EQUIDECOMPOSABILITY, VOLUME FORMULAE AND ORTHOSPECTRA

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Abstract. Bridgeman-Kahn and Calegari derived formulae for the volumes of compact hyperbolic n-manifolds with totally geodesic boundary in terms of the orthospectrum. Their methods are apparently different from each other, and computing the volume of different subspaces of unit tangent bundle of hyperbolic n-space. In this paper, we show that the two volume formulae coincide. We also derive a closed form of the formula in dimension 3.

Bridgeman-Kahn and Calegari derived formulae for the volumes of compact hyperbolic n-manifolds with totally geodesic boundary in terms of the orthospectrum of the manifold. Both methods for producing the formulae are based on decomposing the unit tangent bundle into countably many pieces, each of which is naturally associated to a unique orthogeodesic. In fact, each of these pieces is congruent to a model piece, respectively $\mathcal{B}(l)$ for the Bridgeman-Kahn decomposition and $\mathcal{C}(l)$ for Calegari's, determined up to isometry by the length l of the corresponding orthogeodesic $\alpha*$. So the volume of the unit tangent bundle can be expressed as a sum of the volumes of these pieces and each volume only depends on the length of an orthogeodesic. The formulae obtained are valid for all compact hyperbolic n-manifolds with totally geodesic boundary, however, the decompositions used by Bridgeman-Kahn and Calegari are quite different. It is natural to ask how the terms in the two formula are related. We show that the two formulae coincide, that is, for each orthogeodesic the associated Bridgeman-Kahn model piece and the Calegari model piece have the same volume regardless of the dimension.

Theorem 1. *For all $n \geq 2$,*

$$\mathrm{vol}_n(\mathcal{B}(l)) = \mathrm{vol}_n(\mathcal{C}(l)).$$

We note that for $n = 2$, Calegari [4] obtained this result by direct computation. Our method is geometric, we will show that the pair of sets $\mathcal{B}(l)$ and $\mathcal{C}(l)$ satisfy a property which we call *countable equidecomposability* (Section 5). This generalises the familiar notion of scissors congruence by allowing decompositions having countably many pieces rather than just finitely many.

Both Bridgeman and Calegari found a closed expressions for the volume of the 2 dimensional pieces and they give integral formulae for $\mathrm{vol}_n(\mathcal{B}(l))$ and $\mathrm{vol}_n(\mathcal{C}(l))$ respectively in all dimensions $n \geq 2$. When n is odd and in particular in dimension 3, Calegari's decomposition is more convenient for purposes of calculation. We exploit this to give a closed form for the volume of the pieces in terms of an ortholength l.

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Theorem 2.

$$\mathrm{vol}_3(\mathcal{C}(l)) = \frac{2\pi(l+1)}{e^{2l}-1}.$$

In two dimensions the volume of each piece turns out to be the Rogers' dilogarithm of a simple function of the ortholength [2], [4]. This case is of particular interest since the deformation theory of convex surfaces leads yields functional relations for the dilogarithm. However, as one sees from the formula above, in three dimensions the volume of each piece can be written in terms of the ortholength and its exponential. The deformation theory of hyperbolic 3 manifolds which have totally geodesic boundary is trivial and no functional relations are to be expected.

More on closed forms for the volume of pieces. Calegari has also used his method to compute an expression for the terms in the 3-dimensional Basmajian identity [1]. In particular, for a compact hyperbolic 3-manifold M with totally geodesic boundary ∂M, one has

$$-\chi(\partial M) = \sum_{\alpha^*} \frac{4}{e^{\ell(2\alpha^*)}-1}$$

where the sum is over all orthogeodesics α^*. Compare this with Bridgeman-Kahn

$$2\mathrm{vol}_3(M) = \sum_{\alpha^*} \frac{\ell(\alpha^*)+1}{e^{2\ell(\alpha^*)}-1}.$$

Thus both the volume of the 3-manifold and the area of the boundary are determined by the orthospectrum. Moreover, these quantities are written as series over the orthospectra and the terms are expressed using just standard functions.

Whilst in all dimensions $n \geq 2$ the term in Basmajian's identity is readily computable in terms of standard functions (see [4]), there is a curious dichotomy for the Bridgeman-Kahn and Calegari identities which we will explain briefly. Both Bridgeman-Kahn and Calegari give integral formulae for the terms in the series that yields the volume of the hyperbolic n-manifold. Bridgeman-Kahn note further that, when n is even, it is possible to find a closed expression for this term involving usual functions and the dilogarithm ([3] Corollary 8). On the other hand, Calegari gives an integral expression which readily yields a closed expression for the terms, again involving usual functions and the dilogarithm, but *only when n is odd* (Proposition 6 below). By Theorem 1 Bridgeman-Kahn's and Calegari's terms are identical so this situation is somewhat puzzling and we can offer no explanation.

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1. Preliminaries

1.1. **Orthogeodesics and the orthospectrum.** We consider compact hyperbolic manifolds M of finite volume with non-empty totally geodesic boundary ∂M. Such a manifold is obtained as the quotient of a convex subset of $\mathbb{H}^n$ by a group of orientation preserving isometries Γ. The limit set Λ of Γ is a non empty Γ-invariant closed subset of $\partial\mathbb{H}^n$ which, by a theorem of Ahlfors, is measure 0. Henceforth, we identify M with the quotient of the convex hull of Λ by Γ.

