Knot invariants defined by counting diagrams

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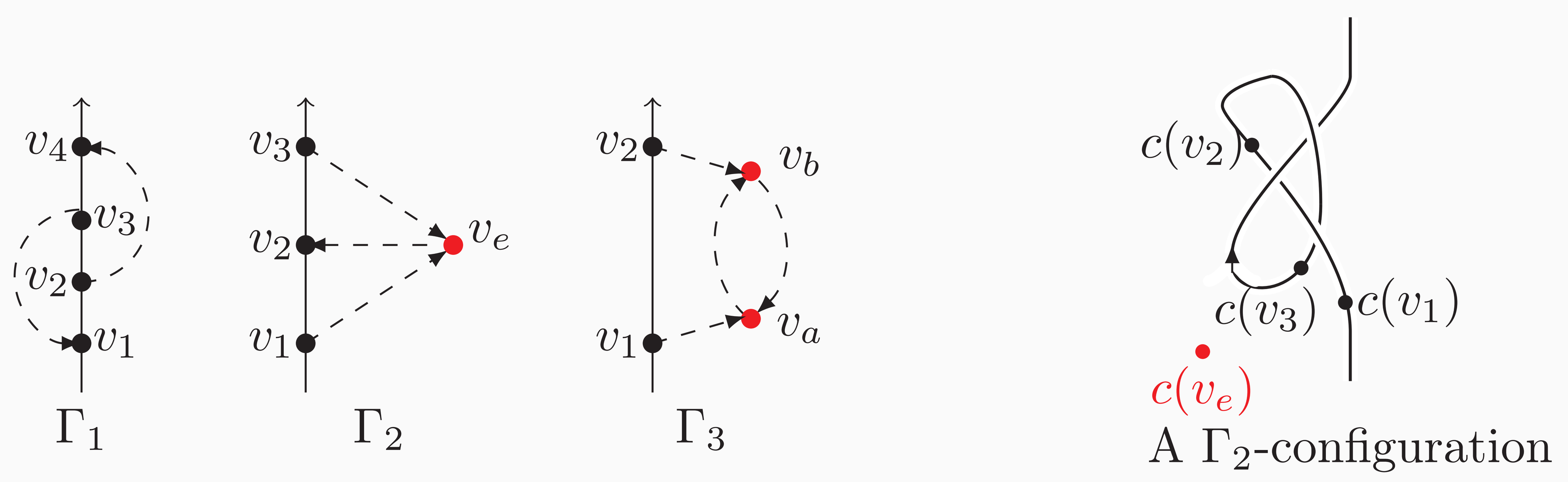
Diagrams and configurations

Fix $\psi\colon \mathbb{R} \hookrightarrow \mathbb{R}^3$ an embedding such that $\psi(x) = (0,0,x)$ for $|x| > 1$ (ψ is a *long knot*).
The degree 2 Jacobi diagrams are the three following graphs:



A Γ-*configuration* is a map $c\colon V(\Gamma) \hookrightarrow \mathbb{R}^3$ such that there exist $t_1 < t_2 < \dots$ such that $c(v_i) = \psi(t_i)$ for any black vertex.
Set $C_\Gamma(\psi) := \{\Gamma - \text{configurations}\}$.

For any edge $v \overset{e}{\dashrightarrow} w$, there is a Gauss map $p_e\colon \begin{array}{ccc} C_\Gamma(\psi) & \to & \mathbb{S}^2 \\ c & \mapsto & \frac{c(w)-c(v)}{||c(w)-c(v)||} \end{array}$.

The degree 2 invariant

Let ω be the standard area form of total area 1 on $\mathbb{S}^2$. Set :

$$\langle \Gamma \rangle = \int_{C_\Gamma(\psi)} \bigwedge_{e \text{ edge of } \Gamma} p_e^*(\omega)$$

Theorem 1 (Bar-Natan, ~90). *$z_2(\psi) := \langle \Gamma_1 \rangle + \langle \Gamma_2 \rangle + \frac{1}{2}\langle \Gamma_3 \rangle$ is a knot invariant.*
$z_2(\psi) = \frac{1}{2}\Delta''_\psi(1)$ where Δ_ψ is the Alexander polynomial of ψ.

