Full asymptotic expansion of transition probabilities in the adiabatic limit*

A Joye† and Ch-Ed Pfister‡
† Département de Physique, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland
‡ Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Lausanne, Switzerland

Abstract. We consider a two-level quantum mechanical system driven by an analytic time-dependent Hamiltonian of the form $H(\varepsilon t)$. In the adiabatic limit, $\varepsilon \ll 1$, the transition probability $P(+, -)$ from one energy level (labelled by $-$) at time $t = -\infty$ to the other (labelled by $+$) at time $t = +\infty$ is known to behave as $P(+, -) = \exp(-\alpha_1 \varepsilon^{-1}) \exp(\alpha_0)(1 + O(\varepsilon))$. Using a simple iterative procedure generating Hamiltonians $H_0 = H, H_1, \ldots, H_{N+1}$, we compute the full asymptotic expansion of the transition probability

$$\ln P(+, -) = -\alpha_1 \varepsilon^{-1} + a_0 + \sum_{j=0}^{N} a_j \varepsilon^j + O(\varepsilon^{N+1}) \quad \forall N \gg 0.$$

1. Introduction

The adiabatic theorem of quantum mechanics has a long history since it had already been established by Born and Fock in 1928 [1]. This theorem describes the asymptotic regime of the slow evolution of a quantum mechanical system. Consider the Schrödinger equation (with $\hbar = 1$)

$$i \frac{d\psi}{dt}(\tau) = H(\varepsilon \tau)\psi(\tau). \tag{1.1}$$

The parameter $1/\varepsilon$ is the characteristic time-scale of the system. In terms of the rescaled time $t = \varepsilon \tau$ the equation reads

$$i \varepsilon \frac{d\psi_r}{dt}(t) = H(t)\psi_r(t). \tag{1.2}$$

The adiabatic limit corresponds to the singular limit $\varepsilon \to 0$ which describes an infinitely slow evolution. In this limit, if $H(t)$ has an energy level $e(t)$ isolated for all $t \in [t_0, t_1]$, then a system in the eigenstate corresponding to $e(t_0)$ at $t_0$ evolves to an eigenstate corresponding to $e(t_1)$ at $t_1$. In particular, a transition to an eigenstate with an energy different from $e(t_1)$ is forbidden.

The natural task which comes next is to estimate this transition probability when $\varepsilon$ is small but finite. It has been shown under a very general hypothesis that the transition probability tends to zero as $\varepsilon^2$ [2]. This gives a bound to the leading term of the asymptotic behaviour.

* Supported by Fonds National Suisse de la Recherche, Grant 2000-5.600.

0305-4470/91/040753+14$03.50 © 1991 IOP Publishing Ltd

753
When the Hamiltonian is $C^\infty$-smooth and $d^n/dt^n H(t)|_{t=t_1} = 0$ for all $n$ one can write an asymptotic expansion in powers of $\varepsilon$ for the transition probability at time $t_1$.

This result was first obtained by Lenard [3] for matrices and has been generalized in a different way by Garrido [4], Nenciu [5] and Joye and Pfister [6] using the iterative scheme of section 2. An important feature of this asymptotic expansion is that the first $N$ terms ($N \geq 2$) vanish if $d^n/dt^n H(t)|_{t=t_1} = 0$ for all $n \leq N$, showing that in such a case the transition probability at time $t_1$ is of order $O(\varepsilon^{N+1})$. An interesting case occurs if $d^n/dt^n H(t)|_{t=t_1} = d^n/dt^n H(t)|_{t=t_1} = 0$ for all $n$, since the transition probability at time $t_1$ is smaller than $O(\varepsilon^N)$ for all $N$.

This result can be improved if the Hamiltonian $H(t)$ depends analytically on time in a region including the real axis. The analyticity assumption and the above conditions on the derivatives of $H(t)$ at $t_0$ and $t_1$ are compatible if we choose $t_0 = -\infty$ and $t_1 = +\infty$. It can be shown that they are realized provided $H(t)$ is analytic in a strip including the real axis and $H(t)$ has well-defined limits $H(\pm \infty)$ with $t \rightarrow +\infty$ and $t \rightarrow -\infty$. Under the weak hypothesis that the limit of $|t|^{1+\alpha}(H(t) - H(\pm \infty))$ is zero when $t \rightarrow \pm \infty$ for some $\alpha > 0$, it has been shown by Joye and Pfister [7] that the transition probability at time $t_1 = +\infty$ is bounded by $e^{-\delta/\varepsilon}$, $\delta > 0$. Similar results have been derived recently in a different way by Nenciu [8]. All these results hold for very general Hamiltonians. We should also mention the paper by Jaksic and Segert [9] on the same subject; however, their results are weaker. This behaviour was expected to be true for a long time. In the case of an analytic $2 \times 2$ matrix, real symmetric on the real axis, Dykhne [10] proposed in 1962 a formula for the exponentially small leading term of the transition probability at $t_1 = +\infty$. A proof of this formula was given in 1977 by Hwang and Pechukas [11]. In the general case of an analytic $2 \times 2$ matrix, Hermitian on the real axis, the Dykhne formula must be completed by a prefactor of geometrical nature as observed independently by Berry [12] and Joye et al [13]. A detailed analysis as well as a geometrical interpretation of the Dykhne formula is given in [13]. Finally we would like to mention two recent works by Hagedorn on related topics [14] and [15]. In [14] the author gives an asymptotic expansion in $\varepsilon$ of the wavefunction in the presence of a real eigenvalue-crossing point. In the second paper the transition probability is computed in the limit $\varepsilon \rightarrow 0$ for a system having an avoided crossing with a gap of order $\sqrt{\varepsilon}$. In this scaling limit the end result is given by the Landau-Zener formula. Notice that there is no geometric factor in this case.

In this paper we derive a generalization of the Dykhne formula which allows us to write an asymptotic expansion in powers of $\varepsilon$ at any order for the logarithm of the transition probability at $t_1 = +\infty$. The main point of the proof is to combine the iterative scheme of [6] in order to get corrections up to order $\varepsilon^N$ and to use the method of [13] to estimate the higher-order terms. The idea to apply the iterative scheme of Garrido to derive an asymptotic expansion of the Berry phase was already used by Berry [16]. This procedure has been further emphasized in a recent work by Berry [17] where it is called superadiabatic renormalization. In these two references the nature of the asymptotic expansion is studied and the results should apply to our expansion (1.12). With the intent of stating our results in a precise way, let us describe what kinds of two-level system are considered. They can be characterized by four conditions.

