HADAMARD PRODUCTS OF ALGEBRAIC FUNCTIONS

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ABSTRACT. Allouche and Mendès-France (in *Hadamard grade of power series*, J. Number Theory **131** (2011)) have defined the grade of a formal power series with algebraic coefficients as the smallest integer k such that this series is the Hadamard product of k algebraic power series. In this paper, we obtain lower and upper bounds for the grade of hypergeometric series by comparing two different asymptotic expansions of their Taylor coefficients, one obtained from their definition and another one obtained when assuming that the grade has a certain value. In such expansions, Gamma values at rational points naturally appear and our results mostly depend on the Rohrlich-Lang Conjecture for polynomial relations in Gamma values. We also obtain unconditional and sharp results when we can apply Diophantine results such as the Wolfart-Wüstholz Theorem for Beta values.

1. Introduction

The main goal of this paper is to study the notion of grade of a power series introduced by Allouche and Mendès-France in [2]. We recall that the Hadamard product $F(z)*G(z) \in \mathbb{C}[[z]]$ of two formal power series $F(z)=\sum_{n\geq 0} f_n z^n \in \mathbb{C}[[z]]$ and $F(z)=\sum_{n\geq 0} g_n z^n \in \mathbb{C}[[z]]$ is defined by

$$F(z)*G(z)=\sum_{n\geq 0} f_n g_n z^n.$$

Definition 1. *A formal power series $F(z)\in\overline{\mathbb{Q}}[[z]]$ has finite grade over $\overline{\mathbb{Q}}$ if there exist $A_1(z),...,A_m(z)\in\overline{\mathbb{Q}}[[z]]$ algebraic over $\overline{\mathbb{Q}}(z)$ such that*

$$F(z)=A_1(z)*\cdots*A_m(z). \tag{1.1}$$

If $F(z)$ has finite grade over $\overline{\mathbb{Q}}$ then the smallest integer $m\geq 1$ such (1.1) holds for some $A_1(z),...,A_m(z)\in\overline{\mathbb{Q}}[[z]]$ algebraic over $\overline{\mathbb{Q}}(z)$ is denoted by $\mathrm{grade}_{\overline{\mathbb{Q}}}(F(z))$ and is called the grade of $F(z)$ over $\overline{\mathbb{Q}}$. If $F(z)$ does not have finite grade over $\overline{\mathbb{Q}}$, then we set $\mathrm{grade}_{\overline{\mathbb{Q}}}(F(z))=\infty$.

From now on, by "algebraic function" or "algebraic series", we will mean a power series in $\overline{\mathbb{Q}}[[z]]$ which is algebraic over $\overline{\mathbb{Q}}(z)$.

If $F(z)=\sum_{n\geq 0} f_n z^n\in\overline{\mathbb{Q}}[[z]]$ has finite grade over $\overline{\mathbb{Q}}$ then it is a globally bounded G-function because algebraic functions are globally bounded G-functions (Abel, Eisenstein) and this property is preserved by Hadamard product (see [1, §VI.4, Corollary]). We recall that $F(z)$ is said to be globally bounded (terminology due to Christol [6]) if all the coefficients belong to some number field K and

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- for every place ν of K, the ν-adic radius of convergence of $F(z)$ is non zero;
- there exists some non zero integer N such that the coefficients of F belong to $\mathcal{O}_K[1/N]$, where $\mathcal{O}_K$ is the ring of integers of K.

Moreover, $F(z)$ is a G-function if all the coefficients belong to some number field K and

- the maximum of the moduli of the conjugates of f_n grows at most geometrically with n;
- there exists a sequence of positive numbers $(d_n)_{n\geq 0}$ which grows at most geometrically with n such that $d_n a_n$ belongs to the ring of integers $\mathcal{O}_K$ of K;
- $F(z)$ satisfies some non trivial homogeneous linear differential equation with coefficients in $K(z)$.

It follows for instance that $\log(1-z)$ and $(1-z)^\alpha$ with $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ do not have finite grade. It is very likely that there exist globally bounded G-functions with infinite grade; see Proposition 1.

It is in general very difficult to estimate the grade of a globally bounded G-function and Allouche-Mendès-France raised a few questions in this respect, from which this paper grew. To estimate the grade of a given globally bounded G-function $F(z)$, our strategy (detailed in Section 4) is to compare two distinct asymptotic expansions of the Taylor coefficients of $F(z)$: one we know a priori from the definition of $F(z)$ and another obtained by assuming that $F(z)$ is the Hadamard product of a certain number of algebraic functions (and based on Theorem 3 in Section 2). In the cases considered in this paper, this leads to a polynomial identity in values of the Euler Gamma function at rational numbers :

$$P\big(\Gamma(a_1), \Gamma(a_2), \dots, \Gamma(a_k)\big) = 0 \tag{1.2}$$

where $P(X_1, \dots, X_k) \in \overline{\mathbb{Q}}[X_1, \dots, X_k]$ and $a_1, \dots, a_k \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Not much is known on the (im)possibility of such a relation, although the Rohrlich-Lang Conjecture tells us what to expect. This conjecture is widely open and most of our results are conditional. But an important case was proved by Wolfart-Wüstholz [21], which will enable us to obtain unconditional results a well. See Section 3 for details on these questions.

We now describe the main results of this paper. We first consider the (special type of) hypergeometric series

$$F_{\mathbf{e},\mathbf{f}}(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{\mu}(e_i n)!}{\prod_{j=1}^{\lambda}(f_j n)!} z^n,$$

where $\mathbf{e} = (e_1, ..., e_\mu) \in \mathbb{Z}_{>0}^{\mu}$ and $\mathbf{f} = (f_1, ..., f_\lambda) \in \mathbb{Z}_{>0}^{\lambda}$. Without loss of generality, we can assume that $e_i \neq f_j$ for all $(i,j) \in \{1, ..., \mu\} \times \{1, ..., \lambda\}$. Moreover, we assume that

$$\sum_{i=1}^{\mu} e_i = \sum_{j=1}^{\lambda} f_j,$$

which is a necessary and sufficient condition for $F_{\mathbf{e},\mathbf{f}}(z)$ to be a G-function.

Theorem 1. *(i) If $\lambda - \mu \leq 0$, then* $\mathrm{grade}_{\overline{\mathbb{Q}}}(F_{\mathbf{e},\mathbf{f}}(z)) = \infty$.

(ii) If $\lambda - \mu \geq 2$, then $\mathrm{grade}_{\overline{\mathbb{Q}}}(F_{\mathbf{e},\mathbf{f}}(z)) \geq 2$.

(iii) If $\lambda-\mu=3$ *then* $\mathrm{grade}_{\overline{\mathbb{Q}}}(F_{\mathbf{e},\mathbf{f}}(z))\geq 3$.
(iv) Assuming the Rohrlich-Lang Conjecture, if $\lambda-\mu\geq 1$ *then* $\mathrm{grade}_{\overline{\mathbb{Q}}}(F_{\mathbf{e},\mathbf{f}}(z))\geq \lambda-\mu$.

This result has the following consequence.

Corollary 1. *Let* $k\geq 1$ *and* $a_1,\dots,a_k$, $b_1,\dots,b_k$ *be integers such that* $a_j>b_j\geq 1$ *for all* $j\in\{1,...,k\}$. *Then under the Rohrlich-Lang Conjecture, the grade of*

$$\sum_{n=0}^{\infty}\left(\prod_{j=1}^{k}\binom{a_jn}{b_jn}\right)z^n$$

is equal to k. *Moreover, this is true unconditionally for* $k\in\{1,2,3\}$.

Remarks. Allouche and Mendès-France wondered if the grade of the series

$$S_k(z)=\sum_{n=0}^{\infty}\binom{2n}{n}^k z^n$$

is always k, for any integer $k\geq 1$. Note that $S_1(z)=\frac{1}{\sqrt{1-4z}}$ is algebraic over $\overline{\mathbb{Q}}(z)$ and that, for any integer $k\geq 1$, we have

$$S_k(z)=\underbrace{S_1(z)*S_1(z)*\cdots*S_1(z)}_{k\text{ times}}.$$

So the answer is positive for $k=1$ and, for all integer $k\geq 1$, we have $\mathrm{grade}_{\overline{\mathbb{Q}}}(S_k(z))\leq k$ with equality conditionally. Moreover, $S_2(z)$ is known to be transcendental over $\overline{\mathbb{Q}}(z)$ hence $\mathrm{grade}_{\overline{\mathbb{Q}}}(S_2(z))=2$. The first open case is for $k=3$, which we solve positively: this is the particular case $k=3$, $a_1=a_2=a_3=2$ and $b_1=b_2=b_3=1$ of Corollary 1.

