TRACE FIELD DEGREES OF ABELIAN DIFFERENTIALS

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ABSTRACT. We prove that every even number $2 \leq 2d \leq 2g$ is realised as the degree of a Thurston–Veech pseudo-Anosov stretch factor in every connected component of every stratum of the moduli space of Abelian differentials.

1. INTRODUCTION

Pseudo-Anosov mapping classes first appeared in Thurston's work in connection to classification of surface homeomorphisms. Nowadays, their study is a theory by itself combining Teichmüller theory, dynamics, flat geometry and number theory. A mapping class f is pseudo-Anosov if and only if it asymptotically stretches every isotopy class of essential simple closed curves by a fixed factor $\lambda(f)$, with respect to any Riemannian metric. Examples arising from Anosov torus covers are abundant, and there are many other constructions, for instance using train-tracks [PP90], Rauzy induction [Rau79, Vee82], veering triangulations [Ago11], Penner's construction [Pen88] or Thurston–Veech's construction[1] [Thu88] and [Vee89, §9].

Algebraic degrees of stretch factors. An important aspect of the theory of pseudo-Anosov mapping classes emerged with Fried's work and concerns the study of the stretch factor $\lambda(f)$. This is a bi-Perron algebraic integer of degree bounded above by the dimension of the Teichmüller space for the underlying surface. The question of realising any bi-Perron algebraic integer as a stretch factor is a major challenge in the theory. Despite recent advances [Pan20, LP21], Fried's question remains widely open. Observe that we cannot hope for a positive answer if we fix the topology of the underlying surface: there are cubic bi-Perron number that are not realised as the stretch factor of any mapping class on a genus three surface, see the work by Thurston [Thu14, Page 6], and more recently [Yaz21].

Thurston–Veech's construction. Given two multicurves $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ and $\beta = \beta_1 \cup \cdots \cup \beta_m$ with n and m components, respectively, that fill a surface Σ and intersect minimally, we let X be their geometric intersection

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[1]In the literature, this construction is often called *Thurston's construction*. We choose the name to include Veech since in its full generality, the construction first appeared, independently, in the two cited articles by Thurston and Veech.

matrix. In his 1988 seminal bulletin paper [Thu88], Thurston proved that all nontrivial products of multitwists in $\langle T_\alpha, T_\beta \rangle$ except powers of conjugates of T_α or T_β are pseudo-Anosov if the Perron-Frobenius eigenvalue of $XX^\top$ is strictly greater than four. In the same article, Thurston provides the upper bound on the algebraic degree of a pseudo-Anosov stretch factor $\lambda(f)$ by the dimension of the Teichmüller space in general, and by $2g$ in the special case of orientable invariant foliations. He also claimed, without proof, that "*the examples of* [Thu88, Theorem 7] *show that this bound is sharp*". The referenced examples are exactly the examples nowadays known as Thurston's construction, described above. More recently, Strenner [Str17] answered the question of which degrees appear for a pseudo-Anosov on a genus g surface, including all nonorientable surfaces, by using Penner's construction [Pen88].

Main results. In this paper, we justify Thurston's remark for pseudo-Anosov mapping classes with orientable invariant foliations. In fact, we obtain a stronger result justifying Thurston's remark in every connected component of every strata of Abelian differentials, not just for a given genus.

Theorem 1. *Every even integer $2 \leq 2d \leq 2g$ is realised as the degree of a stretch factor of a product of two affine multitwists on a surface in every connected component of every stratum of Abelian differentials on Riemann surfaces of genus g.*

The terminology "connected component" can be skipped on a first reading of this paper, and we refer to §4.1 and [KZ03] for more details. A stratum is the set of Abelian differentials having prescribed singularity multiplicities $(k_1, \dots, k_n)$, where $\sum_{i=1}^n k_i = 2g-2$.

The extension field $K = \mathbb{Q}(\lambda+\lambda^{-1})$ is important in Teichmüller dynamics and is called the trace field. It is an invariant of Abelian differentials and has degree at most g over $\mathbb{Q}$ [KS00]. We will deduce Theorem 1 from the following result asserting that choosing a connected component of a stratum of Abelian differentials poses no restriction on the degree of trace fields.

Theorem 2. *Every integer $1 \leq d \leq g$ is realised as the degree of the trace field of a product of two affine multitwists on a surface in every connected component of every stratum of Abelian differentials on Riemann surfaces of genus g.*

Stretch factor degrees and trace field degrees are closely related. Since λ is a root of the polynomial $t^2 - (\lambda + \lambda^{-1})t + 1$, the degree of λ over K is either one or two. The degree one case corresponds to pseudo-Anosov homeomorphisms with vanishing SAF invariant by a result of Calta and Schmidt [CS13], see also Strenner's article [Str18]. As a key step in proving Theorem 1 we present a novel nonsplitting criterion stating that the degree of the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda+\lambda^{-1})$ equals two under certain conditions in Thurston–Veech's construction, see Theorem 4 in §2.

Square-tiled surfaces. The case when $[\mathbb{Q}(\lambda+\lambda^{-1}) : \mathbb{Q}] = 1$ plays a special role in Teichmüller theory, and our theorems are well-known in this context. The translation surfaces admitting such pseudo-Anosov maps are also called arithmetic surfaces, or square-tiled surfaces, since they are torus coverings [GJ00]. In particular, this implies that the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda+\lambda^{-1})$ has degree two.

Outline of the proof. Let $\mathcal{H}(k_1, k_2, \dots, k_m)$ be a given stratum of Abelian differentials in genus g. Fix some $2 \leq d \leq g$. This is the degree of a trace field we want to construct. In Thurston–Veech's construction, the stretch factor λ of $T_\alpha \circ T_\beta$ is related to the geometric intersection matrix of α and β as follows: the number $\lambda + \lambda^{-1} + 2$ equals the Perron-Frobenius eigenvalue of $XX^\top$. In order to control the degree of $\lambda + \lambda^{-1}$, we therefore need to control the degree of the Perron-Frobenius eigenvalue of $XX^\top$. Roughly, our strategy consists of the following four steps.

Step 1) construct examples. For positive integers y, y_i, $i = 1, \dots, g-1$, we start by constructing a square-tiled surface $(X, \omega) \in \mathcal{H}(k_1, k_2, \dots, k_m)$ depending on y, y_i. We think of the numbers y, y_i as variables that we specify in the following. Applying Thurston–Veech's construction using the core curves of the horizontal and vertical annuli of (X, ω) gives us a $g \times g$ matrix $XX^\top$.

Step 2) specify the y_i. The characteristic polynomial $p_g(t, y) \in \mathbb{Z}[t, y]$ of the matrix $XX^\top$ satisfies $p_g(t, y) = (t-1)^{g-d} p_d(t, y)$ if we set $g - d + 1$ of the $g - 1$ parameters y_i equal to 1. Furthermore, if all the other y_i are pairwise different, then $p_d(t, y)$ is shown to be irreducible in $\mathbb{Z}[t, y]$ in §3.2.

Step 3) specify y. Hilbert's irreducibility theorem [Lan60] furnishes infinitely many integer specifications of y such that $p_d(t, y) \in \mathbb{Z}[t]$ is irreducible. By our construction, all these choices of parameters correspond to surfaces in $\mathcal{H}(k_1, k_2, \dots, k_m)$. Furthermore, the trace field is generated by the Perron-Frobenius eigenvalue of $XX^\top$, which has degree d as desired.

Step 4) Apply the nonsplitting criterion. Finally, we apply Theorem 4 to deduce that the stretch factor λ of $T_\alpha \circ T_\beta$ is of degree $2d$ for all the specifications of y_i and y as above.

This description of the strategy does not yet take into account the connected components we want to reach, but basically the same idea can be applied in order to deal with all connected components. However, we need to take variations of the families of examples we consider in order to find surfaces belonging to all of them. This is dealt with in §4.

