CONNECTED COMPONENTS OF PRYM EIGENFORM LOCI IN GENUS THREE

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ABSTRACT. This paper is devoted to the classification of connected components of Prym eigenform loci in the strata $\mathcal{H}(2,2)^{\rm odd}$ and $\mathcal{H}(1,1,2)$ of the Abelian differential bundle $\Omega\mathcal{M}_3$ over $\mathcal{M}_3$. These loci, discovered by McMullen [Mc06], are $\mathrm{GL}^+(2,\mathbb{R})$-invariant submanifolds of complex dimension 3 of $\Omega\mathcal{M}_g$ that project to the locus of Riemann surfaces whose Jacobian variety has a factor admitting real multiplication by some quadratic order $\mathcal{O}_D$.

It turns out that these subvarieties can be classified by the discriminant D of the corresponding quadratic orders. However there algebraic varieties are not necessarily irreducible. The main result we show is that for each discriminant D the corresponding locus has one component if $D \equiv 0, 4 \bmod 8$, two components if $D \equiv 1 \bmod 8$, and is empty if $D \equiv 5 \bmod 8$.

Surprisingly our result contrasts with the case of Prym eigenform loci in the strata $\mathcal{H}(1,1)$ (studied by McMullen [Mc07]) which is connected for every discriminant D.

1. INTRODUCTION

Since the work of McMullen [Mc03] it has been known that the properties of $\mathrm{SL}_2(\mathbb{R})$-orbit closure of translation surfaces are strongly related to the endomorphisms rings of the Jacobian of the underlying Riemann surfaces (see also [Möl06]). The algebro-geometric approach emphasized by McMullen is to detect affine homeomorphisms of the flat metric on the level of the first homology group as affine homeomorphisms induce self-adjoint endomorphisms of the Jacobian variety.

Recall that an Abelian variety $\mathbb{A} \in \mathcal{A}_g$ admits real multiplication by a totally real number field K of degree g over $\mathbb{Q}$ if there exists an inclusion $K \hookrightarrow \mathrm{End}(\mathbb{A}) \otimes \mathbb{Q}$ such that for any $k \in K$, the action of k is self-adjoint with respect to the polarization of $\mathbb{A}$. Equivalently, $\mathrm{End}(\mathbb{A})$ contains a copy of an order $\mathcal{O} \subset K$ acting by self-adjoint endomorphism.

1.1. Brief facts summary in the genus 2 case.

The locus

$$\mathcal{E}_2 = \{(X,\omega) \in \Omega\mathcal{M}_2 : \mathrm{Jac}(X) \text{ admits real multiplication with } \omega \text{ as an eigenform}\},$$

plays an important role in the classification of $\mathrm{SL}(2,\mathbb{R})$-orbit closures in $\Omega\mathcal{M}_2$. Here $\mathbb{A} = \mathrm{Jac}(X) \in \mathcal{A}_2$, K is a real quadratic field, and the endomorphism ring is canonically isomorphic to the ring of homomorphisms of $H_1(X,\mathbb{Z})$ that preserve the Hodge decomposition. The polarization comes from the intersection form $\left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ on the homology.

The locus $\mathcal{E}_2$ is actually a (disjoint) union of subvarieties indexed by the discriminants of the orders $\mathcal{O} \subset \mathrm{End}(\mathrm{Jac}(X))$. Since orders in quadratic fields (quadratic orders) are classified by their discriminant, the *unique* quadratic order with discriminant D is denoted by $\mathcal{O}_D$. We then define

$$\Omega E_D = \{(X,\omega) \in \mathcal{E}_2 : \omega \text{ is as an eigenform for real multiplication by } \mathcal{O}_D\}.$$

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The subvarieties ΩE_D are of interest since they are $\mathrm{GL}^+(2,\mathbb{R})$-invariant submanifolds of $\Omega\mathcal{M}_2$ (see [Mc07, Mc06]). We can further stratified ΩE_D by defining $\Omega E_D(\kappa) = \Omega E_D \cap \mathcal{H}(\kappa)$ for $\kappa = (2)$ or $\kappa = (1,1)$. This defines complex submanifolds of dimension 2 and 3, respectively. Hence $\Omega E_D(2)$ projects to a union of algebraic curves (Teichmüller curves) in the moduli space $\mathcal{M}_2$.

1.2. **Components of $\Omega E_D(1,1)$ and $\Omega E_D(2)$.** It is well known that the set of Abelian varieties $\mathbb{A} \in \mathcal{A}_2$ admitting real multiplication by $\mathcal{O}_D$ with a specified faithful representation $\mathfrak{i} : \mathcal{O}_D \to \mathrm{End}(\mathbb{A})$ is parametrized by the Hilbert modular surface $X_D := (\mathbb{H} \times -\mathbb{H})/\mathrm{SL}(\mathcal{O}_D)$. In [Mc07], it has been shown that each ΩE_D can be viewed as a $\mathbb{C}^*$-bundle over a Zariski open subset of X_D, and we have

$$\mathcal{E}_2 = \bigcup_{D\geq 4, D\equiv 0,1 \bmod 4} \Omega E_D$$

In particular ΩE_D is a connected, complex suborbifold of $\Omega\mathcal{M}_2$ of dimension 3. The fact that there is only one (connected) eigenform locus for each D follows from the fact that there is only one faithful, proper, self-adjoint representation $\mathfrak{i} : \mathcal{O}_D \to M_4(\mathbb{Z})$ up to conjugation by $\mathrm{Sp}(4,\mathbb{Z})$ (see [Mc07] Theorem 4.4). It follows that $\Omega E_D(1,1)$ is a Zariski open set in ΩE_D. In particular $\Omega E_D(1,1)$ is connected for any quadratic discriminant D.

The classification of components of $\Omega E_D(2)$ has also been obtained by McMullen [Mc05].

1.3. **Higher genera.** In [Mc06] it is shown that analogues of ΩE_D exist in higher genus (up to 5). These subvarieties of $\Omega\mathcal{M}_g$ are called *Prym eigenform loci*. Surfaces in a Prym eigenform locus are pairs (X,ω) such that there exists a holomorphic involution $\tau : X \to X$ such that $g(X) - g(Y) = 2$, where $Y = X/\langle\tau\rangle$, $\tau^*\omega = -\omega$, and the *Prym variety* $\mathrm{Prym}(X,\tau)$ admits a real multiplication with ω an eigenform (see Section 2 for precise definitions). Note that the condition $g(X)-g(Y) = 2$ is needed for our discussion. For any genus, the set of Prym eigenforms whose Prym variety admits a multiplication by $\mathcal{O}_D$ will be denoted by ΩE_D, and the intersection of ΩE_D with a stratum $\mathcal{H}(\kappa)$ is denoted by $\Omega E_D(\kappa)$.

The goal of this paper is to investigate the topology of the Prym eigenform loci in the strata $\mathcal{H}(2,2)$ and $\mathcal{H}(1,1,2)$ of $\Omega\mathcal{M}_3$. It is well known that the stratum $\mathcal{H}(2,2)$ also has two components $\mathcal{H}(2,2)^{\mathrm{odd}}$ and $\mathcal{H}(2,2)^{\mathrm{hyp}}$. It is not difficult to see that Prym eigenforms in $\mathcal{H}(2,2)^{\mathrm{hyp}}$ are double covers of surfaces in $\Omega E_D(2)$ (see Proposition 2.3). Thus $\dim \Omega E_D(2,2)^{\mathrm{hyp}} = 2$, and $\Omega E_D(2,2)^{\mathrm{hyp}}$ is a (finite) union of $\mathrm{GL}^+(2,\mathbb{R})$ closed orbits. On the other hand, we have $\dim \Omega E_D(2,2)^{\mathrm{odd}} = 3$. The stratum $\mathcal{H}(1,1,2)$ is connected and we also have $\dim \Omega E_D(1,1,2) = 3$ (see Proposition 2.6).

Our main result reveals that the situation in genus three is quite different from the one in genus. More precisely:

Theorem A. *Let $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$. For any discriminant $D \geq 8$, with $D \equiv 0,1 \bmod 4$, the loci $\Omega E_D(\kappa)$ are non empty if and only if $D \equiv 0,1,4 \bmod 8$, and in this case they are pairwise disjoint. Moreover the following dichotomy holds:*

1. *If D is even then $\Omega E_D(\kappa)$ is connected,*
2. *If D is odd then $\Omega E_D(\kappa)$ has exactly two connected components.*

Remark 1.1. *One of the main differences between the cases of genus two and genus three is that the polarization of the Prym variety in genus three has the form $\left(\begin{smallmatrix} J & 0 \\ 0 & 2J \end{smallmatrix}\right)$, which is the reason why $\Omega E_D(\kappa)$ is empty if $D \equiv 5 \bmod 8$.*

In genus two we have Prym$(X,\tau) = \mathrm{Jac}(X)$ and the Prym involution τ must be the hyperelliptic involution which is unique, in genus three Prym(X,τ) is only a factor of $\mathrm{Jac}(X)$, and there may be more than one Prym involution as we will see in Section 3. Thus it is not obvious that one can use the discriminant to distinguish different Prym eigenform loci.

It is also worth noticing that while $\Omega E_9(4)$ and $\Omega E_{16}(4)$ are empty (see [LN11]*), the loci $\Omega E_9(\kappa)$ and $\Omega E_{16}(\kappa)$ do exist for $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$.*

1.4. **Triple tori.** An important tool in our proof is the use of *triple tori*:

(1) We say that $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ admits a *three tori decomposition* if there exists a triple of homologous saddle connections $\{\sigma_0,\sigma_1,\sigma_2\}$ on X joining the two distinct zeros of ω.
(2) We say that $(X,\omega) \in \Omega E_D(1,1,2)$ admits a *three tori decomposition* if there exist two pairs of homologous saddle connections $\{\sigma_0,\sigma_1\}$ and $\{\sigma'_0,\sigma'_1\}$ on X joining the double zero to the simple zeros such that $\{\sigma'_0,\sigma'_1\} = \tau(\{\sigma_0,\sigma_1\})$.

If (X,ω) admits a three tori decomposition then it can be viewed as a connected sum of three slit tori (X_j,ω_j), $j = 0,1,2$, (see Figure 1). We will always assume that X_0 is preserved by the Prym involution τ and X_1, X_2 are exchanged by τ.

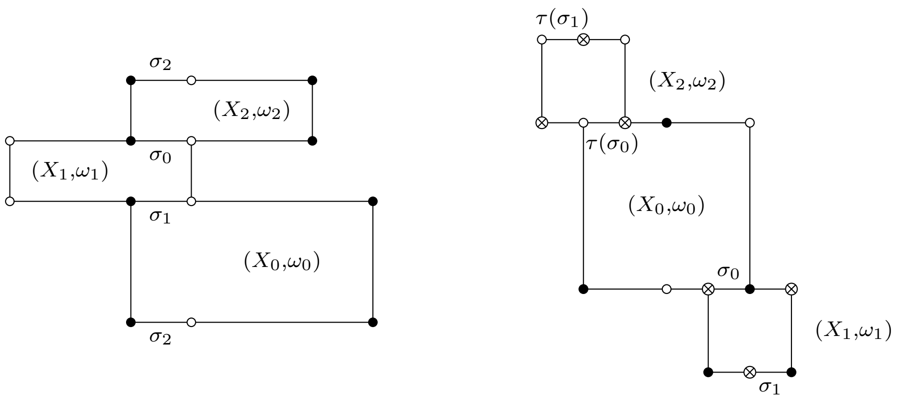

FIGURE 1. Decomposition of $(X,\omega) \in$ Prym$(2,2)^{\mathrm{odd}}$ (left) and $(X,\omega) \in$ Prym$(1,1,2)$ (right) into three tori.

As a corollary of our main result, we prove the following theorem, which is used in the paper [LN13b]:

Theorem B. *For any discriminant D such that $\Omega E_D(\kappa) \neq \varnothing$, there exist in any component of $\Omega E_D(\kappa)$ surfaces which admit three-tori decompositions.*

Theorem B is proved in Section 7.

1.5. **Strategy of the proof.** The important ingredients of the proof of the main theorem is the use of surgeries (see Section 5). The core of Theorem A are Theorem 6.1 on admissible saddle connection and Theorem 4.1 on non-connectedness. The proofs of Theorem 6.1 and Theorem 4.1 appear in Section 6 and 4 respectively.

(1) An elementary way to get Prym eigenforms in $\mathcal{H}(2,2)^{\rm odd}$ and $\mathcal{H}(1,1,2)$ is given by Lemma 2.4. Another way is the use of the surgery "Breaking up a zero" on a Prym eigenforms in $\mathcal{H}(4)$ (see [KZ03]). We deduce that $\Omega E_D(2,2)^{\rm odd}$ and $\Omega E_D(1,1,2)$ are non-empty whenever $\Omega E_D(4)$ is non-empty.
(2) In Section 3 we prove that the loci $\Omega E_D(\kappa)$ are pairwise disjoint (Lemma 3.3 and Theorem 3.1). As we have noticed in Remark 1.1 a surface $(X,\omega) \in \mathcal{H}(2,2)^{\rm odd}$ may have more than one Prym involution. However we then show that *all* Prym varieties admit real multiplication by $\mathcal{O}_9$. This proves in particular that $\Omega E_9(2,2)^{\rm odd}$ is non-empty despite the fact that $\Omega E_9(4) = \varnothing$.
(3) To get an upper bound of the number of components of $\Omega E_D(\kappa)$ our strategy is to find in each component $\mathcal{C}$ of $\Omega E_D(\kappa)$ a surface (X,ω) such that we can collapse the zeros of ω along some saddle connections to get a surface in $\Omega E_D(4)$. Such saddle connections are called *admissible* (see Section 5). In this situation, the component $\mathcal{C}$ is adjacent to the locus $\Omega E_D(4)$ *i.e.* $\Omega E_D(4) \subset \overline{\mathcal{C}}$. We then prove that the number of components of $\Omega E_D(\kappa)$ that are adjacent to $\Omega E_D(4)$ can not exceed the number of components of $\Omega E_D(4)$. Surprisingly, it turns out that there exist components that are *not* adjacent to $\Omega E_D(4)$. These are precisely the components of the loci $\Omega E_9(\kappa)$ and $\Omega E_{16}(\kappa)$. This fact is proved in Theorem 6.1. This result plus the fact that $\Omega E_D(4)$ has at most two connected components (by [LN11]) furnish the desired upper bound.
(4) Finally, to get the exact count of the number of components we will show in Section 4 that if $\Omega E_D(4)$ is not connected then $\Omega E_D(\kappa)$ is not connected either (in contrast with the situation in genus two). This difference comes from the invariant defined in [LN11].

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2. Background

We review the necessary tools and results involved in the proof of our main result. For an introduction to translation surfaces in general, and a nice survey on this topic, see *e.g.* [Zor06, MaTa02].

2.1. Prym eigenform.

Let X be a Riemann surface and τ an involution of X. We define the Prym variety of (X,τ) to be

$$\mathrm{Prym}(X,\tau) = (\Omega(X,\tau)^-)^*/H_1(X,\mathbb{Z})^-$$

where $\Omega(X,\tau)^- = \ker(\tau + \mathrm{id}) \subset \Omega(X)$, $\Omega(X)$ is the space of holomorphic one forms on X), and $H_1(X,\mathbb{Z})^-$ is the anti-invariant homology of X with respect to τ. Remark that $\mathrm{Prym}(X,\tau)$ has naturally a polarization: the lattice $H_1(X,\mathbb{Z})^-$ is equipped with the restriction of the intersection form on $H_1(X,\mathbb{Z})$.

Following [Mc06] we will call a translation surface (X,ω) a *Prym form* if there exists an involution τ of X such that $\dim_{\mathbb{C}} \Omega(X,\tau)^- = 2$, and $\omega \in \Omega(X,\tau)^-$. Note that the condition $\dim_{\mathbb{C}} \Omega^-(X,\tau) = 2$ is equivalent to $g(X) - g(Y) = 2$, where $Y := X/\langle\tau\rangle$. In this situation, we will call τ a *Prym involution* of X. Note that a Riemann surface may have more than one Prym involution (see Theorem 3.1).

Recall that a (real) quadratic order is a ring isomorphic to $\mathbb{Z}[x]/(x^2 + bx + c)$, the *discriminant* of the order is defined by $D = b^2 - 4c$. Orders with the same discriminants are isomorphic. Thus for any

$D \in \mathbb{N}$, $D \equiv 0, 1 \bmod 4$, we will write $\mathcal{O}_D$ to designate the *unique* quadratic order of discriminant D. When D is not a square, $\mathcal{O}_D$ is a finite index subring of the integer ring in the quadratic field $K := \mathbb{Q}(\sqrt{D})$.

