# Introduction to actions of algebraic groups

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Abstract. These notes present some fundamental results and examples in the theory of algebraic group actions, with special attention to the topics of geometric invariant theory and of spherical varieties. Their goal is to provide a self-contained introduction to more advanced lectures.

## Introduction

These notes are based on lectures given at the conference "Hamiltonian actions: invariants and classification" (CIRM Luminy, April 6 - April 10, 2009). They present some fundamental results and examples in the theory of algebraic group actions, with special attention to the topics of geometric invariant theory and of spherical varieties.

Geometric invariant theory provides very powerful tools for constructing and studying moduli spaces in algebraic geometry. On the other hand, spherical varieties form a remarkable class of algebraic varieties with algebraic group actions. They generalize several important subclasses such as toric varieties, flag varieties and symmetric varieties, and they satisfy many stability and finiteness properties. The classification of spherical varieties by combinatorial invariants is an active research domain, and one of the main topics of the conference.

The goal of these notes is to provide a self-contained introduction to more advanced lectures by Paolo Bravi, Ivan Losev and Guido Pezzini on spherical and wonderful varieties, and by Chris Woodward on geometric invariant theory and its relation to symplectic reduction.

Here is a brief overview of the contents. In the first part, we begin with basic definitions and properties of algebraic group actions, including the construction of homogeneous spaces under linear algebraic groups. Next, we introduce and discuss geometric and categorical quotients, in the setting of reductive group actions on affine algebraic varieties. Then we adapt the construction of categorical quotients to the projective setting.

The prerequisites for this part are quite modest: we assume familiarity with fundamental notions of algebraic geometry, but not with algebraic groups. It should also be emphasized that we only present the most basic notions and results of the theory; for example, we do not present the Hilbert-Mumford criterion. We refer to the notes of Woodward for this and further developments; the books by Dolgachev (see [1]) and Mukai (see [8]) may also be recommended, as well as the classic by Mumford, Kirwan and Fogarty (see [9]).

The second part is devoted to spherical varieties, and follows the same pattern as the first part: after some background material on representation theory of connected reductive groups (highest weights) and its geometric counterpart ($U$-invariants), we obtain fundamental characterizations and finiteness properties of affine spherical varieties. Then we deduce analogous properties in the projective setting, and we introduce some of their combinatorial invariants: weight groups, weight cones and moment polytopes. The latter also play an important role in Hamiltonian group actions.

In this second part, we occasionally make use of some structure results for reductive groups (the open Bruhat cell, minimal parabolic subgroups), for which we refer to Springer's book [13]. But apart from that, the prerequisites are still minimal. The books by Grosshans (see [3]) and Kraft (see [5]) contain a more thorough treatment of $U$-invariants; the main problems and latest developments on the classification of spherical varieties are exposed in the notes by Bravi, Losev and Pezzini.

# 1. Geometric invariant theory

## 1.1. Algebraic group actions: basic definitions and properties

Throughout these notes, we consider algebraic varieties (not necessarily irreducible) over the field $\mathbb{C}$ of complex numbers. These will just be called *varieties*, and equipped with the Zariski topology (as opposed to the complex topology) unless otherwise stated. The algebra of regular functions on a variety $X$ is denoted by $\mathbb{C}[X]$; if $X$ is affine, then $\mathbb{C}[X]$ is also called the *coordinate ring.* The field of rational functions on an irreducible variety $X$ is denoted by $\mathbb{C}(X)$.

**Definition 1.1.** An *algebraic group* is a variety $G$ equipped with the structure of a group, such that the multiplication map

$$\mu : G \times G \longrightarrow G, \quad (g, h) \longmapsto gh$$

and the inverse map

$$\iota : G \longrightarrow G, \quad g \longmapsto g^{-1}$$

are morphisms of varieties.

The *neutral component* of an algebraic group $G$ is the connected component $G^0 \subset G$ that contains the neutral element $e_G$.

**Examples 1.2.** 1) Any *finite group* is algebraic.

2) The *general linear group* $\mathrm{GL}_n$, consisting of all invertible $n \times n$ matrices with complex coefficients, is the open subset of the space $\mathrm{M}_n$ of $n \times n$ complex matrices (an affine space of dimension $n^2$) where the determinant $\Delta$ does not vanish. Thus, $\mathrm{GL}_n$ is an affine variety, with coordinate ring generated by the matrix coefficients $a_{ij}$, where $1 \leq i, j \leq n$, and by $\frac{1}{\Delta}$. Moreover, since the coefficients of the product $AB$ of two matrices (resp. of the inverse of $A$) are polynomial functions of the coefficients of $A$, $B$ (resp. of the coefficients of $A$ and $\frac{1}{\Delta}$), we see that $\mathrm{GL}_n$ is an affine algebraic group.

3) More generally, any closed subgroup of $\mathrm{GL}_n$ (i.e., defined by polynomial equations in the matrix coefficients) is an affine algebraic group; for example, the *special linear group* $\mathrm{SL}_n$ (defined by $\Delta = 1$), and the other classical groups. Conversely, all affine algebraic groups are linear, see Corollary 1.13 below.

3) The *multiplicative group* $\mathbb{C}^*$ is an affine algebraic group, as well as the *additive group* $\mathbb{C}$. In fact, $\mathbb{C}^* \cong \mathrm{GL}_1$ whereas $\mathbb{C}$ is isomorphic to the closed subgroup of $\mathrm{GL}_2$ consisting of matrices of the form $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.

4) Let $\mathrm{T}_n \subset \mathrm{GL}_n$ denote the subgroup of diagonal matrices. This is an affine algebraic group, isomorphic to $(\mathbb{C}^*)^n$ and called an $n$-dimensional *torus.*

Also, let $\mathrm{U}_n \subset \mathrm{GL}_n$ denote the subgroup of upper triangular matrices with all diagonal coefficients equal to 1. This is a closed subgroup of $\mathrm{GL}_n$, isomorphic as a variety to the affine space $\mathbb{C}^{n(n-1)/2}$. Moreover, $\mathrm{U}_n$ is a nilpotent group, its ascending central series consists of the closed subgroups $Z_k(U_n)$ defined by the vanishing of the matrix coefficients $a_{ij}$, where $1 \leq j - i \leq n - k$, and each quotient $Z_k(U_n)/Z_{k-1}(U_n)$ is isomorphic to $\mathbb{C}^k$.

The closed subgroups of $\mathrm{U}_n$ are called *unipotent.* Clearly, any unipotent group is nilpotent; moreover, each successive quotient of its ascending central series is a closed subgroup of some $\mathbb{C}^k$, and hence is isomorphic to some $\mathbb{C}^\ell$.

5) Every smooth curve of degree 3 in the projective plane $\mathbb{P}^2$ has the structure of an algebraic group (see e.g. [4, Proposition IV.4.8]). These *elliptic curves* yield examples of projective, and hence non-affine, algebraic groups.

We now gather some basic properties of algebraic groups:

**Lemma 1.3.** *Any algebraic group $G$ is a smooth variety, and its (connected or irreducible) components are the cosets $gG^0$, where $g \in G$. Moreover, $G^0$ is a closed normal subgroup of $G$, and the quotient group $G/G^0$ is finite.*

*Proof.* The variety $G$ is smooth at some point $g$, and hence at any point $gh$ since the multiplication map is a morphism. Thus, $G$ is smooth everywhere.

Since $G$ is a disjoint union of cosets $gG^0$, and each of them is connected, they form the connected components of $G$; in particular, there are finitely many cosets. Also, the inverse $\iota(G^0)$ is a connected component (since $i$ is an isomorphism of varieties), and hence equals $G^0$. Thus, for any $g \in G^0$, the coset $gG^0$ contains $e_G$; hence $gG^0 = G^0$. Therefore, $G^0$ is a closed subgroup of $G$. Likewise, $gG^0g^{-1} = G^0$ for all $g \in G$. ☐

**Definition 1.4.** A *$G$-variety* is a variety $X$ equipped with an action of the algebraic group $G$,

$$\alpha : G \times X \longrightarrow X, \quad (g, x) \longmapsto g \cdot x$$

which is also a morphism of varieties. We then say that $\alpha$ is an *algebraic $G$-action.*

Any algebraic action $\alpha : G \times X \to X$ yields an action of $G$ on the coordinate ring $\mathbb{C}[X]$, via

$$(g \cdot f)(x) := f(g^{-1} \cdot x)$$

for all $g \in G$, $f \in \mathbb{C}[X]$ and $x \in X$. This action is clearly linear.

**Lemma 1.5.** *With the preceding notation, the complex vector space $\mathbb{C}[X]$ is a union of finite-dimensional $G$-stable subspaces on which $G$ acts algebraically.*

*Proof.* The action morphism $\alpha$ yields an algebra homomorphism

$$\alpha^{\#} : \mathbb{C}[X] \longrightarrow \mathbb{C}[G \times X], \quad f \longmapsto \big((g, x) \mapsto f(g \cdot x)\big),$$

the associated coaction. Since $\mathbb{C}[G \times X] = \mathbb{C}[G] \otimes \mathbb{C}[X]$, we may write

$$f(g \cdot x) = \sum_{i=1}^{n} \varphi_i(g)\psi_i(x),$$

where $\varphi_1, \dots, \varphi_n \in \mathbb{C}[G]$ and $\psi_1, \dots, \psi_n \in \mathbb{C}[X]$. Then

$$g \cdot f = \sum_{i=1}^{n} \varphi_i(g^{-1})\psi_i$$

and hence the translates $g \cdot f$ span a finite-dimensional subspace $V \subset \mathbb{C}[G]$. Clearly, $V$ is $G$-stable. Moreover, we have $h \cdot (g \cdot f) = \sum_{i=1}^{n} \varphi_i(g^{-1}h^{-1})\psi_i$ for any $g, h \in G$, and the functions $h \mapsto \varphi_i(g^{-1}h^{-1})$ are all regular. Thus, the $G$-action on $V$ is algebraic. ☐

This result motivates the following:

**Definition 1.6.** A *rational $G$-module* is a complex vector space $V$ (possibly of infinite dimension) equipped with a linear action of $G$, such that every $v \in V$ is contained in a finite-dimensional $G$-stable subspace on which $G$ acts algebraically.

Examples of rational $G$-modules include coordinate rings of $G$-varieties, by Lemma 1.5. Also, note that the finite-dimensional $G$-modules are in one-to-one correspondence with the homomorphisms of algebraic groups $f : G \to \mathrm{GL}_n$ for some $n$, i.e., with the finite-dimensional algebraic representations of $G$.

Some linear actions of an algebraic group $G$ do not yield rational $G$-modules; for example, the $G$-action on $\mathbb{C}(G)$ via left multiplication, if $G$ is irreducible and non-trivial. However, we shall only encounter rational $G$-modules in these notes, and just call them *$G$-modules* for simplicity. Likewise, the actions of algebraic groups under consideration will be assumed to be algebraic as well.

**Example 1.7.** Let $G = \mathbb{C}^*$; then

$$\mathbb{C}[G] = \mathbb{C}[t, t^{-1}] = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}\, t^n.$$

Given a $\mathbb{C}^*$-variety $X$, any $f \in \mathbb{C}[X]$ satisfies

$$(t \cdot f)(x) = f(t^{-1} \cdot x) = \sum_{n=-\infty}^{\infty} t^n\, f_n(x),$$

where the $f_n \in \mathbb{C}[X]$ are uniquely determined by $f$. In particular, $f = \sum_n f_n$. Since $tt' \cdot f = t \cdot (t' \cdot f)$ for all $t, t' \in \mathbb{C}^*$, we obtain

$$t \cdot f_n(x) = t^n f_n(x)$$

for all $t \in \mathbb{C}^*$ and $x \in X$. This yields a decomposition

$$\mathbb{C}[X] = \bigoplus_{n=-\infty}^{\infty} \mathbb{C}[X]_n,$$

where each $t \in \mathbb{C}^*$ acts on $\mathbb{C}[X]_n$ via multiplication by $t^n$. It follows that the product in $\mathbb{C}[X]$ satisfies

$$\mathbb{C}[X]_m \mathbb{C}[X]_n \subset \mathbb{C}[X]_{m+n}$$

for all $m, n$, i.e., the preceding decomposition is a $\mathbb{Z}$-*grading* of the algebra $\mathbb{C}[X]$.

Conversely, any finitely generated $\mathbb{Z}$-graded algebra without non-zero nilpotent elements yields an affine $\mathbb{C}^*$-variety, by reversing the preceding construction. Also, the $\mathbb{C}^*$-modules correspond to $\mathbb{Z}$-graded vector spaces.

More generally, the coordinate ring of the $n$-dimensional torus $\mathrm{T}_n$ is the algebra of Laurent polynomials,

$$\mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}] = \bigoplus_{(a_1,\dots,a_n)\in\mathbb{Z}^n} \mathbb{C} t_1^{a_1} \cdots t_n^{a_n},$$

and the actions of $\mathrm{T}_n$ on affine varieties (resp. the $\mathrm{T}_n$-modules) correspond to $\mathbb{Z}^n$-graded affine algebras (resp. vector spaces).

