THE NUMBER e

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1. The Greek beginning. The distinguished American mathematician, Benjamin Peirce, was wont to find all of analysis in the equation

$$i^{-i} = \sqrt{e^{\pi}}.$$

In fact, he had his picture taken in front of a blackboard on which this mystic formula, in somewhat different shape, was inscribed. He would say to his hearers, "Gentlemen, we have not the slightest idea of what this equation means, but we may be sure that it means something very important."

With regard to the symbols which appear in this charm, there is a vast literature connected with π; and i, when written $\sqrt{-1}$, leads into the broad field of analysis in the complex domain; but it seems surprisingly difficult to find a connected account of e.

I think we may make a fair beginning with the twelfth proposition of the Second Book of Apollonius, *Conics*, which tells us that if from a point on a hyperbola lines be drawn in given directions to meet the asymptotes, the product of the two distances is independent of the position of the point chosen on the curve. This theorem is more general than we shall need to arrive at the number e, and it is not original with Apollonius. Let us confine ourselves to the very special case where the hyperbola is rectangular, and we draw to each asymptote a line parallel to the other. When x and y are distances, we may write

$$xy = 1. \tag{1}$$

It is intriguing to inquire who first discovered the theorem which leads to this equation. In the commentary of Eutocius on the Sphere and Cylinder of Archimedes [1], we come to a discussion of the classical problem of inserting two mean proportionals between two given lengths. In one solution, which he labels "ut Menaechmus," we have what amounts to the equations

$$\begin{gathered} a/x = x/y = y/b; \\ y^2 = bx; \qquad xy = ab. \end{gathered} \tag{2}$$

He goes on to seek the intersection of a parabola and a hyperbola.

Eutocius' statement would place the theorem very early in the history of the conics, for Menaechmus is usually regarded as the discoverer or inventor of these curves, although this ascription is by no means certain. Allman writes [2], "It is much to be regretted that the two solutions of Menaechmus have not been transmitted to us in their original form. That they have been altered either by Eutocius or by some author whom he followed appears not only in the employment in these solutions of the terms parabola and hyperbola, as has frequently been pointed out, but more from the fact that the language used in them is, in character, altogether that of Apollonius." A similar doubt is shown in Loria [3]. On the other hand, Heath is perfectly definite on this point; he states, "This property in the particular case of the rectangular hyperbola was known to Menaechmus" [4].

But there is another reason for doubting the ascription to Menaechmus, aside from the linguistic objection. The classical Greek discussion of the conics always corresponds to our analysis when the axes are a tangent and the diameter through the point of contact, and with these data proofs are not simple. Heath, following Zeuthen, shows the fact that the hyperbola can be written immediately in the form (1) if we start with a technique like ours, that is, when the axes are a pair of conjugate diameters [5]. That is perfectly true, but the Greeks made surprisingly little study of the conics when expressed in this form more familiar to us; Apollonius comes to it quite late. It seems to me altogether doubtful that the first discoverer of the curves should have been able to make the transition.

2. Grégoire de St. Vincent. If we grant that the Greek mathematicians, perhaps Menaechmus, were familiar with the fundamental property of the recangular hyperbola expressed in (1), what has this to do with e? We must look ahead some two thousand years to that original writer whose name appears at the head of this paragraph. In 1647, he published his fundamental *Prologomena a Santo Vincento, Opus geometricum quadraturae circuli et sectionum coni.* This I have not seen in its original form, but the content is given at great length by Bopp in [6]. Here is the general scheme. We take the hyperbola

$$xy = 1. \tag{1}$$

On the x axis we take n equivalent rectangles whose bases are

$$P_0P_1,\ P_1P_2,\ \cdots,\ P_{n-1}P_n,$$

while each has an upper vertex on the curve Q_i. Then,

$$P_0P_1 \cdot P_0Q_0 = P_1P_2 \cdot P_1Q_1 = P_2P_3 \cdot P_2Q_2 = \cdots,$$

and

$$\frac{P_0Q_0}{P_1Q_1} = \frac{P_1P_2}{P_0P_1}; \qquad \frac{P_1Q_1}{P_2Q_2} = \frac{P_2P_3}{P_1P_2} \tag{2}$$

but

$$OP_0 \cdot P_0Q_0 = OP_1 \cdot P_1Q_1 = OP_2 \cdot P_2Q_2 \cdots,$$

so that

$$\frac{OP_0}{OP_1} = \frac{P_1Q_1}{P_0Q_0} = \frac{P_0P_1}{P_1P_2} = \frac{OP_1}{OP_2}, \quad \text{by composition.}$$

If

$$OP_1 = \rho OP_0 \qquad \text{then} \qquad OP_j = \rho^j OP_0. \tag{3}$$

St. Vincent even treats the case where OP_0 and OP_j are incommensurable, but we need not follow him here.

