On linearization of line bundles

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Abstract

We study the linearization of line bundles and the local properties of actions of connected linear algebraic groups, in the setting of seminormal varieties. We show that several classical results on normal varieties extend to that setting, if the Zariski topology is replaced with the étale topology.

1 Introduction

Linearization of line bundles in the presence of algebraic group actions is a basic notion of geometric invariant theory; it also has applications to the local properties of such actions. For example, given an action of a connected linear algebraic group G on a normal variety X over a field k, and a line bundle L on X, some positive power $L^{\otimes n}$ admits a G-linearization (as shown by Mumford [MFK94, Cor. 1.6] when X is proper, and by Sumihiro [Su75, Thm. 1.6] in a more general setting of group schemes; when k is algebraically closed of characteristic 0, we may take for n the order of the Picard group of G as shown by Knop, Kraft, Luna and Vust [KKLV89, 2.4]). It can be inferred that X is covered by G-stable Zariski open subsets U_i, equivariantly isomorphic to G-stable subvarieties of projectivizations of finite-dimensional G-modules; if G is a split torus, then the U_i may be taken affine (see [Su74, Cor. 2], [Su75, Thm. 3.8, Cor. 3.11], [KKLV89, Thm. 1.1]).

The latter result does not extend to non-normal varieties, a classical example being the nodal curve X obtained from the projective line $\mathbb{P}^1$ by identifying 0 and ∞: the natural action of the multiplicative group $\mathbb{G}_m$ on $\mathbb{P}^1$ yields an action on X, and every $\mathbb{G}_m$-stable open neighborhood of the node is the whole X. Yet X admits an equivariant étale covering by an affine variety, namely, the union of two affine lines meeting at their origin, where $\mathbb{G}_m$ acts by scalar multiplication on each line.

In this article, we show that the above results on linearization of line bundles and the local properties of algebraic group actions hold under weaker assumptions than normality, if the Zariski topology is replaced with the étale topology. For simplicity, we state our main result in the case where k is algebraically closed:

Theorem 1.1. *Let X be a variety equipped with an action of a connected linear algebraic group G.*

(i) *If X is seminormal, then there exists a torsor $\pi : Y \to X$ under the character group of G, and a positive integer n (depending only on G) such that $\pi^*(L^{\otimes n})$ is G-linearizable for any line bundle L on X.*

(ii) *If in addition X is quasi-projective, then it admits an equivariant étale covering by G-stable subvarieties of projectivizations of finite-dimensional G-modules.*

(iii) *If G is a torus and X is quasi-projective, then X admits an equivariant étale covering by affine varieties.*

We now provide details on the notions occuring in the above statement. By a variety, we mean a reduced separated scheme of finite type over the ground field (in particular, varieties need not be irreducible). Also, a line bundle L on a G-variety X is called G-linearizable, if L admits a G-action which lifts the action on X and is linear on fibers (see [MFK94, Sec. 1.3] or Subsection 2.2 for further details on linearization). Finally, recall the definition of seminormality: a reduced scheme X is seminormal if every integral bijective morphism $f : X' \to X$ which induces an isomorphism on all residue fields, is an isomorphism. Every reduced scheme X has a seminormalization map $\sigma : X^+ \to X$, which factors the normalization map $\eta : \tilde{X} \to X$ (see [AB69, GT80, Sw80]). Nodal curves are seminormal, but cuspidal curves are not. Any projective cuspidal cubic curve X has an action of the multiplicative group $\mathbb{G}_m$ for which most line bundles on X are not linearizable; see [Al02, 4.1.5] or Example 2.16 for details. Thus, Theorem 1.1 (i) does not extend to arbitrarily singular varieties.

That result takes a much simpler form when the character group of G is trivial, e.g., if G is unipotent or semisimple. Then any line bundle on a seminormal G-variety X has a G-linearizable positive power (depending only on G). As a consequence, if X is quasi-projective, then it admits an equivariant embedding in the projectivization of a finite-dimensional G-module. Yet when the character group of G is nontrivial, the cover $\pi : Y \to X$ is infinite and hence Y is not of finite type; but its irreducible components are varieties, as follows from Proposition 4.3.

In the case where X is proper and G is a torus, Theorem 1.1 (i) has been obtained by Alexeev (see [Al02, Thm. 4.3.1]) in the process of the construction of certain moduli spaces. His proof is based on the representability of the Picard functor, and hence does not extend to our general setting. We rather rely on results and methods from algebraic K-theory, taken from an article of Weibel (see [We91]) which is chiefly concerned with the Picard group of Laurent polynomial rings over commutative rings. The connection with linearization will hopefully be clear from the following overview of the present article.

We work over an arbitrary field k; this raises some technical issues, as for example there exist connected unipotent groups having an infinite Picard group, whenever k is imperfect (see [KMT74, Sec. 6.12]).

In Section 2, we gather preliminary results on the Picard group of linear algebraic groups, and on the equivariant Picard group $\mathrm{Pic}^G(X)$ which classifies G-linearized line bundles on a G-scheme X; these results are variants of those in [Su75, KKV89, KKLV89]. In particular, when X is reduced, we obtain an exact sequence

$$\mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X),$$

where φ denotes the forgetful map, and the obstruction map ψ arises from the pull-back under the action morphism $G \times X \to X$ (see Proposition 2.10). Finally, we show that the obstruction group $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ is n-torsion if X is normal, where n is a positive integer depending only on G (Theorem 2.14).

The obstruction group is studied further in Section 3. We construct an injective map $c : H^1_{\text{ét}}(X, \widehat{G}) \to \text{Pic}(G \times X)/p_2^*\text{Pic}(X)$, where the left-hand side denotes the first étale cohomology group with coefficients in the character group of G (viewed as an étale sheaf); recall that this cohomology group classifies $\widehat{G}$-torsors over X.

Our main technical result (Theorem 3.3) asserts in particular that the cokernel of c is n-torsion for n as above, if X is a geometrically seminormal variety. When $G = \mathbb{G}_m$, so that $\widehat{G} = \mathbb{Z}$ and $G \times X = X[t, t^{-1}]$, the map $c : H^1_{\text{ét}}(X, \mathbb{Z}) \to \text{Pic}(X[t, t^{-1}])/\text{Pic}(X)$ is a key ingredient of [We91, Sec. 7], where it is shown that c is an isomorphism if X is seminormal. For an arbitrary G, our proof proceeds via a reduction to $\mathbb{G}_m$ by analyzing the behavior of the Picard group under various fibrations.

In Section 4, we present several applications of our analysis of the obstruction group. We first show that linearizability is preserved under algebraic equivalence (Proposition 4.1). Then we obtain a version of Theorem 1.1 over an arbitrary base field (Theorems 4.4, 4.7 and 4.8). Finally, we show that the seminormality assumption in Theorem 1.1 (i) and (ii) may be suppressed in prime characteristics (Subsection 4.3).

Further applications, to the theorem of the square and the local properties of nonlinear group actions, will be presented in the follow-up article [Br14].

Notation and conventions. We consider schemes, their morphisms and their products over an arbitrary field k, with algebraic closure $\bar{k}$. All schemes are assumed to be separated and locally noetherian. For any such scheme X, we denote by $\mathcal{O}(X)$ the k-algebra of global sections of the structure sheaf, and by $\mathcal{O}(X)^*$ the group of units (i.e., invertible elements) of that algebra. The scheme obtained from X by base change via a field extension K/k is denoted by X_K.

A smooth group scheme of finite type will be called an algebraic group. Throughout this article, G denotes a connected algebraic group, and e_G its neutral element; a G-scheme is a scheme X equipped with a G-action $\alpha : G \times X \to X$. We denote by $\widehat{G} = \text{Hom}_{\text{gp}}(G, \mathbb{G}_m)$ the character group scheme of G. We will view $\widehat{G}$ as an étale sheaf of free abelian groups of finite rank on $\text{Spec}(k)$, and denote by $\widehat{G}(S)$ the abelian group of sections of $\widehat{G}$ over a scheme S.

2 Preliminary results

2.1 The Picard group of a linear algebraic group

Recall that a connected unipotent group (resp. a torus) is said to be split, if it is an iterated extension of copies of the additive group $\mathbb{G}_a$ (resp. of the multiplicative group $\mathbb{G}_m$). Also, a connected reductive group is said to be split, if it has a split maximal torus. We now introduce a direct generalization of these notions:

Definition 2.1. We say that G is *split* if there is an exact sequence of algebraic groups

$$1 \longrightarrow U \longrightarrow G \longrightarrow H \longrightarrow 1, \tag{1}$$

where U is a split connected unipotent group, and H is a split connected reductive group.

Remarks 2.2. (i) The exact sequence (1) is unique if it exists, since U is the unipotent radical of G. Also, the class of split algebraic groups is stable under quotients by closed normal subgroups and under base change by field extensions.
(ii) The split solvable groups in the above sense are exactly the extensions of split tori by split connected unipotent groups. This is equivalent to the usual notion of split solvable groups (iterated extensions of copies of $\mathbb{G}_a$ and $\mathbb{G}_m$) in view of [Bo91, Thm. 15.4].
(iii) If k is perfect, then any connected unipotent group is split (see e.g. [Bo91, Cor. 15.5]); moreover, the unipotent radical of $G_{\bar{k}}$ is defined over k. It follows that G is split if and only if it has a split maximal torus. As a consequence, the class of split algebraic groups is also stable under group extensions.
(iv) If k is imperfect, then there exist nontrivial forms of $\mathbb{G}_a$, i.e., nonsplit connected unipotent groups of dimension 1 (see e.g. [Ru70, Thm. 2.1]).
(v) Clearly, any split group is connected and linear. Also, $G_{\bar{k}}$ is split for any connected linear algebraic group G; thus, $G_{k'}$ is split for some finite extension of fields k'/k. But such an extension may not be chosen separable; for example, when G is a nontrivial form of $\mathbb{G}_a$ (see e.g. [Ru70, Lem. 1.1, Lem. 1.2]).

Lemma 2.3. *If G is split, then the sheaf $\widehat{G}$ is constant. Moreover,* $\mathrm{Pic}(G)$ *is finite and the natural map* $\mathrm{Pic}(G) \to \mathrm{Pic}(G_K)$ *is an isomorphism for any field extension K/k.*

Proof. With the notation of Definition 2.1, the sheaf $\widehat{U}$ is trivial, and hence the pull-back map $\widehat{H} \to \widehat{G}$ is an isomorphism. Choose a split maximal torus T of H. Then the étale sheaf of abelian groups $\widehat{T}$ on $\mathrm{Spec}(k)$ is constant, and the pull-back map $\widehat{H} \to \widehat{T}$ is injective; it follows that $\widehat{G}$ is constant as well.