The complement of Λ consists of countably many $n-1$ dimensional balls and the convex hull of each ball is a half space bounded by a totally geodesic $n-1$-dimensional hyperplane which we call a *side of the convex hull*. To each side D_a there is a subgroup $\Gamma_{D_a} < \Gamma$ consisting of all the elements such that $g(D_a) = D_a$. The quotient of the side by Γ_{D_a} is a compact, totally geodesic, $n-1$-dimensional manifold embedded in the boundary of ∂M.

Let D_a, D_b be a pair of sides then the *orthogeodesic connecting* D_a *and* D_b is the shortest geodesic arc $\hat{\alpha}^*$ with an endpoint in D_a and the other in D_b. The length of the orthogeodesic $\ell(\hat{\alpha}^*)$ is the minimum distance between D_a and D_b. If $\hat{\alpha}^*$ is the orthogeodesic joining a pair of sides of the convex hull then its image α^* in the quotient $\mathrm{CH}(\Lambda)/\Gamma$ is an orthogeodesic of length $\ell(\hat{\alpha}^*)$ joining a pair of totally geodesic boundary components. We call the set of ortholengths $\ell(\alpha^*)$ (with multiplicities in $\mathrm{CH}(\Lambda)/\Gamma$) the *orthospectrum of* $CH(\Lambda)/\Gamma$. Less formally the orthospectrum of the manifold $\mathrm{CH}(\Lambda)/\Gamma$ is the set of lengths of common perpendiculars between (not necessarily distinct) boundary components.

1.2. **The unit tangent bundle.** Our arguments are based on elementary hyperbolic geometry. We shall use the following (standard) notation throughout.

We denote $p : T\mathbb{H}^n \to \mathbb{H}^n$ the canonical map that associates to a tangent vector its basepoint. Let A be an isometry (diffeomorphism) of $\mathbb{H}^n$ then it induces a diffeomorphism of the tangent bundle which we continue to denote by A.

If $v \in T\mathbb{H}^n$ is a (non zero) tangent vector then

$$\gamma_v : \mathbb{R} \to \mathbb{H}^n$$

is the unique geodesic parameterised by arclength such that $\dot{\gamma}_v(0)$ is a positive multiple of v. The geodesic γ_v determines a pair of distinct points $\gamma_v(\pm\infty)$ in the ideal boundary of $\mathbb{H}^n$. Observe that the map

$$\begin{array}{ccc} v & \mapsto & \gamma_v(-\infty) \\ T\mathbb{H}^n & \to & \partial\mathbb{H}^n \end{array}$$

is continuous and, in particular, the preimage of any measurable subset of $\partial\mathbb{H}^n$ is a measurable subset of the tangent bundle.

Whenever we speak of a geodesic $\hat{\alpha}$ in $\mathbb{H}^n \cup \partial\mathbb{H}^n$ we mean the union of a geodesic $\hat{\alpha}$ and its ideal endpoints $\alpha^{\pm}$.

As discussed in [3], the unit tangent bundle $T_1\mathbb{H}^n$ has a standard volume form Ω, which is just the product of the standard volume forms on $\mathbb{H}^n$ and S^{n-1}. To obtain an explicit formula for $d\Omega$, it is convenient to try to parametrize unit tangent vectors by triples

$$(x, y, t) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}.$$

We consider the upper half space model of $\mathbb{H}^n$ so that the ideal boundary is identified with $\mathbb{R}^{n-1} \cup \{\infty\}$. A point $v \in T_1\mathbb{H}^n$ determines a unique directed geodesic γ_v and so an ordered pair of points $(\gamma_v(-\infty), \gamma_v(\infty))$ in the ideal boundary $\mathbb{R}^{n-1} \cup \{\infty\}$ and, provided neither of these points is ∞, we may set $(x, y) = (\gamma_v(-\infty), \gamma_v(\infty))$. The last coordinate $t \in \mathbb{R}$ is the signed hyperbolic length between the highest point of γ_v and $p(v)$. Our parametrization is defined on a open dense subset of $T_1\mathbb{H}^n$ and it is easy to check that the complement has measure zero so we may ignore its

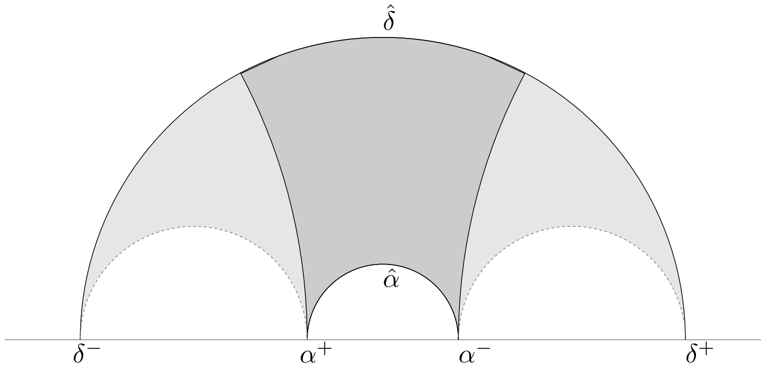

Figure 1: The quadrilateral $\mathcal{Q}$ and its chimney.

contribution when we compute volumes in $T_1\mathbb{H}^n$. With this parametrization, we have

$$d\Omega = \frac{2dV(x)dV(y)dt}{|x-y|^{2n-2}},$$

where $dV(x) = dx_1 dx_2 \cdots dx_{n-1}$ for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^{n-1}$.

