QUANTIZATION OF EXACT LAGRANGIAN SUBMANIFOLDS IN A COTANGENT BUNDLE

LECTURES AT THE 2016 IMJ SUMMER SCHOOL "SYMPLECTIC TOPOLOGY, SHEAVES AND MIRROR SYMMETRY"

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The problem addressed in these lectures is the following. Let M be a manifold and let $\widetilde{\Lambda} \subset T^*M$ be a compact connected exact Lagrangian submanifold. Choosing a primitive of the Liouville form $f\colon \Lambda \to \mathbb{R}$, we define a closed conic connected Lagrangian submanifold $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$, where $T^*_{\tau>0}(M \times \mathbb{R}) = \{(x,t;\xi,\tau);\ \tau > 0\}$ and

$$\Lambda = \{(x,t;\xi,\tau);\ (x;\xi/\tau) \in \widetilde{\Lambda} \text{ and } t = f(x;\xi/\tau)\}. \tag{0.1}$$

We will prove that there exists a sheaf F on $M \times \mathbb{R}$ with microsupport Λ and simple along Λ (which we call a "quantization" of Λ) and use it to recover a previous result of Abouzaid and Kragh that the projection $\Lambda \to M$ is a homotopy equivalence.

The lectures mainly follow the preprint [5]. However in *loc. cit.* we work with the derived category of sheaves and we need to check some criterion to glue locally defined sheaves. In these notes we work in the framework of infinity categories of Lurie and Toën (see [14, 15, 20, 21]). We thus obtain the existence of a quantization in a more straightforward way.

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Part 1. Simple sheaves, the Kashiwara-Schapira stack, microlocalization

In these notes the coefficient ring $\mathbf{k}$ is $\mathbb{Z}$ or a field. We denote by M a manifold of class C^∞. We let $\dot{T}^*M = T^*M \setminus M$ be the cotangent bundle with the zero-section removed. We denote by $\pi_M \colon T^*M \to M$ the projection.

Following the notations of [19] we let $\underline{\mathbb{I}}(\mathbf{k}_M)$ be the full dg-subcategory of the category of complexes of sheaves of $\mathbf{k}$-modules on M consisting of the h-injective complexes of injective sheaves. In these notes we only need complexes with bounded or locally bounded cohomology and we restrict to this case (locally bounded means that the restriction of the complex to any compact subset has bounded cohomology). Indeed the microsupport of sheaves was defined only in the bounded case because at the time of [8] it was not known how to deal with unbounded complexes (however it is likely that most results in [10] extend to the unbounded case).

The homotopy category of $\underline{\mathbb{I}}(\mathbf{k}_M)$ is the (locally bounded) derived category of sheaves on M, denoted $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$. As in [19] we denote by $\underline{\otimes}$ and $\underline{\mathcal{H}om}$ the tensor product and internal Hom in $\underline{\mathbb{I}}(\mathbf{k}_M)$. For a morphism of manifolds $f \colon M \to N$ we denote by $\underline{f}_*$, $\underline{f}_!$, $\underline{f}^{-1}$ and $\underline{f}^!$ the direct image, proper direct image, inverse image and extraordinary inverse image between the dg-categories $\underline{\mathbb{I}}(\mathbf{k}_M)$ and $\underline{\mathbb{I}}(\mathbf{k}_N)$. They lift the similar functors between the derived categories $\mathsf{D}(\mathbf{k}_M)$ and $\mathsf{D}(\mathbf{k}_N)$.

1. Microsupport

The fundamental notion used in these notes is the microsupport of a sheaf, defined by Kashiwara and Schapira in [8]. Let us recall the definition.

Definition 1.1. (see [10, Def. 5.1.2]) Let $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$. We define $\mathrm{SS}(F) \subset T^*M$ as the closure of the set of points $(x_0;\xi_0) \in T^*M$ such that there exists a real C^1-function ϕ on M satisfying $d\phi(x_0) = \xi_0$ and $(\mathrm{R}\Gamma_{\{x;\,\phi(x)\geq\phi(x_0)\}}(F))_{x_0} \not\simeq 0$.

For $F \in \underline{\mathbb{I}}(\mathbf{k}_M)$ we let $\mathrm{SS}(F)$ be the microsupport of F viewed as an object of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ (the dg-category $\underline{\mathbb{I}}(\mathbf{k}_M)$ and the category $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ have the same objects). We set $\dot{\mathrm{SS}}(F) = \mathrm{SS}(F) \cap \dot{T}^*M$.

We do not recall here the properties of the microsupport and we refer to [10]. We only mention that there exist natural bounds for the microsupports of a direct or inverse image of a sheaf F in terms of the microsupport of F; we can also bound the microsupports of the tensor product and the internal $\mathcal{H}om$. The microsupport is closed and $\mathbb{R}^+$-conic by definition. Its intersection with the zero section is the support of the sheaf. It also satisfies a "triangular inequality" with respect to the distinguished triangles.

A deep result of [9] says that the microsupport is a coisotropic subset. Here we deal with sheaves with a Lagrangian microsupport. This condition is related with the constructiblity: by [9], if M is a real analytic manifold, a sheaf F on M is constructible if and only if $\mathrm{SS}(F)$ is subanalytic and Lagrangian and $\mathrm{R}\Gamma_{\{x\}}F$ is perfect, for all $x \in M$.

We take the following definitions and notations from [10], with a slight modification: our $\mathsf{D}^{\mathrm{lb}}_S(\mathbf{k}_M)$ is the same as $\mathsf{D}^{\mathrm{lb}}_{S\cup T^*_M M}(\mathbf{k}_M)$ in [10].

Definition 1.2. For a subset S of $\dot{T}^*M$ we denote by $\mathsf{D}^{\mathrm{lb}}_S(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ of the F such that $\dot{\mathrm{SS}}(F) \subset S$. We denote by $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; S)$ the quotient of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ by $\mathsf{D}^{\mathrm{lb}}_{\dot{T}^*M\setminus S}(\mathbf{k}_M)$. If $p \in \dot{T}^*M$, we write $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; p)$ for $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; \{p\})$.

We also denote by $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M)$ the full triangulated subcategory of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ of the F for which there exists a neighborhood Ω of S in T^*M such that $\mathrm{SS}(F) \cap \Omega \subset S$. Finally we let $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M; S)$ be the quotient of $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M)$ by $\mathsf{D}^{\mathrm{lb}}_{\dot{T}^*M\setminus S}(\mathbf{k}_M)$.

We define in the same way the dg enhancements $\underline{\mathbb{I}}_S(\mathbf{k}_M)$, $\underline{\mathbb{I}}(\mathbf{k}_M; S)$, $\underline{\mathbb{I}}_{(S)}(\mathbf{k}_M)$, $\underline{\mathbb{I}}_{(S)}(\mathbf{k}_M; S)$, with homotopy categories $\mathsf{D}^{\mathrm{lb}}_S(\mathbf{k}_M)$, $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; S)$, $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M)$, $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M; S)$, respectively (see [20] Cor. 8.7 or [21] §4.3 for the quotient).