**Definition 1.8.** Given two $G$-varieties $X$, $Y$, a morphism of varieties $f : X \to Y$ is *equivariant* if it satisfies $f(g \cdot x) = g \cdot f(x)$ for all $g \in G$ and $x \in X$. We then say that $f$ is a *$G$-morphism.*

**Proposition 1.9.** *Let $G$ be an affine algebraic group, and $X$ an affine $G$-variety. Then $X$ is equivariantly isomorphic to a closed $G$-subvariety of a finite-dimensional $G$-module.*

*Proof.* We may choose finitely many generators $f_1, \dots, f_n$ of the algebra $\mathbb{C}[X]$. By Lemma 1.5, the translates $g \cdot f_i$, where $g \in G$ and $i = 1, \dots, n$, are all contained in a finite-dimensional $G$-submodule $V \subset \mathbb{C}[X]$. Then $V$ also generates the algebra $\mathbb{C}[X]$, and hence the associated evaluation map

$$\iota : X \longrightarrow V^*, \quad x \longmapsto \big(v \mapsto v(x)\big)$$

is a closed immersion; $\iota$ is equivariant by construction. □

**Definition 1.10.** Given a $G$-variety $X$ and a point $x \in X$, the *orbit* $G \cdot x \subset X$ is the set of all $g \cdot x$, where $g \in G$. The *isotropy group* (also called the *stabilizer*) $G_x \subset G$ is the set of those $g \in G$ such that $g \cdot x = x$; it is a closed subgroup of $G$.

Here are some fundamental properties of orbits and their closures:

**Proposition 1.11.** *With the preceding notation, the orbit $G \cdot x$ is a locally closed, smooth subvariety of $X$, and every component of $G \cdot x$ has dimension* $\dim(G) - \dim(G_x)$. *Moreover, the closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and of orbits of strictly smaller dimension. Any orbit of minimal dimension is closed; in particular, $\overline{G \cdot x}$ contains a closed orbit.*

*Proof.* By Lemma 1.3, $G \cdot x$ consists of finitely many orbits of $G^0$; moreover, $(G_x)^0 \subset (G^0)_x \subset G_x$ and these closed subgroups have all the same dimension. As a consequence, we may assume $G$ to be connected.

Consider the orbit map

$$\alpha_x : G \longrightarrow X, \quad g \longmapsto g \cdot x.$$

Clearly, $\alpha_x$ is a morphism with fiber $G_{g \cdot x} = g G_x g^{-1}$ at any $g \in G$, and with image $G \cdot x$. Thus, $G \cdot x$ is a constructible subset of $X$, and hence contains a dense open subset of $\overline{G \cdot x}$. Since $G$ acts transitively on $G \cdot x$, this orbit is open in its closure, and is smooth. The formula for its dimension follows from a general result on the dimension of fibers of morphisms (see e.g. [14, Corollary 15.5.5]), and the remaining assertions are easily checked. □

**Examples 1.12.** 1) Consider the action of the multiplicative group $\mathbb{C}^*$ on the affine $n$-space $\mathbb{C}^n$ by scalar multiplication:

$$t \cdot (x_1, \dots, x_n) := (t x_1, \dots, t x_n).$$

Then the origin 0 is the unique closed orbit, and the orbit closures are exactly the lines through 0.
2) Let $\mathbb{C}^*$ act on $\mathbb{C}^2$ via

$$t \cdot (x, y) := (t x, t^{-1} y).$$

Then the closed orbits are the origin and the "hyperbolae" $(xy = c)$, where $c \neq 0$. The other orbit closures are the coordinate axes.
3) The natural action of $\mathrm{SL}_2$ on $\mathbb{C}^2$ has 2 orbits: the origin and its complement. The isotropy group of the first basis vector $e_1$ is $\mathrm{U}_2 \cong \mathbb{C}$, and the open orbit $\mathrm{SL}_2 \cdot e_1 = \mathbb{C}^2 \setminus \{0\}$ is a classical example of a non-affine variety.
4) Consider the action of the product $\mathrm{GL}_m \times \mathrm{GL}_n$ on the space $\mathrm{M}_{m,n}$ of $m \times n$ matrices, via

$$(A, B) \cdot C := BCA^{-1}.$$

Then the orbits are exactly the matrices of a prescribed rank $r$, where $0 \leq r \leq \min(m, n)$. In particular, there is an open orbit, consisting of matrices of maximal rank, and the origin is the unique closed orbit.

Applying Proposition 1.11 to algebraic group homomorphisms yields the following:

**Corollary 1.13.** (i) *Let $f : G \to H$ be a homomorphism of algebraic groups. Then the image of $f$ is a closed subgroup. If the kernel of $f$ is trivial, then $f$ is a closed immersion.*
(ii) *Any affine algebraic group is linear.*

*Proof.* (i) Consider the action of $G$ on $H$ via $g \cdot h := f(g)h$. Then there exists a closed orbit by Proposition 1.11. But the orbits are all isomorphic via the action of $H$ by right multiplication; hence $f(G) = G \cdot e_H$ is closed.

If $f$ has a trivial kernel, then it yields a bijective morphism $\varphi : G \to f(G)$. Since $f(G)$ is a smooth variety, it follows that $\varphi$ is an isomorphism by a corollary of Zariski's Main Theorem (see e.g. [14, Corollary 17.4.7]).

(ii) Let $G$ be an affine algebraic group, acting on itself by left multiplication. For the corresponding action on the algebra $\mathbb{C}[G]$, we may find a finite dimensional $G$-submodule $V$ which generates that algebra. The induced homomorphism $G \to \mathrm{GL}(V)$ is injective, and thus a closed immersion by (i). $\square$

Another useful observation on orbits is the following semi-continuity result:

**Lemma 1.14.** *Let $G$ be an algebraic group and $X$ a $G$-variety. Then the set*

$$\{x \in X \mid \dim(G \cdot x) \leq n\}$$

*is closed in $X$ for any integer $n$. Equivalently, the sets*

$$\{x \in X \mid \dim(G_x) \geq n\}$$

*are all closed in $X$.*

*Proof.* Consider the morphism

$$\beta : G \times X \longrightarrow X \times X, \quad (g, x) \longmapsto (x, g \cdot x).$$

Then the fiber of $\beta$ at any point $(g, x)$ is $(gG_x, x)$; thus, all irreducible components of this fiber have the same dimension, $\dim(G_x)$. Now the second assertion follows from semi-continuity of the dimension of fibers of a morphism (see e.g. [14, Theorem 15.5.7]). $\square$

Next, we obtain an important result due to Chevalley:

**Theorem 1.15.** *Let $G$ be a linear algebraic group, and $H \subset G$ a closed subgroup. Then there exists a finite-dimensional $G$-module $V$ and a line $\ell \subset V$ such that the stabilizer $G_\ell$ is exactly $H$.*

*Proof.* Consider the action of $G$ on itself by left multiplication. Then the stabilizer of the closed subvariety $H$ is $H$ itself. Thus, $H$ is also the stabilizer of the ideal $I(H) \subset \mathbb{C}[G]$. We may choose a finite-dimensional vector space $W \subset I(H)$ which generates that ideal; since $I(H)$ is an $H$-module, we may further assume that $W$ is $H$-stable. Then $W$ is contained in a finite-dimensional $G$-submodule $V \subset \mathbb{C}[G]$. Clearly, $H$ is the stabilizer of $W$; thus, $H$ is also the stabilizer of the line $\wedge^n W$ of the $G$-module $\wedge^n V$, where $n := \dim(W)$. $\square$

The preceding result may be rephrased in terms of the natural action of $G$ on the projective space $\mathbb{P}(V)$ (the space of lines in $V$): *any closed subgroup of $G$ is the stabilizer of a point in the projectivization of a $G$-module.* In turn, this is the starting point for the construction of quotients of linear algebraic groups by closed subgroups:

**Theorem 1.16.** *Let $G$ be a linear algebraic group, and $H$ a closed subgroup. Then the coset space $G/H$ has a unique structure of $G$-variety that satisfies the following properties:*

(i) *The quotient map $\pi : G \to G/H$, $g \mapsto gH$ is a morphism.*

(ii) *A subset $U \subset G/H$ is open if and only if $\pi^{-1}(U)$ is open.*

(iii) *For any open subset $U \subset G/H$, the comorphism $\pi^{\#}$ yields an isomorphism $\mathbb{C}[U] \cong \mathbb{C}[\pi^{-1}(U)]^H$ (the algebra of $H$-invariant regular functions on $\pi^{-1}(U)$).*

*Moreover, $G/H$ is smooth and quasi-projective.*

*Proof.* We use ideas and results from the theory of schemes (see e.g. [4, Chapter III], especially III.9 and III.10) which are quite relevant in this setting. By Theorem 1.15, we may choose a $G$-module $V$ and a point $x \in \mathbb{P}(V)$ such that $H = G_x$. Let $X := G \cdot x$, and $p : G \to X$, $g \mapsto g \cdot x$ the orbit map. Then $p$ is a surjective $G$-morphism, and its fibers are the cosets $gH$, where $g \in G$. By generic smoothness and equivariance, $\pi$ is a smooth morphism. Hence $\pi$ is open: it satisfies (i) and (ii).

We now show that $p$ satisfies (iii); equivalently, the natural map $\mathcal{O}_X \to (p_*\mathcal{O}_G)^H$ is an isomorphism. Consider the diagram

$$\begin{tikzcd} G \times H \arrow[r, "\mu"] \arrow[d, "p_1"'] & G \arrow[d, "p"] \\ G \arrow[r, "p"] & X \end{tikzcd}$$

where $\mu$ denotes the multiplication map $(g, h) \mapsto gh$, and $p_1$ stands for the first projection. Clearly, this square is commutative; this yields a morphism

$$f : G \times H \longrightarrow G \times_X G$$

(where $G \times_X G$ denotes the fibred product), which is easily seen to be bijective. Moreover, the first projection $G \times_X G \to G$ is smooth, since it is obtained by base change from the smooth morphism $p$. As $G$ is smooth, $G \times_X G$ is smooth as well, and hence $f$ is an isomorphism (by [14, Corollary 17.4.7] again). Since $p$ is flat, this yields an isomorphism of sheaves

$$p^*(p_*\mathcal{O}_G) \cong p_{1*}(\mu^*\mathcal{O}_G).$$

But $p_{1*}(\mu^*\mathcal{O}_G) = p_{1*}\mathcal{O}_{G\times H} = \mathcal{O}_G \otimes \mathbb{C}[H]$. Taking $H$-invariants yields the isomorphism

$$p^*(p_*\mathcal{O}_G)^H \cong \mathcal{O}_G = p^*\mathcal{O}_X.$$

Since $p$ is faithfully flat, this yields in turn the desired isomorphism $(p_*\mathcal{O}_G)^H \cong \mathcal{O}_X$. □

**Definition 1.17.** A variety $X$ is *homogeneous* if it is equipped with a transitive action of an algebraic group $G$. A *homogeneous space* is a pair $(X, x)$, where $X$ is a homogeneous variety, and $x$ a point of $X$ called the *base point.*

By Theorem 1.16, the homogeneous spaces $(X, x)$ under a linear algebraic group $G$ are exactly the quotient spaces $G/H$, where $H := G_x$, with base point the coset $H$.

## 1.2. Quotients of affine varieties by reductive group actions

**Definition 1.18.** Given an algebraic group $G$ and a $G$-variety $X$, a *geometric quotient* of $X$ by $G$ consists of a morphism $\pi : X \to Y$ satisfying the following properties:

(i) $\pi$ is surjective, and its fibers are exactly the $G$-orbits in $X$.

(ii) A subset $U \subset Y$ is open if and only if $\pi^{-1}(U)$ is open.

(iii) For any open subset $U \subset Y$, the comorphism $\pi^{\#}$ yields an isomorphism $\mathbb{C}[U] \cong \mathbb{C}[\pi^{-1}(U)]^H$.

Under these assumptions, the topological space $Y$ may be identified with the orbit space $X/G$ equipped with the quotient topology, in view of (i) and (ii). Moreover, the structure of variety on $Y$ is uniquely defined by (iii) (which may be rephrased as the equality of sheaves $\mathcal{O}_Y = (\pi_*\mathcal{O}_X)^G$). In particular, if $X$ is irreducible, then so is $Y$, and we have the equality of function fields $\mathbb{C}(Y) = \mathbb{C}(X)^G$.

For example, the geometric quotient of a linear algebraic group by a closed subgroup (acting via right multiplication) exists by Theorem 1.16. In general, a geometric quotient need not exist, as seen from the following :

**Examples 1.19.** 1) As in Example 1.12.1, consider the action of $G = \mathbb{C}^*$ on $X = \mathbb{C}^n$ by scalar multiplication. Then there is no geometric quotient, since 0 lies in every orbit closure. But the open subset $\mathbb{C}^n \setminus \{0\}$ admits a geometric quotient, the natural map to the projective space $\mathbb{P}^{n-1}$.
2) Let $G = \mathbb{C}^*$ act on $\mathbb{C}^2$ via $t \cdot (x, y) := (tx, t^{-1}y)$, as in Example 1.12.2. Then $X := \mathbb{C}^2 \setminus \{0\}$ is a $G$-stable open subset in which all orbits are closed, but which admits no geometric quotient. Indeed, $\mathbb{C}(X)^G = \mathbb{C}(xy)$; thus, the $G$-orbits $(x = 0)$ and $(y = 0)$ are not separated by $G$-invariant rational functions.
3) Let $G = \mathbb{C}$ act on $X = \mathbb{C}^3$, viewed as the space of polynomials of degree at most 2 in a variable $x$, by translation on $x$:

$$t \cdot (ax^2 + 2bx + c) := a(x + t)^2 + 2b(x + t) + c = ax^2 + 2(ax + b)t + at^2 + 2bt + c.$$

Then all orbits are closed, and contained in the fibers of the map

$$\pi : \mathbb{C}^3 \longrightarrow \mathbb{C}^2, \quad (a, b, c) \longmapsto (a, ac - b^2).$$

Specifically, the fiber over $(x, y)$ consists of one orbit if $x \neq 0$, and of two orbits if $x = 0$ but $y \neq 0$; all these orbits have trivial isotropy group. Moreover, the fiber over $(0, 0)$ is the line $\ell$ defined by $b = c = 0$, and consisting of the $G$-fixed points. It follows that $\mathbb{C}(X)^G = \mathbb{C}(a, ac - b^2)$ and that $X$ admits no geometric quotient, nor does the $G$-stable open subset $X \setminus \ell$ consisting of orbits with trivial isotropy group.