2. Subsets of the tangent bundle

Let $\hat{\delta}$ and $\hat{\alpha}$ be a pair of disjoint geodesics in the $\mathbb{H} \cup \partial\mathbb{H}$. The convex hull of $\hat{\delta}$ and $\hat{\alpha}$ is an ideal quadrilateral $\mathcal{Q}$ (see figure). The problem is to show that a certain pair of subsets of the unit tangent bundle of $\mathbb{H}$, have the same volume.

- *Bridgeman's set* $\mathcal{B}(l)$ is the set of unit vectors v tangent to geodesic segments joining $\hat{\alpha}$ to $\hat{\delta}$. More formally, it is the set of $v \in p^{-1}(\mathcal{Q})$ satisfying
  (1) the ray $\gamma_v(\mathbb{R}_+)$ meets $\hat{\delta}$,
  (2) the ray $\gamma_v(\mathbb{R}_-)$ meets $\hat{\alpha}$.
  We shall call this set the set of *bridging vectors*.
- *Calegari's set* $\mathcal{C}(l)$ is the set of unit vectors v such that
  (1) the ray $\gamma_v(\mathbb{R}_+)$ meets $\hat{\delta}$,
  (2) the point $p(v)$ is in the chimney (see below) of the quadrilateral $\mathcal{Q}$.

The *chimney* is the dark subset of the ideal quadrilateral in Figure (1) it is the convex hull of $\hat{\alpha} \cup \{\alpha^+, \alpha^-\}$ and the nearest point retraction of $\hat{\alpha}$ to $\hat{\delta}$. Following Calegari, we say that $\hat{\alpha}$ is the *top of the chimney* and the nearest point retraction of $\hat{\alpha}$ to $\hat{\delta}$ is its *base* which we will denote by B. The chimney is a convex quadrilateral with the top and the base forming a pair of sides and we refer to the remaining pair of sides as the *walls*.

The top of the chimney bounds an interval $T_\infty = [\alpha^+, \alpha^-] \subset \mathbb{R}$ and it is easy to see that the nearest point retraction

$$\pi_{\hat{\delta}} : T_\infty \to \hat{\delta}$$

is a homeomorphism.

Our Theorem 1 says that the sets $\mathcal{B}(l), \mathcal{C}(l)$ have the same volume. We will begin by discussing in detail the case $n = 2$ in the next section (compare Calegari in [5].)

3. Comparing the sets $\mathcal{B}(l), \mathcal{C}(l)$

Calegari [4] defines a positive function $h : \Delta \to \mathbb{R}^+$ on the chimney whose value is the density in the unit tangent sphere at a point x of vectors $v \in \mathcal{C}(l)$. Thus, by Fubini's theorem, the volume of $\mathcal{C}(l)$ is the integral of h over the chimney Δ. In two dimensions the value of $h(x)$ is the visual measure of $\hat{\delta}$ at x, and this is just the angle between the pair of geodesics passing through x and the ideal points $\delta^{\pm}$ (Figure 3).

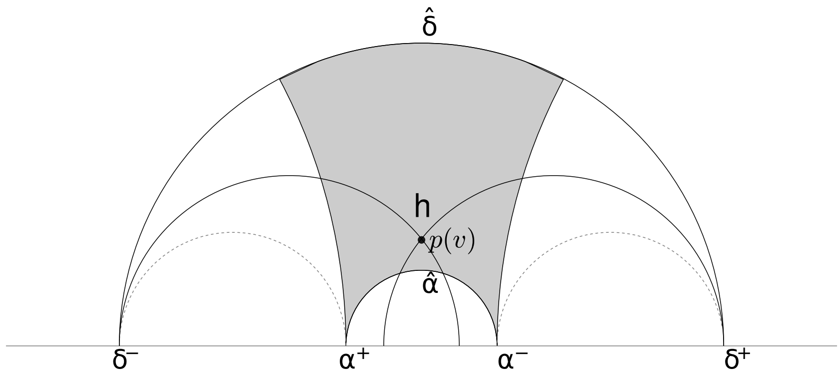

Figure 2: Calegari's angle

To see that the sets $\mathcal{B}(l), \mathcal{C}(l)$ have the same volume we show how one can associate to a vector in Calegari's set $\mathcal{C}(l)$ a unique vector tangent to one of Bridgeman's geodesic arcs: more formally that there is a measure preserving bijection ϕ between a subset of full measure of Brigeman's set and a subset of full measure of Calegari's set.

In the next figure we see one of Bridgeman's arcs γ that passes through $p(v)$ and exits the chimney by a side. By the uniqueness of geodesics in $\mathbb{H}$, that the tangent vector to γ at $x \in \gamma \cap$ chimney is in $\mathcal{C}(l)$; so on this portion we can take ϕ to be the identity. The problem is thus to define such a map for vectors v tangent to γ at points $p(v)$ in the complement of the chimney.

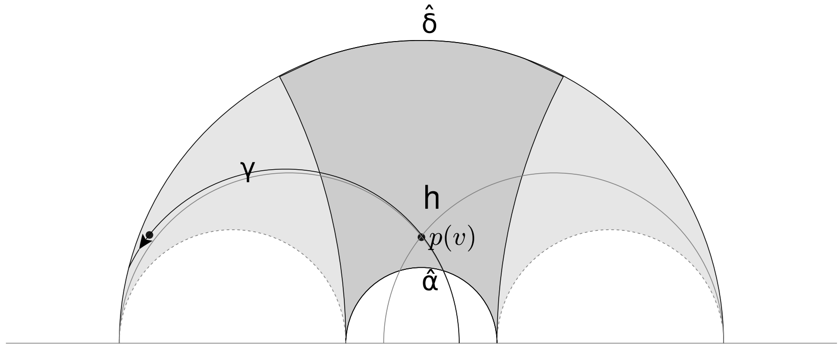

Figure 3: How do I account for all v tangent to the geodesic γ?

