EQUIVARIANT CHOW RING AND CHERN CLASSES OF WONDERFUL SYMMETRIC VARIETIES OF MINIMAL RANK

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Abstract. We describe the equivariant Chow ring of the wonderful compactification X of a symmetric space of minimal rank, via restriction to the associated toric variety Y. Also, we show that the restrictions to Y of the tangent bundle T_X and its logarithmic analogue S_X decompose into a direct sum of line bundles. This yields closed formulae for the equivariant Chern classes of T_X and S_X, and, in turn, for the Chern classes of reductive groups considered by Kiritchenko.

0. Introduction

The purpose of this article is to describe the equivariant intersection ring and equivariant Chern classes of a class of almost homogeneous varieties, namely, wonderful symmetric varieties of minimal rank; these include the wonderful compactifications of semi-simple groups of adjoint type.

The main motivation comes from questions of enumerative geometry on a spherical homogeneous space G/K. As shown by De Concini and Procesi, these questions find their proper setting in the ring of conditions $C^*(G/K)$, isomorphic to the direct limit of cohomology rings of G-equivariant compactifications X of G/K (see [DP83, DP85]). Recently, the Euler characteristic of any complete intersection of hypersurfaces in G/K has been expressed by Kiritchenko (see [Ki06]), in terms of the Chern classes of the logarithmic tangent bundle S_X of any "regular" compactification X. As shown in [Ki06], these Chern classes are independent of the choice of X, and hence yield elements of $C^*(G/K)$; moreover, their determination may be reduced to the case where X is a "wonderful variety".

In fact, it is more convenient to work with the rational equivariant cohomology ring $H^*_G(X)$, from which the ordinary rational cohomology ring $H^*(X)$ is obtained by killing the action of generators of the polynomial ring $H^*(BG)$; the Chern classes of S_X have natural representatives in $H^*_G(X)$, the equivariant Chern classes. When X is a complete symmetric variety, the ring $H^*_G(X)$ admits algebraic descriptions by work of Bifet, De Concini, Littelman, and Procesi (see [BDP90, LP90]).

Here we consider the case where X is the wonderful compactification of a symmetric space G/K of minimal rank, that is, G is semi-simple of adjoint type and $\mathrm{rk}(G/K) = \mathrm{rk}(G) - \mathrm{rk}(K)$; the main examples are the groups $G = (G \times G)/\operatorname{diag}(G)$ and the spaces $\mathrm{PSL}(2n)/\mathrm{PSp}(2n)$. Moreover, we follow a purely algebraic approach: we work over an arbitrary algebraically closed field, and replace the equivariant cohomology

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ring with the equivariant intersection ring $A^*_G(X)$ of [EG98] (for wonderful varieties over the complex numbers, both rings are isomorphic over the rationals).

We show in Theorem 2.2.1 that the pull-back map

$$r : A^*_G(X) \to A^*_T(Y)^{W_K}$$

is an isomorphism over the rationals. Here $T \subset G$ denotes a maximal torus containing a maximal torus $T_K \subset K$ with Weyl group W_K, and Y denotes the closure in X of $T/T_K \subset G/K$, so that Y is the toric variety associated with the Weyl chambers of the restricted root system of G/K.

We also determine the images under r of the equivariant Chern classes of the tangent bundle T_X and its logarithmic analogue S_X. For this, we show in Theorem 3.1.1 that the normal bundle $N_{Y/X}$ decomposes (as a T-linearized bundle) into a direct sum of line bundles indexed by certain roots of K; moreover, any such line bundle is the pull-back of $O_{\mathbb{P}^1}(1)$ under a certain T-equivariant morphism $Y \to \mathbb{P}^1$. By Proposition 1.1.1, the product of these morphisms yields a closed immersion of the toric variety Y into a product of projective lines, indexed by the restricted roots.

In the case of regular compactifications of reductive groups, Theorem 2.2.1 is due to Littelmann and Procesi for equivariant cohomology rings (see [LP90]); it has been adapted to equivariant Chow ring in [Br98]. Here, as in the latter paper, we rely on a precise version of the localization theorem in equivariant intersection theory inspired, in turn, by a similar result in equivariant cohomology, see [GKM99]. The main ingredient is the finiteness of T-stable points and curves in X; this also plays an essential role in Tchoudjem's description of cohomology groups of line bundles on wonderful varieties of minimal rank, see [Tc05].

For wonderful group compactifications, a more precise, "additive" description of the equivariant cohomology ring is due to Strickland, see [St06]; an analogous description of the equivariant Grothendieck group has been obtained by Uma in [Um07]. Both results may be generalized to our setting of minimal rank. However, determining generators and relations for the equivariant cohomology or Grothendieck ring is still an open question; see [Br07, Um07] for some steps in this direction.

Our determination of the equivariant Chern classes seems to be new, already in the group case; it yields a closed formula for the image under r of the equivariant Todd class of X, analogous to the well-known formula expressing the Todd class of a toric variety in terms of boundary divisors. The toric variety Y associated to Weyl chambers is considered in [Pr90, DL94], where its cohomology is described as a graded representation of the Weyl group; its realization as a general orbit closure in a product of projective lines seems to have been unnoticed.

Our results extend readily to all regular compactifications of symmetric spaces of minimal rank. Specifically, the description of the equivariant Chow ring holds unchanged, with a similar proof, and the determination of equivariant Chern classes follows from the wonderful case by the results of [Ki06, Sec. 5]. Another direct generalization concerns the spherical (not necessarily symmetric) varieties of minimal rank considered in [Tc05]. Indeed, the structure of such varieties may be reduced to the symmetric case, as shown by Ressayre in [Re04].

This article is organized as follows. Section 1 gathers preliminary notions and results on symmetric spaces, their wonderful compactifications, and the associated toric varieties. In particular, for a symmetric space G/K of minimal rank, we study the relations between the root systems and Weyl groups of G, K, and G/K; these are our main combinatorial tools. In Section 2, we first describe the T-stable points and curves in a wonderful symmetric variety X of minimal rank; then we obtain our main structure result for $A^*_G(X)$, and some useful complements as well. Section 3 contains the decompositions of $N_{Y/X}$ and of the restrictions $T_X|_Y$, $S_X|_Y$, together with their applications to equivariant Chern and Todd classes.

Throughout this article, we consider algebraic varieties over an algebraically closed field k of characteristic $\neq 2$; by a point of such a variety, we mean a closed point. As general references, we use [Ha77] for algebraic geometry, and [Sp98] for algebraic groups.

1. Preliminaries

1.1. **The toric variety associated with Weyl chambers.** Let Φ be a root system in a real vector space V (we follow the conventions of [Bo81] for root systems; in particular, Φ is finite but not necessarily reduced). Let W be the Weyl group, Q the root lattice in V, and $Q^\vee$ the dual lattice (the co-weight lattice) in the dual vector space V^*. The Weyl chambers form a subdivision of V^* into rational polyhedral convex cones; let Σ be the fan of V^* consisting of all Weyl chambers and their faces. The pair $(Q^\vee, \Sigma)$ corresponds to a toric variety

$$Y = Y(\Phi)$$

equipped with an action of W via its action on $Q^\vee$ which permutes the Weyl chambers. The group W acts compatibly on the associated torus

$$T := \mathrm{Hom}(Q, \mathbb{G}_m) = Q^\vee \otimes_{\mathbb{Z}} \mathbb{G}_m.$$

Thus, Y is equipped with an action of the semi-direct product $T\,W$. Note that the character group $\mathcal{X}(T)$ is identified with Q; in particular, we may regard each root α as a homomorphism

$$\alpha : T \to \mathbb{G}_m.$$

The choice of a basis of Φ,

$$\Delta = \{\alpha_1, \ldots, \alpha_r\},$$

defines a positive Weyl chamber, the dual cone to Δ. Let $Y_0 \subset Y$ be the corresponding T-stable open affine subset. Then Y_0 is isomorphic to the affine space $\mathbb{A}^r$ on which T acts linearly with weights $-\alpha_1, \ldots, -\alpha_r$. Moreover, the translates $w \cdot Y_0$, where $w \in W$, form an open covering of Y.

In particular, the variety Y is nonsingular. Also, Y is projective, as Σ is the normal fan to the convex polytope with vertices $w \cdot v$ ($w \in W$), where v is any prescribed regular element of V. The following result yields an explicit projective embedding of Y:

Proposition 1.1.1. (i) *For any $\alpha \in \Phi$, the morphism $\alpha : T \to \mathbb{G}_m$ extends to a morphism*

$$f_\alpha : Y \to \mathbb{P}^1.$$

Moreover, f_α and $f_{-\alpha}$ differ by the inverse map $\mathbb{P}^1 \to \mathbb{P}^1$, $z \mapsto z^{-1}$.