Also, U (viewed as a variety) is isomorphic to an affine space $\mathbb{A}^n$, and G (viewed as a variety again) is isomorphic to $U \times H$, since the U-torsor $G \to G/U \cong H$ is trivial. It follows that the pull-back map $\mathrm{Pic}(H_K) \to \mathrm{Pic}(G_K)$ is an isomorphism for any field extension K/k. Thus, we may assume that G is reductive. Choose a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. By [Iv76, Prop. 1.9], we have an exact sequence

$$0 \longrightarrow \widehat{G} \xrightarrow{i^*} \widehat{T} \xrightarrow{\gamma} \mathrm{Pic}(G/B) \xrightarrow{f^*} \mathrm{Pic}(G) \longrightarrow 0, \tag{2}$$

where $i : T \to G$ denotes the inclusion, γ the characteristic homomorphism, and $f : G \to G/B$ the quotient map. Furthermore, $\mathrm{Pic}(G/B)$ and γ can be explicitly described in terms of the root datum of (G, T), in view of [Iv76, Prop. 5.2, Thm. 5.3]. It follows that $\mathrm{Pic}(G)$ depends only on this root datum (as already observed in [Ra70, Rem. VII.1.7.a)]); moreover, this datum is unchanged under field extensions. □

We also record the following observation, implicit in [Ra70, Lem. VII.1.6.1]:

Lemma 2.4. *Let X be a scheme, and k'/k a finite extension of fields. Then the kernel of the pull-back map* $\mathrm{Pic}(X) \to \mathrm{Pic}(X_{k'})$ *is killed by* $[k' : k]$.

Proof. Consider a line bundle L on X such that $L_{k'}$ is trivial. Then the norm $\mathrm{N}(L_{k'})$ (defined in [EGA, II.6.5]) is trivial as well. But $\mathrm{N}(L_{k'}) \cong L^{\otimes [k':k]}$ by [EGA, II.6.5.2.4]; this yields the assertion. □

We now obtain a refinement of a result of Raynaud (see [Ra70, Cor. VII.1.6]). Assume that G is linear, and consider a finite extension k'/k of fields such that $G_{k'}$ is split. By Lemma 2.3, the group $\mathrm{Pic}(G_{k'})$ is finite and independent of k'. Denote by $m = m(G)$ the exponent of that group (i.e., the smallest positive integer such that $\mathrm{Pic}(G_{k'})$ is m-torsion) and by $d = d(G)$ the greatest common divisor of the degrees $[k' : k]$ of splitting fields for G. Finally, set $n = n(G) := d\,m$.

Proposition 2.5. *With the above notation and assumptions, the abelian group* $\mathrm{Pic}(G_K)$ *is killed by* n *for any field extension* K/k.

Proof. Choose a splitting field k' and a maximal ideal $\mathfrak{m}$ of the algebra $K \otimes_k k'$. Then the quotient field $K' := (K \otimes_k k')/\mathfrak{m}$ is a finite extension of K of degree dividing $[k' : k]$. By Lemma 2.4, it follows that the kernel of the pull-back map $\mathrm{Pic}(G_K) \to \mathrm{Pic}(G_{K'})$ is killed by $[k' : k]$. Moreover, K' contains k' and hence $\mathrm{Pic}(G_{K'}) \cong \mathrm{Pic}(G_{k'})$ in view of Lemma 2.3. Thus, $\mathrm{Pic}(G_K)$ is killed by $[k' : k]\,m$. □

We say that n is the *stable exponent* of $\mathrm{Pic}(G)$. When G is split (e.g., when k is algebraically closed), n is just the exponent of that group. For an arbitrary connected linear algebraic group G, we do not know any example where the stable exponent differs from the exponent of $\mathrm{Pic}(G)$.

2.2 A criterion for linearizability

We first obtain a variant of another result of Raynaud (see [Ra70, Cor. VII.1.2]):

Lemma 2.6. *Let X be a reduced scheme. Then the multiplication map*

$$\mu : \widehat{G}(X) \times \mathcal{O}(X)^* \longrightarrow \mathcal{O}(G \times X)^*, \quad (\chi, f) \longmapsto ((g, x) \mapsto \chi(x)(g)\, f(x))$$

is an isomorphism.

Proof. Clearly, μ is a group homomorphism. If $\mu(\chi, f) = 1$ then pulling back to $\{e_G\} \times X$, we get $f = 1$ and hence $\chi = 1$; thus, μ is injective.

To show the surjectivity, consider $f \in \mathcal{O}(G \times X)^*$. Replacing f with the map $(g, x) \mapsto f(g, x)\, f(e_G, x)^{-1}$, we may assume that $f(e_G, x) = 1$ identically. Then the map $g \mapsto f(g, x)$ is a character for any point x, by [Ro57, Prop. 3]. Therefore, $f \in \widehat{G}(X)$. □

Lemma 2.7. *Let X be a reduced G-scheme. Then for any $f \in \mathcal{O}(X)^*$, there exists a unique $\chi = \chi(f) \in \widehat{G}(X)$ such that $f(\alpha(g, x)) = \chi(x)(g)f(x)$ identically. Moreover, the assignement $f \mapsto \chi(f)$ yields an exact sequence*

$$0 \longrightarrow \mathcal{O}(X)^{*G} \longrightarrow \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G}(X), \tag{3}$$

where $\mathcal{O}(X)^{*G}$ denotes the subgroup of G-invariants in $\mathcal{O}(X)^*$.

Proof. Applying Lemma 2.6 to $f \circ \alpha \in \mathcal{O}(G \times X)^*$ yields $\chi \in \widehat{G}(X)$ and $\varphi \in \mathcal{O}(X)^*$ such that $f(\alpha(g, x)) = \chi(x)(g)\, \varphi(x)$ identically. By evaluating at $g = e_G$, we obtain $\varphi = f$. This yields the first assertion; the second one follows readily. □

Lemma 2.8. *Let X be a reduced scheme, and $f \in \mathcal{O}(G \times G \times X)^*$ such that $f(e_G, g, x) = 1 = f(g, e_G, x)$ identically. Then $f = 1$.*

Proof. By Lemma 2.6 and the isomorphism $\widehat{G \times G} \cong \widehat{G} \times \widehat{G}$, there exist $\chi, \eta \in \widehat{G}(X)$ and $\varphi \in \mathcal{O}(X)^*$ such that $f(g, h, x) = \chi(x)(g)\, \eta(x)(h)\, \varphi(x)$ identically. Then the assumption means that $\eta(x)(h)\, \varphi(x) = 1 = \chi(x)(g)\, \varphi(x)$ identically, and hence $\varphi = 1 = \chi = \eta$. □

Let X be a G-scheme, and $\pi : L \to X$ a line bundle. Recall from [MFK94, Def. 1.6] that a G-linearization of L is an action of G on the scheme L which lifts the given G-action α on X, and commutes with the $\mathbb{G}_m$-action on L by multiplication on fibers. Equivalently, a G-linearization of L is an isomorphism $\Phi : \alpha^*(L) \to p_2^*(L)$ of line bundles on $G \times X$, which satisfies the cocycle condition $\Phi_{gh} = \Phi_h \circ h^*(\Phi_g)$ for all points g, h of G.

When X is reduced, this cocycle condition may be omitted, as shown by the following result (implicit in [Su75, p. 577]; for the case where k is algebraically closed of characteristic 0, see [KKLV89, Lem. 2.3]):

Lemma 2.9. *Let X be a reduced G-scheme, and L a line bundle on X. Then L admits a G-linearization if and only if the line bundles $\alpha^*(L)$ and $p_2^*(L)$ on $G \times X$ are isomorphic.*

Proof. Let $\Phi : \alpha^*(L) \to p_2^*(L)$ be an isomorphism. Since $\alpha \circ (e_G \times \mathrm{id}_X) = p_2 \circ (e_G \times \mathrm{id}_X) = \mathrm{id}_X$, the pull-back $(e_G \times \mathrm{id}_X)^*(\Phi)$ is identified with an automorphism of the line bundle $\pi : L \to X$, i.e., with the multiplication by some $f \in \mathcal{O}(X)^*$. Replacing Φ with $\Phi \circ p_2^*(f)^{-1}$, we may assume that $f = 1$. Then Φ corresponds to a morphism $\beta : G \times L \to L$ such that the diagram

$$\begin{array}{ccc} G \times L & \xrightarrow{\beta} & L \\ {\scriptstyle \mathrm{id}_G \times \pi}\downarrow & & {\scriptstyle \pi}\downarrow \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

commutes; moreover, $\beta(e_G, z) = z$ identically. It remains to show that β satisfies the associativity condition of a group action. But the obstruction to associativity is an automorphism of the line bundle $\mathrm{id}_{G \times G} \times \pi : G \times G \times L \to G \times G \times X$, i.e., the multiplication by some $\varphi \in \mathcal{O}(G \times G \times X)^*$. Moreover, since $\beta(g, \beta(e_G, z)) = \beta(g, z) = \beta(e_G, \beta(g, z))$ identically, we have $\varphi(g, e_G, x) = 1 = \varphi(e_G, g, x)$. By Lemma 2.8, it follows that $\varphi = 1$, i.e., β is associative. □

2.3 The equivariant Picard group

The isomorphism classes of G-linearized line bundles on a given G-scheme X form an abelian group that we will call the equivariant Picard group, and denote by $\mathrm{Pic}^G(X)$. This group is equipped with a homomorphism

$$\varphi : \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}(X)$$

which forgets the linearization. Also, we have a homomorphism

$$\gamma : \widehat{G}(X) \longrightarrow \mathrm{Pic}^G(X)$$

which assigns to any $\chi \in \widehat{G}(X)$, the class of the trivial line bundle $p_1 : X \times \mathbb{A}^1 \to X$ on which G acts by $\beta(g, x, t) := (\alpha(g, x), \chi(x)(g)t)$.

With this notation, we may now state the following result (a version of [KKLV89, Lem. 2.2, Prop. 2.3]):

Proposition 2.10. *Let X be a reduced G-scheme. Then there is an exact sequence*

$$0 \to \mathcal{O}(X)^{*G} \to \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{G}(X) \xrightarrow{\gamma} \mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X),$$

where $\psi(L)$ denotes the image of $\alpha^*(L)$ in $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$*, for any $L \in \mathrm{Pic}(X)$.*

Proof. In view of Lemma 2.7, it suffices to show that the above sequence is exact at $\widehat{G}(X)$, $\mathrm{Pic}^G(X)$ and $\mathrm{Pic}(X)$.

Exactness at $\widehat{G}(X)$: Let $\lambda \in \widehat{G}(X)$. Then $\gamma(\lambda) = 0$ if and only if there is an isomorphism of G-linearized line bundles $F : X \times \mathbb{A}^1 \to X \times \mathbb{A}^1$, where the left-hand side denotes the trivial G-linearized bundle, and the right-hand side, that linearized via λ. Then F is given by $F(x, z) = (x, f(x)z)$ for some $f \in \mathcal{O}(X)^*$; moreover, the equivariance of F translates into the condition that $f(\alpha(g, x)) = \lambda(x)(g)\, f(x)$ on $G \times X$, i.e., $\lambda = \chi(f)$.

Exactness at $\mathrm{Pic}^G(X)$: Let L be a G-linearized line bundle on X. Then $\varphi(L) = 0$ if and only if L is trivial as a line bundle. Moreover, any G-linearization of the trivial line bundle $p_1 : X \times \mathbb{A}^1 \to X$ is of the form

$$\beta(g, x, z) = (\alpha(g, x), f(g, x)z),$$

where $f \in \mathcal{O}(G \times X)^*$; moreover, $f(e_G, x) = 1$ identically. By Lemma 2.6, it follows that $f(g, x) = \chi(x)(g)$ for a unique $\chi \in \widehat{G}(X)$.

