, $\rho \in L^\infty([0,T], L^1 \cap L^p(\mathbb{R}^n))$ for some $p > n$. By potential estimates we have $E = c(n)\nabla\Delta^{-1}\rho \in L^\infty([0,T] \times \mathbb{R}^n)$, and

$$\|E\|_{L^\infty([0,T],L^\infty)} \leq C_p \|\rho\|_{L^\infty([0,T],L^1\cap L^p)}. \tag{2.1}$$

Moreover, $\nabla E \in L^\infty([0,T], L^p(\mathbb{R}^n))$ by virtue of the Caldéron-Zygmund inequality, see [7, Theo. 4.12]. Therefore it follows from DiPerna and Lions theory on transport equations [5, Theo. III2] that there exists a map $\Phi = (X, V) \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n \times \mathbb{R}^n)$ such that for a.e. $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$,

$t \mapsto (X,V)(t,x,v)$ is an absolutely continuous integral solution of the ODE

$$\begin{cases} \dot{X}(t,x,v) = V(t,x,v), \quad X(0,x,v) = x \\ \dot{V}(t,x,v) = E(t,X(t,x,v)), \quad V(0,x,v) = v. \end{cases} \tag{2.2}$$

Moreover,

$$\forall t \in [0,T], \quad f(t) = \Phi(t)_{\#} f_0 \tag{2.3}$$

which means that $f(t)(B) = f_0\left((\Phi(t,\cdot,\cdot)^{-1}(B)\right)$ for all Borel set $B \subset \mathbb{R}^n$. Such a map is unique and is called Lagrangian flow associated to E. We refer also to [1, Theo. 5.7] for a more recent statement and for further developments on the theory.

We note that (2.1) implies that $t \mapsto \Phi(t,x,v) \in W^{1,\infty}([0,T])$ for a.e. $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$.

As a byproduct of our analysis we shall see in Paragraph 2.4 that under the assumptions of Theorem 1.1 the Lagrangian flow actually corresponds to the classical notion of flow.

2.2. **Estimate on the Lagrangian trajectories.** We consider two solutions f_1 and $f_2 \in C([0,T], \mathcal{M}_+(\mathbb{R}^n \times \mathbb{R}^n) - w^*)$ such that ρ_1 and ρ_2 belong to $L^\infty([0,T], L^p(\mathbb{R}^n))$ for some $p > n$. Denoting by $\Phi_1 = (X_1, V_1)$ and $\Phi_2 = (X_2, V_2)$ the corresponding Lagrangian flows, we introduce the distance

$$\mathcal{D}(t) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |X_1(t,x,v) - X_2(t,x,v)| f_0(x,v)\, dx\, dv.$$

We infer from (2.2) that

$$|X_1(t,x,v) - X_2(t,x,v)| \le \int_0^t \int_0^s |E_1(\tau, X_1(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))|\, d\tau\, ds. \tag{2.4}$$

In particular, $\sup_{(x,v)} |X_1(t,x,v) - X_2(t,x,v)| \le CT^2(\|E_1\|_{L^\infty} + \|E_2\|_{L^\infty})$, which shows that $\mathcal{D}$ defines a continuous function on $[0,T]$. The purpose of this paragraph is to establish the estimate

Proposition 2.1. *For all $t \in [0,T]$ and for all $p > n$,*

$$\mathcal{D}(t) \le C\, p \max\left(1 + \|\rho_1\|_{L^\infty([0,T],L^p)}, \|\rho_2\|_{L^\infty([0,T],L^p)}\right) \int_0^t \int_0^s \mathcal{D}^{1-\frac{n}{p}}(\tau)\, d\tau\, ds.$$

The proof of Proposition 2.1 relies on the following potential estimate, the proof of which is postponed at the end of this paragraph.