(ii) *The product morphism*

$$f := \prod_{\alpha \in \Phi} f_\alpha : Y \to \prod_{\alpha \in \Phi} \mathbb{P}^1$$

is a closed immersion. It is equivariant under TW, where T acts on the right-hand side via its action on each factor $\mathbb{P}^1_\alpha$ through the character α, and W acts via its natural action on the set Φ of indices.

(iii) *Conversely, the T-orbit closure of any point of $\prod_{\alpha \in \Phi}(\mathbb{P}^1 \setminus \{0, \infty\})$ is isomorphic to Y.*

(iv) *Any non-constant morphism $F : Y \to C$, where C is an irreducible curve, factors through $f_\alpha : Y \to \mathbb{P}^1$ where α is an indivisible root, unique up to sign. Moreover,*

$$(f_\alpha)_* \mathcal{O}_Y = \mathcal{O}_{\mathbb{P}^1}. \tag{1.1.1}$$

Proof. (i) Since α has a constant sign on each Weyl chamber, it defines a morphism of fans from Σ to the fan of $\mathbb{P}^1$, consisting of two opposite half-lines and the origin. This implies our statement.

(ii) The equivariance property of f is readily verified. Moreover, the product map

$$\prod_{i=1}^{r} f_{\alpha_i} : Y \to (\mathbb{P}^1)^r$$

restricts to an isomorphism $Y_0 \to (\mathbb{P}^1 \setminus \{\infty\})^r$, since each f_{α_i} restricts to the i-th coordinate function on $Y_0 \cong \mathbb{A}^r$. Since $Y = W \cdot Y_0$, it follows that f is a closed immersion.

(iii) follows from (ii) by using the action of tuples $(t_\alpha)_{\alpha \in \Phi}$ of non-zero scalars, via component-wise multiplication.

(iv) Taking the Stein factorization, we may assume that $F_* \mathcal{O}_Y = \mathcal{O}_C$. Then C is normal, and hence nonsingular. Moreover, the action of T on Y descends to a unique action on C such that F is equivariant (indeed, F equals the canonical morphism $Y \to \operatorname{Proj} R(Y, F^*\mathcal{L})$, where $\mathcal{L}$ is any ample invertible sheaf on C, and $R(Y, F^*\mathcal{L})$ denotes the section ring $\bigoplus_n \Gamma(Y, F^*\mathcal{L}^n)$. Furthermore, $F^*\mathcal{L}$ admits a T-linearization, and hence T acts on $R(Y, \mathcal{L})$). It follows that $C \cong \mathbb{P}^1$ where T acts through a character χ, uniquely defined up to sign. Thus, F induces a morphism from the fan of Y to the fan of $\mathbb{P}^1$; this morphism is given by the linear map $\chi : V^* \to \mathbb{R}$. In other words, χ has a constant sign on each Weyl chamber. Thus, χ is an integral multiple of an indivisible root α, uniquely defined up to sign. Since F has connected fibers, then $\chi = \pm\alpha$.

Conversely, if α is an indivisible root, then the fibers of the morphism $\alpha : T \to \mathbb{G}_m$ are irreducible. This implies (1.1.1). □

Next, for later use, we determine the divisor of each f_α regarded as a rational function on Y. Since f_α is a T-eigenvector, its divisor is a linear combination of the T-stable prime divisors $Y_1, \ldots, Y_m$ of the toric variety Y, also called its *boundary divisors*. Recall that $Y_1, \ldots, Y_m$ correspond bijectively to the rays of the Weyl chambers, i.e., to the W-translates of the fundamental co-weights $\omega_1^\vee, \ldots, \omega_r^\vee$ (which form the dual basis of the basis of simple roots). The isotropy group of each $\omega_i^\vee$ in W is the maximal parabolic subgroup W_i generated by the reflections associated with the simple roots α_j, $j \neq i$. Thus, the orbit $W\omega_i^\vee \cong W/W_i$ is in bijection with the subset

$$W^i := \{w \in W \mid w\alpha_j \in \Phi^+ \text{ for all } j \neq i\}$$

of minimal representatives for the coset space W/W_i. So the boundary divisors are indexed by the set

$$E(\Phi) := \{(i, w) \mid 1 \leq i \leq r,\ w \in W^i\} \cong \{w\omega_i^\vee \mid 1 \leq i \leq r,\ w \in W\};$$

we will denote these divisors by $Y_{i,w}$. Furthermore, we have

$$\operatorname{div}(f_\alpha) = \sum_{(i,w) \in E(\Phi)} \langle \alpha, w\omega_i^\vee \rangle \, Y_{i,w} \tag{1.1.2}$$

by Proposition 1.1.1 and the classical formula for the divisor of a character in a toric variety (see e.g. [Od88, Prop. 2.1]). Also, note that $\langle \alpha, w\omega_i^\vee \rangle$ is the i-th coordinate of $w^{-1}\alpha$ in the basis of simple roots.

1.2. **Symmetric spaces.** Let G be a connected reductive algebraic group, and

$$\theta : G \to G$$

an involutive automorphism. Denote by

$$K = G^\theta \subset G$$

the subgroup of fixed points; then the homogeneous space G/K is a *symmetric space.*

We now collect some results on the structure of symmetric spaces, referring to [Ri82, Sp85] for details and proofs. The identity component K^0 is reductive, and non-trivial unless G is a *θ-split torus*, i.e., a torus where θ acts via the inverse map $g \mapsto g^{-1}$.

A parabolic subgroup $P \subseteq G$ is said to be *θ-split* if the parabolic subgroup $\theta(P)$ is opposite to P. The minimal θ-split parabolic subgroups are all conjugate by elements of K^0; we choose such a subgroup P and put

$$L := P \cap \theta(P),$$

a θ-stable Levi subgroup of P. The intersection $L \cap K$ contains the derived subgroup $[L, L]$; thus, every maximal torus of L is θ-stable. We choose such a torus T, so that

$$T = T^\theta T^{-\theta} \quad \text{and} \quad T^\theta \cap T^{-\theta} \text{ is finite.} \tag{1.2.1}$$

Moreover, the identity component

$$A := T^{-\theta,0}$$

is a maximal θ-split subtorus of G. All such subtori are conjugate in K^0; their common dimension is the *rank* of the symmetric space G/K, denoted by $\operatorname{rk}(G/K)$. Moreover,

$$C_G(A) = L = (L \cap K)A$$

(where $C_G(A)$ denotes the centralizer of A in G), and $(L \cap K) \cap A = A \cap K$ consists of all elements of order 2 of A.

The product $PK^0 \subseteq G$ is open, and equals PK; thus, PK/K is an open subset of G/K, isomorphic to $P/P \cap K = P/L \cap K$. Let P_u be the unipotent radical of P, so that $P = P_u L$. Then the map

$$\iota : P_u \times A/A \cap K \to PK/K, \quad (g, x) \mapsto g \cdot x \tag{1.2.2}$$

is an isomorphism.

The character group $\mathcal{X}(A/A \cap K)$ may be identified with the subgroup $2\mathcal{X}(A) \subset \mathcal{X}(A)$. On the other hand, $A/A \cap K \cong T/T \cap K$ and hence $\mathcal{X}(A/A \cap K)$ may be identified with the subgroup of $\mathcal{X}(T)$ consisting of those characters that vanish on $T \cap K$, i.e.,

$$\mathcal{X}(A/A \cap K) = \{\chi - \theta(\chi) \mid \chi \in \mathcal{X}(T)\}. \tag{1.2.3}$$

Here θ acts on $\mathcal{X}(T)$ via its action on T.

Denote by

$$\Phi_G \subset \mathcal{X}(T)$$

the root system of (G, T), with Weyl group

$$W_G = N_G(T)/T.$$

Choose a basis Δ_G consisting of roots of P. Let $\Phi_G^+ \subset \Phi_G$ be the corresponding subset of positive roots and let $\Delta_L \subset \Delta_G$ be the subset of simple roots of L. The natural action of the involution θ on Φ fixes point-wise the subroot system Φ_L. Moreover, θ exchanges the subsets $\Phi_G^+ \setminus \Phi_L^+$ and $\Phi_G^- \setminus \Phi_L^-$ (the sets of roots of P_u and of $\theta(P_u) = \theta(P)_u$).

