Knot invariants defined by counting diagrams

David Leturcq

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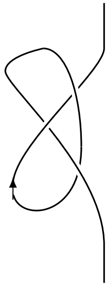

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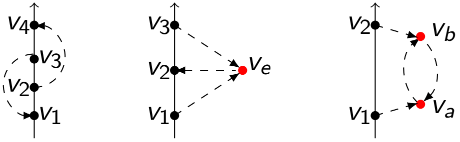
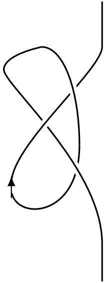


Γ_1 Γ_2 Γ_3



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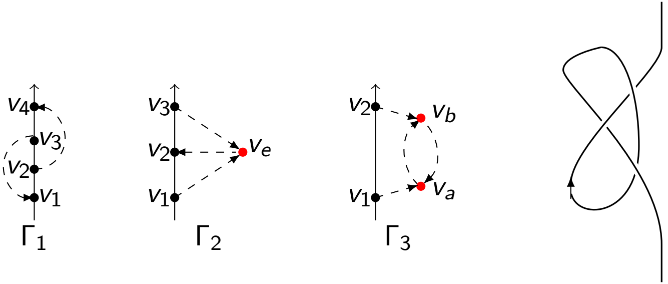

Γ-configuration : $c\colon V(\Gamma) \hookrightarrow \mathbb{R}^3$, such that there exist $t_1 < t_2 < \ldots$ such that for any black dot $c(v_i) = \psi(t_i)$.

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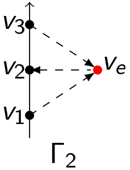

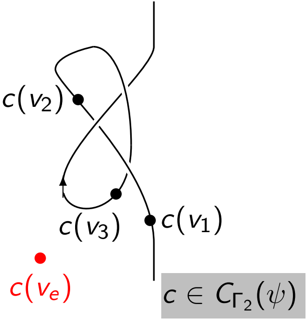

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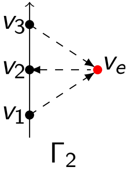

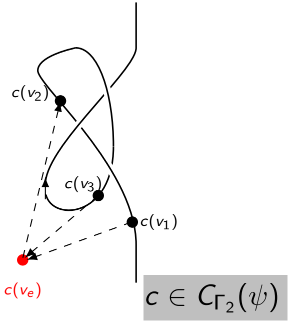

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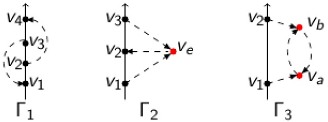

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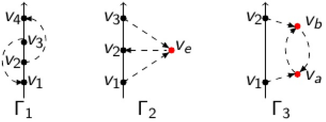

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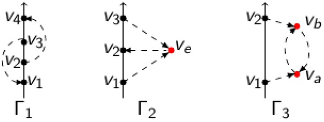

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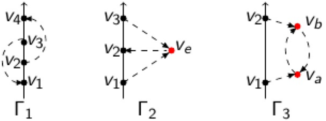

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- For $k = 2$, and for any odd dimension n, the invariant z_2 is related to some second order derivative of Alexander polynomials for all ribbon knots (Watanabe, 2007), and for all knots when $n \equiv 1 \mod 4$ (Leturcq, 2019).