Lemma 2.2. *There exists $C > 0$ such that for all $p > n$ and $g \in L^1 \cap L^p(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right| |g(z)|\, dz \le Cp(\|g\|_{L^p} + \|g\|_{L^1})|x-y|^{1-\frac{n}{p}}.$$

Remark 2.3. Setting $E[g] = x/|x|^n * g = c(n)\nabla\Delta^{-1} g$ we observe that Lemma 2.2 implies the estimate

$$|E[g](x) - E[g](y)| \le Cp(\|g\|_{L^p} + \|g\|_{L^1})|x-y|^{1-n/p}. \tag{2.5}$$

This latter inequality can be obtained by combining Morrey's inequality, which implies that $|E[g](x)-E[g](y)| \leq C\|\nabla E[g]\|_{L^p}|x-y|^{1-\frac{n}{p}}$, and Calderón-Zygmund inequality, see [7, Theo. 4.12], which implies that $\|\nabla E[g]\|_{L^p} \leq Cp\|g\|_{L^p}$.

Proof of Proposition 2.1.

By (2.4), we have

$$\begin{aligned}\mathcal{D}(t) &\leq \int_0^t\int_0^s \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_1(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\right) d\tau\,ds\\ &\leq \int_0^t\int_0^s \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_1(\tau,x,v)) - E_1(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\right) d\tau\,ds\\ &+ \int_0^t\int_0^s \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_2(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\right) d\tau\,ds\\ &\leq I + J.\end{aligned}$$

First, applying (2.5) to E_1 and using that $\rho_1 \in L^\infty([0,T], L^1(\mathbb{R}^n))$ we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_1(\tau,x,v)) - E_1(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\\ &\qquad\leq Cp\left(1+\|\rho_1\|_{L^\infty(L^p)}\right)\iint_{\mathbb{R}^n\times\mathbb{R}^n} |X_1(\tau,x,v)-X_2(\tau,x,v)|^{1-\frac{n}{p}} f_0(x,v)\,dx\,dv.\end{aligned}$$

Therefore by Jensen's inequality we find

$$I \leq Cp\left(1+\|\rho_1\|_{L^\infty(L^p)}\right)\int_0^t\int_0^s \mathcal{D}(\tau)^{1-\frac{n}{p}}\,d\tau\,ds. \tag{2.6}$$

Next, inserting that $f_2(\tau) = \Phi_2(\tau)_\# f_0$, we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_2(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\\ &\qquad= \iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau,x) - E_2(\tau,x)| f_2(\tau,x,v)\,dx\,dv.\end{aligned}$$

On the other hand, since $f_1(\tau) = \Phi_1(\tau)_\# f_0$ and $f_2(\tau) = \Phi_2(\tau)_\# f_0$,

$$E_1(\tau,x) - E_2(\tau,x) = \gamma\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{x-X_1(\tau,y,w)}{|x-X_1(\tau,y,w)|^n} - \frac{x-X_2(\tau,y,w)}{|x-X_2(\tau,y,w)|^n}\right) f_0(y,w)\,dy\,dw. \tag{2.7}$$

Therefore, we obtain by Fubini's theorem

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}^n} |E_1(\tau, X_2(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))| f_0(x,v)\,dx\,dv\\ &\leq \int_{\mathbb{R}^n}\left|\iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\frac{x-X_1(\tau,y,w)}{|x-X_1(\tau,y,w)|^n} - \frac{x-X_2(\tau,y,w)}{|x-X_2(\tau,y,w)|^n}\right) f_0(y,w)\,dy\,dw\right|\rho_2(\tau,x)\,dx\\ &\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n}\left(\int_{\mathbb{R}^n}\left|\frac{x-X_1(\tau,y,w)}{|x-X_1(\tau,y,w)|^n} - \frac{x-X_2(\tau,y,w)}{|x-X_2(\tau,y,w)|^n}\right|\rho_2(\tau,x)\,dx\right) f_0(y,w)\,dy\,dw\\ &\leq \iint_{\mathbb{R}^n\times\mathbb{R}^n} Cp\left(\|\rho_2(\tau)\|_{L^1} + \|\rho_2(\tau)\|_{L^p}\right)|X_1(\tau,y,w)-X_2(\tau,y,w)|^{1-\frac{n}{p}} f_0(y,w)\,dy\,dw,\end{aligned}$$

where we have applied Lemma 2.2 in the last inequality. Hence Jensen's inequality yields

$$J \leq C\,p\left(1+\|\rho_2\|_{L^\infty(L^p)}\right)\int_0^t\int_0^s \mathcal{D}(\tau)^{1-\frac{n}{p}}\,d\tau\,ds. \tag{2.8}$$

The conclusion follows from (2.6) and (2.8).