Also, denote by

$$p : \mathcal{X}(T) \to \mathcal{X}(A)$$

the restriction map from the character group of T to that of A. Then $p(\Phi_G) \setminus \{0\}$ is a (possibly non-reduced) root system called the *restricted root system*, that we denote by $\Phi_{G/K}$. Moreover,

$$\Delta_{G/K} := p(\Delta_G \setminus \Delta_L)$$

is a basis of $\Phi_{G/K}$. The corresponding Weyl group is

$$W_{G/K} = N_G(A)/C_G(A) \cong N_{K^0}(A)/C_{K^0}(A). \tag{1.2.4}$$

Also, $W_{G/K} \cong N_W(A)/C_W(A)$, and $N_W(A) = W_G^\theta$ whereas $C_W(A) = W_L$. This yields an exact sequence

$$1 \to W_L \to W_G^\theta \to W_{G/K} \to 1. \tag{1.2.5}$$

1.3. **The wonderful compactification of an adjoint symmetric space.** We keep the notation and assumptions of Subsec. 1.2 and we assume, in addition, that G is semi-simple and adjoint; equivalently, Δ_G is a basis of $\mathcal{X}(T)$. Then the symmetric space G/K is said to be *adjoint* as well.

By [DP83, DS99], G/K admits a canonical compactification: the *wonderful compactification* X, which satisfies the following properties.

(i) X is a nonsingular projective variety.

(ii) G acts on X with an open orbit isomorphic to G/K.

(iii) The complement of the open orbit is the union of $r = \mathrm{rk}(G/K)$ nonsingular prime divisors $X_1, \ldots, X_r$ with normal crossings.
(iv) The G-orbit closures in X are exactly the partial intersections

$$X_I := \bigcap_{i \in I} X_i$$

where I runs over the subsets of $\{1, \ldots, r\}$.
(v) The unique closed orbit, $X_1 \cap \cdots \cap X_r$, is isomorphic to $G/P \cong G/\theta(P)$.

We say that X is a *wonderful symmetric variety* with *boundary divisors* $X_1, \ldots, X_r$. By (iii) and (iv), each orbit closure X_I is nonsingular.

Let Y be the closure in X of the subset

$$A/A \cap K \cong AK/K = LK/K \subseteq G/K.$$

Then Y is stable under the action of the subgroup $LN_K(A) \subseteq G$. Since $L \cap N_K(A) = L \cap K = C_K(A)$, and $N_K(A)/C_K(A) \cong W_{G/K}$ by (1.2.4), we obtain an exact sequence

$$1 \to L \to LN_K(A) \to W_{G/K} \to 1.$$

Moreover, since Y is fixed point-wise by $L \cap K$, the action of $LN_K(A)$ factors through an action of the semi-direct product

$$(L/L \cap K)\, W_{G/K} \cong (A/A \cap K)\, W_{G/K}.$$

The adjointness of G and (1.2.3) imply that $\mathcal{X}(A/A \cap K)$ is the restricted root lattice, with basis

$$\Delta_{G/K} = \{\alpha - \theta(\alpha) \mid \alpha \in \Delta_G \setminus \Delta_L\}.$$

Moreover, Y is the toric variety associated with the Weyl chambers of the restricted root system $\Phi_{G/K}$ as in Subsec. 1.1. This defines the open affine toric subvariety $Y_0 \subset Y$ associated with the positive Weyl chamber dual to $\Delta_{G/K}$. Note that

$$Y = W_{G/K} \cdot Y_0. \tag{1.3.1}$$

Also, recall the local structure of the wonderful symmetric variety X: the subset

$$X_0 := P \cdot Y_0 = P_u \cdot Y_0$$

is open in X, and the map

$$\iota : P_u \times Y_0 \to X_0, \quad (g, x) \mapsto g \cdot x \tag{1.3.2}$$

is a P-equivariant isomorphism. Moreover, any G-orbit in X meets X_0 along a unique orbit of P, and meets transversally Y_0 along a unique orbit of $A/A \cap K$.

It follows that the G-orbit structure of X is determined by that of the associated toric variety Y: any G-orbit in X meets transversally Y along a disjoint union of orbit closures of $A/A \cap K$, permuted transitively by $W_{G/K}$. As another consequence, $X_0 \cap G/K = PK/K$ and $Y_0 \cap G/K = AK/K$, so that ι restricts to the isomorphism (1.2.2).

Finally, the closed G-orbit $X_1 \cap \cdots \cap X_r$ meets Y_0 transversally at a unique point z, the T-fixed point in Y_0. The isotropy group G_z equals $\theta(P)$, and the normal space to $G \cdot z$ at z is identified with the tangent space to Y at that point. Hence the weights of T in the tangent space to X at z are the positive roots $\alpha \in \Phi_G^+ \setminus \Phi_L^+$ (the contribution of the tangent space to $G \cdot z$), and the simple restricted roots $\gamma = \alpha - \theta(\alpha)$, where $\alpha \in \Delta_G \setminus \Delta_L$ (the contribution of the tangent space to Y).

1.4. **Symmetric spaces of minimal rank.** We return to the setting of Subsect. 1.2. In particular, we consider a connected reductive group G equipped with an involutive automorphism θ, and the fixed point subgroup $K = G^\theta$.

Let T be any θ-stable maximal torus of G. Then (1.2.1) implies that

$$\mathrm{rk}(G) \geq \mathrm{rk}(K) + \mathrm{rk}(G/K)$$

with equality if and only if the identity component $T^{\theta,0}$ is a maximal torus of K^0, and $T^{-\theta,0}$ is a maximal θ-split subtorus. We then say that the symmetric space G/K is *of minimal rank*; equivalently, all θ-stable maximal tori of G are conjugate in K^0.

Throughout this subsection, we assume that G/K is of minimal rank, and put

$$T_K := T^{\theta,0} = (T \cap K)^0.$$

The following auxiliary result is proved in [Br07, Subsec. 3.2]:

Lemma 1.4.1. (i) *The roots of* (K^0, T_K) *are exactly the restrictions to* T_K *of the roots of* (G, T).

(ii) *The Weyl group of* (K^0, T_K) *may be identified with* W_G^θ.

In particular, $C_G(T_K) = T$ by (i) (this may also be seen directly). We put

$$W_K := W_G^\theta \quad \text{and} \quad N_K := N_{K^0}(T) = N_{K^0}(T_K).$$

By (ii), this yields an exact sequence

$$1 \to T_K \to N_K \to W_K \to 1. \tag{1.4.1}$$

Moreover, by (1.2.5), W_K fits into an exact sequence

$$1 \to W_L \to W_K \to W_{G/K} \to 1. \tag{1.4.2}$$

The group W_K acts on Φ_G and stabilizes the subset $\Phi_G^\theta = \Phi_L$; the restriction map

$$q : \mathcal{X}(T) \to \mathcal{X}(T_K)$$

is W_K-equivariant and θ-invariant. Denoting by Φ_K the root system of (K^0, T_K), Lemma 1.4.1 (i) yields the equality

$$\Phi_K = q(\Phi_G).$$

We now obtain two additional auxiliary results:

Lemma 1.4.2. *Let* $\beta \in \Phi_K$. *Then one of the following cases occurs:*

(a) $q^{-1}(\beta)$ *consists of a unique root,* $\alpha \in \Phi_L$.

(b) $q^{-1}(\beta)$ *consists of two strongly orthogonal roots* α, $\theta(\alpha)$, *where* $\alpha \in \Phi_G^+ \setminus \Phi_L^+$ *and* $\theta(\alpha) \in \Phi_G^- \setminus \Phi_L^-$.

In particular, q *induces a bijection* $q^{-1}q(\Phi_L) = \Phi_L \cong q(\Phi_L)$. *Moreover,* α *and* $\theta(\alpha)$ *are strongly orthogonal for any* $\alpha \in \Phi_G \setminus \Phi_L$; *then* $s_\alpha s_{\theta(\alpha)} \in W_G^\theta = W_K$ *is a representative of the reflection of* $W_{G/K}$ *associated with the restricted root* $\alpha - \theta(\alpha)$.

