Cluster structures associated to open Richardson varieties: a Sage algorithm

arXiv:2201.10292 [1], framagit.org/EtienneMenard/TheseSage

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Context

We want to define seed for cluster algebras on the coordinate ring of an open Richardson variety $\mathcal{R}_{v,w} = C_w \cap C^v$ of a flag variety of type A, D or E.

In order to do so, we need to find a seed for the cluster structure on the associated category $\mathcal{C}_{v,w} = \mathcal{C}_w \cap \mathcal{C}^v$ of modules over the preprojective algebra Λ. [2] proved the existence but no explicit way to build it.

Seeds

A seed is the datum of

- a cluster: a lineary free family of vectors of a given space. Variables can be mutated (cluster variable) or not (frozen variable). Here : direct summands of rigid modules
- a quiver: an oriented graph
- a mutation operation: gives another seed by changing in the cluster only one cluster variable, in a way prescribed by the quiver and changing the quiver around the corresponding vertex

Δ-vectors

Every rigid indecomposable module can be defined by an integer tuple called Δ-vector ([3]).

In fact, for any reduced representative $\overline{w_0}$ of w_0 element of longest length (r) of W, to one rigid indecomposable $M \in \text{mod}(\Lambda)$ we can define one datum : $\Delta_{\overline{w_0}}(M) \in \mathbb{N}^r$

$\Rightarrow$ mutation of modules $\leftrightarrow$ integer computation

Belonging criteria ([1])

Given $\dot{w} = [i_r, \ldots, i_{\ell(w)+1}, \underbrace{i_{\ell(w)}, \ldots, i_1}_{\overline{w}}]$ and $\dot{v} = [i_r, \ldots, i_{\ell(v)+1}, \underbrace{i_{\ell(v)}, \ldots, i_1}_{\overline{v}}]$ reduced representatives of w_0, we have two criteria :

- $M \in \mathcal{C}_w$ iff $\Delta_{\dot{w}}(M) = (a_1, \ldots, a_{\ell(w)}, \underbrace{a_{\ell(w+1)}, \ldots, a_r}_{0})$
- $M \in \mathcal{C}^v$ iff $\Delta_{\dot{v}}(M) = (\underbrace{a_1, \ldots, a_{\ell(v)}}_{0}, a_{\ell(v+1)}, \ldots, a_r)$

Input data

As initial input we have :

- A Weyl group W (Sage predefined class)
- A reduced representant $\overline{w}$ of $w \in W$ (tuple of integer)
- An element $v \in W$ (Sage predefined class) such that $v \leq w$ for Bruhat order

Data preprocessing

We compute the following data before starting the mutation algorithm itself:

- The right-most reduced representative $\overline{v}$ of v among subwords of $\overline{w}$ (tuple of integer)
- The initial seed $(V_{\overline{w}}, \Gamma_{\overline{w}})$ as :
  - Rigid indecomposable summands described by their $\Delta_{\dot{v}}$-vectors (not so easy to determine) (vector of integer)
  - Rigid indecomposable summands described by their $\Delta_{\dot{w}}$-vectors (combitorially determined from $\overline{w}$) (vector of integer)
  - Quiver $\Gamma_{\overline{w}}$ as an adjacency matrix (combitorially determined from $\overline{w}$) (integer matrix)