Let A be an Abelian variety of (complex) dimension 2. We say that A admits a real multiplication by $\mathcal{O}_D$ if there exists an injective ring morphism $\mathfrak{i} : \mathcal{O}_D \to \mathrm{End}(A)$ such that $\mathfrak{i}(\mathcal{O}_D)$ is a self-adjoint proper subring of $\mathrm{End}(A)$ (properness means that if $f \in \mathrm{End}(A)$ and there exists $n \in \mathbb{N}^*$ such that $nf \in \mathfrak{i}(\mathcal{O}_D)$, then $f \in \mathfrak{i}(\mathcal{O}_D)$).

Definition 2.1. *We will call a translation surface (X, ω) a* Prym eigenform*, if there exists a Prym involution τ of X such that*

- *Prym(X, τ) admits a real multiplication by some quadratic order $\mathcal{O}_D$,*
- *$\omega \in \Omega(X, \tau)^-$ is an eigenvector of $\mathcal{O}_D$.*

The set of Prym eigenforms admitting real multiplication by $\mathcal{O}_D$ will be denoted by ΩE_D. In [Mc05] it is showed that ΩE_D are closed, $\mathrm{GL}^+(2, \mathbb{R})$-invariant submanifolds of the bundle $\Omega\mathcal{M}_g$. Up to now, these are the only known $\mathrm{GL}^+(2, \mathbb{R})$-invariant submanifolds of $\Omega\mathcal{M}_g$ which are neither closed orbits nor covers of Abelian differentials or quadratic differentials in lower genus. The intersection of ΩE_D with a stratum $\mathcal{H}(\kappa)$ will be denoted by $\Omega E_D(\kappa)$. Clearly, $\Omega E_D(\kappa)$ are $\mathrm{GL}^+(2, \mathbb{R})$-invariant submanifolds of $\mathcal{H}(\kappa)$.

Any translation surface in genus two is a Prym form (the Prym involution being the hyperelliptic involution). It turns out that the locus $\mathcal{E}_2$ of Prym eigenforms in genus two is a disjoint union of ΩE_D for $D \equiv 0, 1 \bmod 4$ and $D \geq 5$. It is a fact that $\Omega E_D(2)$ is connected if $D \equiv 0, 4, 5 \bmod 8$, and has two components otherwise ($D \equiv 1 \bmod 8$). On the other hand $\Omega E_D(1, 1)$ is connected for all D (see [Mc07, Mc05]).

McMullen [Mc06] proved the existence of Prym eigenforms in genus 3 and 4, and in particular that $\Omega E_D(4)$ and $\Omega E_D(6)$ are non-empty for infinitely many D. It is well known that the minimal stratum $\mathcal{H}(4)$ of $\Omega\mathcal{M}_3$ has two components $\mathcal{H}^{\mathrm{hyp}}(4)$ and $\mathcal{H}^{\mathrm{odd}}(4)$ (see [KZ03] for precise definitions) and the loci $\Omega E_D(4)$ are contained in $\mathcal{H}^{\mathrm{odd}}(4)$. In [LN11] the authors gave a complete classification of $\Omega E_D(4)$, namely:

Theorem 2.2 (Lanneau-Nguyen [LN11])**.** *For $D \geq 17$, $\Omega E_D(4)$ is non empty if and only if $D \equiv 0, 1, 4$ mod 8. All the loci $\Omega E_D(4)$ are pairwise disjoint. Moreover, for the values $0, 1, 4$ of discriminants, the following dichotomy holds. Either*

1. *D is odd and then $\Omega E_D(4)$ has exactly two connected components, or*
2. *D is even and $\Omega E_D(4)$ is connected.*

In addition, each connected component of $\Omega E_D(4)$ corresponds to a closed $\mathrm{GL}^+(2, \mathbb{R})$-orbit.

For $D < 17$, only $\Omega E_{12}(4)$ and $\Omega E_8(4)$ are non-empty, each of which consists of a unique closed $\mathrm{GL}^+(2, \mathbb{R})$-orbit.

The first striking fact about Prym eigenforms in $\mathcal{H}(4)$ is that $\Omega E_D(4) = \varnothing$ if $D \equiv 5 \bmod 8$, this is actually due to the signature of the polarization of the Prym variety in genus three which is different from the one in genus two. The second remarkable fact is that $\Omega E_9(4) = \Omega E_{16}(4) = \varnothing$ even though 9 and 16 are not "forbidden values" of D. It is worth noticing that even though we have the same statement in the case $D \equiv 1 \bmod 8$ as McMullen's result in $\Omega E_D(2)$ (namely, $\Omega E_D(4)$ has two components), the reason for this disconnectedness is different. Roughly speaking, the two components of $\Omega E_D(4)$ correspond to two distinct complex lines in the space $\Omega(X, \tau)^- \simeq H^1(X, \mathbb{R})^-$, whereas in the case $\Omega E_D(2)$, the two components correspond to the same complex line (this is actually a consequence of the fact that $\Omega E_D(1, 1)$ is connected), they can only be distinguished by the spin invariant (see [Mc05, Section 5]).

2.2. **Prym eigenforms in $\mathcal{H}(2,2)$ and $\mathcal{H}(1,1,2)$.** The stratum $\mathcal{H}(1,1,2)$ is connected while the stratum $\mathcal{H}(2,2)$ has two connected components: $\mathcal{H}(2,2)^{\rm hyp}$ and $\mathcal{H}(2,2)^{\rm odd}$ (see [KZ03]). We will not use this classification in the sequel.

Proposition 2.3. *If $(X,\omega) \in \Omega E_D(2,2)^{\rm hyp}$ then there exists a Prym eigenform $(X',\omega') \in \Omega E_{D'}(2)$ and an unramified double cover $\rho : X \to X'$ such that $\rho^*\omega' = \omega$. In particular $\Omega E_D(2,2)^{\rm hyp}$ is a finite union of $\mathrm{GL}^+(2,\mathbb{R})$ closed orbits, each of which is a copy of a $\mathrm{GL}^+(2,\mathbb{R})$-orbit in $\Omega E_{D'}(2)$.*

Proof of Proposition 2.3. By definition X is a hyperelliptic Riemann surface, and the hyperelliptic involution ι exchanges the zeros of ω. Since ι commutes with all automorphisms of X, we have $\tau' := \tau\circ\iota$ is also an involution of X (where τ is the Prym involution of X). Set $X' := X/\langle\tau'\rangle$. Note that $\ker(\tau'-\mathrm{id}) = \ker(\tau+\mathrm{id})$, thus we have $\dim\ker(\tau'-\mathrm{id}) = 2$ and X' is a Riemann surface of genus two. Let $\rho : X \to X'$ be the associated double cover. Using Riemann-Hurwitz formula, it is easy to see that ρ is unramified. Since ${\tau'}^*\omega = \omega$, there exists a holomorphic one-form ω' on X' such that $\rho^*\omega' = \omega$. Since $(X,\omega) \in \mathcal{H}(2,2)$ and ρ is unramified, we conclude that $(X',\omega') \in \mathcal{H}(2)$.

Remark that ρ^* is an isomorphism between $\ker(\tau'-\mathrm{id}) = \ker(\tau'+\mathrm{id})$ and $\Omega(X')$, and ρ maps $H_1(X,\mathbb{Z})^-$ to a sublattice of index two in $H_1(X',\mathbb{Z})$, therefore ρ induces a two-to-one covering from $\mathrm{Prym}(X,\tau)$ to $\mathrm{Jac}(X')$. By assumption $\mathrm{Prym}(X,\tau)$ admits a real multiplication by the order $\mathcal{O}_D$ for which ω is an eigenvector. It follows that $\mathrm{Jac}(X')$ also admits a real multiplication by $\mathcal{O}_D\otimes\mathbb{Q}$ for which ω' is an eigenvector. Thus there exists a discriminant D' satisfying $D'|D$ such that $(X',\omega') \in \Omega E_{D'}(2)$. This shows the first part of the proposition.

But we know from [Mc07] that $\Omega E_{D'}(2)$ is a union of $\mathrm{GL}^+(2,\mathbb{R})$ closed orbits, and since the map $(X,\omega) \mapsto (X',\omega')$ is clearly $\mathrm{GL}^+(2,\mathbb{R})$-equivariant, it follows that (X,ω) belongs to a $\mathrm{GL}^+(2,\mathbb{R})$-closed orbit. Since any Riemann surface X in $\mathcal{M}_g$ admits only finitely may unramified double covers, we derive that there are only finitely many closed orbits in $\Omega E_D(2,2)^{\rm hyp}$. The proposition is then proved. □

Because of Proposition 2.3, in the rest of the paper we will focus on $\Omega E_D(2,2)^{\rm odd}$ and $\Omega E_D(1,1,2)$. Observe that if $(X,\omega) \in \mathrm{Prym}(2,2)^{\rm odd}$ then the Prym involution τ exchanges the two zeros of ω, and if $(X,\omega) \in \mathrm{Prym}(1,1,2)$ then τ exchanges the simple zeros ω.

The next lemma provides us with examples of Prym eigenforms in $\mathrm{Prym}(2,2)^{\rm odd}$ and $\mathrm{Prym}(1,1,2)$.

Lemma 2.4 (Real multiplication by $\mathcal{O}_D$)**.** *Let $(w,h,e) \in \mathbb{Z}^3$ be an integral vector satisfying*

$$\begin{cases} w>0, h>0, \ \gcd(w,h,e)=1, \\ D = e^2+8wh, \ e+\sqrt{D}>0. \end{cases}$$

Let $\lambda := \frac{e+\sqrt{D}}{2}$. Note that $\lambda^2 = e\lambda + 2wh$. We denote by $S_\kappa(w,h,e)$ the surface defined in Figure 2 below. Then

$$S_\kappa(w,h,e) \in \Omega E_D(\kappa).$$

Proof of Lemma 2.4. Each surface in Figure 2 is a connected some of three slit tori, and admits an involution τ which fixes one torus and exchanges the other two (see also Section 1.4). It is not difficult to see that τ is a Prym involution, and that $S_\kappa(w,h,e) \in \mathrm{Prym}(\kappa)$. Let (X,ω) be one of the surfaces in Figure 2. Let X_0 be the torus invariant by τ, and X_1, X_2 be the other two tori. By construction, there are bases (a_i,b_i) of $H_1(X_i,\mathbb{Z})$, $i=0,1,2$, such that

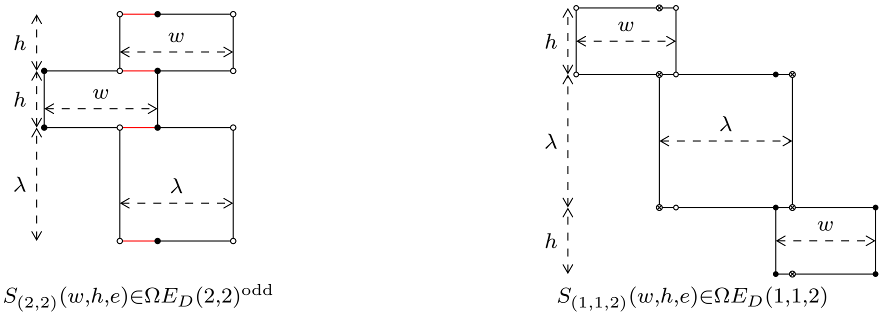

FIGURE 2. Real multiplication by $\mathcal{O}_D$

- $\tau(a_0)=-a_0, \tau(b_0)=-b_0, \tau(a_1)=-a_2, \tau(b_1)=-b_2$,
- $\omega(a_0)=\lambda, \omega(b_0)=\imath\lambda$,
- For $i=1,2$ one has $\omega(a_i)=w$ and $\omega(b_i)=\imath h$.

Set $a=a_1+a_2, b=b_1+b_2$. Then $\{a_0,b_0,a,b\}$ is a symplectic basis of $H_1(X,\mathbb{Z})^-$ in which the intersection form is given by the matrix $\left(\begin{smallmatrix} J & 0\\ 0 & 2J\end{smallmatrix}\right)$. Let T be the endomorphism of $H_1(X,\mathbb{Z})$ which is given in the basis (a_0,b_0,a,b) by the matrix

$$T=\begin{pmatrix} e\mathrm{Id}_2 & \left(\begin{smallmatrix} 2w & 0\\ 0 & 2h\end{smallmatrix}\right)\\ \left(\begin{smallmatrix} h & 0\\ 0 & w\end{smallmatrix}\right) & 0\end{pmatrix}.$$

Since the restriction of the intersection form on $H_1^-(X,\mathbb{Z})$ is given by $\left(\begin{smallmatrix} J & 0\\ 0 & 2J\end{smallmatrix}\right)$, it is easy to check that T is self-adjoint with respect this form. Note that in this basis ω is given by the vector $(\lambda,\imath\lambda,2w,2h\imath)$, therefore we have $T^*\omega=\lambda\omega$. It follows that $T\in\mathrm{End}(\mathrm{Prym}(X,\tau))$. Since T satisfies $T^2=eT+2wh\mathrm{Id}_4$, T generates a self-adjoint proper subring of $\mathrm{End}(\mathrm{Prym}(X,\tau))$ isomorphic to $\mathcal{O}_D$ for which ω is an eigenvector. Thus $S_\kappa(w,h,w)\in\Omega E_D(\kappa)$. The lemma is proved. □

Corollary 2.5. *For any $D\geq 8$, $D\equiv 0,1,4$ mod 8, the loci $\Omega E_D(2,2)^{\rm odd}$ and $\Omega E_D(1,1,2)$ are non-empty.*

Proof of Corollary 2.5. Apply Lemma 2.4 for some $(w,h,e)\in\mathbb{Z}$ with $D=e^2+8wh$. □

2.3. **Kernel foliation.** To investigate the topology of these loci we first recall the notion of the kernel foliation. Let $(X,\omega)\in\mathcal{H}(\kappa)$ be a translation surface. In a neighborhood of (X,ω) the kernel foliation leaf of (X,ω) consists of surfaces having the same absolute periods as (X,ω) and the relatives periods slightly different. This foliation has already appeared in several papers (see for example [EMZ03, Cal04, MZ08, HMSZ, MW14].

For $\kappa\in\{(2,2)^{\rm odd},(1,1,2)\}$, the intersection with the kernel foliation leaves gives rise to a foliation of the Prym eigenform loci $\Omega E_D(\kappa)$, the leaves of this foliation have complex dimension one. Constructions of surfaces in the intersection of the kernel foliation and Prym eigenform loci can be found in [LN13a, LN13b]. Since the leaves of this foliation has dimension one, for any $(X,\omega)\in\Omega E_D(\kappa)$, we can use the notation $(X,\omega)+w$, with $w\in\mathbb{C}$ and $|w|$ small, to denote surfaces in the same leaf and close to (X,ω) (see [LN13b], Section 3). Moreover up to the action of $\mathrm{GL}^+(2,\mathbb{R})$, a neighborhood of (X,ω) in $\Omega E_D(\kappa)$ consists of surfaces in the same kernel foliation leaf as (X,ω). Namely, we have

Proposition 2.6 ([LN13a], Corollary 3.2)**.** *Let $(X',\omega') \in \Omega E_D(\kappa)$ close enough to $(X,\omega) \in \Omega E_D(\kappa)$. Then there exists a unique pair (g,w), where $g \in \mathrm{GL}^+(2,\mathbb{R})$ close to* id*, and $w \in \mathbb{C}$ with $|w|$ small, such that $(X',\omega') = g \cdot ((X,\omega)+w)$. In particular, we have $\dim \Omega E_D(\kappa) = 3$.*

3. UNIQUENESS

In genus two Prym involution and hyperelliptic involution coincide, so it is unique. In higher genus, surfaces may have more than one Prym involution (see *e.g.* the Appendix). In [LN11], we showed that if $(X,\omega) \in \Omega E_D(4)$ then the Prym involution is unique. This is no longer true in $\mathcal{H}(2,2)^{\mathrm{odd}}$, nevertheless, if a surface in $\mathcal{H}(2,2)^{\mathrm{odd}}$ has two Prym involutions, then both Prym varieties admit real multiplication by $\mathcal{O}_9$ as we will see in Theorem 3.1. It follows in particular that if $D_1 \neq D_2$ and $\kappa \in \{(2,2)^{\mathrm{odd}},(1,1,2)\}$ then $\Omega E_{D_1}(\kappa) \cap \Omega E_{D_2}(\kappa) = \varnothing$.