More generally, we consider the generalized hypergeometric series with parameters $\mathbf{a}=(a_1,...,a_p)\in\mathbb{Q}^p$ and $\mathbf{b}=(b_1,...,b_q)\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^q$ defined by

$${}_pF_q(\mathbf{a},\mathbf{b};z):={}_pF_q\left[\begin{matrix}a_1, & a_2, & \dots, & a_p\\ & b_1, & \dots, & b_q\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(1)_n(b_1)_n\cdots(b_q)_n}z^n \tag{1.3}$$

where $(x)_n$ denotes the Pochhammer symbol defined by $(x)_0=1$ and, for all integer $n\geq 1$, $(x)_n=x(x+1)\cdots(x+n-1)$. We assume that $q=p-1$, which is a necessary and sufficient condition for ${}_pF_q(z)$ to be a G-function. Whether or not ${}_pF_{p-1}(\mathbf{a},\mathbf{b};z)$ is globally bounded can be decided by studying some Landau type functions according to the work of Christol [6, 9]. Following Christol [6], we define the height of ${}_pF_{p-1}(\mathbf{a},\mathbf{b};z)$ by

$$h(\mathbf{a},\mathbf{b}):=\#\{1\leq j\leq p\mid b_j\in\mathbb{Z}\}-\#\{1\leq j\leq p\mid a_j\in\mathbb{Z}\},$$

with $b_p:=1$.

Theorem 2. *Consider* $\mathbf{a}=(a_1,...,a_p)\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^p$ *and* $\mathbf{b}=(b_1,...,b_{p-1})\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^{p-1}$. *Then, under the Rohrlich-Lang Conjecture, we have* $\mathrm{grade}_{\overline{\mathbb{Q}}}({}_pF_{p-1}(\mathbf{a},\mathbf{b};z))\geq|h(\mathbf{a},\mathbf{b})|$.

Remarks. This is a generalization of Theorem 1(iv). Indeed, using the equality

$$(an)! = a^{an}(1)_n(1/a)_n(2/a)_n \cdots ((a-1)/a)_n,$$

we see that $F_{\mathbf{e},\mathbf{f}}(z) = {}_pF_{p-1}(\mathbf{a},\mathbf{b};Cz)$ for some $C \in \mathbb{Q}^\times$ and some $\mathbf{a} = (a_1,...,a_p) \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^p$ and $\mathbf{b} = (b_1,...,b_{p-1}) \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^{p-1}$ with $h(\mathbf{a},\mathbf{b}) = \lambda - \mu$.

This result has the following consequence.

Corollary 2. *Assume that* $\mathbf{a} = (a_1,...,a_p) \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^p$ *and* $b_1 = \cdots = b_{p-1} = 1$*. Then, under the Rohrlich-Lang Conjecture, we have* $\mathrm{grade}_{\overline{\mathbb{Q}}}({}_pF_{p-1}(\mathbf{a},\mathbf{b};z)) = p$.

In certain cases, our method may lead to stronger results than the lower bound given by the height in Theorem 2. For instance, we have the following result about an hypergeometric series considered by Christol in [7] in relation with its conjecture about globally bounded G-functions and diagonals of multivariate rational functions.

Proposition 1. *Under the Rohrlich-Lang conjecture, the globally bounded* G*-function (of height* 2*)*

$${}_3F_2\left[\begin{matrix}1/7,\, 2/7,\, 4/7\\ 1/2,\, 1,\end{matrix};z\right]$$

has infinite grade.

Remarks. Another interesting example mentioned by Christol in [6, 7] is the globally bounded

$${}_3F_2\left[\begin{matrix}1/9,\, 4/9,\, 5/9\\ 1/3,\, 1,\end{matrix};z\right].$$

Unfortunately, it seems that our method cannot prove anything beyond the fact that its grade is ≥ 2. Proving that it is ≥ 3 would answer a long standing question of Christol [6] concerning the impossibility of writing this function as the Hadamard product of two algebraic hypergeometric series. This example shows that the *ad hoc* arithmetic proof of Proposition 1 cannot be repeated on a general basis.

Theorem 2 leads to the following question. Consider $\mathbf{a} = (a_1,...,a_p) \in \mathbb{Q}^p$ and $\mathbf{b} = (b_1,...,b_{p-1}) \in (\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^{p-1}$ such that, for all $(i,j) \in \{1,...,p\}\times\{1,...,p-1\}$, $a_i - b_j \notin \mathbb{Z}$. Is the grade of ${}_pF_{p-1}(\mathbf{a},\mathbf{b};z)$ finite if and only if it is the Hadamard product of $|h(\mathbf{a},\mathbf{b})|$ algebraic hypergeometric series? Note that Christol proved [6] that a globally bounded hypergeometric series has grade 1 (i.e. is algebraic) if and only if it has height 1.

Of course, hypergeometric functions are not the only class of globally bounded G-functions whose grade can be estimated by our method. We illustrate this with our last result, which concerns the G-functions

$$M_{\mathbf{r}}(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}^{r_0}\binom{n+k}{k}^{r_1}\binom{n+2k}{k}^{r_2}\cdots\binom{n+mk}{k}^{r_m}\right)z^n \in \mathbb{Z}[[z]],$$

where $\mathbf{r} = (r_0, r_1, \ldots, r_m) \in \mathbb{N}^{m+1}$. Such binomial sums have been considered by McIntosh [14] who obtained a result that proves to be useful to estimate the grade of $M_{\mathbf{r}}(z)$.

When $m = 1$, the cases $r_0 = 2, r_1 = 1$ and $r_0 = r_1 = 2$ correspond to the famous generating functions of Apéry's numbers for $\zeta(2)$ and $\zeta(3)$ respectively. We set $R = \sum_{j=0}^{m} r_j$.

Proposition 2. *For any $m \geq 0$ and any $\mathbf{r} \in \mathbb{N}^{m+1}$ such that $r_0 \geq 1$, the following properties hold :*

(i) If $R \geq 3$, then $\mathrm{grade}_{\overline{\mathbb{Q}}}(M_{\mathbf{r}}(z)) \geq 2$.

(ii) If $R = 4$, then $\mathrm{grade}_{\overline{\mathbb{Q}}}(M_{\mathbf{r}}(z)) \geq 3$.

(iii) Assuming the Rohrlich-Lang Conjecture, if $R \geq 2$, then $\mathrm{grade}_{\overline{\mathbb{Q}}}(M_{\mathbf{r}}(z)) \geq R - 1$.

Remarks. Under the conditions of the proposition, if $R = 1$, then $M_{\mathbf{r}}(z) = \frac{1}{1-2z}$, which has grade 1. For $R \geq 2$, the lower bound in *(iii)* is best possible in general because $M_{\mathbf{r}}(z)$ is algebraic for $m = 1$ and $r_0 = r_1 = 1$: it is the algebraic function that generates the Legendre polynomials on $[0, 1]$ evaluated at -1.

The paper is organised as follows. In Section 2, we obtain the asymptotic expansion of the sequence of Taylor coefficients of any algebraic function. In Section 3, we recall various conjectures and theorems on algebraic relations about Gamma values at rational points. In Section 4, we explain our strategy for proving Theorems 1 and 2. The former is proved in Section 5 and the latter in Section 7. Corollaries 1 and 2 are proved in Sections 6 and 8. In Sections 9 and 11, we prove Propositions 1 and 2, while in Section 10 we discuss Christol's hypergeometric function.

2. Taylor coefficients of algebraic functions

Our study of the grade of globally bounded G-functions relies on the asymptotic behavior of the Taylor coefficients of algebraic functions.

Consider a power series

$$A(z) = \sum_{n=0}^{\infty} A_n z^n \in \mathbb{C}[[z]],$$

algebraic over $\overline{\mathbb{Q}}(z)$, which is not a polynomial. We denote its singularities at finite distance by $\xi_1, \ldots, \xi_p \in \overline{\mathbb{Q}}$. We consider the slit plane $\mathbb{C} \setminus \cup_{j=1}^{p} [\xi_j, \infty)$, where the cuts $[\xi_j, \infty)$ are pairwise without intersection. In this slit plane, $A(z)$ is univalued and, around each singularity ξ_j, we have the convergent Puiseux expansion

$$A(z) = (z - \xi_j)^{s_j} \sum_{k=0}^{\infty} \phi_{k,j} (z - \xi_j)^{k/d_j} \tag{2.1}$$

where $\phi_{k,j} \in \overline{\mathbb{Q}}$, $\phi_{0,j} \neq 0$, $s_j = c_j/d_j \in \mathbb{Q}$ and $d_j \geq 1$.

Theorem 3. *Let $A(z)$ be an algebraic function given as above. There exist p sequences $(B_{1,n})_{n\geq 0}, \ldots, (B_{p,n})_{n\geq 0}$ such that, for any integer n large enough,*

$$A_n = \sum_{j=1}^{p} B_{j,n}$$

and such that :

(*i*) *for any large enough real algebraic number* ω*, any* $j \in \{1,\dots,p\}$ *and any integer* n *large enough,* $B_{j,n}$ *can be represented as a sum of convergent* séries de facultés*:*

$$B_{j,n} = \xi_j^{-n} \sum_{\ell=0}^{d_j-1} \frac{\Gamma\big(\frac{n}{\omega}\big)}{\Gamma\big(\frac{n}{\omega}+\frac{\ell}{d_j}+s_j\big)} \sum_{k=0}^{\infty} \frac{\beta_{j,k,\ell}(\omega)}{\big(\frac{n}{\omega}+\frac{\ell}{d_j}+s_j\big)_{k+1}\Gamma(-k-\frac{\ell}{d_j}-s_j\big)} \tag{2.2}$$

where $\beta_{j,k,\ell}(\omega) \in \overline{\mathbb{Q}}$*;*

(*ii*) *for any* $j \in \{1,\dots,p\}$*,* $B_{j,n}$ *has an asymptotic expansion of the form*

$$B_{j,n} \sim \xi_j^{-n} \sum_{\ell=0}^{d_j-1} \frac{1}{\Gamma\big(-\frac{\ell}{d_j}-s_j\big) n^{\ell/d_j+s_j}} \sum_{q=1}^{\infty} \frac{\widetilde{\beta}_{j,\ell,q}}{n^q}, \tag{2.3}$$

where $\widetilde{\beta}_{j,\ell,q} \in \overline{\mathbb{Q}}$ *and* $\widetilde{\beta}_{j,0,1} \neq 0$.