Proof. Note that any $\alpha \in q^{-1}(\beta)$ is a root of $(C_G(S), T)$, where $S \subseteq T_K$ denotes the identity component of the kernel of β, and $C_G(S)$ stands for the centralizer of S in G. Also, $C_G(S)$ is a connected reductive θ-stable subgroup of G, and the symmetric space $C_G(S)/C_K(S)$ is of minimal rank. Moreover, θ yields an involution of the quotient group $C_G(S)/S$, and the corresponding symmetric space is still of minimal rank. So we may reduce to the case where S is trivial, i.e., K has rank 1. Since β is a root of K^0, it follows that $K^0 \cong \mathrm{SL}(2)$ or $\mathrm{PSL}(2)$. Together with the minimal rank assumption, it follows that one of the following cases occurs, up to an isogeny of G:
(a) $K^0 = G = \mathrm{PSL}(2)$; then θ is trivial.
(b) $K^0 = \mathrm{PSL}(2)$ and $G = \mathrm{PSL}(2) \times \mathrm{PSL}(2)$; then θ exchanges both factors.
This implies our assertions. □

We will identify $q(\Phi_L)$ with Φ_L in view of Lemma 1.4.2.

Lemma 1.4.3. (i) *q induces bijections*

$$\Phi_G^+ \setminus \Phi_L^+ \cong \Phi_K \setminus \Phi_L \cong \Phi_G^- \setminus \Phi_L^-.$$

(ii) *$\Phi_K \setminus \Phi_L$ is a root system, stable under the action of W_K on Φ_K.*

(iii) *The restricted root system $\Phi_{G/K}$ is reduced.*

Proof. (i) follows readily from Lemma 1.4.2.
(ii) Since Φ_L is stable under W_K, then so is $\Phi_K \setminus \Phi_L$. In particular, the latter is stable under any reflection s_β, where $\beta \in \Phi_K \setminus \Phi_L$. It follows that $\Phi_K \setminus \Phi_L$ is a root system.
(iii) We have to check the non-existence of roots $\alpha_1, \alpha_2 \in \Phi_G \setminus \Phi_L$ such that $\alpha_2 - \theta(\alpha_2) = 2(\alpha_1 - \theta(\alpha_1))$. Considering the identity component of the intersection of kernels of α_1, $\theta(\alpha_1)$ and α_2 and arguing as in the proof of Lemma 1.4.2, we may assume that $\mathrm{rk}(G) \leq 3$. Clearly, we may further assume that G is semi-simple and adjoint. Then the pair (G, K) is either $(\mathrm{PSL}(2) \times \mathrm{PSL}(2), \mathrm{PSL}(2))$ or $(\mathrm{PSL}(4), \mathrm{PSp}(4))$, and the assertion follows by inspection. □

Also, note that the pull-back under q of any system of positive roots of Φ_K is a θ-stable system of positive roots of Φ_G. Let Σ_G be the corresponding basis of Φ_G; then $q(\Sigma_G)$ is a basis of Φ_K. If G is semi-simple, then the involution θ is uniquely determined by the its restriction to Σ_G, and the latter is an involution of the Dynkin diagram of G. It follows that the symmetric spaces of minimal rank under an adjoint semi-simple group are exactly the products of symmetric spaces that occur in the following list.

Examples 1.4.4. 1) Let $G = \mathbf{G} \times \mathbf{G}$, where $\mathbf{G}$ is an adjoint semi-simple group, and let θ be the involution of G such that $\theta(x, y) = (y, x)$; then $K = \mathrm{diag}(\mathbf{G})$ and $\mathrm{rk}(G/K) = \mathrm{rk}(\mathbf{G})$.

The maximal θ-stable subtori $T \subset G$ are exactly the products $\mathbf{T} \times \mathbf{T}$, where $\mathbf{T}$ is a maximal torus of $\mathbf{G}$; then $T_K = \mathrm{diag}(\mathbf{T})$, and $A = \{(x, x^{-1}) \mid x \in \mathbf{T}\}$. Thus, $L = T$, and $W_K = W_{G/K} = \mathrm{diag}(W_{\mathbf{G}}) \subset W_{\mathbf{G}} \times W_{\mathbf{G}} = W_G$.

Moreover, Φ_G is the disjoint union of two copies of $\Phi_{\mathbf{G}}$, each of them being mapped isomorphically to $\Phi_K = \Phi_{G/K}$ by q.

2) Consider the group $G = \mathrm{PSL}(2n)$ and the involution θ associated with the symmetry of the Dynkin diagram; then $K = \mathrm{PSp}(2n)$ and $\mathrm{rk}(G/K) = n - 1$.

The Levi subgroup L is the image in $G = \mathrm{PGL}(2n)$ of the product $\mathrm{GL}(2) \times \cdots \times \mathrm{GL}(2)$ (n copies). The Weyl group W_G is the symmetric group S_{2n}, and W_K is the subgroup preserving the partition of the set $\{1, 2, \ldots, 2n\}$ into the n subsets $\{1, 2\}$, $\{3, 4\}$, …, $\{2n - 1, 2n\}$. Thus, W_K is the semi-direct product of $S_2 \times \cdots \times S_2$ (n copies) with S_n. Moreover, $W_L = S_2 \times \cdots \times S_2$, so that $W_{G/K} = S_n$.

For the root systems, we have $\Phi_G = A_{2n-1}$, $\Phi_K = C_n$, $\Phi_L = A_1 \times \cdots \times A_1$ (the subset of long roots of Φ_K), $\Phi_K \setminus \Phi_L = D_n$ (the short roots), and $\Phi_{G/K} = A_{n-1}$.

3) Consider the group $G = \mathrm{PSO}(2n)$ and the involution θ associated with the symmetry of the Dynkin diagram (of type D_n); then $K = \mathrm{PSO}(2n-1)$ and $\mathrm{rk}(G/K) = 1$.

The Levi subgroup L is the image in G of $\mathrm{SO}(2) \times \mathrm{SO}(2n-2)$. The Weyl group W_G is the semi-direct product of $\{\pm 1\}^{n-1}$ with S_n, and W_K is the semi-direct product of $\{\pm 1\}^{n-1}$ with S_{n-1}. Moreover, W_L is the semi-direct product of $\{\pm 1\}^{n-2}$ with S_{n-1}, so that $W_{G/K} = \{\pm 1\}$.

We have $\Phi_G = D_n$, $\Phi_K = B_{n-1}$, $\Phi_L = D_{n-1}$ (the subset of long roots of Φ_K), $\Phi_K \setminus \Phi_L = A_1 \times \cdots \times A_1$ ($n - 1$ copies), and $\Phi_{G/K} = A_1$.

4) Let G be an adjoint simple group of type E_6, and θ the involution associated with the symmetry of the Dynkin diagram; then K is a simple group of type F_4, and $\mathrm{rk}(G/K) = 2$.

The Levi subgroup L has type D_4, and $W_{G/K} = S_3$. We have $\Phi_G = E_6$, $\Phi_K = F_4$, $\Phi_L = D_4$ (the subset of long roots of Φ_K), $\Phi_K \setminus \Phi_L = D_4$, and $\Phi_{G/K} = A_2$.

2. Equivariant Chow ring

2.1. **Wonderful symmetric varieties of minimal rank.** From now on, we consider an adjoint semi-simple group G equipped with an involutive automorphism θ such that the corresponding symmetric space G/K is of minimal rank. Then the group K is connected, semi-simple and adjoint; see [Br07, Lem. 3.2].

We choose a θ-stable maximal torus $T \subseteq G$, so that $A := T^{-\theta,0}$ is a maximal θ-split subtorus. Also, we put $T_K := T^\theta$; this group is connected by [Br07, Lem. 3.2] again. Thus, T_K is a maximal torus of K. In agreement with the notation of Subsec. 1.4, we denote by N_K the normalizer of T_K in K, and by W_K the Weyl group of (K, T_K).

As in Subsec. 1.3, we denote by X the wonderful compactification of G/K, also called a *wonderful symmetric variety of minimal rank*. The associated toric variety Y is the closure in X of $T/T_K \cong A/A \cap K$. Recall that Y is stable under the subgroup $LN_K \subseteq G$, and fixed point-wise by $L \cap K$. Thus, LN_K acts on Y via its quotient group

$$LN_K/(L \cap K) \cong (T/T_K)\, W_{G/K} \cong TN_K/T_K.$$

We will mostly consider Y as a TN_K-variety.

By [Tc05, Sec. 10], X contains only finitely many T-stable curves. We now obtain a precise description of all these curves, and of those that lie in Y. This may be deduced from the results of [loc. cit.], which hold in the more general setting of wonderful varieties of minimal rank, but we prefer to provide direct, somewhat simpler arguments.