Theorem 3.1. *Let $(X,\omega) \in \mathcal{H}(2,2)^{\mathrm{odd}}$ be a surface having two Prym involutions $\tau_1 \neq \tau_2$ such that $\tau_1^*\omega = \tau_2^*\omega = -\omega$. Then there exist $\mathfrak{i}_1 : \mathcal{O}_9 \to \mathrm{End}(\mathrm{Prym}(M,\tau_1))$ and $\mathfrak{i}_2 : \mathcal{O}_9 \to \mathrm{End}(\mathrm{Prym}(M,\tau_2))$ such that $\mathfrak{i}_i(\mathcal{O}_9)$ is a self-adjoint proper subring of $\mathrm{End}(\mathit{Prym}(X,\tau_i))$, and ω is an eigenform for both subrings $\mathfrak{i}_1(\mathcal{O}_9)$ and $\mathfrak{i}_2(\mathcal{O}_9)$. In particular $(X,\omega) \in \Omega E_9(2,2)^{\mathrm{odd}}$.*

If $(X,\omega) \in \mathcal{H}(1,1,2)$, then there exists at most one Prym involution τ such that $\tau^*\omega = -\omega$.

Corollary 3.2. *For $\kappa \in \{(2,2)^{\mathrm{odd}},(1,1,2)\}$, if $D_1 \neq D_2$ then $\Omega E_{D_1}(\kappa) \cap \Omega E_{D_2}(\kappa) = \varnothing$.*

Proof of Corollary 3.2. Let $(X,\omega) \in \Omega E_{D_1}(\kappa) \cap \Omega E_{D_2}(\kappa)$. Let τ_1 and τ_2 be the corresponding Prym involutions of X. If $\tau_1 \neq \tau_2$ then by Theorem 3.1 one has $\kappa = (2,2)^{\mathrm{odd}}$ and $D_1 = D_2 = 9$. If $\tau_1 = \tau_2$ then Lemma 3.3, applied to $\mathbb{A} = \mathrm{Prym}(X,\tau_1) = \mathrm{Prym}(X,\tau_2)$, gives the uniqueness of the self-adjoint proper subring $\mathcal{O} \subset \mathrm{End}(\mathbb{A})$, which implies $D_1 = D_2$. The corollary is then proved. □

We will need the following two elementary lemmas. The first one proves the uniqueness of the proper subring, once the Prym variety and the eigenform are given (see also [LN11, Section 5] for related results).

Lemma 3.3. *Let $\mathbb{A}$ be an Abelian variety of dimension two. We regard $\mathbb{A}$ as a quotient $\mathbb{C}^2/L$, where L is a lattice isomorphic to $\mathbb{Z}^4$ equipped with a non-degenerate skew-symmetric inner product $\langle , \rangle : L \times L \to \mathbb{Z}$ which is compatible with the complex structure. Let $v \neq 0$ be a vector in $\mathbb{C}^2$. Assume that there exists a self-adjoint endomorphism φ of $\mathbb{A}$ such that $\varphi(v) = \lambda v$, with $\lambda \in \mathbb{R}$, $\varphi \neq \lambda \cdot \mathrm{Id}$. Then there exists a unique discriminant D and a unique self-adjoint proper subring $\mathcal{O}$ of $\mathrm{End}(\mathbb{A})$ isomorphic to $\mathcal{O}_D$ for which v is an eigenvector.*

Proof. Let $S = \mathbb{C} \cdot v$ be the complex line generated by v, and let S' denote the orthogonal complement of S with respect to $\langle , \rangle$ in $\mathbb{C}^2$. Note that S' is also a complex line in $\mathbb{C}^2$. Set $w = \imath v$. Since φ is an endomorphism of $\mathbb{A}$, we have $\varphi(w) = i\varphi(v) = \lambda w$. In other words $\varphi_{|S} = \lambda \cdot \mathrm{id}_S$. Since φ is self-adjoint, it also preserves the complex line S'. Thus $\varphi_{|S'} = \lambda' \cdot \mathrm{id}_{S'}$ where $\lambda' \neq \lambda$.

Since the self-adjoint endomorphism φ preserves the lattice L its minimal polynomial $\chi_\varphi \in \mathbb{Z}[X]$ has degree 2. By definition λ is a real root of χ_φ. Hence λ', that is a root, is also real. Moreover, since v is an eigenvector of φ, up to a real scalar, all the coordinates of v in a basis of L belong to $K = \mathbb{Q}(\lambda)$. Remark that either $K = \mathbb{Q}$, or $K \subset \mathbb{R}$ and $[K : \mathbb{Q}] = 2$.

Let K_v be the subring of $\mathrm{End}(\mathbb{A}) \otimes \mathbb{Q}$ consisting of self-adjoint endomorphisms of $\mathbb{A}$ for which v is an eigenvector. For any $f \in K_v$, the matrix of f in the decomposition $\mathbb{C}^2 = S \oplus S'$ has the form $\begin{pmatrix} \lambda(f) & 0 \\ 0 & \lambda'(f) \end{pmatrix}$.

We claim that K_v is either isomorphic to K or to $\mathbb{Q}^2$. To see this, it suffices to notice that each element of K_v is uniquely determined by its eigenvalues on S and S'. If $\lambda \in \mathbb{Q}$ then we can assume that all the coordinates of v belong to $\mathbb{Q}$, hence both $\lambda(f)$ and $\lambda'(f)$ belong to $\mathbb{Q}$ as f is defined of over $\mathbb{Q}$. Thus $\Lambda : K_v \to \mathbb{Q}^2, \quad f \mapsto (\lambda(f), \lambda'(f))$ is an isomorphism of $\mathbb{Q}$ vector spaces. If $\lambda \notin \mathbb{Q}$, then $\lambda \in K = \mathbb{Q}(\sqrt{d})$, with $d \in \mathbb{N}$, d is not a square. It follows that $\lambda'(f)$ is the Galois conjugate of $\lambda(f)$ in K. Consequently, $\Lambda : K_v \to K, \quad f \mapsto \lambda(f)$ is an isomorphism of $\mathbb{Q}$-vector spaces.

Set $\mathcal{O} = K_v \cap \mathrm{End}(\mathbb{A})$. By definition, $\mathcal{O}$ is the unique self-adjoint proper subring of $\mathrm{End}(\mathbb{A})$ for which v is an eigenvector. It remains to show that $\mathcal{O} \simeq \mathbb{Z}[X]/(X^2 + bX + c)$ for some $b, c \in \mathbb{Z}$, such that $b^2 - 4c > 0$. We have $\dim_{\mathbb{Q}} K_v = 2$. For any $f \in \mathcal{O}$ such that (f, id) is a basis of K_v as a $\mathbb{Q}$-vector space, we denote by $\Delta(f)$ the discriminant of the minimal polynomial of f. Note that we have $\Delta(f) > 0$, and $\Delta(f) = (\lambda(f) - \lambda'(f))^2$. Set $D = \min\{\Delta(f) : \ f \in \mathcal{O}, \ (f, \mathrm{id}) \text{ is a basis of } K_v\}$. Let ψ be an element of $\mathcal{O}$ such that $\Delta(\psi) = D$. Let us show that $\mathcal{O} = \mathbb{Z}\psi + \mathbb{Z}\mathrm{id}$. Indeed, let f be an element of $\mathcal{O}$, then we can write $f = x\psi + y\mathrm{id}$, with $(x, y) \in \mathbb{Q}^2$, and $\Delta(f) = x^2 D$. If $x \notin \mathbb{N}$, by replacing ψ by $f - [x]\psi$, we can find $\psi' \in \mathcal{O}$ such that $0 < \Delta(\psi') < D$, therefore we must have $x \in \mathbb{N}$. It follows that $y \in \mathbb{N}$. Finally, since $\psi \in \mathrm{End}(\mathbb{A})$, the minimal polynomial of ψ has the the form $\psi^2 + b\psi + c\mathrm{id}$, with $b, c \in \mathbb{Z}$ such that $D = b^2 - 4c$. The proof of the lemma is now complete. □

Lemma 3.4. *Let $(X, \omega) \in \mathcal{H}(2, 2) \sqcup \mathcal{H}(1, 1, 2)$, and τ_1, τ_2 be two Prym involutions of X such that $\tau_1^*\omega = \tau_2^*\omega = -\omega$.*

(a) *If $(X, \omega) \in \mathcal{H}(1, 1, 2)$ then $\tau_1 = \tau_2$.*
(b) *If $(X, \omega) \in \mathcal{H}(2, 2)^{\mathrm{odd}}$ and $\tau_1 \neq \tau_2$, then there exists a branched cover $p : X \to Y$ of degree three, where Y is a torus, which is ramified only at the zeros of ω and satisfies $\omega = p^*\xi$, where ξ is a holomorphic one-form on Y. Moreover, the involutions τ_1, τ_2 descend to the unique involution of Y which acts by $-\mathrm{id}$ on the homology and exchanges the images by p of the zeros of ω.*

Proof of Lemma 3.4. Set $\tau = \tau_1 \circ \tau_2$, one has $\tau^*\omega = \omega$ and τ fixes all the zeros of ω. We identify the neighborhood of a double zero P of ω with the unit disk $\Delta \subset \mathbb{C}$ such that P is mapped to 0. In this local chart, $\omega = z^2 dz$. If $\omega \in \mathcal{H}(1, 1, 2)$ then P is the unique double zero of ω, therefore $\tau_1(P) = \tau_2(P) = P$. Since both τ_1, τ_2 are involutions, their restrictions to this neighborhood of P read $\tau_i(z) = -z$. Thus $\tau(z) = z$, which implies that $\tau = \mathrm{id}_X$ and $\tau_1 = \tau_2$.

From now on, we will assume that $\omega \in \mathcal{H}(2, 2)^{\mathrm{odd}}$ and $\tau_1 \neq \tau_2$. The restriction of τ to the local chart of P (defined above) can be written as $\tau(z) = \zeta z$. Since $\tau^*\omega = \omega$, it follows that $\zeta^3 = 1$. Obviously $\zeta \neq 1$, otherwise τ is the identity map in a neighborhood of P and hence it is the identity on X implying $\tau_1 = \tau_2$. Let $p : X \to Y = X/\langle\tau\rangle$ be the quotient map. Since τ has order three, p is a ramified covering of degree 3. Clearly, the two zeros of ω are branched points of p of order 3. Moreover since $\dim \ker(\tau - \mathrm{id}) \geq 1$, one has $\mathrm{genus}(Y) \geq 1$. These two facts, combined with the Riemann-Hurwitz formula

$$-4 = 2 - 2 \cdot \mathrm{genus}(X) = 3 \cdot (2 - 2 \cdot \mathrm{genus}(Y)) - \sum_{x \in X}(e_p(x) - 1) \leq -\sum_{x \in X}(e_p(x) - 1) \leq -4$$

implies that $\mathrm{genus}(Y) = 1$ and the two zeros of ω are the only branched points of p. Since $\tau^*\omega = \omega$, the form ω descends to a holomorphic 1-form ξ on Y *i.e.* $\omega = p^*\xi$.

Now the subgroup of $\mathrm{Aut}(X)$ generated by τ, namely $\{\mathrm{id}, \tau_1 \circ \tau_2, \tau_2 \circ \tau_1\}$, is clearly invariant by the conjugations by τ_1 and τ_2. Therefore τ_1 and τ_2 induces two involutions, say ι_1 and ι_2, on Y. Since τ_i permutes the zeros of ω, the equality $\tau_i^*\omega = -\omega$ reads $\iota_i^*\xi = -\xi$, and ι_i exchanges the images of the zeros of ω by p. Now Y is a torus: there exists only one such involution. Hence $\iota_1 = \iota_2$. The lemma is then proved. □

Proof of Theorem 3.1. From Lemma 3.4 it is sufficient to assume that $(X, \omega) \in \mathcal{H}(2,2)^{\mathrm{odd}}$. For simplicity we continue with the notions of the previous lemma. Let $P, Q \in X$ denote the zeros of ω. We claim that there exists a basis of $H_1(Y, \mathbb{Z})$ given by a pair of simple closed geodesics $\{\alpha, \beta\}$ that are invariant by ι and that do not contain $p(P)$ and $p(Q)$. Indeed one can pick a fixed point of ι and take a pair of simple closed geodesics passing through this point and missing the point $p(P)$ and $p(Q) = \iota(p(P))$.

The next goal is to construct a symplectic basis of X and a self-adjoint endomorphism of Prym(X, τ_1). We lift the curves α, β to X in the following manner. Let $R \in Y$ be the (unique) intersection point of α with β. Hence $\iota(R) = R$ and R is a regular point of the covering p. Since $p \circ \tau_1 = \iota \circ p$, the involution τ_1 induces a permutation (of order two) of $p^{-1}(R) = \{R_0, R_1, R_2\}$. Therefore τ_1 fixes some point, say R_0.

Let α_0 (respectively, β_0) be the pre-image of α (respectively, β) passing through R_0. For $j = 1, 2$ let $\alpha_j = \tau^j(\alpha_0), \beta_j = \tau^j(\beta_0)$, where $\tau = \tau_1 \circ \tau_2$. By construction:

$$\begin{cases} \langle \alpha_i, \alpha_j \rangle = \langle \beta_i, \beta_j \rangle = 0 & \text{for} \quad i, j \in \{0,1,2\}, i \neq j, \\ \langle \alpha_i, \beta_j \rangle = \delta_{ij} & \text{for} \quad i, j \in \{0,1,2\}. \end{cases}$$

Hence $(\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2)$ is a symplectic basis of $H_1(X, \mathbb{Z})$. This allows us to construct a symplectic basis (a_0, b_0, a_1, b_1) of $H_1(X, \mathbb{Z})^- = \ker(\tau_1 + \mathrm{id})$ as usual:

$$\begin{cases} a_0 = \alpha_0 & b_0 = \beta_0 \\ a_1 = \alpha_1 + \alpha_2 & b_1 = \beta_1 + \beta_2 \end{cases}$$

The intersection form is given by the matrix $\left(\begin{smallmatrix} J & 0 \\ 0 & 2J \end{smallmatrix}\right)$. One can normalize by using $\mathrm{GL}^+(2, \mathbb{R})$ so that $\xi(\alpha) = 1$ and $\xi(\beta) = \imath \in \mathbb{C}$. Then ω (viewed as an element of $H^1(X, \mathbb{C})^-$) is the vector (in the basis dual to (a_0, b_0, a_1, b_1))

$$(\omega(a_0), \omega(b_0), \omega(a_1), \omega(b_1)) = (1, \imath, 2, 2\imath).$$

It is straightforward to check that (X, ω) coincide with $S_{(2,2)}(1, 1, -1) \in \Omega E_9(2, 2)$.

Let us consider the matrix

$$T = \left(\begin{smallmatrix} -\mathrm{id}_2 & 2\cdot\mathrm{id}_2 \\ \mathrm{id}_2 & 0 \end{smallmatrix}\right)_{(a_0,b_0,a_1,b_1)}$$

It is straightforward to check that T is self-adjoint with respect to the restriction of the intersection form on $H_1(M, \mathbb{Z})^-$. Moreover $(1, \imath, 2, 2\imath) \cdot T = (1, \imath, 2, 2\imath)$ thus ω is an eigenform for T: hence $(X, \omega) \in \Omega E_D(2, 2)^{\mathrm{odd}}$. Since $T^2 = -T + 2$, the endomorphism T generates a proper subring of $\mathrm{End}(\mathrm{Prym}(M, \tau_1))$ isomorphic to $\mathcal{O}_D$ where $D = 1 + 4 \cdot 2 = 9$. By Lemma 3.3, this subring is unique.

The same argument shows that $\mathrm{End}(\mathrm{Prym}(M, \tau_2))$ also contains a unique self-adjoint proper subring isomorphic to $\mathcal{O}_9$, for which ω is an eigenform. The proof of the theorem is now complete. □

4. NON CONNECTEDNESS OF $\Omega E_D(\kappa)$

In this section we will show that when $D \equiv 1 \bmod 8$, the number of components of $\Omega E_D(2, 2)^{\mathrm{odd}}$ and $\Omega E_D(1, 1, 2)$ is at least two. It is worth noticing that this is not true in genus two, namely, $\Omega E_D(2)$ has two connected components, while $\Omega E_D(1, 1)$ is connected (see [Mc05]).

