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An adiabatic theorem for singularly perturbed hamiltonians*

by

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ABSTRACT. – The adiabatic approximation in quantum mechanics is considered in the case where the self-adjoint hamiltonian $H_0(t)$, satisfying the usual spectral gap assumption in this context, is perturbed by a term of the form $\varepsilon H_1(t)$. Here $\varepsilon \rightarrow 0$ is the adiabaticity parameter and $H_1(t)$ is a self-adjoint operator defined on a smaller domain than the domain of $H_0(t)$. Thus the total hamiltonian $H_0(t) + \varepsilon H_1(t)$ does not necessarily satisfy the gap assumption, $\forall \varepsilon > 0$. It is shown that an adiabatic theorem can be proven in this situation under reasonable hypotheses. The problem considered can also be viewed as the study of a time-dependent system coupled to a time-dependent perturbation, in the limit of large coupling constant.

RÉSUMÉ. – On considère l'approximation adiabatique en mécanique quantique dans le cas où l'hamiltonien auto-adjoint $H_0(t)$, satisfaisant l'hypothèse habituelle de lacune spectrale dans ce contexte, est perturbé par un terme de la forme $\varepsilon H_1(t)$. Ici $\varepsilon \rightarrow 0$ est le paramètre d'adiabaticité et $H_1(t)$ est un opérateur auto-adjoint défini sur un domaine plus petit que celui de $H_0(t)$. Ainsi l'hamiltonien total $H_0(t) + \varepsilon H_1(t)$ ne satisfait pas

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nécessairement l'hypothèse de lacune spectrale, $\forall \varepsilon > 0$. On montre qu'un théorème adiabatique peut être démontré dans cette situation moyennant des hypothèses raisonnables. Le problème considéré peut aussi être vu comme l'étude d'un système dépendant du temps couplé à une perturbation dépendante du temps, dans la limite de grande constante de couplage.

1. INTRODUCTION

Due to its central role in Quantum Mechanics, the time-dependent Schrödinger equation has been the object of numerous studies. Since exact solutions to that equation are rather scarce, several asymptotic regimes governed by a set of suitable parameters were considered. Among these asymptotic conditions, the so-called adiabatic regime is a widely used limit in physics. The adiabatic limit describes the evolution in time of a system when the governing hamiltonian is a slowly varying function of time. A typical example is the slow, with respect to the time scale of the system, switching on and off of time dependent exterior perturbations. This regime also plays an important role in the study of models involving slow and fast variables, like molecular systems. Mathematically speaking, the adiabatic limit is a singular limit, as is the semi-classical limit. It corresponds to the limit $\varepsilon \rightarrow 0$ of the equation

$$i \varepsilon \psi'(t) = H_0(t) \psi(t), \quad \psi(0) = \psi_0 \quad (1.1)$$

where the prime denotes the time derivative, $\psi(t)$ is a vector valued function in a separable Hilbert space \mathcal{H} and $H_0(t)$ is a smooth enough family of self adjoint operators, bounded from below and defined on some time independent dense domain D_0 . We denote by $U_\varepsilon(t)$ the associated unitary evolution operator that $U_\varepsilon(0) = I$. Let us further assume that there is a gap in the spectrum of $H_0(t)$ for any $t \in [0, 1]$ and denote by $P_0(t)$ the spectral projector corresponding to the bounded part of the spectrum. Then, the adiabatic theorem of quantum mechanics asserts that there exists a unitary operator $V(t)$ satisfying the intertwining property

$$V(t) P_0(0) = P_0(t) V(t), \quad \forall t \in [0, 1] \quad (1.2)$$

which approximates $U_\varepsilon(t)$ in the limit $\varepsilon \rightarrow 0$

$$\sup_{t \in [0, 1]} \|U_\varepsilon(t) - V(t)\| = \mathcal{O}(\varepsilon). \quad (1.3)$$

This means in particular that a solution of (1.1) such that

$$\psi(0) = P_0(0)\psi(0) \quad (1.4)$$

will satisfy

$$\psi(t) = P_0(t)\psi(t) + \mathcal{O}(\varepsilon) \quad (1.5)$$

for any $t \in [0, 1]$. In other words, such a solution follows the spectral subspace $P_0(t)\mathcal{H}$ up to negligible corrections in the limit $\varepsilon \rightarrow 0$. This formulation is a generalization of the early works [BF] and [Ka1] and can be found in [N1], [ASY]. Recently, several refinements and further generalisations of this result were proven on a rigorous basis, see [N2], [JP1] and the references therein. Most of these works deal with the construction of higher and higher order approximate solutions to (1.1) in the limit $\varepsilon \rightarrow 0$, even up to exponential order, under hypotheses similar to those loosely given above. In particular, the spectral gap hypothesis is crucial for these results to hold. Another generalisation of the adiabatic theorem consists in trying to get rid of the gap hypothesis. Such a result was obtained by Avron, Howland and Simon [AHS] who considered the case of hamiltonians with dense pure point spectrum. Under certain conditions on the mismatch of resonances they could prove similar results to (1.5) for two kinds of hamiltonians, with $P_0(t)$ the corresponding, either finite or infinite dimensional, spectral projector.

1.1. The problem

In that paper we consider a related natural generalisation of the adiabatic theorem which consists in adding to the hamiltonian a perturbation of order ε . We simply replace the hamiltonian $H_0(t)$ satisfying a gap hypothesis by the perturbed hamiltonian $H_0(t) + \varepsilon H_1(t)$, where the self-adjoint perturbation $H_1(t)$ is defined on some time independent dense domain D_1 . This corresponds to the first order correction stemming from a formal hamiltonian $H(t, \varepsilon) = H(t) + \varepsilon H_1(t) + \varepsilon^2 H_2(t) + \dots$, for example. If $D_1 \supseteq D_0$, regular perturbation theory shows that the total hamiltonian satisfies the gap assumption for ε small enough, so that the usual adiabatic theorem applies. However, if $D_1 \subset D_0$, the term $\varepsilon H_1(t)$ becomes very singular even though $\varepsilon \rightarrow 0$ in the adiabatic limit. In particular, if the gap hypothesis holds for $H_0(t)$, it doesn't hold in general for the (suitable extension of the) operator $H_0(t) + \varepsilon H_1(t)$, no matter how small ε is. We are thus in another case where the driving hamiltonian has no gap in its spectrum. We consider the equation

$$i\varepsilon U'_\varepsilon(t) = (H_0(t) + \varepsilon H_1(t))U_\varepsilon(t) \quad (1.6)$$

on the suitable domain for the hamiltonian in the limit $\varepsilon \rightarrow 0$, under the main assumption that $H_0(t)$ satisfies the gap hypothesis. Our main result, see Proposition 2.1, is the construction of an approximate evolution operator $V(t)$ satisfying (1.2) and (1.3), provided some regularity conditions are satisfied. This result shows that the adiabatic theorem survives some singular perturbations or it can be viewed as providing another situation where the adiabatic theorem holds although the driving hamiltonian does not satisfy the gap assumption. If the hamiltonians $H_0(t)$ and $H_1(t)$ are both time-independent, a very complete discussion of this problem can be found in [N3].