4. Constructing a pair of measurable bijections

We begin by covering by the quadrilateral with copies of the chimney. Let A denote the sidepairing that takes one of the walls of the chimney to the other. We may suppose that the attracting fixed point of A is δ^+ and the repelling fixed point δ^-. Note that the isometry A is *not* an element of the covering group Γ.

Now we have a cover of the quadrilateral by translates of the chimney

$$\mathcal{Q} \subset \bigcup_k A^k(\text{chimney}).$$

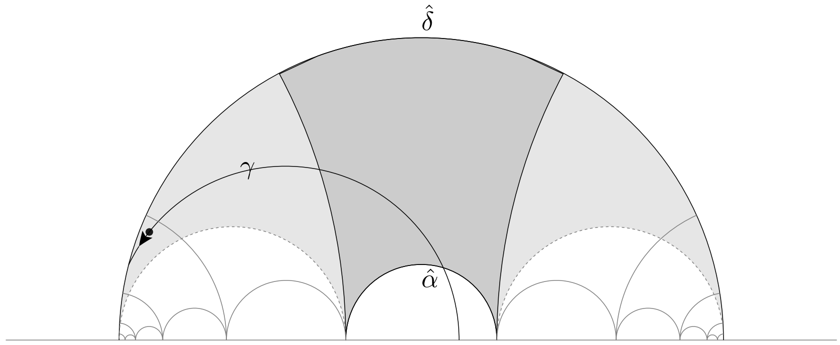

Figure 4: Covering using the chimney, the point $x = p(v)$ is marked on the left.

If $x = p(v)$ is a point of γ then (up to a set of measure 0) there is a unique power of A such that $A^n(x)$ is in the chimney. Since A preserves $\hat{\delta}$ we have $A^n(\gamma) \cap \hat{\delta} \neq \emptyset$ so the image of v by A^n is tangent to a geodesic, namely $A^n(\gamma)$, that exits via $\hat{\delta}$. This of course means that $A^n(v)$ is in Calegari's set $\mathcal{C}(l)$.

We define a map

$$\begin{array}{rcl} f : v & \mapsto & A^n(v),\ p(v) \in A^{-n}(\text{chimney}) \\ \mathcal{B}(l) & \rightarrow & \text{tangent vectors to chimney} \end{array}$$

Note that, for each $k \in \mathbb{Z}$, the set $p^{-1}(A^k(\text{chimney}))$ is a measurable subset of the tangent bundle so the map is

- well defined on a subset of full measure in $\mathcal{B}(l)$
- measure preserving where it is defined since A is an isometry.

To show that it is a bijection (on a set of full measure) we construct an (almost everywhere defined) inverse. Let v be a vector in $\mathcal{C}(l)$, by definition $\gamma_v(\mathbb{R}_+)$ meets $\hat{\delta}$, and consider the ideal point $\gamma_v(-\infty)$. Since geodesics meet at most once this is a point of the interval $]\delta^-, \delta^+[$. This interval is tessellated by the intervals $A^k([\alpha^+, \alpha^-])$ so one can define a map :

$$\begin{array}{rcl} \phi : v & \mapsto & A^n(v),\ \text{ if } \gamma_v(-\infty) \in A^{-n}(]\alpha^+, \alpha^-[) \\ \mathcal{C}(l) & \rightarrow & \text{tangent vectors to the ideal quadrilateral } \mathcal{Q}. \end{array}$$

Since v is tangent to a geodesic that joins $\hat{\delta}$ to $A^{-n}(\hat{\alpha})$, its image $\phi(v)$ is tangent to a vector that joins $\hat{\delta}$ to $\hat{\alpha}$.

4.1. **Inverses.** It is easy to check that these maps are inverses by examining the last figure.

- the map f breaks our geodesic arc γ into 3 pieces by translation by A and intersection with the chimney.
- ϕ takes each of these pieces, extends it to find an ideal endpoint in the interval $]\delta^-, \delta^+[\subset \mathbb{R}$, then glues them back using this data to make the original arc.

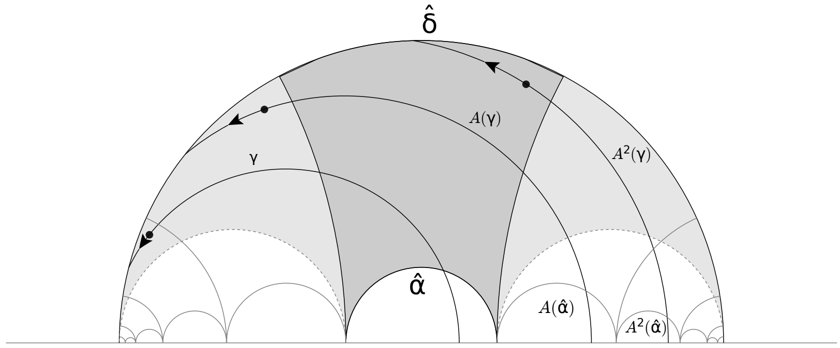

Figure 5: Translates of the geodesic γ and the point $p(v)$.