2. Simple sheaves

In this section we assume that Λ is a locally closed conic Lagrangian submanifold of $\dot{T}^*M$. We recall the definition of simple and pure sheaves along Λ and give some of their properties.

2.1. **Definition and first properties.**

We first recall some notations from [10] (in particular from §7). For a function $\varphi\colon M \to \mathbb{R}$ of class C^∞ we define

$$\Lambda_\varphi = \{(x; d\varphi(x));\ x \in M\}. \tag{2.1}$$

For a given point $p = (x; \xi) \in \Lambda \cap \Lambda_\varphi$ we have the following Lagrangian subspaces of $T_p(T^*M)$

$$\lambda_0(p) = T_p(T^*_xM), \qquad \lambda_\Lambda(p) = T_p\Lambda, \qquad \lambda_\varphi(p) = T_p\Lambda_\varphi. \tag{2.2}$$

We recall the definition of the inertia index (see for example §A.3 in [10]). Let (E, σ) be a symplectic vector space and let $\lambda_1, \lambda_2, \lambda_3$

be three Lagrangian subspaces of E. We define a quadratic form q on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ by $q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1)$. Then $\tau_E(\lambda_1, \lambda_2, \lambda_3)$ is defined as the signature of q, that is, $p_+ - p_-$, where $p_\pm$ is the number of ± 1 in a diagonal form of q. We set

$$\tau_\varphi = \tau_{p,\varphi} = \tau_{T_pT^*M}(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p)). \tag{2.3}$$

Proposition 2.1 (Prop. 7.5.3 of [10]). *Let $\varphi_0, \varphi_1 \colon M \to \mathbb{R}$ be functions of class C^∞, let $p = (x;\xi) \in \Lambda$ and let $F \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda)}(\mathbf{k}_M)$. We assume that Λ and Λ_{φ_i} intersect transversally at p and that $\varphi_i(x) = 0$, for $i = 0, 1$. Then $(\mathrm{R}\Gamma_{\{\varphi_1 \geq 0\}}(F))_x$ is isomorphic to $(\mathrm{R}\Gamma_{\{\varphi_0 \geq 0\}}(F))_x[\frac{1}{2}(\tau_{\varphi_0} - \tau_{\varphi_1})]$.*

Definition 2.2 (Def. 7.5.4 of [10]). In the situation of Proposition 2.1 we say that F is pure at p with shift $s \in \frac{1}{2}\mathbb{Z}$ if $(\mathrm{R}\Gamma_{\{\varphi_0 \geq 0\}}(F))_x \simeq L[s - \frac{1}{2}d_M - \frac{1}{2}\tau_{\varphi_0}]$ for some $L \in \mathrm{Mod}(\mathbf{k})$, where d_M is the dimension of M. If moreover $L \simeq \mathbf{k}$, we say that F is simple with shift s at p.

We only say F is pure at p to mean that $(\mathrm{R}\Gamma_{\{\varphi_0 \geq 0\}}(F))_x$ is concentrated in a single degree.

If F is pure (resp. simple) at all points of Λ we say that it is pure (resp. simple) along Λ.

We know from [10] that, if Λ is connected and $F \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda)}(\mathbf{k}_M)$ is pure at some $p \in \Lambda$, then F is in fact pure along Λ. Moreover the $L \in \mathrm{Mod}(\mathbf{k})$ in the above definition is the same at every point.

Example 2.3. Let us describe the generic case, that is, when Λ is half part of the conormal bundle of a hypersurface. We consider the hypersurface $X = \mathbb{R}^{n-1} \times \{0\}$ in $M = \mathbb{R}^n$. We let $\Lambda = \{(\underline{x}, 0; 0, \xi_n); \xi_n > 0\}$ be the "positive" half part of $T^*_X M$. We set $Z = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$. Let $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda \cup T^*_M M}(\mathbf{k}_M)$. We assume that F is simple along Λ.

Let $x \in M \setminus Z$ and $y \in \mathrm{Int}(Z)$. For any ball B around y we have $\mathrm{R}\Gamma(B; F) \xrightarrow{\sim} F_y$ and we deduce a morphism $F_y \to F_x$. Then there exists $i \in \mathbb{Z}$ and a distinguished triangle

$$F_y \to F_x \to \mathbf{k}[i] \xrightarrow{+1} . \tag{2.4}$$

The shift of F is $i - 1/2$.

If $\mathbf{k}$ is a field, we can be more precise: there exists $L \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k})$ such that $F \simeq L_M \oplus \mathbf{k}_Z[i-1]$ or $F \simeq L_M \oplus \mathbf{k}_{M \setminus Z}[i]$.

For any $p \in \Lambda$ we can find an integral transform that sends a neighborhood of p in Λ to the conormal bundle of a smooth hypersurface. Then, Theorem 7.2.1 of [10] reduces the general case to Example 2.3 and we can deduce:

Lemma 2.4. *Let $p = (x;\xi)$ be a given point of Λ. Then there exists a neighborhood Λ_0 of p in Λ such that*

(i) *there exists* $F \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda_0)}(\mathbf{k}_M)$ *which is simple along* Λ_0,

(ii) *for any contractible open subset* $\Lambda_1 \subset \Lambda_0$ *and* $G \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda_1)}(\mathbf{k}_M)$, *there exist a neighborhood* Ω *of* Λ_1 *in* T^*M *and* $L \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k})$ *such that* $F \overset{\mathrm{L}}{\otimes} L_M \simeq G$ *in* $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M;\Omega)$.

The F given in Lemma 2.4 may have a microsupport bigger than Λ_0 in general. When Λ is in a good position we can give a more precise result using a "cut-off" lemma (see [10]).

Proposition 2.5 (Prop. 6.1.4 of [10])**.** *We set* $E = \mathbb{R}^n$. *Let* $\gamma_0, \gamma_1 \subset E^*$ *be two open convex cones such that* $\overline{\gamma_0} \subset \gamma_1$ *and* $\overline{\gamma_1}$ *is proper. Let* U *be an open neighborhood of* 0 *in* E. *Then there exist a smaller neighborhood* V *and a functor* $p\colon \mathbb{I}(\mathbf{k}_U) \to \mathbb{I}(\mathbf{k}_V)$ *together with a morphism of functors* $u\colon p \to r_U^V$, *where* r_U^V *is the restriction, such that, for any* $F \in \mathbb{I}(\mathbf{k}_U)$,

(i) $\mathrm{SS}(p(F)) \subset V \times \overline{\gamma_1}$,

(ii) $u(F)$ *is an isomorphism over* $V \times \gamma_1$, *that is,* $\mathrm{SS}(G) \cap (V \times \gamma_1) = \emptyset$ *for a cone* G *of* $u(F)$ *in* $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_V)$,

(iii) *if* $\dot{\mathrm{SS}}(F) \cap (U \times (\overline{\gamma_1} \setminus \gamma_0)) = \emptyset$, *then* $\dot{\mathrm{SS}}(p(F)) = \dot{\mathrm{SS}}(F) \cap (V \times \overline{\gamma_0})$ *and* $\dot{\mathrm{SS}}(G) = \dot{\mathrm{SS}}(F) \cap (V \times (E^* \setminus \gamma_1))$, *where* G *is as in (ii).*

Proposition 2.5 implies in particular Lemma 2.6. This last result can also be deduced from the local description of Lagrangian submanifolds by generating functions. However Proposition 2.5 gives a functorial way to eliminate extra components of a microsupport and also implies Proposition 3.3 below.