It should also be noticed that the equation (1.6) rewritten as

$$i U_\varepsilon(t) = \left(H_1(t) + \frac{1}{\varepsilon} H_0(t) \right) U_\varepsilon(t), \quad \varepsilon \rightarrow 0 \quad (1.7)$$

can be viewed as describing the evolution of a system driven by the hamiltonian $H_1(t)$ coupled to the perturbation $H_0(t)$ in the limit of infinite coupling constant. We give below an application of our result in this setting.

1.2. Heuristics

We want to give here a formal argument explaining why this result should hold and what the approximate evolution $V(t)$ should be. Let $U_1(t)$ be defined by

$$i U_1'(t) = H_1(t) U_1(t), \quad U_1(0) = \mathbb{I} \quad (1.8)$$

and let us consider the corresponding interaction picture. The operator $\tilde{U}(t) \equiv U_1^{-1}(t) U_\varepsilon(t)$ then satisfies

$$i \varepsilon \tilde{U}'(t) = U_1^{-1}(t) H_0(t) U_1(t) \tilde{U}(t) \equiv \tilde{H}(t) \tilde{U}(t). \quad (1.9)$$

The new hamiltonian $\tilde{H}(t)$ is ε -independent and satisfies a gap hypothesis if $H_0(t)$ does with corresponding spectral projector given by

$$\tilde{P}(t) = U_1^{-1}(t) P_0(t) U_1(t). \quad (1.10)$$

In consequence, we can apply the standard adiabatic theorem to $\tilde{U}(t)$. Thus there exists an approximate evolution up to order ε , $\tilde{V}(t)$, of $\tilde{U}(t)$ such that

$$\tilde{V}(t) \tilde{P}(0) = \tilde{P}(t) \tilde{V}(t), \quad \forall t \in [0, 1]. \quad (1.11)$$

Coming back to the actual evolution this implies that

$$U_\varepsilon(t) = U_1(t) \tilde{U}(t) = U_1(t) \tilde{V}(t) + \mathcal{O}(\varepsilon) \quad (1.12)$$

where the approximation $V(t) \equiv U_1(t) \tilde{V}(t)$ is such that

$$\begin{aligned} V(t) P_0(0) &= U_1(t) \tilde{V}(t) \tilde{P}(0) \\ &= U_1(t) \tilde{P}(t) \tilde{V}(t) \\ &= P_0(t) U_1(t) \tilde{V}(t) \\ &= P_0(0) V(t). \end{aligned} \quad (1.13)$$

It is known [N1], [ASY] that the approximate evolution $\tilde{V}(t)$ is given by the solution of

$$i \varepsilon \tilde{V}'(t) = (\tilde{H}(t) + i \varepsilon [\tilde{P}'(t), \tilde{P}(t)]) \tilde{V}(t), \quad \tilde{V}(0) = \mathbb{I}. \quad (1.14)$$

Now, we compute

$$\begin{aligned} [\tilde{P}'(t), \tilde{P}(t)] \\ = U_1^{-1}(t) ([P_0'(t), P_0(t)] + i [[H_1(t), P_0(t)], P_0(t)]) U_1(t) \end{aligned} \quad (1.15)$$

so that, $V(t)$ should satisfy

$$\begin{aligned} i \varepsilon V'(t) &= (\varepsilon H_1(t) + H_0(t) + i \varepsilon [P_0'(t), P_0(t)] \\ &\quad - \varepsilon [[H_1(t), P_0(t)], P_0(t)]) V(t) \\ &= (H_0(t) + \varepsilon P_0(t) H_1(t) P_0(t) \\ &\quad + \varepsilon (\mathbb{I} - P_0(t)) H_1(t) (\mathbb{I} - P_0(t)) \\ &\quad + i \varepsilon [P_0'(t), P_0(t)]) V(t). \end{aligned} \quad (1.16)$$

This equation is to be compared with (2.17). Of course, the main problem to turn this sketch into a proof is the question of domains. We want to stress the fact that our goal is to give here reasonable hypotheses under which the adiabatic theorem holds, although it should be possible to obtain higher order approximations in this case too, provided additional assumptions are made.

In the next section we provide a precise statement of our main result in Proposition 2.1 and give a proof of it. Further remarks on $V(t)$ and a simple application are given at the end of the section. The appendix contains the proofs of technical lemmas.

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2. MAIN RESULTS

2.1. Hypotheses

We start by expressing here the hypotheses we need in order to prove our main Proposition 2.1.

HYPOTHESIS D . – Let $H_0(t)$ and $H_1(t)$ be two time-dependent self-adjoint operators in a separable Hilbert space \mathcal{H} which are densely defined on their respective domains D_0 and D_1 for all $t \in [0, 1]$. These domains are assumed to be independent of t . We introduce the operator

$$H(t, \varepsilon) = H_0(t) + \varepsilon H_1(t), \quad D_0 \cap D_1 \rightarrow \mathcal{H} \quad (2.1)$$

and we further assume that there exists a dense domain $D \subseteq D_0 \cap D_1$ on which $H(t, \varepsilon)$ is essentially self-adjoint. Thus there exists a dense domain $D' \supseteq D_0 \cap D_1 \supseteq D$ on which the closure $\overline{H(t, \varepsilon)}$ of $H(t, \varepsilon)$ is self-adjoint. This domain D' is also supposed to be independent of t .