Exactness at $\mathrm{Pic}(X)$: Let L be a line bundle on X. By Lemma 2.9, L is G-linearizable if and only if $\alpha^*(L) \cong p_2^*(L)$. This is equivalent to the condition that $\alpha^*(L) \cong p_2^*(M)$ for some line bundle M on X, since that condition implies $L \cong M$ by pulling back to $\{e_G\} \times X$. ☐

Remark 2.11. The obstruction map $\psi = \psi_{G,X}$ is compatible with pull-backs in the following sense: given a homomorphism $h : G \to G'$ of connected algebraic groups, a reduced G'-scheme X' and a morphism $f : X \to X'$ of schemes such that $f(g \cdot x) = h(g) \cdot f(x)$ identically, the diagram

$$\begin{array}{ccc} \mathrm{Pic}(X') & \xrightarrow{\psi_{G',X'}} & \mathrm{Pic}(G' \times X')/p_2^*\mathrm{Pic}(X') \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle (h\times f)^*} \\ \mathrm{Pic}(X) & \xrightarrow{\psi_{G,X}} & \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X) \end{array}$$

commutes. (This follows readily from the definition of ψ as a pull-back).

Next, we consider linearization of line bundles over normal G-schemes. We will need two lemmas:

Lemma 2.12. *Let X be a normal integral scheme, η its generic point, and Y a smooth integral scheme. Then we have an exact sequence*

$$\mathrm{Pic}(X) \xrightarrow{p_1^*} \mathrm{Pic}(X \times Y) \xrightarrow{(\eta\times \mathrm{id}_Y)^*} \mathrm{Pic}(\eta \times Y) = \mathrm{Pic}(Y_{k(X)}).$$

Proof. This can be extracted from [EGA, Cor. II.21.4.13], but we prefer to give a direct proof. Note that $X \times Y$ is normal. Let L be a line bundle on $X \times Y$ such that $(\eta \times \mathrm{id}_Y)^*(L)$ is trivial. Then $L \cong \mathcal{O}_{X\times Y}(D)$ for some Cartier divisor D on $X \times Y$, and there exists a rational function f on $X \times Y$ such that $D - \mathrm{div}(f)$ vanishes on $\{\eta\} \times Y$. Thus, $D - \mathrm{div}(f) = p_1^*(E)$ for some Weil divisor E on X. Then $L \cong p_1^*(\mathcal{O}_X(E))$; by descent, it follows that E is a Cartier divisor. □

Lemma 2.13. *Let X be a normal integral G-scheme, and η its generic point. Then we have an exact sequence*

$$\mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\rho} \mathrm{Pic}(G_{k(X)}), \tag{4}$$

where ρ denotes the composition $\mathrm{Pic}(X) \xrightarrow{\alpha^*} \mathrm{Pic}(G \times X) \xrightarrow{(\mathrm{id}_G\times\eta)^*} \mathrm{Pic}(G_{k(X)})$.

Proof. This follows by combining Proposition 2.10 and Lemma 2.12 (with $Y = G$). □

Lemma 2.13 together with Proposition 2.5 imply the following:

Theorem 2.14. *Let G be a connected linear algebraic group, and X a normal G-scheme. Then $L^{\otimes n}$ is G-linearizable for any line bundle L on X, where n denotes the stable exponent of* $\mathrm{Pic}(G)$.

As mentioned in the introduction, the normality assumption in Theorem 2.14 cannot be omitted in view of examples of nodal or cuspidal curves. We now provide details on these:

Example 2.15. Let X be the curve obtained from $\mathbb{P}^1$ by identifying the points 0 and ∞ to the nodal point z. Denote by $\eta : \mathbb{P}^1 \to X$ the normalization. Then we have an exact sequence

$$0 \longrightarrow k^* \xrightarrow{\delta} k^* \times k^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{\eta^*} \mathrm{Pic}(\mathbb{P}^1) \longrightarrow 0,$$

where δ denotes the diagonal (this follows e.g. from the Units-Pic sequence of [We91, Prop. 7.8]). This yields an exact sequence (of abstract groups)

$$0 \longrightarrow \mathbb{G}_m(k) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

where the degree map is identified with η^*; the corresponding sequence of group schemes is exact in view of [BLR90, Sec. 9.2]).

Also, the automorphism group of X is isomorphic to the stabilizer of $\{0, \infty\}$ in $\mathrm{Aut}(\mathbb{P}^1)$; this is the semi-direct product of $\mathbb{G}_m$ (fixing 0 and ∞) with the group generated by an involution exchanging these points. In particular, the connected automorphism group of X is just $\mathbb{G}_m =: G$; it acts on the Picard group by preserving the degree.

Let L be a line bundle on X of degree $n \neq 0$. Then L is not G-linearizable. Indeed, any linearization of $\eta^*(L) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ differs from the standard linearization (arising from the linear action of G on $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$ with weights 0 and 1) by a character of G, i.e., an integer m. Thus, the linear action of G on the fiber $\eta^*(L)_0$ (resp. $\eta^*(L)_\infty$) has weight $m+n$ (resp. m). But $\eta^*(L)_0 \cong L_z \cong \eta^*(L)_\infty$ as G-modules, if L is G-linearized.

By Proposition 2.10, we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\gamma} \mathrm{Pic}^G(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \xrightarrow{\psi} \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X).$$

Using the Units-Pic exact sequence again, one may check that $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X) \cong \mathbb{Z}$ and this identifies ψ with the degree map. Thus, the G-linearizable line bundles on X are exactly those of degree 0.

Example 2.16. Let X be the curve obtained from $\mathbb{P}^1$ by pinching the fat point $Z := \mathrm{Spec}(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)$ to the cuspidal point z. Then the normalization $\eta : \mathbb{P}^1 \to X$ yields an exact sequence

$$0 \longrightarrow k^* \longrightarrow (\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)^* \longrightarrow \mathrm{Pic}(X) \xrightarrow{\eta^*} \mathrm{Pic}(\mathbb{P}^1) \longrightarrow 0.$$

In view of the isomorphisms $(\mathcal{O}_{\mathbb{P}^1,\infty}/\mathfrak{m}^2)^*/k^* \cong (1+\mathfrak{m})/(1+\mathfrak{m}^2) \cong \mathfrak{m}/\mathfrak{m}^2$, this may be identified with the exact sequence

$$0 \longrightarrow \mathbb{G}_a(k) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

The corresponding sequence of group schemes is exact again.

Also, the automorphism group of X is isomorphic to $\mathrm{Aut}(\mathbb{P}^1, \infty)$, i.e., to the automorphism group of the affine line; this is the semi-direct product $\mathbb{G}_a \ltimes \mathbb{G}_m$, where $\mathbb{G}_m$ acts on $\mathbb{G}_a$ by scalar multiplication. This group acts on $\mathrm{Pic}(X)$ by preserving the degree, and on $\mathfrak{m}/\mathfrak{m}^2$ via the linear action of the quotient $\mathbb{G}_m$ with weight -1. In particular, no nontrivial line bundle of degree 0 is $\mathbb{G}_m$-linearizable. But there exists a $\mathbb{G}_m$-linearizable line bundle L of degree 1: indeed, the map $\mathbb{P}^1 \to \mathbb{P}^2$, $[x:y] \mapsto [x^3 : xy^2 : y^3]$ factors through an embedding $F : X \to \mathbb{P}^2$, equivariant for the action of $\mathbb{G}_m$ on $\mathbb{P}^2$ with weights $0, 2, 3$; thus, we may take $L = F^*\mathcal{O}_{\mathbb{P}^2}(1)$. In view of Proposition 2.10, this yields an exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\gamma} \mathrm{Pic}^{\mathbb{G}_m}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$

We now consider the action of $G := \mathbb{G}_a$ on X. In characteristic 0, no line bundle L of nonzero degree is G-linearizable. Indeed, we may assume that L has positive degree n, and (replacing L with a positive power) that L is very ample. Then one may check that the image of the pull-back map $H^0(X, L) \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ is a hyperplane, say H. If L is G-linearizable, then H is stable under the representation of G in $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$, the space of homogeneous polynomials of degree n in x, y on which G acts via $t \cdot (x, y) := (x, y + tx)$. It follows that $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ contains a unique G-stable hyperplane: the span of the monomials $x^n, x^{n-1}y, \ldots, xy^{n-1}$, or equivalently, the kernel of the evaluation map at ∞. But this contradicts the assumption that L is very ample.

In contrast, if k has prime characteristic p, then X has a G-linearizable line bundle L_p of degree p. Consider indeed the morphism

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^{p-1}, \quad [x:y] \longmapsto [x^p : x^{p-2}y^2 : x^{p-3}y^3 : \cdots : y^p].$$

Then f factors through a morphism $F : X \to \mathbb{P}^{p-1}$, and its image generates a G-stable hyperplane of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p))$, viewed as the space of homogeneous polynomials of degree p in x, y with the natural action of G. Thus, we may take $L_p = F^*\mathcal{O}_{\mathbb{P}^{p-1}}(1)$. Note that F is an embedding if $p \geq 3$; if $p = 2$, then the map

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3, \quad [x : y] \longmapsto [x^4 : x^2y^2 : xy^3 : y^4]$$

yields an embedding of X. In any case, X admits an equivariant embedding in the projectivization of some finite-dimensional G-module.

We now describe the obstruction to linearization, in arbitrary characteristic. Proposition 2.10 yields an exact sequence

$$0 \longrightarrow \mathrm{Pic}^G(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X).$$

Moreover, one may check that $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X) \cong \mathcal{O}(G \times Z)^*/\mathcal{O}(Z)^* \cong k[t]/k$ and this identifies ψ with the map $L \mapsto \deg(L)\, t$. In particular, a line bundle on X is G-linearizable if and only if its degree is a multiple of the characteristic.

3 The obstruction to linearization on seminormal varieties

3.1 The obstruction group

Given a scheme X, we analyze the quotient $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$. Using the section $e_G \times \mathrm{id}_X : X \to G \times X$ of p_2, we may identify $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$ with the kernel of $(e_G \times \mathrm{id}_X)^* : \mathrm{Pic}(G \times X) \to \mathrm{Pic}(X)$. This identifies the obstruction map ψ with $\alpha^* - p_2^* : \mathrm{Pic}(X) \to \mathrm{Pic}(G \times X)$.