Lemma 2.6. *Let* M *be a manifold and* Λ *a closed conic Lagrangian submanifold of* $\dot{T}^*M$ *such that the projection* $\Lambda/\mathbb{R}_{>0} \to M$ *is finite. Then, for any* $x \in M$, *there exist a neighborhood* U *of* x *and* $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_U)$ *such that* $\dot{\mathrm{SS}}(F) \subset \Lambda \cap T^*U$ *and* F *is simple along* $\Lambda \cap T^*U$.

3. Definition of the Kashiwara-Schapira stack

Let M be a manifold and Λ a locally closed conic Lagrangian submanifold of $\dot{T}^*M$. We define the Kashiwara-Schapira stack of Λ by taking the stack associated with the prestack of microlocal categories $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M;S)$ introduced in [10] and recalled in Definition 1.2.

A stack is roughly a sheaf of categories. A prestack $\mathcal{C}$ on a topological space X consists of the data of a category $\mathcal{C}(U)$, for each open subset U of X, restriction functors $r_{V,U}\colon \mathcal{C}(U) \to \mathcal{C}(V)$, for $V \subset U$, and isomorphisms of functors $r_{W,V} \circ r_{V,U} \simeq r_{W,U}$, for $W \subset V \subset U$, satisfying compatibility conditions. A stack is a prestack satisfying some gluing conditions. We refer to [11] Chap. 19 for a definition. For a given prestack we can construct its associated stack, similar to the associated sheaf of a presheaf (see [11] Ex.19.7). The same notions hold in the framework of dg-categories, where the limits used to define a stack or construct an associated stack are replaced by homotopy limits in the model category of dg-categories (see [21] §5.3).

The basic example of a stack is the stack of sheaves on a manifold M, $U \mapsto \mathrm{Mod}(\mathbf{k}_U)$, for U open in M. The derived categories give a prestack on M, $U \mapsto \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_U)$, but it is not a stack. However $U \mapsto \underline{\mathbb{I}}(\mathbf{k}_U)$ is a stack in the dg-sense. In the same way, for our given Λ,

$$\underline{\mathbb{I}}_{\Lambda}(\mathbf{k}_M), U \mapsto \underline{\mathbb{I}}_{T^*U\cap\Lambda}(\mathbf{k}_U), \quad \text{and} \quad \underline{\mathbb{I}}_{(\Lambda)}(\mathbf{k}_M), U \mapsto \underline{\mathbb{I}}_{(T^*U\cap\Lambda)}(\mathbf{k}_U),$$

are also stacks.

Definition 3.1. We define the Kashiwara-Schapira stack of T^*M, denoted $\mu\mathrm{Sh}(\mathbf{k}_{T^*M})$, as the stack on T^*M associated with the prestack $\Omega \mapsto \mathbb{I}(\mathbf{k}_M;\Omega)$, for Ω open subset of T^*M.

More generally, if $\Lambda \subset T^*M$ is a locally closed conic subset, we define $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ as the stack on Λ which is the restriction to Λ of the stack on T^*M associated with the prestack $\Omega \mapsto \mathbb{I}_{\Lambda\cap\Omega}(\mathbf{k}_M;\Omega)$.

The Kashiwara-Schapira stack is conic in the sense that it is the inverse image of a stack on the topological space $(T^*M)_{conic}$, whose open sets are the conic open sets of T^*M, by the obvious map $T^*M \to (T^*M)_{conic}$.

Alternatively we can define $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ as the stack on Λ associated with the prestack $\Lambda_0 \mapsto \mathbb{I}_{(\Lambda_0)}(\mathbf{k}_M;\Lambda_0)$, for Λ_0 open in Λ. The quotient functor gives a functor of stacks

$$\mathfrak{m}_\Lambda \colon \underline{\mathbb{I}}_{(\Lambda)}(\mathbf{k}_M) \to \pi_{\Lambda *}(\mu\mathrm{Sh}(\mathbf{k}_\Lambda)), \tag{3.1}$$

where $\pi_\Lambda \colon \Lambda \to M$ is the projection.

We will also use the following partial localization where we only localize outside the zero section.

Definition 3.2. Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic subset. We define $\underline{\mathbb{I}}^{pl}_{\Lambda}(\mathbf{k}_M)$ as the stack on M associated with the prestack $U \mapsto \mathbb{I}_{\Lambda\cap T^*U}(\mathbf{k}_M;\dot{T}^*U)$. We write for short $\underline{\mathbb{I}}^{pl}(\mathbf{k}_M) = \underline{\mathbb{I}}^{pl}_{\dot{T}^*M}(\mathbf{k}_M)$.

As in (3.1) we have a functor

$$\mathfrak{m}^{pl}_\Lambda \colon \underline{\mathbb{I}}^{pl}_{\Lambda}(\mathbf{k}_M) \to \pi_{\Lambda *}(\mu\mathrm{Sh}(\mathbf{k}_\Lambda)). \tag{3.2}$$

We are mainly interested in the case where Λ is a smooth conic Lagrangian manifold. Using Proposition 2.5 we can prove:

Proposition 3.3. *Let M be a manifold and let Λ be a closed conic Lagrangian submanifold of $\dot{T}^*M$ such that the projection $\Lambda/\mathbb{R}_{>0} \to M$ is finite. Then the functor* (3.2) *is a quasi-equivalence.*

Idea of proof. Since the statement is local in M we can assume that M is a vector space. Let x be a given point of M and let $L_1, \dots, L_k$ be half-lines in $T^*_x M \cap \Lambda$. For each i we choose two open convex cones $\gamma_{i,0}$, $\gamma_{i,1} \subset M^*$ as in Proposition 2.5 such that $L_i \subset \gamma_{i,0}$. We let $p_i \colon \mathbb{I}(\mathbf{k}_U) \to \mathbb{I}(\mathbf{k}_V)$ be the functor given by the proposition, where $V \subset U$ are neighborhoods of x. We can assume that the cones $\overline{\gamma_{i,1}}$ are disjoints and that U, V do not depend on i. Then $F \mapsto \bigoplus p_i(F)$ gives a quasi-inverse to the functor (3.2). □

Definition 3.4. Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold. Let $d \colon \Lambda \to \frac{1}{2}\mathbb{Z}$ be a given function. We let $\mu\mathrm{Sh}^p(\mathbf{k}_\Lambda)$ (resp. $\mu\mathrm{Sh}^{p,d}(\mathbf{k}_\Lambda)$, $\mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)$, $\mu\mathrm{Sh}^{s,d}(\mathbf{k}_\Lambda)$) be the substack of $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ formed by the pure sheaves along Λ (resp. pure with shift d, simple, simple with shift d).

In this definition "shift d" means " shift $d(p)$ at any $p \in \Lambda$". We also recall that the shift is locally constant on the open subset of Λ of points where $\pi_M|_\Lambda$ is of maximal rank.