Lemma 2.1.1. (i) *The T-fixed points in X (resp. Y) are exactly the points $w \cdot z$, where $w \in W$ (resp. W_K), and z denotes the unique T-fixed point of Y_0. These fixed points are parametrized by W_G/W_L (resp. $W_K/W_L \cong W_{G/K}$).*

(ii) *For any $\alpha \in \Phi_G^+ \setminus \Phi_L^+$, there exists a unique irreducible T-stable curve $C_{z,\alpha}$ which contains z and on which T acts through its character α. The T-fixed points in $C_{z,\alpha}$ are exactly z and $s_\alpha \cdot z$.*

(iii) *For any $\gamma = \alpha - \theta(\alpha) \in \Delta_{G/K}$, there exists a unique irreducible T-stable curve $C_{z,\gamma}$ which contains z and on which T acts through its character γ. The T-fixed points in $C_{z,\gamma}$ are exactly z and $s_\alpha s_{\theta(\alpha)} \cdot z$.*

(iv) *The irreducible T-stable curves in X are the W_G-translates of the curves $C_{z,\alpha}$ and $C_{z,\gamma}$. They are all isomorphic to $\mathbb{P}^1$.*

(v) *The irreducible T-stable curves in Y are the $W_{G/K}$-translates of the curves $C_{z,\gamma}$.*

Proof. The assertions on the T-fixed points in X are proved in [Br07, Lem. 3.4]. And since Y is the toric variety associated with the Weyl chambers of $\Phi_{G/K}$, the group $W_{G/K}$ acts simply transitively on its T-fixed points. This proves (i).

Let $C \subset X$ be an irreducible T-stable curve. Replacing C with a W_G-translate, we may assume that it contains z. Then $C \cap X_0$ is an irreducible T-stable curve in X_0, an affine space where T acts linearly with weights the positive roots $\alpha \in \Phi_G^+ \setminus \Phi_L^+$, and the simple restricted roots $\gamma = \alpha - \theta(\alpha)$, $\alpha \in \Delta_G \setminus \Delta_L$. Since all these weights have multiplicity 1, it follows that $C \cap X_0$ is a coordinate line in X_0. Thus, C is isomorphic to $\mathbb{P}^1$ where T acts through α or γ. In the former case, C is contained in the closed G-orbit $G \cdot z$; it follows that its other T-fixed point is $s_\alpha \cdot z$. In the latter case, C is contained in Y, and hence its other T-fixed point corresponds to a simple reflection in $W_{G/K}$. By considering the weight of the T-action on C, this simple reflection must be the image in $W_{G/K}$ of $s_\alpha s_{\theta(\alpha)} \in W_K$. This implies the remaining assertions (ii)-(v). $\square$

2.2. **Structure of the equivariant Chow ring.** We will obtain a description of the G-equivariant Chow ring of X with rational coefficients. For this, we briefly recall some properties of equivariant intersection theory, referring to [Br97, EG98] for details.

To any nonsingular variety Z carrying an action of a linear algebraic group H, one associates the equivariant Chow ring $A_H^*(Z)$. This is a positively graded ring with degree-0 part $\mathbb{Z}$, and degree-1 part the equivariant Picard group $\mathrm{Pic}_H(Z)$ consisting of isomorphism classes of H-linearized invertible sheaves on Z.

Every closed H-stable subvariety $Y \subseteq Z$ of codimension n yields an equivariant class

$$[Y]_H \in A_H^n(Z).$$

The class $[Z]_H$ is the unit element of $A_H^*(Z)$.

Any equivariant morphism $f : Z \to Z'$, where Z' is a nonsingular H-variety, yields a pull-back homomorphism

$$f^* : A_H^*(Z') \to A_H^*(Z).$$

In particular, $A_H^*(Z)$ is an algebra over $A_H^*(pt)$, where pt denotes $\mathrm{Spec}\, k$.

The equivariant Chow ring of Z is related to the ordinary Chow ring $A^*(Z)$ via a homomorphism of graded rings

$$\varphi_H : A^*_H(Z) \to A^*(Z)$$

which restricts trivially to the ideal of $A^*_H(Z)$ generated by $A^+_H(pt)$ (the positive part of $A^*_H(pt)$). If H is connected, then φ_H induces an isomorphism over the rationals:

$$A^*_H(Z)_{\mathbb{Q}}/A^+_H(pt)A^*_H(Z)_{\mathbb{Q}} \cong A^*(Z)_{\mathbb{Q}}.$$

More generally, there is a natural homomorphism of graded rings

$$\varphi^{H'}_H : A^*_H(Z) \to A^*_{H'}(Z)$$

for any closed subgroup $H' \subset H$. If $H' = H^0$, the neutral component of H, then the group of components H/H^0 acts on the graded ring $A^*_{H^0}(Z)$, and the image of $\varphi^{H'}_H$ is contained in the invariant subring $A^*_{H^0}(Z)^{H/H^0}$. Moreover, $\varphi^{H^0}_H$ induces an isomorphism of rational equivariant Chow rings

$$A^*_H(Z)_{\mathbb{Q}} \cong A^*_{H^0}(Z)^{H/H^0}_{\mathbb{Q}}.$$

If H is a connected reductive group, and $T \subseteq H$ is a maximal torus with normalizer N and associated Weyl group W, then the composite of the canonical maps

$$A^*_H(Z) \to A^*_N(Z) \to A^*_T(Z)^W$$

is an isomorphism over the rationals. In particular, we obtain an isomorphism

$$A^*_H(pt)_{\mathbb{Q}} \cong A^*_T(pt)^W_{\mathbb{Q}}.$$

Furthermore, $A^*_T(pt)$ is canonically isomorphic to the symmetric algebra (over the integers) of the character group $\mathcal{X}(T)$. This algebra will be denoted by S_T, or just by S if this yields no confusion.

Returning to the G-variety X, we may now state our structure result:

Theorem 2.2.1. *The map*

$$r : A^*_G(X) \to A^*_T(X)^{W_G} \to A^*_T(X)^{W_K} \to A^*_T(Y)^{W_K} \tag{2.2.1}$$

obtained by composing the canonical maps, is an isomorphism over the rationals.

Proof. We adapt the arguments of [Br98, Sec. 3.1] regarding regular compactifications of reductive groups; our starting point is the precise version of the localization theorem obtained in [Br97, Sec. 3.4]. Together with Lemma 2.1.1, it implies that the T-equivariant Chow ring $A^*_T(X)$ may be identified as an S-algebra to the space of tuples $(f_{w\cdot z})_{w\in W_G/W_L}$ of elements of S such that

$$f_{v\cdot z} \equiv f_{w\cdot z} \pmod{\chi}$$

whenever the T-fixed points $v \cdot z$ and $w \cdot z$ are joined by an irreducible T-stable curve where T acts through its character χ. This identification is obtained by restricting to the fixed points. The ring structure on the above space of tuples is given by pointwise addition and multiplication; moreover, S is identified with the subring of constant tuples (f).

It follows that $A^*_G(X)_{\mathbb{Q}} \cong A^*_T(X)^{W_G}_{\mathbb{Q}}$ may be identified, via restriction to z, with the subring of $S^{W_L}_{\mathbb{Q}}$ consisting of those f such that

$$v^{-1} \cdot f \equiv w^{-1} \cdot f \pmod{\chi} \tag{2.2.2}$$

for all v, w and χ as above. By Lemma 2.1.1 again, it suffices to check the congruences (2.2.2) when $v = 1$. Then either we are in case (ii) of that lemma, and $w = s_\alpha$, or we are in case (iii) and $w = s_\alpha s_{\theta(\alpha)}$. In the former case, (2.2.2) is equivalent to the congruence

$$f \equiv s_\alpha \cdot f \pmod{\alpha},$$

which holds for any $f \in S_{\mathbb{Q}}$. In the latter case, we obtain

$$f \equiv s_\alpha s_{\theta(\alpha)} \cdot f \pmod{\alpha - \theta(\alpha)}. \tag{2.2.3}$$

Thus, $A^*_G(X)_{\mathbb{Q}}$ is identified with the subring of $S^{W_L}_{\mathbb{Q}}$ defined by the congruences (2.2.3) for all $\alpha \in \Delta_G \setminus \Delta_L$.

On the other hand, we may apply the same localization theorem to the T-variety Y. Taking invariants of W_K and using the exact sequence (1.4.2), we see that $A^*_T(Y)^{W_K}_{\mathbb{Q}}$ may be identified with the same subring of $S_{\mathbb{Q}}$, by restricting to the same point z. This implies our statement. □

2.3. **Further developments.** We will obtain a more precise description of the ring $A^*_T(Y)^{W_K}_{\mathbb{Q}}$ that occurs in Theorem 2.2.1. This will not be used in the sequel of this article, but has its own interest.