HYPOTHESIS R_0 . – The operator $H_0(t)$ is strongly C^2 on D_0 and its spectrum $\sigma_0(t)$ is divided into two parts $\sigma_0^a(t)$ and $\sigma_0^b(t)$ separated by a finite gap for all $t \in [0, 1]$ such that $\sigma_0^b(t)$ is bounded. Let $P_0(t)$ be the spectral projector associated with the part $\sigma_0^b(t)$ constructed by means of the Riesz formula

$$P_0(t) = -\frac{1}{2\pi i} \oint_{\Gamma} R_0(t, \lambda) d\lambda \quad (2.2)$$

where $R_0(t, \lambda) = (H_0(t) - \lambda)^{-1}$ and Γ is a path in the resolvent set of $H_0(t)$ which encircles $\sigma_0^b(t)$. Note that both $R_0(t, \lambda)$, $\lambda \in \Gamma$ and $P_0(t)$ are strongly C^2 on \mathcal{H} . Moreover, for any $t \in [0, 1]$,

$$\text{Ran } P_0(t) \subseteq D_1 \quad (2.3)$$

$$\text{Ran } P_0'(t) \subseteq D_1 \quad (2.4)$$

HYPOTHESIS R_1 . – The operator $H_1(t)$ is strongly C^1 on D_1 and $H_1(0)P_0'(t)$ is strongly C^0 on \mathcal{H} . Moreover, for all $(\lambda, t) \in \Gamma \times [0, 1]$,

$$\text{Ran } R_0(t, \lambda) H_1(t) P_0(t) \subseteq D_1 \quad (2.5)$$

$$\text{Ran } R_0(t, \lambda) P_0'(t) P_0(t) \subseteq D_1 \quad (2.6)$$

$$\sup_{(\lambda, t) \in \Gamma \times [0, 1]} \|H_1(0) R_0(t, \lambda) H_1(t) P_0(t)\| < b < \infty \quad (2.7)$$

$$\sup_{(\lambda, t) \in \Gamma \times [0, 1]} \|H_1(0) R_0(t, \lambda) P_0'(t) P_0(t)\| < b < \infty. \quad (2.8)$$

HYPOTHESIS \overline{R} . – We finally require the self-adjoint $\overline{H}(t, \varepsilon)$ to be strongly C^1 on D' and bounded from below for all $t \in [0, 1]$.

Remarks.

– It follows from hypothesis D that $\overline{H}(t, \varepsilon)$ converges strongly to $H_0(t)$ as $\varepsilon \rightarrow 0$ in the generalized sense so that the spectrum of $\overline{H}(t, \varepsilon)$ is asymptotically concentrated on any neighbourhood of the spectrum of $H_0(t)$ ([Ka2], p. 475).

– Hypothesis D is of course also satisfied in the trivial case $D_1 \supseteq D_0$.

– To check assumption (2.4), it is enough to show that

$$\text{Ran } P'_0(t) P_0(t) \subseteq D_1 \tag{2.9}$$

due to the identity

$$P_0(t) P'_0(t) P_0(t) \equiv 0. \tag{2.10}$$

– It is also actually enough in \overline{R} to require the existence for all $t \in [0, 1]$ of a real number in the resolvent set of $\overline{H}(t, \varepsilon)$ instead of asking $\overline{H}(t, \varepsilon)$ to be bounded from below.

2.2. Preliminaries

As a consequence of hypothesis \overline{R} , we have a well defined unitary evolution $U_\varepsilon(t)$ which together with its inverse, is strongly differentiable on D' , maps D' into D' and $U_\varepsilon(t)$ satisfies the Schrödinger equation

$$i \varepsilon \frac{\partial}{\partial t} U_\varepsilon(t) = \overline{(H_0(t) + \varepsilon H_1(t))} U_\varepsilon(t), \quad U_\varepsilon(0) = \mathbb{I} \tag{2.11}$$

(see [RS]). Another direct consequence of our hypotheses is the following technical lemma, the proof of which is given in appendix.

LEMMA 2.1. – Under conditions D , R_0 and R_1 , we have

a) $H_1(t) P_0(t)$ is bounded and

$$P_0(t) H_1(t) = (H_1(t) P_0(t))^* \tag{2.12}$$

$$H_1(t) P_0(t) = (P_0(t) H_1(t))^* \tag{2.13}$$

b) $H_1(t) P_0(t)$ is strongly C^1 on \mathcal{H}

c) $P_0(t) H_1(t)$ is strongly C^1 on \mathcal{H} .

Remark. – We actually don't need conditions (2.5) to (2.8) to prove this result.

We set

$$H_1^a(t) = Q_0(t) H_1(t) P_0(t) + P_0(t) H_1(t) Q_0(t) \tag{2.14}$$

where $Q_0(t) = (\mathbb{I} - P_0(t))$. We have the immediate

COROLLARY 2.1. – *The operator $H_1^\alpha(t)$ is self-adjoint, bounded and strongly C^1 on \mathcal{H} .*

We also introduce

$$K_0(t) = i[P_0'(t), P_0(t)] \quad (2.15)$$

which is bounded, self-adjoint and strongly C^1 as well. Hence the perturbed operator

$$\overline{H_0(t) + \varepsilon H_1(t)} - \varepsilon H_1^\alpha(t) + \varepsilon K_0(t) \quad (2.16)$$

is self-adjoint, bounded from below and strongly C^1 on D' (see [Ka2]). Thus there exists a unitary $V(t)$ which together with its inverse maps D' to D' , is strongly differentiable on D' and $V(t)$ satisfies

$$\left. \begin{aligned} i \varepsilon V'(t) &= (\overline{H_0(t) + \varepsilon H_1(t)} - \varepsilon H_1^\alpha(t) + \varepsilon K_0(t)) V(t), \\ V(0) &= \mathbb{I}. \end{aligned} \right\} \quad (2.17)$$

Let us check that

$$[\overline{H_0(t) + \varepsilon H_1(t)} - \varepsilon H_1^\alpha(t), P_0(t)] = 0. \quad (2.18)$$

Consider $\varphi \in D \subseteq D_0 \cap D_1$. Since $P_0(t)\varphi \in D_0 \cap D_1$ as well, we can write

$$\begin{aligned} & [\overline{H_0(t) + \varepsilon H_1(t)} - \varepsilon H_1^\alpha(t), P_0(t)] \varphi \\ &= [H_0(t) + \varepsilon H_1(t) - \varepsilon H_1^\alpha(t), P_0(t)] \varphi \\ &= \varepsilon [H_1(t) - H_1^\alpha(t), P_0(t)] \varphi \\ &= \varepsilon (H_1(t) P_0(t) - Q_0(t) H_1(t) P_0(t)) \varphi \\ &\quad - \varepsilon (P_0(t) H_1(t) - P_0(t) H_1(t) Q_0(t)) \varphi \equiv 0 \end{aligned} \quad (2.19)$$

using $Q_0(t) = \mathbb{I} - P_0(t)$. Hence, (2.18) holds as D is dense. It follows from classical results (see [Kr]) that

LEMMA 2.2. – *Let $V(t)$ be defined by (2.17). Then*

$$V(t) P_0(0) = P_0(t) V(t) \quad (2.20)$$

for all $t \in [0, 1]$.

