# Multiscale à Lyon

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GDT MathsinFluids, ENS Lyon, 15 fevrier 2019

## 1. Discovery method; exact formulas

$$\Box\, u^\epsilon = 0, \quad u^\epsilon_t\big|_{t=0} = 0, \quad u^\epsilon\big|_{t=0} := f(y)\, e^{iy_1/\epsilon}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

$$u^\epsilon = u^\epsilon_+ + u^\epsilon_-$$

$$\mathbf{u}^\epsilon_\pm(\mathbf{t},\mathbf{y}) := \int \hat{\mathbf{f}}\Big(\eta - \frac{\mathbf{e}_1}{\epsilon}\Big)\, \mathbf{e}^{\mathbf{i}(\mathbf{y}\eta \mp \mathbf{t}|\eta|)}\, \mathbf{d}\eta\,.$$

New variable $\zeta := \eta - \frac{\mathbf{e}_1}{\epsilon}, \quad \eta = (\mathbf{e}_1 + \epsilon\zeta)/\epsilon$

$$\mathbf{u}^\epsilon_+(\mathbf{t},\mathbf{y}) = \int \hat{\mathbf{f}}(\zeta)\, \mathbf{e}^{\mathbf{i}\mathbf{y}(\mathbf{e}_1+\epsilon\zeta)/\epsilon}\, \mathbf{e}^{-\mathbf{i}\mathbf{t}|\mathbf{e}_1+\epsilon\zeta|/\epsilon}\, \mathbf{d}\zeta\,.$$

Taylor approximation

$$|\,\mathbf{e}_1 + \epsilon\zeta\,| = 1 + \epsilon\zeta_1 + \ \mathbf{O}(\,\epsilon^2\,|\zeta|^2\,) \ \approx \ 1 + \epsilon\zeta_1 \quad \text{yields}$$

$$\mathbf{u}^\epsilon(\mathbf{t},\mathbf{y}) \ \approx \ \mathbf{e}^{\mathbf{i}(\mathbf{y}_1-\mathbf{t})/\epsilon}\, \mathbf{b}(\mathbf{t},\mathbf{y})\,,$$

$$\mathbf{b}(\mathbf{t},\mathbf{y}) := \int \hat{\mathbf{f}}(\zeta)\, \mathbf{e}^{\mathbf{i}(\mathbf{y}_1\zeta_1 - \mathbf{t}\zeta_1)}\, \mathbf{d}\zeta\,, \qquad \frac{\partial \mathbf{b}}{\partial \mathbf{t}} + \frac{\partial \mathbf{b}}{\partial \mathbf{y}_1} = 0\,.$$

*Propagation on rays. Oscillations yield direction of propagation.*

## 2. A standard recipe is dangerous

$$\mathbf{u_t}+\mathbf{u_x}+\mathbf{u}=0\,,\quad \mathbf{u}(0,\mathbf{x})=\mathbf{a}(\mathbf{x})\,\mathbf{e}^{\mathbf{i}\mathbf{x}/\epsilon},\quad \mathbf{u}(\mathbf{t},\mathbf{x})=\mathbf{e}^{-\mathbf{t}}\,\mathbf{a}(\mathbf{x}-\mathbf{t})\,\mathbf{e}^{\mathbf{i}(\mathbf{x}-\mathbf{t})/\epsilon}$$

Standard recipe has two elements.

- Examine size terms.
- You can ignore all but largest terms.

Size of terms for problem on first line are,

$$\mathbf{u_t}\ \sim\epsilon^{-1},\quad \mathbf{u_x}\sim\epsilon^{-1},\quad \mathbf{u}\sim\epsilon^{0}\ll\epsilon^{-1}\,.$$

Drop the small one. Yields equation $\mathbf{u_t}+\mathbf{u_x}=0$ with solution $\mathbf{a}(\mathbf{x}-\mathbf{t})\,\mathbf{e}^{\mathbf{i}(\mathbf{x}-\mathbf{t})/\epsilon}$. Error is 100 percent!