Remark 2.4. A similar function can be introduced to establish uniqueness for (1.5) with bounded vorticity, see e.g. [13, Theo. 3.1, Chapter 2],

$$\tilde{\mathcal{D}}(t) = \int_{\mathbb{R}^2} |X_1(t,x)-X_2(t,x)||\omega_0(x)|\,dx,$$

where X_1 and X_2 denote the Lagrangian flows

$$\dot{X}_i(t,x) = u_i(t, X_i(t,x)), \quad X(0,x) = x.$$

By the same arguments as in the proof of Proposition 2.1, it satisfies

$$\tilde{\mathcal{D}}(t) \leq C\,p\max\left(\|\omega_1\|_{L^\infty(L^1\cap L^p)}, \|\omega_2\|_{L^\infty(L^1\cap L^p)}\right)\int_0^t \tilde{\mathcal{D}}^{1-\frac{2}{p}}(s)\,ds$$

therefore, by conservation of the L^p norms of the vorticity,

$$\tilde{\mathcal{D}}(t) \leq C\,p\|\omega_0\|_{L^1\cap L^p}\int_0^t \tilde{\mathcal{D}}^{1-\frac{2}{p}}(s)\,ds.$$

Proof of Lemma 2.2.

The proof for $p=\infty$ is well-known, see e.g. [14, Chapter 8] for the case $n=2$. When $p<+\infty$ it is obtained by very similar arguments, but we provide the full details because we are not aware of any reference in the literature. Let $p_0>n$. By Hölder inequality, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\frac{x-z}{|x-z|^n}\right||g(z)|\,dz &\leq \sup_{x\in\mathbb{R}^n}\left(\int_{|x-z|\leq 1}\frac{|g(z)|}{|x-z|^{n-1}}\,dz + \int_{|x-z|\geq 1}\frac{|g(z)|}{|x-z|^{n-1}}\,dz\right)\\ &\leq \|g\|_{L^{p_0}}\||z|^{-n+1}\|_{L^{p_0'}(B(0,1))} + \|g\|_{L^1}\\ &\leq C_{p_0}(\|g\|_{L^1}+\|g\|_{L^{p_0}}),\end{aligned}$$

with C_{p_0} depending only on p_0. Hence it suffices to establish Lemma 2.2 for $|x-y|<1$. Let us introduce $d=|x-y|$ and $A=(x+y)/2$. We split the integral as

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|\frac{x-z}{|x-z|^n}-\frac{y-z}{|y-z|^n}\right||g(z)|\,dz = \int_{\mathbb{R}^n\setminus B(A,1)}\left|\frac{x-z}{|x-z|^n}-\frac{y-z}{|y-z|^n}\right||g(z)|\,dz\\ &+\int_{B(A,1)\setminus B(A,d)}\left|\frac{x-z}{|x-z|^n}-\frac{y-z}{|y-z|^n}\right||g(z)|\,dz + \int_{B(A,d)}\left|\frac{x-z}{|x-z|^n}-\frac{y-z}{|y-z|^n}\right||g(z)|\,dz\\ &= I+J+K.\end{aligned}$$

For $|z-A|\geq 1$ we have $|u-z|\geq 1-d/2\geq 1/2$ for any $u\in[x,y]$, hence by the mean-value theorem we find

$$I \leq Cd\int_{\mathbb{R}^n\setminus B(A,1)}\sup_{u\in[x,y]}\frac{|g(z)|}{|u-z|^n} \leq Cd\,\|g\|_{L^1}.$$

