Exercise sheet 11
Zeta functions

Exercise 1. [Elementary properties of the zeta function]

(a) Prove that ζ does not vanish on the domain $\mathrm{Re}(s) > 1$.

(b) Prove the following equalities of holomorphic functions

$$\begin{aligned}\frac{1}{\zeta(s)} &= \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s} \quad (\mathrm{Re}(s) > 1)\\ \frac{\zeta(s-1)}{\zeta(s)} &= \sum_{n=1}^{+\infty} \frac{\varphi(n)}{n^s} \quad (\mathrm{Re}(s) > 2)\end{aligned}$$

where μ is the Möbius function and φ the Euler totient function.

Exercise 2. [Dedekind zeta functions]

Let $K = \mathbb{Q}(\sqrt{d})$.

(a) Recall the discriminant of K and the number of roots of unity in K.

(b) For $|d| \leq 7$, recall the class number h_K.

(c) For $2 \leq d \leq 7$, give the fundamental unit of $\mathcal{O}_K^*$.

(d) Using the previous questions, give the residue at $s = 1$ of ζ_K.

(e) In the quadratic imaginary case, devise a (crude) approach to estimate h_K.

Exercise 3. [Functional equation for zeta]

The *Jacobi theta function* is defined over $]0, +\infty[$ as the sum

$$\theta(t) := \sum_{n=1}^{+\infty} e^{-\pi n^2 t}.$$

(a) Prove that for every $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, one has the integral formula

$$\xi(s) := (2\pi)^{-s}\Gamma(s/2)\,\zeta(s) = \int_0^{+\infty} \theta(t) t^{s/2} \frac{dt}{t}.$$

(b) We admit the Jacobi identity

$$\theta(1/t) = \sqrt{t}\theta(t) + \frac{\sqrt{t}-1}{2}.$$

Use this formula to prove that

$$\xi(s) = \frac{1}{s(s-1)} + f(s) + f(1-s),$$

where

$$f(s) = \int_1^{+\infty} \theta(t) t^{s/2} \, \frac{dt}{t}.$$

(c) Deduce that ξ extends to a meromorphic function on $\mathbb{C}$, whose only (simple) poles are 0 and 1, and such that $\xi(s) = \xi(1-s)$.

(d) Finally, prove (assuming the properties of the Gamma function) that ζ extends to a meromorphic function whose only pole is at $s = 1$, and whose zeroes, except $-2, -4, \cdots$, are in the strip $0 \leq \mathrm{Re}(s) \leq 1$.

Exercise 4. [Analytic density]

Let K be any number field.

(a) Using the Dedekind zeta function ζ_K, prove that

$$\sum_{\mathfrak{p}} \frac{1}{N(\mathfrak{p})^s} \sim \log\left(\frac{1}{s-1}\right)$$

when $s \to 1^+$ by real values (and $\mathfrak{p}$ goes through all the maximal ideals of K).

This leads to the notion of *analytic density* of a set A of maximal ideals of $\mathcal{O}_K$: such a set is said to have analytic density $d(A) \in [0,1]$ when

$$\lim_{\substack{s \to 1^+ \\ s \in \mathbb{R}}} \frac{\sum_{\mathfrak{p} \in A} \frac{1}{N(\mathfrak{p})^s}}{\log(1/(s-1))} = d(A).$$

(b) Prove that if A is finite or contains no ideal whose norm is a prime number, it has analytic density 0.

(c) Explain how the analytic density behaves with respect to disjoint union and complement.

(d) Assume $K/\mathbb{Q}$ is a Galois extension. Let S be the set of prime numbers p which are totally split in K and T the set of prime ideals of $\mathcal{O}_K$ above such primes. Prove that T has density 1 and that S has density $1/[K:\mathbb{Q}]$. What does it imply for $K = \mathbb{Q}(\zeta_n)$?

(e) Using analytic densities, prove that two Galois extensions of $\mathbb{Q}$ having exactly the same totally split primes are equal.