Compare with one of Kruskal's rules of asymptology. *In an expansion there must be more than one term of leading order*. Otherwise there is no way that the leading term can be compensated.

The leading terms must compensate. They compensate exactly in this example. So you *cannot ignore the lower order terms*!. Beware!

## 3. My projection obsession

$$\mathbf{L}(\mathbf{y},\partial_{\mathbf{y}}) \;=\; \partial_{\mathbf{t}} \;+\; \sum_{\mathbf{j}=1}^{\mathbf{d}} \mathbf{A}_{\mathbf{j}}\partial_{\mathbf{j}} + \mathbf{L}_0 \;:=\; \mathbf{L}_1(\partial) + \mathbf{L}_0, \quad \mathbf{A}_{\mathbf{j}} = \mathbf{A}_{\mathbf{j}}^{*}\,.$$

$$\mathbf{u}^{\epsilon} \;\sim\; \mathbf{e}^{\mathbf{i}\phi(\mathbf{y})/\epsilon} \sum_{\mathbf{j}=0}^{\infty} \epsilon^{\mathbf{j}}\, \mathbf{a}_{\mathbf{j}}(\mathbf{y}), \quad \text{want} \quad \mathbf{L}\,\mathbf{u}^{\epsilon} \;\sim\; 0\,.$$

$$\partial_{\mu}\big(\mathbf{e}^{\mathbf{i}\phi/\epsilon}\, \mathbf{a}_{\mathbf{j}}\big) \;=\; \mathbf{e}^{\mathbf{i}\phi/\epsilon}\, \Big(\frac{\mathbf{i}}{\epsilon}\partial_{\mu}\phi\; \mathbf{a}_{\mathbf{j}} \;+\; \partial_{\mu}\mathbf{a}_{\mathbf{j}}\Big)$$

*N.b. One order on left. Two on right. Typical multiscale analysis.*
Multiply by $\epsilon^{\mathbf{j}}\mathbf{A}_{\mu}$. Sum on $\mu$. Add $\mathbf{L}_0\mathbf{u}^{\epsilon}$ to find

$$\mathbf{L}(\mathbf{u}^{\epsilon}) \;=\; \mathbf{e}^{\mathbf{i}\phi/\epsilon}\Big(\frac{\mathbf{i}}{\epsilon}\,\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}))\,\mathbf{a}_0 + \Big(\mathbf{i}\,\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}))\,\mathbf{a}_1 + \mathbf{L}(\partial)\mathbf{a}_0\Big) + \cdots\Big)$$

$$\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}))\,\mathbf{a}_0 \;=\; 0, \qquad \mathbf{i}\,\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}))\,\mathbf{a}_1 + \mathbf{L}(\partial)\mathbf{a}_0 \;=\; 0 \quad \textbf{tricky!}$$

The matrix valued function $\mathbf{L}_1(\mathbf{d}\phi)$ must have a kernel so that there can be $\mathbf{a}_0 \neq 0$. It must not be surjective. Conclude

$$\mathbf{a}_0 \in \mathrm{Ker}\;\mathbf{L}_1(\mathbf{d}\phi), \qquad \mathbf{L}(\partial)\mathbf{a}_0 \;\in\; \mathrm{Range}\;\mathbf{L}_1(\mathbf{d}\phi)\,.$$

4. Deciphering $a_0 \in \mathrm{Ker}\ L_1(d\phi), \quad L(\partial)a_0 \in \mathrm{Range}\ L_1(d\phi)$.

$\det \mathbf{L}_1(\mathbf{d}\phi) = 0$ on a nice open set meeting $\mathbf{t} = 0$.

**Constant rank hypothesis.** $\mathrm{Rank}\,\mathbf{L}_1(\mathbf{d}\phi)$ is constant. ( **Ex.** $\phi$ is a linear function $\tau\mathbf{t} + \xi\mathbf{x}$.)

