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My work is concerned with surfaces and representations of surface groups, a subject which has interactions with many other branches of mathematics. The results I have obtained have connections to sevral of these : algebraic geometry, dynamics, number theory , three dimensional topology and geometric group theory.

Contents

1. Introduction : moduli space

1.1. **Definitions.** To a surface Σ of genus g with $n > 0$ marked points one associates its *Teichmller space*$\mathcal{T}(\Sigma)$ that is the space of marked conformal structures on Σ ; from a purely topological point of view this space is homeomorphic to $\mathbb{R}^{6g-6+2n}$. Each point of the Teichmller space is a surface which admits a unique metric of constant curvature -1 that is a *hyperbolic surface.* The Uniformization Theorem says that there is a Fuchsian group $\Gamma < PSL(2,\mathbb{R}) \simeq \mathrm{isom}^+(\mathbb{H})$, isomorphic to the fundamental group of the surface Σ, . So we think of Σ as the quotient of $\mathbb{H}/\Gamma$ and the map associating to a conformal structure a Fuchsian group provides an embedding of Teichmller space in $\chi(\pi_1(\Sigma))$ – that is the space of represenations (modulo conjugation) of the fundamental of Σ in $SL(2,\mathbb{R})$. This space $\chi(\pi_1(\Sigma))$ is the character variety and since $SL(2,\mathbb{R})$ is an algebraic group, this space inherits the structure of an algebraic variety. The character variety s (and Teichmller space) has a natural symplectic structure which comes from the Killing form on the Lie algebra of $SL(2,\mathbb{R})$.

Finally the group$\pi_0(\mathrm{Diff}^+(\Sigma))$, called *the Teichmller modular group*, acts properly discontinuously on Teichmller space; *the moduli space of the surface* is the quotient.

1.2. **Geometric identities.** In the early 90s I discovered a class of dentities (so-called geometric identities) for the lengths of closed simple geodesics in a hyperbolic surface. Here, by an identity, we mean a series constant sur $\mathcal{T}(\Sigma)$. The simplest case is that of the punctured torus [**?**] :

Theorem 1.1. *Let $\Sigma_{1,1}$ be a once punctured torus with a complete hyperbolic metric of finite area. Then*

$$\sum \frac{2}{1+\exp(\ell_\alpha)} = 1 \tag{1}$$

where the sum is taken over all closed simple geodesics $\Sigma_{1,1}$.

Following this I obtained similar results for surfaces with cone points [**?**] and closed surfaces [**?**]. The method of the proof of all these results is based on a careful study of leaves of non-compact geodesic laminations and is essentially "topological". Each term in the series is associated to a special kind of non-compact simple geodesic, namely one which has both its ends at marked point(s). One then shows that these are the only isolated points in a certain space of laminations endowed with a natural metric. The term can be interpreted as the radius of a ball that contains just the isolated point. Troels Jorgensen asked if it was possible to give a different proof, for example, using the work of Wolpert. In [**?**] I managed to do this using the Kerckoff-Wolpert formula for the first variation of the length of a geodesic under a Dehn-twist.

1.3. **Symplectic Volume.** An important application of these identities is to calculate sympectic volumes. In [**?**], Wolpert gives an expression for the Weil-Petersson form ω_{WP}, in *normalised Fenchel-Nielsen coordinates* :

$$\omega_{WP} = \sum_i d\ell_{\gamma_i} \wedge d\tau_{\gamma_i},$$

where the sum is over a family of closed simple geodesics that form a pants decomposition of M, ℓ_{γ_i} is the length of the geodesic γ_i and $\tau_{\gamma_i}$the associated twist parameter γ_i . Wolpert went on to show that ω_{WP} is a non degenerate $\mathcal{MCG}$-invariant 2-form. By finding an explicit fundamental domain for the modular group of the punctured torus, Wolpert [**?**] showed that the volume of $\mathcal{M}_{1,1} = \mathcal{T}_{1,1}/\mathcal{MCG}$ for $\wp = d\ell_\alpha \wedge d\tau_\alpha$ is $\frac{\pi^2}{6}$.

