

LINKING NUMBER AND MILNOR INVARIANTS

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ABSTRACT. We give a quick overview of the definitions and properties of the linking number and its higher-order generalization, Milnor invariants. Preliminary version of a section for the *Concise Encyclopedia of Knot theory*.

1. INTRODUCTION

The linking number is probably the oldest invariant of knot theory. In 1833, several decades before the seminal works of physicists Tait and Thompson on knot tabulation, the work of Gauss on electrodynamics led him to formulate the number of “intertwinings of two closed or endless curves” as

$$(1) \quad \frac{1}{4\pi} \iint \frac{(x' - x)(dydz' - dzdy') + (y' - y)(dzdx' - dxdz') + (z' - z)(dxdy' - dydx')}{[(x' - x)^2 + (y' - y)^2 + (z' - z)^2]^{3/2}},$$

where x, y, z (resp. x', y', z') are coordinates on the first (resp. second) curve. This number, now called *linking number*, is an invariant of 2–component links,¹ which typically distinguishes the trivial link and the Hopf link, shown below.



Alternatively, the linking number can be simply defined in terms of the first homology group of the link complement (see Thm. 2.1). But, rather than the abelianization of the fundamental group, one can consider finer quotients, for instance by the successive terms of its lower central series, to construct more subtle link invariants. This is the basic idea upon which Milnor based his work on higher order linking numbers, now called *Milnor $\bar{\mu}$ -invariants*.

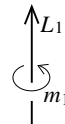
We review in Section 2 several definitions and key properties of the linking number. In Section 3, we shall give a precise definition of Milnor invariants and explore some of their properties, generalizing those of the linking number. In the final Section 4, we briefly review further known results on Milnor invariants.

2. THE LINKING NUMBER

The linking number has been studied from multiple angles and has thus been given many equivalent definitions of various natures. We already saw in (1) the original definition due to Gauss ; let us review below a few others.

2.1. Definitions. For the rest of this section, let $L = L_1 \cup L_2$ be a 2 component link, and let $\text{lk}(L)$ denote the linking number of L .

Recall that the first homology group of the complement $S^3 \setminus L_1$ of L_1 is the infinite cyclic group generated by the class $[m_1]$ of a meridian m_1 , as shown on the right.



Theorem 2.1. *The linking number of L is the integer k such that $[L_2] = k[m_1] \in H_1(S^3 \setminus L_1; \mathbb{Z})$.*

Recall now that any knot bounds orientable surfaces in S^3 , called Seifert surfaces. Given a Seifert surface S for L_2 , one may assume up to isotopy that L_1 intersects S at finitely many transverse points. At each intersection point, the orientation of L_1 and that of S form an oriented 3–frame, hence a sign using the right-hand rule ; the *algebraic intersection* of L_1 and S is the sum of these signs over all intersection points.

Theorem 2.2. *The linking number of L is the algebraic intersection of L_1 with a Seifert surface for L_2 .*

¹In this note, by “link” we will always mean an ordered and oriented tame link in the 3–sphere, up to ambient isotopy.

Given a regular diagram of L , the linking number can also simply be computed as the number of crossings where L_1 passes over L_2 , counted with signs as follow.

Theorem 2.3.

$$lk(L) = \left(\text{number of } L_1 \begin{array}{c} \nearrow \\ \searrow \end{array} L_2 \right) - \left(\text{number of } L_2 \begin{array}{c} \nearrow \\ \searrow \end{array} L_1 \right)$$

The proof of the above results, *i.e.* the fact that these definitions are all equivalent and are equivalent to (1), can be found for example in [41, §. 5.D], along with several further definitions. See also [40] for details.

Remark 2.4. As part of Theorems 2.2 and 2.3, the linking number does not depend on the choices involved in these definitions, namely the choice of a Seifert surface for L_2 and of a diagram of L , respectively.

2.2. Basic properties. It is rather clear, using any of the above definitions, that reversing the orientation of either component of L changes the sign of $lk(L)$.

Also clear from Gauss formula (1) is the fact that the linking number does not depend on the order of the link components, that is :

$$(2) \quad lk(L_1 \cup L_2) = lk(L_2 \cup L_1).$$

This can also be seen, for example, from the diagrammatic definition (Thm. 2.3), by considering two projections, on two parallel planes which are ‘on either sides’ of L , so that crossings where 1 overpasses 2 in one diagram are in one-to-one correspondence with crossings where 2 overpasses 1 in the other, with same sign.