**Definition.** $\pi(\mathbf{y})$ is orthogonal projection on $\mathrm{Ker}\,\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}))$. $\mathbf{Q}(\mathbf{y})$ is a partial inverse defined by

$$\mathbf{Q}(\mathbf{y})\,\pi(\mathbf{y}) = 0, \qquad \mathbf{Q}(\mathbf{y})\,\mathbf{L}_1(\mathbf{d}\phi(\mathbf{y}) = \mathbf{I} - \pi(\mathbf{y})\,.$$

The equations to be unraveled are equivalent to

$$\big(\mathbf{I} - \pi(\mathbf{y})\big)\mathbf{a}_0 \;=\; 0, \qquad \pi\mathbf{L}(\partial)\pi\mathbf{a}_0 \;=\; 0\,.$$

Differentiate the first with respect to time and add to find that

$$\big(\mathbf{I} - \pi(\mathbf{y})\big)\partial_{\mathbf{t}}\mathbf{a}_0 \;+\; \pi(\mathbf{y})\mathbf{L}(\partial)\,\big(\pi(\mathbf{y})\mathbf{a}_0\big) \;-\; (\partial_{\mathbf{t}}\pi(\mathbf{y}))\mathbf{a}_0 \;=\; 0$$

A symmetric hyperbolic equation for $\mathbf{a}_0$. $\mathbf{a}_0$ determined from its initial values with $\pi\mathbf{a}_0 = 0$.

## 5. The Borel summation trick

More generally, given values of $\pi a_j(0,x)$ the $a_j$ are uniquely determined so that formally $L(\partial)\, e^{i\phi/\epsilon}\big(a_0 + \epsilon a_1 + \cdots\big) \sim 0$.

So what?

Choose a function $a(\epsilon, y)$ smooth in $y$ so the Taylor series at $\epsilon = 0$ is equal to $a_0 + \epsilon a_1 + \cdots$. Then $\forall N$

$$L\big(e^{i\phi/\epsilon}\, a(\epsilon, y)\big) \;=\; O(\epsilon^N) \quad \text{in} \quad C^\infty\,.$$

**Lax's Theorem (1957).** *Define $u(\epsilon, y)$ to be the solution of the Cauchy Problem with initial datum $e^{i\phi(0,x)/\epsilon}\, a(0,x)$. Then $\forall N$*

$$e^{i\phi/\epsilon}\, a - u(\epsilon, y) \;=\; O(\epsilon^N) \quad \text{in} \quad C^\infty.$$

Immediate from the energy estimates for hyperbolic operators applied to $u(\epsilon, y) - e^{i\phi/\epsilon}\, a(\epsilon, y)$. Data and $L$ are $O(\epsilon^N)$ all N.

## 6. Another failure of standard proceedure

- When you propose an approximate solution $u_{approx}$ to an equation $F(u) = 0$ you should *always* test to see how small $F(u_{approx})$ is.

- In multiscale analysis the leading approximation *always* fails this test.

- For this reason this test is *never* performed.

**Example.** If you keep only the leading term $u_{approx} := e^{i\phi/\epsilon}\, a_0$, then from $Lu$ the $\epsilon^{-1}$ is eliminated as well as "half" of the $\epsilon^0$ term. Therefore $Lu_{approx} = O(\epsilon^0)$ is comparable in size to $u$. Not small.

When the slowly vaying amplitude assumption is emplyed rather than a systematic two scale expansion, the smallness of the residual is never tested. It would fail.

## 7. Decyphering $\pi L \pi a_0 = 0$, rays and transport

$$\mathrm{Char}\, L := \left\{(\tau,\xi) \in \mathbb{R}^{1+d} \setminus 0 : \det L_1(\tau,\xi) = 0\right\}$$

$$\iff \exists \ \text{ plane wave solution } \ u = e^{i(\tau t + \xi x)} r, \quad L_1(\partial) u = 0$$

Mostly smooth points where locally $\tau = \tau(\xi)$, $\tau \in C^\omega$.