5. Equidecomposability and volume

A more succinct way of thinking of this is that $\mathcal{B}(l)$ and $\mathcal{C}(l)$ contain sets of full measure that are countable equidecomposable (we define this below). Recall that a pair of subsets X, Y of a metric space are *equidecomposable* if there exists a pair of decompositions

$$X = \sqcup_{k=0}^{N} X_k,\ Y = \sqcup_{k=0}^{N} Y_k$$

and isometries ϕ_k such that $Y_k = \phi_k(X_k)$.

We extend this definition slightly and say that subset X, Y of a metric space are *countable equidecomposable* if there exists a pair of decompositions

$$X = \sqcup_{k=0}^{\infty} X_k,\ Y = \sqcup_{k=0}^{\infty} Y_k$$

and isometries ϕ_k such that

$$Y_k = \phi_k(X_k).$$

If the metric space is equipped with a countably additive measure μ, such that the isometries are measure preserving maps, then we conclude that X and Y have the same μ-volume provided the pieces X_k, Y_k are measurable.

5.1. **Equidecomposability in dimension 2.** Using this notion we reformulate the ideas of the previous section.

Theorem 3. *Let $\Delta \subset \mathcal{Q}$ be a chimney and $\dot{\Delta}$ its interior. Then there are subsets of full measure $X \subset \mathcal{B}(l)$ and $Y \subset \mathcal{C}(l)$ that are countable equidecomposable.*

Proof: Define

$$\begin{aligned} X_k &:= \{v \in \mathcal{B}(l),\, p(v) \in A^{-k}(\dot{\Delta})\} \\ Y_k &:= \{v \in \mathcal{C}(l),\, \gamma_v(\mathbb{R}_-) \cap A^k(\hat{\alpha}) \neq \emptyset\}. \end{aligned}$$

It is easy to see that the sets

$$X = \sqcup X_k,\, Y = \sqcup Y_k$$

are full measure in $\mathcal{B}(l)$ and $\mathcal{C}(l)$ respectively.

Finally we show that $\phi_k = A^k$. Suppose $v \in \mathcal{B}(l)$ then $\gamma_v(\mathbb{R}_-) \cap \hat{\alpha} \neq \emptyset$ so that $A^k(v)$ satisfies

$$\gamma_{A^k(v)}(\mathbb{R}_-) \cap A^k(\hat{\alpha}) \neq \emptyset. \tag{1}$$

Further, if $v \in X_k$ then

$$p(A^k(v)) = A^k(p(v)) \in \dot{\Delta}. \tag{2}$$

From (1) and (2) one has $A^k(X_k) \subset Y_k$. The other inclusion follows by a similar argument.

□

5.2. **Equidecomposability from covers.** In fact, to construct appropriate decompositions X_i, Y_i, one only really needs a countable cover of the interval $]\delta^-, \delta^+[$ by A-translates of $T_\infty = [\alpha^+, \alpha^-]$. Note that such a cover determines a countable cover of the geodesic $\hat{\delta}$ via nearest point retraction to $\hat{\delta}$.

Theorem 4. *Let A_k, $A_0 = I$ be hyperbolic isometries*

1. *preserving the components of $\partial\mathbb{H} \setminus \{\delta^+, \delta^-\}$*
2. *such that $\{A_k(B)\}_{k=0}^{\infty}$ cover $\hat{\delta}$.*

Then there are measurable sets X_k, Y_k such that

$$\mathcal{B}(l) = \sqcup_k X_k,\, \mathcal{C}(l) = \sqcup_k Y_k$$

and $A_k(X_k) = Y_k$, $\forall k \geq 0$.

Proof: The base B is the image under the nearest point retraction to $\hat{\delta}$ of a disc $T_\infty \subset \partial\mathbb{H}$. Since the $A_k(B)$ cover $\hat{\delta}$, the translates $\{A_k(T_\infty)\}_{k=0}^{\infty}$ cover the component of $\partial\mathbb{H} \setminus \{\delta^+, \delta^-\}$ containing the endpoints $\gamma_v(-\infty)$, $v \in \mathcal{C}(l)$. We define a map

$$\begin{array}{rcl} \sigma : \mathcal{C}(l) & \rightarrow & \mathbb{Z}^+ \\ \sigma(v) & = & \inf\{k : \gamma_v(-\infty) \in A_k(T_\infty)\}. \end{array}$$

By construction σ has the property that $\sigma^{-1}(\{0, \dots k\})$ is precisely the union $\cup_{j=1}^{k} A_k(T_\infty)$ which is a measurable set. It follows that

$$Y_k := \sigma^{-1}(\{k\}) = \sigma^{-1}(\{0, \dots k\}) \setminus \sigma^{-1}(\{0, \dots k-1\}),$$

is measurable too. Analogously we define the set X_k to be $\tau^{-1}(\{k\})$ where

$$\begin{array}{rcl} \tau : \mathcal{B}(l) & \rightarrow & \mathbb{Z}^+ \\ \tau(v) & = & \inf\{k : p(v) \in A_k^{-1}(\Delta)\}. \end{array}$$

Exactly same argument as before shows that $A_k(X_k) = Y_k$.

□

6. Equidecomposability in higher dimensions

Let D_b and D_a be a pair of disjoint totally geodesic hyperplanes $\mathbb{H}^n$ and consider the unit tangent bundle over their convex hull $\mathcal{Q}$. The walless chimney is the convex hull of D_a and its image under nearest point retraction to D_b which we still denote B. The *top of the chimney* is D_a and its *base* is B. The top of the chimney bounds a round disc $T_\infty \subset \partial\mathbb{H}^n$ and the base B is an open round ball.