Theorem 4.1. *For any $D \geq 9$ satisfying $D \equiv 1 \bmod 8$, the loci $\Omega E_D(2, 2)^{\mathrm{odd}}$ and $\Omega E_D(1, 1, 2)$ are not connected.*

Proof of Theorem 4.1. First of all by Corollary 2.5, $\Omega E_D(\kappa)$ is non-empty. Before going into the details, we first explain why $\Omega E_D(4)$ is not connected when $D \equiv 1 \bmod 8$ (see [LN11, Theorem 6.1]). For every surface $(X,\omega) \in \Omega E_D(4)$ we denote by S the subspace of $H^1(X,\mathbb{R})^- = \ker(\tau + \mathrm{Id}) \subset H^1(X,\mathbb{R})$ generated by $\{\mathrm{Re}(\omega), \mathrm{Im}(\omega)\}$ and by S' the orthogonal complement of S with respect to the intersection form in $H^1(X,\mathbb{R})^-$. By definition there is an injective ring morphism $\mathfrak{i} : \mathcal{O}_D \to \mathrm{End}(\mathrm{Prym}(X,\tau))$ such that $\mathfrak{i}(\mathcal{O}_D)$ is a self-adjoint proper subring of $\mathrm{End}(\mathrm{Prym}(X,\tau))$, and for any $T \in \mathfrak{i}(\mathcal{O}_D)$, S is an eigenspace of T. It turns out that an element $T \in \mathrm{Im}(\mathfrak{i})$ is uniquely determined by its minimal polynomial and by the eigenvalue of its restriction to S. Indeed, the minimal polynomial of T has degree two; thus if $T_{|S} = \lambda \mathrm{id}_S$ then $T_{|S'} = \lambda' \mathrm{id}_{S'}$, where λ and λ' are the roots of the minimal polynomial of T. Therefore T is given by the matrix $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda' \end{smallmatrix}\right)$ in the decomposition $H^1(X,\mathbb{R})^- = S \oplus S'$. In [LN11, Section 6] for each $D \equiv 1 \bmod 8$, with $D \geq 17$, we constructed two surfaces $(X_i,\omega_i) \in \Omega E_D(4)$, $i = 0,1$, where the corresponding generators of the order $T_0 \in \mathrm{Im}(\mathfrak{i}_0)$ and $T_1 \in \mathrm{Im}(\mathfrak{i}_1)$ satisfies:

- T_0 and T_1 have the same minimal polynomial,
- $T_{0|S_0} = \lambda \mathrm{id}_{S_0}$ and $T_{1|S_1} = \lambda \mathrm{id}_{S_1}$,
- $\langle , \rangle_{|\mathrm{Rg}(T_0)} \neq 0 \bmod 2$ and $\langle , \rangle_{|\mathrm{Rg}(T_1)} = 0 \bmod 2$.

Now if (X_0,ω_0) and (X_1,ω_1) belong to the same connected component (*i.e.* $A \cdot (X_0,\omega_0) = (X_1,\omega_1)$ where $A \in \mathrm{GL}^+(2,\mathbb{R})$) then there exists an isomorphism $f : H_1(X_0,\mathbb{Z})^- \to H_1(X_1,\mathbb{Z})^-$ such that $T_1' := f^{-1} \circ T_1 \circ f$ defines an endomorphism of $\mathrm{Prym}(X_0,\tau_0)$. By uniqueness of the Prym involution and the map $\mathfrak{i}_0 : \mathcal{O}_D \to \mathrm{End}(X_0,\tau_0)$, it follows that both T_0 and T_1' belong to $\mathfrak{i}_0(\mathcal{O}_D)$. By construction T_0 and T_1' have the same minimal polynomial and $T_{0|S_0} = (T_1')_{|S_0} = \lambda \mathrm{id}_{S_0}$. Thus in view of the above remark, $T_0 = T_1'$. This is a contradiction since $\langle , \rangle_{|\mathrm{Rg}(T_0)} \neq \langle , \rangle_{|\mathrm{Rg}(T_1')} \bmod 2$.

We now go back to the proof of Theorem 4.1 and apply similar ideas. Let $(w,h,e) \in \mathbb{Z}^3$ be as in Lemma 2.4 where $D = e^2 + 8wh \equiv 1 \bmod 8$. Note that e is odd. We will show that the two surfaces

$$(X_0,\omega_0) := S_\kappa(w,h,-e) \in \Omega E_D(\kappa) \qquad \text{and} \qquad (X_1,\omega_1) := S_\kappa(w,h,e) \in \Omega E_D(\kappa)$$

do not belong to the same component. Recall that by construction, for $j = 0,1$, we have associated to (X_j,ω_j) a generator T_j of the order $\mathfrak{i}_j(\mathcal{O}_D) \subset \mathrm{End}(\mathrm{Prym}(X_j,\tau_j))$. Recall that, in the symplectic basis (a_0,b_0,a,b) of $H_1(X_j,\mathbb{Z})$ given in Lemma 2.4, the endomorphism is given by the matrix

$$T_0 = \begin{pmatrix} -e\mathrm{Id}_2 & \left(\begin{smallmatrix} 2w & 0 \\ 0 & 2h \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} h & 0 \\ 0 & w \end{smallmatrix}\right) & 0 \end{pmatrix} \qquad \text{and} \qquad T_1 = \begin{pmatrix} e\mathrm{Id}_2 & \left(\begin{smallmatrix} 2w & 0 \\ 0 & 2h \end{smallmatrix}\right) \\ \left(\begin{smallmatrix} h & 0 \\ 0 & w \end{smallmatrix}\right) & 0 \end{pmatrix} \text{ respectively .}$$

Let us assume that there is a continuous path $\gamma : [0,1] \to \Omega E_D(\kappa)$ such that $\gamma(i) = (X_i,\omega_i)$ for $i = 0,1$, we will draw a contradiction. Let $\tilde{\gamma}$ be a lift of γ to the vector bundle $\Omega\mathcal{T}_3$ over the Teichmüller space $\mathcal{T}_3$. We will denote by (X_s,ω_s) the image of $s \in [0,1]$ by $\tilde{\gamma}$. Let Σ be the base surface of the Teichmüller space. By construction the path $\tilde{\gamma}$ induces a continuous map which sends every $s \in [0,1]$ to a tuple $(\mathbf{J}_s, \tau_s, L_s, \mathfrak{i}_s, S_s)$, where

- $\mathbf{J}_s$ is the complex structure of $H^1(\Sigma,\mathbb{R})$, induced by the complex structure of X_s,
- $\tau_s \in \mathrm{Sp}(6,\mathbb{Z})$ is the matrix which gives the action of the Prym involution of X_s on $H^1(\Sigma,\mathbb{R})$,
- L_s is the lattice $H^1(\Sigma,\mathbb{Z}) \cap \ker(\tau_s + \mathrm{id})$,
- $\mathfrak{i}_s : \mathcal{O}_D \to \mathrm{End}(H^1(\Sigma,\mathbb{R})^-_s)$, where $H^1(\Sigma,\mathbb{R})^-_s = \ker(\tau_s + \mathrm{id}) \subset H^1(\Sigma,\mathbb{R})$, is an injective ring morphism where $\mathfrak{i}_s(\mathcal{O}_D)$ is a self-adjoint proper subring of $\mathrm{End}(H^1(\Sigma,\mathbb{R})^-_s)$ that preserves L_s.
- S_s is the subspace of $H^1(X,\mathbb{R})^- = \ker(\tau_s + \mathrm{Id})$ generated by $\{\mathrm{Re}(\omega_s), \mathrm{Rg}(\omega_s)\}$.

Remark that since ω is holomorphic, S_s is invariant by $\mathbf{J}_s$. The action of $\mathrm{GL}^+(2,\mathbb{R})$ preserves the subspace $S_s \subset H^1(\Sigma,\mathbb{R})^-$, and the kernel foliation leaves invariant $[\mathrm{Re}(\omega)]$ and $[\mathrm{Im}(\omega)]$. Therefore S_s is invariant along the path $\tilde{\gamma}$. Clearly, the matrix τ_s is also invariant along the deformation $\tilde{\gamma}$. This implies that L_s and $\mathfrak{i}_s$ are also invariant along $\tilde{\gamma}$. In particular $S_0 = S_1 = S$ and $\mathfrak{i}_0 = \mathfrak{i}_1$.

There exists an isomorphism $f : H_1(X_0,\mathbb{Z})^- \longrightarrow H_1(X_1,\mathbb{Z})^-$ satisfying $f(S_0) = S_1$ and such that $T_1' = f^{-1} \circ T_1 \circ f$ belongs to $\mathfrak{i}_0(\mathcal{O}_D) \subset \mathrm{End}(\mathrm{Prym}(X_0,\tau_0))$. Remark that T_1' and $T_0 + e\cdot\mathrm{Id}$ have the same minimal polynomial $X^2 - eX - 2wh$. In addition the eigenvalues of T_1' on $S_0 = \mathbb{C}\cdot\omega_0$, and the eigenvalue of $T_0 + e\cdot\mathrm{Id}$ on $\mathbb{C}\cdot\omega_0$ are both equal to $\lambda = (e+\sqrt{D})/2$. Hence $T_1' = T_0 + e\cdot\mathrm{Id}$.

Now $\mathrm{Rg}(T_0 + e\cdot\mathrm{Id})$ mod 2 is generated by $\{a,b\}$. The restriction of the intersection form $\langle,\rangle$ to this subspace is equal to 0 mod 2. On the other hand the restriction of the intersection form to $\mathrm{Rg}(T_1')$ does not vanish modulo 2:

$$\langle T_1(a_0), T_1(b_0)\rangle \equiv \langle a_0, b_0\rangle \equiv 1 \bmod 2.$$

This is a contradiction, and the theorem follows. □

In Section 5.4 we will give a topological argument for the non connectedness of $\Omega E_9(2,2)^{\mathrm{odd}}$.

5. Collapsing zeros along a saddle connection

In this section we describe a surgery on collapsing several zeros of Prym eigenforms together such that the resulting surface is still a non degenerate Prym eigenform. This can be thought as the converse of the surgery "breaking up a zero" (see [KZ03]).

In what follows, all the zeros are *labelled* and all the saddle connections are *oriented*: if $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, we label the zeros by P and Q and if $(X,\omega) \in \Omega E_D(1,1,2)$, we label the simple zeros by R_1, R_2 and the double zero by Q. Let σ_0 be a saddle connection on X.

Convention 1. *We will always assume that:*

(1) *If $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$ then σ_0 is a saddle connection from P to Q that is invariant by τ.*
(2) *If $(X,\omega) \in \Omega E_D(1,1,2)$ then σ_0 is a saddle connection from R_1 to Q.*

Observe that such saddle connections always exist on any $(X,\omega) \in \Omega E_D(\kappa)$: for $\kappa = (2,2)^{\mathrm{odd}}$, take σ_0 to be the union of a path of minimal length from a regular fixed point of τ to a zero of ω and its image by τ, for $\kappa = (1,1,2)$, take a path of minimal length from the set $\{R_1,R_2\}$ to $\{Q\}$.

5.1. Admissible saddle connections.

We begin with the following definition.

Definition 5.1. *Let $(X,\omega) \in \mathrm{Prym}(\kappa)$ be a Prym form.*

(1) *$\kappa = (2,2)^{\mathrm{odd}}$: we say that σ_0 is admissible if for any saddle connection $\sigma \neq \sigma_0$ from P to Q satisfying $\omega(\sigma) = \lambda\omega(\sigma_0)$, with $\lambda \in \mathbb{R}_+$, one has $\lambda > 1$.*
(2) *$\kappa = (1,1,2)$: we say that σ_0 is admissible if, for any saddle connection $\sigma \neq \sigma_0$ starting from R_1 and satisfying $\omega(\sigma) = \lambda\omega(\sigma_0)$, with $\lambda \in \mathbb{R}_+$, either $\lambda > 1$ if σ ends at Q, or $\lambda > 2$ if σ ends at R_2.*

Observe that by definition the subset consisting of surfaces having an admissible saddle connection is an open $\mathrm{GL}^+(2,\mathbb{R})$-invariant subset.

Lemma 5.2. *Let $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$. For any $(X,\omega) \in \Omega E_D(\kappa)$ and any σ_0 satisfying Convention 1, there exists in a neighborhood of (X,ω) a surface $(X',\omega') \in \Omega E_D(\kappa)$ with a saddle connection σ_0' (corresponding to σ_0 and also satisfies Convention 1) such that any saddle connection σ' on X' in the same direction as σ_0', if it exists, satisfies*

(1) *Case* $\kappa = (2,2)^{\mathrm{odd}}$*:* $\omega'(\sigma') = \omega'(\sigma'_0)$.
(2) *Case* $\kappa = (1,1,2)$*:*
   (a) *If* σ' *connects a simple zero to the double zero then* $\omega'(\sigma') = \omega'(\sigma'_0)$.
   (b) *If* σ' *connects two simple zeros then* $\omega'(\sigma') = 2\omega'(\sigma'_0)$.

Proof of Lemma 5.2. For any vector $v \in \mathbb{R}^2$ small enough we denote by σ'_0 the saddle connection on $(X',\omega') = (X,\omega) + v$ corresponding to σ_0. Observe that the set

$$\mathrm{Slope}(X,\omega) = \{s \in \mathbb{R} \cup \{\infty\} : s \text{ is the slope of } \omega(\gamma) \neq 0, \text{ with } [\gamma] \in H_1(X,\mathbb{Z})\}$$

is countable. Hence there exists a vector $v \in \mathbb{R}^2$ small enough so that the slope of $\omega'(\sigma'_0)$ does not belong to the set $\mathrm{Slope}(X,\omega) = \mathrm{Slope}(X',\omega')$.

Case $\kappa = (2,2)^{\mathrm{odd}}$. Let σ' be a saddle connection starting from P in the same direction as σ'_0. If σ' ends at P then $[\sigma'] \in H_1(X',\mathbb{Z})$ and $0 \neq \omega'(\sigma')$ has the same slope as $\omega'(\sigma'_0)$. This is a contradiction. Thus σ' ends at Q and $[\gamma'] = [\sigma'_0 * (-\sigma')] \in H_1(X',\omega')$. If $\omega'(\gamma') \neq 0$ then we get again a contradiction. Therefore $\omega'(\gamma') = 0$ *i.e.* $\omega'(\sigma') = \omega'(\sigma'_0)$.

Case $\kappa = (1,1,2)$. Let σ' be a saddle connection starting from R_1 in the same direction as σ'_0. If σ' ends at R_1 we run into the same contradiction. If σ' ends at Q then we also run into the same conclusion namely $\omega'(\sigma') = \omega'(\sigma'_0)$. Hence let us assume that σ' ends at R_2. Thus $\sigma'_0 * \tau(\sigma'_0) * (-\sigma')$ is a closed path from R_1 to R_1 (through Q and R_2). Therefore $[\gamma'] = [\sigma'_0 * \tau(\sigma'_0) * (-\sigma')] \in H_1(X',\omega')$. The same contradiction shows that $\omega'(\gamma') = 0$, *i.e.* $\omega'(\sigma') = 2\omega'(\sigma'_0)$. The lemma is proved. □

5.2. **Non-admissible saddle connection and twin/double-twin.** Lemma 5.2 leads to the following natural definition.

Definition 5.3. *Let* $(X,\omega) \in \Omega E_D(\kappa)$ *and let* σ_0 *be a saddle connection on* X *satisfying Convention 1.*
If $\kappa = (2,2)^{\mathrm{odd}}$*: a saddle connection* σ *is a* twin *of* σ_0 *if* σ *joins* P *to* Q *and* $\omega(\sigma) = \omega(\sigma_0)$.
If $\kappa = (1,1,2)$*:*

(1) *a saddle connection* σ *is a twin of* σ_0 *if it has the same endpoints and* $\omega(\sigma) = \omega(\sigma_0)$.
(2) *a saddle connection* σ *is a double-twin of* σ_0 *if* σ *joins* R_1 *to* R_2 *and* $\omega(\sigma) = 2\omega(\sigma_0)$.

When $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$, since the angle between two twin saddle connections is a multiple of 2π and the angle at P is 6π, we see that each saddle connection σ_0 has at most two twins. When $(X,\omega) \in \Omega E_D(1,1,2)$, the same remark shows that σ_0 has at most one twin or one double-twin. Moreover the midpoint of any double twin saddle connection is fixed by the Prym involution.