We illustrate how this result can be proved using the identity (1). Let G be a discrete grouop acting properly by diffeomorphism on a manifold X and $H < G$. Let ω is a G-invariant differentiakl form on X integrable on Δ_H a fundamental domain for H. We write G/H for a family of representatives for the cosets of H in G. Now if Δ_G is a fundamental domain for G, then $\cup_{f\in G/H} f(\Delta_G)$ is full measure in a fundamental domain for H and we have an "unwinding formula":

$$\int_{\Delta_G} \sum_{f\in G/H} f^*\omega = \int_{\cup_{f\in G/H} f(\Delta_G)} \omega = \int_{\Delta_H} \omega. \tag{2}$$

We are free to choose any fundamental domain for Δ_H since ω is H-invariant.

Let M be a punctured torus and $\alpha \subset M$ a closed simple geodesic. For the punctured torus $\mathcal{MCG}$ acts transitively on the set of homotopy classes of (homotopically non trivial) closed simple curves. In Fenchel-Nielsen coordinates a fundamental domain for for the Dehn twist $\langle T_\alpha \rangle < \mathcal{MCG}$ is

$$\Delta_{\langle T_\alpha\rangle} = \{0 < \ell_\alpha < \infty,\ 0 < \tau_\alpha < \ell_\alpha\}.$$

Using (2) – $(G,H)=(\mathcal{MCG},\langle T_\alpha\rangle)$– one gets the volume $\mathcal{M}_{1,1}$:

$$\int_{\Delta_{\mathcal{MCG}}}\sum_{f\in\mathcal{MCG}} f^*\left(\frac{2}{1+\exp(\ell_\alpha)}\times dl_\alpha\wedge d\tau_\alpha\right)=\int_0^\infty\int_0^l\frac{dl\,dt}{1+e^l}=\frac{\pi^2}{6}.$$

Starting with this observation Mirzakhani [**?**] constructs an integration scheme for the moduli space of a surface with boundary. She showed that the symplectic volumes satisfy a certain recurrence relation and this leads to a new proof of a *conjecture of E. Witten.* Of fundamental importance in this approach is a generalization of the geometric identities described above to surfaces with boundary.

2. Hitchin component

2.1. **Identities for the Hitchin component, with F. Labourie.** With Francois Labourie, we extended Mirzakhani's results to the Hitchin component $\mathcal{H}(N)$. The *Hitchin component* is a component of the SL$(N,\mathbb{R})$-character variety of a surface (compact, connected, orientable) surface M. The space $\mathcal{H}(N)$ is the analog of Teichmller space "in higher rank" ; Hitchin [**?**] used Higg's bundle techniques to show that if $\rho\in\mathcal{H}(N)$ then ρ is discrete and faithful. Francois Labourie [**?**] we show that the identity above has a natural formulation in terms of (generalized) cross ratios. Then, using this formulation, we study identities arising from the cross ratios constructed for representations in $SL(N,\mathbb{R})$ by Labourie [**?**]. The associated moduli space is called the *Hitchin component*, for $N=2$ this is just the Teichmueller space and for $N>2$ it is homeomorphic to an open ball of dimension

$$(2g-2+n)\times 1/2N(N+1).$$

We give a brief overview of the main ideas.

Let Σ be a closed surface. and $\partial_\infty\pi_1(\Sigma)$ be the *boundary at infinity* of the fundamental group $\pi_1(\Sigma)$ of Σ. A *cross ratio* on $\partial_\infty\pi_1(\Sigma)$ is a $\pi_1(\Sigma)$-invariant Hlder function on

$$\partial_\infty\pi_1(\Sigma)^{4*}=\{(x,y,z,t)\in\partial_\infty\pi_1(\Sigma)^4\ x\neq t,\ \text{and}\ y\neq z\},$$

satisfying certain rules, the most significant being the *multiplicative cocycle identities.* To every non trivial element γ of the group $\partial_\infty\pi_1(\Sigma)$ we associate a positive number, $\ell_b(\gamma)$, called the *period* of γ