The sets $\mathcal{B}(l)$ and $\mathcal{C}(l)$ are defined as before:

- $\mathcal{B}(l)$ is the set of unit tangent vectors v
  (1) the ray $\gamma_v(\mathbb{R}_+)$ meets D_b,
  (2) the ray $\gamma_v(\mathbb{R}_-)$ meets D_a.
- $\mathcal{C}(l)$ is the set of unit tangent vectors v such that
  (1) the ray $\gamma_v(\mathbb{R}_+)$ meets D_b,
  (2) the point $p(v)$ is in the walless chimney.

If $n > 2$ then the plane D_b does not admit a tiling by copies of B. However, it does admit a covering by open discs congruent to B; for example choose any countable dense subset $P \subset D_b$ and take the cover by discs with centers in P and radius equal to that of B. Any such cover yields a covering of T_∞ by taking preimages under the nearest point retraction.

Theorem 5. *Let A_k, $A_0 = I$ be hyperbolic isometries*

(1) *preserving the components of $\partial\mathbb{H} \setminus \partial D_b$*
(2) *such that $\{A_k(B)\}_{k=0}^\infty$ cover D_b.*

Then there are measurable sets X_k, Y_k such that

$$\mathcal{B}(l) = \sqcup_k X_k,\ \mathcal{C}(l) = \sqcup_k Y_k$$

and $A_k(X_k) = Y_k$, $\forall k \geq 0$.

Proof: Note now that D_b is a totally geodesic hyperplane, that is a copy of $\mathbb{H}^{n-1}$ embedded in $\mathbb{H}^n$, so that its boundary ∂D_b is a round sphere in $\partial\mathbb{H}^n$. The sphere separates $\partial\mathbb{H}^n$ into two round balls, one of which is disjoint from Λ.

The base B is the image under the nearest point retraction to D_b of a disc $T_\infty \subset \partial\mathbb{H}^n$. Since the $A_k(B)$ cover D_b, the translates $\{A_k(T_\infty)\}_{k=0}^\infty$ cover the component of $\partial\mathbb{H}^n \setminus \partial D_b$ whose closure contains the limit set and hence the endpoints $\gamma_v(-\infty)$, $v \in \mathcal{C}(l)$.

The rest of the proof is as before.

□

7. Three dimensional case

In this section we show that the volume of Calegari's piece for dimension 3 can be expressed in terms of the ortholength and its exponential. See also [6] for elementary hyperbolic geometry.

7.1. **Integral formula for the volume of Calegari's piece** [4]**.** Recall that $h : \mathcal{C}(l) \to \mathbb{R}_+$ denotes Calegari's angle function (see Figure 3) and that $\mathrm{vol}_n(\mathcal{C}(l))$ is the integral of h over a chimney Δ. Consider the level sets and value of h on the level sets. Let C_t denote the subset of the chimney at distance t from the base, it is easy to see that the value of h is constant on C_t. We begin by discussing the geometry of these level sets and for this we need to study Lambert quadrilaterals.

7.1.1. *Lambert quadrilaterals.* Let $Q(\phi)$ be a *Lambert quadrilateral*, that is a hyperbolic quadrilateral with three right angles and an angle $\phi < \pi/2$, (figure 6). Observe that there is a pair of edges which have both endpoints at right angles. If l and $\iota_\phi(l)$ denote the lengths of these edges then, from elementary hyperbolic geometry, we have:

$$\sinh(\iota_\phi(l)) = \cos(\phi)/\sinh(l).$$

Further, if $\phi = 0$, then we write $\iota(l) := \iota_0(l)$, and we have following equivalent formulae

$$1/\cosh^2(l) + 1/\cosh^2(\iota(l)) = 1 \Leftrightarrow \cosh(\iota(l)) = 1/\tanh(l) \Leftrightarrow \sinh(\iota(l)) = 1/\sinh(l).$$

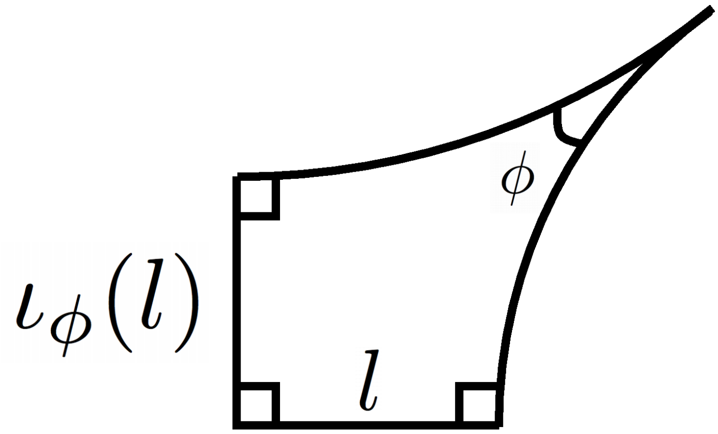

Figure 6: A Lambert quadrilateral

Now the nearest point retraction of C_t to the base of the chimney is surjective if $t \leq l$, and otherwise the image is an annulus with outer radius $\iota(l)$, and inner radius $\iota_\phi(l)$, where ϕ is defined implicitly by $\sin(\phi) = \cosh(l)/\cosh(t)$. This latter formula in turn yields that $\cosh(\iota_\phi(l)) = \frac{\tanh(t)}{\tanh(l)}$ since

$$\begin{aligned}
\cosh^2(\iota_\phi(l)) &= \sinh^2(\iota(l))\cos^2(\phi) + 1 \\
&= \sinh^2(\iota(l))\left(1 - \frac{\cosh^2(l)}{\cosh^2(t)}\right) + 1 \\
&= \frac{\cosh^2(t) - \cosh^2(l) + \sinh^2(l)\cosh^2(t)}{\sinh^2(l)\cosh^2(t)} \\
&= \frac{\cosh^2(t)(1+\sinh^2(l)) - \cosh^2(l)}{\sinh^2(l)\cosh^2(t)} \\
&= \frac{(\cosh^2(t) - 1)\cosh^2(l)}{\sinh^2(l)\cosh^2(l)} \\
&= \frac{\tanh^2(t)}{\tanh^2(l)}.
\end{aligned}$$