By a theorem of Rosenlicht (see e.g. [11, Section 2.3]), *any irreducible $G$-variety $X$ contains a non-empty open $G$-stable subset $X_0$ which admits a geometric quotient $Y_0 = X_0/G$.* Then

$$\dim(Y_0) = \dim(X) - \max_{x \in X} \dim(G \cdot x) = \dim(X) - \dim(G) + \min_{x \in X} \dim(G_x)$$

in view of Lemma 1.14 and of Proposition 1.11.

However, as shown by the preceding example, there is no obvious choice for $X_0$. Also, one may look for a quotient of the whole $X$ in a weaker sense; for example, parametrizing the closed orbits. Such a space of closed orbits exists in Examples 1 (where it is just a point) and 2 (the affine line with coordinate $xy$), but not in Example 3.

More generally, we shall show that the space of closed orbits exists for algebraic groups that are reductive in the following sense:

**Definition 1.20.** A linear algebraic group $G$ is *reductive* if it does not contain any closed normal unipotent subgroup.

We shall need a representation-theoretic characterization of reductive groups, based on the following:

**Definition 1.21.** Let $G$ be an algebraic group, and $V$ a (rational) $G$-module. Then $V$ is *simple* (also called *irreducible*) if it has no proper non-zero submodule. $V$ is *semi-simple* (also called *completely reducible*) if it satisfies one of the following equivalent conditions:
(i) $V$ is the sum of its simple submodules.
(ii) $V$ is isomorphic to a direct sum of simple $G$-modules.
(iii) Any submodule $W \subset V$ admits a $G$-stable complement, i.e., a submodule $W'$ such that $V = W \oplus W'$.

**Example 1.22.** Let $G$ be a unipotent group. Then *every simple $G$-module is trivial*, i.e., is isomorphic to $\mathbb{C}$ where $G$ acts trivially. Otherwise, replacing the (nilpotent) group $G$ with a quotient, we may assume that the centre $Z(G)$ acts non-trivially. By Schur's lemma, each $g \in Z(G)$ acts via multiplication by some scalar $\chi(g) \in \mathbb{C}^*$. The assignement $g \mapsto \chi(g)$ yields a group homomorphism $Z(G) \to \mathbb{C}^*$, which must be constant since $Z(G) \cong \mathbb{C}^n$ as varieties. Thus, $Z(G)$ acts trivially, a contradiction.

It follows that *any non-zero module under a unipotent group contains non-zero fixed points.* This is a version of the Lie-Kolchin theorem, see e.g. [13, Theorem 6.3.1].

**Theorem 1.23.** *The following assertions are equivalent for a linear algebraic group $G$:*
(i) *$G$ is reductive.*
(ii) *$G$ contains no closed normal subgroup isomorphic to the additive group $\mathbb{C}^n$ for some $n \geq 1$.*
(iii) *$G$ (viewed as a Lie group) has a compact subgroup $K$ which is dense for the Zariski topology.*
(iv) *Every finite-dimensional $G$-module is semi-simple.*
(v) *Every $G$-module is semi-simple.*

*Proof.* (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) is a deep result whose proof lies beyond the scope of these notes; see for example [12, Chapter 5] and its references.

(iii) $\Rightarrow$ (iv): Let $V$ be a finite-dimensional $G$-module, and $W \subset V$ a submodule. Since $K$ is compact, $W$ admits a $K$-stable complement $W'$. But since $K$ is Zariski dense in $G$, it follows that $W'$ is $G$-stable.

(iv) $\Rightarrow$ (v) easily follows from the fact that each $G$-module is an increasing union of finite dimensional $G$-submodules, and (v) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i): Let $H$ be a closed normal unipotent subgroup of $G$, and consider a non-zero finite-dimensional $G$-module $V$. Then $V^H$ is non-zero as well, by the preceding example. But $V^H$ is stable by $G$, since $H$ is a normal subgroup; thus, $V^H$ admits a $G$-stable complement $W$. Clearly, $W^H = 0$; therefore, $W = 0$ by the preceding argument. In other words, $H$ fixes $V$ pointwise. Since $G \hookrightarrow \mathrm{GL}(V)$ for some $G$-module $V$, it follows that $H$ is trivial. ☐

We now come to the main result of this section:

**Theorem 1.24.** *Let $G$ be a reductive algebraic group, and $X$ an affine $G$-variety. Then:*
(i) *The subalgebra $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ (consisting of regular $G$-invariant functions) is finitely generated.*
(ii) *Let $f_1, \ldots, f_n$ be generators of the algebra $\mathbb{C}[X]^G$. Then the image of the morphism*

$$X \longrightarrow \mathbb{C}^n, \quad x \longmapsto \big(f_1(x), \ldots, f_n(x)\big)$$

*is closed and independent of the choice of $f_1, \ldots, f_n$.*
(iii) *Denote by*

$$\pi = \pi_X : X \longrightarrow X/\!/G$$

*the surjective morphism defined by* (ii). *Then every $G$-invariant morphism $f : X \to Y$, where $Y$ is an affine variety, factors through a unique morphism $\varphi : X/\!/G \to Y$.*
(iv) *For any closed $G$-stable subset $Y \subset X$, the induced morphism $Y/\!/G \to X/\!/G$ is a closed immersion. In other words, the restriction of $\pi_X$ to $Y$ may be identified with $\pi_Y$. Moreover, given another closed $G$-stable subset $Y' \subset X$, we have $\pi_X(Y \cap Y') = \pi_X(Y) \cap \pi_X(Y')$.*
(v) *Each fiber of $\pi_X$ contains a unique closed $G$-orbit.*
(vi) *If $X$ is irreducible, then so is $X/\!/G$. If in addition $X$ is normal, then so is $X/\!/G$.*

*Proof.* The main ingredient is the *Reynolds operator*, defined as follows. For any $G$-module $V$, the invariant subspace $V^G$ admits a unique $G$-stable complement $V_G$, the sum of all non-trivial $G$-submodules of $V$. The Reynolds operator

$$R_V : V \longrightarrow V^G$$

is the projection associated with the decomposition $V = V^G \oplus V_G$. If $f : V \to W$ is a morphism of $G$-modules, and $f^G : V^G \to W^G$ denotes the induced linear map, then clearly $R_W \circ f = f^G \circ R_V$. In particular, if $f$ is surjective, then so is $f^G$.

When $V = \mathbb{C}[X]$, we set

$$R_X := R_{\mathbb{C}[X]} : \mathbb{C}[X] \longrightarrow \mathbb{C}[X]^G.$$

Then $R_X$ *is* $\mathbb{C}[X]^G$*-linear*, i.e., we have for any $a \in \mathbb{C}[X]^G$ and $b \in \mathbb{C}[X]$:

$$R_X(ab) = aR_X(b),$$

as follows by considering the morphism of $G$-modules $\mathbb{C}[X] \to \mathbb{C}[X]$, $b \mapsto ab$.

We now claim that *the ring* $\mathbb{C}[X]^G$ *is Noetherian.* To see this, consider an ideal $I$ of $\mathbb{C}[X]^G$, and the associated ideal $J := I\mathbb{C}[X]$ of $\mathbb{C}[X]$. Then $J$ is $G$-stable, and $J^G = R_X(J) = IR_X\big(\mathbb{C}[X]\big) = I$. Since $\mathbb{C}[X]$ is Noetherian, this implies readily our claim.

In turn, the claim implies assertion (i) in the case that $X$ is a finite-dimensional $G$-module, say $V$. Indeed, the $\mathbb{C}^*$-action on $V$ by scalar multiplication yields a positive grading

$$\mathbb{C}[V] = \bigoplus_{n=0}^{\infty} \mathbb{C}[V]_n$$

where $\mathbb{C}[V]_n$ denotes the space of homogeneous polynomial functions of degree $n$. This grading is clearly $G$-stable, and hence restricts to a positive grading of the subalgebra $\mathbb{C}[V]^G$. Since the latter is Noetherian, it is finitely generated in view of the graded Nakayama lemma.

In the general case, we may equivariantly embed $X$ into a $G$-module $V$; then the surjective $G$-homomorphism $\mathbb{C}[V] \to \mathbb{C}[X]$ induces a surjective homomorphism $\mathbb{C}[V]^G \to \mathbb{C}[X]^G$. Thus, $\mathbb{C}[X]^G$ is finitely generated; this completes the proof of (i).

For (ii), let $I \subset \mathbb{C}[X]^G$ be a maximal ideal, and $J = I\mathbb{C}[X]$ as above. Recall that $J \cap \mathbb{C}[X]^G = I$; in particular, $J \neq \mathbb{C}[X]$. Thus, $J$ is contained in some maximal ideal $M$, and $I = M \cap \mathbb{C}[X]^G$. This algebraic statement translates into the surjectivity of the morphism of affine varieties associated with the inclusion $\mathbb{C}[X]^G \subset \mathbb{C}[X]$.

(iii) The morphism $p$ yields a homomorphism $p^{\#} : \mathbb{C}[Y] \to \mathbb{C}[X]$ with image contained in $\mathbb{C}[X]^G$; this translates into our assertion.

(iv) As above, the surjective $G$-homomorphism $\mathbb{C}[X] \to \mathbb{C}[Y]$ induces a surjective homomorphism $\mathbb{C}[X]^G \to \mathbb{C}[Y]^G$; this implies the first assertion. For the second assertion, denote by $J$ (resp. $J'$) the ideal of $Y$ (resp. $Y'$) in $\mathbb{C}[X]$. Then the ideal of $Y \cap Y'$ is $I + I'$, and the ideal of $Y/\!/G$ (resp. $Y'/\!/G$, $Y/\!/G \cap Y'/\!/G$) in $X/\!/G$ is $I^G$ (resp. $I'^G$, $I^G + I'^G$). But $I^G + I'^G = R_X(I + I') = (I + I')^G$, i.e., $Y/\!/G \cap Y'/\!/G = (Y \cap Y')/\!/G$.

(v) By (iv), $\pi$ maps any two distinct closed orbits $Y, Y' \subset X$ to distinct points of $X/\!/G$.

(vi) The first assertion is obvious. For the second assertion, it suffices to show that $\mathbb{C}[X]^G$ is integrally closed in $\mathbb{C}(X)^G$, since the latter contains the fraction field of $\mathbb{C}[X]^G$. But this follows readily from the assumption that $\mathbb{C}[X]$ is integrally closed in $\mathbb{C}(X)$. □

Note that the above map $\pi$ is uniquely determined by the universal property (iii); it is called a *categorical quotient* (for affine varieties). Also, $X/\!/G$ may be viewed as the space of closed orbits by (v). We now define an open subset of $X$ that turns out to admit a geometric quotient:

**Definition 1.25.** Let $G$ be a reductive group, and $X$ an affine $G$-variety. A point $x \in X$ is *stable* if the orbit $G \cdot x$ is closed in $X$ and the isotropy group $G_x$ is finite. The (possibly empty) set of stable points is denoted by $X^{\mathrm{s}}$.

**Proposition 1.26.** *With the preceding notation and assumptions, $\pi(X^{\mathrm{s}})$ is open in $X/\!/G$, we have $X^{\mathrm{s}} = \pi^{-1}\pi(X^{\mathrm{s}})$ (in particular, $X^{\mathrm{s}}$ is an open $G$-stable subset of $X$), and the restriction $\pi^{\mathrm{s}} : X^{\mathrm{s}} \to \pi(X^{\mathrm{s}})$ is a geometric quotient.*

*Proof.* Let $x \in X^{\mathrm{s}}$ and consider the subset $Y \subset X$ consisting of those points $y$ such that $G_y$ is infinite; equivalently, $\dim(G_y) > 0$. Then $Y$ is closed, $G$-stable and disjoint from $G \cdot x$. Thus, there exists $f \in \mathbb{C}[X]^G$ such that $f(x) \neq 0$ and $f|_Y$ is identically 0. Then the open subset

$$X_f := \{y \in X \mid f(y) \neq 0\} \subset X$$

satisfies $X_f = \pi^{-1}\pi(X_f)$; in particular, $X_f$ is $G$-stable. Moreover, $x \in X_f$, and $G_y$ is finite for any $y \in X_f$. It follows that $G \cdot y$ is closed in $X$ for any such $y$ (otherwise, let $z$ lie in the unique closed $G$-orbit in $\overline{G \cdot y}$. Then $f(z) = f(y)$ by invariance, and hence $z \in X_f$. But $\dim(G_z) = \dim(G) - \dim(G \cdot z) > \dim(G) - \dim(G \cdot y) = 0$, a contradiction). Hence $X_f \subset X^{\mathrm{s}}$; since $\pi(X_f) = (X/\!/G)_f$ is open in $X/\!/G$, it follows that $\pi(X^{\mathrm{s}})$ is open as well, and satisfies $\pi^{-1}\pi(X^{\mathrm{s}}) = X^{\mathrm{s}}$.