1) Analyticity. There exists a closed strip $S_\alpha = \{ z = t + is \in \mathbb{C} | |s| \leq a \}$ such that the Hamiltonian $H(t)$, $t \in \mathbb{R}$ is a $2 \times 2$ Hermitian traceless matrix which has an analytic extension on some domain containing $S_\alpha$. 
Remark. The important properties of $S_\alpha$ which we use are that it is a connected, simply connected set containing $\mathbb{R}$ and coincides with a strip when $|t|$ is large enough. It is convenient to suppose that $S_\alpha$ is closed.

(II) Behaviour at infinity. There exist $2 \times 2$ non-degenerate Hermitian matrices $H(+)$ and $H(-)$ such that
\[
\lim_{r \to +\infty} \sup_{|s| < a} \|H(t + is) - H(\pm)| |t|^{1+\alpha} = 0
\]
for some positive $\alpha$.

(III) Separation of the spectrum. For each $t \in \mathbb{R}$ the spectrum of $H(t)$ consists of two separated eigenvalues $e^+(t)$ and $e^-(t)$ such that $e^+(t) - e^-(t) \geq \delta$, $\delta > 0$.

The condition on the trace can always be satisfied by an energy shift. A convenient way of looking at such systems is to write them as spin systems in a time-dependent magnetic field
\[
H(z) = B(z) \cdot s
\]
\[
= B_1(z) \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + B_2(z) \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + B_3(z) \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
the functions $B_k$ being analytic in $S_\alpha$, $B_k(\bar{z}) = B_k(z)$ and for some positive $\alpha$
\[
\lim_{r \to +\infty} \sup_{|s| < a} |B_k(t + is) - B_k(\pm)| |t|^{1+\alpha} = 0.
\]
By the Cauchy formula we have for any $a' < a$ and integer $n \geq 1$
\[
\lim_{r \to +\infty} \sup_{|s| < a'} \left| \frac{d^n}{dz^n} B_k(t + is) \right| |t|^{1+\alpha} = 0.
\]
Remark. By choosing the constant $a$ slightly smaller we may suppose that (1.5) is true with $a' \leq a$.

The eigenvalues on the real axis are
\[
e^\pm(t) = \pm \frac{1}{2} \sqrt{\rho(t)}
\]
where
\[
\rho(t) = B_1^2(t) + B_2^2(t) + B_3^2(t)
\]
is strictly positive. By convention we choose in (1.6) the branch of the square root which is positive on the positive real axis. The corresponding eigenprojections are
\[
P^\pm(t) = \frac{1}{2} \pm \frac{B(t) \cdot s}{\sqrt{\rho(t)}}.
\]
The eigenvalues and eigenprojections on $S_\alpha$ are defined by the analytic continuations of (1.6) and (1.8). They are multivalued and singular at the eigenvalue crossings which coincide with the zeros of the analytic continuation $\rho(z)$ of $\rho(t)$ in $S_\alpha$. Notice that $\rho(z)$ is single valued in $S_\alpha$. We suppose furthermore:

(IV) Eigenvalue crossings. The set $X$ of zeros of $\rho(z)$ in $S_\alpha$ consists of $2n$ interior points $z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n$ and each zero is simple. By convention $\text{Im} \ z_k > 0$, $k = 1, \ldots, n$.  

Let $\psi_r$ be the solution of the Schrödinger equation (1.2) satisfying the boundary condition

$$\lim_{t \to -\infty} \|P^-(t)\psi_r(t)\| = 1$$

(1.9)

which is equivalent, since $\psi_r$ is normalized, to the condition

$$\lim_{t \to -\infty} \|P^+(t)\psi_r(t)\| = 0.\quad (1.10)$$

The problem which is solved in this paper is the following: to find an asymptotic expansion in powers of $\varepsilon$ for the logarithm of the transition probability

$$\mathcal{P}(+, -) = \lim_{t \to +\infty} \|P^+(t)\psi_r(t)\|^2$$

(1.11)

at time $t = +\infty$. We show in section 2 that for any integer $N \geq 0$ there exists $\varepsilon^*(N)$ such that

$$\ln \mathcal{P}(+, -) = -\alpha_{-1}\varepsilon^{-1} + \sum_{k=0}^{N} \alpha_k\varepsilon^k + O(\varepsilon^{N+1})$$

(1.12)

provided $0 \leq \varepsilon \leq \varepsilon^*(N)$. The term $\alpha_{-1}$ is positive and both terms $\alpha_{-1}$ and $\alpha_0$ can be interpreted geometrically. They are given explicitly in (2.35) and (2.37). When we put $N = 0$ in (1.12) we recover the generalized Dykhne formula found by Berry [12] and Joye et al [13]. The result (1.12) is proved in all cases for which we can prove the generalized Dykhne formula as in [13]. This means that another additional condition (condition V) is needed. This condition is analysed thoroughly in [13] and is recalled below in its geometrical version for the sake of completeness.

For each Hamiltonian $H(z) = B(z) \cdot s$ on $S_o$, we can associate a distance $d_{\rho}$ on $S_o$, where $\rho(z) = B_1^2(z) + B_2^2(z) + B_3^2(z)$. Let $\gamma$ be some rectifiable curve in $S_o$. The $\rho$-length of $\gamma$ is

$$|\gamma|_{\rho} = \int_{\gamma} |\rho(z)|^{1/2}|dz|$$

(1.13)

and the $\rho$-distance $d_{\rho}(z_1, z_2)$ between two points $z_1$ and $z_2$ of $S_o$ is given by the infimum of $|\gamma|_{\rho}$, where $\gamma$ is a rectifiable curve in $S_o$ from $z_1$ to $z_2$. Having a distance $d_{\rho}$, we can introduce the concept of a $\rho$-geodesic which is a curve $t \mapsto \gamma(t)$ which is locally shortest for the $\rho$-length: if $z_1 = \gamma(t_1)$ and $z_2 = \gamma(t_2)$ and $|t_1 - t_2|$ is small enough, then the curve between $z_1$ and $z_2$ which has the minimal $\rho$-length is given by $\gamma(t)$, $t \in [t_1, t_2]$. The supplementary condition is:

(V) Existence of an infinite geodesic. There exist an eigenvalue crossing, say $z_1$, and a geodesic $t \mapsto \gamma(t)$, $t \in \mathbb{R}$ in $S_o$, passing through $z_1$ such that $\lim_{t \to \pm \infty} \Re \gamma(t) = \pm \infty$ and $|\Im \gamma(t)| < a$ for large enough $|t|$.