Non admissible saddle connection does not necessarily imply the existence of a twin or double twin (see Remark 5.6 below). However Lemma 5.2 shows that, under mild assumption, this dichotomy holds. As an immediate corollary, we draw

Corollary 5.4. *Let* $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$*. For any connected component* $\mathcal{C}$ *of* $\Omega E_D(\kappa)$*, there exist* $(X,\omega) \in \mathcal{C}$ *and a saddle connection* σ_0 *on* (X,ω) *satisfying Convention 1 such that either* σ_0 *is admissible, or* σ_0 *has a twin or a double twin.*

5.3. **Collapsing admissible saddle connections.** We have the following proposition that is a converse to the surgery "breaking up a zero" (see [KZ03]). We prove the proposition only in the setting of Corollary 5.4. A more general statement holds but it is not needed in this paper.

Proposition 5.5. *Let $(X,\omega) \in \Omega E_D(\kappa)$ be a Prym form and σ_0 a saddle connection satisfying Convention 1. We assume that σ_0 is admissible. Then one can collapse the zeros of ω along σ_0 by using the kernel foliation so that the resulting surface belongs to $\Omega E_D(4)$.*

In particular, if there is no saddle connection in the same direction as σ_0 connecting two different zeros, then one can collapse σ_0 to get a surface in $\Omega E_D(4)$.

Proof. We first consider the case when $(X,\omega) \in \Omega E_D(2,2)^{\mathrm{odd}}$. The proof we describe is constructive. Set $\ell = |\sigma_0|$. As usual we assume that σ_0 is horizontal. By definition of admissible saddle connection, any other horizontal saddle connection from P to Q has length $> \ell$.

For any horizontal geodesic ray emanating from a zero of ω we say that the ray is *positive* if it has direction $(1,0)$ and *negative* if it has direction $(-1,0)$. For instance by convention σ_0 is a positive ray for P and a negative ray for Q. Since the conical angle at P and Q is 6π, there are two other positive horizontal rays emanating from P, say $\sigma^+_{P,1}$ and $\sigma^+_{P,2}$, as well as two other negative rays for Q, say $\sigma^-_{Q,1}$ and $\sigma^-_{Q,2}$. We parametrize each ray by its length to the zero where it emanates. Obviously, if a positive ray intersects a negative ray then it corresponds to a (horizontal) saddle connection.

We will first prove the proposition under a slightly stronger condition

$$(\mathcal{C}) \text{ any horizontal saddle connection other than } \sigma_0 \text{ has length } > 2\ell.$$

We will construct a set $\mathcal{T}$ which is a union of horizontal rays as follows: the first element of $\mathcal{T}$ is σ_0. Now if a ray $\sigma^+_{P,1}$ or $\sigma^+_{P,2}$ intersects $\sigma^-_{Q,1}$ or $\sigma^-_{Q,2}$ by assumption, the associated saddle connection has length $\lambda > 2\ell$. Hence we can choose $\varepsilon > 0$ such that positive rays $\sigma^+_{P,1}$ and $\sigma^+_{P,2}$ at time $\ell+\varepsilon$ do not intersect $\sigma^-_{Q,1}$ and $\sigma^-_{Q,2}$ at time $\ell+\varepsilon$ in their interior. These are the next elements of $\mathcal{T}$. Finally we consider the negative rays from P and positive rays from Q at time ε. By the assumption, the union $\mathcal{T}$ of all of these rays is an embedded tree in X (see Figure 3).

We now consider a neighborhood $\mathcal{T}(\delta) = \{p \in X;\ \mathrm{h}(p,\mathcal{T}) < \delta\}$ of $\mathcal{T}$, where h is the distance measured in the vertical direction. For $\delta > 0$ small enough, $\mathcal{T}$ is a retract by deformation of $\mathcal{T}(\delta)$. We can easily construct $\mathcal{T}(\delta)$ from 10 Euclidian rectangles whose heights are equal to δ and widths are equal to $\ell + 2\varepsilon$.

We will now change the flat metric of $\mathcal{T}(\delta)$ without changing the metric outside of this neighborhood. Given any $\ell' \in]0,\ell[$, by varying the points where the rectangles are sewn, we can obtain a new surface (X',ω') in $\Omega E_D(2,2)^{\mathrm{odd}}$ with a saddle connection invariant by the involution whose length is equal to ℓ'. Note that the surfaces obtained from this construction belong the same leaf of the kernel foliation as (X,ω).

When $\ell' = 0$, we get a new closed surface $(X_0,\omega_0) \in \mathcal{H}(4)$ sharing the same absolute periods as (X,ω). Moreover there exists an involution τ_0 on X_0 such that $\tau_0^*\omega_0 = -\omega_0$. Hence $(X_0,\omega_0) \in \Omega E_D(4)$.

Let us now give the proof of the lemma without the additional condition $(\mathcal{C})$. Using $\left(\begin{smallmatrix} 1 & 0 \\ 0 & e^t \end{smallmatrix}\right)$, $t > 0$, we can assume that any non-horizontal saddle connection has length $> 4\ell$. Set

$$\ell_0 = \min\{|\sigma|,\ \sigma \text{ is a simple horizontal geodesic loop at } P \text{ or } Q\}.$$

Choose any $\delta \leq 1/2\min\{\ell,\ell_0\}$, and consider $B(P,\delta) := \{x \in X,\ \mathbf{d}(x,P) < \delta\}$ and $B(Q,\delta) := \{x \in X,\ \mathbf{d}(x,Q) < \delta\}$, where $\mathbf{d}$ is the distance defined by flat metric. By assumption, $B(P,\delta)$ and $B(Q,\delta)$ are two embedded topological disks in X which are disjoint. Therefore, the surface $(X,\omega) + (-\delta, 0)$, which is

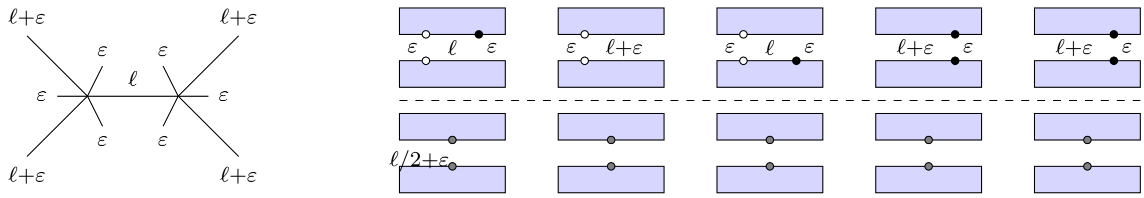

FIGURE 3. Collapsing two zeros along a saddle connection invariant by τ.

obtained by moving P by $\delta/2$ to the right, and Q by $\delta/2$ to the left, is well defined. In the new surface, any horizontal saddle connection from P to Q has length reduced by δ, but the lengths of all horizontal geodesic loops are unchanged since they are absolute periods of ω. Note also that if σ is another horizontal saddle connection joining P to Q, then $|\sigma| - |\sigma_0|$ is also unchanged. It follows that after finite steps, we can find a surface in the horizontal leaf of (X, ω) such that $|\sigma_0| < 1/2\ell_0$ and $|\sigma_0| < 1/2|\sigma|$ for any other horizontal saddle connection σ from P to Q. We can now apply the above arguments to conclude.

We now turn into the case when $(X, \omega) \in \Omega E_D(1,1,2)$. The construction is similar, we keep the same convention: σ_0 is a positive ray for R_1 and a negative ray for Q. Note that $\tau(\sigma_0)$ is a horizontal saddle connection from Q to R_2, it is a positive ray for Q and negative ray for R_2. We denote by $\sigma^{\pm}_{Q,i}$, $i = 1, 2$, the two other positive/negative rays from Q, and $\sigma^+_{R_1}$ (resp. $\sigma^-_{R_2}$) the other positive (resp. negative) ray from R_1 (resp. from R_2).

We again prove the proposition with a slightly stronger assumption that for any other horizontal saddle connection σ, one has $|\sigma| > 4|\sigma_0|$. We will construct a set $\mathcal{T}$ which is a union of positive/negative rays parametrized by the length to its origin. The first elements of $\mathcal{T}$ are σ_0 and $\tau(\sigma_0)$. We then add to $\mathcal{T}$

- the rays $\sigma^+_{R_1}$ and $\sigma^-_{R_2}$ a time $2\ell + \varepsilon$
- the negative rays from R_1 and positive rays from R_2 at time ε.
- the positive and negative rays from Q other than σ_0 and $\tau(\sigma_0)$ at time $\ell + \varepsilon$.

with $\varepsilon > 0$ small. Now if the ray $\sigma^+_{R_1}$ intersects any negative horizontal ray then, by assumption, the associated saddle connection has length $> 4\ell$. Hence we can choose $\varepsilon > 0$ such that positive ray $\sigma^+_{R_1}$ at time $2\ell + \varepsilon$ does not intersect any negative ray from Q at time $\ell + \varepsilon$, nor any negative ray from R_1 or R_2 at time $2\ell + \varepsilon$. Similar arguments apply for other positive rays. It follows that $\mathcal{T}$ is a tree.

We now consider a neighborhood $\mathcal{T}(\delta) = \{p \in X;\ \mathrm{h}(p, \mathcal{T}) < \delta\}$ of $\mathcal{T}$ where h is the distance measured in the vertical direction. For $\delta > 0$ small enough, $\mathcal{T}$ is a retract by deformation of $\mathcal{T}(\delta)$. We can easily construct $\mathcal{T}(\delta)$ from 10 Euclidian rectangles whose heights are equal to δ and widths are equal to $2\ell + 2\varepsilon$. As above we can change the flat metric of $\mathcal{T}(\delta)$ without changing the metric outside of this neighborhood. The rest of the proof follows the same lines as the case $\kappa = (2,2)^{\mathrm{odd}}$. □

5.4. **Twins and non connectedness of Prym eigenform loci when $D = 9$.** In this section we give another elementary proof of the non connectedness of the loci $\Omega E_9(2,2)^{\mathrm{odd}}$ (Theorem 4.1).

Another proof of Theorem 4.1, case $\Omega E_9(2,2)$. Set $X_0 := S_{(2,2)}(1,1,-1)$ and $X_1 := S_{(2,2)}(1,1,1)$ (see Lemma 2.4). For $i = 0, 1$, let $\mathcal{C}_i$ be the connected component of (X_i, ω_i).

We claim that on any surface in $\mathcal{C}_0$, any saddle connection which connects the two zeros of ω_0 has exactly two twins. Since this property is not satisfied by (X_1, ω_1) (the longest horizontal saddle connection on (X_1, ω_1) connects the zeros of ω_1 and has no other twins) this will prove the theorem for $\Omega E_9(2,2)^{\rm odd}$.

By construction the surface (X_0, ω_0) has three distinct Prym involutions, each of which preserves exactly one of the tori in the decomposition shown in Figure 2. By Lemma 3.4 there exists a ramified covering $p : X_0 \to N_0$ of degree three (ramified at the zeros of ω_0) where N_0 is a torus. Hence any saddle connection on X_0 which connect the zeros of ω_0 has two other twins. For any surface in the kernel foliation leaf or in the $\mathrm{GL}^+(2,\mathbb{R})$-orbit of (X_0, ω_0), this property is clearly preserved (since surfaces still have three Prym involutions). Thus (X_1, ω_1) does not belong to the component of (X_0, ω_0). $\square$

Remark 5.6. *On the surface $S_{(2,2)}(1,1,1)$, the longest horizontal saddle connection (the one that is contained in the boundary components of the bottom cylinder) satisfies Convention 1, and has no twins. It is not admissible since there are two other saddle connections in the same direction with smaller length. If we move this surface slightly in the kernel foliation leaf to break this parallelism, we will find a twin of this saddle connection (see Figure 4). This example shows that having a saddle connection satisfying Convention 1 which has no twins does not imply the existence of an admissible one. On the other hand, we know that $S_{(2,2)}(1,1,1)$ belongs to $\Omega E_9(2,2)^{\rm odd}$ and $\Omega E_9(4) = \varnothing$, so there exists no admissible saddle connection on $S_{(2,2)}(1,1,1)$.*

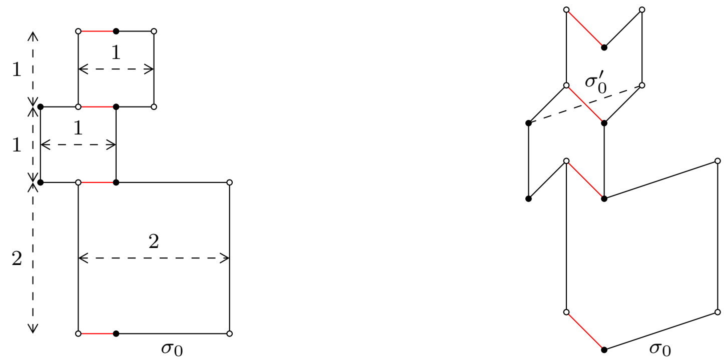

FIGURE 4. On the left: $(X,\omega) = S_{(2,2)}(1,1,1)$, on the right: $(X,\omega) + (0,\varepsilon)$. In (X,ω), σ_0 has no twin, but in $(X,\omega)+(0,\varepsilon)$ it has one.

5.5. **Twins and triple tori.** The next lemma provides a useful criterion to have triple tori from twin saddle connections (see Section 1.4 for the definition of triple tori).

Lemma 5.7. *Let (X,ω) be a translation surface and let σ_0 be a saddle connection on X satisfying Convention 1. We assume that σ_1 is a twin of σ_0 that is not invariant by τ.*

1. *If $(X,\omega) \in$ Prym$(2,2)^{\rm odd}$ and $\sigma_1 \cup \tau(\sigma_1)$ is separating then the triple of saddle connections $\sigma_0, \sigma_1, \tau(\sigma_1)$ decomposes (X,ω) into a triple of flat tori.*
2. *If $(X,\omega) \in$ Prym$(1,1,2)$ and $\sigma_0 \cup \sigma_1$ is separating then the pairs (σ_0,σ_1) and $(\tau(\sigma_0),\tau(\sigma_1))$ decomposes (X,ω) into a triple of flat tori.*

Proof of Lemma 5.7. As usual we assume first that $(X,\omega) \in \mathrm{Prym}(2,2)^{\rm odd}$. Let $\sigma_2 = \tau(\sigma_1)$. We first begin by observing that (X,ω) is the connected sum of a flat torus (X_0,ω_0) with a surface $(X',\omega') \in \mathcal{H}(1,1)$, along $\sigma_1 \cup \sigma_2$. Indeed the saddle connections σ_1 and σ_2 determine a pair of angle $(2\pi, 4\pi)$ at P and Q. Since τ permutes P and Q, and preserves the orientation of X, it turns out that the angles 2π at P and the angle 2π at Q belong to the same side of $\sigma_1 \cup \sigma_2$.

As subsurfaces of X, X_0 and X' have a boundary that consists of the saddle connections σ_1 and σ_2. We can glue σ_1 and σ_2 together to obtain two closed surfaces that we continue to denote by X_0 and X'. We now have on X_0 (respectively, on X') a marked geodesic segment σ (respectively, a saddle connection σ') that corresponds to the identification of σ_1 and σ_2. Note also that σ_0 is contained in X'.

The involution τ induces two involutions: τ_0 on X_0 and τ' on X'. The involution τ_0 is uniquely determined by the properties $\tau_0(\omega_0) = -\omega_0$ and τ_0 permutes the endpoints of σ (namely, P and Q). Hence τ_0 is the elliptic involution and has in particular 4 fixed points: the midpoint of σ and three fixed points of τ. Since τ has 4 fixed points, τ' has exactly 2 fixed points: the midpoint of σ_0 (σ_0 is invariant by τ) and the midpoint of σ'.