This Lemma shows that $V(t)$ follows the decomposition of the Hilbert space \mathcal{H} into $\mathcal{H} = P_0(t)\mathcal{H} \oplus Q_0(t)\mathcal{H}$. Our goal is now to show that this evolution is an approximation of the actual evolution as $\varepsilon \rightarrow 0$:

PROPOSITION 2.1 (Adiabatic Theorem). – *Assume D , R_0 , R_1 and \overline{R} and let $U_\varepsilon(t)$ be defined by (2.11). The unitary $V(t)$ defined by (2.17) enjoying the intertwining property (2.20) is such that*

$$\sup_{t \in [0, 1]} \|U_\varepsilon(t) - V(t)\| = \mathcal{O}(\varepsilon). \quad (2.21)$$

2.3. Proof of Proposition 2.1

In order to compare $U_\epsilon(t)$ and $V(t)$, we introduce the unitary operator $A(t) = V^{-1}(t)U_\epsilon(t)$. It maps D' to D' and it satisfies for any $\varphi \in D'$

$$\left. \begin{aligned} i A'(t) \varphi &= i V^{-1}(t) U_\epsilon(t) \varphi + i V^{-1}(t) U'_\epsilon(t) \varphi \\ &= V^{-1}(t) (H_1^a(t) - K_0(t)) V(t) A(t) \varphi, \\ A(0) &= I. \end{aligned} \right\} \quad (2.22)$$

We have used the unitary of $V^{-1}(t)$ and the property $U_\epsilon(t) \varphi \in D'$ to derive (2.22). We want to perform an integration by parts on the Volterra equation corresponding to (2.22), as in [ASY] and [AHS]. Let $B(t)$ be a bounded, strongly C^1 operator and $\mathcal{R}(B)(t)$ be defined by

$$\mathcal{R}(B)(t) = \frac{1}{2\pi i} \oint_\Gamma R_0(t, \lambda) B(t) R_0(t, \lambda) d\lambda, \quad (2.23)$$

where the path Γ is as in (2.2).

LEMMA 2.3. – *If hypothesis R_0 holds, the operator $\mathcal{R}(B)(t)$ is bounded, strongly C^1 and maps \mathcal{H} into D_0 . Moreover,*

$$a) \quad [\mathcal{R}(B)(t), H_0(t)] = -[P_0(t), B(t)] \quad (2.24)$$

$$b) \quad \begin{aligned} \mathcal{R}(B)(t) &= P_0(t) \mathcal{R}(B)(t) Q_0(t) \\ &\quad + Q_0(t) \mathcal{R}(B)(t) P_0(t) \end{aligned} \quad (2.25)$$

$$c) \quad \mathcal{R}^*(B)(t) = \mathcal{R}(B^*)(t) \quad (2.26)$$

The first identity is proven in [ASY] and the second, the proof of which is given in appendix, is mentioned in [JP2]. The proof of the last assertion is straightforward and will be omitted. Hypothesis (2.3) and (2.4) are of course not required to get this Lemma.

We need to control the range of $\mathcal{R}(B)(t)$ when $B(t) = H_1^a(t) - K_0(t)$.

LEMMA 2.4. – *If hypotheses D , R_0 and R_1 hold, then the bounded operator $\mathcal{R}(H_1^a - K_0)(t)$ maps \mathcal{H} into $D_0 \cap D_1$. Moreover*

$$\sup_{t \in [0, 1]} \|H_1(t) \mathcal{R}(H_1^a - K_0)(t)\| < \infty. \quad (2.27)$$

Proof. – Consider the first statement. It follows from the first assertion of Lemma 2.3 that it is sufficient to show that $\text{Ran } \mathcal{R}(H_1^a - K_0)(t) \subseteq D_1$ and it follows from the part *b)* of it that it is actually enough to look at

$$\text{Ran } Q_0(t) \mathcal{R}(H_1^a - K_0)(t) P_0(t) = \text{Ran } \mathcal{R}((H_1^a - K_0) P_0)(t). \quad (2.28)$$

Consider now the operator

$$H_1(t) R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) R_0(t, \lambda), \quad \lambda \in \Gamma. \quad (2.29)$$

This operator is well defined because, see (2.10),

$$\begin{aligned} & R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) \\ &= R_0(t, \lambda) (Q_0(t) H_1(t) P_0(t) - i P_0'(t) P_0(t)) \\ &= R_0(t, \lambda) H_1(t) P_0(t) - P_0(t) R_0(t, \lambda) H_1(t) P_0(t) \\ &\quad - i R_0(t, \lambda) P_0'(t) P_0(t) \end{aligned} \quad (2.30)$$

maps \mathcal{H} into D_1 by hypothesis R_1 . Moreover, (2.29) is uniformly bounded in $(\lambda, t) \in \Gamma \times [0, 1]$. Indeed,

$$\begin{aligned} & H_1(t) R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) \\ &= H_1(t) (H_1(0) + i)^{-1} (H_1(0) + i) \\ &\quad \times R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) \end{aligned} \quad (2.31)$$

where $H_1(t) (H_1(0) + i)^{-1}$ is bounded due to the closed graph theorem and strongly C^1 and

$$\begin{aligned} & \|H_1(0) R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t)\| \\ &\leq \|H_1(0) R_0(t, \lambda) H_1(t) P_0(t)\| \\ &\quad + \|H_1(0) P_0(t) R_0(t, \lambda) H_1(t) P_0(t)\| \\ &\quad + \|H_1(0) R_0(t, \lambda) P_0'(t) P_0(t)\| \\ &\leq 2b + \sup_{(\lambda, t) \in \Gamma \times [0, 1]} \|H_1(0) P_0(t)\| \\ &\quad \times \|R_0(t, \lambda)\| \|H_1(t) P_0(t)\| < \infty \end{aligned} \quad (2.32)$$

(since $H_1(0) P_0(t)$ is C^1 , see (A.5).) This implies that both strong integrals

$$\frac{1}{2\pi i} \oint_{\Gamma} R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) R_0(t, \lambda) \psi d\lambda \quad (2.33)$$

$$\frac{1}{2\pi i} \oint_{\Gamma} H_1(t) R_0(t, \lambda) (H_1^a(t) - K_0(t)) P_0(t) R_0(t, \lambda) \psi d\lambda \quad (2.34)$$

exist for any $\psi \in \mathcal{H}$. Hence, since $H_1(t)$ is closed, the vector (2.33) belongs to D_1 and (2.34) equals $H_1(t)$ applied on (2.33). The second statement follows the fact that (2.34) is uniformly bounded in $t \in [0, 1]$. ♠