The symmetry property (2) allows for a symmetrized version of Theorem 2.3:

$$lk(L) = \frac{1}{2} \left(\left(\text{number of } \begin{array}{c} \nearrow \\ \searrow \end{array} \right) - \left(\text{number of } \begin{array}{c} \searrow \\ \nearrow \end{array} \right) \right),$$

where one *only* counts crossings involving a strand of component 1 and a strand of component 2.

2.3. Two classification results. The following notion was introduced by Milnor in the fifties [30].

Definition 2.5. Two links are *link-homotopic* if they are related by a sequence of ambient isotopies and self-crossing changes, *i.e.* crossing changes involving two strands of a same component (see Figure 1).

The idea behind this notion is that, as a first approximation to the general study of links, working up to link-homotopy allows to unknot each individual component of a link, and only record their ‘mutual interactions’ – this is in a sense studying ‘linking modulo knotting’.

Observe that the linking number is invariant under link-homotopy. Furthermore, it is not too difficult to check that any 2–component link is link-homotopic to an iterated Hopf link, *i.e.* the closure of the pure braid σ_1^{2n} for some $n \in \mathbb{Z}$. This number n is precisely the linking number of the original link, thus showing:

Theorem 2.6. *The linking number classifies 2–component links up to link-homotopy.*

Generalizing this result to a higher number of components, however, requires additional invariants, and this was one of the driving motivations that led Milnor to develop his $\bar{\mu}$ –invariants, see Section 3.3.2. There is, however, an equivalence relation on links with an arbitrary number of components which is classified by the linking number; this is the Δ –*equivalence*, which is generated by the Δ –move, shown on the right-hand side of Figure 1. Indeed, the following is due H. Murakami and Y. Nakanishi [33], see also [27].

Theorem 2.7. *Two links $\cup_{i=1}^n L_i$ and $\cup_{i=1}^n L'_i$ ($n \geq 2$) are Δ –equivalent if, and only if $lk(L_i \cup L_j) = lk(L'_i \cup L'_j)$ for all i, j such that $1 \leq i < j \leq n$.*

Notice in particular that the Δ –move is an unknotting operation, meaning that any knot can be made trivial by a finite sequence of isotopies and Δ –moves.



FIGURE 1. A self-crossing change (left) and a Δ –move (right)

3. MILNOR INVARIANTS

In this whole section, let $L = L_1 \cup \cdots \cup L_n$ be an n -component link, for some fixed $n \geq 2$.

3.1. Definition of Milnor $\bar{\mu}$ -invariants. In his master's and doctoral theses, supervised by R. Fox, Milnor defined numerical invariants extracted from the peripheral system of a link, which widely generalize the linking number.

Denote by X the complement of an open tubular neighborhood of L . Pick a point in the interior of X , and denote by π the fundamental group of X based at this point. It is well-known that, given a diagram of L , we can write an explicit presentation of π , called the Wirtinger presentation, where each arc in the diagram provides a generator, and each crossing yields a relation. Such presentations, however, are in general difficult to work with, owing to their large number of generators. We introduce below a family of quotients of π for which a much simpler presentation can be given, and which still retain rich topological informations on L .

The *lower central series* $(\Gamma_k G)_{k \geq 1}$ of a group G is the nested family of subgroups defined inductively by

$$\Gamma_1 G = G \quad \text{and} \quad \Gamma_{k+1} G = [G, \Gamma_k G].$$

The k th nilpotent quotient of the group G is the quotient $G/\Gamma_k G$.

Let us fix, from now on, a value of $k \geq 2$. For each $i \in \{1, \dots, n\}$, consider two elements $m_i, \lambda_i \in \pi/\Gamma_k \pi$, where m_i represents a *choice* of a meridian for L_i in X , and where λ_i is a word representing the i th *preferred longitude* of L_i , i.e. a parallel copy of L_i in X having linking number zero with L_i . Denote also by F the free group $F(m_1, \dots, m_n)$.