Next, applying first Hölder inequality, then the mean-value theorem, we obtain

$$J \leq \|g\|_{L^p} \left(\int_{B(A,1)\setminus B(A,d)} \left| \frac{x-z}{|x-z|^n} - \frac{y-z}{|y-z|^n} \right|^{p'} dz \right)^{1/p'}$$
$$\leq \|g\|_{L^p}\, d \left(\int_{B(A,1)\setminus B(A,d)} \sup_{u\in[x,y]} \frac{1}{|u-z|^{np'}}\, dz \right)^{1/p'}.$$

Now, for $|z-A|\geq d$ we have $|u-z|\geq |z-A|-|u-A|\geq |z-A|/2$ for any $u\in[x,y]$. Therefore

$$J \leq Cd\|g\|_{L^p} \left(\int_{B(A,1)\setminus B(A,d)} \frac{1}{|z-A|^{np'}}\, dz \right)^{1/p'} \leq Cd\,\|g\|_{L^p} d^{n(\frac{1}{p'}-1)}(p'-1)^{-\frac{1}{p'}}$$

hence

$$J \leq Cp\,\|g\|_{L^p} d^{1-\frac{n}{p}}.$$

Applying again Hölder inequality, we obtain

$$K \leq \|g\|_{L^p} \left(\int_{B(A,d)} \frac{1}{|x-z|^{p'(n-1)}}\, dz \right)^{1/p'} + \|g\|_{L^p} \left(\int_{B(A,d)} \frac{1}{|y-z|^{p'(n-1)}}\, dz \right)^{1/p'}.$$

Since for $|z-A|\leq d$ we have $\max(|x-z|,|y-z|)\leq 3d/2$, we finally obtain

$$K \leq 2\|g\|_{L^p} \left(\int_{B(0,3d/2)} \frac{1}{|u|^{p'(n-1)}} \right)^{1/p'} \leq C\|g\|_{L^p}\, d^{1-\frac{n}{p}}.$$

2.3. **Proof of Theorem 1.1.** Given two solutions f_1 and f_2 of (1.1) satisfying the assumptions of Theorem 1.1, let $\mathcal{D}$ be the corresponding distance function. Since $\max(\|\rho_1\|_{L^\infty(L^p)}, \|\rho_2\|_{L^\infty(L^p)}) \leq Cp$ by assumption, Proposition 2.1 implies that

$$\mathcal{D}(t) \leq C\,p^2 \int_0^t \int_0^s \mathcal{D}^{1-\frac{n}{p}}(\tau)\, d\tau\, ds.$$

Let $\mathcal{F}_p(t) = \int_0^t\int_0^s \mathcal{D}^{1-\frac{n}{p}}(\tau)\,d\tau\,ds \to \mathcal{F}(t) = \int_0^t\int_0^s \mathcal{D}(\tau)\,d\tau\,ds$ for all $t\in[0,T]$ as $p\to+\infty$ by Lebesgue's dominated convergence theorem. Since $\mathcal{D}\in C([0,T])$ we have $\mathcal{F}_p\in C^2([0,T])$, with

$$\forall t\in[0,T],\quad \mathcal{F}_p''(t) \leq C\,p^2 \mathcal{F}_p^{1-\frac{n}{p}}(t).$$

We next argue similarly as in the proof of Lemma 4 in [4]. We multiply the previous inequality by $\mathcal{F}_p'(t)\geq 0$ and integrate on $[0,t]$. We obtain

$$\forall t\in[0,T],\quad (\mathcal{F}_p'(t))^2 \leq C\,p^2 \mathcal{F}_p(t)^{2-\frac{n}{p}}$$

therefore

$$\forall t\in[0,T],\quad \mathcal{F}_p'(t) \leq C\,p\mathcal{F}_p(t)^{1-\frac{n}{2p}}.$$