We can now construct a modified integration by parts formula which is the main tool to prove the adiabatic theorem:

LEMMA 2.5. – For any $\psi \in \mathcal{H}$, we can write

$$\begin{aligned} & Q_0(0) V^{-1}(s) (H_1^a(s) - K_0(s)) V(s) P_0(0) \psi \\ &= i\varepsilon \frac{d}{ds} (Q_0(0) V^{-1}(s) \mathcal{R}(H_1^a - K_0)(s) V(s) P_0(0) \psi) \\ &\quad - i\varepsilon Q_0(0) V^{-1}(s) \mathcal{R}'(H_1^a - K_0)(s) V(s) P_0(0) \psi \\ &\quad - \varepsilon Q_0(0) V^{-1}(s) [\mathcal{R}(H_1^a - K_0)(s), H_1(s)] V(s) P_0(0) \psi \\ &\equiv i\varepsilon \frac{d}{ds} (Q_0(0) V^{-1}(s) \mathcal{R}(H_1^a - K_0)(s) V(s) P_0(0) \psi) \\ &\quad + \varepsilon Q_0(0) C(s) P_0(0) \psi \end{aligned} \tag{2.35}$$

where $C(s)$ is bounded and satisfies

$$\sup_{\substack{s \in [0, 1] \\ \varepsilon \geq 0}} \|C(s)\| < \infty. \tag{2.36}$$

Proof. – On the one hand, using the definition of $H_1^a(s)$ and the property (2.10), we can write using lemmas 2.2 and 2.3

$$\begin{aligned} & Q_0(0) V^{-1}(s) (H_1^a(s) - K_0(s)) V(s) P_0(0) \\ &= Q_0(0) V^{-1}(s) [(H_1^a(s) - K_0(s)), P_0(s)] V(s) P_0(0) \\ &= Q_0(0) V^{-1}(s) [\mathcal{R}(H_1^a - K_0)(s), H_0(s)] V(s) P_0(0). \end{aligned} \tag{2.37}$$

On the other hand, for any $\varphi \in D \subset D'$,

$$\begin{aligned} & i\varepsilon \frac{d}{ds} (V^{-1}(s) \mathcal{R}(H_1^a - K_0)(s) V(s) \varphi) \\ &= i\varepsilon V^{-1'}(s) \mathcal{R}(H_1^a - K_0)(s) V(s) \varphi \\ &\quad + i\varepsilon V^{-1}(s) \mathcal{R}(H_1^a - K_0)(s) V(s)' \varphi \\ &\quad + i\varepsilon V^{-1}(s) \mathcal{R}'(H_1^a - K_0)(s) V(s) \varphi \\ &= V^{-1}(s) [\mathcal{R}(H_1^a - K_0)(s), \overline{H_0(s) + \varepsilon H_1(s)} \\ &\quad - \varepsilon H_1^a(s) + \varepsilon K_0(s)] V(s) \varphi \\ &\quad + i\varepsilon V^{-1}(s) \mathcal{R}'(H_1^a - K_0)(s) V(s) \varphi. \end{aligned} \tag{2.38}$$

Note that $\mathcal{R}(H_1^a - K_0)(s)$ maps \mathcal{H} into $D_0 \cap D_1$ (Lemma 2.4), so that differentiation of the operator $V^{-1}(s)$ on the left of (2.38) is justified. Replacing φ by $P_0(0)\psi$ and using the property

$$V(s)P_0(0)\psi = P_0(s)V(s)P_0(0)\psi \in D_0 \cap D_1 \subset D' \quad (2.39)$$

we can expand the commutator

$$\begin{aligned} & [\mathcal{R}(H_1^a - K_0)(s), \overline{H_0(s) + \varepsilon H_1(s)} - \varepsilon H_1^a(s) + \varepsilon K_0(s)] \\ & \quad \times V(s)P_0(0)\psi \\ &= [\mathcal{R}(H_1^a - K_0)(s), H_0(s) + \varepsilon H_1(s) - \varepsilon H_1^a(s) + \varepsilon K_0(s)] \\ & \quad \times V(s)P_0(0)\psi \\ &= [\mathcal{R}(H_1^a - K_0)(s), H_0(s)]V(s)P_0(0)\psi \\ & \quad + \varepsilon [\mathcal{R}(H_1^a - K_0)(s), H_1(s)]V(s)P_0(0)\psi \\ & \quad - \varepsilon [\mathcal{R}(H_1^a - K_0)(s), H_1^a(s) - K_0(s)]V(s)P_0(0)\psi. \end{aligned} \quad (2.40)$$

By Lemma 2.3, part *b*) again, it is easily checked that

$$Q_0(s)[\mathcal{R}(H_1^a - K_0)(s), H_1^a(s) - K_0(s)]P_0(s) \equiv 0 \quad (2.41)$$

so that the formula of Lemma 2.5 follows. The commutator

$$[\mathcal{R}(H_1^a - K_0)(s), H_1(s)] \quad (2.42)$$

is uniformly bounded in s , as seen from Lemma 2.4 and the identity

$$\mathcal{R}(H_1^a - K_0)(s)H_1(s) = (H_1(s)\mathcal{R}(H_1^a - K_0)(s))^* \quad (2.43)$$

which is proven as part *a*) of Lemma 2.1. Indeed, it is readily checked from Lemmas 2.3 *c*) and 2.4 that $\mathcal{R}(H_1^a - K_0)(s)$ has the required properties. The uniform boundedness in s and ε of $C(s)$ then follows from the strong continuity of $\mathcal{R}'(H_1^a - K_0)(s)$ and from the unitarity of $V(s)$. ♠

Remark. – It is essential to consider $P_0(0)\psi$ instead of $\varphi \in D$ in (2.40) to expand the operator $\overline{H_0 + \varepsilon H_1}$ because $V(s)\varphi$ does not necessarily belong to $D_0 \cap D_1$.