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2. A nonsplitting criterion

The goal of this section is to present an algebraic criterion that allows us to deduce that the degree of the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda+\lambda^{-1})$ equals two for certain products of multitwists. Let $\alpha = \alpha_1 \cup \cdots \cup \alpha_n$ and $\beta = \beta_1 \cup \cdots \cup \beta_m$ be two multicurves with n and m components, respectively, that fill a surface Σ and intersect minimally. Let X be their geometric intersection matrix, that is, the $n \times m$ matrix whose ij-th coefficient equals the geometric intersection number of α_i and β_j. We assume that the Perron-Frobenius eigenvalue μ^2 of $XX^\top$ is of degree d. Furthermore, we let $\Omega = \left(\begin{smallmatrix} 0 & X \\ X^\top & 0 \end{smallmatrix}\right)$. For a symmetric matrix A, we denote by $\sigma(A)$ its signature *i.e.* the number of positive eigenvalues minus the number of negative eigenvalues. We will also denote by $\mathrm{null}(A)$ its nullity *i.e.* the dimension of its kernel.

Lemma 3. *The following properties hold.*

1. *The number* $\sigma(\Omega+2I)+\mathrm{null}(\Omega+2I)$ *equals the number of eigenvalues of* Ω *in the interval* $[-2,2]$.
2. *The eigenvalues* λ_i *of* $M = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -X^\top & I \end{pmatrix}$ *are related to the eigenvalues* μ_i *of* Ω *by the equation* $\mu_i^2 = 2 - \lambda_i - \lambda_i^{-1}$.

Proof. The first property is exactly Lemma 3.7 in [Lie17]. The second property is Proposition 3.3(b) in [Lie17]; as the proof in [Lie17] does not explicitly deal with the case where M is not diagonalisable, we present a complete argument here. We first calculate

$$M = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -X^\top & I \end{pmatrix} = \begin{pmatrix} I - XX^\top & X \\ -X^\top & I \end{pmatrix}$$

and note that its inverse is given by

$$M^{-1} = \begin{pmatrix} I & -X \\ X^\top & I - X^\top X \end{pmatrix}.$$

One directly verifies the equation $\Omega^2 = 2I - M - M^{-1}$. In order to obtain the same equation for all the eigenvalues (counting multiplicity), we change basis such that M is in Jordan normal form. Note that in the new basis, also the matrix M^{-1} becomes a block diagonal matrix, where all the blocks are of upper triangular form and correspond to the Jordan blocks of M. In particular, also the matrix Ω^2 becomes upper triangular in the new basis, and the equation for the eigenvalues, $\mu_i^2 = 2 - \lambda_i + \lambda_i^{-1}$ (counting multiplicity), can be read off from the diagonal entries of the matrix equation. □

Our criterion for the construction of pseudo-Anosov maps with stretch factors of controlled degree is the following.

Theorem 4. *If $n+m > \sigma(\Omega+2I) + \mathrm{null}(\Omega+2I) > m+n-2d$, then the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.*

Remark 5. This criterion is particularly strong in case $n = m = d$, that is, when α and β have the same number of components and if the characteristic polynomial of the matrix $XX^\top$ is irreducible. In this case,

$$2n > \sigma(\Omega+2I) + \mathrm{null}(\Omega+2I) > 0$$

is sufficient to ensure that the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$.

Proof. We first ensure that the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov. If $n+m > \sigma(\Omega+2I)+\mathrm{null}(\Omega+2I)$, then Ω has an eigenvalue outside the interval $[-2,2]$ by (1) of Lemma 3. In particular, the dominating eigenvalue μ of Ω is larger than 2 and the matrix product $\left(\begin{smallmatrix} 1 & \mu \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -\mu & 1 \end{smallmatrix}\right)$ is hyperbolic, as its trace $2-\mu^2$ is larger than 2 in modulus. Hence, the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov by Thurston–Veech's construction [Thu88, Vee89].

Now, let λ be the stretch factor of the mapping class $T_\alpha \circ T_\beta$. By Thurston–Veech's construction, we have $\lambda + \lambda^{-1} = \mu^2 - 2$. In particular, we directly observe $\mathbb{Q}(\lambda+\lambda^{-1}) = \mathbb{Q}(\mu^2)$. Furthermore, the degree of the field extension $\mathbb{Q}(\lambda) : \mathbb{Q}(\lambda+\lambda^{-1})$ is either 1 or 2. It equals 2, which is what we want to show, exactly if λ and λ^{-1} are Galois conjugates.

We now finish the proof by arguing that λ and λ^{-1} are indeed Galois conjugates. By (2) of Lemma 3, the dilatation λ is also the leading eigenvalue of $-M$, where M is the matrix product given in the (2) of Lemma 3. In particular, the Galois conjugates of λ are among the eigenvalues $-\lambda_i$ of the matrix $-M$. These eigenvalues are in turn related to the eigenvalues μ_i of Ω by the equation $\mu_i^2 = 2 + \lambda_i + \lambda_i^{-1}$, again by Lemma 3. Since we have $\sigma(\Omega+2I)+\mathrm{null}(\Omega+2I) > n+m-2d$, the matrix Ω has at most $2d-1$ eigenvalues outside the interval $[-2,2]$. Via the correspondence in Lemma 3, the matrix $-M$ hat at most $2d-1$ eigenvalues that do not lie on the unit circle. In particular, one of the $2d$ Galois conjugates of λ or λ^{-1} (including λ and λ^{-1} themselves) must be on the unit circle by the pigeonhole principle. Thus the minimal polynomial of λ or λ^{-1} (and hence of both) is reciprocal and it follows that λ and λ^{-1} are Galois conjugates. □

3. Strata of Abelian differentials

In this section, we present a proof of our main results, Theorem 2 and Theorem 1, for each stratum. We postpone the more intricate analysis of the connected components to §4.

Let $\mathcal{H}(k_1, k_2, \ldots, k_m)$ be a stratum of Abelian differentials. Recall that the number of odd k_i must itself be even, say $2l$. Furthermore, if g is the genus of the underlying topological surface, we have the equality $2g-2 = \sum_{i=1}^m k_i$.

3.1. **Constructing a surface.** We start by constructing a square-tiled surface. First, we ensure that we land in the stratum $\mathcal{H}(k_1, k_2, \ldots, k_m)$. We start out with a long horizontal square-tiled surface with some large number $y^2 - g + 1$ of squares and opposite side identifications, see Figure 1. The surface obtained by identifying the sides is a torus, and there are no

FIGURE 1. A horizontal square-tiled surface.

singularities of the flat structure. We can add an angle of 4π to some cone point (or marked point) by inserting a vertical strip of $y_i + 1$ square tiles, as in Figure 2. We treat the $y_i \geq 1$ as variables that we will need to specify later on. Like this, we can reach all strata with even multiplicities. In

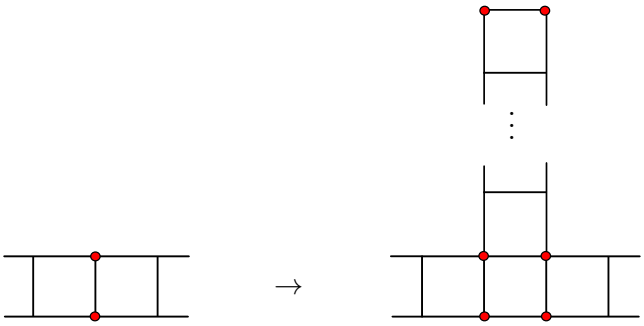

FIGURE 2. Inserting a vertical strip of squares adds 4π to the angle around a point.

order to create odd multiplicities, we insert an L-shaped square-tiled surface with $y_i + 1$ tiles, as shown in Figure 3. This creates two cone points

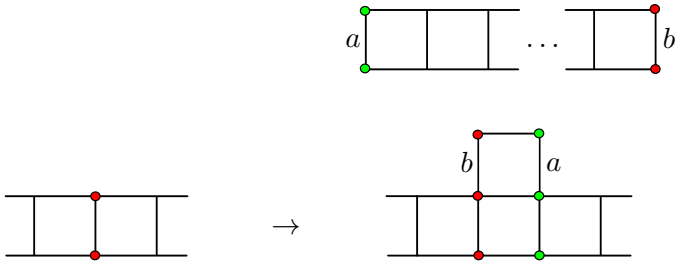

FIGURE 3. Inserting an L-shaped square-tiled surface creates two cone angles of 4π out of one marked point.

of angle 4π, which is multiplicity one. Recall that there must be an even number $2l$ of odd multiplicities k_i, so we can repeat this step l times to

have the right number of odd multiplicities, and then successively add two to the multiplicities by inserting vertical strips as above, until we reach the stratum $\mathcal{H}(k_1, k_2, \ldots, k_m)$. Following this procedure, we need to add a total of l L-shapes and $g - l - 1$ vertical strips.