7.1.2. *Volumes of balls in hyperbolic or spherical space.* Let $V_n^H(r)$ (resp. $V_n^S(r)$) denote the volume of a ball of radius r in hyperbolic (resp. spherical) n-space. The following integral fomula for V_n^H and V_n^S are well known [7]:

$$V_n^H(r) = \Omega_{n-1}\int_0^r \sinh^{n-1}(t)dt,$$
$$V_n^S(r) = \Omega_{n-1}\int_0^r \sin^{n-1}(t)dt$$

where $\Omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the volume of a euclidean sphere of dimension n-1 and radius 1.

7.1.3. *Calegari's integral formula.* We now state the integral formula which will allow us to calculate $\mathrm{vol}_3(\mathcal{C}(l))$. Observe that, for $q \in \mathcal{C}(l)$, $h(q)$ depends only on the distance t between the base of chimney and q. Writing h as a function of t yields

$$h(q) = \Omega_{n-1}^{-1}V_{n-1}^S(\arcsin(1/\cosh(t))),$$

where $\arcsin(1/\cosh(t))$ is the angle between vertical line and the boundary of its t-neighborhood. Using this, Calegari [4] obtains a formula of the volume of the piece $\mathcal{C}(l)$ with ortholength l, namely:

$$\begin{aligned}\frac{1}{2}\mathrm{vol}_n(\mathcal{C}(l)) = &\int_0^l \cosh^{n-1}(t)V_{n-1}^H(\iota(l))V_{n-1}^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}dt\\ &+\int_l^\infty \cosh^{n-1}(t)(V_{n-1}^H(\iota(l)) - V_{n-1}^H(\iota_\phi(l)))V_{n-1}^S(\arcsin(1/\cosh(t)))\Omega_{n-1}^{-1}dt.\end{aligned}$$

7.2. **The volume of Calegari's piece of dimension 3.** We are ready to give an explicit expression for the volume of a piece in dimesion three. It is more convenient to work with Calegari's decomposition (compare [3] Section 4.2) and we have to evaluate the following integral:

(3)
$$\begin{aligned}\frac{1}{2}\mathrm{vol}_3(\mathcal{C}(l)) = &\int_0^l \cosh^2(t)V_2^H(\iota(l))V_2^S(\arcsin(1/\cosh(t)))\Omega_2^{-1}dt\\ &+\int_l^\infty \cosh^2(t)(V_2^H(\iota(l)) - V_2^H(\iota_\phi(l)))V_2^S(\arcsin(1/\cosh(t)))\Omega_2^{-1}dt.\end{aligned}$$

By easy computation, we have

$$V_2^H(r) = 2\pi(\cosh(r) - 1),$$
$$V_2^S(r) = 2\pi(1 - \cos(r)).$$

Then by substitution, we have

$$V_2^S(\arcsin(1/\cosh(t))) = 2\pi(1 - \sqrt{1 - 1/\cosh^2(t)} = 2\pi(1 - \tanh(t)).$$

Now we compute the first term of (3),

$$\begin{aligned}
&\int_0^l \cosh^2(t) V_2^H(\iota(l)) V_2^S(\arcsin(1/\cosh(t)))\Omega_2^{-1} dt \\
&= \pi \int_0^l \cosh^2(t)(\cosh(\iota(l)) - 1))(1 - \tanh(t))dt \\
&= \pi(\cosh(\iota(l)) - 1) \int_0^l \cosh^2(t)(1 - \tanh(t))dt \\
&= \frac{\pi}{2}(\cosh(\iota(l)) - 1) \int_0^l (e^{-2t} + 1)dt \\
&= \frac{\pi}{4}(\cosh(\iota(l)) - 1)(-e^{2l} + 2l + 1).
\end{aligned}$$

We calculate the second term of (3) using $\cosh(\iota_\phi(l)) = \frac{\tanh(t)}{\tanh(l)}$, and $\cosh(\iota(l)) = 1/\tanh(l)$:

$$\begin{aligned}
&\int_l^\infty \cosh^2(t)(V_2^H(\iota(l)) - V_2^H(\iota_\phi(l))) V_2^S(\arcsin(1/\cosh(t)))\Omega_2^{-1} dt \\
&= \pi \int_l^\infty \cosh^2(t)(\cosh(\iota(l)) - \frac{\tanh(t)}{\tanh(l)})(1 - \tanh(t))dt \\
&= \pi \cosh(\iota(l)) \int_l^\infty \cosh^2(t) + \sinh^2(t) - 2\cosh(t)\sinh(t)dt \\
&= \pi \cosh(\iota(l)) \int_l^\infty e^{-2t} dt \\
&= \frac{\pi}{2} \cosh(\iota(l)) e^{-2l}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\frac{\mathrm{vol}_3(\mathcal{C}(l))}{2} &= \frac{\pi}{4}(\cosh(\iota(l)) - 1)(-e^{2l} + 2l + 1) + \frac{\pi}{2}\cosh(\iota(l))e^{-2l} \\
&= \frac{\pi}{4}\left((\frac{1}{\tanh(l)} + 1)e^{-2l} + (\frac{1}{\tanh(l)} - 1)(2l + 1) \right) \\
\iff \frac{\mathrm{vol}_3(\mathcal{C}(l))}{2} &= \frac{\pi(l+1)}{e^{2l} - 1}.
\end{aligned}$$

and this completes the proof of Theorem 2.