Theorem 3.1 (Chen-Milnor Theorem [31]). *The k th nilpotent quotient of π has a presentation given by*

$$\pi/\Gamma_k \pi = \langle m_1, \dots, m_n \mid m_i \lambda_i m_i^{-1} \lambda_i^{-1} \ (i = 1, \dots, n), \Gamma_k F \rangle.$$

In order to extract numerical invariants from the nilpotent quotients of π , we consider the *Magnus expansion* [25] of each λ_j , which is an element $E(\lambda_j)$ of $\mathbb{Z}\langle\langle X_1, \dots, X_n \rangle\rangle$, the ring of formal power series in non commuting variables X_1, \dots, X_n , obtained by the substitution

$$m_i \mapsto 1 + X_i \quad \text{and} \quad m_i^{-1} \mapsto 1 - X_i + X_i^2 - X_i^3 + \dots$$

We denote by $\mu_L(i_1 i_2 \dots i_p j)$ the coefficient of $X_{i_1} X_{i_2} \dots X_{i_p}$ in the Magnus expansion of λ_j :

$$(3) \quad E(\lambda_j) = 1 + \sum_{i_1 i_2 \dots i_p} \mu_L(i_1 i_2 \dots i_p j) X_{i_1} X_{i_2} \dots X_{i_p}.$$

This coefficient is *not*, in general, an invariant of the link, as it depends upon the choices made in this construction (essentially, picking a system of meridians and longitudes for L). We can, however, promote these numbers to genuine invariants by regarding them modulo the following indeterminacy.

Definition 3.2. Given a sequence I of indices in $\{1, \dots, n\}$, let $\Delta_L(I)$ be the greatest common divisor of the coefficients $\mu_L(I')$, for all sequences I' obtained from I by deleting at least one index and permuting cyclicly.

For example, $\Delta_L(123) = \gcd\{\mu_L(12), \mu_L(21), \mu_L(13), \mu_L(31), \mu_L(23), \mu_L(32)\}$.

Theorem-Definition (Milnor [31]). *The residue class*

$$\bar{\mu}_L(I) \equiv \mu_L(I) \text{ mod } \Delta_L(I)$$

is an invariant of ambient isotopy of L , for any sequence I of at most k integers, called a Milnor invariant of L . The number of indices in I is called the length of the invariant.

Before giving some examples in Section 3.2, a few remarks are in order.

Remark 3.3. Observe from (3) that $\mu_L(I)$ is only defined for a sequence I of *two* or more indices. As a convention, we set $\mu_L(i) = 0$ for all i .

Remark 3.4. If all Milnor invariants of L of length $< k$ are zero, then those of length k are well defined integers. These *first non-vanishing Milnor invariants* are thus much easier to study in practice and, as a matter of fact, a large proportion of the litterature on Milnor link invariants focusses on them.

Remark 3.5. Working with the k th nilpotent quotient of π only allows to define Milnor invariants up to length k . But this k can be chosen arbitrarily large, so this is no restriction. Also, the invariant $\bar{\mu}_L(I)$ does not depend on the choice of k , as long as k is no greater than the length of I .

3.2. First examples. By Remark 3.3, length 2 Milnor invariants are well-defined over \mathbb{Z} . In order to define them, it suffices to work in the second nilpotent quotient, *i.e.* in $\pi/\Gamma_2\pi \cong H_1(X; \mathbb{Z})$. But the (image of the) j th preferred longitude λ_j in $H_1(X; \mathbb{Z})$ writes $\sum_{i \neq j} \text{lk}(L_i, L_j)m_i$, and we have

$$E(\lambda_j) = 1 + \sum_{i \neq j} \text{lk}(L_i, L_j)X_i + \text{terms of degree} \geq 2.$$

Hence length 2 Milnor invariants are exactly the pairwise linking numbers of a link. This justifies regarding length ≥ 3 Milnor invariants as ‘higher order linking numbers’. In order to get a grasp on these, let us consider an elementary example.

Example 3.6. Consider the Borromean rings B , illustrated below.² The Wirtinger presentation for π is

$$\pi = \left\langle \begin{array}{c|ccc} m_1, m_2, m_3 & m_3m_1m_3^{-1}n_1^{-1} & m_1m_2m_1^{-1}n_2^{-1} & m_2m_3m_2^{-1}n_3^{-1} \\ n_1, n_2, n_3 & n_1^{-1}n_2n_1m_2^{-1} & n_2^{-1}n_3n_2m_3^{-1} & n_3^{-1}n_1n_3m_1^{-1} \end{array} \right\rangle.$$