Also, recall that $\mathrm{Pic}(X)$ is isomorphic to the étale cohomology group $H^1_{\text{ét}}(X, \mathbb{G}_m)$ (see [Mi80, Prop. II.4.9]). Thus, the Leray spectral sequence for p_2 yields an exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(X, p_{2*}(\mathbb{G}_m)) \longrightarrow \mathrm{Pic}(G \times X) \longrightarrow H^0_{\text{ét}}(X, R^1p_{2*}(\mathbb{G}_m)). \tag{5}$$

When X is reduced, Lemma 2.6 gives an isomorphism of étale sheaves on X

$$\mu : \widehat{G} \times \mathbb{G}_m \xrightarrow{\cong} p_{2*}(\mathbb{G}_m). \tag{6}$$

In particular, the map $\widehat{G} \to p_{2*}(\mathbb{G}_m)$ defines a map $H^1_{\text{ét}}(X, \widehat{G}) \to H^1_{\text{ét}}(X, p_{2*}(\mathbb{G}_m))$ and hence a map

$$c = c_{G,X} : H^1_{\text{ét}}(X, \widehat{G}) \longrightarrow \mathrm{Pic}(G \times X). \tag{7}$$

Lemma 3.1. *Let X be a reduced scheme.*

(i) *The above map c sits in an exact sequence*

$$0 \longrightarrow H^1_{\text{ét}}(X, \widehat{G}) \times \mathrm{Pic}(X) \xrightarrow{c \times p_2^*} \mathrm{Pic}(G \times X) \longrightarrow H^0_{\text{ét}}(X, R^1p_{2*}(\mathbb{G}_m)). \tag{8}$$

(ii) *c is compatible with pull-backs on G and X in the following sense: for any homomorphism $h : G \to G'$ of connected algebraic groups, and any morphism $f : X \to X'$ of reduced schemes, we have a commutative diagram*

$$\begin{array}{ccc} H^1_{\text{ét}}(X', \widehat{G'}) & \xrightarrow{c_{G',X'}} & \text{Pic}(G' \times X') \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(X, \widehat{G}) & \xrightarrow{c_{G,X}} & \text{Pic}(G \times X), \end{array}$$

where the vertical arrows are pull-backs.
(iii) *For any geometric point $\bar{x}$ of X, there is an isomorphism*

$$R^1 p_{2*}(\mathbb{G}_m)_{\bar{x}} \cong \text{Pic}(G \times \text{Spec}(\mathcal{O}_{X,\bar{x}})), \tag{9}$$

where $\mathcal{O}_{X,\bar{x}}$ denotes the strict henselization of the local ring $\mathcal{O}_{X,x}$; this identifies the natural map $\text{Pic}(G) \to R^1 p_{2*}(\mathbb{G}_m)_{\bar{x}}$ with the pull-back map $p_1^* : \text{Pic}(G) \to \text{Pic}(G \times \text{Spec}(\mathcal{O}_{X,\bar{x}}))$.

Proof. (i) is obtained by combining (5) and (6).

(ii) By construction of c, it suffices to check that both squares in the diagram

$$\begin{array}{ccccc} H^1_{\text{ét}}(X', \widehat{G'}) & \longrightarrow & H^1_{\text{ét}}(X', p'_{2*}(\mathbb{G}_m)) & \longrightarrow & \text{Pic}(G' \times X') \\ \downarrow & & \downarrow & & \downarrow \\ H^1_{\text{ét}}(X, \widehat{G}) & \longrightarrow & H^1_{\text{ét}}(X, p_{2*}(\mathbb{G}_m)) & \longrightarrow & \text{Pic}(G \times X) \end{array}$$

commute. We may view each group as a Čech cohomology group by [Mi80, Cor. III.2.10]. Then the commutativity of the left square follows from the compatibility of the map $\widehat{G} \to p_{2*}(\mathbb{G}_m)$ with pull-backs, and that of the right square is obtained by viewing $H^1_{\text{ét}}(X, p_{2*}(\mathbb{G}_m))$ as the subgroup of $\text{Pic}(G \times X)$ consisting of those line bundles that are trivial on each $G \times U_i$ for some étale covering $(U_i \to X)$.

(iii) is a consequence of [Mi80, Thm. III.1.15]. □

When $G = \mathbb{G}_m$ (so that $\widehat{G}$ is the constant sheaf $\mathbb{Z}$) and X is of finite type, the abelian group $H^1_{\text{ét}}(X, \mathbb{Z})$ is free of finite rank in view of [We91, Thm. 7.9]. Moreover, we have $H^0_{\text{ét}}(X, \mathbb{Z}) = H^0_{\text{Zar}}(X, \mathbb{Z}) =: \mathbb{Z}(X)$, and this abelian group is free of finite rank as well (see e.g. [We91, Ex. 7.1]. Also, $H^1_{\text{ét}}(X, \mathbb{Z}) = 0$ if X is normal, by [We91, Prop. 7.4, Thm. 7.5]. For a reduced scheme X such that the normalization $\eta : \tilde{X} \to X$ is finite, the group $H^1_{\text{ét}}(X, \mathbb{Z})$ may be determined from η: consider indeed the conductor square

$$\begin{array}{ccc} Y' & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \eta \\ Y & \longrightarrow & X \end{array}$$

(see e.g. [Fe03]). Then we have an exact sequence

$$0 \to \mathbb{Z}(X) \to \mathbb{Z}(\tilde{X}) \oplus \mathbb{Z}(Y) \to \mathbb{Z}(Y') \to H^1_{\text{ét}}(X, \mathbb{Z}) \to H^1_{\text{ét}}(\tilde{X}, \mathbb{Z}) \oplus H^1_{\text{ét}}(Y, \mathbb{Z}) \to H^1_{\text{ét}}(Y', \mathbb{Z}) \tag{10}$$

as follows from [We91, Thm. 7.6, Prop. 7.8]. For instance, when X is the nodal curve as in Example 2.15, this yields $H^1_{\text{ét}}(X,\mathbb{Z}) \cong \mathbb{Z}$.

These results extend readily to the case where $\mathbb{Z}$ is replaced with any constant sheaf Λ of free abelian groups of finite rank. Indeed, the natural map $H^i_{\text{ét}}(X,\mathbb{Z})\otimes_{\mathbb{Z}}\Lambda \to H^i_{\text{ét}}(X,\Lambda)$ is then an isomorphism for all $i \geq 0$, since $H^i_{\text{ét}}(X,\mathbb{Z}^r) \cong H^i_{\text{ét}}(X,\mathbb{Z})^r$.

We now record variants of some of these results for locally constant sheaves:

Lemma 3.2. *Let X be a scheme of finite type, and Λ an étale sheaf of free abelian groups of finite rank on* $\mathrm{Spec}(k)$*.*

(i) *The abelian group $\Lambda(X)$ is free of finite rank. Moreover, $H^1_{\text{ét}}(X,\Lambda)$ is finitely generated.*

(ii) *If X is normal, then $H^1_{\text{ét}}(X,\Lambda)$ is finite.*

Proof. (i) We may choose a finite Galois extension of fields k'/k such that $\Lambda_{k'}$ is constant. Denoting by Γ the Galois group, we have an isomorphism $H^0(X,\Lambda) \cong H^0(X_{k'},\Lambda_{k'})^{\Gamma}$. Moreover, the Hochschild-Serre spectral sequence (see [Mi80, Thm. III.2.20]) yields an exact sequence

$$0 \longrightarrow H^1(\Gamma, H^0_{\text{ét}}(X_{k'},\Lambda_{k'})) \longrightarrow H^1_{\text{ét}}(X,\Lambda) \longrightarrow H^1_{\text{ét}}(X_{k'},\Lambda_{k'})^{\Gamma}.$$

Since both abelian groups $H^0_{\text{ét}}(X_{k'},\Lambda_{k'})$ and $H^1_{\text{ét}}(X_{k'},\Lambda_{k'})$ are free of finite rank, this yields the assertions.

(ii) follows from the above exact sequence in view of the vanishing of $H^1_{\text{ét}}(X_{k'},\Lambda_{k'})$. □

Next, recall that a ring R is called Nagata, or pseudo-geometric, if R is noetherian and for any prime ideal $\mathfrak{p}$ of R and any finite extension L of the fraction field of $R/\mathfrak{p}$, the integral closure of R in L is a finite R-module. (This is equivalent to R being noetherian and universally japanese, in view of [Na62, Thm. 36.5]; see also [EGA, Thm. IV.7.7.2]). A scheme X is Nagata if X has an affine open covering by spectra of Nagata rings; this holds for example when X is locally of finite type.

We may now state our main technical result:

Theorem 3.3. *Let X be a Nagata scheme.*

(i) *If G is split and X is connected and seminormal, then the map*

$$p_1^* \times c \times p_2^* : \mathrm{Pic}(G) \times H^1_{\text{ét}}(X,\widehat{G}) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)$$

is an isomorphism.

(ii) *If G is linear and X is geometrically seminormal, then the cokernel of the map*

$$c \times p_2^* : H^1_{\text{ét}}(X,\widehat{G}) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)$$

is killed by the stable exponent of $\mathrm{Pic}(G)$*.*

The proof will be given in the next two subsections. When k has characteristic $p > 0$, the assumptions of (geometric) seminormality can be suppressed by tensoring with $\mathbb{Z}[\frac{1}{p}]$, see Theorem 4.13.

When $G = \mathbb{G}_a$, Theorem 3.3 boils down to the isomorphism $\mathrm{Pic}(X \times \mathbb{A}^1) \cong \mathrm{Pic}(X)$, which characterizes seminormality for reduced affine schemes (see [Tr70, Thm. 3.6] and [Sw80, Thm. 1]).

3.2 Some fibrations of seminormal schemes

From now on, we assume that all schemes under consideration are Nagata. This will allow us to apply results from [GT80], where schemes are assumed to be locally noetherian and with finite normalization; also, we will use inductive arguments based on the Units-Pic sequence for a finite morphism of schemes, and the conductor square of the normalization.

We first record two easy observations:

Lemma 3.4. *Let $f : X \to Y$ be a smooth surjective morphism. Then X is seminormal if and only if Y is seminormal.*

Proof. By [GT80, Prop. 5.1], there is a cartesian square

$$\begin{array}{ccc} X^+ & \xrightarrow{f^+} & Y^+ \\ {\scriptstyle \sigma_X}\big\downarrow & & {\scriptstyle \sigma_Y}\big\downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

where the vertical arrows are the seminormalization maps. So the assertion follows by descent of isomorphisms (see [SGA1, Exp. VIII, Cor. 5.4]). □

Lemma 3.5. *Let X be a reduced G-scheme, and $\sigma : X^+ \to X$ the seminormalization. Then X^+ has a unique structure of G-scheme such that σ is equivariant.*

Proof. The action map $\alpha : G \times X \to X$ is smooth and surjective. Thus, we obtain a cartesian square as in the proof of Lemma 3.4

$$\begin{array}{ccc} (G \times X)^+ & \longrightarrow & X^+ \\ \big\downarrow & & \big\downarrow \\ G \times X & \xrightarrow{\alpha} & X, \end{array}$$

where the vertical maps are the seminormalizations. Replacing α with the projection $p_2 : G \times X \to X$ and arguing similarly, we obtain an isomorphism $(G \times X)^+ \xrightarrow{\cong} G \times X^+$. Thus, α lifts to a morphism $\alpha^+ : G \times X^+ \to X^+$. Since σ restricts to an isomorphism on an open dense subscheme of X, we see that α^+ is unique and satisfies the properties of an algebraic group action. □

Lemma 3.4 applies to any torsor under an algebraic group, and also to any affine bundle, i.e., a morphism $f : X \to Y$, where Y is covered by Zariski open subsets U_i such that $f^{-1}(U_i) \cong U_i \times \mathbb{A}^n$ as schemes over U_i. We now record an easy property of such bundles:

Lemma 3.6. *Let $f : X \to Y$ be an affine bundle, where Y is seminormal. Then the pull-back maps $\mathcal{O}(Y)^* \to \mathcal{O}(X)^*$ and $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ are isomorphisms.*

Proof. Recall that $\mathcal{O}(X)^* = H^0_{\mathrm{Zar}}(X, \mathcal{O}_X^*)$, $\mathrm{Pic}(X) = H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*)$, and likewise for Y. In view of the Leray spectral sequence for f in the Zariski topology, it suffices to show that the natural map $\mathcal{O}_Y^* \to f_*(\mathcal{O}_X^*)$ is an isomorphism, and $R^1 f_*(\mathcal{O}_X^*) = 0$. For this, we may assume that f is the projection $p_1 : Y \times \mathbb{A}^n \to Y$. Let U be an open subscheme of Y, and $A := \mathcal{O}(U)$. Since A is reduced, we obtain $\mathcal{O}(f^{-1}(U))^* \cong A[t_1, \ldots, t_n]^* = A^*$. If U is affine, then the pull-back map $\mathrm{Pic}(A) \to \mathrm{Pic}(A[t_1, \ldots, t_n])$ is an isomorphism, by seminormality and [Tr70, Thm. 3.6]. So the presheaf on Y given by $U \mapsto \mathrm{Pic}(U \times \mathbb{A}^n)$ has trivial stalks (since every line bundle is trivial on some affine neighborhood of each point). Thus, the sheaf associated with this presheaf is trivial as well, i.e., $R^1 p_{1*}(\mathcal{O}^*_{Y \times \mathbb{A}^n}) = 0$. □

Next, we consider torsors under split tori; recall that any such torsor for the fppf topology is locally trivial for the Zariski topology.

Lemma 3.7. *Let $f : X \to Y$ be a torsor under a split torus T, where X is seminormal and connected. Then there is an exact sequence*

$$0 \longrightarrow \mathcal{O}(Y)^* \xrightarrow{f^*} \mathcal{O}(X)^* \xrightarrow{\chi} \widehat{T} \xrightarrow{\gamma} \mathrm{Pic}(Y) \xrightarrow{f^*} \mathrm{Pic}(X) \xrightarrow{\chi} H^1_{\text{ét}}(Y, \widehat{T}), \tag{11}$$

where γ denotes the characteristic homomorphism that assigns to any character of T, the class of the associated line bundle on Y.

Proof. By Lemma 2.7, we have a short exact sequence of étale sheaves on Y

$$0 \longrightarrow \mathbb{G}_m \longrightarrow f_*(\mathbb{G}_m) \xrightarrow{\chi} \widehat{T} \longrightarrow 0.$$

The associated long exact sequence of étale cohomology begins with (11), except that $\mathrm{Pic}(X)$ is replaced with $H^1_{\text{ét}}(Y, f_*(\mathbb{G}_m))$. Also, the Leray spectral sequence for f yields an exact sequence

$$0 \longrightarrow H^1_{\text{ét}}(Y, f_*(\mathbb{G}_m)) \longrightarrow \mathrm{Pic}(X) = H^1_{\text{ét}}(X, \mathbb{G}_m) \longrightarrow H^0_{\text{ét}}(Y, R^1 f_*(\mathbb{G}_m)).$$

To complete the proof, it suffices to show that $R^1 f_*(\mathbb{G}_m) = 0$. As in the proof of Lemma 3.1, this is equivalent to the assertion that $\mathrm{Pic}(T \times \mathrm{Spec}(\mathcal{O}_{Y,\bar{y}})) = 0$ for any geometric point $\bar{y}$ of Y. Since $T \cong \mathbb{G}_m^n$ for some positive integer n, this amounts to $\mathrm{Pic}(A[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]) = 0$, where A denotes the henselian local ring $\mathcal{O}_{Y,\bar{y}}$.

This vanishing follows by combining results of [We91]. More specifically, there is an exact sequence for any commutative ring R

$$0 \longrightarrow \mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R[t]) \times \mathrm{Pic}(R[t^{-1}]) \longrightarrow \mathrm{Pic}(R[t, t^{-1}]) \longrightarrow \mathrm{LPic}(R) \longrightarrow 0,$$

where $\mathrm{LPic}(R) \cong H^1_{\text{ét}}(\mathrm{Spec}(R), \mathbb{Z})$ (see [We91, Lem. 1.5.1, Thm. 5.5]). If R is seminormal, then the maps $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t])$ and $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t^{-1}])$ are isomorphisms by [Tr70, Thm. 3.6] again; thus, we obtain an isomorphism

$$\mathrm{Pic}(R[t, t^{-1}])/\mathrm{Pic}(R) \cong \mathrm{LPic}(R).$$

Since the (injective) map $\mathrm{Pic}(R) \to \mathrm{Pic}(R[t, t^{-1}])$ is split by the evaluation at $t = 1$, this yields an isomorphism

$$\mathrm{Pic}(R[t, t^{-1}]) \cong \mathrm{Pic}(R) \oplus \mathrm{LPic}(R).$$

Also, $\mathrm{LPic}(R) \cong \mathrm{LPic}(R[t, t^{-1}])$ by [We91, Thm. 2.4]. Since $R[t, t^{-1}]$ is seminormal as well, we obtain by induction

$$\mathrm{Pic}(R[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]) \cong \mathrm{Pic}(R) \oplus \bigoplus_{i=1}^{n} \mathrm{LPic}(R).$$

Moreover, $\mathrm{Pic}(A) = 0$ since A is local, and $\mathrm{LPic}(A) = 0$ by [We91, Thm. 2.5]. □

Remarks 3.8. (i) The argument of Lemma 3.7 yields a map $\delta : H^1_{\text{ét}}(Y, \widehat{T}) \to H^2_{\text{ét}}(Y, \mathbb{G}_m)$ such that the sequence

$$\operatorname{Pic}(X) \xrightarrow{\chi} H^1_{\text{ét}}(Y, \widehat{T}) \xrightarrow{\delta} H^2_{\text{ét}}(Y, \mathbb{G}_m)$$

is exact; recall that $H^2_{\text{ét}}(Y, \mathbb{G}_m)$ is the cohomological Brauer group $\operatorname{Br}'(Y)$. One may check that δ is the composition

$$H^1_{\text{ét}}(Y, \widehat{T}) = H^1_{\text{ét}}(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{T} \xrightarrow{\text{id} \otimes \gamma} H^1_{\text{ét}}(Y, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1_{\text{ét}}(Y, \mathbb{G}_m) \xrightarrow{\cup} H^2_{\text{ét}}(Y, \mathbb{G}_m),$$

where $\cup$ denotes the cup product.

(ii) For a torsor $f : X \to Y$ under an arbitrary connected linear algebraic group G (assumed in addition to be reductive if k is imperfect), one has a longer exact sequence

$$0 \to \mathcal{O}(Y)^* \to \mathcal{O}(X)^* \to \widehat{G}(k) \to \operatorname{Pic}(Y) \to \operatorname{Pic}(X) \to \operatorname{Pic}(G) \to \operatorname{Br}'(Y) \to \operatorname{Br}'(X)$$

when Y is a smooth variety (see [Sa81, Prop. 6.10]).

(iii) Returning to the setting of seminormal schemes, the exact sequence (11) can be generalized to torsors under connected algebraic groups. This will not be needed here, and is postponed to [Br14].

3.3 Proof of the main result

Recall our standing assumption that all schemes are Nagata. A key ingredient of the proof of Theorem 3.3 is the following invariance property:

Proposition 3.9. *Let $f : X \to Y$ be a locally trivial fibration for the étale topology, with smooth and geometrically connected fibers. Then $f^* : H^i_{\text{ét}}(Y, \Lambda) \to H^i_{\text{ét}}(X, \Lambda)$ is an isomorphism for $i = 0, 1$ and any étale sheaf Λ of free abelian groups of finite rank on* $\operatorname{Spec}(k)$.

Proof. Note that the assumptions on f still hold after base change by a finite Galois extension of fields k'/k. Choose such an extension so that $\Lambda_{k'}$ is trivial, and denote by Γ the Galois group. Then the Hochschild-Serre spectral sequence yields a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \to H^1(\Gamma, H^0_{\text{ét}}(Y_{k'}, \Lambda_{k'})) & \to & H^1_{\text{ét}}(Y, \Lambda) & \to & H^0(\Gamma, H^1_{\text{ét}}(Y_{k'}, \Lambda_{k'})) & \to & H^2(\Gamma, H^0_{\text{ét}}(Y_{k'}, \Lambda_{k'})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 \to H^1(\Gamma, H^0_{\text{ét}}(X_{k'}, \Lambda_{k'})) & \to & H^1_{\text{ét}}(X, \Lambda) & \to & H^0(\Gamma, H^1_{\text{ét}}(X_{k'}, \Lambda_{k'})) & \to & H^2(\Gamma, H^0_{\text{ét}}(X_{k'}, \Lambda_{k'})),
\end{array}$$

where the vertical arrows are pull-backs by f. As a consequence, we may assume that Λ is constant. Clearly, we may further assume that $\Lambda = \mathbb{Z}$.

Next, using the Leray spectral sequence for f, it suffices to show that the natural map $\mathbb{Z} \to f_*(\mathbb{Z})$ is an isomorphism, and $R^1 f_*(\mathbb{Z}) = 0$. In view of the local triviality assumption, we may assume that f is the projection $F \times Y \to Y$, where F is smooth and geometrically connected. By [Mi80, Thm. III.1.15], we are reduced to checking that whenever $Y = \mathrm{Spec}(A)$ for a henselian local ring A, we have

$$\mathbb{Z}(F \times Y) = \mathbb{Z} \quad \text{and} \quad H^1_{\text{ét}}(F \times Y, \mathbb{Z}) = 0.$$

Since Y is connected and F is geometrically connected, $F \times Y$ is connected by [EGA, Cor. II.4.5.8]; this yields the former displayed equality. The latter displayed equality is proved in [We91, 2.5, 5.5] in the case where $F = \mathbb{G}_m$; we argue along the same lines in our setting. We may assume that Y is reduced by using [Mi80, Rem. III.1.6]. Consider the normalization map $\eta : \tilde{Y} \to Y$ and its conductor square

$$\begin{array}{ccc} S' & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ S & \longrightarrow & Y. \end{array}$$

Note that $F \times \tilde{Y}$ is normal, since F is smooth; then one easily checks that the normalization of $F \times Y$ is $\mathrm{id}_F \times \eta : F \times \tilde{Y} \to F \times Y$, with conductor $F \times S$. In other words, the conductor square of the normalization of $F \times Y$ is

$$\begin{array}{ccc} F \times S' & \longrightarrow & F \times \tilde{Y} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{id}_F \times \eta} \\ F \times S & \longrightarrow & F \times Y. \end{array}$$

So (10) yields an exact sequence

$$\mathbb{Z}(F \times \tilde{Y}) \oplus \mathbb{Z}(F \times S) \to \mathbb{Z}(F \times S') \to H^1_{\text{ét}}(F \times Y, \mathbb{Z}) \to H^1_{\text{ét}}(F \times \tilde{Y}, \mathbb{Z}) \oplus H^1_{\text{ét}}(F \times S, \mathbb{Z}).$$