3.2. **Calculating the polynomial.** The square-tiled surface we construct in Section 3.1 naturally decomposes into horizontal and vertical annuli that are one square wide. Let X be the intersection matrix for the core curves α_i of the horizontal annuli and the core curves β_j of the vertical annuli. We index the rows by horizontal curves and the columns by vertical curves. We now describe the matrix $XX^\top$. Since the curves α_i and β_j pairwise intersect in a tree-like fashion, we use the following way of looking at the computation. The i-th diagonal coefficient equals the number of vertical curves intersecting the i-th horizontal curve α_i. Furthermore, an off-diagonal ij-th coefficient is equal to 1 if there exists a vertical curve intersecting both horizontal curves α_i and α_j. Otherwise, it equals 0.

In order to write down the matrix $XX^\top$, we quickly recall our construction. We have one horizontal curve that we start with. It intersects y^2 vertical curves. We further have one horizontal curve for each L-shaped surface we inserted, which are l in total. These curves respectively intersect y_i vertical curves, for $i = 1, \ldots, l$, and are linked to the starting horizontal curve via an intersecting vertical curve. Finally, for each vertical surface we inserted, we get another y_i horizontal curves, all intersecting a single vertical curve that also intersects the starting horizontal curve. Here, i runs from $l+1$ to $g-1$. We present the matrix using parameters $b, b_i \in \mathbb{R}$. These parameters are helpful in the proof of Lemma 7, and later in §4.3. For the purpose of the calculation of $XX^\top$ in this section, we simply have $b = b_i = 1$ for all i. We write $\mathbf{b}_{n\times m}$ for the $n \times m$ matrix with all entries equal to $b \in \mathbb{R}$. In case $n = m$, we simplify and write $\mathbf{b}_n$.

Definition 1. *For parameters $b, b_i \in \mathbb{R}$, $i = 1, \ldots, l$, we consider the matrix*

$$XX^\top = \left(\begin{array}{cccc|ccc} y^2 & b_1 & \cdots & b_l & \mathbf{b}_{1\times y_{l+1}} & \mathbf{1}_{1\times y_{l+2}} & \cdots & \mathbf{1}_{1\times y_{g-1}} \\ b_1 & y_1 & & & & & & \\ \vdots & & \ddots & & & & & \\ b_l & & & y_l & & & & \\ \hline \mathbf{b}_{y_{l+1}\times 1} & & & & \mathbf{1}_{y_{l+1}} & & & \\ \mathbf{1}_{y_{l+2}\times 1} & & & & & \mathbf{1}_{y_{l+2}} & & \\ \vdots & & & & & & \ddots & \\ \mathbf{1}_{y_{g-1}\times 1} & & & & & & & \mathbf{1}_{y_{g-1}} \end{array}\right).$$

Lemma 6. *The characteristic polynomial of $XX^\top$ equals*

$$p(t, y, \mathbf{y}) = t^a \left(-y^2 \prod_{i=1}^{g-1}(t - y_i) + t \prod_{i=1}^{g-1}(t - y_i) - \sum_{i=1}^{g-1} c_i \prod_{j\neq i}(t - y_j) \right),$$

where $a = \sum_{i=l+1}^{g-1}(y_i - 1)$, $c_{l+1} = y_{l+1}b^2$, $c_i = y_i$ *for* $i \geq l+2$ *and* $c_i = b_i^2$ *otherwise.*

Proof. This calculation is slightly tedious, but obtained in a fairly straightforward manner by developing the first column of $(tI - XX^\top)$. We begin by observing that the determinants of the $y_i \times y_i$ matrices

$$tI_{y_i} - \mathbf{1}_{y_i} = \begin{pmatrix} t-1 & -1 & \cdots & -1 \\ -1 & t-1 & & \\ \vdots & & \ddots & \\ -1 & & & t-1 \end{pmatrix},$$

$$M_{y_i}(t) = \begin{pmatrix} -1 & -1 & \cdots & -1 \\ -1 & t-1 & & \\ \vdots & & \ddots & \\ -1 & & & t-1 \end{pmatrix}$$

are respectively given by the polynomials $t^{y_i-1}(t - y_i)$ and $-t^{y_i-1}$. The former calculation follows by inspecting the eigenvalues of the matrix $\mathbf{1}_{y_i}$, and the latter is derived by solving the equation

$$\det(tI_{y_i} - \mathbf{1}_{y_i}) = t\det(tI_{y_i-1} - \mathbf{1}_{y_i-1}) + \det(M_{y_i}(t)).$$

We note and will later use that changing the diagonal coefficient (-1) of the matrix $M_{y_i}(t)$ with some other diagonal coefficient $(t-1)$ does not change the determinant.

Now, by developing the first column of $(tI - XX^\top)$, we get that the characteristic polynomial of $XX^\top$ has the following summands. The first summand (obtained by deleting the first row and the first column when developing) equals

$$(t-y^2)\prod_{i=1}^{l}(t-y_i)\prod_{i=l+1}^{g-1}\det(tI_{y_i} - \mathbf{1}_{y_i}) = t^a(t-y^2)\prod_{i=1}^{g-1}(t-y_i),$$

where $a = \sum_{i=l+1}^{g-1}(y_i - 1)$. The rest of the summands are obtained as follows. Assume that in the development we delete the first column and the k-th row, where $k \geq 2$. We have to take the determinant of the matrix obtained by deleting the k-th row of the matrix

$$\left(\begin{array}{ccc|ccccc} -b_1 & \cdots & -b_l & -\mathbf{b}_{1\times y_{l+1}} & & -\mathbf{1}_{1\times y_{l+2}} & \cdots & -\mathbf{1}_{1\times y_{g-1}} \\ t-y_1 & & & & & & & \\ & \ddots & & & & & & \\ & & t-y_l & & & & & \\ \hline & & & tI_{y_{l+1}} - \mathbf{1}_{y_{l+1}} & & & & \\ & & & & tI_{y_{l+2}} - \mathbf{1}_{y_{l+2}} & & & \\ & & & & & & \ddots & \\ & & & & & & & tI_{y_{g-1}} - \mathbf{1}_{y_{g-1}} \end{array}\right).$$

After switching adjacent rows (a total of $k-2$ times) to move the first row to be the $k-1$st one, the matrix obtained is almost of block diagonal form and we can read off the determinant. For the rows $k = 2, \dots, l+1$, we obtain the summand

$$(-b_{k-1})(-1)^{1+k}(-1)^{k-2}\left(\prod_{j\neq k-1,\ 1\leq j\leq l}(t-y_j)\prod_{i=l+1}^{g-1}\det(tI_{y_i}-\mathbf{1}_{y_i})\right)(-b_{k-1}) = \\ = -b_{k-1}^2 t^a \prod_{j\neq k-1}(t-y_j).$$

For the rows $k > l + y_{l+1}$, we obtain summands of the form

$$(-1)(-1)^{1+k}(-1)^{k-2}\left(\prod_{j=1}^{l}(t-y_j)\prod_{j\neq i,\ l+1\leq j\leq g-1}\det(tI_{y_j}-\mathbf{1}_{y_j})\right)\det(M_{y_i}(t)) = \\ = -t^a\prod_{j\neq i}(t-y_j).$$

Here, we assume for the calculation that the k-th row intersects the diagonal block $tI_{y_i} - \mathbf{1}_{y_i}$, where $i \geq l+2$. There are a total of y_i summands of this type. If the k-th row intersects the block $tI_{y_{l+1}} - \mathbf{1}_{y_{l+1}}$, the corresponding constant vectors of the first row and the first column have coefficients $b \in \mathbb{R}$. In this case, we obtain y_{l+1} times the summand

$$-b^2 t^a \prod_{j\neq l+1}(t-y_j).$$

Adding all summands, we finally obtain the polynomial

$$t^a\left((t-y^2)\prod_{i=1}^{g-1}(t-y_i) - \sum_{i=1}^{g-1}c_i\prod_{j\neq i}(t-y_j)\right),$$

where $a = \sum_{i=l+1}^{g-1}(y_i-1)$, $c_{l+1} = y_{l+1}b^2$, $c_i = y_i$ for $i \geq l+2$ and $c_i = b_i^2$ otherwise. □

Lemma 7. *Let $k \geq 1$, and let $y_i, c_i \in \mathbb{Z}$ for $i = 1, \dots, k$ such that all y_i are pairwise distinct and all c_i are positive. Then the polynomial*

$$p(t,y) = -y^2\prod_{i=1}^{k}(t-y_i) + t\prod_{i=1}^{k}(t-y_i) - \sum_{i=1}^{k}c_i\prod_{j\neq i}(t-y_j)$$

is irreducible in $\mathbb{Z}[t,y]$.