Remark 3.5. It follows from Section 10 and Remark 10.2 below that $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ and $\mathbb{I}_\Lambda(\mathbf{k}_M)$ are invariant by Hamiltonian deformation of Λ. Hence we can assume that the hypothesis "$\Lambda/\mathbb{R}_{>0} \to M$ finite" of Proposition 3.3 is satisfied.

4. Microlocalization

The microlocal categories $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; S)$ are introduced in [10] but not the stack $\mu\mathrm{Sh}(\mathbf{k}_{T^*M})$ because dg-categories were not available at this time. However it turns out that the $\mathcal{H}om$ sheaf in $\mu\mathrm{Sh}(\mathbf{k}_{T^*M})$ was already defined in [10] in an indirect way and denoted μhom. This is important since μhom is the only way we have to compute this $\mathcal{H}om$ sheaf.

The functor μhom is itself a variant of Sato's microlocalization functor. It is a bifunctor from $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ to $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{T^*M})$. We refer to *loc. cit.* for the definition and only recall some properties. Since it is defined using a geometric construction (deformation to the normal cone) and the Grothendieck six operations, it can be extended to the dg-category $\mathbb{I}(\mathbf{k}_M)$. We have natural isomorphisms

$$\mathrm{R}\pi_{M*}\mu hom(F, G) \simeq \mathrm{R}\mathcal{H}om(F, G), \tag{4.1}$$

$$\mathrm{R}\pi_{M!}\mu hom(F, G) \simeq \delta_M^{-1}\mathrm{R}\mathcal{H}om(q_2^{-1}F, q_1^{-1}G), \tag{4.2}$$

where $\delta_M\colon M \to M \times M$ is the diagonal embedding. If F is cohomologically constructible, then $\delta_M^{-1}\mathrm{R}\mathcal{H}om(q_2^{-1}F, q_1^{-1}G) \simeq \mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} G$ and we have "Sato's distinguished triangle"

$$\mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} G \to \mathrm{R}\mathcal{H}om(F,G) \to \mathrm{R}\dot{\pi}_{M*}(\mu hom(F,G)|_{\dot{T}^*M}) \xrightarrow{+1} . \tag{4.3}$$

Proposition 4.1. (Cor. 6.4.3 of [10].) *Let $F, G \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$. Then*

$$\operatorname{supp} \mu hom(F,G) \subset \mathrm{SS}(F) \cap \mathrm{SS}(G), \tag{4.4}$$

$$\mathrm{SS}(\mu hom(F,G)) \subset -H^{-1}(C(\mathrm{SS}(G), \mathrm{SS}(F))). \tag{4.5}$$

where $H\colon T^*T^*M \xrightarrow{\sim} TT^*M$ is the natural isomorphism given in local coordinates by $H(dx_i) = -\partial/\partial\xi_i$ and $H(d\xi_i) = \partial/\partial x_i$.

Using (4.1) and (4.4) we can define a morphism, for $\Omega \subset T^*M$,

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M;\Omega)}(F,G) \to H^0(\Omega; \mu hom(F,G)|_\Omega). \tag{4.6}$$

Theorem 4.2 (Thm. 6.1.2 of [10])**.** *If $\Omega = \{p\}$ for some $p \in T^*M$, then* (4.6) *is an isomorphism.*

Corollary 4.3. *Let $F, G \in \mathbb{I}(\mathbf{k}_M)$. We let $\mathfrak{m}(F), \mathfrak{m}(G)$ be their images in $\mu\mathrm{Sh}(\mathbf{k}_{T^*M})$. Let Ω be an open subset of T^*M. Then the complexes* $\mathrm{Hom}(\mathfrak{m}(F)|_\Omega, \mathfrak{m}(G)|_\Omega)$ *and* $\underline{\Gamma}(\Omega; \mu hom(F,G))$ *are quasi-isomorphic.*

By (4.5) we see that, if $\Lambda \subset \dot{T}^*M$ is a conic smooth Lagrangian submanifold and $F, G \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda)}(\mathbf{k}_M)$, then $\mathrm{SS}(\mu hom(F,G))$ is contained in $T^*_\Lambda(T^*M)$. Since $\mu hom(F,G)$ has support in Λ, this implies that $\mu hom(F,G)$ is locally constant on Λ. Hence μhom induces a functors of stacks

$$\overline{\mu hom}\colon \mu\mathrm{Sh}(\mathbf{k}_\Lambda)^{opp} \times \mu\mathrm{Sh}(\mathbf{k}_\Lambda) \to \mathsf{Loc}(\mathbf{k}_\Lambda), \tag{4.7}$$

where, for a topological space X we denote by $\mathsf{Loc}(\mathbf{k}_X)$ the substack of $\mathbb{I}(\mathbf{k}_X)$ formed by the complexes with locally constant sheaves.

We can deduce from Lemma 2.4 (ii) and Theorem 4.2:

Proposition 4.4. *Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic Lagrangian submanifold. We assume that there exists a globally defined simple sheaf $F_0 \in \mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)(\Lambda)$. Then the functor $\overline{\mu hom}$ of* (4.7) *induces a quasi-equivalence of dg-stacks*

$$\overline{\mu hom}(F_0, \cdot)\colon \mu\mathrm{Sh}(\mathbf{k}_\Lambda) \to \mathsf{Loc}(\mathbf{k}_\Lambda), \qquad G \mapsto \overline{\mu hom}(F_0, G).$$

We can also be more precise in Lemma 2.4 (ii): the $L \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k})$ which appears there is in fact $L = \mu hom(F,G)_p$.

We can also compute the germs of μhom for sheaves in $\mathsf{D}^{\mathrm{lb}}_{(\Lambda)}(\mathbf{k}_M)$. Let $p = (x;\xi) \in \Lambda$ and $\varphi_0\colon M \to \mathbb{R}$ be as in Proposition 2.1. For $F, G \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda)}(\mathbf{k}_M)$, we have

$$\mu hom(F,G)_p \simeq \mathrm{RHom}((\mathrm{R}\Gamma_{\{\varphi_0 \geq 0\}}(F))_x, (\mathrm{R}\Gamma_{\{\varphi_0 \geq 0\}}(G))_x). \tag{4.8}$$

5. Global sections of $\mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)$

Let $\Lambda \subset \dot{T}^*M$ be a locally closed conic connected Lagrangian submanifold. We have two obstructions to the existence of a globally defined simple sheaf $F \in \mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)(\Lambda)$. One is deduced easily from the results in [10] (see Proposition 2.1): it is the Maslov class of Λ. The other one depends on the coefficient rings. It is the image by the morphism $\mathbb{Z}/2\mathbb{Z} \to \mathbf{k}^\times$ of a kind of relative Stiefel-Whitney class of Λ.