Since S is local henselian and strictly contained in Y, we may argue by noetherian induction and assume that $H^1_{\text{ét}}(F \times S, \mathbb{Z}) = 0$. Also, since $F \times \tilde{Y}$ is normal, we have $H^1_{\text{ét}}(F \times \tilde{Y}, \mathbb{Z}) = 0$. Thus, it suffices to show that the pull-back map $\mathbb{Z}(F \times \tilde{Y}) \to \mathbb{Z}(F \times S')$ is surjective. As $F \times Z$ is connected for any connected scheme Z, the pull-back maps $\mathbb{Z}(\tilde{Y}) \to \mathbb{Z}(F \times \tilde{Y})$ and $\mathbb{Z}(S') \to \mathbb{Z}(F \times S')$ are isomorphisms. Thus, we are reduced to checking the surjectivity of the pull-back map $\mathbb{Z}(\tilde{Y}) \to \mathbb{Z}(S')$. But this is established in [We91] at the end of the proof of Theorem 2.5, p. 360. □

Remarks 3.10. (i) The above proposition applies to any G-torsor for the étale topology. For example, under the assumptions of Lemma 3.7, this yields an isomorphism $H^1_{\text{ét}}(Y, \widehat{T}) \cong H^1_{\text{ét}}(X, \widehat{T})$.
(ii) The assertion of that proposition also holds for a morphism of schemes $f : X \to Y$ which is a universal homeomorphism, in view of [Mi80, Rem. II.3.17, Rem. II.1.6]. In particular, we obtain isomorphisms $H^1_{\text{ét}}(X, \Lambda) \cong H^1_{\text{ét}}(X_{\text{red}}, \Lambda) \cong H^1_{\text{ét}}(X^+, \Lambda)$, where $X_{\text{red}} \subset X$ denotes the reduced subscheme, and X^+ its normalization. These isomorphisms also follow from [We91, Thm. 7.6, Cor. 7.6.1]).

Next, we obtain a local version of Theorem 3.3. To motivate its statement, recall from [Na62, Thm. 44.2] that the henselian local ring $\mathcal{O}_{X,\bar{x}}$ is Nagata for any geometric point $\bar{x}$ of a Nagata scheme X. By using [GT80, Prop. 5.1, Prop. 5.2], it follows that $\mathcal{O}_{X,\bar{x}}$ is seminormal if so is $\mathcal{O}_{X,x}$. Also, recall that $H^1_{\text{ét}}(\mathrm{Spec}(\mathcal{O}_{X,\bar{x}}), \mathbb{Z}) = 0$ by [We91, Thm. 2.5].

Lemma 3.11. *Let A be a henselian local ring, and $X := \mathrm{Spec}(A)$.*
(i) *If G is split and X is seminormal, then the pull-back map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times X)$ is an isomorphism.*
(ii) *If G is linear and X is geometrically seminormal, then $\mathrm{Pic}(G \times X)$ is killed by the stable exponent of $\mathrm{Pic}(G)$.*

Proof. (i) We may choose a pair (T, B), where $T \subset G$ is a split maximal torus, and $B \subset G$ is a Borel subgroup containing T. This defines morphisms

$$G \times X \xrightarrow{\varphi} G/U \times X \xrightarrow{f} G/B \times X,$$

where φ is a torsor under the unipotent part U of B, and f is a torsor under $B/U \cong T$. Since G is split, so is U, since it is an extension of a maximal connected unipotent subgroup of a split connected reductive group, by the unipotent radical of G which is split as well. Thus, φ is an affine bundle. Note that $G \times X$, $G/U \times X$ and $G/B \times X$ are seminormal by Lemma 3.4. Hence φ^* induces an isomorphism $\mathrm{Pic}(G/U \times X) \cong \mathrm{Pic}(G \times X)$, in view of Lemma 3.6. Next, $G/U \times X$ is connected, since X is connected and G/U is geometrically connected. Thus, Lemma 3.7 yields an exact sequence

$$\widehat{T} \xrightarrow{\gamma} \mathrm{Pic}(G/B \times X) \xrightarrow{f^*} \mathrm{Pic}(G/U \times X) \longrightarrow H^1_{\text{ét}}(G/B \times X, \widehat{T}).$$

Moreover, we have $H^1_{\text{ét}}(G/B \times X, \widehat{T}) \cong H^1_{\text{ét}}(G/B \times X, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{T} \cong H^1_{\text{ét}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \widehat{T}$ by Proposition 3.9; thus, $H^1_{\text{ét}}(G/B \times X, \widehat{T}) = 0$ by [We91, Thm. 2.5] again. Also, we have $\mathrm{Pic}(G/B \times X) \cong \mathrm{Pic}(G/B)$, since X is local and the Picard functor of G/B is representable by a constant group scheme. Thus, the above exact sequence reduces to

$$\widehat{T} \xrightarrow{\gamma} \mathrm{Pic}(G/B) \xrightarrow{\delta} \mathrm{Pic}(G \times X) \longrightarrow 0,$$

where γ is the characteristic homomorphism, and δ denotes the pull-back via the natural map $G \times X \to G/B$. But the cokernel of γ is isomorphic to $\mathrm{Pic}(G)$ in view of the exact sequence (2).

(ii) By Remark 2.2 (v), we may choose a finite extension of fields k'/k such that $G_{k'}$ is split. Then $A \otimes_k k'$ is the product of finitely many seminormal henselian local rings. In view of (i), it follows that $\mathrm{Pic}(G \times X)_{k'})$ is the product of finitely many copies of $\mathrm{Pic}(G_{k'})$. This yields the assertion in view of Proposition 2.5. □

Proof of Theorem 3.3.

(i) In view of Lemmas 3.1 and 3.11, the map $\mathrm{Pic}(G) \to R^1 p_{2*}(\mathbb{G}_m)$ is an isomorphism on stalks. Thus, $R^1 p_{2*}(\mathbb{G}_m)$ is the constant sheaf $\mathrm{Pic}(G)$; as a consequence, the composition of the maps $\mathrm{Pic}(G) \xrightarrow{p_1^*} \mathrm{Pic}(G \times X) \longrightarrow H^0(X, R^1 p_{2*}(\mathbb{G}_m))$ is an isomorphism. So the assertion follows from the exact sequence (8).

(ii) Let n denote the stable exponent of $\mathrm{Pic}(G)$. Then $\mathrm{Pic}(G \times \mathrm{Spec}(\mathcal{O}_{X,\bar{x}}))$ is n-torsion for any geometric point $\bar{x}$ of X, in view of Lemma 3.11. By Lemma 3.1, it follows that $R^1 p_{2*}(\mathbb{G}_m)$ is n-torsion as well; hence so is the cokernel of $c \times p_2^*$, in view of the exact sequence (8) again.

4 Some applications

4.1 Linearization, algebraic equivalence and free abelian covers

We first show that linearizability of line bundles is preserved in an algebraic family:

Proposition 4.1. *Consider a split algebraic group G, a connected seminormal G-variety X, a smooth connected variety Y and a line bundle L on $X \times Y$. Assume that the pull-back of L to $X \times \{y\}$ is G-linearizable for some $y \in Y(k)$. Then L is G-linearizable.*

Proof. By Proposition 2.10, we have $\psi_{G,X}(\mathrm{id}_X \times y)^*(L) = 0$ in $\mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$; also, note that $\psi_{G,X}(\mathrm{id}_X \times y)^*(L) = (\mathrm{id}_{G\times X} \times y)^* \psi_{G,X\times Y}(L)$ in view of Remark 2.11.

Also, we have a commutative square by Lemma 3.1 and Theorem 3.3:

$$\begin{array}{ccc} \mathrm{Pic}(G) \times H^1_{\text{ét}}(X \times Y, \widehat{G}) & \xrightarrow{(\mathrm{id}_X \times y)^*} & \mathrm{Pic}(G) \times H^1_{\text{ét}}(X, \widehat{G}) \\ {\scriptstyle p_1^* \times c_{G,X\times Y}} \downarrow & & {\scriptstyle p_1^* \times c_{G,X}} \downarrow \\ \mathrm{Pic}(G \times X \times Y)/p_{23}^*\mathrm{Pic}(X \times Y) & \xrightarrow{(\mathrm{id}_{G\times X} \times y)^*} & \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X), \end{array}$$

where the vertical arrows are isomorphisms. Moreover, Y is geometrically connected, since it is connected and has a k-rational point (see [EGA, Cor. 4.5.14]). By Proposition 3.9, it follows that the map $p_1^* : H^1_{\text{ét}}(X, \widehat{G}) \to H^1_{\text{ét}}(X \times Y, \widehat{G})$ is an isomorphism. Since $\mathrm{id}_X \times y$ is a section of p_1, we see that $(\mathrm{id}_X \times y)^* : H^1_{\text{ét}}(X \times Y, \widehat{G}) \to H^1_{\text{ét}}(X, \widehat{G})$ is the inverse isomorphism. As a consequence, both horizontal arrows in the above diagram are isomorphisms. Thus, $\psi_{G,X\times Y}(L) = 0$, i.e., L is linearizable. □

Remarks 4.2. (i) The seminormality assumption cannot be suppressed in the above proposition. Consider indeed the cuspidal curve X with its action of $\mathbb{G}_m$ as in Example 2.16, and the line bundle L on $X \times \mathbb{A}^1$ associated with the isomorphism $\mathbb{G}_a \cong \mathrm{Pic}^0(X)$. Denote by L_t the pull-back of L to $X \times \{t\}$, where $t \in k$. Then L_0 is linearizable, but L_t is not for $t \neq 0$.

(ii) The following variant of that proposition is obtained by similar arguments: let X be a connected, geometrically seminormal variety equipped with an action of a connected linear algebraic group G. Let Y be a smooth connected variety equipped with a k-rational point y, and L a line bundle on $X \times Y$ which pulls back to a G-linearizable line bundle on $X \times \{y\}$. Then $L^{\otimes n}$ is G-linearizable, where n denotes the stable exponent of $\mathrm{Pic}(G)$.

Next, we obtain a lifting property for actions of connected algebraic groups:

Proposition 4.3. *Let X be a G-variety, Λ an étale sheaf of free abelian groups of finite rank on* $\mathrm{Spec}(k)$, *and $\pi : Y \to X$ a Λ-torsor.*

(i) *There is a unique action of G on Y such that π is equivariant; moreover, this action commutes with that of Λ.*

(ii) *The scheme Y is a union of closed G-stable subvarieties Y_i such that π restricts to finite surjective maps $\pi_i : Y_i \to X$ for all i, and every Y_i meets only finitely many Y_j's. If in addition Λ is constant, then we may take for Y_i the Λ-translates of some closed G-stable subvariety $Z \subset Y$.*

Proof. (i) Recall that the isomorphism classes of Λ-torsors over X are in bijection with $H^1_{\text{ét}}(X, \Lambda)$ (see [Mi80, Prop. III.4.6, Rem. III.4.8]).