In the Dykhne formula, the exponential rate of the transition probability is computed using one particular complex eigenvalue crossing (see (2.35)). In [13] we show that whenever condition V holds, the relevant eigenvalue crossing which is used in the Dykhne formula is the $z_1$ of condition V. We emphasize that $z_1$ is not necessarily the closest eigenvalue crossing to the real axis in the Euclidean distance, it is the closest eigenvalue crossing to the real axis in the $\rho$-distance. There is no known proof of the Dykhne formula without condition V, which is a condition of global character. There
is another formulation of it in terms of Stokes lines and anybody who is familiar with WKB analysis will recognize that our condition is typical in such a context. We refer to [13] for more details and examples. We finish this section with one remark. The introduction of the \( \rho \)-distance and the \( \rho \)-geometry is one new important result of [13]. It allows, in particular, the eigenvalue crossing which governs the transition probability \( \mathcal{P}(+, -) \) to be distinguished.

2. Asymptotic expansion of \( \mathcal{P}(+, -) \)

Let us now outline the strategy which is used for the proof of (1.12) and show how one computes the coefficients \( \alpha_k \). The technical details are treated in the next section. We assume that \( z_i \) is the closest zero of \( \rho(z) \) to the real axis in the \( \rho \)-distance and that condition V is fulfilled. (If we want to analyse the transition probability \( \mathcal{P}(-, +) \) where the roles of the projections \( P^+ \) and \( P^- \) are exchanged, then \( z_i \) is the relevant eigenvalue crossing.) The main result of this section is formula (2.34).

We first introduce a sequence of Hamiltonians \( H_0 = H, H_1, \ldots \) as in the papers of Garrido [4] and Nenciu [5]; however, our construction is different and simpler.

Let

\[
K_0(z) = i(P^+(z)P^+(z) + P^-(z)P^-(z)) = -\frac{B(z) \wedge B'(z)}{\rho(z)} \cdot s \tag{2.1}
\]

where ' denotes here and henceforth in this paper \( d/dz \) and \( B \wedge B' \) is the vector product of \( B \) and \( B' \). We define

\[
H_1(z, \varepsilon) = H(z) - \varepsilon K_0(z) = B_1(z, \varepsilon) \cdot s. \tag{2.2}
\]

The \( 2 \times 2 \) matrix \( H_1(z, \varepsilon) \) is meromorphic in \( S_\delta \), all its poles, if any, are of first order and at points of \( X \). It is still traceless since for any projection \( Q(z) \) we have

\[
Q(z)Q'(z)Q(z) = 0 \tag{2.3}
\]

and it satisfies condition II with the same \( H(\pm) \), as well as condition III with \( \delta/2 \) instead of \( \delta \), provided \( \varepsilon \) is small enough (see lemma 2.2 below). Let \( r \) be some fixed small positive number (whose definite value is fixed in the proof of lemma 2.3), and let \( D(z_0, r) = \{ z | |z - z_0| < r \} \) be the disc of centre \( z_0 \) and radius \( r \). We define \( \Omega \) by the closed set

\[
\Omega = S_\delta \cup \bigcup_{z_0 \in X} D(z_0, r) \tag{2.4}
\]

and choose \( r \) so small that for any two points \( z_k \) and \( z_l \) of \( X \) the discs \( D(z_k, r) \) and \( D(z_l, r) \) are disjoint, each disc \( D(z_k, r) \subset S_\delta \) and does not intersect the real axis.

The eigenvalues of \( H_1(z, \varepsilon) \) at \( z \in \Omega \) are \( \pm \frac{1}{2} \sqrt{\rho_1(z, \varepsilon)} \) where

\[
\rho_1(z, \varepsilon) = B_{1,1}^2(z, \varepsilon) + B_{1,2}^2(z, \varepsilon) + B_{1,3}^2(z, \varepsilon) \tag{2.5}
\]

is analytic and single valued in \( \Omega \). We define \( e_1^\pm(t, \varepsilon) = \pm \frac{1}{2} \sqrt{\rho_1(t, \varepsilon)} \) for all \( t \in \mathbb{R} \) and then \( e_1^+(z, \varepsilon) \) by analytic continuation. The corresponding eigenprojections are \( P_1^\pm(z, \varepsilon) \).

**Lemma 2.1.** If \( \varepsilon \) is small enough, then \( H_1(z, \varepsilon) \) is analytic on \( \Omega \). There are no eigenvalue crossings of \( H_1 \) in \( \Omega \). The variation (in the positive sense) of the argument of \( \rho_1(z, \varepsilon) \) around the boundary of any disc \( D(z_0, r), z_0 \in X \), is \( 2\pi \).
The last property mentioned in lemma 2.1 implies that under analytic continuation around a simple loop in $\Omega$ encircling exactly one disc $D(z_0, r)$, $z_0 \in X$, the eigenvalues of $H_1(z, \epsilon)$ are exchanged. Let

$$K_1(z, \epsilon) = i \left( P^+_1(z, \epsilon) P^+_1(z, \epsilon) + P^-_1(z, \epsilon) P^-_1(z, \epsilon) \right).$$

We define

$$H_2(z, \epsilon) = H(z) - \epsilon K_1(z, \epsilon).$$

By recurrence we introduce as above

$$K_{q-1}(z, \epsilon) = i \left( P^+_{q-1}(z, \epsilon) P^+_{q-1}(z, \epsilon) + P^-_{q-1}(z, \epsilon) P^-_{q-1}(z, \epsilon) \right)$$

$$= - \frac{B_{q-1}(z, \epsilon) \cdot B'_{q-1}(z, \epsilon)}{\rho_{q-1}(z, \epsilon)} \cdot \mathbb{1}$$

and

$$H_q(z, \epsilon) = H(z) - \epsilon K_{q-1}(z, \epsilon).$$

**Lemma 2.2.** For any $q \geq 0$ there exists a positive $\epsilon_\star(q)$ and an integrable function $\beta_q(t)$ such that for any $z = t + is \in \Omega$ and any $\epsilon$, $0 \leq \epsilon \leq \epsilon_\star(q)$

$$\|K_q(t + is, \epsilon)\| \leq \beta_q(t)$$

and

$$\|K_q(t + is, \epsilon) - K_{q-1}(t + is, \epsilon)\| \leq \epsilon^q \beta_q(t)$$

where $K_{-1} = 0$ by convention.