If $y \in \pi^{-1}\pi(x)$, then $\overline{G \cdot y} \supset G \cdot x$ and hence $y \in G \cdot x$ by the above argument. In other words, the fibers of $\pi^{\mathrm{s}}$ are exactly the $G$-orbits. This shows property (i) of Definition 1.18. For (ii), it suffices to check that $\pi(U)$ is open in $X/\!/G$ for any open subset $U \subset X^{\mathrm{s}}$. Replacing $U$ with $G \cdot U$, we may assume that $U$ is $G$-stable. Then $Y := X \setminus G \cdot U$ is a closed $G$-stable subset of $X$, and hence $\pi(F)$ is closed in $X/\!/G$; this implies our assertion. Finally, (iii) holds for those open subsets $U \subset \pi(X^{\mathrm{s}})$ of the form $(X/\!/G)_f$; since these form a basis of the topology of $\pi(X^{\mathrm{s}})$, it follows that (iii) holds for an arbitrary $U$. □

**Examples 1.27.** 1) For the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ by multiplication, the quotient variety is just a point; there are no stable points.

2) For the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ as in Example 1.12.2, the quotient morphism is the map $\mathbb{C}^2 \to \mathbb{C}$, $(x, y) \mapsto xy$. The set of stable points is the complement of the union of coordinate lines.

3) Let $G = \mathrm{SL}_n$ act on the space $\mathrm{Q}_n$ of quadratic forms in $n$ variables, by linear change of variables. Then $\mathrm{Q}_n$ is a $G$-module, and one checks that the algebra $\mathbb{C}[\mathrm{Q}_n]^G$ is generated by the discriminant $\Delta$. In other words, the quotient morphism is just $\Delta : \mathrm{Q}_n \to \mathbb{C}$. The fiber at any $c \in \mathbb{C}^*$ is a closed $G$-orbit, with stabilizer isomorphic to the special orthogonal group $\mathrm{SO}_n$. The fiber at 0 consists of $n$ orbits: the quadratic forms of ranks $n-1, n-2, \ldots, 0$. There are no stable points, but the restriction of $\pi$ to the open subset $(\Delta \neq 0)$, consisting of non-degenerate forms, is a geometric quotient.

4) More generally, let $G = \mathrm{SL}_n$ act on the space $V = V_{d,n}$ of homogeneous polynomials of degree $d$ in $n$ variables by linear change of variables. If $d = 1$ then the quotient is a point, and if $d = 2$ then $V = \mathrm{Q}_n$, so that we may assume $d \geq 3$. Then the discriminant $\Delta$ is a homogeneous invariant, but does not generate the algebra $\mathbb{C}[V]^G$ unless $n = 2$ (in fact, the structure of this algebra is unknown apart from some small values of $n$ and $d$). By a theorem of Jordan and Lie (see [10, Theorem 2.1] for a modern proof), the stabilizer $G_f$ is finite for any $f \in V$ such that $\Delta(f) \neq 0$. As in the proof of Proposition 1.26, it follows that the open subset $(\Delta \neq 0)$ consists of stable points.

5) Let $G = \mathrm{GL}_n$ act on the space $\mathrm{M}_n$ of $n \times n$ matrices by conjugation. Then one checks that the quotient morphism is the map $\pi : \mathrm{M}_n \to \mathbb{C}^n$ that associates with any matrix $A$ the coefficients of its characteristic polynomial, $\det(tI_n - A)$. Moreover, the closed orbits are exactly those of diagonalizable matrices. Also, the subgroup $\mathbb{C}^* I_n$ of scalar matrices acts trivially on $\mathrm{M}_n$, and hence there are no stable points. If one replaces $G$ with its quotient $\mathrm{PGL}_n = \mathrm{GL}_n / \mathbb{C}^* I_n$, then every non-trivial $g \in G$ acts non-trivially, but the stabilizers are again infinite: there are still no stable points.

## 1.3. Quotients of projective varieties by reductive group actions

Consider a linear algebraic group $G$ and a finite-dimensional $G$-module $V$. Recall the positive grading

$$\mathbb{C}[V] = \bigoplus_{n=0}^{\infty} \mathbb{C}[V]_n$$

by $G$-submodules, used in the proof of Theorem 1.24. Any non-zero $f \in \mathbb{C}[V]_n$ defines an affine open subset $\mathbb{P}(V)_f$ of $\mathbb{P}(V)$ (the projectivization of $V$), where $f \neq 0$. If $f$ is $G$-invariant, then $\mathbb{P}(V)_f$ is $G$-stable for the induced action of $G$ on $\mathbb{P}(V)$. If in addition $G$ is reductive, then we have a categorical quotient

$$\pi_f : \mathbb{P}(V)_f \longrightarrow \mathbb{P}(V)_f$$

by Theorem 1.24. The algebra $\mathbb{C}[\mathbb{P}(V)_f]$ is the subalgebra of the graded algebra $\mathbb{C}[V][\frac{1}{f}]$ consisting of homogeneous elements of degree 0. In other words,

$$\mathbb{C}[\mathbb{P}(V)_f] = \bigcup_{m=0}^{\infty} \frac{\mathbb{C}[\mathbb{P}(V)]_{mn}}{f^m}$$

and therefore

$$\mathbb{C}[\mathbb{P}(V)_f]^G = \bigcup_{m=0}^{\infty} \frac{\mathbb{C}[\mathbb{P}(V)]_{mn}^G}{f^m}.$$

Recall that $\mathbb{P}(V)$ is obtained from its open subsets $\mathbb{P}(V)_f$ by glueing them according to the identifications $\mathbb{P}(V)_{ff'} = \mathbb{P}(V)_f \cap \mathbb{P}(V)_{f'}$. Likewise, the quotients $\pi_f$ may be glued together into a morphism

$$\pi : \mathbb{P}(V)^{\mathrm{ss}} \longrightarrow \mathbb{P}(V)^{\mathrm{ss}} /\!/ G,$$

where $\mathbb{P}(V)^{\mathrm{ss}}$ is a $G$-stable subset of $\mathbb{P}(V)$: the union of the subsets $\mathbb{P}(V)_f$ over all $f \in \mathbb{C}[V]_n^G$, $n \geq 1$.

This construction motivates the following:

**Definition 1.28.** A non-zero $v \in V$, or its image $[v] \in \mathbb{P}(V)$, is *semi-stable* if $f(v) \neq 0$ for some $f \in \mathbb{C}[V]^G$, homogeneous of positive degree.

We denote by $V^{\mathrm{ss}}$ the set of all semi-stable points; this is the preimage of $\mathbb{P}(V)^{\mathrm{ss}}$ in $V \setminus \{0\}$.

The *nilcone* $\mathcal{N}(V)$ is the complement of $V^{\mathrm{ss}}$ in $V$.

Observe that $\mathcal{N}(V)$ is the set of common zeroes of all homogeneous invariants of positive degree, i.e., the fiber at 0 of the quotient map $\pi_V : V \to V/\!/G$. By Theorem 1.24, it follows that $\mathcal{N}(V)$ consists of those $v \in V$ such that the orbit closure $\overline{G \cdot v}$ contains 0. In other words, *$v$ is semi-stable if and only if $0 \notin \overline{G \cdot v}$.*

**Proposition 1.29.** *With the preceding notation, $\mathbb{P}(V)^{\mathrm{ss}}/\!/G$ is a normal, projective variety. Moreover, for any open affine subset $U \subset \mathbb{P}(V)^{\mathrm{ss}}/\!/G$, the preimage $\pi^{-1}(U)$ is an open affine $G$-stable subset of $\mathbb{P}(V)$, and $\pi$ induces an isomorphism $\mathbb{C}[U] \cong \mathbb{C}[\pi^{-1}(U)]^G$.*

*Proof.* By construction, $\mathbb{P}(V)^{\mathrm{ss}}/\!/G$ is the Proj of the positively graded algebra $\mathbb{C}[V]^G$, and the latter is integrally closed in is fraction field by Theorem 1.24 (vi); this yields the first assertion.

For the second assertion, note that there exists a multiplicative subset $S \subset \mathbb{C}[V]^G$ consisting of homogeneous elements, such that $\mathbb{C}[U] = \mathbb{C}[V]^G[\frac{1}{S}]_0$ (the subalgebra of homogeneous elements of degree 0 in the localization $\mathbb{C}[V]^G[\frac{1}{S}]$). It follows that $\pi^{-1}(U)$ is affine with coordinate ring $\mathbb{C}[V][\frac{1}{S}]_0$; thus, $\mathbb{C}[\pi^{-1}(U)]^G = \mathbb{C}[U]$. $\square$

The second assertion of Proposition 1.29 may be rephrased as follows: $\pi$ is an affine morphism and yields an isomorphism

$$\mathcal{O}_{\mathbb{P}(V)^{\mathrm{ss}}/\!/G} \cong \big(\pi_* \mathcal{O}_{\mathbb{P}(V)^{\mathrm{ss}}}\big)^G.$$

Such a morphism is called a *good quotient*.

Next, we adapt the notion of stable points to this projective setting:

**Definition 1.30.** A point $x \in \mathbb{P}(V)$ is *stable* if $x$ is semi-stable, the orbit $G \cdot x$ is closed in $\mathbb{P}(V)^{\mathrm{ss}}$, and the isotropy group $G_x$ is finite. We denote by $\mathbb{P}(V)^{\mathrm{s}}$ the set of stable points.

**Proposition 1.31.** *With the preceding notation, we have $\mathbb{P}(V)^{\mathrm{s}} = \mathbb{P}(V^{\mathrm{s}})$, where $V^{\mathrm{s}} \subset V$ denotes the subset of stable points. Moreover, $\pi\big(\mathbb{P}(V)^{\mathrm{s}}\big)$ is open in $\mathbb{P}(V)^{\mathrm{ss}}/\!/G$, we have $\mathbb{P}(V)^{\mathrm{s}} = \pi^{-1}\pi\big(\mathbb{P}(V)^{\mathrm{s}}\big)$, and the restriction $\pi^{\mathrm{s}} : \mathbb{P}(V)^{\mathrm{s}} \to \pi\big(\mathbb{P}(V)^{\mathrm{s}}\big)$ is a geometric quotient.*

*Proof.* Let $v \in V^{\mathrm{s}}$; then $v \notin \mathcal{N}(V)$, and hence we may choose a homogeneous $f \in \mathbb{C}[V]^G$ such that $f(v) \neq 0$. The hypersurface $Y := \{w \in V \mid f(w) = f(v)\}$ is $G$-stable, and contained in $V^{\mathrm{s}}$ by Proposition 1.26. Moreover, the natural map $V \setminus \{0\}$ restricts to a finite surjective $G$-morphism $Y \to \mathbb{P}(V)_f$ (of degree equal to the degree of $f$). Since $\mathbb{P}(V)_f = \pi^{-1}\pi\big(\mathbb{P}(V)_f\big)$, it follows that $\mathbb{P}(V)_f \subset \mathbb{P}(V^{\mathrm{s}})$. In particular, $[v] \in \mathbb{P}(V^{\mathrm{s}})$.

Conversely, given $[v] \in \mathbb{P}(V^{\mathrm{s}})$, one checks that $v \in V^{\mathrm{s}}$ by reversing the preceding arguments. This shows the equalities $\mathbb{P}(V)^{\mathrm{s}} = \mathbb{P}(V^{\mathrm{s}})$ and $\mathbb{P}(V)^{\mathrm{s}} = \pi^{-1}\pi\big(\mathbb{P}(V)^{\mathrm{s}}\big)$. The assertion on $\pi^{\mathrm{s}}$ is checked as in the proof of Proposition 1.26. $\square$

**Examples 1.32.** 1) For the action of $\mathbb{C}^*$ on $\mathbb{C}^n$ by multiplication, there are no semi-stable points.

2) For the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ as in Example 1.12.2, every semi-stable point is stable, and the projective quotient variety is just a point.

3) Let $G = \mathrm{SL}_n$ act on $V = \mathrm{Q}_n$ as in Example 1.27.3. Then $\mathbb{P}(V)^{\mathrm{ss}} = \mathbb{P}(V)_\Delta$ and $\mathbb{P}(V)^{\mathrm{s}}$ is empty; the projective quotient variety is again a point.

4) For the action of $G = \mathrm{SL}_n$ on the space $V = V_{d,n}$ of homogeneous polynomials of degree $d$ in $n$ variables as in Example 1.27.4, the points of $\mathbb{P}(V)$ may be viewed as the hypersurfaces of degree $d$ in $\mathbb{P}^{n-1}$, and the $G$-orbits are the isomorphism classes of such hypersurfaces. We saw that every smooth hypersurface is stable, if $d \geq 3$. It follows that the quotient variety is a normal compactification of the moduli space of smooth hypersurfaces.

5) Consider again the action of $\mathrm{GL}_n$ on the space $\mathrm{M}_n$ of $n \times n$ matrices by conjugation. Then the semi-stable points are exactly the lines of non-nilpotent matrices, and there are no stable points. Since the algebra $\mathbb{C}[M_n]^G$ is generated by the coefficients of the characteristic polynomial, and hence by homogeneous invariants of degrees $1, 2, \ldots, n$, the quotient variety is a weighted projective space with weights $1, 2, \ldots, n$. In particular, that variety is singular if $n \geq 3$.

The preceding constructions and results are readily generalized to any closed $G$-subvariety $X \subset \mathbb{P}(V)$, where $V$ is a finite-dimensional $G$-module. We now sketch how to adapt them to the setting of arbitrary projective $G$-varieties.