We now conclude as in the proof of the uniqueness of bounded solutions of the 2D Euler equations, see e.g. [23, 14]: integrating the above inequality yields

$$\forall p>n,\quad \forall t\in[0,T],\quad \mathcal{F}_p(t) \leq (Ct)^{\frac{2p}{n}}.$$

Letting $p \to +\infty$ we obtain that $\mathcal{F}(t) = 0$ for $t \in [0, 1/C]$. Repeating the argument of intervals of length $1/C$ we finally prove that $\mathcal{F}$, therefore also $\mathcal{D}$, vanishes on $[0, T]$. This implies that for all $t \in [0, T]$ we have $X_1(t, \cdot, \cdot) = X_2(t, \cdot, \cdot)$ $f_0 \, dx \, dv$ - a.e. We infer from (2.7) that for all $t \in [0, T]$, $E_1(t, \cdot) = E_2(t, \cdot)$ on $\mathbb{R}^n$. By (2.2), it follows that $V_1(t, \cdot, \cdot) = V_2(t, \cdot, \cdot)$ on $\mathbb{R}^n \times \mathbb{R}^n$. We conclude that for all $t \in [0, T]$ we have $f_1(t, \cdot, \cdot) = f_2(t, \cdot, \cdot)$ a.e. on $\mathbb{R}^n \times \mathbb{R}^n$.

Remark 2.5. In the setting of (1.5), the estimate obtained for $\tilde{\mathcal{D}}$ in Remark 2.4 yields

$$\forall p > 2, \quad \tilde{\mathcal{D}}(t) \leq (C\|\omega_0\|_{L^p} \, t)^p,$$

which does not enable to conclude that $\mathcal{D} = 0$ as above unless $\omega_0 \in L^\infty$.

2.4. **The Lagrangian flow is the classical flow.** We conclude this section with the following remark: let f be a weak solution of (1.1) satisfying the assumptions of Theorem 1.1. In view of Remark 2.3 we have

$$\forall p > n, \sup_{t \in [0,T]} |E(t, x) - E(t, y)| \leq Cp^2 |x - y|^{1 - \frac{n}{p}}.$$

By space continuity of E, Ascoli-Arzela's theorem implies that for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ there exists a curve $\gamma \in W^{1,\infty}([0, T]; \mathbb{R}^n \times \mathbb{R}^n)$ which is a solution to the ODE (2.2). Moreover, if γ_1 and γ_2 are two such integral curves then $d(t) = \int_0^t \int_0^s |\gamma_1 - \gamma_2|(\tau) \, d\tau \, ds$ satisfies $d'' \leq Cp^2 d^{1-n/p}$. So by exactly the same arguments as in the proof of Theorem 1.1 above, $d = 0$ on $[0, T]$. This means that the ODE (2.2) is well-posed for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ and that the Lagrangian flow actually is a classical flow.

3. Proof of Theorem 1.2

We start by recalling an elementary inequality, which can be found in [11, (14)] for the case $n = 3$, and which can be easily adapted to the case $n = 2$. Let $f \in L^1 \cap L^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be nonnegative and $\rho_f(x) = \int f(x, v) \, dv$. Then

(3.1)
$$\forall k \geq 1, \quad \|\rho_f\|_{L^{\frac{k+n}{n}}(\mathbb{R}^n)} \leq C\|f\|_{L^\infty}^{\frac{k}{k+n}} \left(\iint_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f(x, v) \, dx \, dv \right)^{\frac{n}{k+n}},$$

where C is a constant independent on k.