Let us consider the projection in $Q_0(0)\mathcal{H}$ of the integral equation corresponding to (2.22).

$$\begin{aligned} & Q_0(0)A(t)\varphi - Q_0(0)\varphi \\ &= -i \int_0^t ds Q_0(0)V^{-1}(s) \\ & \quad \times (H_1^a(s) - K_0(s))V(s)P_0(0)A(s)\varphi. \end{aligned} \quad (2.44)$$

We can apply Lemma 2.5 and make use of (2.22) to write

$$\begin{aligned}
 & Q_0(0) A(t) \varphi - Q_0(0) \varphi \\
 &= \varepsilon \int_0^t ds \frac{d}{ds} (Q_0(0) V^{-1}(s) \mathcal{R}(H_1^\alpha - K_0)(s) V(s) P_0(0)) A(s) \varphi \\
 &\quad - i \varepsilon \int_0^t ds Q_0(0) C(s) P_0(0) A(s) \varphi \\
 &= \varepsilon Q_0(0) V^{-1}(s) \mathcal{R}(H_1^\alpha - K_0)(s) V(s) P_0(0) A(s) \varphi|_0^t \\
 &\quad - \varepsilon \int_0^t ds Q_0(0) V^{-1}(s) \mathcal{R}(H_1^\alpha - K_0)(s) V(s) P_0(0) A'(s) \varphi \\
 &\quad - i \varepsilon \int_0^t ds Q_0(0) C(s) P_0(0) A(s) \varphi \\
 &= \varepsilon Q_0(0) V^{-1}(s) \mathcal{R}(H_1^\alpha - K_0)(s) V(s) P_0(0) A(s) \varphi|_0^t \\
 &\quad + i \varepsilon \int_0^t ds Q_0(0) V^{-1}(s) \mathcal{R}(H_1^\alpha - K_0)(s) \\
 &\quad \times (H_1^\alpha(s) - K_0(s)) V(s) Q_0(0) A(s) \varphi \\
 &\quad - i \varepsilon \int_0^t ds Q_0(0) C(s) P_0(0) A(s) \varphi. \tag{2.45}
 \end{aligned}$$

Since the only ε -dependent operators $A(s)$ and $V(s)$ are unitary and the above operators are all uniformly bounded in $s \in [0, 1]$, we have arrived to

$$\sup_{t \in [0, 1]} \|Q_0(0) (A(t) - \mathbb{I})\| = \mathcal{O}(\varepsilon). \tag{2.46}$$

Similarly,

$$\begin{aligned}
 & P_0(0) A(t) \varphi - \mathbb{I} P_0(0) \varphi \\
 &= -i \int_0^t ds P_0(0) V^{-1}(s) (H_1^\alpha(s) - K_0(s)) \\
 &\quad \times V(s) Q_0(0) (A(s) - \mathbb{I}) P_0(0) \varphi. \tag{2.47}
 \end{aligned}$$

Taking (2.46) into account we get

$$\sup_{t \in [0, 1]} \|P_0(0) (A(t) - \mathbb{I}) P_0(0)\| = \mathcal{O}(\varepsilon), \tag{2.48}$$

hence

$$\sup_{t \in [0, 1]} \|(A(t) - \mathbb{I}) P_0(0)\| = \mathcal{O}(\varepsilon). \quad (2.49)$$

Finally, since the projector $P_0(0)$ is self-adjoint and $A(t)$ is unitary we can write

$$\begin{aligned} \|P_0(0)(A(t) - \mathbb{I})\| &= \|(P_0(0)(A(t) - \mathbb{I}))^*\| \\ &= \|(A^{-1}(t) - \mathbb{I}) P_0(0)\| \\ &= \|(\mathbb{I} - A(t)) P_0(0)\| \\ &= \mathcal{O}(\varepsilon) \end{aligned} \quad (2.50)$$

uniformly in $t \in [0, 1]$. This last equation and (2.46) imply $A(t) = \mathbb{I} + \mathcal{O}(\varepsilon)$, uniformly in $t \in [0, 1]$. The definition of $A(t)$ eventually yields

$$\begin{aligned} \|(A(t) - \mathbb{I})\| &= \|V^{-1}(t)(U_\varepsilon(t) - V(t))\| \\ &= \|(U_\varepsilon(t) - V(t))\|. \quad \spadesuit \end{aligned} \quad (2.51)$$

2.4. Factorization of $V(t)$

We can now decompose the approximate evolution $V(t)$ as in the usual adiabatic theorem. Let $W(t)$ be defined by

$$i W'(t) = K_0(t) W(t), \quad W(0) = \mathbb{I} \quad (2.52)$$

and consider the unitary operator $\Phi(t) = W^{-1}(t) V(t)$. Since $W(t) P_0(0) = P_0(t) W(t)$ for all t , we have

$$[\Phi(t), P_0(0)] \equiv 0, \quad \forall t \in [0, 1]. \quad (2.53)$$

The ε -independent operator $W(t)$ has a geometrical meaning and describes a parallel transport of $P_0(t) \mathcal{H}$ and $\Phi(t)$ is the analog of a dynamical phase which is singular in the limit $\varepsilon \rightarrow 0$. When restricted to $P_0(0) \mathcal{H}$, the operator $\Phi(t)$ satisfies a simple linear differential equation with bounded generator.

LEMMA 2.6. – *Under hypotheses D , R_0 , R_1 and \bar{R} , we have*

$$V(t) P_0(0) = W(t) \Phi(t) P_0(0) = P_0(t) W(t) \Phi(t) P_0(0) \quad (2.54)$$

where $\Phi(t) P_0(0)$ satisfies

$$\begin{aligned} i \varepsilon \Phi'(t) P_0(0) &= W^{-1}(t) (H_0(t) P_0(t) \\ &+ \varepsilon P_0(t) H_1(t) P_0(t)) W(t) \Phi(t) P_0(0). \end{aligned} \tag{2.55}$$

Accordingly, if $P_0(0) \mathcal{H}$ is one dimensional and $\psi = P_0(0) \psi$,

$$V(t) \psi = \exp \left\{ -i \int_0^t (e_0(s)/\varepsilon + e_1(s)) ds \right\} W(t) \psi, \tag{2.56}$$

where $e_0(t)$ is the associated eigenvalues of $H_0(t)$ and $e_1(t)$ is obtained by first order perturbation theory

$$e_1(t) = \text{Trace } P_0(t) H_1(t) P_0(t). \tag{2.57}$$

Proof. – Using (2.53), the intertwining property, and hypothesis R_0 , we have

$$\begin{aligned} i \varepsilon \Phi'(t) P_0(0) &= i \varepsilon P_0(0) (W^{-1}(t) V(t))' P_0(0) \\ &= W^{-1}(t) P_0(t) \overline{(H_0(t) + \varepsilon H_1(t))} - \varepsilon H^a(t) \\ &\quad \times P_0(t) V(t) P_0(0) \\ &= W^{-1}(t) P_0(t) (H_0(t) + \varepsilon H_1(t)) \\ &\quad \times P_0(t) W(t) W^{-1}(t) V(t) P_0(0). \spadesuit \end{aligned} \tag{2.58}$$