By assumption, any such variety $X$ admits a line bundle $L$ which is *ample*, i.e., there exist a closed immersion

$$\iota : X \to \mathbb{P}(V),$$

where $V$ is a finite-dimensional vector space, and a positive integer $m$ such that $L^m \cong \iota^*\mathcal{O}_{\mathbb{P}(V)}(1)$ (the pull-back of the tautological line bundle on $\mathbb{P}(V)$). Equivalently, the natural rational map

$$f_m : X - \to \mathbb{P}\big(\Gamma(X, L^m)^*\big),$$

that associates with $x \in X$ the hyperplane consisting of those sections $\sigma \in \Gamma(X, L^m)$ such that $\sigma(x) = 0$, is in fact a closed immersion.

We shall assume that $L$ is *$G$-linearized* in the sense of the following:

**Definition 1.33.** Let $X$ be a $G$-variety, and $p : L \to X$ a line bundle. A *$G$-linearization* of $L$ consists of a $G$-action on $L$ such that $p$ is equivariant and the induced map $g : L_x \to L_{g\cdot x}$ is linear for any $g \in G$ and $x \in X$.

**Lemma 1.34.** *Let $L$ be a $G$-linearized line bundle on a $G$-variety $X$. Then for any integer $n$, the tensor power $L^n$ inherits a $G$-linearization, and the space $\Gamma(X, L^n)$ is a $G$-module.*

*Proof.* The first assertion is obvious, and the second one is proved by adapting the argument of Lemma 1.5. □

Some naturally defined line bundles do not admit any linearization: for example, $\mathcal{O}_{\mathbb{P}(V)}(1)$ for the natural action of $\mathrm{PGL}(V)$ on $\mathbb{P}(V)$ (since that action does not lift to an action on $\Gamma\big(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1)\big) \cong V^*$). But given a connected linear algebraic group $G$, and a line bundle $L$ on a *normal* $G$-variety $X$, some positive power $L^n$ admits a linearization; also, there exists a finite covering $p : G' \to G$ of algebraic groups such that $L$ admits a $G'$-linearization (see [6]).

By Lemma 1.34, given a projective $G$-variety $X$ equipped with an ample $G$-linearized line bundle $L$, there exists a positive integer $m$ such that the above map $f_m$ is an equivariant embedding into the projectivization of a $G$-module $V$. We may now define the sets of semi-stable, resp. stable points by $X^{\mathrm{ss}}(L) := X \cap \mathbb{P}(V)^{\mathrm{ss}}$, resp. $X^{\mathrm{s}}(L) := X \cap \mathbb{P}(V)^{\mathrm{s}}$.

**Proposition 1.35.** *With the preceding notation and assumptions, we have*

$$X^{\mathrm{ss}}(L) = \{x \in X \mid \sigma(x) \neq 0 \text{ for some } n \geq 1 \text{ and } \sigma \in \Gamma(X, L^n)^G\},$$

$$X^{\mathrm{s}}(L) = \{x \in X \mid G \cdot x \text{ is closed in } X^{\mathrm{ss}}(L) \text{ and } G_x \text{ is finite }\}.$$

*In particular, $X^{\mathrm{ss}}(L)$ and $X^{\mathrm{s}}(L)$ are open $G$-stable subsets of $X$, and they are unchanged when $L$ is replaced with a positive power $L^n$. Moreover, there exists a good quotient $\pi : X^{\mathrm{ss}}(L) \to Y$, where $Y$ is a projective variety, and $\pi$ restricts to a geometric quotient $X^{\mathrm{s}}(L) = \pi^{-1}(Y^{\mathrm{s}}) \to Y^{\mathrm{s}}$, where $Y^{\mathrm{s}} := \pi(X^{\mathrm{s}})$ is open in $Y$.*

*Proof.* We may assume that $X \subset \mathbb{P}(V)$ and $L^m = \mathcal{O}_X(1)$. Then $X$ corresponds to a closed subvariety $\tilde{X} \subset V$, stable by the natural action of $G \times \mathbb{C}^*$ on $V$ (where $\mathbb{C}^*$ acts by scalar multiplication). The positively graded algebra

$$\mathbb{C}[\tilde{X}] = \bigoplus_{n=0}^{\infty} \mathbb{C}[\tilde{X}]_n$$

is the homogeneous coordinate ring of $X$; the restriction map

$$\mathbb{C}[V]_n = \Gamma\big(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(n)\big) \longrightarrow \Gamma\big(X, \mathcal{O}_X(n)\big) = \Gamma(X, L^{mn})$$

factors through a map

$$\mathbb{C}[\tilde{X}]_n \longrightarrow \Gamma(X, L^{mn})$$

which is an isomorphism for $n \gg 0$. Thus, the same properties hold for the induced maps on invariants,

$$\mathbb{C}[V]_n^G \longrightarrow \mathbb{C}[\tilde{X}]_n^G \longrightarrow \Gamma(X, L^{mn})^G.$$

Also, note any $\sigma \in \Gamma(X, L^n)$ satisfies $\sigma^m \in \Gamma(X, L^{mn})$ and $X_\sigma = X_{\sigma^m}$ (here $X_\sigma$ denotes the open subset of $X$ where $\sigma \neq 0$). These observations imply easily our statements. □

## 2. Spherical varieties

### 2.1. Representations of connected reductive groups and $U$-invariants

Given an algebraic group $G$ and two $G$-modules $V, W$, we denote by $\mathrm{Hom}^G(V, W)$ the vector space of morphisms of $G$-modules (i.e., equivariant linear maps) $f : V \to W$. Observe that

$$\mathrm{Hom}^G(V, \mathbb{C}[X]) \cong \big(\mathbb{C}[X] \otimes V^*\big)^G \cong \mathrm{Mor}^G(X, V^*),$$

for any $G$-variety $X$, where $\mathrm{Mor}^G$ denote the space of $G$-morphisms (of varieties). Moreover, the left-hand side has a natural structure of a module over the invariant algebra $\mathbb{C}[X]^G$; these modules are called *modules of covariants*.

We now assume that $G$ is reductive, and denote by $\mathrm{Irr}(G)$ the set of isomorphism classes of simple $G$-modules.

**Lemma 2.1.** *Any $G$-module $M$ admits a canonical decomposition*

$$M \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Hom}^G(V, M) \otimes V,$$

*where the map from the left-hand side to the right-hand side is given by*

$$f \otimes v \in \mathrm{Hom}^G(V, M) \otimes V \longmapsto f(v) \in M.$$

*In particular, for any $G$-variety $X$, we have a canonical decomposition*

$$\mathbb{C}[X] \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Mor}^G(X, V^*) \otimes V.$$

*Moreover, each $\mathbb{C}[X]^G$- module $\mathrm{Mor}^G(X, V^*)$ is finitely generated.*

*Proof.* For the first assertion, since $M$ is a direct sum of simple $G$-modules, it suffices to treat the case that $M$ is simple. Then, by Schur's lemma, $\mathrm{Hom}^G(V, M)$ is a line if $V \cong M$, and is zero otherwise; this yields the statement.

To show the finite generation of $\mathrm{Mor}^G(X, V^*)$, note that the algebra $\mathbb{C}[X \times V]^G$ is finitely generated and graded via the $\mathbb{C}^*$-action on $V$:

$$\mathbb{C}[X \times V]^G = \bigoplus_{n=0}^{\infty} (\mathbb{C}[X] \otimes \mathbb{C}[V]_n)^G.$$

Thus, we may choose homogeneous generators $f_1, \ldots, f_n$. Denote their degrees by $d_1, \ldots, d_n$; then the algebra $\mathbb{C}[X]^G$ is generated by those $f_i$ such that $d_i = 0$, and $\mathrm{Mor}^G(X, V^*) = \mathbb{C}[X \times V]_1$ is generated over $\mathbb{C}[X]^G$ by those $f_i$ such that $d_i = 1$. $\square$

**Lemma 2.2.** *There is a canonical decomposition of $G \times G$-modules*

$$\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} V^* \otimes V \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{End}(V),$$

*where $G \times G$ acts on $\mathbb{C}[G]$ via its action on $G$ by $(g, h) \cdot x := gxh^{-1}$.*

*Proof.* Lemma 2.1 yields an isomorphism

$$\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} \mathrm{Mor}^G(G, V^*) \otimes V$$

which is easily seen to be $G \times G$-equivariant. Here the right copy of $G$ acts on each $\mathrm{Mor}^G(G, V^*) \otimes V$ via its action on $V$, and the left copy of $G$ acts on $\mathrm{Mor}^G(G, V^*)$ via left multiplication on $G$. Moreover, we have an isomorphism of $G$-modules

$$\mathrm{Mor}^G(G, V^*) \cong V^*, \quad \varphi \mapsto \varphi(e_G),$$

where the inverse isomorphism is given by $f \mapsto \big(g \mapsto f(g)\big)$. $\square$

**Examples 2.3.** 1) If $G$ is finite, then Lemma 2.2 gives back the classical decomposition of the regular representation.
2) Let $T \cong (\mathbb{C}^*)^n$ be a torus; then

$$\mathbb{C}[T] \cong \mathbb{C}[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \cong \bigoplus_{(a_1,\ldots,a_n)\in\mathbb{Z}^n} \mathbb{C}t_1^{a_1}\cdots t_n^{a_n},$$

where each Laurent monomial $t_1^{a_1}\cdots t_n^{a_n}$ is an eigenvector of $T \times T$. By Lemma 2.2, each simple $T$-module is a line where $T$ acts via a *character*

$$\chi = \chi_{a_1,\ldots,a_n} : (t_1, \ldots, t_n) \longmapsto t_1^{a_1}\cdots t_n^{a_n},$$

i.e., a homomorphism of algebraic groups $T \to \mathbb{C}^*$. The characters form an abelian group for pointwise multiplication: the *character group* $\Lambda(T)$, isomorphic to $\mathbb{Z}^n$. Lemma 2.1 gives back the correspondence between $T$-modules and $\Lambda$-graded vector spaces, already noted in Example 1.7.

We now consider a *connected* reductive group $G$, and choose a *Borel subgroup* $B \subset G$, i.e., a maximal connected solvable subgroup (all such subgroups are conjugate in $G$). Also, choose a *maximal torus* $T \subset B$ (all such subgroups are conjugate in $B$). Then $B = TU$ where $U \subset B$ denotes the largest unipotent subgroup; moreover, $U$ is a *maximal unipotent subgroup* of $G$. Since $U$ is a normal subgroup of $B$ and is isomorphic as a variety to an affine space, the character group $B$ is isomorphic to that of $T \cong B/U$; we denote that group by $\Lambda$. This is a free abelian group of finite rank $r := \dim(T)$, the *rank of G*.

With this notation at hand, we obtain a parametrization of the simple $G$-modules:

**Theorem 2.4.** (i) *For any simple G-module $V$, the fixed point subspace $V^U$ is a line, where $B$ acts via a character $\lambda(V)$. Moreover, $V$ is uniquely determined by $\lambda(V)$ up to G-isomorphism.*
(ii) *The set*

$$\Lambda^+ := \{\lambda \in \Lambda \mid \lambda = \lambda(V) \text{ for some } V \in \mathrm{Irr}(G)\}$$

*is the intersection of $\Lambda$ with a rational polyhedral convex cone in the associated vector space $\Lambda_{\mathbb{R}}$ over the real numbers. In particular, $\Lambda^+$ is a finitely generated submonoid of $\Lambda$.*

*Proof.* (i) The main ingredient is the structure of the open Bruhat cell of $G$. Specifically, there exists a unique Borel subgroup $B^-$ which is opposite to $B$, i.e., satisfies $B^- \cap B = T$. Moreover, denoting by $U^-$ the largest unipotent subgroup of $B^-$, the multiplication map

$$U^- \times T \times U \longrightarrow G, \quad (x, y, z) \longmapsto xyz$$

is an open immersion. In particular, the product $B^-B = U^-TU$ is open in $G$ (for these facts, see e.g. [13, Section 8.3]).

Next, consider non-zero points $v \in V^U$ and $f \in V^*$. Then the map

$$a_{f,v} : G \longrightarrow \mathbb{C}, \quad g \longmapsto f(g \cdot v)$$

is non-zero, since the (simple) $G$-module $V$ is spanned by $G \cdot v$. Moreover, $a_{f,v} \in \mathbb{C}[G]^U$. By the Lie-Kolchin theorem, we may choose $f$ to be an eigenvector of $B^-$, of some weight $\mu \in \Lambda$; then $a_{f,v}$ is also an eigenvector of $B^-$ acting on $\mathbb{C}[G]$ via left multiplication. Thus, given another non-zero $v' \in V^U$, the map

$$g \longmapsto \varphi(g) := \frac{f(g \cdot v')}{f(g \cdot v)}$$

is a non-zero rational function on $G$, invariant under $B^- \times U$. Thus, $\varphi$ is constant, i.e., there exists $t \in \mathbb{C}^*$ such that $f\big(g \cdot (v' - tv)\big) = 0$ for all $g \in G$. But then $v' = tv$: we have shown that $V^U$ is a line. It follows that $V^U$ consists of $B$-eigenvectors of some weight $\lambda$, and hence that $a_{f,v}$ is a $B$-eigenvector for that weight. Restricting $a_{f,v}$ to $B \cap B^- = T$, we see that $\lambda = -\mu$. Likewise, $(V^*)^{U^-}$ is a line spanned by $f$. In view of Lemma 2.2, this yields the decomposition of $T \times G$-modules

$$\mathbb{C}[G]^{U^-} \cong \bigoplus_{V\in\mathrm{Irr}(G)} (V^*)^{U^-} \otimes V \cong \bigoplus_{V\in\mathrm{Irr}(G)} V,$$

where the $U^-$-invariants are relative to the action via left multiplication. This identifies $V$ with the $T$-eigenspace in $\mathbb{C}[G]^{U^-}$ with weight $\lambda = \lambda(V)$.