We emphasize that the arithmetic nature of the coefficients will be of first importance in the rest of the paper.

We don't claim any real originality for Theorem 3. For instance, in [17], Orlov provides the leading term of the asymptotic expansion of $B_{j,n}$. In [11, p. 501], Flajolet and Sedgewick provide the asymptotic expansion of A_n where only the singularities of smallest modulus of $A(z)$ are taken into account, and without any explicit description of the coefficients. Actually, Theorem 3 is essentially due to Norlünd in its classical book [15], the main difference being that he does not pay attention to the arithmetic nature of the coefficients. We now sketch Norlünd's proof of the above theorem.

Sketch of the proof of Theorem 3. In order to simplify the notations, we assume that any two distinct singularities ξ_i and ξ_j of $A(z)$ do not belong to the same half-line issued from 0. In this case, we can assume that $[\xi_j,\infty) = \xi_j[1,+\infty)$. For the general case, and for further details, we refer to [15, Section II of Chapitre III].

(*i*) For any integer n, we have

$$A_n = \frac{1}{2i\pi}\int_{\mathcal{G}} \frac{A(z)}{z^{n+1}}\mathrm{d}z$$

where $\mathcal{G}$ is the contour that surrounds 0 and avoids the p cuts $[\xi_j,\infty)$, as in Figure 1. Since $|A(z)| \ll |z|^\delta$ as $z\to\infty$, for some $\delta>0$, we can "send" the contour at infinity provided that $n \gg 0$ and we obtain, for $n\gg 0$,

$$A_n = \sum_{j=1}^{p} B_{j,n} \text{ with } B_{j,n} = \frac{\xi_j^{-n}}{2i\pi}\int_{\widetilde{\gamma}_j} \frac{A(\xi_j z)}{z^{n+1}}\mathrm{d}z \tag{2.4}$$

where $\widetilde{\gamma}_j$ is a contour composed of two half-lines parallel to $[1,+\infty)$, and joined by a half-circle of center 1. Locally around $z=1$, we have the convergent expansion

$$A(\xi_j z) = \xi^{s_j}(z-1)^{s_j}\sum_{k=0}^{\infty}\phi_{k,j}\xi_j^{k/d_j}(z-1)^{k/d_j}$$

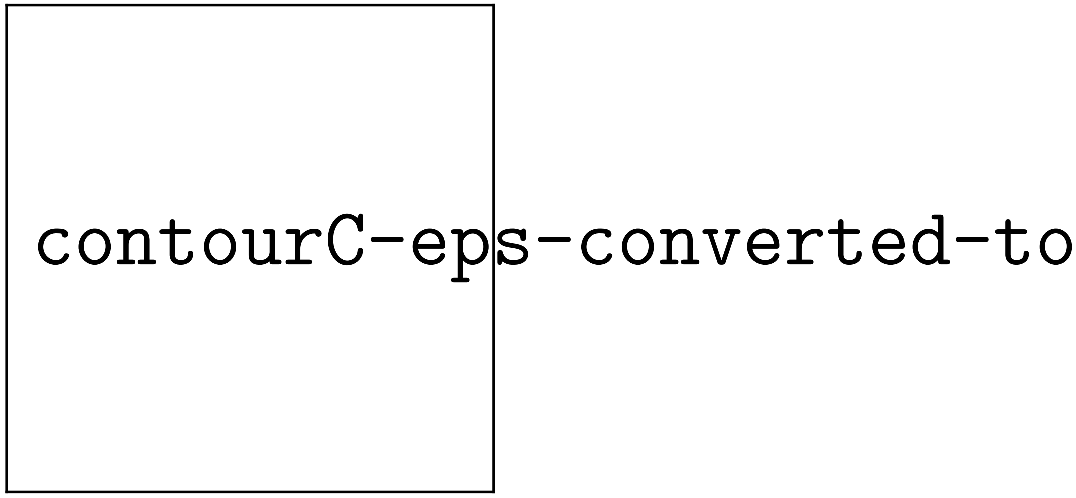

FIGURE 1. The cuts $[\xi_j, \infty)$ and the contour $\mathcal{G}$.

but in general we cannot exchange series and integral in (2.4). This difficulty can be overcome by using the following trick, described in [15, Section II of Chapitre III] (see also [16, Chapitre VI]). We set $x = 1 - z^{-\omega}$, where ω a positive algebraic number to be specified later, so that, for $n \gg 0$,

$$B_{j,n} = \frac{\xi_j^{-n}\omega^{-1}}{2i\pi} \int_{\widehat{\gamma}_j} (1-x)^{n/\omega - 1} A\left(\frac{\xi_j}{(1-x)^{1/\omega}}\right) \mathrm{d}x. \tag{2.5}$$

Here $\widehat{\gamma}_j$ is a closed path surrounding $[0,1]$, that contains 1. We also have

$$\begin{aligned} \omega^{-1} A\left(\frac{\xi_j}{(1-x)^{1/\omega}}\right) &= \omega^{-1}\xi_j^{s_j}\left((1-x)^{-1/\omega}-1\right)^{s_j} \sum_{k=0}^{\infty} \phi_{k,j}\xi_j^k \left((1-x)^{-1/\omega}-1\right)^{k/d_j} \\ &= x^{s_j} \sum_{k=0}^{\infty} \widehat{\phi}_{k,j}(\omega) x^{k/d_j} \end{aligned}$$

where $\widehat{\phi}_{k,j}(\omega) \in \overline{\mathbb{Q}}$ and $\widehat{\phi}_{0,j}(\omega) \neq 0$. If we choose ω large enough, the above expansion holds in the disk $|x| \leq 1$. We can assume that $\widehat{\gamma}_j$ is contained in this disk. Using this expansion in (2.5), we can now invert the sum and integral signs:

$$B_{j,n} = \frac{\xi_j^{-n}}{2i\pi} \sum_{k=0}^{\infty} \widehat{\phi}_{k,j}(\omega) \int_{\widehat{\gamma}_j} x^{k/d_j + s_j}(1-x)^{n/\omega - 1}\mathrm{d}x.$$

This integral is easily evaluated (using the Beta function):

$$\frac{1}{2i\pi} \int_{\widehat{\gamma}_j} x^{k/d_j + s_s}(1-x)^{n/\omega-1}\mathrm{d}x = \frac{e^{i\pi(k/d_j + s_j)}\Gamma\left(\frac{n}{\omega}\right)}{\Gamma\left(\frac{k}{d_j} + s_j + \frac{n}{\omega} + 1\right)\Gamma\left(-\frac{k}{d_j} - s_j\right)}.$$

Finally, after some simple rearrangements, we obtain the following convergent expansion for $B_{j,n}$:

$$B_{j,n} = \frac{e^{i\pi s_j}}{\xi_j^n}\sum_{k=0}^{\infty} e^{i\pi k/d_j}\widehat{\phi}_{k,j}(\omega)\frac{\Gamma\big(\frac{n}{\omega}\big)}{\Gamma\big(\frac{k}{d_j}+s_j+\frac{n}{\omega}+1\big)\Gamma\big(-\frac{k}{d_j}-s_j\big)} \tag{2.6}$$

$$= \frac{e^{i\pi s_j}}{\xi_j^n}\sum_{\ell=0}^{d_j-1}\frac{\Gamma\big(\frac{n}{\omega}\big)}{\Gamma\big(\frac{n}{\omega}+\frac{\ell}{d_j}+s_j\big)}\sum_{k=0}^{\infty}\frac{e^{i\pi k/d_j}\widehat{\phi}_{kd_j+\ell,j}(\omega)}{(\ell/d_j+s_j+n/\omega)_{k+1}\Gamma\big(-k-\frac{\ell}{d_j}-s_j\big)}. \tag{2.7}$$

(ii) From the convergent expression (2.6), we can deduce the asymptotic expansion of $B_{j,n}$. We start from the general asymptotic expansion (as $z\to+\infty$):

$$\frac{\Gamma(z)}{\Gamma(z+x)} \sim z^{-x}\sum_{k=0}^{\infty}\frac{\binom{-x}{k}P_k(x)}{z^k},$$

where $P_k(x)$ is a polynomial in $\mathbb{Q}[x]$ of degree k defined by the expansion

$$\Big(\frac{t}{e^t-1}\Big)^x = \sum_{k=0}^{\infty}P_k(x)\frac{t^k}{k!}.$$

(They are not the Bernoulli polynomials.)