Consider the third nilpotent quotient of π , *i.e.* fix $k = 3$. We choose the meridians m_i as (representatives of) generators, and express preferred longitudes λ_i : we have $\lambda_3 = m_2^{-1}n_2 = m_2^{-1}m_1m_2m_1^{-1}$ (the other two are obtained by cyclic permutation of the indices, owing to the symmetry of the link). Taking the Magnus expansion gives

$$E(\lambda_3) = 1 + X_1X_2 - X_2X_1 + \text{terms of degree} \geq 3.$$

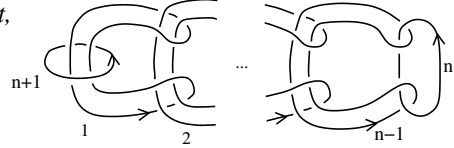
We thus obtain that $\bar{\mu}_B(I) = 0$ for any sequences I of two indices (*i.e.* all linking numbers are zero), and that

$$\bar{\mu}_B(123) = \bar{\mu}_B(231) = \bar{\mu}_B(312) = 1 \quad \text{and} \quad \bar{\mu}_B(132) = \bar{\mu}_B(213) = \bar{\mu}_B(321) = -1.$$

Example 3.6 illustrates the fact that, just like the linking number detects (and actually, counts copies of) the Hopf link, the *triple linking number* $\bar{\mu}(123)$ detects the Borromean rings. This generalizes to the following realization result, due to Milnor [30].

Lemma 3.7. *Let M_n be the n -component link shown on the right, and let σ be any permutation in S_{n-2} . Then*

$$\bar{\mu}_{M_n}(\sigma(1) \cdots \sigma(n-2)n-1n) = \begin{cases} 1 & \text{if } \sigma = \text{Id}, \\ 0 & \text{otherwise.} \end{cases}$$



3.3. Some properties. We gather here several well-known properties of Milnor $\bar{\mu}$ -invariants, most being due to Milnor himself [30, 31] (unless otherwise specified).

3.3.1. Symmetry and Shuffle. It is rather clear that, if L' is obtained from L by reversing the orientation of the i th component, then $\bar{\mu}_{L'}(I) = (-1)^{i(I)}\bar{\mu}_L(I)$, where $i(I)$ denotes the number of occurrences of the index i in the sequence I . The next two relations were shown by Milnor [31], using properties of the Magnus expansion:

Cyclic symmetry: For any sequence $i_1 \cdots i_m$ of m indices in $\{1, \dots, n\}$, we have

$$\bar{\mu}_L(i_1 \cdots i_m) = \bar{\mu}_L(i_m i_1 \cdots i_{m-1}).$$

Shuffle: For any $k \in \{1, \dots, n\}$ and any two sequences I, J of indices in $\{1, \dots, n\}$, we have

$$\sum_{H \in S(I, J)} \bar{\mu}_L(Hk) \equiv 0 \pmod{\text{gcd}\{\Delta(Hk) \mid H \in S(I, J)\}},$$

where $S(I, J)$ is the set of all sequences obtained by inserting the indices of J into I , preserving order.

It follows in particular that there is essentially only one triple linking number: for any 3-component algebraically split link L , we have $\bar{\mu}_L(123) = \bar{\mu}_L(231) = \bar{\mu}_L(312) = -\bar{\mu}_L(132) = -\bar{\mu}_L(213) = -\bar{\mu}_L(321)$. In general, the number of independent $\bar{\mu}$ -invariants was given by K. Orr, see [37, Thm. 15].

² There, we use the usual convention that a small arrow underpassing an arc of the diagram represents a loop, based at the reader's eye and going straight down to the projection plane, enlacing positively the arc and going straight back to the basepoint.

3.3.2. *Link-homotopy and concordance.* Milnor invariants are not only invariants of ambient isotopy: they are actually invariants of *isotopy*, *i.e.* homotopy through embeddings. In particular, Milnor invariants do not see ‘local knots’, as those can be shrunk to a point by a general isotopy.

As mentioned in Section 2.3, the notion of link-homotopy was introduced by Milnor himself, who proved the following in [31].

Theorem 3.8. *For any sequence I of pairwise distinct indices, $\bar{\mu}(I)$ is a link-homotopy invariant.*

Milnor invariants are sharp enough to detect link-homotopically trivial links, and to classify links up to link-homotopy for ≤ 3 components [31]. The case of 4–component links was only completed thirty years later by J. Levine, using a refinement of Milnor invariants [20]. The general case is discussed in Section 4.1. Milnor link-homotopy invariants also form a complete set of link-homotopy invariants for *Brunnian* links, *i.e.* links which become trivial under removal of any component (see Section 3.2 for examples) [30].