Proposition 2.3.1. (i) *We have compatible isomorphisms of graded rings*

$$A^*_T(Y)^{W_K}_{\mathbb{Q}} \cong (S^{W_L}_{T_K} \otimes A^*_{T/T_K}(Y))^{W_{G/K}}_{\mathbb{Q}}$$

and

$$A^*_T(pt)^{W_K}_{\mathbb{Q}} \cong S^{W_K}_{T,\mathbb{Q}} \cong (S^{W_L}_{T_K} \otimes S_{T/T_K})^{W_{G/K}}_{\mathbb{Q}}.$$

(ii) *The image in $A^*_G(X)_{\mathbb{Q}} \cong A^*_T(Y)^{W_K}_{\mathbb{Q}}$ of the subring*

$$A^*_K(pt)_{\mathbb{Q}} \cong S^{W_K}_{T_K,\mathbb{Q}} \cong (S^{W_L}_{T_K} \otimes \mathbb{Q})^{W_{G/K}} \subseteq A^*_T(pt)^{W_K}_{\mathbb{Q}}$$

is mapped isomorphically to $A^*_G(G/K)_{\mathbb{Q}} \cong A^*_K(pt)_{\mathbb{Q}}$ under the pull-back from X to the open orbit G/K.

(iii) *We have isomorphisms*

$$\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{Pic}_G(X)_{\mathbb{Q}} \cong \mathrm{Pic}_T(Y)^{W_K}_{\mathbb{Q}} \cong \mathrm{Pic}_{T/T_K}(Y)^{W_K}_{\mathbb{Q}} \tag{2.3.1}$$

that identify the class $[X_i]$ of any boundary divisor with

$$[X_i \cap Y]_{T/T_K} = \sum_{w \in W^i_{G/K}} [Y_{i,w}]_{T/T_K} \tag{2.3.2}$$

where $Y_{i,w}$ denote the boundary divisors of Y, indexed as in Subsec. 1.1.

Proof. (i) Lemma 2.3.2 below yields a W_K-equivariant isomorphism of graded S_T-algebras

$$A_T^*(Y) \cong S_T \otimes_{S_{T/T_K}} A_{T/T_K}^*(Y),$$

where W_K acts on S_T via its action on T, and on $A_{T/T_K}^*(Y)$ via its compatible actions on T/T_K and Y. Moreover, $\mathcal{X}(T)_{\mathbb{Q}} \cong \mathcal{X}(T_K)_{\mathbb{Q}} \oplus \mathcal{X}(T/T_K)_{\mathbb{Q}}$ as W_K-modules, so that

$$S_{T,\mathbb{Q}} \cong S_{T_K,\mathbb{Q}} \otimes S_{T/T_K,\mathbb{Q}}$$

as graded W_K-algebras. It follows that

$$A_T^*(Y)_{\mathbb{Q}} \cong S_{T_K,\mathbb{Q}} \otimes A_{T/T_K}^*(Y)_{\mathbb{Q}}$$

as graded $S_{T,\mathbb{Q}}$-W_K-algebras. Taking W_K-invariants and observing that the action of $W_L \subseteq W_K$ on the right-hand side fixes pointwise $A_{T/T_K}^*(Y)_{\mathbb{Q}}$, we obtain the desired isomorphisms in view of the exact sequence (1.4.2).

(ii) Since $(G/K) \cap Y = T/T_K$, we obtain a commutative square

$$\begin{array}{ccc} A_G^*(X) & \xrightarrow{r} & A_T^*(Y)^{W_K} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle t} \\ A_G^*(G/K) & \xrightarrow{s} & A_T^*(T/T_K)^{W_K} \end{array}$$

where the vertical arrows are pull-backs, and s is defined analogously to r. Moreover, $A_G^*(G/K) \cong A_K^*(pt)$, $A_T^*(T/T_K) \cong A_{T_K}^*(pt)$, and this identifies $s_{\mathbb{Q}}$ with the isomorphism $A_K^*(pt)_{\mathbb{Q}} \to S_{T_K,\mathbb{Q}}^{W_K}$. Likewise, $t_{\mathbb{Q}}$ is identified with the map

$$(S_{T_K}^{W_L} \otimes A_{T/T_K}^*(Y))_{\mathbb{Q}}^{W_{G/K}} \to (S_{T_K}^{W_L} \otimes \mathbb{Q})^{W_{G/K}}$$

induced by the natural map $A_{T/T_K}^*(Y)_{\mathbb{Q}} \to \mathbb{Q}$. This implies our assertion.

(iii) Let $\mathcal{L}$ be an invertible sheaf on X, then some positive tensor power $\mathcal{L}^n$ admits a G-linearization, and such a linearization is unique since G is semi-simple. This implies the first isomorphism of (2.3.1). The second isomorphism is a consequence of Theorem 2.2.1. To show the third isomorphism, recall that $\mathrm{Pic}^T(Y) \cong A_T^1(Y)$, so that

$$\mathrm{Pic}_T(Y)_{\mathbb{Q}}^{W_K} \cong (\mathcal{X}(T_K)^{W_L} \oplus \mathrm{Pic}_{T/T_K}(Y))_{\mathbb{Q}}^{W_{G/K}} \cong \mathcal{X}(T_K)_{\mathbb{Q}}^{W_K} \oplus \mathrm{Pic}_{T/T_K}(Y)_{\mathbb{Q}}^{W_{G/K}}$$

by (i); moreover, $\mathcal{X}(T_K)_{\mathbb{Q}}^{W_K} = 0$ since the group K is semi-simple. Finally, (2.3.2) follows from the decomposition of $X_i \cap Y$ into irreducible components $Y_{i,w}$, each of them having intersection multiplicity one. □

Lemma 2.3.2. *Let Z be a nonsingular variety carrying an action of a torus T and let $T' \subset T$ be a closed subgroup acting trivially on Z. Then there is a natural isomorphism of graded S_T-algebras*

$$A_T^*(Z) \cong S_T \otimes_{S_{T/T'}} A_{T/T'}^*(Z),$$

where $S_{T/T'}$ is identified with a subring of S_T via the inclusion of $\mathcal{X}(T/T')$ into $\mathcal{X}(T)$.

In particular, if T' is finite then there is a natural isomorphism of graded algebras over $S_{T,\mathbb{Q}} \cong S_{T/T',\mathbb{Q}}$:

$$A_T^*(Z)_{\mathbb{Q}} \cong A_{T/T'}^*(Z)_{\mathbb{Q}}.$$

Proof. We begin by constructing a morphism of graded $S_{T/T'}$-algebras

$$f : A^*_{T/T'}(Z) \to A^*_T(Z)$$

such that $f([Y]_{T/T'}) = [Y]_T$ for any T-stable subvariety $Y \subseteq Z$.

For this, we work in a fixed degree n and consider a pair (V, U), where V is a finite-dimensional T-module, $U \subset V$ is a T-stable open subset such that the quotient $U \to U/T$ is a principal T-bundle, and the codimension of $V \setminus U$ is sufficiently large; see [EG98] for details. Then we can form the mixed quotient

$$Z \times^T U := (Z \times U)/T,$$

and we obtain

$$A^n_T(Z) = A^n(Z \times^T U) \cong A^n(Z \times^{T/T'} U/T') \cong A^n_{T/T'}(Z \times U/T'), \tag{2.3.3}$$

where the latter isomorphism follows from the freeness of the diagonal T/T'-action on $Z \times U/T'$. Composing (2.3.3) with the pull-back under the projection $Z \times U/T' \to Z$ yields a morphism

$$f_n : A^n_{T/T'}(Z) \to A^n_T(Z).$$

One may check as in [EG98] that f_n is independent of the choice of (V, U), and hence yields the desired morphism f.