$$\ell_b(\gamma)=\log b(\gamma^-,\gamma y,\gamma^+,y),$$

where γ^+ and γ^- are respectively the attractive and repulsive fixed points of γ in $\partial_\infty\pi_1(\Sigma)$ and where y is any point of $\partial_\infty\pi_1(\Sigma)$ such that $\gamma(y)\neq y$. A complete hyperbolic metric on Σ gives rise to an identification of $\partial_\infty\pi_1(\Sigma)$ with the real projective line, the classical cross ratio on the projective line then gives rise to a cross ratio on $\partial_\infty\pi_1(\Sigma)$ The period of γ is just the hyperbolic length of the closed geodesic freely homotopic to γ.

A pair of pants P with marked boundary in Σ corresponds to a triple (α,β,γ) of elements of $\pi_1(\Sigma)$, unique up to conjugation, such that $\alpha\gamma\beta=1$. Tthe *pant gap function* G_b at P to be the positive number

$$G_b(P)=\log(b(\alpha^+,\gamma^-,\alpha^-,\beta^+),$$

where b a cross ratio on $\partial_\infty\pi_1(\Sigma)$.

The general form of the McShane identity is:

Theorem 2.1. *Let Σ be closed surface. Let b be a cross ratio on $\partial_\infty \pi_1(\Sigma)$. Let α be a non trivial element of $\pi_1(\Sigma)$. Let $\mathcal{P}$ be the set of homotopy classes of pair of pants with marked boundary in Σ whose first boundary component is α, then*

$$\ell_b(\alpha) = \sum_{P \in \mathcal{P}} G_b(P).$$

For $SL(2,\mathbb{R})$ the pant gap function can be computed in terms of the length of the boundary components of pants using hyperbolic trigonometry [**?**]. We show in [**?**] how to determine G_b using just Thurston's *shear coordinates* and elementary manipulations involving the classical cross ratio. In [**?**] Fock and Goncharov introduced a far reaching generalisations of Thurston's shear coordinates, which we call FGT coordinates, on the (augmented) moduli space of positive representations. We compute the gap functions for Hitchin representations using FGT coordinates for the moduli of pants; since the augmented moduli space is a "covering" of the space of Hitchin representations, we obtain $(n!)^3$ different answers because of the action of the Weyl group. It turns out that, for a suitable choice of FGT coordinates, the pant gap function has a nice expression. However, using the explicit description of the holonomies due to Fock and Goncharov in the case of $n = 3$, we see the pant gap function has in general a very complicated expression for some choices of coordinates.

3. Behavior of simple geodesics on moduli space

3.1. **Multiplicities for lengths of simple geodesics, with Hugo Parlier.**

Let $\Sigma_{g,n}$ be a surface of genus g with n boundary components together with a hyperbolic structure σ, that is a metric of constant curvature -1, of finite area such that the boundary curves are totally geodesic. Define the *full length spectrum* to be the collection of all lengths of primitive closed geodesics on the surface counted with multiplicities and the *simple length spectrum* to be the set of lengths of all simple closed geodesics counted with multiplicities. As the hyperbolic structure varies, the length spectrum changes.

In his thesis Margulis gave a solution to counting problem for the full length spectrum for a closed surface. The number of primitive geodesics of length less than L is

$$e^{\delta L}/L,$$

where δ is the entropy of the (recurrent part of) the geodesic flow. In an article with Igor Rivin I gave a complete solution to the counting problem for the simple length spectrum was solved for the once punctured torus with a hyperbolic metric: The number of primitive geodesics of length less than L is

$$A(\sigma)L^2$$

where $A(\sigma)$ is a (non constant) real analytic function on the Teichmueller space of the punctured torus. Building on our work Mirzakhani proved that there is an asymptotic for a surface of genus g with n holes equipped with a hyperbolic structure: The number of primitive geodesics of length less than L on $\Sigma_{g,n}$ is

$$A(\sigma)L^{6g-6+2n}.$$

The significance of $6g - 6 + 2n$ is that it is the dimension of the Teichmueller space. Evidently the simple spectrum is a very different object from the full length spectrum: Whilst the full length spectrum encodes information about the geodesic flow on the unit tangent bundle of the surface the simple length spectrum encodes information about the point in the moduli space determined any the surface.