Let ι be the hyperelliptic involution of X'. Since ι has 6 fixed points, we derive $\tau' \neq \iota$. We claim that $\iota(\sigma_0) = -\sigma'$. Indeed, $\iota(\sigma_0)$ is a saddle connection such that $\omega'(\iota(\sigma_0)) = -\omega'(\sigma_0)$. Hence

$$\iota(\sigma_0) = -\sigma_0 \qquad \text{or} \qquad \iota(\sigma_0) = -\sigma'.$$

If $\iota(\sigma_0) = -\sigma_0$ then $\tau' \circ \iota(\sigma_0) = \sigma_0$, hence $\tau' \circ \iota$ is the identity map in the neighborhoods of P and Q. Therefore $\tau' \circ \iota = \mathrm{id}_{X'}$: this is a contradiction since we know that $\tau' \neq \iota$. Now the closed curve $\sigma_0 * (-\sigma')$ is preserved by ι, hence it is separating. Cut X' along this closed curve we obtain two flat tori (X_1, ω_1) and (X_2, ω_2). It is not difficult to see that X_1 and X_2 are exchanged by τ'. By construction, X is the connected sum of X_0, X_1, and X_2 which are glued together along the slits corresponding to $\sigma_0, \sigma_1, \sigma_2$.

The proof for the case $(X, \omega) \in \mathrm{Prym}(1, 1, 2)$ is similar, it follows the same lines as the above discussion. □

6. Collapsing surfaces to $\Omega E_D(4)$

The goal of this section is to establish the following theorem, which is a key step in the proof of Theorem A.

Theorem 6.1. *Let $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$. Let $\mathcal{C}$ be a connected component of $\Omega E_D(\kappa)$. If for every surface $(X,\omega) \in \mathcal{C}$ there is no admissible saddle connection on (X,ω) then $D \in \{9, 16\}$. More precisely, under this assumption, $\mathcal{C}$ contains one of the following surfaces (see Lemma 2.4):*

$$S_\kappa(1,1,-1), S_\kappa(1,1,1) \in \Omega E_9(\kappa), \qquad \textit{or} \qquad S_\kappa(1,2,0), S_{(2,2)}(2,1,0) \in \Omega E_{16}(\kappa).$$

Recall that as an immediate consequence we draw an upper bound on the number of connected components of $\Omega E_D(2,2)^{\mathrm{odd}}$ and $\Omega E_D(1,1,2)$ (see Section 7).

6.1. **Strategy of the proof of Theorem 6.1.** Let (X, ω) be a Prym eigenform and let σ_0 be a saddle connection satisfying Convention 1. In view of Corollary 5.4 we can assume that σ_0 has a twin or a double twin, say σ_1, otherwise the theorem is proved. Depending the strata, we will distinguish three cases as follows:

- $\kappa = (2,2)^{\mathrm{odd}}$:

Case A $\tau(\sigma_1) \neq -\sigma_1$ and $\sigma_1 \cup \tau(\sigma_1)$ is separating.
Case B $\tau(\sigma_1) \neq -\sigma_1$ and $\sigma_1 \cup \tau(\sigma_1)$ is non-separating.
Case C $\tau(\sigma_1) = -\sigma_1$.

- $\kappa = (1,1,2)$:

Case A σ_1 is a twin and $\sigma_0 \cup \sigma_1$ is separating.

Case B σ_1 is a twin and $\sigma_0 \cup \sigma_1$ is non-separating.
Case C σ_1 is a double twin ($\tau(\sigma_1) = -\sigma_1$).

We will first prove Theorem 6.1 under the assumption of **Case A**. This case is simpler since Lemma 5.7 applies: (X, ω) admits a three tori decomposition. In the other two cases, by Lemma 6.2 D is a square. We then prove that **Case B** and **Case C** can be reduced to **Case A**: this corresponds, respectively, to Sections 6.3 and 6.4, respectively.

6.2. **Proof of Theorem 6.1 under the assumption of Case A.** From now on we will assume that for any $(X, \omega) \in \mathcal{C} \subset \Omega E_D(\kappa)$, there exists no admissible saddle connection. Let σ_2 be the image of σ_1 by the Prym involution τ. Thanks to Lemma 5.7 (X, ω) admits a three tori decomposition: X_0 is preserved and X_1, X_2 are exchanged by the Prym involution τ.

Claim 1. *There exists $(X, \omega) \in \mathcal{C}$ such that the horizontal direction is periodic on the tori (X_0, X_1, X_2).*

Proof of Claim 1. By moving in the kernel foliation leaf and using $\mathrm{GL}^+(2, \mathbb{R})$ action, we can assume that the slits σ_i are parallel to a simple closed geodesic in (X_0, ω_0) which is horizontal. Since (X, ω) is completely periodic in the sense of Calta (see [LN13a]), the claim follows. □

In the sequel we assume that (X, ω) is decomposed into three horizontal cylinders, say C_0, C_1, C_2, along the saddle connections $\sigma_0, \sigma_1, \sigma_2$, where C_0 is fixed, and C_1, C_2 are exchanged by the Prym involution. We let $s = |\sigma_i|$. We denote by ℓ_i, h_i the width and the height of the cylinder C_i. Obviously $\ell_1 = \ell_2$ and $h_1 = h_2$ (see Figure 1 for the notations).

6.2.1. *Case $\kappa = (2, 2)^{\mathrm{odd}}$.*

Claim 2. *One of the following two equalities holds: $h_0 = h_1$ or $h_0 = 2h_1$.*

Proof of Claim 2. Let δ be a saddle connection in C_0 joining P to Q which crosses the core curve of C_0 only once. Note that δ is anti-invariant by τ. Using $U = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right),\ t \in \mathbb{R}\}$, we can assume that δ is vertical. By assumption, δ is not admissible. Changing the length of the slits (the length of σ_i) if necessary and using the argument in Lemma 5.2, we can assume that δ has a twin δ'. Remark that δ' must intersect $C_1 \cup C_2$, therefore we have $|\delta'| = mh_1 + nh_0$ with $m \in \mathbb{Z}$, $m \geq 1$. The condition $|\delta'| = |\delta|$ implies $n = 0$. Thus $h_0 = mh_1$.

Assume that $h_0 > 2h_1$. Let η_1 be a geodesic segment in C_1 joining P to the midpoint of σ_0. Set $\eta_2 = \tau(\eta_1)$ and $\eta = \eta_1 \cup \eta_2$. Observe that η is a saddle connection invariant by τ. Again, by using the subgroup U we assume that η is vertical. Hence $|\eta| = 2h_1 < h_0$. Clearly any other vertical (upward) geodesic ray starting from P must intersect C_0. Thus, if there exists another vertical saddle connection η' joining P to Q, we must have $|\eta'| \geq h_0 > 2h_1 = |\eta|$. Hence η is admissible and the claim is proved. □

Claim 3. *If $h_0 = h_1$ then, either:*

(1) *$\ell_0 = \ell_1$ and (X, ω) is contained in the same component as $S_\kappa(1, 1, -1) \in \Omega E_9(\kappa)$, or*
(2) *$\ell_0 = 2\ell_1$ and (X, ω) is contained in the same component as $S_\kappa(1, 2, 0) \in \Omega E_{16}(\kappa)$.*

Proof of Claim 3. Let $h := h_0 = h_1$. Up to the action of the horocycle flow we assume that δ_0 is a saddle connection in C_0 joining P to Q such that $\omega(\delta_0) = (0, h)$. Since δ_0 is not admissible, there exists a vertical saddle connection δ_1 joining P and Q such that $\omega(\delta_1) = (0, \lambda)$, where $0 < \lambda \leq h$. Actually $\omega(\delta_1) = (0, h)$. Since δ_1 cannot be contained in C_0, δ_1 is contained in C_1. Thus $\delta_2 = \tau(\delta_1)$ is contained in C_2. Let γ be the saddle connection contained in $\overline{C}_1 \cup \overline{C}_2$ joining P to Q and passing through the midpoint of σ_0 as shown in Figure 5. By assumption there exists another saddle connection γ' joining P to Q parallel to γ such that $|\gamma'| \leq |\gamma|$. We claim that either γ' or $\tau(\gamma')$ starts in C_0. This is clear if γ' is not invariant by τ. If γ' is invariant by τ and starts from C_2 then it must end in C_1 (since $\tau(C_2) = C_1$). In particular γ' must cross C_0 at least once. Hence the vertical coordinate $\omega(\gamma')$ is greater than $2h$. Therefore $|\gamma'| > |\gamma|$ that is a contradiction.

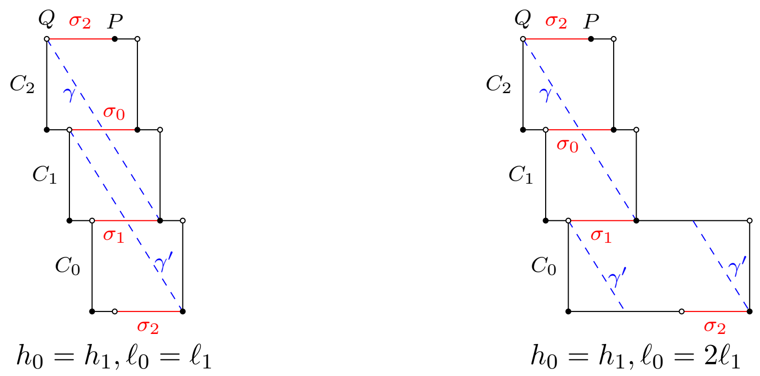

FIGURE 5. Claim 3: $h_1 = h_0$: the surfaces $S_\kappa(1,1,-1)$ and $\mathrm{diag}(1, \frac{1}{2}) \cdot S_\kappa(1,2,0)$.

We can now suppose that γ' starts in C_0. Observe that $\omega(\gamma) = (s - 2\ell_1, 2h)$.

If γ' is not contained in $\overline{C}_0 = X_0$ (Figure 5, left), γ' must end up in C_1. Since $|\gamma'| \leq |\gamma|$ elementary calculation shows that $\omega(\gamma') = (s - \ell_0 - \ell_1, 2h)$. Now γ and γ' are parallel, thus $\ell_0 = \ell_1$ and $(X, \omega) = S_\kappa(1,1,-1)$.

If γ' is contained in $\overline{C}_0$ (Figure 5, right), γ' must intersect twice the core curve of C_0. Thus $\omega(\gamma') = (s - \ell_0, 2h)$, from which we deduce $\ell_0 = 2\ell_1$ and $(X, \omega) = \mathrm{diag}(1, \frac{1}{2}) \cdot S_\kappa(1,2,0)$. The claim is proved. □

Claim 4. *If $h_0 = 2h_1$ then, either:*

(1) *$\ell_0 = \ell_1$ and (X, ω) is contained in the same component as $S_\kappa(2,1,0) \in \Omega E_{16}(\kappa)$, or*
(2) *$\ell_0 = 2\ell_1$ and (X, ω) is contained in the same component as $S_\kappa(1,1,1) \in \Omega E_9(\kappa)$.*

Proof of Claim 4. Let $h := h_1 = h_0/2$. Let δ_1 be a geodesic segment, contained in C_1, joining the midpoint of σ_0 to P. Using the horocycle flow we assume δ_1 to be vertical. Set $\delta_2 = \tau(\delta_1)$ and $\delta = \delta_1 \cup \delta_2$. By construction δ is a saddle connection which is invariant under τ. By assumption, there exists another vertical saddle connection δ' joining P to Q such that $|\delta'| \leq |\delta|$. It is easy to see that any other vertical geodesic ray emanating from P intersects the core curve of C_0. Since $\omega(\delta) = (0, 2h)$ and $h_0 = 2h$, δ' is contained in C_0.

Now let γ be the saddle connection in $\overline{C}_1 \cup \overline{C}_2$ passing through the midpoint of σ_0, joining P to Q such that $\omega(\gamma) = (2\ell_1, 2h)$ (see Figure 6 below).

By assumption, there exists a saddle connection γ' in the same direction as γ such that $|\gamma'| \leq |\gamma|$. As in Claim 3 either γ' or $\tau(\gamma')$ starts in C_0. The proof follows the same lines. Up to permutation, γ' starts in C_0. Since $|\gamma'| \leq |\gamma|$ and γ is parallel to γ', γ' is actually contained in C_0. In particular $\omega(\gamma') = (k\ell_0, 2h)$ for some $k \in \mathbb{Z}$, $k \geq 1$ and $\omega(\gamma') = \omega(\gamma)$. We draw $2\ell_1 = k\ell_0$. Now we claim that the inequality

$$\ell_0 \geq \ell_1$$

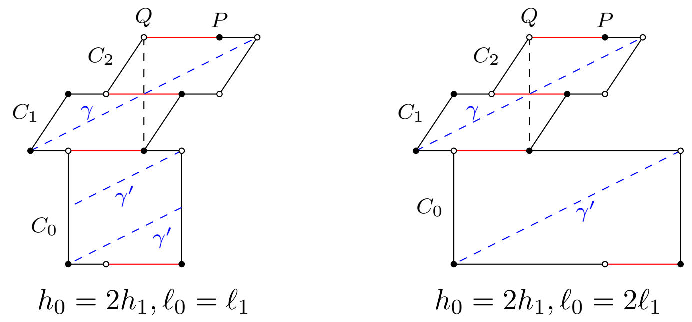

FIGURE 6. Claim 4: $h_0 = 2h_1$: the surface on the left belongs to component of $S_\kappa(2,1,0)$, and on the right belongs to the component of $S_\kappa(1,1,1)$.

holds. Indeed there exists a horizontal saddle connection σ'_0 in X_0 such that $\sigma_1 \cup \sigma'_0$ and $\sigma_2 \cup \sigma'_0$ are the two boundaries components of the cylinder C_0. Similarly, there exists a pair of horizontal saddle connections σ'_1, σ'_2 where σ'_i is contained in X_i such that $\sigma'_i \cup \sigma_0$ is a boundary component of C_i. By construction we have $\tau(\sigma'_0) = -\sigma'_0$, $\tau(\sigma'_1) = -\sigma'_2$, and

$$\ell_0 = |\sigma_1| + |\sigma'_0| \qquad \text{and} \qquad \ell_1 = |\sigma_0| + |\sigma'_1| = |\sigma_0| + |\sigma'_2|.$$

If $\ell_0 < \ell_1$ then $|\sigma'_0| < |\sigma'_1| = |\sigma'_2|$. Hence σ'_0 is admissible, contradicting our assumption.

In conclusion $2\ell_1 = k\ell_0 \geq \ell_1$ implies $\ell_0 = \ell_1$, or $\ell_0 = 2\ell_1$. The corresponding surfaces are represented in Figure 6. It is not hard to check that those two surfaces belong to the same connected component that $S_\kappa(2,1,0)$ and $S_\kappa(1,1,1)$, respectively. The claim is proved. □

6.2.2. *Case* $\kappa = (1,1,2)$.

Claim 5. *One of the following two equalities holds:* $h_0 = h_1$, *or* $h_0 = 2h_1$.

Proof of Claim 5. The proof of this claim follows the same lines as the proof of Claim 2. □

Claim 6. *If* $h_0 = h_1$ *then, either*

(1) $\ell_0 = \ell_1$, *and* (X,ω) *is contained in the same component as* $S_\kappa(1,1,-1) \in \Omega E_9(\kappa)$, *or*
(2) $\ell_0 = 2\ell_1$, *and* (X,ω) *is contained in the same component as* $S_\kappa(1,2,0) \in \Omega E_{16}(\kappa)$.

Proof of Claim 6. Set $h := h_0 = h_1$. We first consider a saddle connection δ contained in C_0, joining R_1 to Q and intersecting the core curve of C_0 only once. As usual we assume δ to be vertical (hence $\omega(\delta) = (0,h)$). By assumption δ has a twin or a double-twin δ_1 (necessarily δ_1 starts in C_1). Clearly δ_1 is a twin: otherwise it must end in C_2, hence it must cross the core curve of C_0 at least once. In particular its length satisfies $|\delta_1| \geq 3h > 2|\delta|$ that is a contradiction.

Let γ be the saddle connection contained in C_1 and joining R_1 to Q, as shown in Figure 7 below.

By assumption $\omega(\gamma) = (\ell_1, h)$. Now there exists another saddle connection γ' starting from R_1 in the same direction as γ. Observe that γ' must start in C_0. Using Lemma 5.2, we can assume that γ' is either a twin or a double-twin of γ.

If γ' is a twin of γ then $\omega(\gamma') = (\ell_1, h)$. Hence $\ell_0 = \ell_1$: $(X,\omega) = S_\kappa(1,1,-1)$.