Recall that two n –component links L and L' are *concordant* if there is an embedding $f : \sqcup_{i=1}^n (S^1 \times [0, 1])_i \rightarrow S^3 \times [0, 1]$ of n disjoint copies of the annulus $S^1 \times [0, 1]$, such that $f((\sqcup_{i=1}^n S_i^1) \times \{0\}) = L \times \{0\}$ and $f((\sqcup_{i=1}^n S_i^1) \times \{1\}) = L' \times \{1\}$. The following was essentially shown by J. Stallings [43] (and also by A. Casson [6] for the more general relation of cobordism).

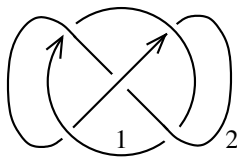
Theorem 3.9. *Milnor invariants are concordance invariants.*

Let us also mention here that Milnor invariants are all zero for a *boundary link*, that is, for a link whose components bound mutually disjoint Seifert surfaces in S^3 [42]. Characterizing geometrically links with vanishing of Milnor invariants is the subject of the k –slice conjecture, proved by K. Igusa and K. Orr in [14], which roughly states that all invariants of length $\leq 2k$ vanish if and only if the link bounds a surface which ‘look like slice disks’ modulo k –fold commutators of the fundamental group of the surface complement.

3.3.3. *Cabling formula.* Milnor invariants indexed by non-repeated sequences are not only topologically relevant, by Theorem 3.8; they also ‘generate’ all Milnor invariants, by the following.

Theorem 3.10. *Let I be a sequence of indices in $\{1, \dots, n\}$, such that some index i appears at least twice in I . Then $\bar{\mu}_L(I) = \bar{\mu}_{\tilde{L}}(\tilde{I})$, where \tilde{I} is obtained from I by replacing one of the occurrences of i by $n + 1$, and where \tilde{L} is obtained from L by adding an $(n + 1)$ th component, which is a parallel copy of L_i having linking number zero with it.*

Example 3.11. A typical example of a link-homotopically trivial link is the Whitehead link W , shown below

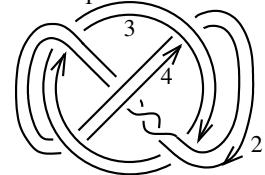


on the left-hand side. But Milnor concordance invariants do not detect W ; specifically, we have $\bar{\mu}_W(1122) = 1$.

As a matter of fact, one can check that

$$\bar{\mu}_W(1122) = \bar{\mu}_{\tilde{W}}(1324) = 1,$$

where \tilde{W} is the 4–component link shown on the right.



3.3.4. *Additivity.* Given two n –component links L and L' , lying in two disjoint 3–balls of S^3 , a *band sum* $L\#L'$ of L and L' is any link made of pairwise disjoint connected sums of the i th components of L and L' ($1 \leq i \leq n$). Although this operation is not well-defined, V. Krushkal showed that, for any sequence I ,

$$\bar{\mu}_{L\#L'}(I) \equiv \bar{\mu}_L(I) + \bar{\mu}_{L'}(I) \pmod{\gcd(\Delta_L(I), \Delta_{L'}(I))},$$

meaning that Milnor invariants are independent of the choice of bands defining $L\#L'$, see [19].

4. FURTHER PROPERTIES

Since their introduction in the fifties, Milnor invariants have been the subject of numerous works. In this section, which claims neither exhaustivity nor precision, we briefly overview some of these results.

4.1. **Refinements and generalizations.** The main difficulty in understanding Milnor $\bar{\mu}$ –invariants lies in these intricate indeterminacies $\Delta(I)$. Multiple attempts have been made to refine this indeterminacy, in order to get more subtle invariants. Milnor link-homotopy invariants were refined by Levine to tackle the 4–component case (see Section 3.3.2), using the notion of ‘reduced link group’, see [20]. A decisive step was taken by Habegger and Lin, who showed that the indeterminacies $\Delta_L(I)$ are equivalent to the indeterminacy

in representing L as the closure of a *string link*, *i.e.* a pure tangle without closed component [11] (see also [21]). This led them to a full link-homotopy classification of (string) links [11].

T. Cochran defined, by considering recursive intersection curves of Seifert surfaces, link invariants which recover (and shed beautiful geometric lights on) $\bar{\mu}$ -invariants, and sometimes refines them [8].