(ii) Consider the algebra $\mathbb{C}[G]^{U^- \times U}$, where the $U$-invariants are relative to the action via right multiplication. Then

$$\mathbb{C}[G]^{U^- \times U} \subset \mathbb{C}[B^- B]^{U^- \times U} \cong \mathbb{C}[T]$$

as a $T$-stable subalgebra. Together with the preceding arguments, it follows that the characters $\lambda \in \Lambda^+$ form a basis of the vector space $\mathbb{C}[G]^{U^- \times U}$.

Next, consider the irreducible components $D_1, \ldots, D_s$ of $G \setminus B^- B$. These are $B^- \times B$-stable prime divisors of $G$ (the closures of the Bruhat cells of codimension 1); denote by $v_1, \ldots, v_s$ the corresponding discrete valuations of the function field $\mathbb{C}(G)$. Since $G$ is smooth, a given function $f \in \mathbb{C}[B^- B] \subset \mathbb{C}(G)$ extends to a regular function on $G$ if and only $v_i(f) \geq 0$ for $i = 1, \ldots, s$. In particular, viewing each $\lambda \in \Lambda$ as a rational $U^- \times U$-invariant function on $G$, we have

$$\Lambda^+ = \{\lambda \in \Lambda \mid v_i(\lambda) \geq 0 \quad (i = 1, \ldots, s)\}.$$

But each $v_i : \mathbb{C}(G)^* \to \mathbb{Z}$ restricts to an additive map $\Lambda \to \mathbb{Z}$, since $v_i(ff') = v_i(f) + v_i(f')$ for all $f, f' \in \mathbb{C}(G)^*$. Thus, $\Lambda^+$ is defined in $\Lambda$ by finitely many linear inequalities. As a consequence, $\Lambda^+$ is a finitely generated monoid (by Gordan's lemma, see e.g. [2, Proposition 1.2.1]).

□

**Definition 2.5.** With the preceding notation and assumptions, $\lambda = \lambda(V)$ is the *highest weight* of the simple $G$-module $V$; we set $V := V(\lambda)$.

The character group $\Lambda$ is called the *weight lattice* of $G$; the weights in $\Lambda^+$ are called *dominant*.

For an arbitrary $G$-module $M$ and $\lambda \in \Lambda$, we denote by $M_\lambda^{(B)} \subset M$ the $B$-eigenspace with weight $\lambda$, also called the set of *highest weight vectors of weight* $\lambda$.

Putting together Lemma 2.1 and Theorem 2.4, we obtain an isomorphism of $G$-modules

$$M \cong \bigoplus_{\lambda \in \Lambda^+} M_\lambda^{(B)} \otimes V(\lambda)$$

and isomorphisms

$$\mathrm{Hom}^G\big(V(\lambda), M\big) \cong M_\lambda^{(B)}$$

for all $\lambda \in \Lambda^+$. As a consequence, *the $G$-module $M$ is uniquely determined by the $T$-module $M^U$.*

**Examples 2.6.** 1) If $G \cong (\mathbb{C}^*)^n$ is a torus with character group $\Lambda \cong \mathbb{Z}^n$, then every weight is dominant.

2) Let $G = \mathrm{GL}_n$; then the subgroup $\mathrm{B}_n$ of upper triangular invertible matrices is a Borel subgroup. We have $\mathrm{B}_n = \mathrm{T}_n\,\mathrm{U}_n$ where $\mathrm{T}_n$ is the diagonal torus, and $\mathrm{U}_n$ is the largest unipotent subgroup. The diagonal coefficients yield a basis $(\varepsilon_1, \ldots, \varepsilon_n)$ of the weight lattice $\Lambda_n$. The opposite Borel subgroup $\mathrm{B}_n^-$ consists of all lower triangular invertible matrices. One checks that the open subset $\mathrm{B}_n^-\,\mathrm{B}_n \subset \mathrm{M}_n$ is defined by the non-vanishing of the principal minors

$$\Delta_k : \mathrm{GL}_n \longrightarrow \mathbb{C}, \quad A = (a_{ij}) \longmapsto \det(a_{ij})_{1 \leq i,j \leq k}$$

for $k = 1, \ldots, n$ (in particular, $\Delta_n$ is the determinant). Each $\Delta_k$ is an eigenvector of $\mathrm{B}_n^- \times \mathrm{B}_n$ with weight $(-\omega_k, \omega_k)$, where

$$\omega_k := \varepsilon_1 + \cdots + \varepsilon_k.$$

Clearly, $\omega_1, \ldots, \omega_n$ form a basis of $\Lambda_n$. Moreover, the eigenvectors of $\mathrm{B}_n^- \times \mathrm{B}_n$ in $\mathbb{C}[\mathrm{GL}_n]$ are exactly the monomials $c\Delta_1^{a_1} \cdots \Delta_n^{a_n}$, where $c \in \mathbb{C}^*$ and $a_1, \ldots, a_{n-1} \geq 0$.

It follows that the monoid $\Lambda_n^+$ is generated by $\omega_1, \ldots, \omega_n$ and by $-\omega_n$. Also, the standard representation $\mathbb{C}^n$ is a simple $\mathrm{GL}_n$-module with highest weight $\omega_1$; the first basis vector $e_1$ is a highest weight vector. More generally, one checks that each $k$-th exterior power $\wedge^k \mathbb{C}^n$ is a simple $\mathrm{GL}_n$-module with highest weight $\omega_k$, and highest weight vector $e_1 \wedge \cdots \wedge e_k$.

3) If $G = \mathrm{SL}_n$, then we may take as opposite Borel subgroups $B := G \cap \mathrm{B}_n$ and $B^- := G \cap \mathrm{B}_n^-$. The weight lattice $\Lambda$ is the quotient of $\Lambda_n$ by the subgroup $\mathbb{Z}\omega_n$; the monoid of dominant weights $\Lambda^+$ is freely generated by the images of $\omega_1, \ldots, \omega_{n-1}$, called the *fundamental weights*. The corresponding simple modules are again the exterior powers $\Lambda^k \mathbb{C}^n$, where $k = 1, \ldots, n-1$.

4) In particular, if $G = \mathrm{SL}_2$, then we may identify the dominant weights with the non-negative integers. The simple $G$-module $V(n)$ with highest weight $n$ is the space $\mathbb{C}[x, y]_n$ of homogeneous

polynomials of degree $n$ in two variables, where $G$ acts via linear change of variables. Indeed, one checks that $\mathbb{C}[x,y]_n$ contains a unique line of $B$-eigenvectors, spanned by $y^n$.

Next, we obtain an important finiteness result due to Hadziev and Grosshans:

**Theorem 2.7.** *Let $G$ be a connected reductive group, $U \subset G$ a maximal unipotent subgroup, and $X$ an affine $G$-variety. Then the algebra $\mathbb{C}[X]^U$ is finitely generated.*

*Proof.* First, we obtain an algebra isomorphism

$$\mathbb{C}[X]^U \cong \mathbb{C}[X \times G/U]^G,$$

where $G$ acts on $\mathbb{C}[X \times G/U]$ via its diagonal action on $X \times G/U$. Indeed, one associates with any $\varphi \in \mathbb{C}[X]^U$ the map $\big((x,g) \mapsto \varphi(g \cdot x)\big) \in \mathbb{C}[X \times G/U]^G$, and with any $f \in \mathbb{C}[X \times G/U]^G$ the map $\big(x \to f(x, e_G)\big) \in \mathbb{C}[X]^U$.

Next, since $\mathbb{C}[X \times G/U]^G \cong \big(\mathbb{C}[X] \otimes \mathbb{C}[G]^U\big)^G$, it suffices to show that the algebra $\mathbb{C}[G]^U$ is finitely generated, in view of Theorem 1.24; equivalently, $\mathbb{C}[G]^{U^-}$ is finitely generated, where $U^- \subset G$ is the maximal unipotent subgroup opposite to $U$ as in the proof of Theorem 2.4. But, as seen in that proof,

$$\mathbb{C}[G]^{U^-} \cong \bigoplus_{\lambda \in \Lambda^+} V(\lambda)$$

as $T$-module, where $T$ acts on each $V(\lambda)$ via its character $\lambda$. This yields a grading of $\mathbb{C}[G]^{U^-}$ by the monoid $\Lambda^+$; moreover, the product $V(\lambda)V(\mu) \subset \mathbb{C}[G]^{U^-}$ equals $V(\lambda+\mu)$, since $\mathbb{C}[G]^{U^-}$ is a domain. Thus, the algebra $\mathbb{C}[G]^{U^-}$ is generated by those $V(\lambda)$ associated with a generating subset $S \subset \Lambda^+$. Moreover, we may choose $S$ to be finite, by Theorem 2.4 again. $\square$

With the preceding notation and assumptions, the subalgebra $\mathbb{C}[X]^U \subset \mathbb{C}[X]$ corresponds to a $U$-invariant morphism of affine varieties $p : X \to X//U$, which is clearly a categorical quotient in the sense of Subsection 1.2.

In contrast with the quotient by $G$, the map $p$ need not be surjective. For example, take $G = X = \mathrm{SL}_2$ where $G$ acts by left multiplication. Then $G/U \cong \mathbb{C}^2 \setminus \{0\}$ (see Example 1.12.3) whereas $G//U \cong \mathbb{C}^2$ by assigning with a matrix its first column.

It turns out that many properties of an affine $G$-variety $X$ may be read off its categorical quotient $X//U$; this is exposed in detail in [3, Chapter 3]. We shall only need a small part of these results:

**Proposition 2.8.** *Let $G$ be a connected algebraic group, $U$ a maximal unipotent subgroup, and $X$ an irreducible affine $G$-variety. Then the following hold:*

(i) *$\mathbb{C}(X)^U$ is the fraction field of $\mathbb{C}[X]^U$. Moreover, any $B$-eigenvector in $\mathbb{C}(X)$ is the quotient of two $B$-eigenvectors in $\mathbb{C}[X]$.*

(ii) *$X$ is normal if and only if $X//U$ is normal.*

*Proof.* (i) Clearly, the fraction field of $\mathbb{C}[X]^U$ is contained in $\mathbb{C}(X)^U$. To show the opposite inclusion, consider $f \in \mathbb{C}(X)^U$. Then the vector space of 'denominators'

$$\{\varphi \in \mathbb{C}[X] \mid f\varphi \in \mathbb{C}[X]\}$$

is non-zero and $U$-stable. Therefore, this $U$-submodule of $\mathbb{C}[X]$ contains a non-zero $U$-invariant.

This proves the first assertion; the second one is checked similarly.

(ii) If $X$ is normal, then so is $X//U$ by Theorem 1.24 (vi).

Conversely, assume that $X//U$ is normal and consider the normalization map

$$\eta : Y \longrightarrow X.$$

We claim that the $G$-action on $X$ lifts uniquely to a $G$-action on the affine variety $Y$ such that $\eta$ is equivariant. Indeed, the normalization of $G \times X$ is the map

$$G \times Y \longrightarrow G \times X, \quad (g,y) \longmapsto \big(g, \eta(y)\big)$$

and the action map $\alpha : G \times X \to X$ lifts to a unique morphism (of varieties) $\beta : G \times Y \to Y$ by the universal property of the normalization. Since $\eta$ is an isomorphism over a $G$-stable open subset, it follows that $\beta$ is a group action; this implies our claim.

Next, the coordinate ring $\mathbb{C}[Y]$, the integral closure of $\mathbb{C}[X]$ in its fraction field $\mathbb{C}(X)$, is a finitely generated algebra, and the "conductor"

$$I := \{f \in \mathbb{C}[X] \mid f\mathbb{C}[Y] \subset \mathbb{C}[X]\}$$

is a non-zero ideal of $\mathbb{C}[X]$. Clearly, $I$ is $G$-stable, and hence is a $G$-submodule of $\mathbb{C}[X]$. Thus, $I$ contains a non-zero $U$-invariant $f$. Then $f \in \mathbb{C}[X]^U$, and $f\mathbb{C}[Y]^U$ is an ideal of $\mathbb{C}[X]^U$; as a consequence, the $\mathbb{C}[X]^U$-module $\mathbb{C}[Y]^U$ is finitely generated. So $\mathbb{C}[Y]^U$ is integral over $\mathbb{C}[X]^U$, and both have the same field of fractions. Since $X//U$ is normal, it follows that $\mathbb{C}[Y]^U = \mathbb{C}[X]^U$, and hence that $\mathbb{C}[Y] = \mathbb{C}[X]$. We conclude that $\eta$ is an isomorphism. □

Also, note the following direct consequence of Lemma 2.1 and Theorem 2.7:

**Corollary 2.9.** *Let $G$ be a connected reductive group, $U$ a maximal unipotent subgroup, and $X$ an affine $G$-variety. Then the space $\mathbb{C}[X]^{(B)}_\lambda$ is a finitely generated module over $\mathbb{C}[X]^G$, for any $\lambda \in \Lambda^+$. If $X$ is irreducible, then the set*

$$\{\lambda \in \Lambda \mid \mathbb{C}[X]^{(B)}_\lambda \neq 0\}$$

*is a finitely generated submonoid of $\Lambda$.*

We shall denote that monoid by $\Lambda^+(X)$; it is called the *weight monoid* of the affine $G$-variety $X$.