Moreover lemma 2.1 is valid for $H_q$ if $0 \leq \epsilon \leq \epsilon_\star(q)$.

We want to prove (1.12). For this purpose we work with $H_q$, $q = N + 1$, and we decompose $\psi(t)$ in a basis of analytic eigenvectors of $H_q(t, \epsilon)$, $\varphi_q^\sigma(t, \epsilon)$ and $\varphi_q^\sigma(t, \epsilon)$. We choose the phase of $\varphi_q^\sigma(t, \epsilon)$ so that $\lim_{t \rightarrow -\infty} \varphi_q^\sigma(t, \epsilon) = \varphi^\sigma(-\infty)$, where $\varphi^\sigma(-\infty)$ are fixed eigenvectors of $H(-\infty)$, and

$$\left\langle \varphi_q^\sigma(t, \epsilon) \left| \frac{d}{dt}\varphi_q^\sigma(t, \epsilon) \right. \right\rangle = 0 \quad \sigma = \pm.$$  

(2.10)

Here $\langle \rangle$ is the usual Hermitian scalar product of $\mathbb{C}^2$. Condition (2.10) is equivalent to (see e.g. [4])

$$\frac{d}{dt}\varphi_q^\sigma(t, \epsilon) = -iK_q(t, \epsilon)\varphi_q^\sigma(t, \epsilon) \quad \sigma = \pm.$$  

(2.11)

Let

$$\lambda_q^\sigma(t, \epsilon) = \int_0^t e_q^\sigma(t', \epsilon) \; dt' \quad \sigma = \pm$$  

(2.12)

and let $\psi(t)$ be decomposed as

$$\psi(t) = \sum_{\sigma = \pm} c_q^\sigma(t, \epsilon) \exp(-i \epsilon^{-1} \lambda_q^\sigma(t, \epsilon)) \varphi_q^\sigma(t, \epsilon).$$

(2.13)

We rewrite the differential equation

$$i \epsilon \frac{d\psi(t)}{dt} = (H_q(t, \epsilon) + \epsilon K_{q-1}(t, \epsilon)) \psi(t)$$

(2.14)
Taking the scalar product of this expression with \( \varphi_q^\nu \) we get

\[
\frac{d}{dt} \varphi_q^\nu(t, \epsilon) = \sum_{\tau = \pm} a_q^{\nu, \tau}(t, \epsilon) \exp\{i\epsilon^{-1}(\lambda_q^\nu(t, \epsilon) - \lambda_q^\tau(t, \epsilon))\} \varphi_q^\tau(t, \epsilon) \tag{2.16}
\]

with

\[
a_q^{\nu, \tau}(t, \epsilon) = i(\varphi_q^\nu(t, \epsilon))(K_q(t, \epsilon) - K_{q-1}(t, \epsilon))\varphi_q^\tau(t, \epsilon). \tag{2.17}
\]

By lemma 2.2

\[
|a_q^{\nu, \tau}(t, \epsilon)| \leq \epsilon^\eta b_q(t) \tag{2.18}
\]

and this implies in particular that the coefficients \( c_q^\nu(t, \epsilon) \) have well-defined limits \( c_q^\nu(\pm \infty, \epsilon) \) when \( t \to \pm \infty \), although \( \psi_\epsilon(t) \) does not have limits. The boundary condition (1.9) which is satisfied by \( \psi_\epsilon \) is equivalent to

\[
c_q^\nu(-\infty, \epsilon) = 1 \quad \text{and} \quad c_q^\nu(-\infty, \epsilon) = 0 \tag{2.19}
\]

and the transition probability \( \mathcal{P}(+, -) \) is given by

\[
\mathcal{P}(+, -) = |c_q^\nu(+\infty, \epsilon)|^2 \tag{2.20}
\]

since \( H_q(t) \) tends to \( H(+) \) when \( t \to \infty \) by lemma 2.2.

We now use the analyticity assumption in an essential way. On the simply connected domain \( S_q \), the solution \( \psi_\epsilon(t) \) has a single-valued analytic extension \( \psi_\epsilon(z) \) which satisfies the equation

\[
 i\epsilon\psi_\epsilon'(z) = H(z)\psi_\epsilon(z). \tag{2.21}
\]

The eigenvalues \( e_q^\nu(t, \epsilon) \), eigenvectors \( \varphi_q^\nu(t, \epsilon) \) as well as the coefficients \( c_q^\nu(t, \epsilon) \) in (2.13) have also analytic extensions on \( \Omega \), but these extensions are multivalued (see the proof of lemma 2.3). This fact is at the basis of the analysis of \( \mathcal{P}(+, -) \) made by Landau and Lifshitz [18]. Let \( z_1 \) be the eigenvalue crossing of \( H \) which is the closest to the real axis in the \( \rho \)-distance. Let \( t_1 \leq t_2 \) be two points of the real axis and \( \gamma \) a path in \( \Omega \) going from \( t_1 \) to \( t_2 \) and such that the path composed of \( \gamma \) and then the portion of the real axis from \( t_2 \) to \( t_1 \) is a simple closed path encircling \( z_1 \) but no other eigenvalue crossing of \( H \) (see figure 1). At \( t_1 \) we have

\[
\psi_\epsilon(t_1) = \sum_{\sigma = \pm} c_q^\nu(t_1, \epsilon) \exp(-i\epsilon^{-1}\lambda_q^\nu(t_1, \epsilon))\varphi_q^\nu(t_1, \epsilon). \tag{2.22}
\]
Each object on the right-hand side has an analytic extension along the path $\gamma$. Coming back to the real axis at $t_2$ we get

$$\psi_r(t_2) = \sum_{\sigma \rightarrow 0} \tilde{c}_\sigma^\sigma(t_2, \epsilon) \exp(-i\epsilon^{-1}\tilde{\lambda}_\sigma^\sigma(t_2, \epsilon))\tilde{\varphi}_q^\sigma(t_2, \epsilon) \quad (2.23)$$

where $\tilde{\cdot}$ means that (2.23) is the analytic extension of (2.22) along $\gamma$. This expression can be compared with