$$e^{i(\tau(\xi) t + \xi x)} r(\xi) \quad \mapsto \quad \text{group velocity} \quad v(\xi) = -\nabla_\xi \tau(\xi) \ \ \textit{Huh??}$$

$$L_1 = \partial_t + w \cdot \partial_x, \quad \tau + w \cdot \xi = 0, \quad \forall \xi, v(\xi) = w$$

$$L_1 = \partial_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_1 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \partial_2, \quad \tau^2 = |\xi|^2, \quad v(\xi) = \frac{\xi}{\tau} \in S^1$$

$$\text{For } \ \phi = \tau t + \xi x, \quad \pi L \pi \, a_0 = 0 \iff \left(\partial_t + v(\xi) \cdot \partial_x + \pi L_0 \pi\right) a_0 \ = \ 0$$

*Blackboard.* Raylike sollutions with initial conditions with distinct $\phi$ for transport and wave. Dispersion.

Application (Halpern-R). Amplifying layers for Berenger's PML. Including $\sigma(x)$.

## 8. Nonlinear interaction one phase

For a quasilinear system seek solutions $\underline{u} + \epsilon e^{i\phi/\epsilon}\ a(\epsilon, y)$.

$\underline{u}$ is constant so the linerization has constant coefficients.

The amplitude chosen so lifetime of smooth solution expected $O(1)$. Time scale of nonlinear interaction is 1.

Nonlinearity will generate harmonics $e^{in\phi/\epsilon}$ and the harmonics will interact. A cascade. Summing on $n \in \mathbb{Z}$ the series

$$\epsilon\ e^{in\phi/\epsilon}\big(a_0^n + \epsilon a_1^n + \cdots\big)$$

yields

$$\underline{u} + \epsilon U\big(y, \phi(y)/\epsilon\big), \quad U(y, \theta) \text{ periodic}$$

The profile equation is transport $\partial_t + v \cdot \partial_x$ along rays plus a quadratic Burgers nonlinear term in $\theta$.

The Burgers term on $S^1$ yields the cascade of energy to higher and higher frequencies. Before shocks. *Blackboard:* Typical spectra.

## 9. Examples of Resonance and Nonresonance

$$(\partial_t + \partial_y)\, u = 0\,, \qquad u(0,y) = a(y)\, e^{iy/\epsilon}\,,$$

$$(\partial_t - \partial_y)\, v = 0\,, \qquad v(0,y) = b(y)\, e^{i\phi(y)/\epsilon}\,,$$

$$\partial_t\, w = uv\,, \qquad w(0,y) = 0\,,$$

$$w = \int_0^t a(y-t)\, b(y+t)\, e^{i((y-t)+\phi(y+t))/\epsilon}\, dt\,.$$

**Example 1.** $\phi(y) = y$ yields

$$w = e^{i2y/\epsilon} \int_0^t a(y-t)\, b(y+t)\, dt\,.$$

Old phases $\hookrightarrow$ new. $2y = (y-t) + (y+t)$.

**Example 2.** Phase in the $w$ integral satisfies

$$\partial_t(\text{phase}) = \partial_t\big(\, y - t + \phi(y+t)\,\big) = -1 + \phi'(y+t)\,.$$

$\phi' \neq 1 \hookrightarrow w = O(\epsilon)$. Ex. 1 is the unique resonance relation.

## 10. Three wave interaction

$$(\partial_t + \partial_x)u_1 = c_1\, u_2\, u_3^*, \quad (\partial_t + \partial_x)u_2 = c_2\, u_1^*\, u_3, \quad \partial_t u_3 = c_3\, u_1\, u_2$$

Homogeneous oscillations

$$u_1 = A_1(t)e^{(t-x)/\epsilon}, \quad u_2 = A_2(t)e^{(t+x)/\epsilon} \quad u_3 = A_3(t)e^{2t/\epsilon}\,.$$

$$A_1' = c_1\, A_2\, A_3^*, \quad A_2' = c_2\, A_1^*\, A_3, \quad A_3' = c_3 A_1\, A_2\,.$$

For later use, compute $A''$

$$A_1'' \;=\; c_1\left((c_2\, A_2\, A_3^*)\, A_3^* + A_2\,(c_3\, A_1\, A_2)^*\right)$$

Cubic terms with diagramatic description. *Blackboard.*

Multidimensional problems can have much richer resonance structure.