It follows that

$$\frac{\Gamma\big(\frac{n}{\omega}\big)}{\Gamma\big(\frac{k}{d_j}+s_j+\frac{n}{\omega}+1\big)} \sim n^{-k/d_j-s_j-1}\sum_{r=0}^{\infty}\omega^r\frac{\binom{-k/d_j-s_j-1}{r}P_r\big(\frac{k}{d_j}+s_j+1\big)}{n^r}.$$

Hence

$$B_{j,n} \sim \xi_j^{-n}\sum_{\ell=0}^{d_j-1}\frac{e^{i\pi(\ell/d_j+s_j)}}{\Gamma\big(-\frac{\ell}{d_j}-s_j\big)n^{\ell/d_j+s_j+1}}\sum_{q=0}^{\infty}\frac{c_q(\ell,j,\omega)}{n^q}, \tag{2.8}$$

where $c_q(\ell,j,\omega)$ is a finite sum of algebraic numbers:

$$c_q(\ell,j,\omega) = \sum_{\substack{k,r\geq 0\\ k+r=q}}\frac{(-1)^k\omega^r\widehat{\phi}_{kd_j+\ell,j}(\omega)\Gamma\big(-\frac{\ell}{d_j}-s_j\big)}{\Gamma\big(-k-\frac{\ell}{d_j}-s_j\big)}\binom{-k-\frac{\ell}{d_j}-s_j-1}{r}P_r\Big(k+\frac{\ell}{d_j}+s_j+1\Big)$$

$$= \Big(\frac{\ell}{d_j}+s_j+1\Big)_q\sum_{\substack{k,r\geq 0\\ k+r=q}}\frac{(-\omega)^r\widehat{\phi}_{kd_j+\ell,j}(\omega)}{r!}P_r\Big(k+\frac{\ell}{d_j}+s_j+1\Big).$$

Moreover,

$$c_0(0,j,\omega) = \widehat{\phi}_{0,j}(\omega) \neq 0,$$

so that the main term of the asymptotic expansion of $B_{j,n}$ is (as expected)

$$\frac{\nu_j}{\Gamma(-s_j)}\cdot\frac{\xi_j^{-n}}{n^{s_j+1}}$$

where $\nu_j \in \overline{\mathbb{Q}}$. □

Remarks. Since $A(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$, it satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$. Therefore, the sequence of its Taylor coefficients satisfies a linear difference equation with coefficients in $\overline{\mathbb{Q}}(n)$. For any $j \in \{1, ..., p\}$, the sequence $(B_{j,n})_{n \geq N}$ constructed in the proof of Theorem 3 satisfies the same difference equation, for N large enough (see the beginning of [15, Chapitre III]).

3. Algebraic relations amongst Gamma values

Let us consider the following normalized Gamma function :

$$G(x) = \Gamma(x)/\sqrt{\pi}.$$

We introduce the equivalence relation on $\mathbb{C}^\times$ defined by

$$a \sim b \Leftrightarrow a/b \in \overline{\mathbb{Q}}^\times.$$

Conjecture 1 (Rohrlich [8, 13, 20, 21])**.** *Consider some rational numbers $a_1, \ldots, a_n \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ with common denominator N. Then, we have*

$$\prod_{j=1}^{n} G(a_j) \sim 1$$

if and only if, for all integer $m \in \{1, \ldots, N-1\}$ coprime to N, we have

$$\sum_{j=1}^{n} \{m a_j\} = \frac{n}{2}$$

where $\{x\} \in [0, 1)$ denotes the fractional part of x.

Moreover, the idea behind this conjecture is that any relation

$$\prod_{j=1}^{n} G(a_j) \sim 1$$

"comes from" the following standard relations:

$$G(x) \sim G(x+1) \text{ for } x \notin \mathbb{Z}_{\leq 0} \qquad \text{(Functional equation)}$$

$$G(x)G(1-x) \sim 1 \text{ for } x \notin \mathbb{Z} \qquad \text{(Complements formula)}$$

$$G(x) \sim \prod_{j=0}^{n-1} G\Big(\frac{x+j}{n}\Big) \text{ for } n \in \mathbb{N} \text{ and } x \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}, \qquad \text{(Distribution relations).}$$

We refer to [8, 13, 20] for details. The "if part" of the conjecture was proved by Koblitz and Ogus; see the appendix of [10] or [21].

In fact, much more is expected.

Conjecture 2 (Rohrlich-Lang [20]). *Consider a non zero polynomial* $P(X_1, \ldots, X_n) \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$ *and* $a_1, \ldots, a_n \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ *such that*

$$P\big(\Gamma(a_1), \Gamma(a_2), \ldots, \Gamma(a_n)\big) = 0.$$

Then we can find two distinct monomials of P*, say* $X_1^{s_1} \cdots X_n^{s_n}$ *and* $X_1^{t_1} \cdots X_n^{t_n}$*, such that*

$$\Gamma(a_1)^{s_1} \cdots \Gamma(a_n)^{s_n} \sim \Gamma(a_1)^{t_1} \cdots \Gamma(a_n)^{t_n}. \tag{3.1}$$

Moreover, the relation (3.1) *can be rewritten as a relation* $\prod_{j=1}^{n} G(b_j) \sim 1$ *that satisfies Rohrlich conjecture above.*

In direction of the conjecture, we will use results obtained by Schneider and Wolfart-Wüstholz concerning linear forms in Beta values. The Beta function is defined by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Theorem 4 (Schneider [19]). *For any* $a, b \in \mathbb{Q} \setminus \mathbb{Z}$ *such that* $a + b \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$*, the number* $B(a, b)$ *is transcendental.*

Theorem 5 (Wolfart-Wüstholz [21]). *For any positive integer* n*, consider* $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Q}^{+\times}$*, and* $c_1, \ldots, c_n \in \overline{\mathbb{Q}}$*, not all zero, such that*

$$\sum_{j=1}^{n} c_j B(a_j, b_j) = 0.$$

Then $n \geq 2$ *and there exist two distinct integers* $p, q \in \{1, 2, \ldots, n\}$ *such that*

$$B(a_p, b_p) \sim B(a_q, b_q).$$

We will also use the following simple result.

Proposition 3. *Assume the Rohrlich Conjecture. Let* $k, n \in \mathbb{Z}$*,* $n \geq 0$ *and let* $a_1, \ldots, a_n \in \mathbb{Q} \setminus \mathbb{Z}$ *be such that*

$$G(1)^k G(a_1) \cdots G(a_n) \sim 1. \tag{3.2}$$

Then $k = 0$*.*

Proof. We can assume that $k \geq 0$, otherwise we use the complements formula to get the similar relation

$$G(1)^{-k} G(-a_1) \cdots G(-a_n) \sim 1.$$

According to Rohrlich Conjecture, the quantity

$$f(m) := k\{m\} + \{ma_1\} + \cdots + \{ma_n\}$$

is equal to $\frac{n+k}{2}$ for all integers m coprime to the common denominator N of $a_1, ..., a_n$. Since $a_j \notin \mathbb{Z}$, we have $\{a_j\} + \{-a_j\} = 1$ for all $j \in \{1, ..., n\}$. Hence,

$$f(1) + f(D-1) = \{a_1\} + \cdots + \{a_n\} + \{-a_1\} + \cdots + \{-a_n\} = n.$$

But, we also have $f(1) = f(D-1) = \frac{n+k}{2}$. Hence $k = 0$. $\square$

4. Strategy to study the grade of a power series f

The typical situation we will encounter is when we have *a priori* informations on the asymptotic behavior of the Taylor coefficients of $F(z) \in \overline{\mathbb{Q}}[[z]]$, namely

$$[z^n]F(z) = \frac{C\omega^n}{n^\kappa}(1 + o(1)) \tag{4.1}$$

for some $C, \omega \in \mathbb{C}^\times$ and $\kappa \in \mathbb{Q}$, where $[z^n]F(z)$ denotes the n-th taylor coefficient of $F(z)$.

Assume that $F(z)$ has grade lower than or equal to k i.e. that

$$F(z) = A_1(z) * \cdots * A_k(z)$$

for some algebraic functions $A_1(z), ..., A_k(z) \in \overline{\mathbb{Q}}[[z]]$.

Remarks. The fact that $F(z) \in \overline{\mathbb{Q}}[[z]]$ has grade $\leq k$ means that

$$F(z) = A_1(z) * A_2(z) * \cdots * A_\ell(z)$$

for some $\ell \leq k$. We can assume that $\ell = k$ because

$$F(z) = A_1(z) * A_2(z) * \cdots * A_k(z)$$

with $A_{\ell+1}(z) = \cdots = A_n(z) = \frac{1}{1-z}$.

Theorem 3 implies that $[z^n]F(z)$ decomposes as a finite sum of the following form:

$$[z^n]F(z) = \sum_{\zeta \in \overline{\mathbb{Q}}} \frac{C_\zeta \zeta^n}{n^{\kappa_\zeta}}(1 + o(1)) \tag{4.2}$$

for some $\kappa_\zeta \in \mathbb{Q}$ and some

$$C_\zeta \in \mathrm{Vect}_{\overline{\mathbb{Q}}}\left\{\frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)} \;\middle|\; s_1, ..., s_k \in \mathbb{Q},\ s_1 + \cdots + s_k = -\kappa_\zeta \mod \mathbb{Z}\right\}.$$

Comparing Eqs. (4.1) and (4.2), we get $\omega \in \overline{\mathbb{Q}}$, $\kappa_\omega = \kappa$ and

$$C = C_\omega \in \mathrm{Vect}_{\overline{\mathbb{Q}}}\left\{\frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)} \;\middle|\; s_1, ..., s_k \in \mathbb{Q},\ s_1 + \cdots + s_k = -\kappa \mod \mathbb{Z}\right\}.$$

If C is a product of values of the Gamma function at rational numbers then, in some cases, Rohrlich-Lang Conjecture (or Schneider and Wolfart-Wüstholz theorems) will lead to a contradiction and hence will prove that $F(z)$ has grade $\geq k + 1$.