Proof. We note that $p(t,y)$ is a polynomial of degree two in the variable y. Furthermore, the coefficient of y^2 and the constant coefficient are relatively prime. This follows from the observation that the complex roots of the coefficient of y^2 are exactly the y_i, while none of those numbers is a root of the constant coefficient. This implies that the only possibility to factorise $p(t,y)$ is by writing it as a product of two factors linear in the variable y. But this

is prohibited by Eisenstein's criterion. Indeed, the constant coefficient has a simple root: the Perron-Frobenius eigenvalue of a matrix of the form $XX^\top$ as in Definition 1, where we set $l = g - 1 = k$, $b_i = \sqrt{c_i}$ and $y = 0$. We can therefore apply Eisenstein's criterion to the irreducible factor in $\mathbb{Z}[t]$ containing this root, and deduce that $p(t, y)$ can not be factored into a product of two factors with positive degree in the variable y. □

3.3. **Main results for strata.** We are now ready to prove the analogues of Theorem 2 and Theorem 1 for strata of Abelian differentials.

Theorem 8. *Every number $1 \leq d \leq g$ is realised as the degree of the trace field of a product of two affine multitwists on a surface in every stratum of Abelian differentials on Riemann surfaces of genus g.*

Proof. Let $\mathcal{H}(k_1, k_2, \ldots, k_m)$ be a stratum of Abelian differentials. We use the surface constructed in §3.1. By Thurston–Veech's construction [Thu88, Vee89], there exists a flat structure on it, obtained by changing the side lengths of the rectangles, such that the multitwists T_α and T_β have affine representatives, and such that the degree of the trace field is given by the degree of the Perron-Frobenius eigenvalue μ^2 of $XX^\top$.

Let $2 \leq d \leq g$ be the some degree of a trace field we want to construct. Set $g - d + 1$ of the $g - 1$ parameters y_i equal to 1 and all others $\neq 1$ and pairwise distinct. In this way, the characteristic polynomial of $XX^\top$ can be factored as $(t-1)^{g-d}p(t, y)$, where the polynomial $p(t, y)$ is of degree d in the variable t and with pairwise distinct y_i. In particular, Lemma 7 implies that $p(t, y)$ is irreducible as a polynomial in $\mathbb{Z}[t, y]$. Now, by Hilbert's irreducibility theorem [Lan60], there are infinitely many integer specifications of y such that the resulting polynomial is irreducible in $\mathbb{Z}[t]$. For $|y|$ large enough, all these specifications can be realised geometrically as in §3.1, since we start with $y^2 - g + 1$ squares in the construction. In particular, for every such y, we obtain an Abelian differential with trace field of degree d. □

Theorem 9. *Every even number $2 \leq 2d \leq 2g$ is realised as the degree of a product of two affine multitwists on a surface in every stratum of Abelian differentials on Riemann surfaces of genus g.*

Proof. In the proof of Theorem 8, we have constructed examples with Perron-Frobenius eigenvalue μ^2 of $XX^\top$ having degree d by letting $g - d + 1$ parameters y_i equal to 1. For these examples, we now bound $\sigma(2I + \Omega)$ in order to apply Theorem 4 to $T_\alpha \circ T_\beta$. Let Ω' be the matrix obtained from Ω by deleting all the rows and all the columns corresponding to y or the $g - 1 - (g - d + 1) = d - 2$ parameters y_i that are not set equal to 1. We have

$$\sigma(\Omega + 2I) \geq \sigma(\Omega' + 2I) - (d - 1).$$

By construction, Ω' is the adjacency matrix of a forest consisting of path graphs (some of which might be of length zero). In particular, one directly

verifies that $2I + \Omega'$ is positive definite. We get

$$\sigma(\Omega + 2I) \geq \sigma(\Omega' + 2I) - (d-1) = n + m - 2d + 2 > n + m - 2d.$$

Furthermore, one directly checks that the matrix $\sigma(\Omega + 2I)$ has a negative eigenvalue as soon as $y > 4$, which we are allowed to assume. This implies

$$n + m > \sigma(\Omega + 2I) + \mathrm{null}(\Omega + 2I).$$

Theorem 4 applies and the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$. □

Remark 10. The examples we construct above can all obtained by capping off monodromies of certain fibred links called positive arborescent Hopf plumbings. The pseudo-Anosov stretch factors therefore appear as the dominating roots of the Alexander polynomials of these links. It is conceivable that our argument, or at least a portion thereof, could be replaced by a careful analysis of these Alexander polynomials using the skein relation. However, the calculations we present here can readily be applied to our examples in §4, which are not necessarily obtained from arborescent Hopf plumbings anymore.

4. Connected components of strata

In this section, we study the connected components of strata of Abelian differentials. After recalling the classification of the connected components, we first analyse to which connected components our examples from §3 belong. We then construct examples covering all remaining connected components, finally proving Theorem 2 and Theorem 1 in full generality.

4.1. **Classification of connected components of strata.** The connected components of the strata of the moduli space of Abelian differentials are classified by [KZ03]. There are at most three connected components, and the classification uses two topological invariants that we describe now.

(1) *Hyperellipticity.* For $g \geq 2$, the strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1, g-1)$ have a component that consists entirely of hyperelliptic Riemann surfaces, where the hyperelliptic involution permutes the two zeros (when there are two).
(2) *Parity of the spin structure.* If the degrees of the singularities of a stratum are all even, then one can define a spin structure, or equivalently a quadratic form q on the first homology group. The parity of this spin structure (or the Arf invariant of the form) is a topological invariant.

Remark 11. If a translation surface belongs to a hyperelliptic component $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ and admits a cylinder decomposition, then all cylinders are fixed by the hyperelliptic involution, and each of them contains exactly two fixed points in its interior. Since the total number of fixed points is $2g+2$, this observation can be used to show that a translation surface does not belong to a hyperelliptic component.

We will use the topological definition of the spin structure (see [KZ03, §3.1] for details) to have an effective way to compute its parity in terms of the Arf invariant of q. Since the flat metric (X,ω) has trivial holonomy, outside of finite number of singularities, we have a well-defined horizontal direction. Consider a smooth simple closed oriented curve α on X which does not contain any singularities. The total change of the angle between the tangent vector to α and the tangent vector to the horizontal is equal to $2\pi\cdot \mathrm{ind}(\alpha)$, where $\mathrm{ind}(\alpha)\in\mathbb{Z}$. Choose any symplectic basis $(a_i,b_i)_{i=1,\dots,g}$ of $H_1(X;\mathbb{Z}/2)$. Then the parity of the spin structure is [KZ03, Equation (4)]:

$$\Phi(\omega)=\sum_{i=1}^{g} q(a_i)q(b_i) \mod 2, \tag{1}$$

where $q(\alpha)=\mathrm{ind}(\alpha)+1$ for an oriented smooth path α. Together with the formula $q(\alpha+\beta)=q(\alpha)+q(\beta)+i(\alpha,\beta)$ for any $\alpha,\beta\in H_1(X;\mathbb{Z}/2)$, it is easy to calculate the parity of the spin structure given in any (non symplectic) basis of the first homology.

Next we explain concretely how to compute $\Phi(\omega)$, where (X,ω) is obtained from the construction in Section 3. Observe that (X,ω) belongs to a non hyperelliptic component if $g>2$. To see this, when $(X,\omega)\in\mathcal{H}^{hyp}(2g-2)$, note that the number of cylinders we have inserted is $g-1$. By Remark 11 they contribute to $2g-2$ fixed points of the hyperelliptic involution (located on the $2g-2$ horizontal core curves), say $p_1,p_1',\dots,p_{g-1},p_{g-1}'$. There are two more fixed points q,q' on the horizontal core curve of the long cylinder $\mathcal{C}$ we start with, and one fixed point on its boundary, say q'', that is on the same vertical closed curve as q'. The last fixed point is the singularity. On the other hand, each inserted cylinder should have two fixed points on its *vertical* core curve: one is p_i, the other one is $p_i''\in\mathcal{C}$. Thus necessarily $p_i''=q$ for all $i=1,\dots,g-1$. This is possible only if $g-1=1$. For $(X,\omega)\in\mathcal{H}^{hyp}(g-1,g-1)$ the situation is similar.