We consider first the case where Λ is constant, and argue as in the proof of Lemma 2.9. By Proposition 3.9, the pull-back map $p_2^* : H^1_{\text{ét}}(X, \Lambda) \to H^1_{\text{ét}}(G \times X, \Lambda)$ is an isomorphism. It follows that the torsor $\alpha^*(\pi)$ (obtained from π by pull-back via $\alpha : G \times X \to X$) is isomorphic to $p_2^*(\eta)$ for some Λ-torsor $\eta : Z \to X$. Pulling back via $e_G \times \mathrm{id}_X$ yields that $\eta \cong \pi$, i.e., $\alpha^*(\pi) \cong p_2^*(\pi)$. This means that there exists a morphism $\beta : G \times Y \to Y$ such that the diagram

$$\begin{array}{ccc} G \times Y & \xrightarrow{\beta} & Y \\ {\scriptstyle \mathrm{id}_G \times \pi} \downarrow & & \downarrow {\scriptstyle \pi} \\ G \times X & \xrightarrow{\alpha} & X \end{array}$$

commutes. In particular, the map $y \mapsto \beta(e_G, y)$ is an automorphism of Y which lifts the identity of X, i.e., which sits in $\mathrm{Aut}_X(Y) \cong \Lambda(X)$. This defines a morphism $\lambda : X \to \Lambda$ such that $\beta(e_G, y) = \lambda(\pi(y))y$ identically. Replacing β with $((-\lambda) \circ \pi)\beta$, we may assume that $\beta(e_G, y) = y$ identically. Then the obstruction to the associativity of β is an automorphism of the Λ-torsor $\mathrm{id}_{G \times G} \times \pi : G \times G \times Y \to G \times G \times X$, i.e., a morphism $\varphi : G \times G \times X \to \Lambda$. Moreover, $\varphi(g, e_G, x) = 0 = \varphi(e_G, g, x)$ identically, and hence $\varphi = 0$ since G is connected. Thus, β is associative.

To show that β is unique, note that any two lifts of α differ by an automorphism of the Λ-torsor $p_2^*(\pi)$, i.e., by a morphism $f : G \times X \to \Lambda$. Since both lifts pull back to the identity on $\{e_G\} \times X$, we see that f restricts to 0 on $\{e_G\} \times X$; thus, $f = 0$.

To show that β commutes with the action of Λ, fix $\lambda \in \Lambda$ and consider the morphism

$$\psi : G \times Y \longrightarrow Y \times Y, \quad (g, y) \longmapsto (y, \lambda g(-\lambda)g^{-1} \cdot y).$$

Clearly, the image of ψ is contained in $Y \times_X Y$. Since the latter is isomorphic to $\Lambda \times Y$ via $(\mu, y) \mapsto (y, \mu \cdot y)$, this yields a map $\gamma : G \times Y \to \Lambda$ such that $g\lambda g^{-1}(-\lambda) \cdot y = \gamma(g, y) \cdot y$ identically. It follows that $\gamma(e_G, y) = 0$ identically, and hence that $\gamma = 0$.

Next, we consider an arbitrary Λ. We may then choose a finite Galois extension of fields k'/k such that $\Lambda_{k'}$ is constant. Form the cartesian square

$$\begin{array}{ccc} Y_{k'} & \xrightarrow{\psi} & Y \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ X_{k'} & \xrightarrow{\varphi} & X. \end{array}$$

Then the G-action on X lifts canonically to an action of $G_{k'}$ on $X_{k'}$; in turn, that action lifts uniquely to an action of $G_{k'}$ on $Y_{k'}$ by the first step. The latter action is equivariant under the Galois group of k'/k by uniqueness, and hence descends to the desired action of G on Y.

(ii) Again, we consider first the case where Λ is constant. Denote by $\eta : \tilde{X} \to X$ the normalization and form the cartesian square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tau} & Y \\ {\scriptstyle \tilde{\pi}} \downarrow & & \downarrow {\scriptstyle \pi} \\ \tilde{X} & \xrightarrow{\eta} & X. \end{array}$$

The Λ-torsor $\tilde{\pi}$ is trivial by Lemma 3.2; in view of Proposition 4.3, it follows that $\tilde{\pi}$ is equivariantly trivial for the action of $\Lambda \times G$. In other words, we may choose a closed G-stable subvariety $\tilde{Z} \subset \tilde{Y}$ such that $\tilde{\pi}$ restricts to an isomorphism $\tilde{Z} \to \tilde{X}$, and the natural map $\Lambda \times \tilde{Z} \to \tilde{Y}$ is an isomorphism. So $Y = \bigcup_{\lambda \in \Lambda} \lambda \cdot \tau(\tilde{Z})$; moreover, $\tau(\tilde{Z})$ is closed in Y (since τ is finite), of finite type, and G-stable. As the restriction $\eta \circ \tilde{\pi} : \tilde{Z} \to X$ is finite and surjective, so is $\pi : \tau(\tilde{Z}) \to X$. Finally, since τ is finite, $\tau^{-1}\tau(\tilde{Z})$ meets only finitely many Λ-translates of $\tilde{Z}$. Equivalently, $\tau(\tilde{Z})$ meets only finitely many translates $\lambda \cdot \tau(\tilde{Z})$, where $\lambda \in \Lambda$. Thus, $Z := \tau(\tilde{Z})$ satisfies the assertion.

We now handle the general case, where Λ is not necessarily constant. Choose a finite Galois extension of fields k'/k such that $\Lambda_{k'}$ is constant. By the above step, $Y_{k'}$ is a locally finite union of closed $G_{k'}$-stable subvarieties Y'_i, finite and surjective over $X_{k'}$; we may further assume that the Y'_i are stable under the Galois group. Then we may take for Y_i the image of Y'_i under the natural map $Y_{k'} \to Y$. □

We may now show that the obstruction to linearizability vanishes by passing to a suitable free abelian cover:

Theorem 4.4. *Let X be a G-variety. Assume either that G is split and X is seminormal, or that G is linear and X is geometrically seminormal. Then any line bundle L on X defines a $\widehat{G}$-torsor $\pi : Y \to X$ such that $\pi^*(L^{\otimes n})$ is G-linearizable, where n denotes the stable exponent of* $\mathrm{Pic}(G)$.

Proof. Consider the obstruction class $\psi(L^{\otimes n}) \in \mathrm{Pic}(G \times X)/p_2^*\mathrm{Pic}(X)$. By Theorem 3.3, there exists a $\widehat{G}$-torsor $\pi : Y \to X$ such that $\psi(L^{\otimes n})$ is the image of $c([\pi]) \in \mathrm{Pic}(G \times X)$, where $[\pi] \in H^1_{\text{ét}}(X, \widehat{G})$ denotes the isomorphism class of π. Also, the G-action on X lifts to an action on Y in view of Proposition 4.3. Since the pull-back torsor of π under itself is trivial, and c is compatible with pull-backs by Lemma 3.1, we see that $(\mathrm{id}_G \times \pi)^*(\psi(L^{\otimes n})) = 0$ in $\mathrm{Pic}(G \times Y)/p_2^*\mathrm{Pic}(Y)$. As ψ is compatible with pull-backs as well, this means that $\psi(\pi^*(L^{\otimes n})) = 0$. So Proposition 2.10 yields the desired statement. □

Remarks 4.5. (i) Given a variety X, there exists a universal torsor $f : \tilde{X} \to X$ among torsors over X with structure group a free abelian group Λ of finite rank. Indeed, these torsors are classified by the free abelian group of finite rank

$$H^1_{\text{ét}}(X, \Lambda) \cong H^1_{\text{ét}}(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \Lambda \cong \mathrm{Hom}_{\mathbb{Z}}(H^1_{\text{ét}}(X, \mathbb{Z})^\vee, \Lambda),$$

where $H^1_{\text{ét}}(X, \mathbb{Z})^\vee := \mathrm{Hom}_{\mathbb{Z}}(H^1_{\text{ét}}(X, \mathbb{Z}), \mathbb{Z})$. Thus, the assertion follows from Yoneda's lemma. (When k is algebraically closed, $H^1_{\text{ét}}(X, \mathbb{Z})^\vee$ may be viewed as the largest free abelian quotient of the "enlarged fundamental group" of [SGA3, Exp. X, §6].)

If X is a seminormal G-variety, where G is split, then $f^*(L^{\otimes n})$ admits a G-linearization for any line bundle L on X, as follows from Theorem 4.4.

(ii) In characteristic 0, the assumption of (geometric) seminormality cannot be omitted in Theorem 4.4, as shown by the example of the cuspidal curve X with its action of $\mathbb{G}_a$ as in Example 2.16. But in characteristic $p > 0$, that theorem may be extended to varieties with arbitrary singularities, as we shall see in Subsection 4.3.

4.2 Local properties of linear algebraic group actions

Definition 4.6. Consider a G-variety X.

We say that X is *G-quasiprojective* if it admits a G-equivariant (locally closed) immersion into the projectivization of a finite-dimensional G-module. Equivalently, X admits an ample G-linearized line bundle.

We say that X is *locally G-quasiprojective* (for the étale topology) if it admits an étale open covering $(f_i : U_i \to X)_{i \in I}$, where the U_i are G-quasiprojective varieties, and the f_i are G-equivariant.

Note that if X is a locally G-quasiprojective variety on which G acts effectively, then G must be linear, since it acts effectively on some projective space.

Theorem 4.7. *Let G be a connected linear algebraic group, and X a quasi-projective, geometrically seminormal G-variety. Then X is locally G-quasiprojective.*

Proof. Choose an ample line bundle L on X. By Theorem 4.4, we may assume (after possibly replacing L with a positive power) that $\pi^*(L)$ is G-linearizable for some $\widehat{G}$-torsor $\pi : Y \to X$. Let (Y_i) be a collection of closed G-stable subvarieties of Y satisfying the finiteness assertions of Proposition 4.3 (ii). For any $y \in Y$, denote by Y_y the (finite) union of those Y_i's that contain y. Then Y_y is a closed G-stable subvariety of Y, finite over X; as a consequence, L pulls back to an ample G-linearizable line bundle on Y_y. Moreover, Y_y contains an open neighborhood of y in Y. Thus, the restriction $\pi_y : Y_y \to X$ is étale at y, and hence over a G-stable neighborhood of y. This yields the desired covering. □

Note again that the geometric seminormality assumption cannot be omitted in the above theorem when $\mathrm{char}(k) = 0$; yet the statement will be extended to all varieties in prime characteristics. For split tori, we now obtain a sharper result:

Theorem 4.8. *Let G be a split torus. Then every quasiprojective G-variety X admits a finite étale G-equivariant cover $f : X' \to X$, where X' is the union of open affine G-stable subvarieties. We may take $X' = Y/n\widehat{G}$ for some $\widehat{G}$-torsor $Y \to X$ and some positive integer n.*

Proof. We adapt the proof of Theorem 4.7. Choose L and $\pi : Y \to X$ as in that proof. By Proposition 4.3 (ii), there exists a closed G-subvariety $Z \subset Y$, finite and surjective over X, such that Y is the union of the translates $\chi \cdot Z$, where $\chi \in \widehat{G}$, and Z meets only finitely many such translates.