Remarks. In [2], Allouche and Mendès-France wondered if any algebraic function is the Hadamard product of a finite number of quadratic functions. The answer is negative because the n-th Taylor coefficients of such a Hadamard product is of the form (4.2) with $\kappa_\zeta \in \frac{1}{2}\mathbb{Z}$, and $[z^n](1 - z)^{1/3}$ cannot be represented in this form.

5. Proof of Theorem 1

We recall that

$$F_{\mathbf{e},\mathbf{f}}(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{\mu}(e_i n)!}{\prod_{j=1}^{\lambda}(f_j n)!} z^n,$$

where $\mathbf{e} = (e_1, ..., e_\mu) \in \mathbb{Z}_{>0}^{\mu}$ and $\mathbf{f} = (f_1, ..., f_\lambda) \in \mathbb{Z}_{>0}^{\lambda}$ are such that

$$\sum_{i=1}^{\mu} e_i = \sum_{j=1}^{\lambda} f_j$$

and $e_i \neq f_j$ for all $(i,j) \in \{1, ..., \mu\} \times \{1, ..., \lambda\}$. We define its associated Landau function $\Delta_{\mathbf{e},\mathbf{f}} : \mathbb{R} \to \mathbb{Z}$ by

$$\Delta_{\mathbf{e},\mathbf{f}}(x) = \sum_{i=1}^{\mu} \lfloor e_i x \rfloor - \sum_{j=1}^{\lambda} \lfloor f_j x \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the floor function. We now summarize a few properties of $F_{\mathbf{e},\mathbf{f}}(z)$.

Proposition 4. (i) *$F_{\mathbf{e},\mathbf{f}}(z) \in \mathbb{Z}[[z]]$ if and only if $\Delta_{\mathbf{e},\mathbf{f}}(x) \geq 0$ for all $x \in [0,1]$.*
(ii) *$F_{\mathbf{e},\mathbf{f}}(z)$ is algebraic over $\overline{\mathbb{Q}}(z)$ if and only if $\Delta_{\mathbf{e},\mathbf{f}}(x) \in \{0,1\}$ for all $x \in [0,1]$.*
(iii) *If $\Delta_{\mathbf{e},\mathbf{f}}(x) \leq -1$ for some $x \in [0,1]$, then $F_{\mathbf{e},\mathbf{f}}(z)$ is not globally bounded.*
(iv) *If $\lambda \leq \mu$, then $F_{\mathbf{e},\mathbf{f}}(z)$ is not globally bounded.*

Proof. (i) It is a classical result proved by Landau [12].

(ii) It is a reformulation, due to Rodriguez-Villegas [18], of the Beukers-Heckman criterion for the algebraicity of hypergeometric series [3].

(iii) It is a refinement of Landau's theorem proved for example in [5].

(iv) It is proved in [4] that $\Delta_{\mathbf{e},\mathbf{f}} \geq 0$ on $[0,1]$ if and only its maximum on $[0,1]$ is $\lambda - \mu > 0$. Therefore, having $\mu \geq \lambda$ implies that $\Delta_{\mathbf{e},\mathbf{f}}$ takes negative values on $[0,1]$ and then, by (iii), that $F_{\mathbf{e},\mathbf{f}}(z)$ is not globally bounded. □

We are now in position to prove Theorem 1. We first observe that, by Stirling's formula, there exists $\omega \in \mathbb{Q}^{+\times}$ such that

$$\frac{\prod_{i=1}^{\mu}(e_i n)!}{\prod_{j=1}^{\lambda}(f_j m)!} = \frac{\omega^n}{(2\pi n)^{\kappa}}\big(1 + o(1)\big) \text{ where } \kappa = \frac{\lambda - \mu}{2}. \tag{5.1}$$

(i) We have to prove that if $\lambda - \mu \leq 0$ then $F_{\mathbf{e},\mathbf{f}}$ has infinite grade. As recalled in the introduction of this paper, any series with finite grade is globally bounded. Therefore, the result follows from Proposition 4(iv).

(*ii*) We have to prove that if $\lambda-\mu\geq 2$ then $F_{\mathbf{e},\mathbf{f}}$ has grade ≥ 2. Assume at the contrary that the grade of $F_{\mathbf{e},\mathbf{f}}$ is ≤ 1. Then, according to Section 4, we have

$$\frac{1}{\pi^\kappa}\in \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s)}\;\Big|\; s\in\mathbb{Q},\ s=-\kappa \mod \mathbb{Z}\Big\}=\begin{cases}\overline{\mathbb{Q}}\pi^{-1/2} \text{ if } \lambda-\mu \text{ is odd,}\\ \\ \overline{\mathbb{Q}} \text{ if } \lambda-\mu \text{ is even.}\end{cases}$$

If $\lambda-\mu$ is even, then the only possibility is $\kappa=0$ i.e. $\lambda-\mu=0$, which is excluded. If $\lambda-\mu$ is odd, then the only possibility is $\kappa=\frac{1}{2}$ i.e. $\lambda-\mu=1$, which is again excluded.

(*iii*) Let us now assume that $\lambda-\mu=3$, so $\kappa=\frac{3}{2}$. We have to prove that $F_{\mathbf{e},\mathbf{f}}$ has grade at least 3. Assume that $F_{\mathbf{e},\mathbf{f}}$ has grade ≤ 2. According to Section 4, we have

$$\begin{aligned}\frac{1}{\pi^{3/2}}\in \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s)\Gamma(t)}\;\Big|\; s,t\in\mathbb{Q},\ s+t=\frac{1}{2} \mod \mathbb{Z}\Big\}&\\ =\overline{\mathbb{Q}}\frac{1}{\Gamma(\frac{1}{2})}+\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s)\Gamma(\frac{1}{2}-s)}\;\Big|\; s\in\mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}\Big\}.&\end{aligned}$$

This decomposition is obtained by distinguishing the case $s\in\mathbb{Z}$ or $t\in\mathbb{Z}$ from the case $s,t\notin\mathbb{Z}$. The functional equation $\Gamma(x+1)=x\Gamma(x)$ ensures that

$$\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s)\Gamma(\frac{1}{2}-s)}\;\Big|\; s\in\mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}\Big\}=\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s)\Gamma(\frac{1}{2}-s)}\;\Big|\; s\in\mathbb{Q}\cap(0,1)\setminus\{1/2\}\Big\}.$$

Using the complements formula, we see that, for all $s\in\mathbb{Q}\cap(0,1)\setminus\{\frac{1}{2}\}$,

$$\frac{1}{\Gamma(s)\Gamma(\frac{1}{2}-s)}\sim\frac{\Gamma(1-s)\Gamma(s+\frac{1}{2})}{\pi^2}\sim\frac{B(1-s,s+\frac{1}{2})}{\Gamma(\frac{1}{2})^3}$$

where $1-s, s+\frac{1}{2}\in\mathbb{Q}^{+\times}$. Hence,

$$\frac{1}{\pi^{3/2}}\in\overline{\mathbb{Q}}\frac{1}{\Gamma(\frac{1}{2})}+\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{B(s,t)}{\Gamma(\frac{1}{2})^3}\;\Big|\; s,t\in\mathbb{Q}^{+\times},\ s+t=\frac{1}{2} \mod \mathbb{Z}\Big\}.$$

Multiplying this relation by $\Gamma\big(\frac{1}{2}\big)^3=\pi^{3/2}$, we get

$$\begin{aligned}1\in\overline{\mathbb{Q}}\Gamma\Big(\frac{1}{2}\Big)^2+\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{B(s,t)\;\Big|\; s,t\in\mathbb{Q}^{+\times},\ s+t=\frac{1}{2} \mod \mathbb{Z}\Big\}&\\ =\overline{\mathbb{Q}}B\Big(\frac{1}{2},\frac{1}{2}\Big)+\mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{B(s,t)\;\Big|\; s,t\in\mathbb{Q}^{+\times},\ s+t=\frac{1}{2} \mod \mathbb{Z}\Big\}.&\end{aligned}$$

By the Wolfart-Wüstholz Theorem, we have either $1=B(1,1)\sim B\big(\frac{1}{2},\frac{1}{2}\big)$ or $1\sim B(s,t)$ for some $s,t\in\mathbb{Q}^{+\times}$ such that $s+t=\frac{1}{2}\mod\mathbb{Z}$. This contradicts Schneider's theorem (Theorem 4 above).

(*iv*) Under the Rohrlich-Lang Conjecture and when $\lambda-\mu\geq 1$, we now prove that $F_{\mathbf{e},\mathbf{f}}(z)$ has grade $\geq\lambda-\mu$. Assume that it has grade $\leq k:=\lambda-\mu-1$. Let us first observe that there is nothing to prove when $\lambda-\mu=1$. We now assume that $\lambda-\mu\geq 2$.