Now, let f_0 satisfy the assumptions of Theorem 1.2 and let f be any weak solution on $[0, T]$ with this initial data given by [11, Theo. 1]. By construction we have

$$\sup_{t \in [0,T]} \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v|^m f(t, x, v) \, dx \, dv < +\infty. \tag{3.2}$$

In view of (3.1), in order to control the norms $\|\rho(t)\|_{L^p}$ for large p it suffices to prove that

$$\forall k > 0, \quad \sup_{t \in [0,T]} \|f(t)\|_{L^\infty}^{\frac{k}{k+n}} M_k(t)^{\frac{n}{k+n}} \leq Ck,$$

where

$$M_k(t) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |v|^k f(t, x, v) \, dx \, dv = \iint_{\mathbb{R}^n \times \mathbb{R}^n} |V(t, x, v)|^k f_0(x, v) \, dx \, dv.$$

Since $f \in L^\infty([0,T], L^\infty(\mathbb{R}^n \times \mathbb{R}^n))$ this amounts to showing that

$$\forall k > 0, \quad \sup_{t \in [0,T]} M_k(t)^{\frac{n}{k+n}} \leq Ck. \tag{3.3}$$

At this stage it is not known whether all the $M_k(t)$ remain finite for $t > 0$. We prove next that this is indeed the case and that (3.3) can be achieved thanks to (3.2) in a much easier way as for the propagation (3.2) itself, which is the heart of the matter of [11]. As a matter of fact, since $m > n^2 - n$ we infer from (3.1) and (3.2) that $\rho \in L^\infty([0,T], L^{p_0}(\mathbb{R}^n))$ with $p_0 = (m+n)/n > n$ depending only on n and m. It follows that $E \in L^\infty([0,T], L^\infty(\mathbb{R}^n))$ by (2.1).

For $k > m$, we have by (2.2)

$$\begin{aligned} |V(t,x,v)|^k &\leq |v|^k + k \int_0^t |V(s,x,v)|^{k-1} |E(s, X(s,x,v)|\, ds \\ &\leq |v|^k + k\|E\|_{L^\infty([0,T]\times\mathbb{R}^n)} \int_0^t |V(s,x,v)|^{k-1}\, ds. \end{aligned}$$

Integrating with respect to $f_0(x,v)\, dx\, dv$ we get

$$M_k(t) \leq M_k(0) + k\|E\|_{L^\infty([0,T]\times\mathbb{R}^n)} \int_0^t M_{k-1}(s)\, ds.$$

By induction, we first infer that $\sup_{t\in[0,T]} M_k(t)$ is finite for any $k > m$. On the other hand, we obtain by Hölder inequality

$$M_{k-1}(s) \leq \|f(s)\|_{L^1}^{\frac{1}{k}} M_k(s)^{1-\frac{1}{k}},$$

therefore, since $\|f(s)\|_{L^1} = \|f_0\|_{L^1}$ by (2.3) we get

$$M_k(t) \leq M_k(0) + Ck \int_0^t M_k(s)^{1-\frac{1}{k}}\, ds.$$

Integrating this Gronwall inequality leads to

$$\sup_{t\in[0,T]} M_k(t)^{\frac{1}{k}} \leq M_k(0)^{\frac{1}{k}} + C.$$

By assumption on $M_k(0)$ we find

$$\sup_{t\in[0,T]} M_k(t)^{\frac{1}{k}} \leq (C_0 k)^{\frac{1}{n}} + C \leq (Ck)^{\frac{1}{n}}$$

therefore, finally,

$$\sup_{t\in[0,T]} M_k(t)^{\frac{n}{n+k}} \leq Ck,$$

and the conclusion follows.

4. Proof of Theorem 1.3

4.1. Seeking for initial data. In this section we construct a collection of initial densities that satisfy the assumptions of Theorem 1.2 and that do not necessarily enter in the framework of Loeper's uniqueness condition. We will consider nonnegative measurable functions φ on $\mathbb{R}$ such that

$$\operatorname{supp}(\varphi) \subset]-\infty, M] \text{ for some } M \in \mathbb{R}. \tag{4.1}$$

Proposition 4.1. *Let $\varphi \in L^\infty(\mathbb{R}, \mathbb{R}_+)$ satisfy (4.1). Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $a : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be two measurable functions. We set*

$$f_0(x, v) = \varphi\left(|v|^2 + \Phi(x) + a(x, v)\right). \tag{4.2}$$

We assume that $\rho_0 = \int f_0 \, dv$ has compact support in $B \subset \mathbb{R}^n$, and that

$$\forall p \geq 1, \quad \int_B (M - \Phi(x))_+^p \, dx \leq (C_0 p)^{\frac{2p}{n}},$$

for some constant C_0. Then any initial density given by

$$f_0 \, h_0, \quad \text{where } h_0 \in L^\infty(\mathbb{R}^n \times \mathbb{R}^n),$$

satisfies the assumptions of Theorem 1.2.