(a-i) We first explain why the Maslov class is an obstruction. We cover Λ by open conic subsets Λ_i, $i \in I$, such that the conclusions of Lemma 2.4 hold for each Λ_i and $\Lambda_i \cap \Lambda_j$ is contractible for each $i, j \in I$. In particular we have $F_i \in \mathbb{I}_{(\Lambda_i)}(\mathbf{k}_M)$ which is simple along Λ_i, for each $i \in I$. We let $d_i\colon \Lambda_i \to \frac{1}{2}\mathbb{Z}$ be the shift of F_i along Λ_i. By (ii) of Lemma 2.4 the difference $\delta_{ji} = d_j - d_i$ is a constant function on $\Lambda_{ij} = \Lambda_i \cap \Lambda_j$. This defines a cocycle on Λ and we can deduce from Proposition 2.1 and Proposition 7.5.9 of [10] that this cocycle gives the Maslov class of Λ.

(a-ii) Let us assume that there exists $F \in \mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)(\Lambda)$. We can again find a covering Λ_i, $i \in I$, of Λ and a simple $F_i \in \mathbb{I}_{(\Lambda_i)}(\mathbf{k}_M)$ for each $i \in I$ such that $\mathfrak{m}_{\Lambda_i}(F_i) \simeq F|_{\Lambda_i}$. Then the shift functions $d_i\colon \Lambda_i \to \frac{1}{2}\mathbb{Z}$ satisfy $d_j - d_i = 0$ on Λ_{ij}. Hence the Maslov class of Λ vanishes.

(b) We still consider Λ_i, F_i as in (a-i) and we also assume that Λ_{ijk} is contractible for all $i, j, k \in I$. The complex

$$\mathrm{Hom}_{ij} = \mathrm{Hom}(\mathfrak{m}_{\Lambda_i}(F_i)|_{\Lambda_{ij}}, \mathfrak{m}_{\Lambda_j}(F_j)|_{\Lambda_{ij}})$$

is quasi-isomorphic to $\mathbf{k}[-\delta_{ji}]$. Let us choose invertible morphisms $u_{ji} \in H^{\delta_{ji}}(\mathrm{Hom}_{ij})$ such that $u_{ij} = u_{ji}^{-1}$. Then $c_{ijk} = u_{ik} \circ u_{kj} \circ u_{ji}$ belongs to $H^0(\mathrm{Hom}_{ii})$ which is canonically isomorphic to $\mathbf{k}$ by the morphism sending id to 1. Hence we can consider $\{c_{ijk}\}_{i,j,k \in I}$ as a 2-cochain on Λ with value in $\mathbf{k}^\times$. We can check that this cochain is a cocycle and thus defines an element

$$c^{\mathbf{k}}_\Lambda \in H^2(\Lambda; \mathbf{k}^\times). \tag{5.1}$$

As in (a-ii) we can see that $c^{\mathbf{k}}_\Lambda = 0$ if there exists $F \in \mu\mathrm{Sh}^s(\mathbf{k}_\Lambda)(\Lambda)$.

We also remark that we can assume that the F_i arise from sheaves with coefficients in $\mathbb{Z}$ by scalar extension. It follows that $c^{\mathbf{k}}_\Lambda$ is the image of $c^{\mathbb{Z}}_\Lambda$ by the map $\mathbb{Z}^\times = \{\pm 1\} \to \mathbf{k}^\times$.

(c) Let us assume that the Maslov class of Λ vanishes. Then, with the notations of (a-i), we can write $\{\delta_{ij}\}$ as a coboundary, that is, $\delta_{ji} = e_j - e_i$ for integers e_i, $i \in I$. Shifting F_i by e_i we can assume from the beginning that the functions d_i and d_j agree on Λ_{ij} for all $i, j \in I$. They define a function $d\colon \Lambda \to \frac{1}{2}\mathbb{Z}$ (a Maslov potential).

We recall the notation $\mu\mathrm{Sh}^{s,d}(\mathbf{k}_\Lambda)$ of Definition 3.4 for the stack of simple sheaves with shift d along Λ. Then $\mu\mathrm{Sh}^{s,d}(\mathbf{k}_\Lambda)$ has locally a unique object up to isomorphism and the $\mathcal{H}om$ sheaf between two locally defined objects is concentrated in degree 0 (recall that this $\mathcal{H}om$ sheaf is quasi-isomorphic to μhom). It follows that $\mu\mathrm{Sh}^{s,d}(\mathbf{k}_\Lambda)$ is quasi-equivalent to a stack of usual categories (considered as dg-categories with a Hom complex in degree 0).

In [5] we identify $\mu\mathrm{Sh}^{s,d}(\mathbf{k}_\Lambda)$ with the stack of $c^{\mathbf{k}}_\Lambda$-twisted local systems of rank 1 on Λ. In particular it has a global object if and only if $c^{\mathbf{k}}_\Lambda = 0$. We also identify $c^{\mathbb{Z}}_\Lambda$ with some relative Stiefel-Whitney class of Λ.

Part 2. Convolution and quantization

Let $\Lambda \subset \dot{T}^*N$ be a smooth closed conic Lagrangian submanifold. In this part we consider the problem of defining a sheaf on N from an object of $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$. For this we first introduce a functor, Ψ, from $\mu\mathrm{Sh}_\Lambda(\mathbf{k}_N)$ to the limit of the categories $\underline{\mathbb{I}}(\mathbf{k}_{N\times]0,\varepsilon[})$ when $\varepsilon \to 0$. The definition of Ψ requires a choice of direction on N. We assume in fact that N is a product $N = M \times \mathbb{R}$ (which is anyway the situation we have after adding a variable to turn an exact Lagrangian into a conic Lagrangian).

6. Notations

We set for short $\mathbb{R}_{>0} =]0, +\infty[$ and $\mathbb{R}_{\geq 0} = [0, +\infty[$. We usually endow $\mathbb{R}$ with the coordinate t and $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$ with the coordinate u. The associated coordinates in the cotangent bundles are $(t;\tau)$ for $T^*\mathbb{R}$ and $(u;\upsilon)$ for $T^*\mathbb{R}_{>0}$. We set $T^*_{\tau\geq 0}\mathbb{R} = \{(t,\tau) \in T^*\mathbb{R};\ \tau \geq 0\}$ and we define $T^*_{\tau>0}\mathbb{R}$ similarly. For a manifold M and an open subset $U \subset M \times \mathbb{R}$ we define

$$
\begin{aligned}
T^*_{\tau\geq 0}U &= (T^*M \times T^*_{\tau\geq 0}\mathbb{R}) \cap T^*U, \\
T^*_{\tau>0}U &= (T^*M \times T^*_{\tau>0}\mathbb{R}) \cap T^*U.
\end{aligned} \tag{6.1}
$$

Definition 6.1. Let U be an open subset of $M \times \mathbb{R}$. We let $\mathsf{D}^{\mathrm{lb}}_{\tau>0}(\mathbf{k}_U)$ (resp. $\mathsf{D}^{\mathrm{lb}}_{\tau\geq 0}(\mathbf{k}_U)$) be the full subcategory of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_U)$ of sheaves F satisfying $\dot{\mathrm{SS}}(F) \subset T^*_{\tau>0}U$ (resp. $\mathrm{SS}(F) \subset T^*_{\tau\geq 0}U$). We use the similar notations $\underline{\mathbb{I}}_{\tau>0}(\mathbf{k}_U)$, $\underline{\mathbb{I}}_{\tau\geq 0}(\mathbf{k}_U)$. We also define

$$
\underline{\mathbb{I}}^+(\mathbf{k}_U) = \varinjlim_V \underline{\mathbb{I}}(\mathbf{k}_V),
$$

where V runs over the open subsets of $U \times \mathbb{R}_{>0}$ such that

$$
U \sqcup V \text{ is a neighborhood of } U \times \{0\} \text{ in } U \times \mathbb{R}_{\geq 0}. \tag{6.2}
$$

An object of $\underline{\mathbb{I}}^+(\mathbf{k}_U)$ is represented by an object of $\underline{\mathbb{I}}(\mathbf{k}_V)$ for some V satisfying (6.2) and $\mathrm{Hom}_{\underline{\mathbb{I}}^+(\mathbf{k}_U)}(F,G) = \varinjlim_W \mathrm{Hom}_{\underline{\mathbb{I}}(\mathbf{k}_W)}(F|_W, G|_W)$ where W runs over the open subsets satisfying (6.2).