Discrete definition: counting diagrams

Fix $(X_i)_{1 \le i \le 4}$ four vectors in $\mathbb{S}^2$.
A *numbering* of Γ is a map $\sigma\colon \{\text{edges of } \Gamma\} \hookrightarrow \{1, \dots, 4\}$. Set $\mathrm{Num}(\Gamma) := \{\text{numberings of } \Gamma\}$.
For any Γ with a numbering σ, we can count the configurations c of $C_\Gamma(\psi)$ such that $p_e(c) = \pm X_{\sigma(e)}$ for any e (with some sign rule). This defines a number $I(\Gamma, \sigma)$.

Theorem 2 (Poirier, Thurston, Polyak-Viro). *For any generic choice of $(X_i)_{1 \le i \le 4}$, the algebraic number $I(\Gamma, \sigma)$ is well-defined for any (Γ, σ) and:*

$$z_2(\psi) = \frac{1}{48} \sum_{\sigma \in \mathrm{Num}(\Gamma_1)} I(\Gamma_1, \sigma) + \frac{1}{192} \sum_{\sigma \in \mathrm{Num}(\Gamma_2)} I(\Gamma_2, \sigma) + \frac{1}{768} \sum_{\sigma \in \mathrm{Num}(\Gamma_3)} I(\Gamma_3, \sigma)$$

Generalization to other spaces

Let M be a compact manifold with the homology of $\mathbb{S}^3$, and set $\check{M} = M \setminus \{\infty\}$.
Let τ be a trivialization $\tau\colon M \times \mathbb{R}^3 \to TM$.
Define $C_2^0(\check{M}) = \{(x,y) \in \check{M}^2 |\ x \neq y\}$.
There exists a compact manifold $C_2(\check{M})$ whose interior is $C_2^0(\check{M})$, with an analogue of a Gauss map on the boundary, i. e. a map:

$$p_\tau\colon \partial C_2(\check{M}) \to \mathbb{S}^2$$

For any edge $v \overset{e}{\dashrightarrow} w$, there is a map $p_{e,2}\colon C_\Gamma(\psi) \to C_2(\check{M})$, defined by the formula $p_{e,2}(c) = (c(v), c(w))$.
A *propagator* is a closed 2-form β on $C_2(\check{M})$ such that there exists an antisymmetric 2-form ω_β on $\mathbb{S}^2$, with area 1, such that $\beta_{|\partial C_2(\check{M})} = p_\tau^*(\omega_\beta)$.

Theorem 3 (Lescop). *Fix four propagators $(\beta_i)_{1 \le i \le 4}$. For any Γ with a numbering σ, set :*

$$\langle \Gamma, \sigma \rangle = \int_{C_\Gamma(\psi)} \bigwedge_{e \text{ edge of } \Gamma} p_{e,2}^*(\beta_{\sigma(e)})$$

$z_2(\psi) := \frac{1}{12} \sum_{\sigma \in \mathrm{Num}(\Gamma_1)} \langle \Gamma_2, \sigma \rangle + \frac{1}{24} \sum_{\sigma \in \mathrm{Num}(\Gamma_2)} \langle \Gamma_2, \sigma \rangle + \frac{1}{48} \sum_{\sigma \in \mathrm{Num}(\Gamma_3)} \langle \Gamma_3, \sigma \rangle$ is a knot invariant.

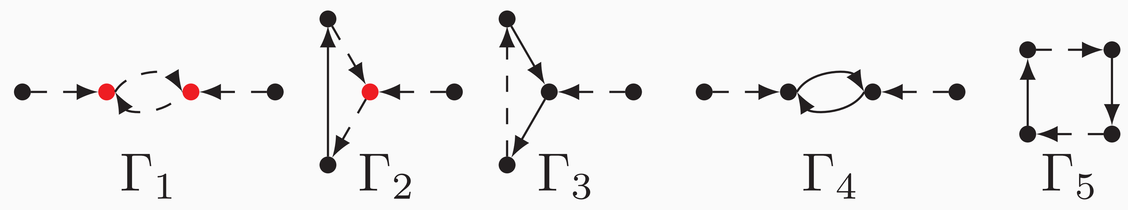
The discrete definition can also be extended using *propagating chains*, which are 4-chains B of $C_2(\check{M})$ such that $\partial B = \frac{1}{2} p_\tau^{-1}(\{-x, +x\})$ for some $x \in \mathbb{S}^2$:

Theorem 4 (Lescop). *Fix four propagating chains $(B_i)_{1 \le i \le 4}$. For any Γ with a numbering σ, we suppose that the algebraic intersection number $I(\Gamma, \sigma)$ of the chains $p_{e,2}^{-1}(B_{\sigma(e)})$ is well-defined. Then:*

$$z_2 = \frac{1}{12} \sum_{\sigma \in \mathrm{Num}(\Gamma_1)} I(\Gamma_1, \sigma) + \frac{1}{24} \sum_{\sigma \in \mathrm{Num}(\Gamma_2)} I(\Gamma_2, \sigma) + \frac{1}{48} \sum_{\sigma \in \mathrm{Num}(\Gamma_3)} I(\Gamma_3, \sigma)$$

Higher dimensional version

Fix an odd integer $n > 1$. Let M be a compact manifold with the homology of $\mathbb{S}^{n+2}$, and set $\check{M} = M \setminus \{\infty\}$.
Fix a long knot $\psi\colon \mathbb{R}^n \hookrightarrow \check{M}$.
We define the following degree 2 Jacobi diagrams:



A Γ-configuration is now a map $c\colon V(\Gamma) \hookrightarrow \check{M}$ that sends the black vertices v to points $\psi(c_i(v))$ of the knot.
Define the spaces $C_\Gamma(\psi)$, $C_2(\check{M})$ and $C_2(\mathbb{R}^n)$ as before.
For any edge, define the map $p_{e,2}$ as $c \in C_\Gamma(\psi) \mapsto (c(v), c(w)) \in C_2(\check{M})$ if e is a dashed edge, and as $c \in C_\Gamma(\psi) \mapsto (c_i(v), c_i(w)) \in C_2(\mathbb{R}^n)$ if e is a solid edge.
Assume that we have a trivialization $\tau\colon \check{M} \times \mathbb{R}^{n+2} \to T\check{M}$. Then we have Gauss maps on the boundaries of the two-point configuration spaces $p_\tau\colon \partial C_2(\check{M}) \to \mathbb{S}^{n+1}$ and $p_\partial\colon \partial C_2(\mathbb{R}^n) \to \mathbb{S}^{n-1}$, as before.
We need two kind of propagators :

- internal propagators are closed $(n-1)$-forms α on $C_2(\mathbb{R}^n)$ such that $\alpha_{|\partial C_2(\mathbb{R}^n)} = p_\partial^*(\omega_\alpha)$ for some antisymmetric volume form ω_α on $\mathbb{S}^{n-1}$ of total mass 1.
- external propagators are closed $(n+1)$-forms β on $C_2(\check{M})$ such that $\beta_{|\partial C_2(\check{M})} = p_\tau^*(\omega_\beta)$ for some antisymmetric volume form ω_β on $\mathbb{S}^{n+1}$ of total mass 1.

Theorem 5 (Bott-Cattaneo-Rossi for $\mathbb{R}^{n+2}$ and standard propagators, Leturcq 2019). *Fix eight propagators $(\alpha_i, \beta_i)_{1 \le i \le 4}$. For any Γ with a numbering σ, set : $\langle \Gamma, \sigma \rangle = \int_{C_\Gamma(\psi)} \bigwedge_{e \text{ dashed}} p_{e,2}^*(\beta_{\sigma(e)}) \wedge \bigwedge_{e \text{ solid}} p_{e,2}^*(\alpha_{\sigma(e)})$*
Then, $z_2(\psi) := \frac{1}{24} \sum_{(\Gamma,\sigma)} \langle \Gamma, \sigma \rangle$ defines a knot invariant.

A combinatorial formula of z_2

Suppose $n \equiv 1 \mod 4$, and $n > 1$.

Theorem 6 (Leturcq, 2019). *Fix a Seifert surface Σ of the knot, and take two homology bases (a_i^k) and (z_i^k) of $H_k(\Sigma)$ for any k such that $\langle a_i^k, z_j^{n+1-k} \rangle = \delta_{i,j}$. Then:*
$z_2(\psi) =$
$\frac{1}{2} \sum_{k=1}^{n} \sum_{i,j} lk(z_i^k, (a_j^{n+1-k})^+) lk(a_i^{n+1-k}, (z_j^k)^+)$
Furthermore, if the $\Delta_{k,\psi}$ are the Alexander polynomials of ψ :

$$z_2(\psi) = \frac{1}{2} \sum_{k=1}^{n} (-1)^k \Delta''_{\psi,k}(1)$$

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