Proof. Since for $(x, v) \in \operatorname{supp} f_0$ we have $|v|^2 \leq M - \Phi(x)$, we obtain

$$\begin{aligned}
\iint_{\mathbb{R}^n \times \mathbb{R}^n} & |v|^k f_0(x, v) h_0(x, v) \, dx \, dv \\
& \leq \omega_n \|h_0\|_{L^\infty} \int_B (M - \Phi(x))_+^{\frac{k}{2}} \rho_0(x) \, dx \\
& \leq \omega_n \|h_0\|_{L^\infty} \|\varphi\|_{L^\infty} \int_B (M - \Phi(x))_+^{\frac{k+n}{2}} \, dx \\
& \leq \omega_n \|h_0\|_{L^\infty} \|\varphi\|_{L^\infty} (C_0(k+n))^{\frac{k+n}{n}}.
\end{aligned}$$

We choose $C_1 > C_0$ sufficiently large such that

$$\omega_n \|h_0\|_{L^\infty} \|\varphi\|_{L^\infty} (C_0(k+n))^{\frac{k+n}{n}} \leq (C_1 k)^{\frac{k}{n}},$$

and the condition of Theorem 1.2 is fulfilled. This concludes the proof. □

4.2. **Proof of Theorem 1.3.** We consider an initial density given by (4.2) with the choice

$$\varphi = \mathbf{1}_{\mathbb{R}_-}, \quad \Phi(x) = -(\ln_- |x|)^{\frac{2}{n}}, \quad a = 0,$$

so that

$$\rho_0(x) = \left| \left\{ v \, : |v|^2 - (\ln_- |x|)^{\frac{2}{n}} \leq 0 \right\} \right| = \omega_n \ln_- |x|, \quad \forall x \in \mathbb{R}^n.$$

Besides, a straightforward computation yields

$$\forall m \geq 0, \quad \int_{\mathbb{R}^n} (\ln_- |x|)^m \, dx = \sigma_n n^{-(m+1)} \Gamma(m+1), \tag{4.3}$$

where σ_n denotes the surface of $\partial B(0, 1)$. Using the asymptotic behavior

$$\Gamma(x+1) \sim x^x \sqrt{2\pi x} e^{-x}, \quad x \to +\infty, \tag{4.4}$$

we obtain

$$\forall p \geq 0, \quad \int_{\mathbb{R}^n} (\ln_- |x|)^{\frac{2p}{n}} \, dx \leq (C_0 p)^{\frac{2p}{n} + \frac{1}{2}}, \tag{4.5}$$

therefore for a convenient choice of $C_1 > C_0$

$$\forall p \geq 0, \quad \int_{\mathbb{R}^n} (\ln_- |x|)^{\frac{2p}{n}} \, dx \leq (C_1 p)^{\frac{2p}{n}}. \tag{4.6}$$

The conclusion follows by invoking Proposition 4.1.

4.3. **Steady states in the two-dimensional gravitational case.** In this last paragraph we focus on the Vlasov-Poisson equation (1.1) in the gravitational case for $n = 2$, which can be rewritten as

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla U \cdot \nabla_v f = 0 \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^2 \\ U(t,x) = \displaystyle\int_{\mathbb{R}^2} \ln|x-y| \rho(t,y)\, dy \\ \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,v)\, dv. \end{cases} \tag{4.7}$$