According to Section 4, we have

$$\frac{1}{\pi^\kappa} \in \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s_1)\cdots\Gamma(s_k)} \;\Big|\; s_1,...,s_k \in \mathbb{Q},\; s_1+\cdots+s_k = -\kappa \mod \mathbb{Z}\Big\}.$$

But, for any $s_1,...,s_k \in \mathbb{Q}$ such that $s_1+\cdots+s_k = -\kappa \mod \mathbb{Z}$, there exists $0 \le \ell \le k$ such that $s_1,..,s_\ell \in \mathbb{Q}\setminus\mathbb{Z}$ and $s_{\ell+1},...,s_k \in \mathbb{Z}$. This leads to:

$$\frac{1}{\pi^\kappa} \in \begin{cases} \displaystyle\sum_{\ell=1}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s_1)\cdots\Gamma(s_\ell)} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell = -\kappa \mod \mathbb{Z}\Big\} \text{ if } \kappa \notin \mathbb{Z} \\ \displaystyle\overline{\mathbb{Q}} + \sum_{\ell=2}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{1}{\Gamma(s_1)\cdots\Gamma(s_\ell)} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell \in \mathbb{Z}\Big\} \text{ otherwise.} \end{cases}$$

Using the complements formula, we obtain

$$\frac{1}{\pi^\kappa} \in \begin{cases} \displaystyle\sum_{\ell=1}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{\Gamma(s_1)\cdots\Gamma(s_\ell)}{\Gamma(\frac{1}{2})^{2\ell}} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell = \kappa \mod \mathbb{Z}\Big\} \text{ if } \kappa \notin \mathbb{Z} \\ \displaystyle\overline{\mathbb{Q}} + \sum_{\ell=2}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{\Gamma(s_1)\cdots\Gamma(s_\ell)}{\Gamma(\frac{1}{2})^{2\ell}} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell \in \mathbb{Z}\Big\} \text{ otherwise.} \end{cases}$$

Multiplying this relation by $\Gamma(\frac{1}{2})^{2\kappa} = \pi^\kappa$, we get

$$1 \in \begin{cases} \displaystyle\sum_{\ell=1}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{\Gamma(s_1)\cdots\Gamma(s_\ell)}{\Gamma(\frac{1}{2})^{2\ell-2\kappa}} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell = \kappa \mod \mathbb{Z}\Big\} \text{ if } \kappa \notin \mathbb{Z} \\ \displaystyle\overline{\mathbb{Q}}\frac{1}{\Gamma(\frac{1}{2})^{-2\kappa}} + \sum_{\ell=2}^{k} \mathrm{Vect}_{\overline{\mathbb{Q}}}\Big\{\frac{\Gamma(s_1)\cdots\Gamma(s_\ell)}{\Gamma(\frac{1}{2})^{2\ell-2\kappa}} \;\Big|\; s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z},\; s_1+\cdots+s_\ell \in \mathbb{Z}\Big\} \text{ otherwise.} \end{cases}$$

By the Rohrlich-Lang Conjecture, we conclude that either $\Gamma\big(\frac{1}{2}\big)^{2\kappa} \sim 1$ or there exist $\ell \in \{1,\dots,k\}$ (with $\ell \ge 2$ if $\kappa \in \mathbb{Z}$) and $s_1,...,s_\ell \in \mathbb{Q}\setminus\mathbb{Z}$ with $s_1+\cdots+s_\ell = \kappa \mod \mathbb{Z}$ such that $\Gamma(s_1)\cdots\Gamma(s_\ell)\Gamma\big(\frac{1}{2}\big)^{2\kappa-2\ell} \sim 1$ i.e.

$$G(s_1)\cdots G(s_\ell)G(1)^{\ell-2\kappa} \sim 1.$$

Therefore, we have either $\kappa = 0$ (by the transcendence of $\Gamma(\frac{1}{2})$), which is excluded because $\lambda-\mu \ge 2$, or $\ell - 2\kappa = 0$ (by Proposition 3) i.e. $\lambda-\mu = \ell$, which is also excluded because $\ell \le \lambda+\mu-1$.

6. proof of Corollary 1

The series

$$\sum_{n=0}^{\infty} \binom{a_1 n}{b_1 n} z^n$$

is algebraic by Proposition 4(ii). Indeed, its Landau function is

$$[a_1x]-[(a_1-b_1)x]-[b_1x]=\{(a_1-b_1)x\}+\{b_1x\}-\{a_1x\},$$

which takes values in $[0,2)$, hence only the values 0 or 1.

It follows that, for any integer $k\geq 1$, the grade of

$$\sum_{n=0}^{\infty}\bigg(\prod_{j=1}^{k}\binom{a_jn}{b_jn}\bigg)z^n=\sum_{n=0}^{\infty}\binom{a_1n}{b_1n}z^n*\cdots*\sum_{n=0}^{\infty}\binom{a_kn}{b_kn}z^n$$

is $\leq k$. That its grade is $\geq k$ under Rohrlich-Lang Conjecture is a consequence of Theorem 1(iv) because this series is an instance of a function $F_{\mathbf{e},\mathbf{f}}(z)$ for which $\lambda-\mu=k$. This is unconditional if $k=1,2$ or 3 by (ii) and (iii) of the same theorem.

7. Proof of Theorem 2

By Stirling's fomula, for any rational number $s\notin\mathbb{Z}_{\leq 0}$,

$$\frac{(s)_n}{n!}=\frac{1+o(1)}{\Gamma(s)n^{s-1}}.$$

It follows that

$$[z^n]_pF_{p-1}(\mathbf{a},\mathbf{b};z)=\frac{\Gamma(b_1)\cdots\Gamma(b_p)}{\Gamma(a_1)\cdots\Gamma(a_p)}\cdot\frac{1+o(1)}{n^{\kappa}}$$

where $b_p:=1$ and

$$\kappa=\sum_{j=1}^{p}a_j-\sum_{j=1}^{p}b_j.$$

We set

$$h:=h(\mathbf{a},\mathbf{b}):=\#\{1\leq j\leq p:b_j\in\mathbb{Z}\}-\#\{1\leq j\leq p:a_j\in\mathbb{Z}\}.$$

If $h=0$, there is nothing to prove. We now consider the case $h\neq 0$. Assume that ${}_pF_{p-1}(\mathbf{a},\mathbf{b};z)$ has grade $\leq k:=|h|-1$. Arguing as for the proof of Theorem 1(iv), we get that either

$$\kappa\in\mathbb{Z}\quad\text{ and }\quad\frac{\Gamma(b_1)\cdots\Gamma(b_p)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sim 1$$

or there exists $1\leq\ell\leq k$, with $\ell\geq 2$ if $\kappa\in\mathbb{Z}$, and there exist $s_1,...,s_\ell\in\mathbb{Q}\setminus\mathbb{Z}$ with $s_1+\cdots+s_\ell=\kappa \mod \mathbb{Z}$ such that

$$\frac{\Gamma(b_1)\cdots\Gamma(b_p)}{\Gamma(a_1)\cdots\Gamma(a_p)}\sim\Gamma(s_1)\cdots\Gamma(s_\ell)\Gamma\Big(\frac{1}{2}\Big)^{-2\ell}.$$

Up to renumbering, we can assume that $a_1,\dots,a_q\in\mathbb{Z}$ and $b_1,\dots,b_{q+h-1},b_p\in\mathbb{Z}$, and none of the other a_i and b_j are integers. Using the complements formula, we get that either $\kappa\in\mathbb{Z}$ and

$$\begin{aligned}1&\sim\Gamma(a_{q+1})\cdots\Gamma(a_p)\Gamma(-b_{q+h})\cdots\Gamma(-b_{p-1})\Gamma(1/2)^{-2(p-q-h)}\\&\sim G(1)^{-h}G(a_1)\cdots G(a_q)G(-b_1)\cdots G(-b_{q+h-1})\end{aligned}$$

or

$$
\begin{aligned}
1 &\sim \Gamma(s_1)\cdots\Gamma(s_\ell)\Gamma(1/2)^{-2\ell}\Gamma(a_{q+1})\cdots\Gamma(a_p)\Gamma(-b_{q+h})\cdots\Gamma(-b_{p-1})\Gamma(1/2)^{-2(p-q-h)}\\
&\sim G(1)^{\ell-h}G(s_1)\cdots G(s_\ell)G(a_1)\cdots G(a_q)G(-b_1)\cdots G(-b_{q+h-1}).
\end{aligned}
$$

Proposition 3 entails that either $h=0$ or $\ell=h$. Both cases are excluded.

8. Proof of Corollary 2

We consider $\mathbf{a}=(a_1,...,a_p)\in(\mathbb{Q}\setminus\mathbb{Z}_{\leq 0})^p$ and $b_1=\cdots=b_{p-1}=1$. Then, we have

$$
{}_pF_{p-1}\left[\begin{matrix}a_1, & a_2, & \ldots, & a_p\\ & 1, & \ldots, & 1\end{matrix};z\right]={}_1F_0\left[\begin{matrix}a_1\\ -\end{matrix};z\right]*\cdots*{}_1F_0\left[\begin{matrix}a_p\\ -\end{matrix};z\right]=\frac{1}{(1-z)^{a_1}}*\cdots*\frac{1}{(1-z)^{a_p}}.
$$

Hence, $\mathrm{grade}_{\overline{\mathbb{Q}}}({}_pF_{p-1}(\mathbf{a},\mathbf{b};z))\leq p$. The result follows from Theorem 2, which gives the other inequality.