2.5. Example

We present here a very simple application of Proposition 2.1. Let \mathcal{H} be the Hilbert space $L^2(\mathbb{R}^n)$ and

$$\left. \begin{aligned} H_1(t) &= -\frac{1}{2} \Delta + w(x, t) \quad \text{on } D_1 \\ H_0(t) &= \beta(t) |\varphi(t)\rangle \langle \varphi(t)| \quad \text{on } \mathcal{H}. \end{aligned} \right\} \tag{2.59}$$

Here $w(x, t)$ is a smooth real valued function with compact support in $\mathbb{R}^n \times [0, 1]$ and D_1 is the domain of the self-adjoint extension of $-\frac{1}{2} \Delta + w(x, t)$ on $C_0^\infty(\mathbb{R}^n)$. The unit vector $\varphi(t) \in \mathcal{H}$ for all $t \in [0, 1]$ and $\beta(t)$ is a real valued function of $[0, 1]$. We consider the equation

$$i \psi'(t) = \left(-\frac{1}{2} \Delta + w(x, t) + \frac{1}{\varepsilon} \beta(t) |\varphi(t)\rangle \langle \varphi(t)| \right) \psi(t) \tag{2.60}$$

on the domain $D' = D_1$ in the limit $\varepsilon \rightarrow 0$. This equation describes the dynamics of a particule in a potential strongly coupled to a rank one operator, in the spirit of equation (1.7). We further assume that $\varphi(t)$ and $\beta(t)$ are C^2 and that

$$\beta(t) \geq g > 0, \quad \forall t \in [0, 1]. \tag{2.61}$$

Thus the spectrum of $H_0(t)$, $\sigma_0(t) = \{0, \beta(t)\}$, is divided into two disjoint parts. Note that the spectrum of $H_1(t)$ consists in the positive real axis plus some negative eigenvalues, depending on $w(x, t)$. Since $H_0(t)$ is rank one, the spectrum of $H_0(t) + \varepsilon H_1(t)$ is of the same nature, so that $\beta(t)$ is not isolated, $\forall \varepsilon > 0$. Under the additionnal hypotheses

$$\Delta\varphi'(t) \in C^1[0, 1] \tag{2.62}$$

$$\sup_{t \in [0, 1]} \|\Delta\Delta\varphi(t)\| < \infty \tag{2.63}$$

$$\sup_{t \in [0, 1]} \|\Delta w(\cdot, t)\varphi(t)\| < \infty \tag{2.64}$$

it is readily checked that our hypotheses D , R_0 , R_1 and \bar{R} are satisfied with

$$R_0(t, \lambda) = \frac{\beta(t)}{(\beta(t) - \lambda)\lambda} |\varphi(t)\rangle \langle \varphi(t)| - \frac{1}{\lambda} \mathbb{I} \tag{2.65}$$

and

$$P_0(t) = |\varphi(t)\rangle \langle \varphi(t)|. \tag{2.66}$$

By virtue of Proposition 2.1 and above computations, if $\psi(0) = \varphi(0)$, then

$$\begin{aligned} \psi(t) &= V(t)\varphi(0) + \mathcal{O}(\varepsilon) \\ &= \exp\{-i\lambda(t, \varepsilon)\}\varphi(t) + \mathcal{O}(\varepsilon), \quad \forall t \in [0, 1]. \end{aligned} \tag{2.67}$$

Here

$$\begin{aligned} \lambda(t, \varepsilon) &= \int_0^t (\beta(s)/\varepsilon - \langle \varphi(s)|\Delta\varphi(s)\rangle/2 \\ &\quad + \langle \varphi(s)|w(\cdot, s)\varphi(s)\rangle - i\langle \varphi(s)|\varphi'(s)\rangle) ds \end{aligned} \tag{2.68}$$

is a real valued function and the last term in the integrant comes from the parallel transport operator $W(t)$.

A. PROOF OF LEMMA 2.1

a) We first show the boundedness of these operators. Since $H_1(t)$ is self-adjoint, it is closed, and hypothesis (2.3) on $P_0(t)$ implies that $H_1(t)P_0(t)$ is well defined on \mathcal{H} and is closed as well. Hence, by the closed graph theorem, it is bounded. Since all operators are densely defined, we have the general relation

$$(H_1(t)P_0(t))^* \supset P_0^*(t)H_1^*(t) \quad (\text{A.1})$$

where $P_0^*(t)H_1^*(t) = P_0(t)H_1(t)$ and $H_1(t)P_0(t)$ is bounded. As D_1 is dense in \mathcal{H} we deduce from the extension principle that

$$P_0(t)H_1(t) = (H_1(t)P_0(t))^* \quad (\text{A.2})$$

is bounded. Finally,

$$(P_0(t)H_1(t))^* = H_1^*(t)P_0^*(t) = H_1(t)P_0(t) \quad (\text{A.3})$$

holds since $P_0(t)$ is bounded, [Ka2] p. 168.

b) Because of assumption (2.4) the operator $H_1(0)P_0'(t)$ is bounded (see above) and it is strongly C^0 on \mathcal{H} by hypothesis. Hence the vector $H_1(0)P_0'(t)\varphi$, where $\varphi \in \mathcal{H}$, is integrable and since $H_1(0)$ is closed we can write

$$\begin{aligned} & \int_0^t H_1(0)P_0'(s)\varphi ds \\ &= H_1(0) \int_0^t P_0'(s)\varphi ds = H_1(0)(P_0(t) - P_0(0))\varphi. \end{aligned} \quad (\text{A.4})$$

Thus the bounded operator $H_1(0)P_0(t)$ is strongly C^1 and

$$(H_1(0)P_0(t))' = H_1(0)P_0'(t). \quad (\text{A.5})$$

Consider

$$H_1(t)P_0(t) = H_1(t)(H_1(0) + i)^{-1}(H_1(0) + i)P_0(t) \quad (\text{A.6})$$

where $H_1(t)(H_1(0) + i)^{-1}$ is also bounded and strongly C^1 by hypothesis. Since $\|H_1(t)(H_1(0) + i)^{-1}\|$ is uniformly bounded in t , we get

$$\begin{aligned} (H_1(t)P_0(t))'\varphi &= H_1(t)(H_1(0) + i)^{-1}((H_1(0) + i)P_0(t))'\varphi \\ &+ (H_1(t)(H_1(0) + i)^{-1})'(H_1(0) + i)P_0(t)\varphi \end{aligned} \quad (\text{A.7})$$

where all operators are bounded and strongly continuous.