For any $z \in Z$, denote by Z_z the (finite) union of the translates $\chi \cdot Z$ which contain z. Then Z_z is a closed G-stable neighborhood of z in Y, finite and surjective over X, and hence G-quasiprojective (since L pulls back to an ample G-linearized line bundle on Z_z). Next, let Z_z^0 be the complement in Z_z of the (finitely many) intersections $Z_z \cap \chi \cdot Z$, where $z \notin \chi \cdot Z$. Then Z_z^0 is an open G-quasiprojective neighborhood of z in Y. Moreover, there are only finitely many such subvarieties Z_z^0, where $z \in Z$, and each of them meets only finitely many of its translates. Thus, we may choose a positive integer n such that each Z_z^0 is disjoint from its translates $\chi \cdot Z_z^0$, where $\chi \in n\widehat{G}$. Then the quotient $Y/n\widehat{G}$ is finite and étale over X, and the natural map $Z_z^0 \to Y/n\widehat{G}$ is an open immersion for any $z \in Z$.

So the G-variety $Y/n\widehat{G}$ is the union of G-quasiprojective open subvarieties, images of the $\chi \cdot Z^0_z$ for $\chi \in \widehat{G}$.

If X is seminormal, then the statement follows, since every G-quasiprojective variety is a union of open affine G-stable subvarieties (see e.g. the proof of [Su74, Cor. 2]).

In the general case, we consider the seminormalization $\sigma : X^+ \to X$. By Lemma 3.5, the action of G on X lifts uniquely to an action on X^+ such that σ is equivariant. Also, the pull-back $\sigma^*(L)$ is ample. Thus, there exists a $\widehat{G}$-torsor $\pi^+ : Y^+ \to X^+$ such that $(\pi^+)^*(L)$ is G-linearizable. Since the pull-back map $H^1_{\text{ét}}(X, \widehat{G}) \to H^1_{\text{ét}}(X^+, \widehat{G})$ is an isomorphism (see Remark 3.10 (ii)), we have a cartesian square

$$\begin{array}{ccc} Y^+ & \xrightarrow{\tau} & Y \\ {\scriptstyle \pi^+}\downarrow & & \downarrow{\scriptstyle \pi} \\ X^+ & \xrightarrow{\sigma} & X, \end{array}$$

where π is a $\widehat{G}$-torsor, and τ the seminormalization. Moreover, G acts on Y and Y^+ so that τ, π and π^+ are equivariant (as follows from Proposition 4.3).

Choose a positive integer n as in the first step of the proof. Then we have a cartesian square

$$\begin{array}{ccc} Y^+/n\widehat{G} & \xrightarrow{\tau_n} & Y/n\widehat{G} \\ {\scriptstyle \pi_n^+}\downarrow & & \downarrow{\scriptstyle \pi_n} \\ X^+ & \xrightarrow{\sigma} & X, \end{array}$$

where π_n^+ and π_n are finite étale, and $Y^+/n\widehat{G}$ is the union of affine G-stable open subvarieties U_i^+. Since τ_n is a universal homeomorphism, the image $\tau_n(U_i^+) =: U_i$ is open in Y for any i, and satisfies $\tau_n^{-1}(U_i) = U_i^+$. Thus, U_i is affine in view of a theorem of Chevalley (see [EGA, Thm. II.6.7.1]). So π_n is the desired morphism. □

Remarks 4.9. (i) We do not know whether the quasi-projectivity assumption is necessary in Theorems 4.7 and 4.8. This assumption can be omitted for a normal G-variety X, since every point admits a Zariski open quasi-projective G-stable neighborhood (see [Su74, Lem. 8], [Su75, Thm. 3.8], [KKLV89, Thm. 1.1]). The proof of that result relies on properties of divisors on normal varieties with an algebraic group action, which do not extend readily to the seminormal setting.

(ii) With the notation and assumptions of Theorem 4.8, the variety X' is not necessarily G-quasiprojective. For example, if X is the nodal curve with its action of $\mathbb{G}_m$ as in Example 2.15, and $f : X' \to X$ has degree n, then X' is a cycle of projective lines $X_1, \ldots, X_n$ as follows e.g. from [SGA3, Exp. X, Ex. 6.4]. Moreover, $\mathbb{G}_m$ acts on X_i so that the point at infinity is identified with the origin of X_{i+1} for any i modulo n. Thus, X' admits no ample $\mathbb{G}_m$-linearized line bundle L (otherwise, the weight of the fiber of L at each 0_i would be greater than the weight at ∞_i, with an obvious notation; but this is impossible, since $0_n = \infty_0$).

4.3 The case of prime characteristic

In this subsection, we assume that k has characteristic $p > 0$, and show how to extend the results of Subsections 4.1 and 4.2 to G-varieties with arbitrary singularities. We will need two preliminary results, probably known but for which we could not find an appropriate reference.

We say that a morphism of schemes $f : X' \to X$ is *subintegral* if f is integral, bijective, and induces isomorphisms on all residue fields. When $X = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(A')$, this corresponds to the notion of subintegral ring extension considered in [Sw80].

Lemma 4.10. *Let $A \subset A'$ be a finite subintegral extension of noetherian rings of prime characteristic p. Then $A'^{p^m} \subset A$ for $m \gg 0$, where $A'^{p^m} := \{x^{p^m} \mid x \in A'\}$.*

Proof. Denote by $N \subset A$ the ideal consisting of nilpotent elements, and define likewise $N' \subset A'$. Then $A/N \subset A'/N'$ is a finite subintegral extension as well. If $(A'/N')^{p^m} \subset A/N$, then of course $A'^{p^m} \subset A + N'$, and hence $A'^{p^{m'}} \subset A$ for $m' \gg 0$, since N' is nilpotent. Thus, we may assume that A, A' are reduced.

Consider the conductor $I \subset A$, i.e., the annihilator of the (finite) A-module A'/A. Then $I \neq 0$, as $(A'/A)_\eta = 0$ for any generic point η of A. Since the conductor square

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ A/I & \longrightarrow & A'/I \end{array}$$

is cartesian, it suffices to show that $(A'/I)^{p^m} \subset A/I$ for $m \gg 0$. Note that $A/I \subset A'/I$ is again a finite subintegral extension of (possibly nonreduced) noetherian rings of characteristic p. Thus, we may conclude by Noetherian induction. $\square$

Lemma 4.11. *Let $f : X' \to X$ be a finite subintegral morphism of noetherian schemes. Then the kernel and cokernel of $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ are killed by a power of p.*

Proof. Consider the reduced subscheme $X'_{\mathrm{red}} \subset X'$. By a standard dévissage (see e.g. [Oo62, §6]), the kernel and cokernel of the pull-back map $\mathrm{Pic}(X') \to \mathrm{Pic}(X'_{\mathrm{red}})$ are killed by a power of p. Thus, it suffices to prove the assertion in the case where X' is reduced. Then f factors through a morphism $X' \to X_{\mathrm{red}}$; by a similar reduction, we may also assume that X is reduced as well. Thus, the natural map $\mathcal{O}_X \to f_*(\mathcal{O}_{X'})$ is injective.

Note that $f^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X')$ is the composition of the natural maps

$$\mathrm{Pic}(X) = H^1_{\mathrm{Zar}}(X, \mathcal{O}_X^*) \xrightarrow{\varphi} H^1_{\mathrm{Zar}}(X, f_*(\mathcal{O}_{X'}^*)) \xrightarrow{\psi} H^1_{\mathrm{Zar}}(X', \mathcal{O}_{X'}^*) = \mathrm{Pic}(X').$$

Moreover, ψ is an isomorphism by the Leray spectral sequence and the vanishing of $R^1 f_*(\mathcal{O}_{X'}^*)$ (for the latter, see [EGA, Cor. IV.21.8.2]). Thus, $\mathrm{Ker}(f^*) = \mathrm{Ker}(\varphi)$ and $\mathrm{Coker}(f^*) \cong \mathrm{Coker}(\varphi)$. Also, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow f_*(\mathcal{O}_{X'}^*) \longrightarrow \mathcal{C} \longrightarrow 0$$

for some sheaf $\mathcal{C}$ on X; this yields a surjective map $H^0(X, \mathcal{C}) \to \mathrm{Ker}(f^*)$ and an injective map $\mathrm{Coker}(f^*) \to H^1(X, \mathcal{C})$. As a consequence, it suffices to show that $\mathcal{C}$ is killed by a power of p.

Since X has a Zariski open covering by finitely many affine schemes, and f is affine, we may assume that X, X' are affine. Then the extension $\mathcal{O}(X) \subset \mathcal{O}(X')$ satisfies the assumptions of Lemma 4.10, and hence there exists $m > 0$ such that $\mathcal{O}(X')^{p^m} \subset \mathcal{O}(X)$. It follows that $(\mathcal{O}_{X',x'})^{p^m} \subset \mathcal{O}_{X,f(x')}$ for any $x' \in X'$. Since f is bijective, we conclude that $\mathcal{C}$ is killed by p^m. □

We may now obtain a variant of Lemma 3.11 without any seminormality assumption:

Lemma 4.12. *Let A be a henselian local ring, and $X := \mathrm{Spec}(A)$.*

(i) *If G is split, then the kernel and cokernel of the pull-back map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times X)$ are killed by p^m for $m \gg 0$.*

(ii) *If G is linear, then $\mathrm{Pic}(G \times X)$ is killed by np^m for $m \gg 0$, where n denotes the stable exponent of $\mathrm{Pic}(G)$.*

Proof. (i) Consider the seminormalization $\sigma : X^+ \to X$; then $\mathrm{id}_G \times \sigma : G \times X^+ \to G \times X$ is the seminormalization by [GT80, Prop. 5.1]. In view of Lemma 4.11, it follows that the kernel and cokernel of the pull-back map $\mathrm{Pic}(G \times X) \to \mathrm{Pic}(G \times X^+)$ are killed by p^m for $m \gg 0$. Also, the pull-back map $\mathrm{Pic}(G) \to \mathrm{Pic}(G \times X^+)$ is an isomorphism by Lemma 3.11. This yields the statement.

(ii) follows from (i) as in the proof of Lemma 3.11. □

By arguing as in the proof of Theorem 3.3, we see that Lemma 4.12 implies the following variant of that theorem:

Theorem 4.13. *Let X be a noetherian scheme.*

(i) *If G is split, then the kernel and cokernel of the map*

$$p_1^* \times c \times p_2^* : \mathrm{Pic}(G) \times H^1_{\text{ét}}(X, \widehat{G}) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)$$

are killed by p^m for $m \gg 0$.

(ii) *If G is linear, then the cokernel of the map*

$$c \times p_2^* : H^1_{\text{ét}}(X, \widehat{G}) \times \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(G \times X)$$

is killed by np^m for $m \gg 0$, where n denotes the stable exponent of $\mathrm{Pic}(G)$.

Using Theorem 4.13, we may extend the results of Subsections 4.1 and 4.2 to arbitrary G-varieties, provided that all the line bundles under consideration are replaced with their p^mth powers for $m \gg 0$. For example, every quasi-projective G-variety is locally G-quasiprojective when G is arbitrary, and is G-quasiprojective when G has trivial character group.

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