If γ' is a double-twin of γ then $\omega(\gamma') = (2\ell_1, 2h)$. Hence γ' crosses twice the core curves of C_0 implying that $\ell_0 = 2\ell_1$: $(X,\omega) = \mathrm{diag}(1,\frac{1}{2}) \cdot S_\kappa(1,2,0)$. This proves the claim. □

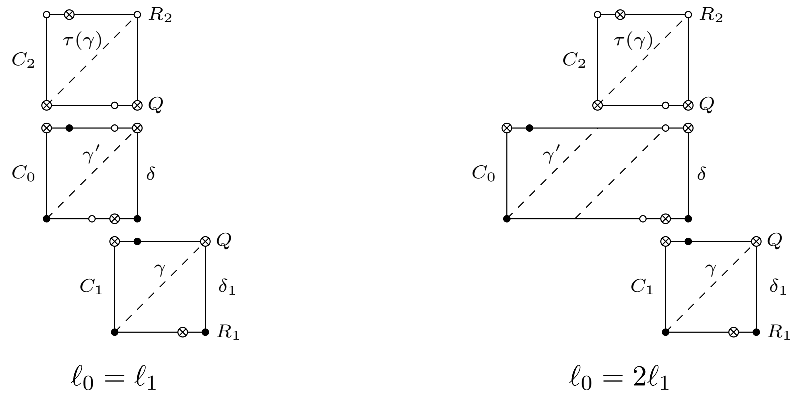

FIGURE 7. Claim 6: $h_0 = h_1$: the surfaces $S_\kappa(1,1,-1)$ and $\mathrm{diag}(1,\frac{1}{2}) \cdot S_\kappa(1,2,0)$.

Claim 7. *If $h_0 = 2h_1$ then $\ell_0 = 2\ell_1$. In addition (X,ω) belongs to the same connected component as $S_\kappa(1,1,1) \in \Omega E_9(\kappa)$.*

Proof. Set $h = h_1$. Let δ_1 be a saddle connection contained in C_1 joining R_1 to Q and intersecting the core curve of C_1 only once. We can suppose that δ_1 is vertical. By assumption, there exists another vertical saddle connection δ starting from R_1. Observe that δ must intersect the core curve of C_0, thus we have $|\delta| \geq h_0 = 2|\delta_1|$. By assumption, δ must be a double-twin of δ_1, which means that δ joins R_1 to R_2 and is contained in C_0.

Now, let γ be the saddle connection in C_1 joining R_1 to Q as shown in Figure 8.

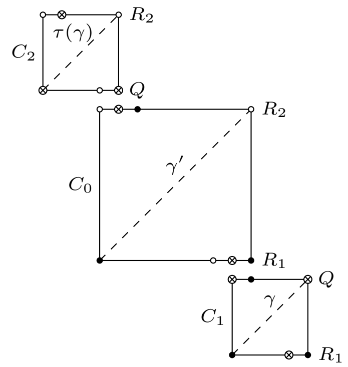

FIGURE 8. Claim 7: $h_0 = 2h_1$: the surface $S_\kappa(1,1,1)$

By assumption, there exists a saddle connection γ' starting from R_1 and parallel to γ. The same argument as above shows that γ' is a double-twin of γ and is contained in C_0. It follows that $\ell_0 = 2\ell_1$. Thus (X,ω) belongs to the component of $S_\kappa(1,1,1)$. The claim is proved. □

6.3. **Reduction from Case B to Case A.** Let $(X,\omega) \in \Omega E_D(\kappa)$ and let σ_0 be a saddle connection in X satisfying Convention 1. We suppose that σ_0 has a twin σ_1. Moreover, if $\kappa = (2,2)^{\mathrm{odd}}$ we assume that

$\sigma_2 := -\tau(\sigma_1) \neq \sigma_1$ and $\sigma_1 \cup \sigma_2$ is non-separating, and if $\kappa = (1,1,2)$ we assume that σ_0, σ_1 is non-separating. Our aim is to show that there exists in the component of (X,ω) a surface having a family of homologous saddle connections satisfying Case A (this is Lemma 6.3). We first show

Lemma 6.2. *Let $(X,\omega) \in \Omega E_D(\kappa)$, where $\kappa \in \{(2,2)^{\mathrm{odd}}, (1,1,2)\}$. If there exists $c \in H_1(X,\mathbb{Z})^-$ satisfying $c \neq 0$ and $\omega(c) = 0$ then D is a square. In particular, up to rescaling by $\mathrm{GL}^+(2,\mathbb{R})$, all the absolute periods of ω belong to $\mathbb{Q} + \imath\mathbb{Q}$.*

Proof of Lemma 6.2. We can assume that c is primitive in $H_1(X,\mathbb{Z})$, that is for any $n \in \mathbb{N}, n > 1, \frac{1}{n}c \notin H_1(X,\mathbb{Z})$. Pick a symplectic basis $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ of $H_1(X,\mathbb{Z})^-$ with $\beta_2 = c$. Set $\mu_i = \langle \alpha_i, \beta_i \rangle$, where $\langle , \rangle$ is the intersection form of $H_1(X,\mathbb{Z})$, and $\omega(\alpha_1) = x_1 + \imath y_1, \omega(\beta_1) = z_1 + \imath t_1, \omega(\alpha_2) = x_2 + \imath y_2$. Since

$$\mathrm{Area}(X,\omega) = \mu_1 \det \begin{pmatrix} x_1 & z_1 \\ y_1 & t_1 \end{pmatrix} + \mu_2 \det \begin{pmatrix} x_2 & 0 \\ y_2 & 0 \end{pmatrix} = \mu_1 \det \begin{pmatrix} x_1 & z_1 \\ y_1 & t_1 \end{pmatrix} > 0$$

it follows that $(x_1 + \imath y_1, z_1 + \imath t_1)$ is a basis of $\mathbb{R}^2$. Using $\mathrm{GL}^+(2,\mathbb{R})$ we can assume that $(x_1, y_1) = (1,0)$ and $(z_1, t_1) = (0,1)$. By [LN11, Proposition 4.2] there exists a unique generator T of $\mathcal{O}_D$ which is given in the basis $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ by a matrix of the form

$$T = \begin{pmatrix} e\mathrm{Id}_2 & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\ \frac{\mu_1}{\mu_2}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) & 0 \end{pmatrix}$$

with $(a,b,c,d,e) \in \mathbb{Z}^5$ such that $T^*\omega = \lambda\omega$ and $\lambda > 0$. Observe that the discriminant D satisfies $D = e^2 - 4\frac{\mu_1}{\mu_2}(bc - ad)$. Since $\mathrm{Re}(\omega) = (1, 0, x_2, 0)$ and $\mathrm{Im}(\omega) = (0, 1, y_2, 0)$ in the above coordinates, direct computations show that $b = d = 0$. Hence $D = e^2$ and $(x_2, y_2) \in \mathbb{Q}^2$. The lemma is proved. □

Lemma 6.3. *Assume that $\kappa = (2,2)^{\mathrm{odd}}$. Then there exists in the component of (X,ω) a surface having a triple of homologous saddle connections $\gamma_0, \gamma_1, \gamma_2$, where γ_0 is invariant, γ_1 and γ_2 are exchanged by the involution.*

Proof. Let c_0, c_1, and c_2 denote the simple closed curves $\sigma_1 * (-\sigma_2), \sigma_0 * (-\sigma_1)$ and $\sigma_0 * (-\sigma_2)$ respectively. Note that we have $c_0 = c_2 - c_1$ and $\tau(c_1) = -c_2$. By assumption $0 \neq c_0 \in H_1(X,\mathbb{Z})$. If $0 = c_1 \in H_1(X,\mathbb{Z})$ then $c_2 = -\tau(c_1) = 0 \in H_1(X,\mathbb{Z})$, which implies that $c_0 = 0 \in H_1(X,\mathbb{Z})$. Thus we can conclude that all of the curves c_0, c_1, c_2 are non-separating.

Cut X along $\sigma_0, \sigma_1, \sigma_2$, we obtain a connected surface whose boundary has three components corresponding to c_0, c_1, c_2. Gluing the pair of geodesic segments in each boundary component together, we get a closed translation surface (X', ω') with three marked geodesic segments. Since the angle between two consecutive twin saddle connections is 2π, we derive that ω' has no zeros, thus (X', ω') must be a torus. We denote the geodesic segments in X' corresponding to c_0, c_1, c_2 by c'_0, c'_1, c'_2 respectively. The involution τ of X induces an involution τ' on X', which leaves c'_0 invariant and exchanges c'_1 and c'_2. Let P'_i and Q'_i, $i = 0,1,2$, denote the endpoints of c'_i, where P'_i (respectively, Q'_i) corresponds to P (respectively, to Q).

Observe that as (X,ω) moves in the leaf of the kernel foliation, the surface (X', ω') is the same, only the segments c'_i vary. Therefore we can assume that (X', ω') is the standard torus $\mathbb{C}/(\mathbb{Z} \oplus \imath\mathbb{Z})$, and the length of c'_0 is small. Let δ_1 (respectively, η_1) denote the geodesic segment of minimal length from P'_0 to P'_1 (respectively, from Q'_0 to Q'_1). Note that as (X,ω) moves in the kernel foliation leaf, $\omega'(\delta_1)$ and $\omega'(\eta_1)$ are invariant. Therefore, we can assume that δ_1 and c'_0 are not parallel.

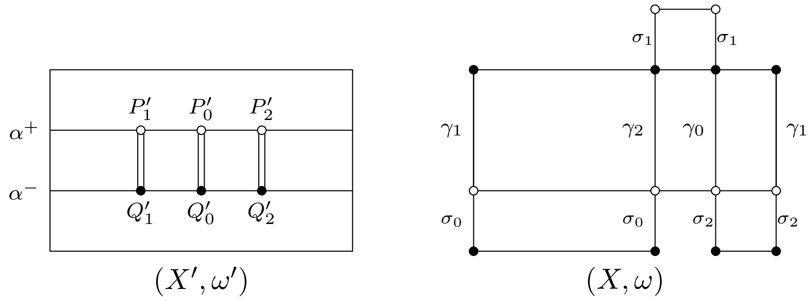

FIGURE 9. Cylinders decomposition in direction of $\alpha^{\pm}$.

Since c'_0 and c'_1 are parallel and have the same length, we see that $c'_0 \cup \delta_1 \cup c'_1 \cup \eta_1$ is the boundary of an embedded parallelogram in X'. It follows in particular that δ_1 and η_1 are parallel and have the same length. Let $\delta_2 = \tau'(\eta_1)$ and $\eta_2 = \tau'(\delta_1)$. We have $\delta = \delta_1 \cup \delta_2$ is a geodesic segment joining P'_1 to P'_2, and $\eta = \eta_1 \cup \eta_2$ is a geodesic segment joining Q'_1 to Q'_2.

Since $\omega(c_0) = 0$, By Lemma 6.2, for any $c \in H_1(X, \mathbb{Z})$, we have $\omega(c) \in \mathbb{Q} + \imath\mathbb{Q}$. Therefore $\omega'(\delta) = \omega'(\eta) \in \mathbb{Q} + \imath\mathbb{Q}$. Since X' is the standard torus, there exists a pair of parallel simple closed geodesics α^+, α^- of X' such that $\delta \subset \alpha^+$, and $\eta \subset \alpha^-$ (see Figure 9). When $|c'_i|$ is small enough and non-parallel to $\alpha^{\pm}$, the geodesics α^+ and α^- cut X' into two cylinders, one of which contains all the segments c'_0, c'_1, c'_2.

Recall that (X, ω) is obtained from (X', ω') by slitting along c'_0, c'_1, c'_2, and regluing the geodesic segments in the boundary. By construction, we see that (X, ω) admits a decomposition into four cylinders in the direction of $\alpha^{\pm}$ as shown in Figure 9. Let C_0 denote the largest cylinder in this decomposition, then it is easy to see that there exist in C_0 three homologous saddle connections $\gamma_0, \gamma_1, \gamma_2$ such that $\tau(\gamma_0) = -\gamma_0$, $\tau(\gamma_1) = -\gamma_2$, and $\gamma_1 \cup \gamma_2$ is a separating curve as desired. □

Lemma 6.4. *Assume that $\kappa = (1,1,2)$ and σ_0 has a twin σ_1 such that the curve $\sigma_0 * (-\sigma_1)$ is non-separating. Then there exists in the component of (X, ω) a surface having two pairs of homologous saddle connections (σ'_1, σ''_1) and (σ'_2, σ''_2), where σ'_i and σ''_i join the simple zero R_i to the double zero Q, and $\{\sigma'_2, \sigma''_2\} = \tau(\{\sigma'_1, \sigma''_1\})$.*

Proof. We will use similar ideas to the proof of Lemma 6.3. Let $c_1 = \sigma_0 * (-\sigma_1)$ and $c_2 = \tau(c_1)$. By the cutting-gluing construction along c_1 and c_2 (using the assumption that c_1 is non-separating), we get a flat torus (X', ω') with three marked geodesic segments c'_1, c'_2, c'' such that $\omega'(c'_1) = \omega'(c'_2) = 1/2\omega'(c'')$ (see Figure 10). For $i = 1, 2$, we denote the endpoints of c'_i by R'_i and Q'_i so that R'_i corresponds to R_i and Q'_i corresponds to Q. We denote the endpoints of c'' by R''_1 and R''_2 such that R''_i corresponds to R_i. The midpoint of c'' corresponds to Q, we denote it by Q''. We denote the subsegment of c'' between Q'' and R''_i by c''_i. The Prym involution of X gives rise to an involution τ' of X' which satisfies $\tau'(c'_1) = -c'_2, \tau'(c''_1) = -c''_2$.

As (X, ω) moves in the leaf of the kernel foliation, the surface (X', ω') remains the same, only the segments c'_1, c'_2, c'' vary. Therefore we can assume that c'_1, c'_2, c'' are contained in three distinct parallel simple closed geodesics of X'. Changing the direction of c'' slightly, we see that there exist geodesic segments σ'_i from Q'_i to R'_i, and σ''_i from Q'' to R''_i, $i = 1, 2$, (see Figure 10) such that

- $\tau'(\sigma'_1) = -\sigma'_2$ and $\tau'(\sigma''_1) = -\sigma''_2$,
- $\sigma'_i * c'_i$ and $\sigma''_i * c''_i$ are homologous.

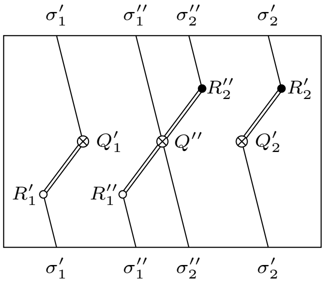

FIGURE 10

Reconstruct (X,ω) from (X',ω') we see that σ'_i and σ''_i are homologous saddle connections, and the pairs (σ'_1,σ''_1) and (σ'_2,σ''_2) have the desired properties. □

6.4. **Reduction from Case C to Case A.** Let (X,ω) be a Prym eigenform in $\mathrm{Prym}(2,2)^{\mathrm{odd}}\sqcup\mathrm{Prym}(1,1,2)$, and let σ_0 be a saddle connection on X satisfying Convention 1. If $(X,\omega)\in\mathrm{Prym}(2,2)^{\mathrm{odd}}$ we suppose that σ_0 has a twin σ_1 which is also invariant by the Prym involution τ, and if $(X,\omega)\in\mathrm{Prym}(1,1,2)$ we suppose σ_0 has a double twin σ_1 (which is invariant by τ). Our aim is to show that there exists in the component of (X,ω) a surface having a family of saddle connections satisfying Case A for both $\kappa=(2,2)^{\mathrm{odd}}$ and $\kappa=(1,1,2)$. We first show

Lemma 6.5. *Define*

- $c=\sigma_0*(-\sigma_1)$, *if* $(X,\omega)\in$ *Prym*$(2,2)^{\mathrm{odd}}$,
- $c=\sigma_0*\tau(s_0)*(-\sigma_1)$, *if* $(X,\omega)\in$ *Prym*$(1,1,2)$.

Then in both cases, we have $c\neq 0\in H_1(X,\mathbb{Z})^-$, *and* $\omega(c)=0$.

Proof. From the definition of c, we have $\tau(c)=-c$, hence $c\in H_1(X,\mathbb{Z})^-$. It is also clear that $\omega(c)=0$. All we need to show is that $c\neq 0\in H_1(X,\mathbb{Z})$.