**Example.** Consider the two dimensional compressible Euler equations linearized at a constant state with zero velocity and sound speed equal to one. The characteristic equation is then $\tau(\tau^2 - |\xi|^2) = 0$. The plane waves corresponding to $\tau = 0$ are called **vorticity waves** and those to $\tau^2 = |\xi|^2$ are **sound waves**. Group velocities are 0 and $\xi/\tau$ respectively. The latter are unit vectors.

Classification of resonance relations. (Blackboard)

- Collinear in $\tau^2 = |\xi|^2$
- In the plane $\tau = 0$
- Ice Cream Cone resonances.

## 12. Characteristic directions in an integer lattice.

Three integer characteristic phases,

$$(1,1,0), \qquad (0,1,0), \quad \text{and}, \quad (0,3,4)$$

Define $\Lambda$ to be the set of integer linear combinations of these.
Define $\Lambda_{char} := \Lambda \cap \mathrm{Char}\, L$

**Thm.** For any rational number $r$ there is a $(\tau, \xi) \in \Lambda_{char}$ with $\xi_2/\xi_1 = r$.

Resonant interaction of frequencies along three lines could possibly generate characteristic phases having all rational slopes.

## 13. Dense oscillations

**Theorem.** $\exists$ *a solution of the leading profile equation*

$$U^0(t,T,X) \;=\; \sum_{p\in\Lambda_{char}} U^0_p(t)\; e^{in\cdot(T,X)}$$

*so that at* $t=0$ *the spectrum of* $U_0$ *is contained in* $\mathbb{Z}(1,1,0)\cup\mathbb{Z}(0,1,0)\cup\mathbb{Z}(0,3,4)$, *and so that for all but a finite set of planes of* $p\in\Lambda_{char}$, $U^0_p(0)=(U^0_p)'(0)=0,\ (U^0_p)''(0)\ \neq\ 0\,.$

**Sketch of Pf.** Choose initial data with Fourier coeffs nonzero exactly at $\mathbb{Z}(1,1,0)\cup\mathbb{Z}(0,1,0)\cup\mathbb{Z}(0,3,4)$. Write

$$\Lambda_{char}\ni p = m_1(1,1,0)+m_2(0,1,0)+m_3(0,3,4) := \alpha+\beta+\gamma\,.$$

Second derivative of $U^0_p(t)$ is the sum of six contributions correponding to the six diagrams leading from $\alpha,\beta,\gamma$ to $p$. Four of them vanish by vanishing interaction coefficients (Blackboard). The other two are equal and not zero for all but a finite number of explicit exceptional planes. $\square$

## 14 . The Euler solution

There are correctors of all orders $U^n(t, T, X)$. Borel's Theorem constructs smooth periodic $U$ with Taylor series

$$U(\epsilon, t, T, X) \sim \sum_{n=0}^{\infty} \epsilon^{n+1} U^n(t, T, X).$$

Define $u_{approx}(t, x) := \epsilon\, U(t, t/\epsilon, x/\epsilon)$.

**Theorem.** *The solution of compressible Euler equations with the same initial data differs from $u_{approx}$ by $O(\epsilon^\infty)$ in $C^\infty$.*

The solution starts with oscillations with spectrum on three lines and immediately launches oscillations in virtually all rational directions.

This is a nice complement to the image that we have of wavefonts marching majestically across $\mathbb{R}^2$ steepening à la Burgers.

## 15. Recapitulation

Rectilinear propagation

Equal angles law of reflection.

Snell's (Descarte's en France) law of refraction

Rays transport, amplification/decay. Amplifying Bérenger layers

Raylike solutions

Group velocity and dispersion

Resonance, one phase cascade, harmonic generation

Resonance, three wave equations

Diagramatic tracing of resonant interactions

Resoances for compressible Euler

Dense oscillations for Euler