The importance of this equation was early recognized, because of its connection with logarithms which were based on the relation of arithmetical and geometrical series. There is a good deal to be said in favor of the thesis that the credit for relating the rectangular hyperbola with logarithms is due to Sarasa. I have not seen his work, but like Cantor, I rely on Kästner. In 1649, Sarasa published *Solutio Problematis a R. P. Marino Mersenno propositi*. This was concerned with the problem: Given three positive quantities and the logarithms of two of them, find the logarithm of the third. Kästner writes [7], "Zu ihrer Beanwortung brang Sarasa drey Saetze aus des Gregorius Buche von der Hyperbel bey, die betreffen Flaechen der Hyperbel an der Aysmptoten, Sarasa erinnert wie das mit Logarithmen zusammenhangt." Cantor's view is similar [8]; he states, "Mit andern Worten, Gregorius hatte das Auftreten von Logarithmen bei der erhähnten Flächenraumen erkannt, wen auch nicht mit Namen genannt. Letzteres that Sarasa, und darin liegt das wirkliche Verdienst seiner Stratschrift."

A contrary view is expressed by Charles Hutton [9] in the words, "As to the first remarks on the analogy between logarithms and hyperbolic spaces, it having been shown by Gregory St. Vincent . . . that if an asymptote be divided into parts in geometrical progression, and from the points of division ordinates be drawn parallel to the other asymptote, they will divide the space between the asymptote and the curve into equal portions, from hence it was shown by Mersenne, that by taking continual sums of these parts there would be obtained areas in arithmetical progression which therefore were analogous to a system of logarithms."

This may be true, but I must point out that whereas St. Vincent published the work referred to above in 1647, Mersenne died in the middle of 1648, and the dates of all of his mathematical writings which I have seen were much earlier. However, St. Vincent's work was certainly well observed. We find Wallis writing in 1658 to Lord Brouncker [10], "Sumptis (in Asymptoto) rectis NH, NI, NK, NQ, NL, NM geometrice proportionalibus, in punctis H, I, K, Q, L, M, ducantur rectae parallelae alteri Asymptoto, spatium Hyperbolicum $A\ B\ H\ M$ in quinque partes dividi ostendit Gregor de Sancto Vincento (si memini) decimo."

3. The introduction of logarithms. The actual word logarithm occurs again in an account of Gregory's *Vera circuli et hyperbolae quadratura*, which was published in Padua in 1667 and laid before the Royal Society [11]. Here we read, "And lastly by the same method he calculates both the logarithm of any natural number, and, vice versa, the natural number of any given logarithm." Perhaps the wisest word on the subject has been pronounced by the kindly old writer Montucla [12], "Au reste la découverte de cette propriété est revindiquée par divers autres géomètres." Among these I surely must mention Christian Huygens, who acknowledges the work of St. Vincent, even though he does not claim for himself the discovery of the relation between the hyperbola and logarithms. This is admirably set forth in [13], first in a French account, then Huygens' own Latin. He finds the areas bounded by the x axis, which is an asymptote, the curve and ordinates. Two such areas terminating by the same ordinate of 1 are

$$\frac{\text{Area } FGDE}{\text{Area } ABDE} = \frac{\log_e FG}{\log_e 10} = \log_{10} FG.$$

Huygens divides numerator and denominator by 32, which amounts to finding the 32nd root of each area, but this has the effect of so far closing up the figure that we may safely replace the hyperbola by a parabola whose outside area is known. He checks by finding a very good value for $\log_{10} 2$.