In an article with Hugo Parlier we study the following three questions:

- Is there a surface for which all the multiplicities are 1?
- How big is the set of such surfaces?
- Is it possible to deform a surface such that the multiplicity stays 1 for all simple geodesics?

It is a surprising result of Randol, following work of Horowitz, that given $N > 0$ there are N distinct closed geodesics of the same length. So Randol's theorem gives a negative response to all three of these questions.

We study the (non-empty) subsets $\mathcal{E}(\alpha, \beta)$ of Teichmüller space where a pair of distinct simple closed geodesics α, β have the same length. When the intersection number $i_{alg}(\alpha, \beta)$ is small, the surfaces $\mathcal{E}(\alpha, \beta)$ play an important part in the theory of fundamental domains for the mapping class group in low genus, see for instance Maskit's description using chains of geodesics and inequalities, but there seems to be little or no work in the literature when $i_{alg}(\alpha, \beta)$ is large. Our main theorem is the following:

Theorem 3.1. *The set of surfaces with simple simple length spectrum is dense and its complement is Baire meagre.*

Any path in the Teichmüller space of the surface passes through a surface which has at least two distinct simple closed geodesics of the same length.

Let $\mathcal{E}$ denote the set of all surfaces with at least one pair of simple closed geodesics of equal length; $\mathcal{E}$ is the union of a countable family of nowhere dense subsets, namely the sets $\mathcal{E}(\alpha, \beta)$ where α, β vary over all distinct simple closed geodesics. The theorem asserts that $\mathcal{E}$ is dense and moreover that the complement contains no arcs and is thus totally disconnected.

We apply this analysis to give an infinite set of counterexamples to a conjecture of Paul Schmutz which we now describe. A *Markoff number* is an integer such that there is a solution $x, y, z \in \mathbb{Z}\, x \geq y \geq z > 2$ of the equation

$$x^2 + y^2 + z^2 - xyz = 0,$$

and we say that (x, y, z) is a *Markoff triple*. The *Markoff number conjecture*, apparently first stated by Frobenius, states that each Markoff number x determines a unique Markoff triple. More recently this problem had a number of authors, notably J. Button, have obtained partial results for x satisfying special arithmetic properties. Following the work of Harvey Cohn and others the Markoff number conjecture is equivalent to the fact that the multiplicity of any number in the simple length spectrum of the modular once punctured torus is at most 6. The *modular once punctured torus* is characterized by the fact that it is a regular 6-fold covering of the modular orbifold and thus it has the maximal possible isometry group, isomorphic to C_6. Schmutz's conjectured that the simple length spectrum of the one holed torus has multiplicity at most 6. Our method yields infinitely many one holed tori with at least one value in the simple spectrum of multiplicity at least 12.

3.2. **Cellulation canonique et Deligne Mumford compactification, avec Bob Penner.** In the 80s R. Penner constructed a natural ($\mathcal{MCG}$-invariant) cellulation of a trivial bundle over Teichmueller space. Each cell corresponds to an embedded fatgraph in the surface and to each edge of the fatgraph is associated a number λ called a *lambda-length* (collectively satisfying certain inequalities.) Penner showed that this data can be interpreted as a system of coordinates on the decorated Teichmueller space – a trivial bundle over Teichmueller space where the fiber is just $\mathbb{R}^n$ where n is the number of marked points. Using a convex hull construction in Minkowski space he shows further that the set of surfaces such that the λ-lengths satisfy the triangle inequality is a codimension 0 cell and that these cells form an $\mathcal{MCG}$ invariant partition of the decorated Teichmueller space. Thus Penner's cellulation descends to a cellulation of a decorated moduli space. Moduli space is not compact but it does have a natural (geometric) compactification – *the Deligne Mumford compactification.* The points that one adds correspond to so-called stable curves i.e. a surface where a family of homotopically distinct simple closed curves have been "pinched". An interesting and difficult problem is to understand how Penner's cellulation extends to a "decorated the Deligne Mumford compactification".