9. Proof of Proposition 1

We want to prove that

$$
{}_3F_2\left[\begin{matrix}1/7,2/7,4/7\\ 1/2,1\end{matrix};z\right]
$$

has infinite grade under the Rohrlich-Lang conjecture. Assume that it has finite grade k. The proof of Theorem 2 given in Section 7 shows that there exist $1\leq\ell\leq k$ and $s_1,...,s_\ell\in\mathbb{Q}\setminus\mathbb{Z}$ with $s_1+\cdots+s_\ell=\frac{1}{2}\mod\mathbb{Z}$ such that

$$
1\sim G(1)^{\ell-2}G(s_1)\cdots G(s_\ell)G\Big(\frac{1}{2}\Big)G\Big(\frac{1}{7}\Big)G\Big(\frac{2}{7}\Big)G\Big(\frac{4}{7}\Big).
$$

Proposition 3 ensures that $\ell=2$. So, there exists $s\in\mathbb{Q}\setminus\frac{1}{2}\mathbb{Z}$ such that

$$
G(s)G\Big(\frac{1}{2}-s\Big)G\Big(\frac{1}{2}\Big)G\Big(\frac{1}{7}\Big)G\Big(\frac{2}{7}\Big)G\Big(\frac{4}{7}\Big)\sim 1.
$$

According to Rohrlich conjecture, the quantity

$$
f(m):=\big\{ms\big\}+\Big\{\frac{m}{2}-ms\Big\}+\Big\{\frac{m}{2}\Big\}+\Big\{\frac{m}{7}\Big\}+\Big\{\frac{2m}{7}\Big\}+\Big\{\frac{4m}{7}\Big\}
$$

is equal to 3 for any integer m coprime to $\mathrm{lcm}\{14,b\}$. We let $a\in\mathbb{Z}$ and $b\in\mathbb{Z}_{>0}$ be such that $s=a/b\in\mathbb{Q}$ and $(a,b)=1$.

Lemma 1. *We have that 7 divides b.*

Proof. Assume that 7 does not divide b. Then by the Chinese remainder theorem, we can find an integer n such that

$$
n\equiv -1\mod b,\quad n\equiv -1\mod 2,\quad n\equiv 1\mod 7.
$$

This n is coprime to $\mathrm{lcm}\{14,b\}$ and we have

$$
f(n)=\{-s\}+\Big\{\frac{1}{2}+s\Big\}+\frac{3}{2}.
$$

On the other hand, we have

$$f(1)=\{s\}+\left\{\frac{1}{2}-s\right\}+\frac{3}{2}.$$

Since s and $s+\frac{1}{2}$ are not integers, we have

$$\{s\}+\left\{\frac{1}{2}-s\right\}+\{-s\}+\left\{\frac{1}{2}+s\right\}=2,$$

so $f(1)+f(n)=5$ and hence either $f(1)$ or $f(m)$ is $\neq 3$. This is a contradiction, and 7 divides b. □

Lemma 2. *We have* $b\neq 7,14,21,28$.

Proof. This was verified with a computer with a case by case inspection. □

We shall now derive a contradiction by proving that there exist two integers m,m' coprime to $\operatorname{lcm}\{14,b\}$ such that $f(m)\neq f(m')$. Note that

$$\left\{\frac{m}{2}\right\}+\left\{\frac{m}{7}\right\}+\left\{\frac{2m}{7}\right\}+\left\{\frac{4m}{7}\right\}$$

only depends on $m \mod 14$. Therefore, it is sufficient to prove that there exist two integers m,m' coprime to $\operatorname{lcm}\{14,b\}$ such that $m=m' \mod 14$ and

$$\{ms\}+\left\{\frac{m}{2}-ms\right\}\neq\{m's\}+\left\{\frac{m'}{2}-m's\right\}.$$

But, for any m coprime to $\operatorname{lcm}\{14,b\}$, we have

$$\{ms\}+\left\{\frac{m}{2}-ms\right\}=\{ms\}+\left\{\frac{1}{2}-ms\right\}=\begin{cases}\frac{1}{2} & \text{if } \{ms\}\in\left(0,\frac{1}{2}\right)\\ \frac{3}{2} & \text{if } \{ms\}\in\left(\frac{1}{2},1\right).\end{cases}$$

Therefore, it is sufficient to prove the following lemma, the proof of which follows an idea of our colleague Emmanuel Peyre.

Lemma 3. *There exist two integers m,m' coprime to* $\operatorname{lcm}\{14,b\}$ *such that* $m=m' \mod 14$ *and*

$$\{ms\}\in\left(0,\frac{1}{2}\right)\quad \textit{and}\quad \{m's\}\in\left(\frac{1}{2},1\right).$$

Proof. We let α and β be the 2-adic and the 7-adic valuations of b and we set $b':=b/(2^\alpha 7^\beta)$, which is coprime to 14. Recall that 7 divides b i.e. $\beta\geq 1$. We split the proof according to the following three cases:

- $\alpha\geq 1$ and $b'\neq 1$,
- $\alpha=0$ and $b'\neq 1$,
- $b'=1$.

We first assume that $\alpha \geq 1$ and $b' \neq 1$ (so $b' \geq 3$, because b' is odd). Note that, for all u coprime to b', we have $(2^\alpha 7^\beta u + b', \mathrm{lcm}\{14, b\}) = 1$. Therefore, it is sufficient to prove that there exists $u \in \mathbb{Z}$ coprime to b' such that

$$(2^\alpha 7^\beta u + b')a \in \left\{1, ..., \frac{b}{2} - 1\right\} \mod b\mathbb{Z}$$

and that there exists $u' \in \mathbb{Z}$ coprime to b' such that

$$(2^\alpha 7^\beta u' + b')a \in \left\{\frac{b}{2} + 1, ..., b - 1\right\} \mod b\mathbb{Z}.$$

More generally, we claim that, for all $c \in \mathbb{Z}$, there exists $u \in \mathbb{Z}$ coprime to b' such that

$$(2^\alpha 7^\beta u + b')a \in I \mod b\mathbb{Z}$$

where $I := \{c + 1, ..., c + \frac{b}{2} - 1\}$ i.e. that

$$X := \#\big\{\overline{u} \in (\mathbb{Z}/b'\mathbb{Z})^\times \;\big|\; 2^\alpha 7^\beta ua \in I' \mod b\mathbb{Z}\big\} \geq 1$$

where $I' = I - b'a$. Since multiplication by a induces a permutation of $(\mathbb{Z}/b'\mathbb{Z})^\times$, we see that

$$X = \#\big\{\overline{u} \in (\mathbb{Z}/b'\mathbb{Z})^\times \;\big|\; 2^\alpha 7^\beta u \in I' \mod b\mathbb{Z}\big\}.$$

By the exclusion-inclusion principle, we have:

$$X = \sum_{d|b'} \mu(d) Y_d$$

where μ is the Möbius function and

$$Y_d := \#\big\{\overline{u} \in d\mathbb{Z}/b'\mathbb{Z} \;\big|\; 2^\alpha 7^\beta u \in I' \mod b\mathbb{Z}\big\}.$$

But Y_d is equal to the number of integers $k \in \{1, ..., b'/d\}$ such that $2^\alpha 7^\beta kd \in I' \mod b\mathbb{Z}$ i.e the number of integers $k \in \{1, ..., b'/d\}$ such that $k \in \frac{1}{2^\alpha 7^\beta d} I' \mod \frac{b'}{d}\mathbb{Z}$. Since $\frac{1}{2^\alpha 7^\beta d} I'$ is included in an interval of length $< \frac{b'}{d}$, we get that Y_d is the number of integers in $\frac{1}{2^\alpha 7^\beta d} I'$ i.e. the number of elements of I' divisible by $2^\alpha 7^\beta d$. Therefore,

$$Y_d = \left\lfloor \frac{\frac{b}{2} - 1}{2^\alpha 7^\beta d} \right\rfloor + \ell_d = \left\lfloor \frac{b}{2d} \right\rfloor + \ell_d = \frac{b'}{2d} - \frac{1}{2} + \ell_d$$

for some $\ell_d \in \{0, 1\}$; we set $\ell'_d = \ell_d - \frac{1}{2} \in \{\pm\frac{1}{2}\}$. Hence,

$$X = \sum_{d|b'} \mu(d)\Big(\frac{b'}{2d} + \ell'_d\Big) = \frac{1}{2}\sum_{d|b'} \mu(d)\frac{b'}{d} + \sum_{d|b'} \mu(d)\ell'_d = \frac{1}{2}\varphi(b') + \sum_{d|b'} \mu(d)\ell'_d,$$

where φ is Euler's totient function. Therefore $X \geq 1$ if and only if

$$\sum_{d|b'} \mu(d)\ell'_d > -\frac{1}{2}\varphi(b'). \tag{9.1}$$

But
$$\sum_{d|b'} \mu(d)\ell'_d \geq -\frac{1}{2}\sum_{d|b'} |\mu(d)|.$$
Therefore (9.1) holds if
$$\sum_{d|b'} |\mu(d)| < \varphi(b'). \tag{9.2}$$
This equality is true because $b' \geq 3$. Indeed if $b' = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$ where $p_1, ..., p_k$ are prime numbers and $\alpha_1, ..., \alpha_k \in \mathbb{Z}_{>0}$ then the left hand side of (9.2) is the number of square free divisors of b', which is equal to 2^k, and the right hand side is equal to $p_1^{\alpha_1-1}(p_1 - 1)\cdots p_k^{\alpha_k-1}(p_k-1)$ and it is easily seen that the former is less than the latter if $b' \geq 3$. This proves our claim.