$$\psi_r(t_2) = \sum_{\sigma \rightarrow 0} c_\sigma^\sigma(t_2, \epsilon) \exp(-i\epsilon^{-1}\lambda_\sigma^\sigma(t_2, \epsilon))\varphi_q^\sigma(t_2, \epsilon) \quad (2.24)$$

defined earlier and which coincides with the analytic extension of (2.22) along the real axis from $t_1$ to $t_2$. By lemma 2.1, which is valid for $H_u$,

$$e_q^+(t_2, \epsilon) = \tilde{e}_q^+(t_2, \epsilon) \quad e_q^+(t_2, \epsilon) = \tilde{e}_q^-(t_2, \epsilon) \quad (2.25)$$

since the eigenvalues of $H_u(z)$ are given by $\pm\sqrt{\rho_u(z)}$, $\rho_u(z) = B_{u,1}^2(z, \epsilon) + B_{u,2}^2(z, \epsilon) + B_{u,3}^2(z, \epsilon)$. The fact that $\varphi_q^\sigma$ is an eigenvector of $H_u$ is not affected by the analytic extension and therefore we can write

$$\tilde{\varphi}_q^+(t_2, \epsilon) = \exp(-i\theta_q^+\epsilon)\varphi_q^+(t_2, \epsilon) \quad (2.26)$$

and

$$\tilde{\varphi}_q^-(t_2, \epsilon) = \exp(-i\theta_q^-\epsilon)\varphi_q^-(t_2, \epsilon). \quad (2.27)$$

In general $\theta_q^{\sigma}$ is a complex number and depends on $t_1$, $t_2$ and $\gamma$. However, if we choose another path in $\Omega$ in the same homotopy class as $\gamma$, then (2.23) is not changed and, in particular, $\theta_q^{\sigma}$ in (2.26) and (2.27) is not modified. Since for any $t_1$ and $t_2$ on the real axis $\varphi_q^\sigma(t_2, \epsilon)$ is the analytic continuation of $\varphi_q^\sigma(t_1, \epsilon)$ along the real axis and since $\|\varphi_q^\sigma(t, \epsilon)\| = 1$ for all $t \in \mathbb{R}$, the imaginary part of $\theta_q^{\sigma}$ is independent of the choice of $t_1$ or $t_2$. Comparing (2.23) and (2.24) using (2.26) and (2.27) we get the fundamental relations

$$c_q^\sigma(t_2, \epsilon) = \exp(-i\epsilon^{-1}\tilde{\lambda}_q^+(t_2, \epsilon) + \epsilon^{-1}\lambda_q^-(t_2, \epsilon)) \exp(-i\theta_q^\sigma)\tilde{z}_q^+(t_2, \epsilon) \quad (2.28)$$

and

$$c_q^\sigma(t_2, \epsilon) = \exp(-i\epsilon^{-1}\tilde{\lambda}_q^-(t_2, \epsilon) + \epsilon^{-1}\lambda_q^+(t_2, \epsilon)) \exp(-i\theta_q^\sigma)\tilde{z}_q^-(t_2, \epsilon). \quad (2.29)$$

As for $\theta_q^{\sigma}$ the imaginary part of $\tilde{\lambda}_q^\sigma$ does not depend on the choice of $t_1$ and $t_2$ and remains unchanged if we choose another path in $\Omega$ in the same homotopy class as $\gamma$. This allows the following expressions to be given for these quantities. Let $\eta$ be any simple closed path in $\Omega$ based at 0 (or any other point of the real axis) which encircles only the eigenvalue crossing $z_1$ of $H$ and which is oriented clockwise. Then, for any $t_1 \in \mathbb{R}$, $t_2 \in \mathbb{R}$ and any path $\gamma$ from $t_1$ to $t_2$ as above, we have

$$\text{Im } \tilde{\lambda}_q^\sigma(t_2, \epsilon) = \text{Im } \int_\eta e_q^\sigma(z, \epsilon) \, dz \quad (2.30)$$

where $\int_\eta e_q^\sigma$ is the integral over $\eta$ of the analytic continuation of $e_q^\sigma$ along $\eta$. Similarly we can show that

$$\text{Im } \theta_q^\sigma = \text{Im } \int_\eta \frac{B_{u,3}(B_{u,1}B_{u,2} - B_{u,2}B_{u,1})}{2\sqrt{\rho_u(B_{u,1}^2 + B_{u,2}^2)}} \quad (2.31)$$
where here we must choose \( \eta \) so that \( B_{\alpha}(z, \epsilon) + B_{\alpha z}(z, \epsilon) \neq 0 \) on \( \eta \). To determine \( \text{Im} \theta_{\alpha}^{-} \) we use the relation

\[
\text{Im} \theta_{\alpha}^{-} = -\text{Im} \theta_{\alpha}^{+}. \tag{2.32}
\]

From (2.29) we have

\[
\mathcal{Q}(+) = \left| e_{\alpha}(+\infty, \epsilon) \right|^2
\]

\[
= \exp \left( 2\epsilon^{-1} \text{Im} \int_{\eta} e_{\alpha}^{-}(z, \epsilon) \, dz \right) \exp \left( 2 \text{Im} \theta_{\alpha}^{+}(\epsilon) \right) \left| \tilde{e}_{\alpha}^{-}(+\infty, \epsilon) \right|^2. \tag{2.33}
\]

We refer the reader to [13] for more details on this first part of the analysis and in particular for a proof of (2.31) and we come to the hard part of the analysis. It amounts to control the behaviour of \( \left| \tilde{e}_{\alpha}^{-}(+\infty, \epsilon) \right|^2 \) as a function of \( \epsilon \). This problem has been solved by Hwang and Pechukas [11]. Using their method we have:

**Lemma 2.3.** If conditions I-V hold and if \( \psi_{\nu}(z) \) satisfies the boundary condition (1.9) (or (2.19)), then

\[
|\tilde{e}_{\alpha}^{-}(+\infty, \epsilon)| = 1 + O(\epsilon^\gamma)
\]

provided \( \epsilon \) is small enough.

This lemma is proved in the next section. From it we get the main result of this paper.