With Penner we developped a theory of screens – combinatorial objects that encode the how the adjacencies between cells in Penner's cellulation. Further we characterize short curves in a surface in terms of the screen:

Theorem 3.2. *For any fatgraph G, the cell $C(G)$ admits as short curves a family K of non-parallel and non-puncture parallel disjointly embedded and essential simple closed curves in F if and only if $K = \partial A$ for some screen A on G.*

This is a first step in understanding how Penner's construction might extend to the compactification. Subsequently, Penner and D. Lafountain (Filtered Screens and Augmented Teichmueller Space) introduced refined the arguments and achieved some progress towards a complete description. Currently, with Pennerwe are working on a related, more geometric approach which will hopefully yield a complete understanding.

4. Dynamics of the mapping class group on the character variety

The Teichmueller space of a surface embeds in the character variety of the fundamental group The modular group acts on the character variety via its action on the fundamenta group:

$$f^* : \rho \mapsto \rho \circ f_*^{-1},$$

and this action extends the usual action on Teichmueller space. Goldman studied this action for the punctured torus giving (essentially) a complete description of the orbit structure.

In this case, and Goldman's work is facilitated by the fact that the modular group is just the group of (outer) automorphisms of a free group on 2-generators, that the character variety can be identified with $\mathbb{R}^3$ and by the existence of an explicit invariant function. This allows one to define a *relative character variety* – essentially a level set of the function– and work in two dimensions. Following a general principle one expects that the action restricted to the relative character variety should be:

- wandering on a subset DF which consists of discrete faithful representations
- ergodic on the complement of DF and, in particular, that there are no "exotic components" that is wnadering sets in the complement of DF.

4.1. **Non Orientable surfaces.** Goldman and his student Stantchev considered the action on a subset of the $SL(2,\mathbb{C})$-character variety of the free group on 2-generators associated to (possibly singular) structures on non orientable surfaces. This space can be decomposed into 4 components $-\chi_{i,j},\ i,j \in \{0,1\}$ and the action of the modular group permutes three of these components composant $\chi_{1,1}, \chi_{0,1}, \chi_{1,0}$. As in the case of the punctured torus, one can identify $\chi_{1,1}$ with $\mathbb{R}^3$ in a natural way so that there is an induced action of the stabiliser $\Gamma_{1,1}$ of $\chi_{1,1}$. Again there is an invariant function κ for which the level sets are planes :

$$\kappa(x,y,z) := -x^2 - y^2 + z^2 - xyz + 2.$$

With Goldman, Stanchev and S.P. Tan we extend preliminary results from Stanchev's thesis. In particular, we study a subset, called the *Fricke space*, containing representations associated to (possibly singular) structures on the the two holed projective plane $C(0,2)$ and the one-holed Klein bottle $C(1,1)$. We prove:

Theorem 4.1. *Suppose that $\kappa < 2$ Then:*

- $\Gamma_{1,1}$ *acts properly on the Fricke space of the two holed projective plane*
- *the action is ergodic on the complement of this set in the relative character variety*

When $\kappa = 2$, the action is ergodic.
Suppose that $\kappa < 2$ Then:

- $\Gamma_{1,1}$ *acts properly on the Fricke space of he one-holed Klein bottle*
- *the complement of this set in the relative character variety is empty*

This result is just part of an extensive study of the dynamics of the character variety.

A notable feature is that the dihedral characters are in the closure of the domain of discontinuity (which is not the case for the representations considered in Goldman's analysis of the punctured torus.) These correspond to strong degenerations of a hyperbolic structure on a Klein bottle with one cone point (as the angle approaches 2π. These degenerations lie on the boundary of the generalized Fricke orbits, but are not generic points on the boundary.