The assignments $U \mapsto \underline{\mathbb{I}}_{\tau>0}(\mathbf{k}_U)$, $\underline{\mathbb{I}}_{\tau\geq 0}(\mathbf{k}_U)$ or $\underline{\mathbb{I}}^+(\mathbf{k}_U)$ define stacks on $M \times \mathbb{R}$.

7. Convolution

The convolution product $\star$ defined below is a variant of the "composition of kernels" considered in [10] (denoted by $\circ$). It is used in [10] to prove the cut-off result stated in Proposition 2.5 above. It is also used by Tamarkin in [18] to study the localization of $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}})$ by the objects with microsupport in $T^*_{\tau\leq 0}(M\times\mathbb{R})$, in a framework similar to the present one. Namely, the convolution gives a projector from $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}})$ to the left orthogonal of the subcategory $\mathsf{D}^{\mathrm{lb}}_{T^*_{\tau\leq 0}(M\times\mathbb{R})}(\mathbf{k}_{M\times\mathbb{R}})$ of objects with microsupport in $T^*_{\tau\leq 0}(M\times\mathbb{R})$. We will use a variant of Tamarkin's definition, where we replace the convolution with $\mathbf{k}_{[0,+\infty[}$ by a convolution with $\mathbf{k}_{[0,u[}$, u a parameter going to 0.

We define the projections

$$\begin{aligned} q\colon M\times\mathbb{R}\times\mathbb{R}_{>0}&\to M\times\mathbb{R}, \qquad (x,t,u)\mapsto(x,t),\\ r\colon M\times\mathbb{R}\times\mathbb{R}_{>0}&\to M\times\mathbb{R}, \qquad (x,t,u)\mapsto(x,t-u)\end{aligned} \tag{7.1}$$

and, for an open subset $U\subset M\times\mathbb{R}$, we set

$$U^+ = q^{-1}(U)\cap r^{-1}(U) \quad q_U = q|_{U^+},\ r_U = r|_{U^+}\colon U^+\to U. \tag{7.2}$$

We define the subsets of $\mathbb{R}\times\mathbb{R}_{>0}$:

$$\begin{aligned}\gamma &= \{(t,u);\ 0\leq t<u\},\\ \gamma_0 &= q^{-1}(0) = \{0\}\times\mathbb{R}_{>0},\\ \gamma_1 &= r^{-1}(0) = \{(t,u)\in\mathbb{R}\times\mathbb{R}_{>0}; t=u\}.\end{aligned} \tag{7.3}$$

Definition 7.1. Let M be a manifold and let $U\subset M\times\mathbb{R}$ be an open subset. For $F\in\mathbb{I}(\mathbf{k}_U)$ and $G\in\mathbb{I}(\mathbf{k}_{\mathbb{R}\times\mathbb{R}_{>0}})$ we define $G\star F\in\mathbb{I}(\mathbf{k}_{U^+})$ by

$$G\star F = \mathrm{R}s_!(F\overset{\mathrm{L}}{\boxtimes} G)|_{U^+}, \tag{7.4}$$

where $s\colon U\times\mathbb{R}\times\mathbb{R}_{>0}\to M\times\mathbb{R}\times\mathbb{R}_{>0}$ is the sum $s(x,t_1,t_2,u) = (x,t_1+t_2,u)$ and we denote by $G\overset{+}{\star}F$ its image in $\mathbb{I}^+(\mathbf{k}_U)$.

We define the functor $\Psi_U\colon\mathbb{I}(\mathbf{k}_U)\to\mathbb{I}^+(\mathbf{k}_U)$ by

$$\Psi_U(F) = \mathbf{k}_\gamma\overset{+}{\star}F. \tag{7.5}$$

Let V be an open subset of U. Let N be a submanifold of M and $U' = U\cap(N\times\mathbb{R})$. We can check

$$\Psi_V(F|_V)\simeq(\Psi_U(F))|_V, \tag{7.6}$$

$$\Psi_{U'}(F|_{U'})\simeq(\Psi_U(F))|_{U'}. \tag{7.7}$$

In particular Ψ is a functor of stacks from $\mathbb{I}(\mathbf{k}_{M\times\mathbb{R}})$ to $\mathbb{I}^+(\mathbf{k}_{M\times\mathbb{R}})$.

Using the notations (7.3) we have $\mathbf{k}_{\gamma_0}\star F\simeq q_U^{-1}(F)$ and $\mathbf{k}_{\gamma_1}\star F\simeq r_U^{-1}(F)$. Since $\gamma_0\subset\gamma$ and $\gamma_1\subset\overline{\gamma}\setminus\gamma$, we have natural morphisms $\mathbf{k}_\gamma\to\mathbf{k}_{\gamma_0}$ and $\mathbf{k}_{\gamma_1}[-1]\to\mathbf{k}_\gamma$. They induce morphisms, for all $F\in\mathbb{I}(\mathbf{k}_U)$,

$$\alpha(F)\colon\mathbf{k}_\gamma\star F\to q_U^{-1}(F), \qquad \beta(F)\colon r_U^{-1}(F)[-1]\to\mathbf{k}_\gamma\star F. \tag{7.8}$$

The morphism $\mathbf{k}_{\gamma_1}[-1]\to\mathbf{k}_\gamma$ factorizes through $\mathbf{k}_{\gamma_1}[-1]\to\mathbf{k}_{\mathrm{Int}(\gamma)}$ and $\mathbf{k}_{\mathrm{Int}(\gamma)}\to\mathbf{k}_\gamma$. These morphisms induce $\beta'(F)\colon r_U^{-1}(F)[-1]\to\mathbf{k}_{\mathrm{Int}(\gamma)}\star F$

and $\beta''(F)\colon \mathbf{k}_{\mathrm{Int}(\gamma)} \star F \to \mathbf{k}_\gamma \star F$. The excision triangle for the inclusion $\gamma_0 \subset \gamma$ induces the distinguished triangle

$$\mathbf{k}_{\mathrm{Int}(\gamma)} \star F \xrightarrow{\beta''(F)} \mathbf{k}_\gamma \star F \xrightarrow{\alpha(F)} q_U^{-1}(F) \xrightarrow{+1} . \tag{7.9}$$