In the same year, 1661, Huygens finds another curve which he calls logarithmic but we should probably call it exponential. This curve has the property that the ordinate corresponding to the point mid-way between two given points of the x-axis is the mean proportional between their ordinates. The equation of the curve is $y=ka^x$. Huygens takes

$$y = 2^{x/x_0}; \qquad x = \frac{\log y}{\log 2} x_0. \tag{4}$$

The constant subtangent is

$$\frac{ydx}{dy} = \frac{x_0}{\log_e 2}. \tag{5}$$

Huygens takes

$$x_0 = 10^n \log_{10} 2.$$

This gives for the constant subtangent

$$\log_{10} e = .43429448190325180,$$

"qualium logarithmus binarij est"

$$.30102995663981195.$$

These numbers had long been known as they had appeared, for instance, in Briggs' *Arithmetica logarithmica* of 1624, pages 10 and 14. As a matter of fact, there appeared in 1618 a second edition of Wright's translation of Napier's *Mirifici Logarithmorum Canonis Descriptio* which contained an appendix, probably written by Oughtred, giving the natural logarithms of various numbers from 100,000 to 900,000. This is probably the earliest table of natural logarithms, although a very similar table by John Spidell appeared in 1619 [**14**].

The astonishing thing about all of those writers who connected logarithms with hyperbolic areas is their lack of interest in what we should call the base. Napier began by considering the relation between an arithmetical and a geometric series. A geometric series consists in successive powers of one number. What is that number? Or given a set of logarithms, what number has the logarithm 1? I mentioned that Briggs gave the logarithm of e, to the base 10 but I find no mention of e itself. Of course, we might write

$$10^n \log_{10} \frac{10^n + \Delta x}{10^n} = 10^n \log_e \left(1 + \frac{\Delta x}{10^n}\right) \log_{10} e = \Delta x \log_{10} e + \cdots,$$

but e itself does not appear. The fact is that there was no comprehension that a logarithm was essentially an exponent. Tropfke is very explicit in this point; he writes, "Freilich dürfen wir nicht an die moderne Erklärung der Logarithmen denken, die in ihnen Potenzexponenten einer bestimmten Grundzahl erkennt. Diese Auffassung machte sich erst um die Mitte des achtenten Jahrhunderts geltend" [**15**]. This is perhaps too strong a statement, for in a note on the same page he quotes James Gregory (whom he calls David Gregory) as saying in his *Exercitationes Geometricae* of 1684, p. 14, "Exponentes sunt ut logarithmi." I have not been able to verify this, but we find in [**16**], "Si seriei Termonorum in Progressione geometrica ab 1 continue proportionalium, puta

1, 2, 4, 8, 16, 32, 64, etc.

accomedetur series Indicum, sive Exponentium, in progressione ab o continue procedentium, puta

0, 1, 2, 3, 4, 5, 6, etc.

Hos exponentes appelabant Logarithmos." We could not well ask for anything clearer or more explicit.

If most writers did not look on logarithms as exponents, how did they consider them? I think we find the clue in St. Vincent's identification of logarithms with hyperbolic areas, remembering that these were the days of Cavalieiri and Roberval, when an area was looked upon as the same thing as an infinite number of line segments, a very helpful if dangerous definition. We find Halley writing [**17**], "They may more properly be said to be numeri rationum exponentes, wherein we consider ratio as a quantity sui generis, beginning from the ratio of equality, or 1 to $1=0$, $\cdots$ and the rationes we suppose to be measured by the number of ratiunculae in each. Now these ratiunculae are in a continued scale of proportionals, infinite in number, between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal ratiunculae between any other two terms as the logarithm of one ratio is to the logarithm of the other. Thus if we suppose there to be between 1 and 10 an infinite scale of mean proportionals whose number is 100000 ad infinitum, between 1 and 2 there shall be 30102 of said proportionals and between 1 and 3, 47712 of them which numbers therefore are the logarithms of the ratio of 1 to 10 1 to 2, and 1 to 3, and so properly called the logarithms of 10, 2, and 3."

It is hard to see how there could be a much worse explanation of logarithms for those who "make constant use of logarithms without having an adequate notion of them." The one certain thing seems to be that a logarithm is an infinite number. I suppose we might translate this into the form

$$\frac{b}{a} = \frac{a+r_1}{a}\cdot\frac{a+r_2}{a+r_1}\cdot\frac{a+r_3}{a+r_2}, \cdots, \frac{a+r_n}{a+r_{n-1}}\cdot\frac{b}{a+r_n}.$$

If

$$\frac{a+r_j}{a+r_{j-1}} = r, \qquad \frac{b}{a} = r^n.$$

Then n would be the logarithm.