The *weight group* $\Lambda(X)$ is defined as the set of weights of $B$-eigenvectors in $\mathbb{C}(X)$; this is a subgroup of $\Lambda$, generated by $\Lambda^+(X)$ in view of Proposition 2.8 (i). In particular, $\Lambda(X)$ is a free abelian group of finite rank: the *rank* of $X$, denoted by $\mathrm{rk}(X)$.

The *weight cone* $C(X)$ is the convex cone in the vector space $\Lambda(X)_\mathbb{R}$ generated by $\Lambda^+(X)$; this is a rational polyhedral cone, which spans $\Lambda(X)_\mathbb{R}$.

Note finally the equalities

$$\Lambda^+(X) = \Lambda^+(X//U), \quad \Lambda(X) = \Lambda(X//U), \quad C(X) = C(X//U)$$

and the inclusion

$$\Lambda^+(X) \subset C(X) \cap \Lambda(X).$$

## 2.2. **Affine spherical varieties**

**Definition 2.10.** An irreducible $G$-variety $X$ is *spherical* if $X$ is normal and contains an open $B$-orbit.

A closed subgroup $H \subset G$ is spherical if so is the homogeneous space $G/H$.

**Examples 2.11.** 1) If $G$ is a torus $T$, then the spherical $G$-varieties are exactly those normal $T$-varieties that contain an open orbit; they are called the *toric varieties.*

Let $X$ be an affine toric $T$-variety, and choose $x \in X$ such that $T \cdot x$ is open in $X$. Then the dominant morphism $T \to X$, $t \mapsto t \cdot x$ yields an injective $T$-homomorphism $\mathbb{C}[X] \hookrightarrow \mathbb{C}[T] = \bigoplus_{\lambda \in \Lambda} \mathbb{C}\lambda$. It follows that

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Lambda^+(X)} \mathbb{C}\lambda$$

as a subalgebra of $\mathbb{C}[T]$. In particular, $X$ is uniquely determined by its weight monoid $\Lambda^+(X)$.

One checks that the normality of $X$ is equivalent to $\Lambda^+(X)$ being *saturated*, i.e., equal to $C(X) \cap \Lambda(X)$. Also, the $T$-stable prime ideals of $\mathbb{C}[X]$ correspond bijectively to the faces of $C(X)$, by assigning with each face $F$ the ideal

$$I_F := \bigoplus_{\lambda \in \Lambda^+(X) \setminus F} \mathbb{C}\lambda$$

(see [2, Sections 1.3, 2.1] for details). Thus, $\mathbb{C}[X]/I_F = \bigoplus_{\lambda \in \Lambda^+(X) \cap F} \mathbb{C}\lambda$ is the coordinate ring of an irreducible $T$-stable subvariety $X_F \subset X$ with weight monoid $F \cap \Lambda(X)$ and weight cone $F$. The weight group of $X_F$ is the intersection of $\Lambda(X)$ with the span of $F$; it is a direct factor of $\Lambda(X)$.

Moreover, the assignement $F \mapsto X_F$ yields a parametrization of the irreducible $T$-stable subvarieties of $X$, which preserves the dimensions and the inclusion relations. As a consequence, *every* *affine toric variety contains only finitely many $T$-orbits, and their closures are toric varieties; they are in bijective correspondence with the faces of the weight cone.*

2) Clearly, a closed subgroup $H \subset G$ is spherical if and only if $H$ has an open orbit in $G/B$, the flag variety of $G$. If $G = \mathrm{SL}_2$, this means that $H$ has an open orbit in the projective line $\mathbb{P}^1$. It follows that *the spherical subgroups of* $\mathrm{SL}_2$ *are exactly the subgroups of positive dimension, and each of them has only finitely many orbits in* $\mathbb{P}^1$.

Specifically, any subgroup $H$ of positive dimension contains a conjugate of $\mathrm{U}_2 \cong \mathbb{C}$, or of the diagonal torus $T \cong \mathbb{C}^*$ of $\mathrm{SL}_2$. Moreover, $\mathrm{U}_2$ has two orbits in $\mathbb{P}^1$: the affine line $\mathbb{C}$ and the point $\infty$, whereas $T$ has three orbits: the punctured line $\mathbb{C}^*$ and the points $0, \infty$.

3) Consider the group $G$ as an affine $G \times G$-variety, for the action via left and right multiplication. Then $G$ is the homogeneous space $(G \times G)/\operatorname{diag}(G)$, with base point $e_G$. Taking as a Borel subgroup of $G \times G$ the product $B^- \times B$ of two opposite Borel subgroups of $G$ and using Lemma 2.2 and Theorem 2.4, we see that $G$ is spherical with weight monoid $\{(-\lambda, \lambda) \mid \lambda \in \Lambda^+\}$. In particular, $\Lambda^+(G) \cong \Lambda^+$. In the case that $G = \mathrm{GL}_n$ or $\mathrm{SL}_n$, this also follows from Examples 2.6 2 and 3.

4) Let $G = \mathrm{GL}_n$ act on the space $\mathrm{Q}_n$ of quadratic forms in $n$ variables, like in Example 1.27.3. For any such form, viewed as a symmetric $n \times n$ matrix $A = (a_{ij})$, consider the principal minors

$$\Delta_k : \mathrm{Q}_n \longrightarrow \mathbb{C}, \quad (a_{ij}) \longmapsto \det(a_{ij})_{1 \le i,j \le k}$$

for $k = 1, \ldots, n$ as in Example 2.6.2. Then one checks that the open subset $(\Delta_1 \neq 0, \ldots, \Delta_n \neq 0)$ is a unique orbit of the Borel subgroup $\mathrm{B}_n$; in particular, $\mathrm{Q}_n$ is spherical. Moreover, each $\Delta_k$ is a $\mathrm{B}_n$-eigenvector with weight $2\omega_k$, with the notation of Example 2.6.2. As in that example, it follows that $\Lambda^+(\mathrm{Q}_n)$ is generated by $2\omega_1, \ldots, 2\omega_n$. Thus, $C(\mathrm{Q}_n)$ is the cone of dominant weights, and $\Lambda(\mathrm{Q}_n)$ is the lattice of even weights.

If $\mathrm{GL}_n$ is replaced with $\mathrm{SL}_n$, then every hypersurface $(\Delta = t)$, where $\Delta = \Delta_n$ is the discriminant and $t \in \mathbb{C}$, is a spherical variety with weight monoid generated by $2\omega_1, \ldots, 2\omega_{n-1}$. For $t \in \mathbb{C}^*$, these hypersurfaces are all isomorphic to $\mathrm{SL}_n / \mathrm{SO}_n$; in particular they are smooth, but the hypersurface $(\Delta = 0)$ is singular.

We now obtain a representation-theoretic characterization of spherical varieties:

**Lemma 2.12.** *For an irreducible affine $G$-variety $X$, the following conditions are equivalent:*

(i) *$X$ contains an open $B$-orbit.*

(ii) *Any $B$-invariant rational function on $X$ is constant.*

(iii) *The $G$-module $\mathbb{C}[X]$ is a direct sum of pairwise distinct simple $G$-modules.*

*Proof.* (i) $\Leftrightarrow$ (ii) follows from Rosenlicht's theorem stated at the beginning of Subsection 1.2.

(ii) $\Rightarrow$ (iii) Assume that the $G$-module $\mathbb{C}[X]$ contains two distinct copies of a simple module $V(\lambda)$. It follows that $\mathbb{C}[X]$ contains two non-proportional $B$-eigenvectors $f_1, f_2$ of the same weight $\lambda$. So the quotient $\frac{f_1}{f_2}$ is a non-constant $B$-invariant rational function, a contradiction.

(iii) $\Rightarrow$ (ii) Let $f \in \mathbb{C}(X)^B$. By Proposition 2.8 (i), we have $f = \frac{f_1}{f_2}$ where $f_1, f_2 \in \mathbb{C}[X]$ are $B$-eigenvectors with the same weight. It follows that $f_1, f_2$ are proportional, i.e., $f$ is constant. □

**Definition 2.13.** A $G$-module $V$ is *multiplicity-free* if $V$ is a direct sum of pairwise non-isomorphic simple $G$-modules. Equivalently, $\dim V_\lambda^{(B)} \le 1$ for all $\lambda \in \Lambda^+$.

**Theorem 2.14.** *For an affine irreducible $G$-variety $X$, the following conditions are equivalent:*

(i) *$X$ is spherical.*

(ii) *The $G$-module $\mathbb{C}[X]$ is multiplicity-free, and the weight monoid $\Lambda^+(X)$ is saturated.*

(iii) *The affine $T$-variety $X/\!/U$ is toric.*

*Then $X$ contains only finitely many $G$-orbits, and their closures are spherical varieties; they correspond bijectively to certain faces of the weight cone $C(X)$, and their weight groups are direct factors of $\Lambda(X)$.*

*Proof.* (i) $\Rightarrow$ (iii) Since $X$ is normal, then so is $X/\!/U$ by Proposition 2.8 (ii). Moreover, the $T$-module $\mathbb{C}[X/\!/U]$ is multiplicity-free. Thus, $X/\!/U$ is toric by Lemma 2.12

(iii) $\Rightarrow$ (ii) follows from the fact that the weight monoid of any affine toric variety is saturated.

(ii) $\Rightarrow$ (i) $X$ contains a dense $B$-orbit by Lemma 2.12, and $X//U$ is normal by Example 2.11.1. In view of Proposition 2.8, it follows that $X$ is normal.

To show the final assertion, note that any irreducible $G$-stable subvariety $Y \subset X$ yields an irreducible $T$-stable subvariety $Y//U \subset X//U$, which determines $Y$ uniquely (since the $G$-stable prime ideal $I(Y) \subset \mathbb{C}[X]$ is uniquely determined by $I(Y)^U \subset \mathbb{C}[X]^U$, a $T$-stable prime ideal). We conclude by combining Exercise 2.11.1 with Proposition 2.8 again. □

In fact, *every affine spherical variety contains only finitely many $B$-orbits*, as a consequence of the preceding result combined with:

**Theorem 2.15.** *Any spherical homogeneous space contains only finitely many $B$-orbits.*

*Proof.* If $G = \mathrm{SL}_2$, then the statement follows from Example 2.11.2. The general case may be reduced to that one as follows.

Let $X$ be a spherical $G$-homogeneous space. It suffices to show that each irreducible $B$-stable subvariety $Y \subset X$ contains an open $B$-orbit. For this, we argue by induction on the codimension $n$ of $Y$; if $n = 0$, the desired statement is just the assumption that $X$ contains an open $B$-orbit.

Let $Y$ be an irreducible $B$-stable subvariety of codimension $n$ in $X$. Since $X$ is homogeneous, we have $G \cdot Y = X$. Now recall that $G$ is generated by its *minimal parabolic subgroups*, i.e., by the closed subgroups properly containing $B$, and minimal for this property. Moreover, every such subgroup $P$ is the semi-direct product of its radical $R(P)$ (the largest connected solvable normal subgroup of $P$) with a subgroup $S$ isomorphic to $\mathrm{SL}_2$ or $\mathrm{PSL}_2$; in particular, $R(P) \subset B$, the quotient $B/R(P)$ is a Borel subgroup of $P$, and $P/B \cong \mathbb{P}^1$ (for these results, see e.g. [13, Section 8.4]). Thus, there exists a minimal parabolic subgroup $P$ such that $P \cdot Y \neq Y$; then $Z := P \cdot Y$ is a closed $P$-stable subvariety of $X$, and $\dim(Z) = \dim(Y) + 1$ so that $\mathrm{codim}(Z) = n - 1$. By the induction assumption, $Z$ contains an open $B$-orbit $Z_0$. The quotient $(P \cdot Z_0)/R(P)$ is an irreducible variety, homogeneous under $S$ (since $P \cdot Z_0$ is homogeneous under $P$) and containing $Z_0/R(P)$ as an open orbit of $B/R(P)$. Therefore,$(P \cdot Z_0)/R(P)$ contains only finitely many orbits of $B/R(P)$, i.e., $P \cdot Z_0$ contains only finitely many $B$-orbits. But $P \cdot Z_0$ is open in $Z = P \cdot Y$, and hence contains a $B$-stable open subset of $Y$; thus, $Y$ contains an open $B$-orbit as desired. □

Also, recall from Examples 2.11 that any affine toric variety $X$ is uniquely determined by its weight monoid $\Lambda^+(X)$ (or, equivalently, by its weight cone and weight group); in contrast, there exist affine spherical varieties having the same weight monoid, but non-isomorphic as varieties. However, *every smooth affine spherical variety is uniquely determined by its weight monoid*, by a recent result of Losev which solves a conjecture of Knop (see [7] for this, and for more on uniqueness properties of spherical varieties).

## 2.3. Projective spherical varieties

**Definition 2.16.** A *polarized variety* is a pair $(X, L)$, where $X$ is an irreducible projective variety, and $L$ an ample line bundle on $X$. If $X$ is equipped with a $G$-action, and $L$ with a $G$-linearization, then $(X, L)$ is called a *polarized $G$-variety.*

To any polarized variety $(X, L)$, one associates the *section ring*

$$R(X, L) := \bigoplus_{n=0}^{\infty} \Gamma(X, L^n).$$

This is a positively graded algebra, equipped with a $G$-action if so is $(X, L)$. Note that $R(X, L)$ is the algebra of regular functions on the total space of the dual line bundle $L^\vee$.