4.2. **Singular structures on a 3 punctured sphere.** Degenerations of a hyperbolic structure on a surface with a single cone point approaching 2π, mentioned in the previous paragraph, are of interest more generally. Do and Norbury found a so-called differential relation which relates the volume of the moduli space of singular surface with a cone point to that of a smooth surface obtained by forgetting the cone point. Their procedure is valid for cone angles less than π by work of Tan, Wong and Zhang. We study the moduli space of a hyperbolic surface with a single cone point of angle ranging from 0 to 2π. Whilst authors such as Schumacher and Trappani have made extensive studies of the complex geometry of singular surfaces there is very liitle in the literature concerning the hyperbolic geometry of surfaces with the exceptions of existence theorems for metrics in a given conformal class (Kahzdan, Troyanov).

Our approach is via a coordinate system closely related to Penner's λ-lengths. We use this to

- compute the action of the mapping class group in these coordinates
- give an explicit expression for Wolpert's symplectic form
- justify Do and Norbury's approach using just hyperbolic geometry.

5. Volumes of hyperbolic manifolds

Whilst visiting Tokyo Tech I began working on volumes of hyperbolic manifolds. With H. Masai we obtained new results on volumes of hyperbolic manifolds with totally geodesic boundary in all dimensions. With S. Kojima we relate the volume of a closed hyperbolic manifold homeomorphic to the suspension of a translation distance of a pseudo-Anasov diffeomorphism ψ of a surface to the translation distance of ψ for the Weil-Petersson and the Teichmueller metrics on Teichmueller space.

5.1. **Identities and volumes.** Bridgeman-Kahn and Calegari derived formulae for the volumes of compact hyperbolic n-manifolds with totally geodesic boundary in terms of the orthospectrum of the manifold. Both methods for producing the formulae are based on decomposing the unit tangent bundle into countably many pieces, each of which is naturally associated to a unique orthogeodesic. In fact, each of these pieces is congruent to a model piece, respectively $\mathcal{B}(l)$ for the Bridgeman-Kahn decomposition and $\mathcal{C}(l)$ for Calegari's, determined up to isometry by the length l of the corresponding orthogeodesic $\alpha*$. So the volume of the unit tangent bundle can be expressed as a sum of the volumes of these pieces and each volume only depends on the length of an orthogeodesic. The formulae obtained are valid for all compact hyperbolic n-manifolds with totally geodesic boundary, however, the decompositions used by Bridgeman-Kahn and Calegari are quite different. It is natural to ask how the terms in the two formula are related. We show that the two formulae coincide, that is, for each orthogeodesic the associated Bridgeman-Kahn model piece and the Calegari model piece have the same volume regardless of the dimension.

Theorem 5.1. *For all $n \geq 2$,*

$$\mathrm{vol}_n(\mathcal{B}(l)) = \mathrm{vol}_n(\mathcal{C}(l)).$$

We note that for $n = 2$, Calegari obtained this result by direct computation. Our method is geometric, we will show that the pair of sets $\mathcal{B}(l)$ and $\mathcal{C}(l)$ satisy a property which we call *countable equidecomposability.* This generalises the familiar notion of scissors congruence by allowing decompositions having countably many pieces rather than just finitely many.

Both Bridgeman and Calegari found a closed expressions for the volume of the 2 dimensional pieces and they give integral formulae for $\mathrm{vol}_n(\mathcal{B}(l))$ and $\mathrm{vol}_n(\mathcal{C}(l))$ respectively in all dimensions $n \geq 2$. When n is odd and in particular in dimension 3, Calegari's decomposition is more convenient for purposes of calculation. We exploit this to give a closed form for the volume of the pieces in terms of an ortholength l.