K. Orr defined invariants, as the class of the ambient S^3 in some third homotopy group, which refines significantly the indeterminacy of Milnor invariants [36, 37]. Orr's invariant is also the first attempt towards a transfinite version of Milnor invariants, a problem posed in [31]; see also the work of J. Levine in [22].

4.2. Link maps and higher dimensional links. There are several higher dimensional versions of the linking number; see [5, § 3.4] for a good survey. Some are invariants of *link maps*, *i.e.* maps sending disjointly a union of two spheres (of various dimensions) to a sphere – these are the natural objects to consider when working up to link-homotopy. In particular, the first example of a link map which is not link-homotopically trivial was given in [9] using an appropriate generalization of the linking number.

Higher dimensional generalizations of Milnor invariants were extensively studied by U. Koschorke for link maps with many components, see [16, 17] and references therein. Note also that Orr's invariants, which generalize Milnor invariants (see Section 4.1), are also defined in any dimensions. But there does not seem to be nontrivial analogues of Milnor invariants for n -dimensional links ($n \geq 2$), *i.e.* for embedded n -spheres in codimension 2; in fact, the latter are all link-homotopically trivial [4]. However, Milnor invariants generalize naturally to 2-string links, *i.e.* knotted annuli in the 4-ball bounded by a prescribed unlink in S^3 , and classify them up to link-homotopy [2], thus providing a higher dimensional version of [11].

4.3. Finite type invariants. The linking number is (up to scalar) the unique degree 1 finite type link invariant. This was generalized independently by D. Bar Natan [3] and X.S. Lin [24], who showed that any Milnor *string link* invariant of length k is a finite type invariant of degree $k - 1$. As a consequence, Milnor string link invariants can be, at least in principle, extracted from the Kontsevich integral, which is universal among finite type invariants. This was made completely explicit by G. Masbaum and N. Habegger in [12].

Note that the finite type property does not make sense for higher order $\bar{\mu}$ -invariants (of *links*), since the indeterminacy $\Delta(I)$ is in general not the same for two links which differ by a crossing change. Nonetheless, $\bar{\mu}$ -invariants of length k are invariants of C_k -equivalence [13], a property shared by all degree $k - 1$ invariants.

4.4. Relations with other invariants. Milnor invariants are not only natural generalizations of the linking number, but are also directly related to this invariant. Indeed, K. Murasugi expressed Milnor $\bar{\mu}$ -invariants of a link as a linking number in certain branched coverings of S^3 along this link [35]. The Alexander polynomial is also rather close in nature to Milnor invariants, being extracted from the fundamental group of the complement. As a matter of fact, there are a number of results relating these two invariants: see for example [7, 23, 26, 34, 44]. On the other hand, the relation to the Kontsevich integral [12], mentioned in § 4.3, hints to potential connections to quantum invariants. Such relations were given with the HOMFLY-PT polynomial [29] and the (colored) Jones polynomial [28]. Milnor string link invariants also satisfy a skein relation [38], which is a typical feature of polynomial and quantum invariants.

There are also known relations outside knot theory. Milnor $\bar{\mu}$ -invariants can be expressed in terms of Massey products of the complement, which are higher order cohomological invariants generalizing the cup product [39, 45]. Also, Milnor string link invariants are in natural correspondence with Johnson homomorphisms of homology cylinders, which are 3-dimensional extensions of certain abelian quotients of the mapping class group [10]. Finally, as part of the deep analogies he established between knots and primes, M. Morishita defined and studied arithmetic analogues of Milnor invariants for prime numbers [32].

4.5. Virtual theory. There are two distinct extensions of the linking number for virtual 2-component links, namely $lk_{1,2}$ and $lk_{2,1}$, where $lk_{i,j}$ is the sum of signs of crossings where i passes over j . Notice that these virtual linking numbers are actually invariants of *welded links*, *i.e.* are invariant under the forbidden move allowing a strand to pass *over* a virtual crossing. Indeed, the linking number is extracted from (a quotient of) the fundamental group, which is a welded invariant [15]. Likewise, extensions of Milnor invariants shall be welded invariants. A general welded extension of Milnor string link invariants is given in [1], which classifies welded string links up to self-virtualization, generalizing the classification of [11]. This extension recovers and extends that of [18], which gives general Gauss diagram formulas for virtual Milnor invariants.

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