We now assume that $\alpha = 0$ and $b' \neq 1$. Note that, for all u coprime to b', we have $(2\cdot 7^\beta u + b', \operatorname{lcm}\{14, b\}) = 1$. Therefore, it is sufficient to prove that there exists $u \in \mathbb{Z}$ coprime to b' such that
$$(2\cdot 7^\beta u + b')a \in \left\{1, ..., \frac{b-1}{2}\right\} \mod b\mathbb{Z}$$
and that there exists $u' \in \mathbb{Z}$ coprime to b' such that
$$(2\cdot 7^\beta u' + b')a \in \left\{\frac{b+1}{2}, ..., b-1\right\} \mod b\mathbb{Z}.$$
More generally, we claim that, for all $c \in \mathbb{Z}$, there exists $u \in \mathbb{Z}$ coprime to b' such that
$$(2\cdot 7^\beta u + b')a \in I \mod b\mathbb{Z}$$
where $I := \{c+1, ..., c+\frac{b-1}{2}\}$ i.e. that
$$X := \#\left\{\overline{u} \in (\mathbb{Z}/b'\mathbb{Z})^\times \;\middle|\; 2\cdot 7^\beta ua \in I' \mod b\mathbb{Z}\right\} \geq 1$$
where $I' = I - b'a$. Since multiplication by $2a$ induces a permutation of $(\mathbb{Z}/b'\mathbb{Z})^\times$, we see that
$$X = \#\left\{\overline{u} \in (\mathbb{Z}/b'\mathbb{Z})^\times \;\middle|\; 7^\beta u \in I' \mod b\mathbb{Z}\right\}.$$
By the exclusion-inclusion principle, we have
$$X = \sum_{d|b'} \mu(d) Y_d$$
where
$$Y_d := \#\left\{\overline{u} \in d\mathbb{Z}/b'\mathbb{Z} \;\middle|\; 7^\beta u \in I' \mod b\mathbb{Z}\right\}.$$
Arguing as above for the case $\alpha \geq 1$ and $b' \neq 1$, we get
$$Y_d = \left\lfloor \frac{b-1}{2\cdot 7^\beta d} \right\rfloor + \ell_d = \left\lfloor \frac{b'}{2d} \right\rfloor + \ell_d = \frac{b'}{2d} - \frac{1}{2} + \ell_d$$
for some $\ell_d \in \{0, 1\}$; we set $\ell'_d = \ell_d - \frac{1}{2} \in \{\pm\frac{1}{2}\}$. Now, the end of the proof is exactly the same that in the previous case $\alpha \geq 1$ and $b' \neq 1$.

It only remains to consider the case $b' = 1$ i.e. $b = 2^\alpha 7^\beta$. Note that, for all $u \in \mathbb{Z}$, we have $(14u+1, b) = 1$. Therefore, it is sufficient to prove that $E := \big\{\{(14u+1)s\} \bigm| u \in \mathbb{Z}\big\}$ satisfies

$$E \cap \big(0, \frac{1}{2}\big) \neq \emptyset \text{ and } E \cap \big(\frac{1}{2}, 1\big) \neq \emptyset.$$

But $14\mathbb{Z}s + \mathbb{Z} = \frac{14a\mathbb{Z}+b\mathbb{Z}}{b}$ contains $\frac{14}{b}\mathbb{Z}$. So E contains the fractional part of any rational number in $\frac{14}{b}\mathbb{Z} + s$. Since $\frac{14}{b} < \frac{1}{2}$ (by Lemma 2), the result follows. $\square$

10. Christol's hypergeometric function

In [6], Christol states that the hypergeometric series (of height 2)

$$C(z) := {}_3F_2\left[\begin{matrix}\frac{1}{9}, \frac{4}{9}, \frac{5}{9}\\ \frac{1}{3}, 1\end{matrix}; z\right]$$

est globalement bornée mais ne semble pas être le produit de Hadamard de deux fonctions de hauteur 1. Let us make a few comments on this statement:

a) From the discussion preceeding his statement, we can infer that he meant "two globally bounded hypergeometric functions of height 1". Indeed, without the "global boundedness" assumption, $C(z)$ is the Hadamard product of the two hypergeometric functions

$${}_2F_1\left[\begin{matrix}\frac{1}{9}, \frac{4}{9}\\ \frac{1}{3}\end{matrix}; z\right] \quad \text{and} \quad {}_1F_0\left[\begin{matrix}\frac{5}{9}\\ -\end{matrix}; z\right] = \frac{1}{(1-z)^{5/9}}$$

which are both of height 1. The former is not algebraic and not even globally bounded.

b) He called *reduced* a hypergeometric series such that $a_i - b_j \notin \mathbb{Z}$ for any $(i,j) \in \{1, 2, \ldots, p\}^2$, with $b_p := 1$. He then proved that a *not necessarily reduced* globally bounded hypergeometric series $F(z)$, with rational parameters and of height 1, is of the form $L(\widetilde{F}(z))$ for some differential operator L with coefficients in $\mathbb{Q}(z)$ and $\widetilde{F}(z)$ a *reduced* globally bounded hypergeometric series of height 1.

c) He proved that reduced globally bounded hypergeometric series with rational parameters and height 1 have a certain interlacing property, which turns out to be the exact necessary and sufficient condition of Beukers-Heckman [3] for algebraicity over $\overline{\mathbb{Q}}(z)$.

From a), b) and c), we can rephrase Christol's statement as follows:

The function $C(z)$ does not seem to be the Hadamard product of two algebraic hypergeometric series with rational coefficients.

This statement would follow if we could prove that $\mathrm{grade}_{\overline{\mathbb{Q}}}(C(z)) \geq 3$. Clearly $C(z)$ has grade ≥ 2 i.e. it is not algebraic over $\overline{\mathbb{Q}}(z)$ because its does not satisfy the necessary interlacing property of Beukers-Heckman. One can try to prove that its grade is not 2 by the strategy used in the previous sections. Since

$$[z^n]C(z) = \frac{\big(\frac{1}{9}\big)_n \big(\frac{4}{9}\big)_n \big(\frac{5}{9}\big)_n}{\big(\frac{1}{3}\big)_n \big(1\big)_n^2} = \frac{\Gamma\big(\frac{1}{3}\big)}{\Gamma\big(\frac{1}{9}\big)\Gamma\big(\frac{4}{9}\big)\Gamma\big(\frac{5}{9}\big)} \cdot \frac{1+o(1)}{n^{11/9}},$$

we are led to decide whether or not there exists some rational number s such that $s, \frac{7}{9}+s \notin \mathbb{Z}$ and

$$G\Big(\frac{1}{3}\Big)G\Big(\frac{8}{9}\Big)G\Big(s+\frac{7}{9}\Big)G\big(-s\big) \sim 1.$$

We would like to show that there is no solution, under the Rohrlich-Lang Conjecture at least. However, it turns out that there are solutions (mod $\mathbb{Z}$): for instance, $s = \frac{1}{3}, \frac{8}{9}$ (explained by the complements formula), $\frac{4}{9}$ (explained by the distribution relations and complements formula) and also $\frac{1}{2}$ and many more. Therefore, this approach does not yield any contradiction and the profusion of solutions does not help to describe in any useful way the putative algebraic functions $A(z)$ and $B(z)$ such that $C(z) = A(z) * B(z)$.

Finally, as for the hypergeometric function of Proposition 1, if one can prove that $\mathrm{grade}_{\overline{\mathbb{Q}}}(C(z)) \geq 3$, then automatically $\mathrm{grade}_{\overline{\mathbb{Q}}}(C(z)) = \infty$ under the Rohrlich-Lang Conjecture.

11. Proof of Proposition 2

The main result of [14] is that, as $n \to +\infty$,

$$[z^n]M_{\mathbf{r}}(z) = \sum_{k=0}^{n} \binom{n}{k}^{r_0} \binom{n+k}{n}^{r_1} \binom{n+2k}{n}^{r_2} \cdots \binom{n+mk}{n}^{r_m} \sim C\frac{\omega^n}{(n\pi)^{\frac{R-1}{2}}}$$

where $R = \sum_{j=0}^{m} r_j$ and $C, \omega \in \overline{\mathbb{Q}}^{\times}$. We are exactly in the same situation as in (5.1), where $R-1$ is replaced by $\lambda - \mu$. Therefore, the proof of Proposition 2 is the same as the proof of Theorem 1 in Section 5, *mutatis mutandis.*

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