c) Let $\varphi \in D_1$. The vector

$$(P_0(t) H_1(t))' \varphi = P_0'(t) H_1(t) \varphi + P_0(t) H_1'(t) \varphi \quad (\text{A.8})$$

is well defined and continuous since $P_0(t)$ is uniformly bounded in t . On the one hand, it follows from the foregoing that

$$P_0'(t) H_1(t) = (H_1(t) P_0'(t))^* \quad (\text{A.9})$$

where $H_1(t) P_0'(t)$ is bounded and strongly C^0 . Hence $P_0'(t) H_1(t)$ is uniformly bounded in t . On the other hand, as $H_1(t)$ is self adjoint, $H_1'(t)$ is symmetric on D_1 . Thus we can write

$$P_0(t) H_1'(t) \subset P_0(t) H_1'^*(t) \subset (H_1'(t) P_0(t))^* \quad (\text{A.10})$$

where

$$H_1'(t) P_0(t) = H_1'(t) (H_1(0) + i)^{-1} (H_1(0) + i) P_0(t). \quad (\text{A.11})$$

Now

$$H_1'(t) (H_0(0) + i)^{-1} = H_1'^*(t) (H_1(0) + i)^{-1} \quad (\text{A.12})$$

so this operator is closed and everywhere defined, hence bounded, and it is strongly C^0 . The same is true for $(H_1(0) + i) P_0(t)$ so that applying the extension principle again we eventually obtain the uniform boundedness in t of the operator $P_0(t) H_1'(t)$. It follows that $(P_0(t) H_1(t))'$ is strongly continuous on a dense domain and is uniformly bounded. Thus by virtue of Theorem 3.5, p. 151 in [Ka2], it is strongly continuous on \mathcal{H} . ♠

B. PROOF OF LEMMA 2.3 b)

By definition

$$\begin{aligned} & P_0(t) \mathcal{R}(B)(t) P_0(t) \\ &= \frac{1}{(2\pi i)^3} \oint_{\Gamma} \left(\oint_{\Gamma'} \left(\oint_{\Gamma''} R_0(t, \lambda') R_0(t, \lambda) \right. \right. \\ & \quad \left. \left. \times B(t) R_0(t, \lambda) R_0(t, \lambda'') d\lambda'' \right) d\lambda' \right) d\lambda \end{aligned} \quad (\text{B.1})$$

where the paths Γ , Γ' and Γ'' are in the resolvent set of $H_0(t)$; do not intersect and Γ surrounds Γ' which surrounds Γ'' which surrounds

$\sigma_0^b(t)$. We can write the integrand under the following form, using the first resolvent equation

$$\begin{aligned} & \frac{R_0(t, \lambda') B(t) R_0(t, \lambda)}{(\lambda' - \lambda)(\lambda - \lambda'')} - \frac{R_0(t, \lambda') B(t) R_0(t, \lambda'')}{(\lambda' - \lambda)(\lambda - \lambda'')} \\ & - \frac{R_0(t, \lambda) B(t) R_0(t, \lambda)}{(\lambda' - \lambda)(\lambda - \lambda'')} + \frac{R_0(t, \lambda) B(t) R_0(t, \lambda'')}{(\lambda' - \lambda)(\lambda - \lambda'')} \end{aligned} \quad (B.2)$$

Now, integrating each term over the variable that does not appear in the resolvents, we obtain the result by the Cauchy formula. For the term $Q_0(t) \mathcal{R}(B)(t) Q_0(t)$ we use the definition of $Q_0(t)$ and the above result to obtain

$$\begin{aligned} & Q_0(t) \mathcal{R}(B)(t) Q_0(t) \\ & = \mathcal{R}(B)(t) - P_0(t) \mathcal{R}(B)(t) - \mathcal{R}(B)(t) P_0(t) \end{aligned} \quad (B.3)$$

With the same paths as above we compute

$$\begin{aligned} & - P_0(t) \mathcal{R}(B)(t) \\ & = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \frac{R_0(t, \lambda') B(t) R_0(t, \lambda)}{(\lambda' - \lambda)} \\ & \quad - \frac{R_0(t, \lambda) B(t) R_0(t, \lambda)}{(\lambda' - \lambda)} d\lambda d\lambda' \end{aligned} \quad (B.4)$$

and

$$\begin{aligned} & - \mathcal{R}(B)(t) P_0(t) \\ & = \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \frac{R_0(t, \lambda) B(t) R_0(t, \lambda')}{(\lambda' - \lambda)} \\ & \quad - \frac{R_0(t, \lambda) B(t) R_0(t, \lambda)}{(\lambda' - \lambda)} d\lambda d\lambda' \end{aligned} \quad (B.5)$$

where the last term in (B.4) and (B.5) drops after an integration over λ' . Let us perform the integration over λ in the first term of (B.4)

$$\begin{aligned} & \frac{1}{2\pi i} \oint_{\Gamma} \frac{R_0(t, \lambda') B(t) R_0(t, \lambda)}{(\lambda' - \lambda)} d\lambda \\ & = \frac{1}{2\pi i} \oint_{\Gamma''} \frac{R_0(t, \lambda') B(t) R_0(t, \lambda'')}{(\lambda' - \lambda'')} d\lambda'' \\ & \quad - R_0(t, \lambda') B(t) R_0(t, \lambda') \end{aligned} \quad (B.6)$$

by the Cauchy formula. Thus it remains

$$\begin{aligned}
 & Q_0(t) \mathcal{R}(B)(t) Q_0(t) \\
 &= \frac{1}{(2\pi i)^2} \oint_{\Gamma'} \oint_{\Gamma''} \frac{R_0(t, \lambda') B(t) R_0(t, \lambda'')}{(\lambda' - \lambda'')} d\lambda' d\lambda'' \\
 &\quad - \frac{1}{(2\pi i)^2} \oint_{\Gamma} \oint_{\Gamma'} \frac{R_0(t, \lambda) B(t) R_0(t, \lambda')}{(\lambda - \lambda')} d\lambda d\lambda' \\
 &\quad - \frac{1}{2\pi i} \oint_{\Gamma'} R_0(t, \lambda') B(t) R_0(t, \lambda') d\lambda' + \mathcal{R}(B)(t) \quad (\text{B.7})
 \end{aligned}$$

where the first two terms vanish by the Cauchy formula and the last two by definition of $\mathcal{R}(B)(t)$. ♠

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