Lemma 7.2. *For $F \in \mathsf{D}^{\mathrm{lb}}_{\tau\geq 0}(\mathbf{k}_U)$ the morphism $\beta'(F)\colon r_U^{-1}(F)[-1] \to \mathbf{k}_{\mathrm{Int}(\gamma)} \star F$ is an isomorphism and* (7.9) *gives the distinguished triangle in $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{U^+})$:*

$$r_U^{-1}(F)[-1] \xrightarrow{\beta(F)} \mathbf{k}_\gamma \star F \xrightarrow{\alpha(F)} q_U^{-1}(F) \xrightarrow{+1} . \tag{7.10}$$

Proof. We set $\gamma' = \overline{\gamma} \setminus \gamma_0$. Applying $\cdot \star F$ to the excision triangle given by $\gamma_1 \subset \gamma'$ gives the distinguished triangle

$$r_U^{-1}(F)[-1] \xrightarrow{\beta'(F)} \mathbf{k}_{\mathrm{Int}(\gamma)} \star F \to \mathbf{k}_{\gamma'} \star F \xrightarrow{+1} .$$

Hence the first assertion follows from the vanishing of $\mathbf{k}_{\gamma'} \star F$, which we prove now.

For $x \in M$ and $u > 0$ we define $i_{(x,u)}\colon \mathbb{R} \to M\times\mathbb{R}\times\mathbb{R}_{>0}$, $t \mapsto (x,t,u)$. To prove that $\mathbf{k}_{\gamma'} \star F \simeq 0$, it is enough to see that $i_{(x,u)}^{-1}(\mathbf{k}_{\gamma'} \star F) \simeq 0$, for all (x,u). By the base change formula we have $i_{(x,u)}^{-1}(\mathbf{k}_{\gamma'} \star F) \simeq \mathrm{R}s_!(i_x^{-1}F \boxtimes \mathbf{k}_{]0,u]})$, where $s\colon \mathbb{R}^2 \to \mathbb{R}$ is the sum and $i_x\colon \mathbb{R} \to M \times \mathbb{R}$ the inclusion. We conclude with Lemma 7.3 below. ☐

Lemma 7.3. *Let $a < b \in \mathbb{R}$ and let $F \in \mathsf{D}^{\mathrm{lb}}_{\tau\geq 0}(\mathbf{k}_{\mathbb{R}})$. Then $\mathrm{R}s_!(i_x^{-1}F \boxtimes \mathbf{k}_{]a,b]}) \simeq 0$.*

Proof. The germs of $\mathrm{R}s_!(i_x^{-1}F \boxtimes \mathbf{k}_{]a,b]})$ at some $x \in \mathbb{R}$ are

$$\mathrm{R}\Gamma_c(s^{-1}(x); (F \boxtimes \mathbf{k}_{]a,b]})|_{s^{-1}(x)}) \simeq \mathrm{R}\Gamma_c(\mathbb{R}; F \otimes \mathbf{k}_{[x-b,x-a[}). \tag{7.11}$$

We set $G = F \otimes \mathbf{k}_{[x-b,x-a[}$. We have $\mathrm{SS}(\mathbf{k}_{[x-b,x-a[}) \subset T^*_{\tau\geq 0}\mathbb{R}$ and the bound for the microsupport of a tensor product gives $\mathrm{SS}(G) \subset T^*_{\tau\geq 0}\mathbb{R}$.

By the "Morse lemma" for sheaves (Cor. 5.4.19 of [10]) the restriction $\mathrm{R}\Gamma(]x-b-1,+\infty[;G) \to \mathrm{R}\Gamma(]x-a+1,+\infty[;G)$ is an isomorphism. Since $\mathrm{supp}(G) \subset [x-b,x-a]$ it follows that (7.11) vanishes.

This holds for all $x \in \mathbb{R}$ and proves the lemma. ☐

Lemma 7.4. *Let $F \in \mathbb{I}(\mathbf{k}_U)$ and let $V \subset U$ be an open subset. We assume that*

$$F|_{V\cap(\{x\}\times\mathbb{R})} \text{ is locally constant, for any } x \in M. \tag{7.12}$$

Then $\Psi_U(F)|_V \simeq 0$. As a special case, if $\mathrm{SS}(F|_V) \subset T^*_V V$, then $\Psi_U(F)|_V \simeq 0$. In particular $\mathrm{supp}(\Psi_U(F)) \subset \dot{\pi}_U(\dot{\mathrm{SS}}(F))$.

Proof. We set $V_x = V \cap (\{x\} \times \mathbb{R})$. By (7.7) we have $\Psi_U(F)|_{V_x} \simeq \Psi_{V_x}(F|_{V_x})$. The set V_x is a disjoint union of open intervals of $\mathbb{R}$ and $F|_{V_x}$ is constant on each of these intervals. A direct computation gives $\Psi_{V_x}(F|_{V_x}) \simeq 0$ and we obtain the result. □

Recall the partial localization $\underline{\mathbb{I}}^{pl}(\mathbf{k}_M)$ introduced in Definition 3.2. Lemma 7.4 says that Ψ factorizes through $\underline{\mathbb{I}}^{pl}(\mathbf{k}_M)$:

Proposition 7.5. *The functors Ψ_U, for $U \subset M \times \mathbb{R}$, induce a functor of stacks $\Psi\colon \underline{\mathbb{I}}^{pl}(\mathbf{k}_{M\times\mathbb{R}}) \to \underline{\mathbb{I}}^{+}(\mathbf{k}_{M\times\mathbb{R}})$.*

8. Microsupport and convolution

Here we take the microsupport into account in the construction of the category $\underline{\mathbb{I}}^{+}(\mathbf{k}_{M\times\mathbb{R}})$ and the functor Ψ. For a map $f\colon M \to N$ we use the notations $f_d\colon M \times_N T^*N \to T^*M$ and $f_\pi\colon M \times_N T^*N \to T^*N$ for its transpose derivative and for the obvious projection. Let $A \subset \dot{T}^*(M \times \mathbb{R})$ be a closed conic subset. We define the subsets A_p, A_r of $\dot{T}^*(M \times \mathbb{R} \times \mathbb{R}_{>0})$ by

$$\begin{aligned} A_p &= p_d(p_\pi^{-1}(A)) = \{(x,t,u;\xi,\tau,0);\ (x,t;\xi,\tau) \in A\}, \\ A_r &= r_d(r_\pi^{-1}(A)) = \{(x,t,u;\xi,\tau,-\tau);\ (x,t-u;\xi,\tau) \in A\}. \end{aligned} \tag{8.1}$$

For $U \subset M \times \mathbb{R}$ we define the subcategory $\underline{\mathbb{I}}^{+}_A(\mathbf{k}_U)$ of $\underline{\mathbb{I}}^{+}(\mathbf{k}_U)$

$$\underline{\mathbb{I}}^{+}_A(\mathbf{k}_U) = \varinjlim_V \mathbb{I}_{(A_p \cup A_r)\cap T^*V}(\mathbf{k}_V),$$

where V runs over the open subsets of $U \times \mathbb{R}_{>0}$ satisfying (6.2). Then $U \mapsto \underline{\mathbb{I}}^{+}_A(\mathbf{k}_U)$ defines a stack on $M \times \mathbb{R}$.