Using the description of the $S_{T/T'}$-module $A^*_{T/T'}(Z)$ and of the S-module $A^*_T(Z)$ in terms of invariant cycles (see [Br97, Thm. 2.1]), we obtain an isomorphism of graded S_T-modules

$$\mathrm{id} \otimes f : S_T \otimes_{S_{T/T'}} A^*_{T/T'}(Z) \to A^*_T(Z).$$

□

Next, we show that $A^*_G(X)_{\mathbb{Q}}$ is a free module over a big polynomial subring:

Proposition 2.3.3. *Let R denote the $\mathbb{Q}$-subalgebra of $A^*_G(X)_{\mathbb{Q}}$ generated by the image of $A^*_K(pt)$ (defined in Proposition 2.3.1(ii)) and by the equivariant classes $[X_1]_G, \ldots, [X_r]_G$ of the boundary divisors. Then R is a graded polynomial ring, and the R-module $A^*_G(X)_{\mathbb{Q}}$ is free of rank $|W_{G/K}|$.*

Proof. By [Br97, 6.7 Corollary] and Lemma 2.1.1, $A^*_G(X)_{\mathbb{Q}}$ is a free module over $A^*_G(pt)_{\mathbb{Q}}$ of rank being the index $|W_G : W_L|$. As a consequence, the ring $A^*_G(X)_{\mathbb{Q}}$ is Cohen–Macaulay of dimension $\mathrm{rk}(G)$.

On the other hand, the R-module $A^*_G(X)_{\mathbb{Q}}$ is finite by [BP02, Lemma 6] (the latter result is proved there in the setting of equivariant cohomology, but the arguments may be readily adapted to equivariant intersection theory).

Since R is a quotient of a polynomial ring in $\mathrm{rk}(K)+r = \mathrm{rk}(G)$ variables, it follows that R equals this polynomial ring. Moreover, $A^*_G(X)_{\mathbb{Q}}$ is a free R-module, since it is Cohen–Macaulay. This proves all assertions except that on the rank of the R-module $A^*_G(X)_{\mathbb{Q}}$, which may be checked by adapting the Poincaré series arguments of [BP02]. □

3. Equivariant Chern classes

3.1. **The normal bundle of the associated toric variety.** We maintain the notation and assumptions of Subsec. 2.1. In particular, X denotes a wonderful symmetric variety of minimal rank, with associated toric variety Y.

Let $\mathcal{N}_{Y/X}$ denote the normal sheaf to Y in X. This is a LN_K-linearized locally free sheaf on Y, which fits into an exact sequence of such sheaves

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0. \tag{3.1.1}$$

Here $\mathcal{T}_X$ denotes the tangent sheaf to X (this is a G-linearized locally free sheaf on X) and $\mathcal{T}_X|_Y$ denotes its pull-back to Y.

The action of G on X yields a morphism of G-linearized sheaves

$$op_X : \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{T}_X, \tag{3.1.2}$$

where $\mathfrak{g}$ denotes the Lie algebra of G. In turn, this yields a morphism of T-linearized sheaves $\mathcal{O}_Y \otimes \mathfrak{g} \to \mathcal{N}_{Y/X}$ which factors through another such morphism

$$\varphi : \mathcal{O}_Y \otimes \mathfrak{g}/\mathfrak{l} \to \mathcal{N}_{Y/X} \tag{3.1.3}$$

(where $\mathfrak{l}$ denotes the Lie algebra of L), since Y is stable under L. Also, note the isomorphism of T-modules

$$\mathfrak{g}/\mathfrak{l} \cong \bigoplus_{\alpha\in\Phi_G\setminus\Phi_L} \mathfrak{g}_\alpha.$$

We may now formulate a splitting theorem for $\mathcal{N}_{Y/X}$:

Theorem 3.1.1. (i) *We have a decomposition of T-linearized sheaves*

$$\mathcal{N}_{Y/X} = \bigoplus_{\beta\in\Phi_K\setminus\Phi_L} \mathcal{L}_\beta \tag{3.1.4}$$

where each $\mathcal{L}_\beta$ is an invertible sheaf on which T_K acts via its character β. The action of N_K on $\mathcal{N}_{Y/X}$ permutes the $\mathcal{L}_\beta$'s according to the action of W_K on $\Phi_K \setminus \Phi_L$.
(ii) *The map (3.1.3) restricts to surjective maps*

$$\varphi_\beta : \mathcal{O}_Y \otimes (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}) \to \mathcal{L}_\beta \quad (\beta = q(\alpha),\ \alpha \in \Phi_G^+ \setminus \Phi_L^+)$$

which induce isomorphisms of T-modules

$$\Gamma(Y, \mathcal{L}_\beta) \cong \mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}.$$

In particular, φ is surjective, and each invertible sheaf $\mathcal{L}_\beta$ is generated by its global sections. Moreover, the corresponding morphism

$$F_\beta : Y \to \mathbb{P}(\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)})^* \cong \mathbb{P}^1$$

equals the morphism $f_{\alpha-\theta(\alpha)}$ (defined in Prop. 1.1.1).

Proof. (i) Since Y is connected and fixed pointwise by T_K, each fiber $\mathcal{N}_{Y/X}(y)$, $y \in Y$, is a T_K-module, independent of the point y. Considering the base point of G/K and denoting by $\mathfrak{t}$ (resp. $\mathfrak{k}$) the Lie algebra of T (resp. K), we obtain an isomorphism of T_K-modules

$$\mathcal{N}_{Y/X}(y) \cong \mathfrak{g}/(\mathfrak{t} + \mathfrak{k}) \cong \bigoplus_{\alpha\in\Phi_G^+\setminus\Phi_L^+} \mathfrak{g}_\alpha$$

which yields the decomposition (3.1.4) of the normal sheaf regarded as a T_K-linearized sheaf. Since the summands $\mathcal{L}_\beta$ are exactly the T_K-eigenspaces, they are stable under T, and permuted by N_K according to their weights.

(ii) Consider the restriction

$$\varphi_0 : \mathcal{O}_{Y_0} \otimes \mathfrak{g} \to \mathcal{N}_{Y_0/X_0}.$$

By the isomorphism (1.3.2), the composite map

$$\mathcal{O}_{Y_0} \otimes \mathfrak{p}_u \to \mathcal{O}_{Y_0} \otimes \mathfrak{g} \to \mathcal{N}_{Y_0/X_0} \tag{3.1.5}$$

is an isomorphism. Thus, φ_0 is surjective; by N_K-equivariance, it follows that φ is surjective as well. Considering T_K-eigenspaces, this implies in turn the surjectivity of each φ_β.

Thus, the T-linearized sheaf $\mathcal{L}_\beta$ is generated by a 2-dimensional T-module of global sections with weights α and $\theta(\alpha)$. This yields a T-equivariant morphism $F_\beta : Y \to \mathbb{P}^1$. Its restriction to the open orbit T/T_K is equivariant of weight $\alpha - \theta(\alpha)$ for the action of T by left multiplication; thus, we may identify this restriction with the character $\alpha - \theta(\alpha)$. Now Proposition 1.1.1 implies that $F_\beta = f_{\alpha-\theta(\alpha)}$. By (1.1.1) and the projection formula, it follows that the map

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)} = \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \to \Gamma(Y, F_\beta^* \mathcal{O}_{\mathbb{P}^1}(1)) = \Gamma(Y, \mathcal{L}_\beta)$$

is an isomorphism. □

3.2. **The (logarithmic) tangent bundle.** Recall that $\mathcal{T}_X$ denotes the tangent sheaf of X, consisting of all k-derivations of $\mathcal{O}_X$. Let $\mathcal{S}_X \subseteq \mathcal{T}_X$ be the subsheaf consisting of derivations preserving the ideal sheaf of the boundary ∂X. Since ∂X is a divisor with normal crossings, the sheaf $\mathcal{S}_X$ is locally free; it is called the *logarithmic tangent sheaf* of the pair $(X, \partial X)$, also denoted by $\mathcal{T}_X(-\log \partial X)$.

Since G acts on X and preserves X, the map op_X of (3.1.2) factors through a map

$$op_{X,\partial X} : \mathcal{O}_X \otimes \mathfrak{g} \to \mathcal{S}_X. \tag{3.2.1}$$

In fact, $op_{X,\partial X}$ is surjective; this follows, e.g., from the local structure of X, see [BB96, Prop. 2.3.1] for details. In other words, $\mathcal{S}_X$ is the subsheaf of $\mathcal{T}_X$ generated by the derivations arising from the G-action.

Clearly, the sheaf $\mathcal{T}_X$ is G-linearized compatibly with the subsheaf $\mathcal{S}_X$. Moreover, the natural maps $\mathcal{T}_X \to \mathcal{N}_{X_i/X} \cong \mathcal{O}_X(X_i)|_{X_i}$ (where $X_1, \ldots, X_r$ denote the boundary divisors) fit into an exact sequence of G-linearized sheaves

$$0 \to \mathcal{S}_X \to \mathcal{T}_X \to \bigoplus_{i=1}^{r} \mathcal{N}_{X_i/X} \to 0, \tag{3.2.2}$$

see e.g. [BB96, Prop. 2.3.2].