4.2. **Non hyperelliptic components, spin 1.** Consider (X,ω) obtained from the construction in §3 when all k_i are even. As a basis of the first homology $H_1(X,\mathbb{Z}/2)$, we take horizontal curves $\gamma_0,\dots,\gamma_{g-1}$ (γ_0 is the horizontal curve that we start with, and γ_i is in the ith vertical cylinder), and vertical curves $\eta_0,\dots,\eta_{g-1}$ (η_0 crosses γ_0 only once, and η_i is the core curve of the ith vertical cylinder for $i>0$). By construction, for every i,j

$$i(\gamma_0,\eta_j)=1,\ i(\gamma_i,\eta_j)=\delta_{ij} \text{ for } i>0, \text{ and } i(\gamma_i,\gamma_j)=i(\eta_i,\eta_j)=0.$$

We can thus form a symplectic basis as follows:

$$\begin{cases} a_1=\gamma_0, & b_1=\eta_0 \\ a_i=\gamma_{i-1}, & b_i=\eta_{i-1}-\eta_0 \text{ for } i\neq 0 \end{cases}$$

Clearly $\mathrm{ind}(\gamma_i) = \mathrm{ind}(\eta_i) = 0$. Substituting in Equation (1), we conclude:

$$\Phi(\omega) = 1 + \sum_{i=2}^{g} q(\gamma_{i-1})q(\eta_{i-1} - \eta_0) = \\ = 1 + \sum_{i=2}^{g} (q(\eta_{i-1}) + q(\eta_0) + i(\eta_0, \eta_{i-1})) \equiv 1 \mod 2.$$

4.3. **Non hyperelliptic components $\mathcal{H}(2k_1, \ldots, 2k_m)$, spin 0, $m > 1$.** We now use a slightly different model defined as follows. Start with the surface depicted in Figure 4, with a long horizontal cylinder made of $y^2 - g + 1$ squares. It belongs to $\mathcal{H}^{\mathrm{hyp}}(2, 2)$. Its spin structure is 0 as we can check directly, or by using the formulae in [KZ03, Corollary 5]. We can

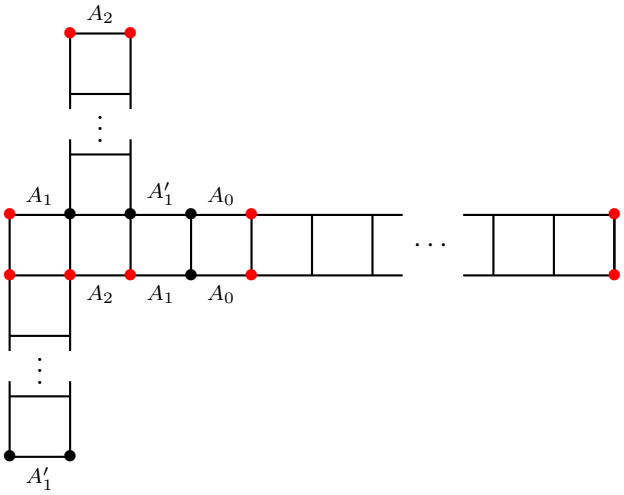

FIGURE 4. A surface in $\mathcal{H}^{\mathrm{hyp}}(2, 2)$ (with even spin structure).

insert $g - 3 \geq 1$ vertical strips of $y_i + 1$ square tiles (for $i = 3, \ldots, g - 1$) as in Section 3 in order to add zeros of even multiplicities and to reach the stratum $\mathcal{H}(2k_1, \ldots, 2k_m)$ where $\sum 2k_i = 2g - 2$. This construction does not change the spin structure as we can see on the computation below. We let γ_0 the horizontal core curve in the long cylinder, and $\gamma_1, \ldots, \gamma_{g-1}$ the other horizontal core curves contained in the ith cylinder. Similarly, we let η_i for $i = 0, \ldots, g - 1$ the vertical core curves: η_0 is the core curve of the vertical cylinder with label A_0 and η_i is the core curve of the ith vertical cylinder for $i > 0$. We have for every i, j

$$i(\gamma_0, \eta_i) = 1 \text{ for } i \neq 1 \text{ and } i(\gamma_0, \eta_1) = 2, \\ i(\gamma_i, \eta_i) = \delta_{ij} \text{ for } i > 0, \\ i(\gamma_i, \gamma_j) = i(\eta_i, \eta_j) = 0.$$

We can thus form a symplectic basis of $H_1(S;\mathbb{Z}_2)$ as follows:

$$\begin{cases} a_1 = \gamma_0, & b_1 = \eta_0 \\ a_2 = \gamma_1, & b_2 = \eta_1 \\ a_i = \gamma_{i-1}, & b_i = \eta_{i-1} - \eta_0 \text{ for } i > 2 \end{cases}$$

By using Equation (1) this leads to

$$\Phi(\omega) = 1 + 1 + \sum_{i=3}^{g} q(\gamma_{i-1})q(\eta_{i-1} - \eta_0) = \\ = 1 + 1 + \sum_{i=3}^{g} (q(\eta_{i-1}) + q(\eta_0) + i(\eta_0, \eta_{i-1})) \equiv 0 \mod 2.$$

We now compute the degree of the trace field. In order to write down the matrix $XX^\top$, we apply the strategy described in §3.2. Observe that the horizontal curve that we start with crosses $y^2 - g + 1 + g - 3 = y^2 - 2$ squares. More precisely it intersects $y^2 - 4$ vertical curves once and one vertical curve twice. We obtain the following matrix, where $\mathbf{b}_{n\times m}$ stands for the $n \times m$ matrix with all entries equal to $b \in \mathbb{Z}$:

$$XX^\top = \left(\begin{array}{c|cccc} y^2 & \mathbf{2}_{1\times y_1} & \mathbf{1}_{1\times y_2} & \cdots & \mathbf{1}_{1\times y_{g-1}} \\ \hline \mathbf{2}_{y_1\times 1} & \mathbf{1}_{y_1\times y_1} & & & \\ \mathbf{1}_{y_2\times 1} & & \mathbf{1}_{y_2\times y_2} & & \\ \vdots & & & \ddots & \\ \mathbf{1}_{y_{g-1}\times 1} & & & & \mathbf{1}_{y_{g-1}\times y_{g-1}} \end{array}\right).$$

From Lemma 6 with $l = 0$, we see that the characteristic polynomial of $XX^\top$ equals $t^a \cdot p(t, y, \mathbf{y})$, where

$$p(t, y, \mathbf{y}) = -y^2 \prod_{i=1}^{g-1}(t - y_i) + t\prod_{i=1}^{g-1}(t - y_i) - \sum_{i=1}^{g-1} c_i \prod_{j\neq i}(t - y_j),$$

for $a = \sum_{i=1}^{g-1}(y_i - 1)$, $c_1 = 4y_1$ and $c_i = y_i$ if $i \geq 2$. From Lemma 7, we deduce that $p(t, y, \mathbf{y})$ is irreducible in $\mathbb{Z}[t, y]$ given that all $y_i \in \mathbb{N}$ are pairwise distinct (here our parameter b in Definition 1 equals 2). As before, we can factor out $(t-1)^{g-d}$ and obtain an irreducible polynomial of degree d by setting $g-d+1$ of the $g-1$ parameters y_i equal to 1. We can then apply the same strategy than the proof of Theorem 8 to get the result.

Corollary 1. *Every number $1 \leq d \leq g$ is realised as the degree of the trace field of a product of two affine multitwists on a surface in every non hyperelliptic connected component with spin 0 of a stratum of Abelian differentials on Riemann surfaces of genus g.*

We further apply the same strategy to realise all even degrees as stretch factors. One can copy the proof of Theorem 9 word for word and obtain the following result.