4. Mercator, Newton, Leibniz. It is fair to say that such a definition of a logarithm was not original with Halley. We find Mercator writing in 1668, [18] "Est enim Logarithmus nihil aliud, quam numerus ratiuncularum contentarum in ratione quam absolutus quisque ad unitatem obtinet." I may mention also that this seems the first place where the words "logarithmus naturalis" are used. But the real significance of the article comes from the fact that instead of studying $\log x$ he takes up $\log(1+x)$, which enables him to start from 0. The article is not clearly written, so I follow the much clearer exposition in Wallis [19], which was published in the same year.

We study the area under the curve (1) from $x=1$ to $x=1+X$. We divide the length on the x-axis into n equal parts, each of length Δx. The abscissas are $1, 1+\Delta x, 1+2\Delta x, \cdots, 1+X$ and the corresponding ordinates are

$$1, \frac{1}{1+\Delta x}, \frac{1}{1+2\Delta x}, \cdots, \frac{1}{1+(n-1)\Delta x}.$$

The infinitesimal, rectangular areas are

$$\Delta x, \Delta x[1 - \Delta x + \Delta x^2 - \Delta x^3 + \cdots],$$
$$\Delta x[1 - (2\Delta x) + (2\Delta x)^2 - (2\Delta x)^3 + \cdots], \cdots.$$

Such infinite expansions were common in Wallis' work. The sum of these rectangular areas may be written

$$\Delta x[1+1+1\cdots] - \Delta x[\Delta x + 2\Delta x + 3\Delta x + \cdots]$$
$$+ \Delta x[(\Delta x)^2 + (2\Delta x)^2 + (3\Delta x)^2 + \cdots] - \cdots.$$

Now $n\Delta x = X$, so we have

$$X - \Delta x^2[1 + 2 + 3 \cdots] + \Delta x^3[1^2 + 2^2 + 3^2 \cdots] - \cdots. \tag{6}$$

With regard to these sums, Wallace says [19], page 222, "quod ostendit ille prop XVI etque a me alibi demonstratum." A reference he makes to Mercator is not conclusive as the statement is sketchy; as to his own work I will follow [20], as I shall need that again. Here he is seeking the area under the curve $y=x^m$ from $x=0$ to $x=X$. His method is not perfectly clear, as he seems merely to generalize by analogy from cases worked out earlier, but what he does is essentially the following:

We take N equal lengths from 0 to $N\Delta x=X$. We have a set of rectangles whose combined areas are

$$\Delta x[0^m + (\Delta x)^m + (2\Delta x)^m + (3\Delta x)^m + \cdots].$$

Let us assume that $0^m+1^m+ \cdots +(N-1)^m=\alpha N^{m+1}+\beta N^m+\gamma N^{m-1}+ \cdots$. Replacing N by $N+1$, and subtracting, we obtain

$$N^m = (m + 1)\alpha N^m + bN^{m-1} + cN^{m-2} \cdots,$$

so

$$\alpha = \frac{1}{m+1}\cdot$$

Substituting, and remembering that $N\Delta x=X$, there results

$$\text{Area} = \frac{X^{m+1}}{m+1} + \beta\Delta x X^m + \gamma\Delta x^2 X^{m-1}.$$

The limit of this as $N\rightarrow\infty$ is $X^{m+1}/m+1$, since $\Delta x\rightarrow 0$. We thus can substitute this result in (6), when $m=1, 2, 3, \cdots$, to obtain Mercator's famous formula:

$$\log(1 + X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \frac{X^4}{4} + \cdots. \tag{7}$$

A good deal has been written about this series, as we see from Mazeres and elsewhere. The obvious way to obtain the equation is to apply the calculus, so we now turn to see how this instrument was brought to bear. In 1669, a year after Mercator had published his work on logarithms [18], Newton sent to Collins his article, *De Analysi per aequationes numero terminorum Infinitas* [21]. This represents his first studies of areas under curves, which he had been working at for a year or two, but had not published. In fact, publication did not occur for a goodly number of years to come; there is, however, no question of giving his results precedence over those of Mercator. It begins as shown below:

Curvarum simplicium Quadratura

Reg. 1 $$\text{Si } ax^{m/n} = y, \quad \text{erit } \frac{an}{m+n} x^{(m+n)/n} = \text{Area } ABD.$$

I must speak further of this. In [22] we read on p. 176, "Dr. Wallis published in his Arithmetica infinitorum in the year 1655 and in the 59th Proposition of that Book, if the Abscissa of any curvilinear figure be called x and m and n be two Numbers, the ordinates erected at right Angles be $x^{n/m}$ the area of the Figure shall be $(n/m+n)x^{m+n/n}$. And this is assumed by Mr. Newton, upon which he founds his Quadrature of Curves. Dr. Wallis demonstrated this by steps in many particular Propositions and then connected all the Propositions into one by a Table of Cases. Mr. Newton reduced all Cases to One, with an indefinite Index, and at the end of his Compendium demonstrated it at once by his method of moments, he being the first who introduced indefinite Indices of Dignites into the Operations of Analysis." This is Newton's own statement of the case and must be taken as final. It is true that Wallis worked out a number of special cases in a manner not exactly like the method followed here, and did not use a literal exponent. The greater generality of Newton's formula is found by replacing x by $x^{1/n}$. Newton's proof by "the method of moments" we should call differentiation, and consisted in showing that if

$$z = \frac{n}{m+n} x^{(m+n)/n}; \quad \text{then } \frac{dz}{dx} = x^{m/n}.$$

It is fair to say also that although he gives Mercator's formula, he gives it as the area under the hyperbola, with no mention of Mercator or of logarithms.

It is time to turn for a moment to the other inventor of the calculus, Gottfried Leibniz. We find him writing in 1677 or 1678 [23],

"In Hyperbol sit $AB = 1$, $BM = x$, $ML = \dfrac{1}{1+x}$,

$$CBMLC = \frac{1}{1}x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots."$$

This is proved by the straight expansion of $1/1+x$, after which there is integration term by term. We find something more interesting a dozen years later, when he writes to Huygens, who is said never to have understood Leibniz calculus of differences [24], "Soit donc x l'abscisse et y l'ordonnée de la courbe, et l'équation comme je vous ay dit

$$\frac{x^3y}{h} = b^{2xy}.$$

Je désignerai le logarithme de x par log x et nous aurons

$$3 \log x + \log y - \log h = 2xy.$$

supposant que le log de l'unite soit 0 et le log $b=1$. Donc par la quadrature de l'hyperbole nous aurons

$$3\int\frac{dx}{x}+\int\frac{dy}{y}-\log h=2xy$$

$$3\frac{dx}{x}+\frac{dy}{y}=2xdy+2ydx,$$

dx sera à dy, on bien DB sera à y comme $2x^2y-x$ est à $3y-2xy$ c'est à dire DB sera

$$\frac{2x^2y-xa^2}{3a^2-2xy}$$

comme vous le demandées, a estant l'unité."

5. Leonhard Euler. It is now time to turn to the man who pulled all this together and who put the number e definitely on the map, Leonhard Euler. This he did in [**25**], beginning in "Caput VII" with the base a. His argument is outlined below:

Since $a^0=1$, we may put $a^w=1+kw$; $w=\log(1+kw)$. Assume w to be very small, and write

$$a^{iw}=(1+kw)^i=1+\frac{i}{1}kw+\frac{i(i-1)}{1\cdot 2}k^2w^2+\frac{i(i-1)(i-2)}{1\cdot 2\cdot 3}k^3w^3+\cdots.$$

Since w is infinitesimally small, and i is infinitely large, we write $iw=z$

$$a^z=\left(1-\frac{kz}{i}\right)^i=1+kz+\frac{(i-1)}{i\cdot 1\cdot 2}k^2z^2+\frac{(i-1)(i-2)}{i\cdot 1\cdot 2\cdot 3}k^3k^3+\cdots.$$