The pull-backs of T_X and $\mathcal{S}_X$ to Y are described by the following:

Proposition 3.2.1. (i) *The exact sequence of TN_K-linearized sheaves*

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0$$

admits a unique splitting.

(ii) *We also have a uniquely split exact sequence of TN_K-linearized sheaves*

$$0 \to \mathcal{S}_Y \to \mathcal{S}_X|_Y \to \mathcal{N}_{Y/X} \to 0, \tag{3.2.3}$$

where $\mathcal{S}_Y$ denotes the logarithmic tangent sheaf of the pair $(Y, \partial Y)$. Moreover, the TN_K-linearized sheaf $\mathcal{S}_Y$ is isomorphic to $\mathcal{O}_Y \otimes \mathfrak{a}$, where TN_K acts on $\mathfrak{a}$ (the Lie algebra of A) via the natural action of its quotient $W_{G/K}$.

Proof. (i) is checked by considering the T_K-eigenspaces as in the proof of Theorem 3.1.1. Specifically, the T_K-fixed part of $\mathcal{T}_X|_Y$ is exactly $\mathcal{T}_Y$, while the sum of all the T_K-eigenspaces with non-zero weights is mapped isomorphically to $\mathcal{N}_{Y/X}$.

(ii) First, note that the natural map

$$\mathcal{O}_Y \otimes \mathfrak{a} \to \mathcal{S}_Y \tag{3.2.4}$$

is an isomorphism, since Y is a nonsingular toric variety under the torus $A/A \cap K$; see e.g. [Od88, Prop. 3.1].

Next, consider the map $\mathcal{T}_X|_Y \to \mathcal{N}_{Y/X}$ and its restriction

$$\pi : \mathcal{S}_X|_Y \to \mathcal{N}_{Y/X}.$$

Clearly, the kernel of π contains the image of the natural map

$$i : \mathcal{O}_Y \otimes \mathfrak{a} \to \mathcal{S}_X|_Y.$$

We claim that the resulting complex of TN_K-linearized sheaves

$$\mathcal{O}_Y \otimes \mathfrak{a} \to \mathcal{S}_X|_Y \to \mathcal{N}_{Y/X} \tag{3.2.5}$$

is exact. By equivariance, it suffices to check this on Y_0. Then the local structure (1.3.2) yields an exact sequence of P-linearized sheaves

$$0 \to \mathcal{O}_{X_0} \otimes \mathfrak{p}_u \to \mathcal{S}_X|_{X_0} \to \mathcal{O}_{X_0} \otimes \mathfrak{a} \to 0,$$

see [BB96, Prop. 2.3.1]. This yields, in turn, an isomorphism

$$\mathcal{O}_{Y_0} \otimes (\mathfrak{p}_u \oplus \mathfrak{a}) \cong \mathcal{S}_X|_{Y_0}$$

which implies our claim by using the isomorphisms (3.1.5) and (3.2.4).

In turn, this implies the exact sequence (3.2.3); its splitting is shown by arguing as in (i). □

Corollary 3.2.2. *We have isomorphisms of TN_K-linearized sheaves*

$$\mathcal{T}_X|_Y \cong \mathcal{T}_Y \oplus \bigoplus_{\beta \in \Phi_K \setminus \Phi_L} \mathcal{L}_\beta, \quad \mathcal{S}_X|_Y \cong (\mathcal{O}_Y \otimes \mathfrak{a}) \oplus \bigoplus_{\beta \in \Phi_K \setminus \Phi_L} \mathcal{L}_\beta, \tag{3.2.6}$$

and an exact sequence of TN_K-linearized sheaves

$$0 \to \mathcal{O}_Y \otimes \mathfrak{a} \to \mathcal{T}_Y \to \bigoplus_{j=1}^{m} \mathcal{O}_Y(Y_j)|_{Y_j} \to 0, \tag{3.2.7}$$

where $Y_1, \dots, Y_m$ denote the boundary divisors of the toric variety Y.

3.3. **Equivariant Chern polynomials.** By [EG98], any G-linearized locally free sheaf $\mathcal{E}$ on X yields *equivariant Chern classes*

$$c_i^G(\mathcal{E}) \in A_G^i(X) \quad (i = 0, 1, \ldots, \mathrm{rk}(\mathcal{E}))$$

which we may encode by the *equivariant Chern polynomial*

$$c_t^G(\mathcal{E}) := \sum_{i=0}^{\mathrm{rk}(\mathcal{E})} c_i^G(\mathcal{E})\, t^i.$$

The map

$$r : A_G^*(X) \to A_T^*(Y)^{W_K}$$

of Theorem 2.2.1 sends $c_t^G(\mathcal{E})$ to $c_t^T(\mathcal{E}|_Y)$, by functoriality of Chern classes.

Together with the decompositions of the restrictions $\mathcal{T}_X|_Y$ and $\mathcal{S}_X|_Y$ (Corollary 3.2.2), this yields product formulae for the equivariant Chern polynomials of the G-linearized sheaves $\mathcal{T}_X$ and $\mathcal{S}_X$:

Proposition 3.3.1. *With the above notation, we have equalities in $A_T^*(Y)$:*

$$r(c_t^G(\mathcal{S}_X)) = \prod_{\beta \in \Phi_K \setminus \Phi_L} (1 + t\, c_1^T(\mathcal{L}_\beta)), \tag{3.3.1}$$

$$r(c_t^G(\mathcal{T}_X)) = \prod_{j=1}^{m} (1 + t\, [Y_j]_T) \times \prod_{\beta \in \Phi_K \setminus \Phi_L} (1 + t\, c_1^T(\mathcal{L}_\beta)), \tag{3.3.2}$$

where $c_1^T(\mathcal{L}_\beta) \in \mathrm{Pic}_T(Y)$ denotes the equivariant Chern class of the T-linearized invertible sheaf $\mathcal{L}_\beta$, and $[Y_j]_T \in \mathrm{Pic}_T(Y)$ denotes the equivariant class of the boundary divisor Y_j.

(Note that the above products are all W_K-invariant, but their linear factors are not.)

Likewise, we may express the image under r of the *equivariant Todd classes* $\mathrm{td}^G(\mathcal{T}_X)$ and $\mathrm{td}^G(\mathcal{S}_X)$:

$$r(\mathrm{td}^G(\mathcal{S}_X)) = \prod_{\beta \in \Phi_K \setminus \Phi_L} \frac{c_1^T(\mathcal{L}_\beta)}{1 - \exp(-c_1^T(\mathcal{L}_\beta))},$$

$$r(\mathrm{td}^G(\mathcal{T}_X)) = \prod_{j=1}^{m} \frac{[Y_j]_T}{1 - \exp(-[Y_j]_T)} \times \prod_{\beta \in \Phi_K \setminus \Phi_L} \frac{c_1^T(\mathcal{L}_\beta)}{1 - \exp(-c_1^T(\mathcal{L}_\beta))}.$$

Finally, we determine the equivariant Chern classes $c_1^T(\mathcal{L}_\beta) \in \mathrm{Pic}^T(Y)$ in terms of the boundary divisors of Y, indexed as in Subsec. 1.1. Here $\beta \in \Phi_K \setminus \Phi_L$ and hence $\beta = q(\alpha)$ for a unique $\alpha \in \Phi_G^+ \setminus \Phi_L^+$.

Proposition 3.3.2. *With the preceding notation, we have*

$$c_1^T(\mathcal{L}_\beta) = \alpha + \sum_{i,w} \langle \alpha - \theta(\alpha), w\omega_i^\vee \rangle\, [Y_{i,w}]_T, \tag{3.3.3}$$

where $\omega_i^\vee$ denote the fundamental co-weights of the restricted root system $\Phi_{G/K}$, and the sum runs over those pairs $(i, w) \in E(\Phi_{G/K})$ such that $w^{-1}(\alpha - \theta(\alpha)) \in \Phi_{G/K}^+$.

Proof. Recall from Theorem 3.1.1 that the T-module of global sections of $\mathcal{L}_\beta$ is isomorphic to $\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\theta(\alpha)}$. Let s be the section of $\mathcal{L}_\beta$ associated with a generator of the line $\mathfrak{g}_\alpha$. Then $c_1^T(\mathcal{L}_\beta) = \alpha + \mathrm{div}_T(s)$, since s is a T-eigenvector of weight α. Moreover, $\mathrm{div}_T(s)$ is the divisor of zeroes of the character $\alpha - \theta(\alpha)$. Together with (1.1.2), this implies the equation (3.3.3). $\square$

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