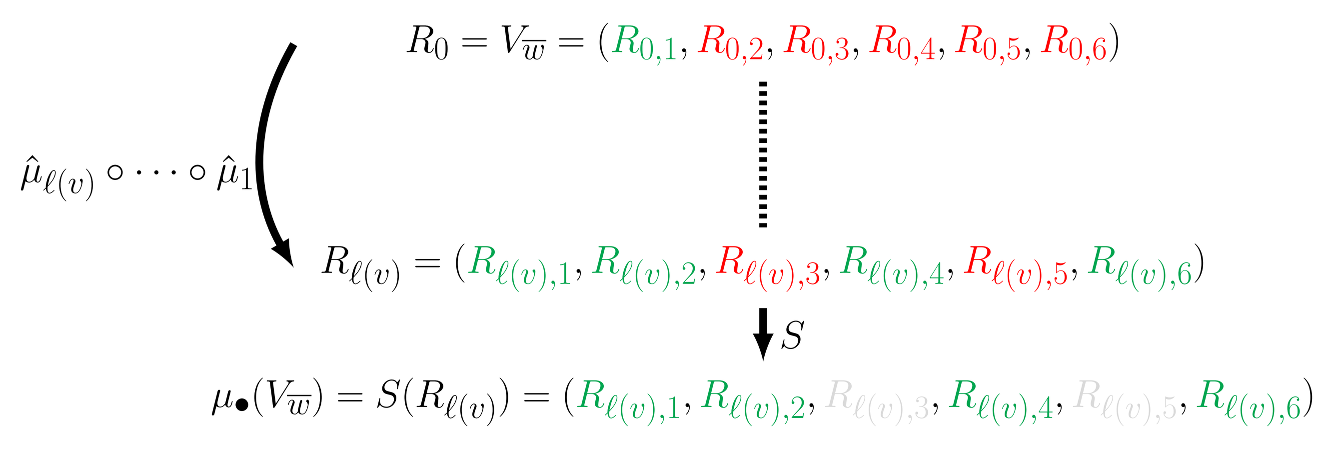
Mutation algorithm

We have two ways to define the mutations to perform :

- A **combinatorial description**: by looking at the letters of $\overline{w}$ used to write $\overline{v}$. *Pros*: knowing all mutations since start. *Cons*: no usable general description of mutations
- Seeing the algorithm as a **greedy algorithm**: mutating shifts the Δ-coordinates toward higher indices: we mutate all modules with $\Delta_{\dot{v},1} \neq 0$ then $\Delta_{\dot{v},2} \neq 0, \ldots$, to reach the $\mathcal{C}^v$ criterion. *Pros* : we understand what is happening. *Cons* : do not have a description of the whole list of mutation a priori.

After mutating : remove cluster variables not complying with the $\mathcal{C}^v$ criterion

Algorithm draft

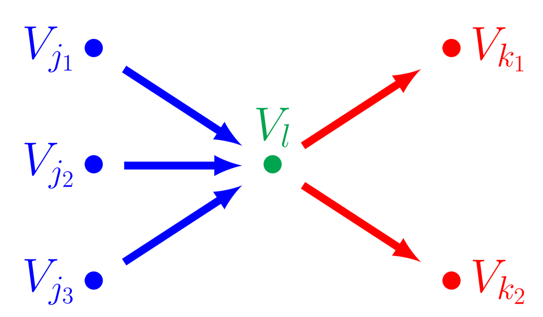


$R_{m,k} \in \mathcal{C}_{v,w}$

$R_{m,k} \in \mathcal{C}_w \setminus \mathcal{C}^v$

Computing mutations

For the quiver : usual rule. For the cluster, rule from [3]: either $\Delta(V_l^*) = \sum_{i=1}^{3} \Delta(V_{j_i}) - \Delta(V_l)$ or $\Delta(V_l^*) = \sum_{i=1}^{2} \Delta(V_{k_i}) - \Delta(V_l)$ (depends on a quantity to compute) if we have the quiver :



In real conditions : only one of the computation has nonnegative coordinates

Output

A seed whose cluster variable all have $\Delta_{\dot{v}}$-vector with $\ell(v)$ first coordinates zero $\Rightarrow$ output $\in \mathcal{C}^v$. Quiver : corresponding quiver with removal of vertices corresponding to deleted cluster variables.

Can also get a slideshow of all the steps of the seed computation.

Possible to get cluster before or after removal of variables, indication of frozen or non-frozen variables

Perspectives

Next projects :

- Study effect of input change (other representant) to the output
- Rewrite algorithms in a cleaner way
- Study [4] and [5] (particular cases of this algorithm) and compare with the algorithm
- Finer description of frozen variables
- Classification of outputs

Bibliography

References

[1] E. Ménard, "Cluster algebras associated with open Richardson varieties: An algorithm to compute initial seeds," *arXiv:2201.10292 [math]*, Jan. 2022.

[2] B. Leclerc, "Cluster structures on strata of flag varieties," *Advances in Mathematics*, vol. 300, pp. 190–228, Sept. 2016.

[3] C. Geiss, B. Leclerc, and J. Schröer, "Kac–Moody groups and cluster algebras," *Advances in Mathematics*, vol. 228, pp. 329–433, Sept. 2011.

[4] P. Galashin and T. Lam, "Positroid varieties and cluster algebras," *arXiv:1906.03501 [math]*, June 2021.

[5] K. Serhiyenko, M. Sherman-Bennett, and L. Williams, "Combinatorics of cluster structures in Schubert varieties," *Séminaire Lotharingien de Combinatoire*, vol. 82B, pp. Art. 8, 12, 2020.

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