Existence and stability of special steady solutions of (4.7) have been studied intensively (see [3, 6, 9, 10] and references therein). By variational methods, Dolbeault, Fernández and Sánchez [6, Theo. 1, Theo. 22] obtained the existence of a steady solution $\overline{f} \in L^1(\mathbb{R}^n \times \mathbb{R}^n)$ having the form

$$\overline{f}(x,v) = \varphi\left(\frac{|v|^2}{2} + U(x)\right), \tag{4.8}$$

with $U = \frac{1}{2\pi} \ln *(\int \overline{f}\, dv)$ and where $\varphi : \mathbb{R} \to \mathbb{R}_+$ is nonincreasing and unbounded. Moreover, φ satisfies (4.1) for some $M \in \mathbb{R}$ and is continuously differentiable on $]-\infty, M[$. Finally, $\overline{\rho} = \int \overline{f}\, dv$ is radially symmetric, compactly supported in $B(0,1)$, and U is continuously differentiable on $\mathbb{R}^2 \setminus \{0\}$. In particular, U has the simple expression, see [6, Lemma 12]

$$\begin{aligned} U(x) &= \ln|x| \int_{|y|\le|x|} \overline{\rho}(|y|)\, dy + \int_{|y|>|x|} \ln|y| \overline{\rho}(|y|)\, dy \\ &= \ln|x| \left(\int_{\mathbb{R}^2} \overline{\rho}(|y|)\, dy\right) + \int_{|y|>|x|} \ln\left(\frac{|y|}{|x|}\right) \overline{\rho}(|y|)\, dy. \end{aligned}$$

Note that U is well defined for all $x \neq 0$ in view of the assumption on the support of $\overline{\rho}$. We remark that $\overline{f}$ may be unbounded so it is not covered by the assumptions of Theorem 1.2. However $\overline{f}$ belongs to $\mathcal{M}_+(\mathbb{R}^2 \times \mathbb{R}^2)$.

Theorem 4.2. *Let $\overline{f}$ be given by (4.8), with φ and U as above. Then for any $K > 0$, any initial density given by*

$$\overline{f}\, \mathbf{1}_{\{\overline{f} \le K\}}\, h_0, \quad \text{where } h_0 \in L^\infty(\mathbb{R}^2 \times \mathbb{R}^2),$$

satisfies the assumptions of Theorem 1.2.

Proof. Note that $\overline{f}\, \mathbf{1}_{\{\overline{f} \le K\}}\, h_0 \in L^1 \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. Since $\overline{\rho}$ is supported in $B(0,1)$ we have $U(x) = \ln|x| (\int \overline{\rho})$ for $|x| \ge 1$, from which we infer that $\overline{f}(x,v) = 0$ whenever $|x| \ge N = \exp(M/(\int \overline{\rho}))$. In addition, we observe that $\overline{f}$ takes the form (4.2), where we have set

$$\Phi(x) = \ln|x| \left(\int_{\mathbb{R}^2} \overline{\rho}(|y|)\, dy\right), \quad a(x,v) = \int_{|y|>|x|} \ln\left(\frac{|y|}{|x|}\right) \rho(|y|)\, dy \ge 0,$$

the only difference with the setting of Proposition 4.1 is that φ is unbounded on $\mathbb{R}$. Mimicking the proof of Proposition 4.1 we still obtain

$$\begin{aligned}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |v|^k\,(\overline{f}\,\mathbf{1}_{\{\overline{f}\leq K\}})(x,v)\,h_0(x,v)\,dx\,dv \\ \leq \|h_0\|_{L^\infty} K \int_{B(0,N)} \left(\int_{B(0,C(M+(\int\overline{\rho})|\ln|x||)^{1/2})} |v|^k\,dv\right) dx \\ \leq C\int_{B(0,N)} \left(M+\Big(\int\overline{\rho}\Big)|\ln|x||\right)^{\frac{k+2}{2}} dx \\ \leq (C_1 k)^{\frac{k+2}{2}} \leq (C_2 k)^{\frac{k}{2}},\end{aligned}$$

where we have used (4.6) in the last inequality, and where $C_2 > C_1$ is a sufficiently large constant.

□

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