**Main result**

\[
\mathcal{Q}(+) = \exp \left( 2\epsilon^{-1} \text{Im} \int_{\eta} e_{\alpha}^{-}(z, \epsilon) \, dz \right) \exp \left( 2 \text{Im} \theta_{\alpha}^{+}(\epsilon) \right) \left( 1 + O(\epsilon^\gamma) \right). \tag{2.34}
\]

From (2.34) it is easy to write the asymptotic expansion (1.12) by writing such an expansion for \( e_{\alpha}^{+}(z, \epsilon) \) and \( \theta_{\alpha}^{+}(\epsilon) \) using the explicit formula (2.31). This creates no difficulty since all expressions to be expanded are analytic in \( z \) and \( \epsilon \) for \( z \in \Omega \) and \( |\epsilon| \) small enough and uniformly bounded in \( z, \epsilon \in \Omega \). Let us finish this section by giving the first two terms \( \alpha_{-1} \) and \( \alpha_0 \) of (1.12). We have

\[
\alpha_{-1} = -2 \text{Im} \int_{\eta} e_{\alpha}^{-}(z) \, dz \tag{2.35}
\]

where \( \int_{\eta} e_{\alpha}^{-} \) is the integral over \( \eta \) of the analytic continuation of the eigenvalue \( e_{\alpha}^{-} \) of \( H \) along \( \eta \). It is shown in [13] that

\[
\text{Im} \int_{\eta} e_{\alpha}^{-}(z) \, dz = -d_{\mu}(z_1, \mathbb{R}) \tag{2.36}
\]

where \( d_{\mu}(z_1, \mathbb{R}) \) is the \( \rho \)-distance to the real axis of the closest eigenvalue crossing of \( H \) to the real axis in this distance. From perturbation theory we have that the first term in the expansion of \( e_{\alpha}^{-}(z, \epsilon) \) in \( \epsilon \) is \( e_{\alpha}^{-}(z) \) and that there is no term proportional to \( \epsilon \). Therefore \( \alpha_0 \) is given by

\[
\alpha_0 = 2 \text{Im} \int_{\eta} \frac{B_2(B_0B_1' - B_1B_0')}{2\sqrt{\rho(B_0^2 + B_1^2)}}. \tag{2.37}
\]
3. Proofs of lemmas

We prove by recurrence on \( q \) the following statements:

(i) for any integer \( n \geq 0 \) and any integer \( q \geq 0 \) there exist constants \( D_{q,n} \) and \( \varepsilon_*(q) \) (independent of \( n \)) such that for all \( z \in \Omega \), all \( \varepsilon \leq \varepsilon_*(q) \) and \( k = 1, 2, 3 \), \( B_{q,k}(z, \varepsilon) \) are analytic on \( \Omega \),

\[
|B_{q,k}(z, \varepsilon)| \leq D_{q,0}
\]

and

\[
(1 + |\text{Re} \ z|)^{1+\alpha} |B^{(n)}_{q,k}(z, \varepsilon)| \leq D_{q,n}
\]

where \( \alpha \) is the constant appearing in condition (11), \( B_{0,k} = B_k, k = 1, 2, 3 \), and \( B^{(n)}_{q,k} = \frac{d^n}{dz^n} B_{q,k} \); (ii) there exists \( \delta > 0 \) such that

\[
\inf_{\varepsilon = \varepsilon_*(q)} \inf_{z \in \Omega} |\rho_q(z, \varepsilon)| \geq \delta;
\]

(iii) the variation of the argument of \( \rho_q \) around the boundary of any disc \( D(z_k, r) \) (in the positive sense) is equal to \( 2\pi \) for any \( \varepsilon \leq \varepsilon_*(q) \); (iv) for any \( q \geq 1 \) and any \( n \geq 0 \) there exists a constant \( F_{q,n} \) such that for \( k = 1, 2, 3 \)

\[
\sup_{\varepsilon = \varepsilon_*(q-1)} \sup_{z \in \Omega} (1 + |z|)^{1+\alpha} |B^{(n)}_{q,k}(z, \varepsilon) - B^{(n)}_{q-1,k}(z, \varepsilon)| \leq \varepsilon^q F_{q,n}
\]

and for any \( q \geq 1 \) there exists a constant \( G_q \) such that

\[
\sup_{\varepsilon = \varepsilon_*(q-1)} \sup_{z \in \Omega} |\rho_q(z, \varepsilon) - \rho_{q-1}(z, \varepsilon)| \leq \varepsilon^q G_q.
\]

The validity of these four statements ensures the validity of lemmas 2.1 and 2.2.

Estimates (1.4) and (1.5) together with the remark following (1.5) imply that (i) is true for \( q = 0 \) with \( \varepsilon_*(0) = \infty \). Clearly there exists \( \delta > 0 \) such that

\[
\inf_{z \in \Omega} |\rho(z)| = \inf_{z \in \Omega} |\rho_0(z)| \geq 2\delta \quad (3.1)
\]

so that (ii) is also verified. Since \( \rho_0 \) has exactly one zero and no pole inside any disc \( D(z_k, r) \) (iii) follows, and finally (iv) is an immediate consequence of (i) and

\[
B_q = B + \varepsilon \frac{B_{q-1} \wedge B'_{q-1}}{\rho_{q-1}} \quad (3.2)
\]

when \( q = 1 \). Let us suppose that the four statements are true for \( q = N - 1 \) and let us prove them for \( q = N \). It is immediate that (i) is true for \( q = N \) and by (3.1), (3.2) and (i) the affirmation (ii) is also correct. Affirmation (iii) is a standard consequence of the argument principle. Indeed

\[
|\rho_q(z, \varepsilon) - \rho(z)| \leq \varepsilon \times \text{constant} \quad (3.3)
\]

uniformly in \( z \in \Omega \) and \( \varepsilon \leq \varepsilon_*(q-1) \). Let \( \gamma \) be the boundary of one disc (positively oriented). If \( \varepsilon \) is small enough we have for all \( z \in \gamma \)

\[
\left| \frac{\rho_q(z, \varepsilon)}{\rho(z)} - 1 \right| < 1. \quad (3.4)
\]

Let \( G(z, \varepsilon) = \frac{\rho_q(z, \varepsilon)}{\rho(z)} \). This is a meromorphic function on some open set containing the disc and which has no zero and no pole on \( \gamma \). The index of the image of \( \gamma \) by
Asymptotics of adiabatic transition probabilities

G with respect to \( z = 0 \) is zero since the image curve is contained in a disc of centre \( z = 1 \) and radius smaller than 1. Thus

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{G'(z, \varepsilon)}{G(z, \varepsilon)} \, dz = \frac{1}{2\pi i} \int_{\gamma} \frac{\rho_q'(z, \varepsilon)}{\rho_q(z, \varepsilon)} \, dz - \frac{1}{2\pi i} \int_{\gamma} \frac{\rho'(z)}{\rho(z)} \, dz = 0. \tag{3.5}
\]

This proves (iii). Finally we have

\[
B_q = B_{q-1} + \varepsilon \left( \frac{B_{q-1} \wedge B'_{q-1} - B_{q-2} \wedge B'_{q-2}}{\rho_{q-1}} \right) \tag{3.6}
\]

and (iv) follows easily from the induction hypothesis.