Theorem 5.2.

$$\mathrm{vol}_3(\mathcal{C}(l)) = \frac{2\pi(l+1)}{e^{2l}-1}.$$

5.2. **Entropy and volumes of 3-manifolds.** A celebrated theorem by Thurston asserts that the suspension of a surface homeomorphism φ N_φ admits a hyperbolic structure iff φ is pseudo-Anosov. By Mostow-Prasad rigidity a hyperbolic structure of finite volume in dimension 3 is unique and geometric invariants are in fact topological invariants. Kin, Takasawa and Kojima compared the hyperbolic volume of N_φ, denoted by $\mathrm{vol}\, N_\varphi$, with the entropy of φ, denoted by $\mathrm{ent}\, \varphi$. By *entropy* we mean the infimum of the topological entropy of automorphisms isotopic to φ. In particular, they proved that there is a constant $C(g,m)>0$ depending only on the topology of Σ such that

$$\operatorname{ent}\varphi \geq C(g,m)\,\mathrm{vol} N_\varphi.$$

This result only asserts the existence of a constant $C(g,m)$ since the proof is based on a result of Brock involving several constants for which, a priori, it appears difficult to compute sharp values. On the other hand, it is well known that the infimum of $\operatorname{ent}\varphi/\mathrm{vol}\,N_\varphi$ is 0. In fact, Penner constructed examples which demonstrate that, as the complexity of surface increases, the entropy of a pseudo-Anosov can be arbitrarily close to 0. By Jørgensen-Thurston Theory, the infimum of volumes of hyperbolic 3-manifolds is strictly positive so $C(g,m)$ necessarily tends to 0 when $g+m\to\infty$.

Our main theorem gives an explicit value for $C(g,m)$

Theorem 5.3. *The inequality,*

$$\operatorname{ent}\varphi \geq \frac{1}{3\pi|\chi(\Sigma)|}\,\mathrm{vol}\,N_\varphi, \tag{3}$$

or equivalently,

$$2\pi|\chi(\Sigma)|\operatorname{ent}\varphi \geq \frac{2}{3}\,\mathrm{vol}\,N_\varphi \tag{4}$$

holds for any pseudo-Anosov φ.

The quantity appearing on the left hand side of (4) is often referred to as the *normalized entropy.* The main theorem can thus be restated informally: "the normalized entropy over the volume is bounded from below by a psoitive constant which does not depend on the topology of Σ".

The value of $C(g,m)$ is close to what is conjectured to be optimal. For example, choose the case that the surface is the punctured torus, so that $g=1$, $m=1$ and $|\chi(\Sigma_{1,1})|=1$. Then the inequality (3) becomes

$$\frac{\operatorname{ent}\varphi}{\mathrm{vol}\,N_\varphi} \geq \frac{1}{3\pi} = 0.10610\ldots$$

In this particular case, it is conjectured that

$$\frac{\operatorname{ent}\varphi}{\mathrm{vol}\,N_\varphi} \geq \frac{\log\frac{3+\sqrt{5}}{2}}{2v_3} = 0.47412\ldots,$$

where $v_3=1.01494\ldots$ is the volume of the hyperbolic regular ideal simplex. The conjectured constant above is known to be attained by the figure eight knot complement which admits a unique $\Sigma_{1,1}$-fibration.

The first application of Theorem 5.3, is an amelioration of the lower bound,

$$\operatorname{ent}\varphi \geq \frac{\log 2}{4(3g-3+m)},$$

for the entropy of pseudo-Anosovs on a surface due to Bob Penner.

Corollary 5.4. *Let φ be a pseudo-Anosov on $\Sigma_{g,m}$ with $m\geq 1$. Then*

$$\operatorname{ent}\varphi \geq \frac{2v_3}{3\pi|\chi(\Sigma)|} = \frac{2v_3}{3\pi(2g-2+m)}.$$

Proof. It is known by Cao and Meryerhoff in that the smallest volume of an orientable noncompact hyperbolic 3-manifold is attained by the figure eight knot complement and it is $2v_3$. Thus replacing $\operatorname{vol} N_\varphi$ in (3) by $2v_3$, we obtain the estimate. $\square$

The second application of our main theorem is the proof of a slightly weaker form of Farb, Leininger and Margalit's finiteness theorem for small dilitation pseudo Anosovs.

Corollary 5.5. *For any $C > 0$, there are finitely many cusped hyperbolic 3-manifolds M_k such that any pseudo-Anosov φ on Σ with $|\chi(\Sigma)| \operatorname{ent} \varphi < C$ can be realized as the monodromy of a fibration on a manifold obtained from one of the M_k by an appropriate Dehn filling.*