Corollary 2. *Every even number $2 \leq 2d \leq 2g$ is realised as the degree of a product of two affine multitwists on a surface in every non hyperelliptic connected component with spin 0 of a stratum of Abelian differentials on Riemann surfaces of genus g.*

4.4. **Reaching non hyperelliptic component of $\mathcal{H}(2g-2)$, spin 0.**

4.4.1. *Degree $d = 2$.* We start with the model presented in Figure 4, and insert $g-3$ vertical cylinders ($g > 3$) with parameters $y_1 = 2$ and $y_i = 1$ for $i > 1$ (see also Figure 5). The number of squares in grey color is $y^2 - 2 - 3 - (g-3) = y^2 - g - 2$. The surface belongs to $\mathcal{H}(2, 2g-4)$. Since it can be continuously deformed to the surface in Figure 4 with spin 0, it also has spin 0. Now we collapse all the grey squares. The resulting surface belongs to the stratum $\mathcal{H}^{\text{non hyp}}(2g-2)$. Again this continuous deformation does not change the parity of the spin structure.

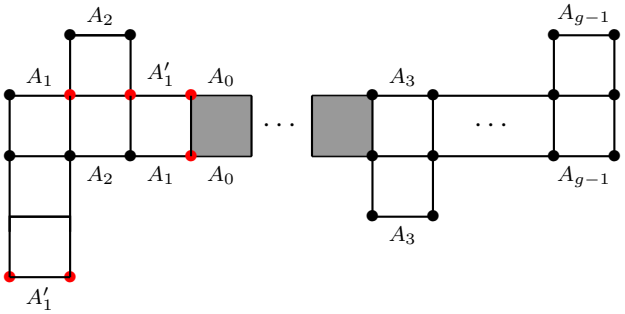

FIGURE 5. A surface in $\mathcal{H}^{\text{non hyp}}(2, 2g-4)$ for $g > 3$. If we collapse the handle (in grey color) we obtain a surface in $\mathcal{H}^{\text{non hyp}}(2g-2)$.

Following the computation in the previous subsection, we now obtain the $(g+1) \times (g+1)$ intersection matrix (recall $y^2 - g - 2 = 0$)

$$XX^\top = \left(\begin{array}{c|cccccc} g+2 & 2 & 2 & 1 & \cdots & 1 \\ \hline 2 & 1 & 1 & & & \\ 2 & 1 & 1 & & & \\ 1 & & & 1 & & \\ \vdots & & & & \ddots & \\ 1 & & & & & 1 \end{array}\right).$$

By Lemma 6, the characteristic polynomial of $XX^\top$ equals

$$t^a \left(-(g+2) \prod_{i=1}^{g-1} (t - y_i) + t \prod_{i=1}^{g-1} (t - y_i) - \sum_{i=1}^{g-1} c_i \prod_{j \neq i} (t - y_j) \right),$$

where $a = 1$, $c_1 = 4y_1$ and $c_i = y_i$ for $i \geq 2$. Thus the polynomial is

$$t(t-1)^{g-3}(-(g+2)(t-2)(t-1)+t(t-2)(t-1)-8(t-1)-(g-2)(t-2)) = \\ = t^2(t-1)^{g-3}(t^2 - t\cdot(g+5) + 2g + 2).$$

The discriminant of $t^2 - t\cdot(g+5) + 2g + 2$ is $D = (g+5)^2 - 8\cdot(g+1) = g^2 + 2g + 17$. We see that $(g+1)^2 < D < (g+5)^2$. If the degree of the trace field is one then D is a square, and one of the following three cases holds:

(1) $D = (g+2)^2$. Then $g^2 + 2g + 17 = g^2 + 4g + 4$. Solving in g we find $2g = 13$ which is a contradiction.
(2) $D = (g+3)^2$. Then $g^2 + 2g + 17 = g^2 + 6g + 9$. Solving in g we find $6g = 1$ which is a contradiction.
(3) $D = (g+4)^2$. Then $g^2 + 2g + 17 = g^2 + 4g + 4$. Solving in g we find $g = 2$ which is again a contradiction with $g > 3$.

4.4.2. *Degree* $2 < d \leq g$. We consider the modified version of our construction as depicted in Figure 6. When $g > 3$ the surface is not hyperelliptic.

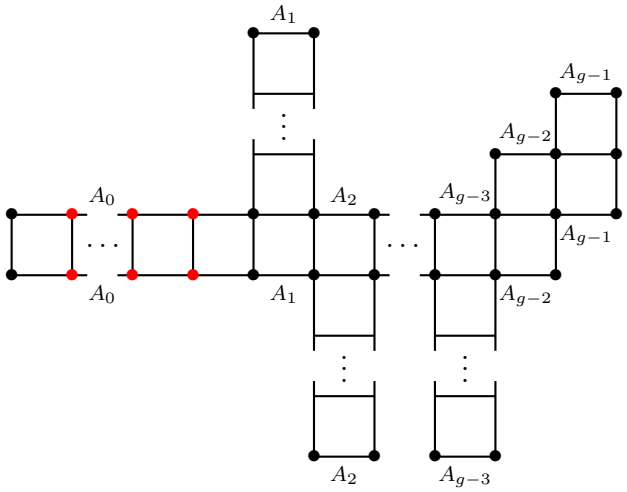

FIGURE 6. A surface in the even spin non hyperelliptic connected component of $\mathcal{H}(2g-2)$ for $g > 3$.

For the computation of the spin structure, we consider the "obvious" core curves γ_i and η_i (for $i = 0, \ldots, g-1$) of the horizontal and vertical cylinders. It forms a (non symplectic) basis of the homology:

$$i(\gamma_0, \eta_i) = 1 \text{ for } i \neq g-1 \text{ and } i(\gamma_0, \eta_{g-1}) = 0,$$
$$i(\gamma_i, \eta_i) = \delta_{ij} \text{ for } i = 0, \ldots, g-1$$
$$i(\gamma_i, \gamma_j) = i(\eta_i, \eta_j) = 0.$$

We can thus form a symplectic basis of $H_1(S;\mathbb{Z}_2)$ as follows:

$$\begin{cases} a_1 = \gamma_0, & b_1 = \eta_0 \\ a_i = \gamma_{i-1}, & b_i = \eta_{i-1} - \eta_0 \ \text{ for } i = 2, \dots, g-1 \\ a_g = \gamma_{g-1}, & b_g = \eta_{g-1} - b_{g-1} = \eta_{g-1} - \eta_{g-2} + \eta_0 \end{cases}$$

Equation (1) reads

$$\Phi(\omega) = q(\gamma_0)q(\eta_0) + \sum_{i=2}^{g-1} q(\gamma_{i-1})q(\eta_{i-1} - \eta_0) + q(\gamma_{g-1})q(\eta_{g-1} - \eta_{g-2} + \eta_0)$$

Since $q(\eta_{i-1} - \eta_0) = q(\eta_{i-1}) + q(\eta_0) + i(\eta_{i-1}, \eta_0) = 1 + 1 + 0 = 0 \mod 2$, the sum with the $g-2$ terms vanishes. For the last term, a direct computation leads to

$$\begin{aligned} q(\eta_{g-1} - \eta_{g-2} + \eta_0) = q(\eta_{g-1}) + q(\eta_{g-2}) + q(\eta_0) + \\ + i(\eta_{g-1}, \eta_{g-2}) + i(\eta_{g-1}, \eta_0) + i(\eta_{g-2} + \eta_0) = \\ = 1 + 1 + 1 + 0 + 0 + 0 = 1 \mod 2. \end{aligned}$$

Finally we get $\Phi(\omega) = 1 + 0 + 1 = 0 \mod 2$.