8. A closed formula in odd dimensions

For completeness we include a proof that the volume of each piece $\mathcal{C}(l)$ is *computable*, that is, can be expressed in terms of standard functions and the dilogarithm.

Proposition 6. $\mathrm{vol}_n(\mathcal{C}(l))$ *can be expressed in terms of elementary functions if the dimension* n *is odd.*

Proof: Since $V_{n-1}^H(\iota(l))$ and Ω_{n-1}^{-1} do not depend on t, it suffices to consider

$$\int_0^l \cosh^{n-1} V_{n-1}^S(\arcsin(1/\cosh(t)))dt, \text{ and} \tag{4}$$

$$\int_l^\infty \cosh^{n-1}(t)(-V_{n-1}^H(\iota_\phi(l))) V_{n-1}^S(\arcsin(1/\cosh(t)))dt. \tag{5}$$

Integrating by parts yields $V^S_{n-1}(r) = \sum_{i=0}^{n-1} q_i \sin(r)^i \cos(r)^{n-1-i}$ for some $q_i \in \mathbb{Q}$. Since $\cos(\arcsin(1/\cosh(t))) = 1/\tanh(t)$, we see that the integrand of (4) is a rational function of e^t, and hence the integral (4) is computable. For (5), we need the assumption that n is odd. Let $n = 2m+1$, then by partial integral,

$$\begin{aligned} V^H_{n-1}(r) = V^H_{2m}(r) &= \int_0^r \sinh^{2m-1}(t)dt \\ &= [\cosh(t)(1-\cosh^2(t))^{m-1}]_0^r - (2m-2)\left(V^H_{2m-2} + V_2m^H\right). \end{aligned}$$

As we see in section 7.1.1, $\cosh(\iota_\phi(l)) = \tanh(t)/\tanh(l)$. (Note that if n is even, the integrand contains $\sinh(\iota_\phi(l))$ and we need square root for $\sinh(\iota_\phi(l))$). Hence if n is odd, the integrand of (5) is also a rational function of e^t. Hence, the integral (5) can be evaluated in terms of usual functions and dilogarithms, just as Bridgeman-Kahn's integral when n is even, but now provided that n is odd.

□

In the light of this proposition, we illustrate the difficulty involved in directly comparing the volumes $\mathrm{vol}_n(\mathcal{B}(l))$ and $\mathrm{vol}_n(\mathcal{C}(l))$ in dimensions $n \geq 3$. We consider just the case of $\mathrm{vol}_3(\mathcal{B}(l))$ for which one has the following formula from Section 4.2 of[3]:

$$\mathrm{vol}_3(\mathcal{B}(l)) = 2\int_0^1 \frac{M_3\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}{\sqrt{1-r^2}}dr,$$

where

$$M_3(x) = \frac{2}{x^2-1}(1-\log(2)) - \frac{1}{2x}\left(\frac{x-1}{x+1}\right)\log(x-1) + \frac{1}{2x}\left(\frac{x+1}{x-1}\right)\log(x+1).$$

By easy computation,

$$\mathrm{vol}_3(\mathcal{B}(l)) = \frac{\pi(1-\log(2))}{a^2-1} + 2\int_0^1 \frac{N_3\left(\sqrt{\frac{a^2-r^2}{1-r^2}}\right)}{\sqrt{1-r^2}}dr,$$

where

$$N_3(x) = -\frac{1}{2x}\left(\frac{x-1}{x+1}\right)\log(x-1) + \frac{1}{2x}\left(\frac{x+1}{x-1}\right)\log(x+1).$$

By changing variables, we have,

$$\mathrm{vol}_3(\mathcal{B}(l)) = \frac{\pi(1-\log(2))}{e^{2l}-1} + \sqrt{e^{2l}-1}\int_0^\infty \left\{\frac{\log(e^l\cosh x+1)}{(e^l\cosh x-1)^2} - \frac{\log(e^l\cosh x-1)}{(e^l\cosh x+1)^2}\right\}dx.$$

Remark. We substitute $a = e^l$, because $a\cosh x$ might be confused with $\operatorname{arccosh} x$.

Now we have as a corollary of Theorem 1 and Theorem 2

Corollary 7.

$$\int_0^\infty \left\{\frac{\log(e^l\cosh x+1)}{(e^l\cosh x-1)^2} - \frac{\log(e^l\cosh x-1)}{(e^l\cosh x+1)^2}\right\}dx = \frac{\pi}{\sqrt{e^{2l}-1}}(l+\log(2)).$$

We note that a simple change of variable $u = \cosh x$ yields an integral which one might be tempted to evaluate using contour integration in $\mathbb{C}$. However, since the derivative of $\cosh x$ is $\sinh x$ there is a factor $(u^2-1)^{-\frac{1}{2}}$ which seems difficult to deal with using standard techniques.

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