It remains to prove lemma 2.3. Let \( \Omega^- = \{ z = t + is, |s| \leq a, t > t^- \} \) with \( t^- \) such that \( \Omega^- \subset \Omega \). Let \( \Omega^+ = \{ z = t + is, |s| \leq a, t > t^+ \} \) with \( \Omega^+ \subset \Omega \) and \( t^+ > t^- \). On \( \Omega^- \) we define \( \Delta_q(z, \varepsilon) \) as the analytic continuation on \( \Omega^- \) of the function

\[
\int_0^t (\varepsilon_q^{\ast}(t', \varepsilon) - \varepsilon_q^{\ast}(t', \varepsilon)) \, dt'. \tag{3.7}
\]

Let us recall that by our convention \( \varepsilon_q^{\ast}(t, \varepsilon) = e''(i), \sigma = +, - \). Condition \( V \) implies the existence of a path in \( S_\sigma, \, r \to \gamma(r) \), parametrized by \( r \in \mathbb{R} \), with the following properties:

(a) \( \gamma(r) \) is contained in the upper half-plane:

\[
\lim_{r \to \pm \infty} \text{Im} \, \gamma(r) = s^+, s^+ < a \quad \lim_{r \to \pm \infty} \text{Re} \, \gamma(r) = \pm \infty
\]

(b) the open region between the real axis and \( \gamma \) contains the eigenvalue crossing \( z_1 \), but no other eigenvalue crossing of \( H \);

(c) \( r \to \text{Im} \, \Delta_\alpha(\gamma(r)) \) is a non-decreasing function of \( r \), where \( \Delta_\alpha \) is defined by analytic continuation along \( \gamma \).

This is the main content of theorem 2.2 of [13]. Actually we can require that the path \( \gamma \) has the property:

(d) there exist \( r_1 \) and \( r_2 > r_1 \) such that on \((-\infty, r_1] \) the function \( \text{Im} \, \Delta_\alpha(\gamma(r)) \) is constant, on \([r_1, r_2] \) its derivative with respect to \( r \) is strictly positive and on \([r_2, \infty) \) the function is again constant.

We can now fix the radius \( r \) of the small discs around the eigenvalue crossings of \( H \). We choose \( r \) so small that the (Euclidian) distance from any eigenvalue crossing to \( \gamma \) is larger than \( 2r \), so that \( \gamma \) is entirely in \( \Omega \). The main step in the proof of lemma 2.3 is to show that there exists a path \( \gamma_0 \) in \( \Omega \) such that for all sufficiently small \( \varepsilon \) properties (a) and (b) are true and property (c) is true with \( \Delta_\alpha \) instead of \( \Delta_\alpha \). Indeed, if we have such a path we proceed as follows. We make an analytic continuation of the coefficients \( c_q^{\ast}(t, \varepsilon) \) and \( c_q^{\ast}(t, \varepsilon) \) in \( \Omega^- \). We get functions \( c_q^{\ast}(z, \varepsilon) \) and \( c_q^{\ast}(z, \varepsilon) \) which are solutions of the differential equation

\[
c_q^{\ast} = a_q^{\ast} c_q^{\ast} + a_q^{\ast} \exp(-i\varepsilon^{-1}\Delta_q) c_q^{\ast}
\]

\[
c_q^{\ast} = a_q^{\ast} \exp(i\varepsilon^{-1}\Delta_q) c_q^{\ast} + a_q^{\ast} c_q^{\ast}
\]

where in (3.8) \( a_q^{\ast} = a_q^{\ast}(z, \varepsilon) \) is the analytic continuation of \( a_q^{\ast}(t, \varepsilon) \) in \( \Omega^- \). From (2.17) it is not completely obvious that \( a_q^{\ast} \) has an analytic continuation. However, we can see that this is the case by remarking that the function \( K_q \) has a single-valued analytic extension on \( \Omega \) (see (2.8)). Let

\[
U_q'(z, \varepsilon) = -iK_q(z, \varepsilon) U_q(z, \varepsilon) \tag{3.9}
\]
with
\[
\lim_{\tau \to -\infty} U_q(t, \varepsilon) = 1.
\] (3.10)

This matrix is invertible on $\Omega^-$ since its determinant is one ($K_q$ is traceless) and it is unitary on the real axis. From (2.11) we have
\[
\varphi''_q(t, \varepsilon) = U_q(t, \varepsilon) \varphi''(-\infty) \quad \sigma = +, -
\] (3.11)
so that we can write
\[
a''_q(t, \varepsilon) = i((U_q(t, \varepsilon)^{-1})^* \varphi''(-\infty))(K_q(t, \varepsilon) - K_{q-1}(t, \varepsilon)) U_q(t, \varepsilon) \varphi'(0^-)).
\] (3.12)

This expression manifestly has an analytic continuation on $\Omega^-$. Then we make an analytic continuation of $c_q$, $\Delta_q$ and $a''_q^\varepsilon$ along $\gamma_q$ and then in $\Omega^+$ since by property $(a)$ $\gamma_q$ starts in $\Omega^-$ and ends in $\Omega^+$. Of course (3.8) still holds. From lemma 2.2 and the definition of $U_q$ we have in $\Omega^-$ or in $\Omega^+$
\[
\lim_{\tau \to -\infty} c''_q(t + i\varepsilon, \varepsilon) = \lim_{\tau \to +\infty} c''_q(t, \varepsilon) = c''_q(\pm\infty, \varepsilon) \quad \sigma = +, -.
\] (3.13)