Since i is very large, we may assume $(i-n)/i=1$, then

$$a^z=1+kz+\frac{k^2z^2}{1\cdot 2}+\frac{k^3z^3}{1\cdot 2\cdot 3}+\cdots.$$

If $z=1$,

$$a=1+k+\frac{k^2}{1\cdot 2}+\frac{k^3}{1\cdot 2\cdot 3}\cdot$$

If we take $a=10$, the base in the logarithm system of Briggs, Euler gives

$$k=2.30238,\quad\text{approximately.}$$

For a natural logarithm we take $k=1$; $a=e$; and

$$e=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots.\tag{8}$$

Euler gives this value to 18 places, without naming the source, namely,

$$e = \lim_{i\to\infty}\left(1+\frac{1}{i}\right)^i. \tag{9}$$

With regard to the use of the letter e, Euler had long employed it, for we find him writing [26], page 80, "scribitur pro numero cujus logarithmus est unitas e, qui est 2.7182817. . . . " Note that this is Leibniz' b.

I pass to Ch. VII of the *Introductio*. Euler assumes for small values of z,

$$\sin z = z; \qquad \cos z = 1.$$

He then, following DeMoivre, writes,

$$\cos nz = \frac{(\cos z + \sqrt{-1}\sin z)^n + (\cos z - \sqrt{-1}\sin z)^n}{2},$$

$$\sin nz = \frac{(\cos z + \sqrt{-1}\sin z)^n - (\cos z - \sqrt{-1}\sin z)^n}{2\sqrt{-1}}.$$

Putting $nz=v$, and remembering that z is small,

$$\cos v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \cdots,$$

$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots.$$

Comparing these with the value given previously for a^z, one obtains

$$\cos v = \frac{e^{v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}; \qquad \sin v = \frac{e^{v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}; \tag{10}$$

and

$$v\sqrt{-1} = \log\left[\cos v + \sqrt{-1}\sin v\right]. \tag{11}$$

This last formula was not, strictly speaking, original. Roger Cotes in [27] sought the area of an ellipsoid of revolution. When the rotation is about the minor axis there is no trouble, but when the motion is about the major axis we find him writing "Posset hujus etiam superficiei per Logometriam designari, sed modo inexplicabili . . . arcus erit rationis inter

$$EX + XC\sqrt{-1} \text{ a}CE \text{ mensura ducta in } \sqrt{-1}."$$

I will leave Euler for a moment to speak of the numerical value of e. William Shanks, who, until quite recently, held the world's record of 707 places for π, had a try at e [28]. Glaisher found an error in this, but Shanks corrected it, and calculated a value which he was sure was right to 205 places. Glaisher verified 137 of them. Boorman [29] calculated e to 346 places. He acknowledged that he and Shanks agreed only up to 187 places. "One is wrong, which one?" Boorman gives the impression of being a rather amateurish mathematician. Adams [30] calculated $\log_{10} e$ to 272 places, probably all correct. Many years ago I knew a youthful teacher of mathematics who had the vaulting ambition to calculate e by long-hand methods to 1,000 places. I lost sight of him over fifty years ago, probably he died early of heart failure.

I return to Euler. In Caput XVIII of *De Fractionibus continuis* [25], he describes methods of expansion into a continued fraction. When it is a question of turning a rational fraction into a continued one, the process is essentially that of finding a highest common factor, and can be done in only one way. Euler writes

$$e = 2.718281828459 \cdots,$$

$$\frac{e-1}{2} - 0.8591409142295.$$

He writes this in the form,

$$\frac{e-1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{etc.}}}}}},$$

and remarks [25], page 388, "Cuius fractio ex Calculo infinitesimali dari potest."

Euler assumes that the quotients will increase by 4 each time, so that the fraction goes on indefinitely. Hence e is not a rational fraction.

As for finding this "ex Calculo infinitesimali" he returns to this very much later in life, "Summatio fractionis continuae cujus indices progressionem arithmeticam constituunt" in Vol. 23 of his Opera mentioned in [25]. The method consists in establishing contact with a Riccati differential equation. For a fuller discussion see [31]. Euler did not complete all the details with modern rigor, but what I have just shown is the first attempt to demonstrate the irrationality of e.

We must wait a whole century for anything really new and startling in this line. This came in 1874 with Hermite's proof that e is not an algebraic number [32], that is, not the root of any equation with integral coefficients. A much simpler demonstration is given by Klein in [33].

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