The intersection matrix (with the parameters $y_{g-2} = y_{g-1} = 1$) is

$$XX^\top = \left(\begin{array}{c|ccc|cc} y^2 & \mathbf{1}_{1\times y_1} & \cdots & \mathbf{1}_{1\times y_{g-3}} & 1 & 0 \\ \hline \mathbf{1}_{y_1\times 1} & \mathbf{1}_{y_1} & & & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ \mathbf{1}_{y_{g-3}\times 1} & & & \mathbf{1}_{y_{g-3}} & 0 & 0 \\ 1 & 0 & \cdots & 0 & 2 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{array}\right).$$

By developing along the last column, its characteristic polynomial equals

$$\begin{aligned} (t-1)\left((t-2)p(t,y,\mathbf{y}) - t^a \prod_{i=1}^{g-3}(t-y_i)\right) - p(t,y,\mathbf{y}) = \\ p(t,y,\mathbf{y})(t^2-3t+1) - t^a(t-1)\prod_{i=1}^{g-3}(t-y_i) = \\ = t^a\left(-y^2(t^2-3t+1)\prod_{i=1}^{g-3}(t-y_i) + (t^3-3t^2+1)\prod_{i=1}^{g-3}(t-y_i)\right. \\ \left. -(t^2-3t+1)\sum_{i=1}^{g-3} c_i \prod_{j\neq i}(t-y_j)\right), \quad (2) \end{aligned}$$

where $p(t,y,\mathbf{y})$ is the degree $g-2$ polynomial in Lemma 6, with the parameters $a = \sum_{i=1}^{g-3}(y_i - 1)$ and $c_i = y_i$ for all $1 \leq i \leq g-3$. Following the same line of proof we used for Lemma 7, we show

Lemma 12. *The polynomial*

$$-y^2(t^2-3t+1)\prod_{i=1}^{g-3}(t-y_i)+(t^3-3t^2+1)\prod_{i=1}^{g-3}(t-y_i)-(t^2-3t+1)\sum_{i=1}^{g-3}y_i\prod_{j\neq i}(t-y_j)$$

is irreducible in $\mathbb{Z}[t,y]$ given that all $y_i \in \mathbb{N}$ are distinct.

Proof. We note that the polynomial has degree two in the variable y with no non trivial common factor between the coefficient of y^2 and the constant coefficient. By Perron-Frobenius theorem, there is a simple irreducible factor of the constant coefficient. Thus Eisenstein's criterion applies in $\mathbb{Z}[t][y]$. □

4.5. **Reaching the hyperelliptic components of strata $\mathcal{H}(2g-2)$ and $\mathcal{H}(g-1,g-1)$.** We start by constructing a square-tiled surface. Pick a long horizontal square-tiled cylinder made of $2n+1$ squares with identifications A_0, $A_1,\dots,A_n$ and $A'_1,\dots,A'_n$ as depicted in Figure 7. We then add a stair case template, made of k steps, using a total of $2k$ squares. Finally we insert a long vertical square-tiled cylinder with some large number y of squares and identifications $C_1,\dots,C_y$ as in Figure 7. We treat y as a variable that we will need to specify later on. This creates a surface $X_{n,k,y}$. Similarly, one can construct a surface $Y_{n,k,y}$ be collapsing one square corresponding to the label A_0.

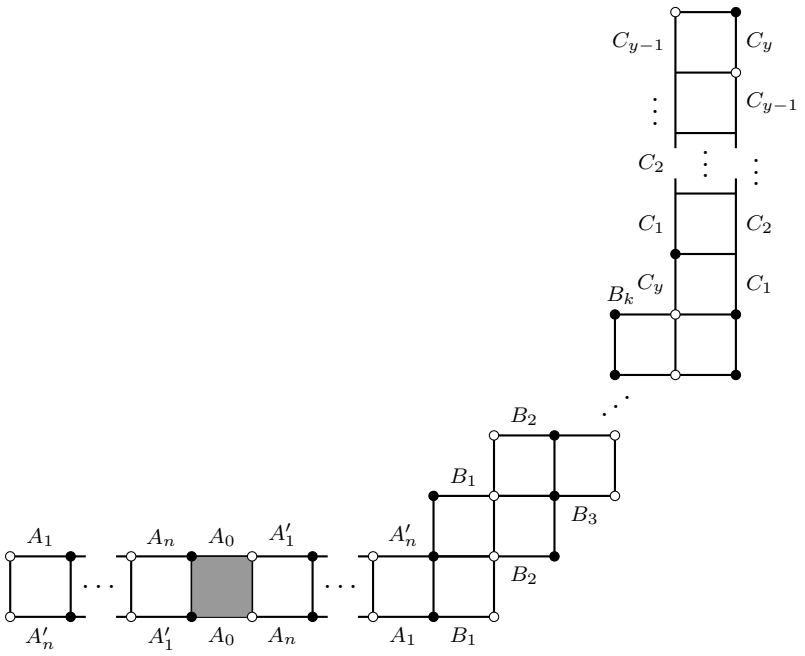

FIGURE 7. The surface $X_{n,k,y} \in \mathcal{H}(g-1,g-1)$ made of $2n+2k+y+2$ squares, and the surface $Y_{n,k,y} \in \mathcal{H}(2g-2)$ obtained from $X_{n,k,y}$ by collapsing the grey square.

Lemma 13. *The genus of $X_{n,k,y}$ and $Y_{n,k,y}$ is $g = n + k + 2$. Moreover $X_{n,k,y}$ belongs to the hyperelliptic connected component of $\mathcal{H}(g-1, g-1)$ while $Y_{n,k,y}$ belongs to the hyperelliptic connected component of $\mathcal{H}(2g-2)$.*

Proof of Lemma 13. Clearly the two square-tiled surfaces are hyperelliptic: the involution fixes the $k+2$ horizontal cylinders. By inspecting the gluing, one sees that $X_{n,k,y}$ has two zeros, each of order $g-1$. The cone angle at each zero is $g \cdot 2\pi$. Since the total number of squares contributing to the cone angle is $2n + 2k + 2 + 2$, we get $(2n + 2k + 4) \cdot 2\pi = g \cdot 2\pi + g \cdot 2\pi$. Hence, $g = n + k + 2$.
Similarly, $Y_{n,k,y}$ has one zero, of order $2g-2$ and cone angle $(2g-1) \cdot 2\pi$. Now the total number of squares contributing to the cone angle is one less: $2n + 2k + 2 + 1$. Thus, $(2n + 2k + 3) \cdot 2\pi = (2g-1) \cdot 2\pi$. □

A quick inspection of the intersections of horizontal curves with vertical curves yields that $XX^\top$ is the following $(k+2) \times (k+2)$ Jacobi matrix:

$$XX^\top = \left(\begin{array}{c|ccccc} \alpha & 1 & & & & \\ \hline 1 & 2 & 1 & & & \\ & 1 & \ddots & \ddots & & \\ & & \ddots & \ddots & 1 & \\ & & & 1 & 2 & y \\ & & & & y & y^2 \end{array}\right),$$

where $\alpha = 2 + 4n$ if one considers $X_{n,k,y}$, and $\alpha = 1 + 4n$ otherwise.

Lemma 14. *If $\alpha \neq 1$, then the characteristic polynomial of $XX^\top$, when regarded as a polynomial in the variables y and t, is irreducible in $\mathbb{Z}[t, y]$.*

Proof of Lemma 14. Let p_k be the characteristic polynomial of $XX^\top$. We will use the characteristic polynomial $q_k(t)$ of the $(k+1) \times (k+1)$ matrix B_k obtained from $2I_{k+1} + \mathrm{Ad}(A_{k+1})$ by adding $\alpha - 2$ to the first diagonal entry, where $\mathrm{Adj}(A_k)$ is the adjacency matrix of the path graph with k vertices. We obtain directly by developing the determinant of $t\mathrm{Id}_{k+2} - XX^\top$ along the last column that $p_k(t, y) = -y^2(q_k(t) + q_{k-1}(t)) + tq_k(t)$. We now claim that the roots of q_k and q_{k-1} are pairwise distinct and simple.

Proof of the claim. We note that B_{k-1} is obtained from B_k by deleting the last row and the last column. Interlacing results for real symmetric matrices tell us that the eigenvalues of B_k and B_{k-1} interlace. This means that if $\lambda_1 \leq \cdots \leq \lambda_{k+1}$ are the eigenvalues of B_k and if $\mu_1 \leq \cdots \leq \mu_k$ are the eigenvalues B_{k-1}, then we have

$$\lambda_i \leq \mu_i \leq \lambda_{i+1}$$

for all $1 \leq i \leq k$. The crucial point is that the inequalities need to be strict. Indeed, Theorem 7 by Gantmacher and Krein [GK37] states that a tridiagonal matrix with positive coefficients on the main diagonal and the adjacent diagonals is oscillatory if and only if all the leading principal

minors are positive. This implies that B_k is oscillatory, since the matrix B_k is positive definite (for $\alpha \geq 1$). Now, Theorem 6.5 by Ando [And87] states that for oscillatory matrices, all the interlacing inequalities are strict. That is, we have

$$\lambda_i < \mu_i < \lambda_{i+1}$$

for all $1 \leq i \leq k$. In particular, the eigenvalues of B_k and the eigenvalues of B_{k-1} are pairwise distinct and simple. □

