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THE MIDTERM COEFFICIENT OF THE CYCLOTOMIC POLYNOMIAL  $F_{pq}(x)$

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**Introduction.** The interested reader will find the background in cyclotomy in [3] and [4] sufficient for the purpose of this note, although the investigation is based on results in [1] and [2].

The monic polynomial whose roots are the primitive  $m$ th roots of unity is defined to be the cyclotomic polynomial  $F_m(x)$ . By Dedekind's inversion formula ([4] p. 114),

$$(1) \quad F_m(x) = \prod_{d|m} (x^d - 1)^{\mu(m/d)}.$$

In [1] it is proved that if  $m$  is a product of two distinct odd primes,  $p$  and  $q$ , then the coefficients of  $F_{pq}(x)$  can equal only  $\pm 1$  or 0.

**General coefficient.** Let  $F_{pq}(x) = \sum_{n=0}^{\phi(pq)} c_n x^n$ .

**THEOREM I.** In  $F_{pq}(x)$

$$(2) \quad c_n = \begin{cases} (-1)^\delta & \text{if } n = \alpha q + \beta p + \delta \text{ in exactly one way,} \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha$  and  $\beta$  are nonnegative integers and  $\delta = 0, 1$ .

*Proof.* From (1) it follows that

$$\begin{aligned} F_{pq}(x) &= (x^{pq} - 1)(x - 1)/(x^p - 1)(x^q - 1) \\ &= (1 - x)(1 + x^q + \dots + x^{(p-1)q})(1 + x^p + x^{2p} + \dots) \\ &= \sum_{\alpha=0}^{p-1} x^{\alpha q} \sum_{\beta=0}^{\infty} x^{\beta p} - \sum_{\alpha=0}^{p-1} x^{\alpha q+1} \sum_{\beta=0}^{\infty} x^{\beta p} \\ &= \sum_{\alpha, \beta, \delta} (-1)^\delta x^{\alpha q + \beta p + \delta}, \end{aligned}$$

where  $\alpha$  runs through the integers from zero to  $p - 1$ ,  $\beta$  is any nonnegative integer, and  $\delta = 0, 1$ . Then  $c_n$  in  $F_{pq}(x)$  is the sum of the coefficients of all terms on the right with exponent  $\alpha q + \beta p + \delta = n$ . Where no such partition exists,  $c_n$  is zero. If there is exactly one partition,  $c_n$  equals  $(-1)^\delta$ .

Assume that  $n$  can be partitioned in two ways:

$$\begin{aligned} n &= \alpha_1 q + \beta_1 p + \delta_1 \\ &= \alpha_2 q + \beta_2 p + \delta_2, \end{aligned}$$

with  $\delta_1 = \delta_2$ . Then  $q(\alpha_1 - \alpha_2) = p(\beta_2 - \beta_1)$ . This implies that  $p$  divides  $\alpha_1 - \alpha_2$ . But since  $\alpha < p$ ,  $|\alpha_1 - \alpha_2| < p$ . Therefore  $\alpha_1 - \alpha_2 = \beta_2 - \beta_1 = 0$ , and the two partitions are identical. Hence, when two distinct partitions of  $n$  in the form (2) exist, in one of them  $\delta = 1$ , in the other  $\delta = 0$ . In this case  $c_n$  is  $(-1)^1 + (-1)^0 = 0$ , and the theorem is proved.

A discussion similar to this occurs in [1]

**Midterm coefficient.** Set  $n = \phi(pq)/2$  in (2). Then

$$\begin{aligned} (p - 1)(q - 1)/2 &= \alpha q + \beta p + \delta, \\ p(2\beta + 1) &\equiv 1 - 2\delta \pmod{q}, \\ px &\equiv \pm 1 \pmod{q}. \end{aligned}$$

Let  $k$  be the solution of  $px \equiv 1 \pmod{q}$ ,  $1 \leq k \leq q - 1$ . Then  $q - k$  is a solution of  $px \equiv -1 \pmod{q}$ .

Consider  $pk \equiv 1 \pmod{q}$ . Then

$$\begin{aligned} pk &= 1 + qh, & h &= (pk - 1)/q, \\ \beta &= (k - 1)/2 & \alpha &= (p - 1)/2 - h/2. \end{aligned}$$

In the case  $k$  is odd, these values of  $\alpha$  and  $\beta$  are integral,  $\delta = 0$ , and the midterm coefficient is 1.

If  $k$  is even,  $q - k$  is odd,  $\delta = 1$ , and the midterm coefficient is  $-1$ . Thus we have

**THEOREM II.** In  $F_{pq}(x)$ , when  $n = \phi(pq)/2$ ,  $c_n = (-1)^{k-1}$ , where  $k$  is the least positive solution of the congruence  $px \equiv 1 \pmod{q}$ .

*Remarks.* In the special case  $q = sp + 1$ ,  $k$  is odd and the midterm coefficient is  $+1$ . Similarly, for  $q = sp - 1$ ,  $k$  is even and the midterm coefficient is  $-1$ .

In any case, the roles of  $p$  and  $q$  in the congruences may be reversed, without affecting the oddness or evenness of  $k$ .

The following table gives the value of the midterm coefficient  $c_n$  of  $F_{pq}(x)$  when  $p$  is 3, 5, or 7. All values of  $m = pq$  and less than 143 reduce to one of these special cases.

$p$	$a$	$c_n$
3	1	} $\pm 1$ according as $q \equiv \pm a \pmod{p}$ .
5	1, 2	
7	1, 3, 5	

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**References**

1. A. S. Bang, Om Ligningen  $\phi_n(x) = 0$ , *Nyt Tidsskrift for Matematik*, 6 (1895) 6-12.
2. A. Migotti, Zur Theorie der Kreisteilungsgleichung, *S.-B. der Math.-Naturwiss. Classe der Kaiserlichen Akademie der Wissenschaften, Wien*, (2) 87 (1883) 7-14.
3. T. Nagell, *Introduction to Number Theory*, Wiley, New York, 1951.
4. B. L. van der Waerden, *Modern Algebra*, transl. Fred Blum, Stechert-Hafner, New York, 1949.