Therefore by property $(b)$ the quantity $c''_q(+\infty, \varepsilon)$ is in fact given by $c''_q(-\infty, \varepsilon)$ as defined above. Thus it is sufficient to study $c''_q(\gamma_q(r), \varepsilon)$ along $\gamma_q$ for large values of $r$. Let $\dot{\gamma}_q(r) = d/dr \gamma_q(r)$. Along $\gamma_q$ we have by writing $c''_q(r, \varepsilon) \equiv c''_q(\gamma_q(r), \varepsilon), a''_q^\varepsilon(r, \varepsilon) \equiv a''_q^\varepsilon(\gamma_q(r), \varepsilon)$ and $\Delta_q(r, \varepsilon) \equiv \Delta_q(\gamma_q(r), \varepsilon)$
\[
c''_q(r, \varepsilon) = \int_{-\infty}^r dr' \dot{\gamma}_q(r') a''_{q^+}(r', \varepsilon) c''_q(r', \varepsilon) + \int_{-\infty}^r dr' \dot{\gamma}_q(r') a''_{q^-}(r', \varepsilon) c''_q(r', \varepsilon)
\times \exp(-i\varepsilon^{-1} \Delta_q(r', \varepsilon)) c''_q(r', \varepsilon)
\] (3.14)
and
\[
c''_q(r, \varepsilon) = 1 + \int_{-\infty}^r dr' \dot{\gamma}_q(r') a''_{q^+}(r', \varepsilon) \exp(i\varepsilon^{-1} \Delta_q(r', \varepsilon)) c''_q(r', \varepsilon)
+ \int_{-\infty}^r dr' \dot{\gamma}_q(r') a''_{q^-}(r', \varepsilon) c''_q(r', \varepsilon).
\] (3.15)

Let $X(r, \varepsilon)$ be the column vector whose components are
\[
X_q^+(r, \varepsilon) = \exp(i\varepsilon^{-1} \Delta_q(r, \varepsilon)) c''_q(r, \varepsilon) \quad X_q^-(r, \varepsilon) = c''_q(r, \varepsilon).
\] (3.16)

We can rewrite (3.14) and (3.15) as
\[
X_q(r, \varepsilon) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^r dr' \dot{\gamma}_q(r') A_q(r, r', \varepsilon) X_q(r', \varepsilon)
\] (3.17)
where the matrix $A_q(r, r', \varepsilon)$ is
\[
A_q(r, r', \varepsilon) = \begin{pmatrix} a''_{q^+}(r, \varepsilon) e^{i\varepsilon^{-1} \Delta_q(r, r', \varepsilon)} a''_{q^-}(r, \varepsilon) e^{i\varepsilon^{-1} \Delta_q(r, r', \varepsilon)} \\ a''_{q^-}(r, \varepsilon) a''_{q^+}(r, \varepsilon) \end{pmatrix}.
\] (3.18)

By lemma 2.2 and property $(c)$ there exists a constant $C$ such that for any $r \geq r'$
\[
\int_{-\infty}^r dr' \| \dot{\gamma}_q(r') \| \| A_q(r, r', \varepsilon) \| \leq \varepsilon^q C
\] (3.19)
and therefore
\[
\lim_{r \to +\infty} c''_q(r, \varepsilon) = 1 + O(\varepsilon^q).
\] (3.20)
We now show that a path $\gamma_q$ in $\Omega$ with properties (a), (b) and (c) always exists for all $\varepsilon$ sufficiently small. Using property (d) of $\gamma$ we choose for $r \in [r_1, r_2]$ $\gamma_q(r) = \gamma(r)$. This is possible if $\varepsilon$ is small enough, since
\[
\frac{d}{dr} \Delta_q(r, \varepsilon) = (e^{-q}_{\varepsilon}(\gamma(r), \varepsilon) - e^{+q}_{\varepsilon}(\gamma(r), \varepsilon)) \gamma_q(r). 
\] (3.21)

Let $\zeta < r$ and
\[
\Gamma_1(\zeta) = \left\{ z \in \Omega; \min_{r \in (-\infty, r_1]} |z - \gamma(r)| \leq \zeta \right\} 
\] (3.22)
and let us consider the analytic continuation of $\Delta_0$ on $\Gamma_1$. If $\zeta$ is small enough the function $\Delta_0$ restricted on $\Gamma_1$ is injective. Let $\hat{\Gamma}_1$ be the image of $\Gamma_1$ by $\Delta_0$. Since the image of the path $r \in (-\infty, r_1] \mapsto \gamma(r)$ by $\Delta_0$ is horizontal, there exists $\eta > 0$ such that the strip $\hat{\Gamma}_1$,
\[
\hat{\Gamma}_1 = \{ z | \Re z \geq \Re \Delta_0(\gamma(r)), |\Im z - \Im \Delta_0(\gamma(r))| \leq \eta \} 
\] (3.23)
is contained in $\hat{\Gamma}_1$. We denote by $\hat{g}^+, \hat{g}^-$ the two horizontal lines
\[
\hat{g}^+ = \{ z | \Im z = \Im \Delta_0(\gamma(r)), \Re z \geq \Re \Delta_0(\gamma(r)) \} 
\] (3.24)
and by $G_1, g_1^+$ and $g_1^-$ the images of these sets by $\Delta_0^{-1}$. We choose $\varepsilon$ small enough such that for all $z \in G_1 \subset \Gamma_1$ we have
\[
|\Delta_q(z, \varepsilon) - \Delta_0(z)| \leq \frac{\eta}{4}. 
\] (3.25)
The path $\gamma_q$ in $G_1$ is defined as the level line in $G_1$ of $\Im \Delta_q(z, \varepsilon), \Im \Delta_q(\gamma(r_1), \varepsilon) = \Im \Delta_q(\gamma(r_1), \varepsilon)$. Notice that
\[
|\Delta_q(\gamma(r_1), \varepsilon) - \Delta_0(\gamma(r_1))| \leq \frac{\eta}{4} 
\] (3.26)
and therefore by (3.25) it is impossible that $\gamma_q$ intersects $g_1^+$ and $g_1^-$. We parametrize this level line by $r \in (-\infty, r_1]$. In a similar way we define $\gamma_q(r), r \geq r_2$. This proves the existence of the path $\gamma_q$ with properties (a), (b) and (c) and therefore the proof of lemma 2.3 is complete.

Acknowledgments

We thank M Berry very much for correspondence and his comments on this work, in particular on the nature of our asymptotic expansion.

References

Commun. Math. Phys. in press
[8] Nenciu G 1990 Private communication
Preprint
    calculation of non-adiabatic transition amplitudes J. Chem. Phys. 67 4640–53
    probabilities in the adiabatic limit Ann. Phys. in press
    Commun. Math. Phys. in press