We now finish the proof the lemma. Let $F \neq t$ be an irreducible factor of q_k. Since the roots of q_k are simple, F^2 is not a factor of tq_k. If F is a factor of $q_k + q_{k-1}$ then q_k and q_{k-1} share a common root, which is not possible by the claim. Hence, by Eisenstein's criterion, $p_k(t, y)$ is irreducible when regarded as a polynomial in the variable y and so can not be factored in the form $(ay+b)(cy+d)$. So, if there is a factorisation of $p_k(t, y)$, then one of the factors must have degree zero in the variable y. But such a factorisation cannot exist, since $q_k(t) + q_{k-1}(t)$ and $tq_k(t)$ are relatively prime in $\mathbb{Z}[t]$. Indeed, since the roots of $q_k(t)$ and $q_{k-1}(t)$ are distinct, the only possible common factor of tq_k and $q_k + q_{k-1}$ is t. But $p_k(0, y)$ is the determinant of $XX^\top$ and equals $y^2 \cdot (\alpha - 1) \neq 0$. This proves the lemma. □

Theorem 15. *For any hyperelliptic connected component $\mathcal{C}$ of $\mathcal{H}(2g-2)$, every number $1 \leq d \leq g-1$ is realised as the degree of the trace field of a product of two affine multitwists on a surface in $\mathcal{C}$.*

Proof of Theorem 15. Since the case $d = 1$ is clear by considering square-tiled surfaces, let us assume $d \geq 2$ and set $k = d - 2 \geq 0$. We construct a surface $X_{n,k,y}$ or $Y_{n,k,y}$ where $n = g - d = g - k - 2 \geq 0$ (see Lemma 13). If $d < g$ then $n \neq 0$ and $\alpha \neq 1$. If $d = g$, that is, $n = 0$, then by assumption we consider only $Y_{n,k,y} \in \mathcal{H}(g-1, g-1)$ so that $\alpha = 4n + 2 = 2 \neq 1$. Thus Lemma 14 applies and the characteristic polynomial of $XX^\top$, viewed as a polynomial in $\mathbb{Z}[y, t]$ is irreducible. Then by Hilbert's irreducibility theorem, there are infinitely many specifications of y so that the resulting polynomial is irreducible as a polynomial in the variable t. Note that all specifications can be realised geometrically. Indeed, one can choose $y > 0$ by symmetry. In particular, applying Thurston–Veech's construction, there exists a product of two multitwists on the surface of genus $g = n+d$ in the desired connected component. □

We also prove the analogous theorem for degrees of stretch factors.

Theorem 16. *For any hyperelliptic connected component $\mathcal{C}$ of $\mathcal{H}(g-1, g-1)$, every even number $2 \leq 2d \leq 2g$ is realised as the degree of the stretch factor of a product of two affine multitwists on a surface in $\mathcal{C}$.*

For any hyperelliptic connected component $\mathcal{C}$ of $\mathcal{H}(2g-2)$, every even number $2 \leq 2d \leq 2g-2$ is realised as the degree of the stretch factor of a product of two affine multitwists on a surface in $\mathcal{C}$.

Proof. We use the same examples as in the proof of Theorem 15. We first deal with the case $d = 2$ by taking the specific example $y = 1$. In this case, we have $k = 0$ and we get $XX^\top = \begin{pmatrix} \alpha & 1 \\ 1 & 1 \end{pmatrix}$. We obtain

$$\mu^2 = \frac{\alpha + 1 + \sqrt{\alpha^2 - 2\alpha + 5}}{2},$$

which is an algebraic number of degree two over $\mathbb{Q}$. Indeed, we have

$$\alpha^2 > \alpha^2 - 2\alpha + 5 > (\alpha - 1)^2$$

in case $\alpha \neq 1, 2$, so this number is not a square and μ^2 is not rational. Neither is it in case $\alpha = 2$, by direct calculation, and the case $\alpha = 1$ is not needed.

We are now ready to apply Theorem 4. Let Ω' be the matrix obtained from Ω by deleting the row and the column corresponding to the cylinder with $2n + 1$ or $2n + 2$ squares. We have

$$\sigma(\Omega + 2I) \geq \sigma(\Omega' + 2I) - 1.$$

By construction, Ω' is the adjacency matrix of a forest consisting of path graphs, so that $2I + \Omega'$ is positive definite. We get

$$\sigma(\Omega + 2I) \geq \dim(\Omega) - 2 > \dim(\Omega) - 4.$$

The criterion applies and the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d = 4$.

For the case $d \geq 3$, we take the examples as in the proof of Theorem 15, without specialising y. Let Ω' be the matrix obtained from Ω by deleting the rows and the columns corresponding to the horizontal cylinder with $2n + 1$ or $2n + 2$ squares, and to the vertical cylinder with $y + 1$ squares. We have

$$\sigma(\Omega + 2I) \geq \sigma(\Omega' + 2I) - 2.$$

By construction, Ω' is the adjacency matrix of a forest consisting of path graphs, so that $2I + \Omega'$ is positive definite. We get

$$\sigma(\Omega + 2I) \geq \dim(\Omega) - 4 > \dim(\Omega) - 2d.$$

The criterion applies and the mapping class $T_\alpha \circ T_\beta$ is pseudo-Anosov with stretch factor λ of degree $2d$. □

4.6. **Reaching hyperelliptic component of $\mathcal{H}(2g - 2)$ with degree g.** Take the stair case model with a "long" stair made of y^2 squares (see Figure 8). The $g \times g$ matrix is

$$XX^\top = \left(\begin{array}{c|cccccc} y^2 & 1 & & & & & \\ \hline 1 & 2 & 1 & & & & \\ & 1 & \ddots & \ddots & & & \\ & & \ddots & \ddots & 1 & & \\ & & & 1 & 2 & 1 & \\ & & & & 1 & 1 & \end{array}\right).$$

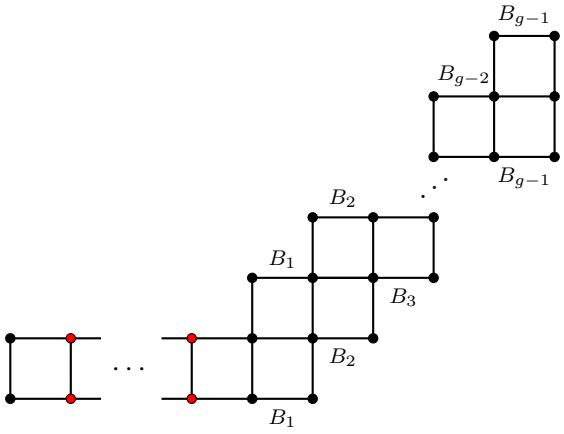

FIGURE 8. A stair case template in the hyperelliptic component of $\mathcal{H}(2g-2)$.

Let $p_g(t,y)$ be the characteristic polynomial of $XX^\top$. We will use the characteristic polynomial $q_g(t)$ of the $g \times g$ matrix B_g obtained from $2I_g + \mathrm{Ad}(A_g)$ by adding -1 to the last diagonal entry, where $\mathrm{Adj}(A_g)$ is the adjacency matrix of the path graph with g vertices.

By developing the determinant of $t\mathrm{Id}_g - XX^\top$ along the first column we get

$$p_g(t,y) = -y^2 q_{g-1}(t) + tq_{g-1}(t) - q_{g-2}(t).$$

We now claim that the polynomials $q_{g-1}(t)$ and $tq_{g-1}(t) - q_{g-2}(t)$ are relatively prime. Using the same argument as in Lemma 14, we get that the matrix B_g is oscillatory, and hence the roots of p_{g-1} and p_{g-2} are all simple and pairwise distinct. Since the minimal polynomial of the Perron-Frobenius eigenvalue of B_g is a simple irreducible factor F of $tq_{g-1}(t) - q_{g-2}(t)$ that is not also a factor of $q_{g-1}(t)$, Eisenstein's criterion applies and $p_g(t,y)$ is irreducible. Thus there are infinitely many specifications of $y > 0$ such that $p_g(t,y) \in \mathbb{Z}[t]$ is irreducible. This yields the degree $d = g$ for the hyperelliptic component of $\mathcal{H}(2g-2)$ for any $g > 1$.

Using this model, it is now straightforward to adapt the proofs of Theorem 15 and Theorem 16 to construct examples where the trace field is of degree g and the stretch factor is of degree $2g$.

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