We first consider the case $(X,\omega)\in\mathrm{Prym}(2,2)^{\mathrm{odd}}$. Remark that the pair of angles at P and Q determined σ_0 and σ_1 is $(2\pi,4\pi)$. Since $\tau(\sigma_0)=-\sigma_0$ and $\tau(\sigma_1)=-\sigma_1$ we see that the angle 2π at P and the angle 4π at Q belong to the same side of c, and vice versa. Cutting X along c we get a surface whose boundary has two components, each of which is a union of two geodesic segments corresponding to σ_0 and σ_1. Since σ_0 and σ_1 are twins, the two segments in each component has the same length, therefore we can glue them together to get a closed (possibly disconnected) translation surface (X',ω') with two marked geodesic segments η_1,η_2. If the new surface is disconnected, then each component is a translation surface with only one singularity of angle 4π. Since such a surface does not exist, we conclude that X' is connected, and hence $c\neq 0$ in $H_1(X,\mathbb{Z})$.

For the case $(X,\omega)\in\mathrm{Prym}(1,1,2)$, by a similar construction, that is cutting along c, then closing the boundary components of the new surface (by gluing the path corresponding to $\sigma_0\cup\tau(\sigma_0)$ and the segment corresponding to σ_1), we also get a translation surface (X',ω') having two singularities with cone angle 4π. The same argument as above shows that this surface belongs to $\mathcal{H}(1,1)$, therefore $c\neq 0\in H_1(X,\mathbb{Z})$. □

Lemma 6.6. *Let* $(X,\omega)\in$ *Prym*$(2,2)^{\mathrm{odd}}$ *be a Prym eigenform having a twin* σ_1 *of* σ_0 *that is invariant by* τ. *Then one can find in the connected component of* (X,ω) *another surface having a triple of homologous saddle connections.*

Proof of Lemma 6.6. Cutting X along $c = \sigma_0 \cup \sigma_1$ and gluing the two segments of each boundary component together, we get a closed translation surface $(X', \omega') \in \mathcal{H}(1,1)$ (see the proof of Lemma 6.5). By construction there exist on X' a pair of disjoint geodesic segments η_1, η_2 such that $\omega'(\eta_1) = \omega'(\eta_2) = \omega(\sigma_0)$ and η_i joins a zero of ω' to a regular point. Let $(P_i, Q_i)_{i=1,2}$ denote the endpoints of η_i, where P_i (respectively, Q_i) corresponds to P (respectively, to Q). The numbering is chosen so that P_1 and Q_2 are the zeros of ω'. The involution τ of X descends to an involution of X' exchanging η_1 and η_2. We denote this involution by τ'. Remark that τ' has two fixed points in X', none of which are contained in the segments η_1, η_2. Note also that as (X, ω) moves in its leaf of the kernel foliation, (X', ω') also moves in its leaf of the kernel foliation in $\mathcal{H}(1,1)$ (only the relative periods change).

Let ι be the hyperelliptic involution of X'. Since ι has six fixed points but τ' has two, we have $\iota \neq \tau$. Remark that $\iota \circ \tau$ is also an involution of X' satisfying $(\iota \circ \tau)^*\omega' = \omega'$. The surface $X'' = X'/\langle \iota \circ \tau \rangle$ is an elliptic curve. Let π is the branched covering $\pi : X' \to X''$, which is is ramified at P_1 and Q_2. Then ω' descends to a holomorphic 1-form ω'' on X'' so that $\omega' = \pi^*\omega''$. For $i = 1, 2$ let P_i'', Q_i'', η_i'' denote the images of P_i, Q_i, η_i in X''. Note that we have $\omega''(\eta_1'') = \omega''(\eta_2'') = \omega(\sigma_0)$.

We consider the tuple $(X'', \omega'', P_1'', P_2'')$ as an element in $\mathcal{H}(0,0)$, that is the moduli space of flat tori with two marked points. We first observe that as (X, ω) moves in its leaf of the kernel foliation, the corresponding surfaces $(X'', \omega'', P_1'', P_2'')$ are the same in $\mathcal{H}(0,0)$ (only $\omega''(\eta_1'') = \omega''(\eta_2'')$ change). Indeed all the coordinates of $(X'', \omega'', P_1'', P_2'')$ are determined by the absolute periods of (X, ω).

Let α_1'' be a simple closed geodesic of (X'', ω'') which passes through P_1'' and does not contain P_2''. Using $\mathrm{GL}(2, \mathbb{R})$, we can assume that α_1'' is horizontal. By moving in the kernel foliation leaf of (X, ω), we can also assume that η_i'' are parallel to α_1'' ($\omega''(\eta_i'') = \lambda\omega''(\alpha_1'')$, with $0 < \lambda < 1$). By construction, the surface (X', ω') admits a decomposition into cylinders in the horizontal direction. Note that X' must have three horizontal cylinders, otherwise there would be horizontal saddle connection joining P_1 to Q_2, which is excluded since α_1'' does not contain η_2'' (see Figure 11).

We can reconstruct (X, ω) from (X', ω'): one sees that (X, ω) also admits a decomposition into three horizontal cylinders (see Figure 11). Consider the surface $(\tilde{X}, \tilde{\omega}) = (X, \omega) + (0, -\epsilon)$, with $\epsilon > 0$ small as shown in Figure 11. We see that $(\tilde{X}, \tilde{\omega})$ admits a decomposition into four horizontal cylinders, three of which are simple. It is easy to check that there exists a triple of twin saddle connections $\gamma_0, \gamma_1, \gamma_2$ in the largest horizontal cylinder of $\tilde{X}$ (which is preserved by the Prym involution) which satisfy $\tau(\gamma_0) = -\gamma_0$, $\tau(\gamma_1) = -\gamma_2$, and $\gamma_1 \cup \gamma_2$ is a separating curve. This proves the lemma. □

Lemma 6.7. *Let $(X, \omega) \in$ Prym$(1,1,2)$ be a Prym eigenform having a double twin σ_1 of σ_0. Then one can find in the connected component of (X, ω) a surface having two pairs of homologous saddle connections (σ_1', σ_1'') and (σ_2', σ_2'') that are exchanged by the Prym involution.*

Proof. Let $(X', \omega') \in \mathcal{H}(1,1)$ be the surface obtained by the "cutting-gluing" construction along $c = \sigma_0 * \tau(\sigma_0) * (-\sigma_1)$ (see Lemma 6.5). Note that we have on X' two marked and disjoint geodesic segments η_1 and η_2 (corresponding to c) such that the midpoint of η_i is a zero of ω', and $\omega'(\eta_i) = \omega(\sigma_1) = 2\omega(\sigma_0)$. We will consider η_i as a slit with two sides η_i' and η_i'', where η_i' corresponds to σ_1, and η_i'' corresponds to the union $\sigma_0 \cup \tau(\sigma_0)$. Remark that the Prym involution τ induces an involution τ' on X' which is not the hyperelliptic involution. It follows that X' is a double cover of a torus.

By construction, as (X, ω) moves in its kernel foliation leaf (X', ω') is fixed, only the marked geodesic segments (slits) η_i are changed. Using $\mathrm{GL}^+(2, \mathbb{R})$, we can assume that X' is horizontally periodic, and it has three horizontal cylinders (one may use the fact that X' is a double cover of a torus to show that a periodic direction with three cylinder does exist). We can arrange so that the slits are also horizontal,

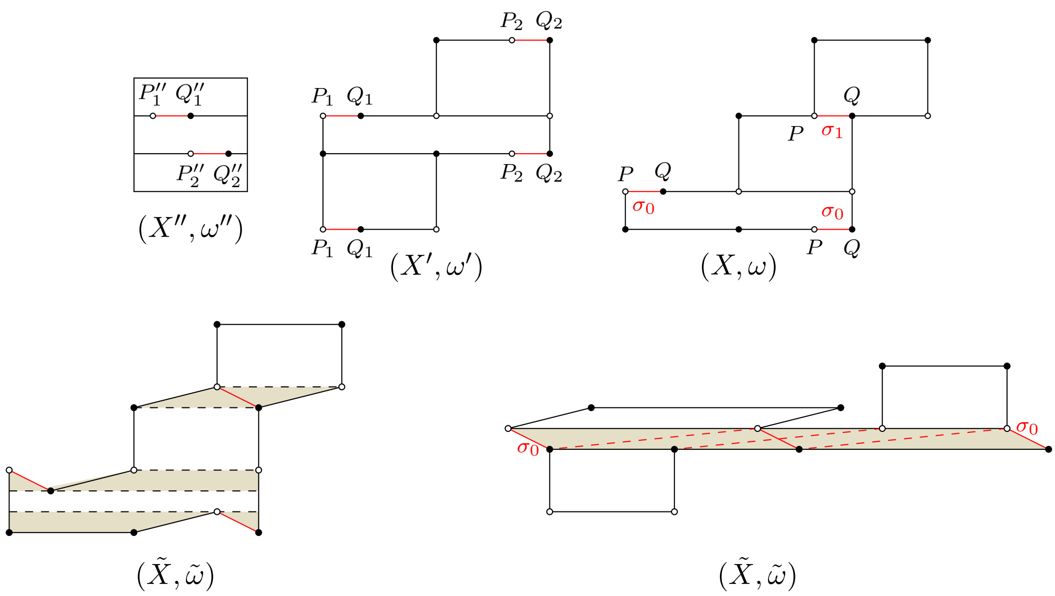

FIGURE 11. σ_0 and σ_1 are invariant by τ, X' admits two involutions: the hyperelliptic one ι which fixes each of the cylinders, and the involution τ' induced by τ which exchanges the pair simple cylinders and fixes the larger one. Observe that τ' exchanges $\eta_1 = \overline{P_1Q_1}$ and $\eta_2 = \overline{P_2Q_2}$.

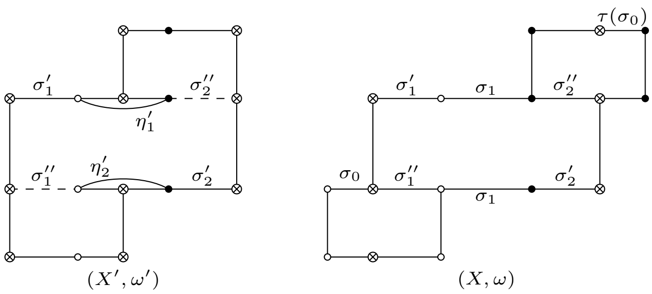

FIGURE 12. $\mathrm{Prym}(1,1,2)$ case C: σ_0 has a double-twin.

and η'_i are contained in the boundary of the largest cylinder (see Figure 12). Reconstruct (X,ω) from (X',ω'), we see that (X,ω) has two pairs of homologous saddle connections (σ'_1,σ''_1) and (σ'_2,σ''_2), such that $\tau(\sigma'_1) = -\sigma'_2, \tau(\sigma''_1) = \sigma''_2$, and $\sigma'_1 * \sigma_1 * \sigma''_2$ is homologous to the core curve of the largest horizontal cylinder in X. The lemma is then proved. □

7. PROOF OF THE MAIN RESULT

Let us now give the proof of our main theorem.

Proof of Theorem A. First of all $D \in \mathbb{N}$, $D \equiv 0, 1, 4 \bmod 8$, the loci $\Omega E_D(\kappa)$ are non empty: this is Corollary 2.5.

We now consider the cases $D = 0, 1, 4 \bmod 8, D \geq 9$, and $D \notin \{9, 16\}$. By Theorem 6.1 and Corollary 5.4, any component of $\Omega E_D(\kappa)$ contains a surface with an admissible saddle connection that collapse to a point in $\Omega E_D(4)$. Recall that $\Omega E_D(4)$ is a finite collection of Teichmüller discs. By Proposition 2.6 for any connected component $\mathcal{C}$ of $\Omega E_D(4)$, there exists at most one component of $\Omega E_D(\kappa)$ adjacent to $\mathcal{C}$ *i.e.* its closure contains $\mathcal{C}$. Therefore, the number of connected components of $\Omega E_D(\kappa)$ is bounded from above by the number of components of $\Omega E_D(4)$.

In particular, when $D \equiv 0, 4 \bmod 8$, since $\Omega E_D(4)$ is connected (see Theorem 2.2), so is $\Omega E_D(\kappa)$. For $D \equiv 1 \bmod 8$, by Theorem 2.2 we know that $\Omega E_D(4)$ has two components, so $\Omega E_D(\kappa)$ has at most two components. On the other hand, Theorem 4.1 tells us that $\Omega E_D(\kappa)$ cannot be connected. Thus we can conclude that $\Omega E_D(\kappa)$ has exactly two connected components.

For $D \equiv 5 \bmod 8$, if $\Omega E_D(\kappa)$ is non-empty then again Theorem 6.1, Corollary 5.4 and Proposition 5.5 implies that $\Omega E_D(4)$ is also non-empty, which contradicts Theorem 2.2.

We now consider the cases $D \in \{9, 16\}$. Since $\Omega E_D(4) = \varnothing$, by Proposition 5.5, there exists no admissible saddle connection on any surface in $\Omega E_9(\kappa) \sqcup \Omega E_{16}(\kappa)$. By Theorem 6.1, we see that any surface in $\Omega E_9(\kappa) \sqcup \Omega E_{16}(\kappa)$ belongs to the same component as one of the surfaces

$$S_\kappa(1,1,-1), S_\kappa(1,1,1), S_\kappa(1,2,0) \text{ or } S_{(2,2)}(2,1,0)$$

(see Lemma 2.4 for the definition).

It follows immediately that $\Omega E_{16}(2,2)$ and $\Omega E_9(\kappa)$ has at most two components and $\Omega E_{16}(1,1,2)$ is connected. The fact that $\Omega E_9(\kappa)$ is not connected is proved in Theorem 4.1. Hence $\Omega E_9(\kappa)$ has exactly two connected components.

It remains to prove that $\Omega E_{16}(2,2)$ is connected. It is sufficient to show that $S_{(2,2)}(1,2,0), S_{(2,2)}(2,1,0) \in \Omega E_{16}(2,2)^{\text{odd}}$ belong to the same component. We consider $(X_\varepsilon, \omega_\varepsilon) = S_{(2,2)}(2,1,0) + (0,\varepsilon)$, with $\varepsilon > 0$ small enough (see Figure 13).

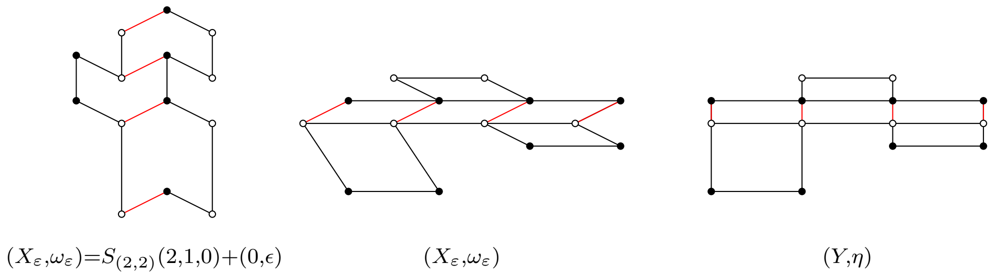

FIGURE 13. Connecting $S_{(2,2)}(2,1,0)$ to $S_{(2,2)}(1,2,0)$.

Observe that $(X_\varepsilon, \omega_\varepsilon)$ admits a decomposition into four horizontal cylinders. Moving horizontally in the kernel foliation leaf of $(X_\varepsilon, \omega_\varepsilon)$ we get a surface $(Y, \eta) = (X_\varepsilon, \omega_\varepsilon) + v$, with $v \in \mathbb{R} \times \{0\}$, which admits a decomposition into three vertical cylinders. It is not difficult to see that (Y, η) can be connected to $S_{(2,2)}(1,2,0)$ by using the action of $\mathrm{GL}^+(2,\mathbb{R})$ and moving in the kernel foliation leaves. The proof of Theorem A is now complete. □

As a direct corollary we prove Theorem B *i.e.* the existence in any component of $\Omega E_D(\kappa)$ of surfaces which admit three-tori decompositions.

Proof of Theorem B. Let $(w,h,e) \in \mathbb{Z}^3$ be as in Lemma 2.4 where $D = e^2 + 8wh$. We consider the corresponding surfaces $(X_\pm, \omega_\pm) := S_\kappa(w,h,\pm e)$. By Lemma 2.4 $(X_\pm,\omega_\pm) \in \Omega E_D(\kappa)$.

If $D \not\equiv 1 \bmod 8$ then by Theorem A, $\Omega E_D(\kappa)$ is connected and $(X_\pm,\omega_\pm)$ admits a three-tori decomposition.

If $D \equiv 1 \bmod 8$ then by Theorem A, $\Omega E_D(\kappa)$ has two connected components and from the proof of Theorem 4.1, (X_+,ω_+) and (X_-,ω_-) do not belong to the same connect component. This ends the proof of Theorem B. □

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