We now gather some basic properties of polarized varieties, their easy proofs being left to the reader:

**Lemma 2.17.** (i) *The algebra $R(X, L)$ is a finitely generated domain.*

(ii) *Let $\widetilde{X}$ denote the affine $\mathbb{C}^*$-variety such that $\mathbb{C}[\widetilde{X}] = R(X, L)$, and let*

$$\varphi : L^\vee \longrightarrow \widetilde{X}$$

*be the natural map. Then $\varphi$ restricts to an isomorphism*

$$L^\vee \setminus s_0(X) \cong \widetilde{X} \setminus \{0\},$$

*where $s_0 : X \to L^\vee$ denotes the zero section, and $0 \in \widetilde{X}$ is the point associated with the maximal homogeneous ideal of $R(X,L)$. As a consequence, $X = \operatorname{Proj} R(X,L)$.*

(iii) *$X$ is normal if and only if $\widetilde{X}$ is normal.*

(iv) *For any irreducible subvariety $Y \subset X$, the restriction map $R(X,L) \to R(Y, L|_Y)$ yields a finite morphism $\widetilde{Y} \to \widetilde{X}$, birational onto its image.*

We say that $\widetilde{X}$ is the *affine cone* over $X$ associated with the ample line bundle $L$.

Next, consider a polarized $G$-variety $(X,L)$. Then the group $\widetilde{G} := G \times \mathbb{G}_m$ acts on $L^\vee$, where $\mathbb{C}^*$ acts by scalar multiplication on fibers. Moreover, the zero section $s_0$ is $\widetilde{G}$-equivariant, and $\widetilde{G}$ also acts on $\widetilde{X}$ so that $\varphi$ is equivariant. Each space $\Gamma(X, L^n)$ is a finite-dimensional $G$-module.

Note that $\widetilde{G}$ is a connected reductive group with Borel subgroup $\widetilde{B} := B \times \mathbb{G}_m$, maximal torus $\widetilde{T} := T \times \mathbb{G}_m$ and weight lattice $\widetilde{\Lambda} := \Lambda \times \mathbb{Z}$. Moreover, the set of dominant weights $\widetilde{\Lambda}^+$ equals $\Lambda^+ \times \mathbb{Z}$.

We may now characterize projective spherical varieties in terms of their affine cones:

**Proposition 2.18.** *The following conditions are equivalent for a polarized $G$-variety $(X,L)$:*

(i) *The $G$-variety $X$ is spherical.*

(ii) *The $\widetilde{G}$-variety $\widetilde{X}$ is spherical.*

(iii) *$X$ is normal, and the $G$-module $\Gamma(X, L^n)$ is multiplicity-free for any integer $n$.*

*Proof.* (i) $\Rightarrow$ (ii) If $B$ has an open orbit in $X$, then the pull-back of this orbit in $L^\vee \setminus s_0(X)$ is an open orbit of $\widetilde{B}$. We conclude by Lemma 2.17 (ii) and (iii).

(ii) $\Rightarrow$ (i) is checked similarly.

(ii) $\Leftrightarrow$ (iii) follows from Theorem 2.14 combined with Lemma 2.17 (iii). $\square$

We say that $(X,L)$ is a *polarized spherical variety* if it satisfies one of these conditions.

Returning to an arbitrary polarized $G$-variety $(X,L)$, let $\widetilde{\Lambda}^+(X,L)$ (resp. $\widetilde{C}(X,L)$, $\widetilde{\Lambda}(X,L)$) be the weight monoid (resp. weight cone, weight group) of the irreducible affine $\widetilde{G}$-variety $\widetilde{X}$. Then $\widetilde{\Lambda}^+(X,L) \subset \Lambda^+ \times \mathbb{Z}$ consists of those pairs $(\lambda, n)$ such that $\Gamma(X, L^n)$ contains a $B$-eigenvector of weight $\lambda$. In particular, $n > 0$ for each non-zero such pair. Thus, each non-zero point of the finitely generated cone $\widetilde{C}(X,L) \subset \Lambda^+_{\mathbb{R}} \times \mathbb{R}$ has a positive coordinate on $\mathbb{R}$. This implies easily the following:

**Lemma 2.19.** (i) *The intersection*

$$Q(X,L) := \widetilde{C}(X,L) \cap (\Lambda_{\mathbb{R}} \times \{1\})$$

*is a rational convex polytope in the affine hyperplane $\Lambda_{\mathbb{R}} \times \{1\}$ of $\widetilde{\Lambda}_{\mathbb{R}}$.*

(ii) *The rational points of $Q(X,L)$ are exactly the quotients $\frac{\lambda}{n}$, where $n$ is a positive integer and $\lambda$ is the weight of a $B$-eigenvector in $\Gamma(X, L^n)$. Moreover, $\widetilde{C}(X,L)$ is the cone over $Q(X,L)$.*

(iii) *$Q(X,L)$ is the convex hull of the points $\frac{\lambda_1}{n_1}, \cdots, \frac{\lambda_N}{n_N}$, where the pairs $(\lambda_i, n_i) \in \widetilde{\Lambda}$ are the weights of $\widetilde{B}$-eigenvectors in $R(X,L)$ which generate the algebra $R(X,L)^U$.*

**Definition 2.20.** With the preceding notation, $Q(X,L)$ is called the *moment polytope* of the polarized $G$-variety $(X,L)$.

As in the affine case, the *weight group* of $X$ is the subgroup $\Lambda(X) \subset \Lambda$ consisting of the weights of $B$-eigenvectors in $\mathbb{C}(X)$.

The *rank* $\mathrm{rk}(X)$ is the rank of its weight group.

**Lemma 2.21.** *Let $(X,L)$ be a polarized $G$-variety. Then the following hold:*

(i) *The second projection $p_2 : \widetilde{\Lambda} \to \mathbb{Z}$ yields an exact sequence*

$$0 \to \Lambda(X) \to \widetilde{\Lambda}(X,L) \to \mathbb{Z} \to 0.$$

(ii) *The vector space $\Lambda(X)_{\mathbb{R}}$ is spanned by the differences of any two points of the moment polytope $Q(X,L)$. As a consequence, $\dim Q(X,L) = \mathrm{rk}(X)$.*

*Proof.* (i) Since $L$ is ample, there exists a positive integer $N$ such that $L^N$ and $L^{N+1}$ are very ample; in particular, the $G$-modules $\Gamma(X, L^N)$ and $\Gamma(X, L^{N+1})$ are both non-zero. It follows that $\widetilde{\Lambda}(X, L)$ contains elements of the form $(\lambda, N)$ and $(\mu, N+1)$. So $p_2$ is surjective.

The elements of $\widetilde{\Lambda}(X, L)$ of the form $(\lambda, 0)$ are exactly the weights of $B$-eigenvectors in the invariant field $\mathbb{C}(\widetilde{X})^{\mathbb{G}_m}$. But the latter field equals $\mathbb{C}(X)$, as follows from Lemma 2.17 (ii). So the kernel of $p_2$ is $\Lambda(X)$.

(ii) Let $x_1, x_2$ be rational points of $Q(X, L)$. By Lemma 2.21, we may write $x_i = \frac{\lambda_i}{n_i}$ where $\lambda_i$ is the weight of a $B$-eigenvector $s_i \in \Gamma(X, L^{n_i})$. Then $n_1 n_2(x_1 - x_2)$ is the weight of $\frac{s_1^{n_2}}{s_2^{n_1}}$, a $B$-eigenvector in $\mathbb{C}(X)$. Thus, $n_1 n_2(x_1 - x_2) \in \Lambda(X)$. Since $Q(X, L)$ is a rational polytope, it follows that the differences of any two of its points lie in $\Lambda(X)_{\mathbb{R}}$.

To show that these differences span $\Lambda(X)_{\mathbb{R}}$, consider $\lambda \in \Lambda$ and let $f \in \mathbb{C}(X)$ be a $B$-eigenvector of weight $\lambda$. Then there exist a positive integer $n$ and two $B$-eigenvectors $s_1, s_2 \in \Gamma(X, L^n)$ such that $f = \frac{s_1}{s_2}$, as follows from Proposition 2.8 (i). Thus, $\frac{\lambda}{n}$ is the difference of two points of $Q(X, L)$. $\square$

Next, we obtain a version of Theorem 2.14 for projective spherical varieties:

**Theorem 2.22.** *Let $(X, L)$ be a polarized spherical variety and $Y \subset X$ an irreducible $G$-stable subvariety. Then the following hold:*

(i) *$(Y, L)$ is a polarized spherical variety.*

(ii) *$\widetilde{\Lambda}(Y, L)$ is a direct summand of $\widetilde{\Lambda}(X, L)$. Thus, $\Lambda(Y)$ is a direct summand of $\Lambda(X)$.*

(iii) *$Q(Y, L)$ is a face of $Q(X, L)$ which determines $Y$ uniquely. Thus, $X$ contains only finitely many $G$-orbits, and these are spherical.*

(iv) *The restriction map $\Gamma(X, L^n) \to \Gamma(Y, L^n)$ is surjective for all $n \geq 0$.*

*Proof.* By Lemma 2.17 (iv), the natural map $\varphi : \widetilde{Y} \to \widetilde{X}$ is finite and birational. Thus, $\varphi(\widetilde{Y})$ is an irreducible $\widetilde{G}$-subvariety of $\widetilde{X}$. By Theorem 2.14, $\varphi(\widetilde{Y})$ is normal, and hence $\varphi$ is a closed immersion. This implies (iv), and also (i) in view of Lemma 2.17 (iii). The remaining assertions follow from Theorem 2.14 again and Theorem 2.22. $\square$

**Corollary 2.23.** *Let $L$ be an ample line bundle on a projective spherical variety $X$. Then $L$ is generated by its global sections.*

*Proof.* Replacing the acting group $G$ with a finite cover, we may assume that $L$ is $G$-linearized. Then its base locus $Z \subset X$ (consisting of common zeroes of all global sections) is a closed $G$-stable subset of $X$. If $Z$ is non-empty, then it contains a closed orbit $Y$. By Theorem 2.22 (iv), it suffices to show that $\Gamma(Y, L)$ is non-zero; in other words, we may assume that $X$ is homogeneous.

By Borel's fixed point theorem, we then have $X = G/P$ for some (parabolic) subgroup $P \supset B$; this yields a morphism $\pi : G/B \to G/P$ with connected fibers. Thus, the natural map $\Gamma(X, L) \to \Gamma(G/B, \pi^* L)$ is an isomorphism. The $G$-linearized line bundle $\pi^* L$ on $G/B$ yields a character $\lambda \in \Lambda$ (the weight of its fibre at the base point), and we have an isomorphism of $G$-modules $\Gamma(G/B, \pi^* L^m) \cong \mathbb{C}[G]^{(B)}_{m\lambda}$ for any integer $m$. Since $L$ is ample, the left-hand side is non-zero for $m \gg 0$. By Theorem 2.4, it follows that $\lambda$ is dominant; we conclude that $\Gamma(G/B, \pi^* L) \neq 0$. $\square$

**Examples 2.24.** 1) Let $G = T$ as in Example 2.11.1 and consider a polarized toric variety $(X, L)$. Then the $T$-orbits in $X$ correspond bijectively to the faces of $Q(X, L)$, and this correspondence preserves the dimensions and inclusions of closures, in view of that example combined with Lemma 2.21. In particular, the closed orbits correspond to the vertices. As a consequence, $Q(X, L)$ *is an integral polytope*, i.e., its vertices are all in $\Lambda$. Moreover, the polarized toric varieties (under an unspecified torus) are in bijective correspondence with the pairs $(\Lambda, Q)$ where $\Lambda$ is a lattice and $Q$ is an integral polytope in $\Lambda_{\mathbb{R}}$.

2) Let Let $G = \mathrm{GL}_n$ act on $\mathrm{Q}_n$ as in Example 2.11.4. Then $X := \mathbb{P}(\mathrm{Q}_n)$ is spherical and its moment polytope $Q$ has vertices the points $\frac{2\omega_k}{k}$ for $k = 1, \ldots, n$. In particular, $Q$ is a simplex, and has non-integral vertices if $n \geq 3$. One checks that the $G$-orbit closures correspond to the simplices over the first $\ell$ vertices, for $\ell = 1, 2, \ldots, n$. Thus, most faces of $Q$ do not arise from orbit closures.

3) Let $G := \mathrm{SL}_2$ act diagonally on $X := \mathbb{P}^1 \times \mathbb{P}^1$. Then $L := \mathcal{O}(1,1)$ is very ample and $G$-linearized; its global sections embed $X$ as a smooth quadric hypersurface in $\mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2) \cong \mathbb{P}^3$. With the notation of Example 2.64., we have isomorphisms of $G$-modules

$$\mathbb{C}^2 \otimes \mathbb{C}^2 = V(1) \otimes V(1) \cong V(0) \oplus V(2)$$

and one checks that the moment polytope $Q(X, L)$ is the interval $[0, 2]$, while $\Lambda(X) = 2\Lambda \cong 2\mathbb{Z}$.

The variety

$$X' := (\Delta = 0) \subset \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2),$$

where $\Delta$ denotes the discriminant of $V(2)$ (the space of quadratic forms in two variables), is a $G$-stable quadratic cone in $\mathbb{P}^3$. One checks that $X'$ is spherical, and the pair $\big(X', L' := \mathcal{O}(1)\big)$ has the same moment polytope and weight lattice as $(X, L)$.

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