If $A \subset \dot{T}^*_{\tau>0}(M \times \mathbb{R})$, then the sets A_p and A_r are disjoints. Hence for any open subset V of $U \times \mathbb{R}_{>0}$ we can consider the functor induced by (3.1)

$$\mathfrak{m}_{A_r \cap T^*V}\colon \mathbb{I}_{(A_p\cup A_r)\cap T^*V}(\mathbf{k}_V) \to \mu\mathrm{Sh}(\mathbf{k}_{A_p \cap T^*V}). \tag{8.2}$$

Lemma 8.1. *(i) Let $U \subset M \times \mathbb{R}$ be an open subset and let V be an open subset of $U \times \mathbb{R}_{>0}$ satisfying* (6.2). *Then the inverse image $F \mapsto (p^{-1}F)|_V$ induces a functor $\mu\mathrm{Sh}(\mathbf{k}_{A\cap T^*U}) \to \mu\mathrm{Sh}(\mathbf{k}_{A_p\cap T^*V})$.*

(ii) If $p(V) = U$ and $p|_V\colon V \to U$ has contractible fibers, then $\mu\mathrm{Sh}(\mathbf{k}_{A\cap T^*U}) \to \mu\mathrm{Sh}(\mathbf{k}_{A_p\cap T^*V})$ is a quasi-equivalence. In particular we have a quasi-equivalence

$$\mu\mathrm{Sh}(\mathbf{k}_{A\cap T^*U}) \to \varinjlim_V \mu\mathrm{Sh}(\mathbf{k}_{A_p\cap T^*V}), \tag{8.3}$$

where V runs over the open subsets of $U \times \mathbb{R}_{>0}$ satisfying (6.2).

If $A \subset \dot{T}^*_{\tau>0}(M\times\mathbb{R})$, the functor (8.2) and the quasi-equivalence (8.3) induce a functor

$$\mathfrak{m}^+_A \colon \underline{\mathbb{I}}^+_A(\mathbf{k}_U) \to \mu\mathrm{Sh}(\mathbf{k}_{A\cap T^*U}) \tag{8.4}$$

in the homotopy category of dg-categories ($Ho(dgCat)$). On the other hand, it follows immediately from Lemma 7.2 and Proposition 7.5 that Ψ induces a functor of stacks

$$\Psi \colon \underline{\mathbb{I}}^{pl}_A(\mathbf{k}_{M\times\mathbb{R}}) \to \underline{\mathbb{I}}^+_A(\mathbf{k}_{M\times\mathbb{R}}). \tag{8.5}$$

Lemma 7.2 also says that this functor is isomorphic to the functor induced by $\underline{q}^{-1}$ and we can deduce the following result.

Proposition 8.2. *Let $A \subset \dot{T}^*_{\tau>0}(M\times\mathbb{R})$ be a closed conic subset. Then the functors*

$$\mathfrak{m}^+_A \circ \Psi,\ \mathfrak{m}^{pl}_A \colon \underline{\mathbb{I}}^{pl}_A(\mathbf{k}_{M\times\mathbb{R}}) \to \mu\mathrm{Sh}(\mathbf{k}_A)$$

are isomorphic.

9. Quantization - I

Here Λ is a closed conic Lagrangian submanifold of $T^*_{\tau>0}(M\times\mathbb{R})$ obtained from a compact connected exact Lagrangian submanifold $\widetilde{\Lambda}$ of T^*M as in (0.1). We assume that the projection $\widetilde{\Lambda} \to M$ is finite; hence $\Lambda/\mathbb{R}_{>0} \to M\times\mathbb{R}$ is also finite (see Remark 3.5).

For $\varepsilon > 0$ we define the translation

$$T_\varepsilon \colon M\times\mathbb{R} \to M\times\mathbb{R}, \qquad (x,t) \mapsto (x, t+\varepsilon).$$

and we let T'_ε be the induced map on $T^*(M\times\mathbb{R})$. The hypothesis that Λ comes from $\widetilde{\Lambda} \subset T^*M$ implies

$$T'_u(\Lambda) \cap T'_v(\Lambda) = \emptyset \quad \text{if } u \neq v. \tag{9.1}$$

Proposition 9.1. *Let $F \in \mu\mathrm{Sh}(\mathbf{k}_\Lambda)(\Lambda)$ be a globally defined object. Then there exist $\varepsilon > 0$ and $F' \in \mathbb{I}_{\Lambda\cup T'_\varepsilon(\Lambda)}(\mathbf{k}_{M\times\mathbb{R}})$ such that $\mathfrak{m}_\Lambda(F')$ is isomorphic to F in the homotopy category of $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)(\Lambda)$.*

Proof. (i) By Proposition 3.3 we have a quasi-equivalence

$$\mathfrak{m}^{pl}_\Lambda \colon (\underline{\mathbb{I}}^{pl}_\Lambda(\mathbf{k}_{M\times\mathbb{R}}))(M\times\mathbb{R}) \to \mu\mathrm{Sh}(\mathbf{k}_\Lambda)(\Lambda)$$

and we can find $F_1 \in (\underline{\mathbb{I}}^{pl}_\Lambda(\mathbf{k}_{M\times\mathbb{R}}))(M\times\mathbb{R})$ such that $\mathfrak{m}^{pl}_\Lambda(F_1) \simeq F$ (in the homotopy category). Applying the functor Ψ of (8.5) we obtain $F_2 = \Psi(F_1) \in \underline{\mathbb{I}}^+_\Lambda(\mathbf{k}_{M\times\mathbb{R}})$ and we have $\mathfrak{m}^+_\Lambda(F_2) \simeq F$ by Proposition 8.2.

(ii) Since $\mathrm{supp}(F_2) \subset \pi_M(\Lambda)$ is compact, we can find $\varepsilon > 0$ such that F_2 is represented by some $F_2^\varepsilon \in \mathbb{I}(\mathbf{k}_{M\times\mathbb{R}\times]0,2\varepsilon[})$. Then $F' = F_2^\varepsilon|_{M\times\mathbb{R}\times\{\varepsilon\}}$ satisfies the conclusion of the proposition. □

10. Deformation of the microsupport

We recall here a result of [6] which says that a deformation of a microsupport of a sheaf by some Hamiltonian isotopy induces a "deformation" of the sheaf.

Let N be a manifold, let I be an open interval of $\mathbb{R}$ and let $u_0 \in I$ be given. We consider a homogeneous Hamiltonian isotopy $\phi\colon \dot{T}^*N \times I \to \dot{T}^*N$ of class C^∞, that is, ϕ is a C^∞-map and, denoting by $\phi_u\colon \dot{T}^*N \times \{u\} \to \dot{T}^*N$ the restriction at time u, we have

(i) $\phi_{u_0} = \mathrm{id}_{\dot{T}^*N}$,
(ii) ϕ_u is a $\mathbb{R}_{>0}$-homogeneous symplectic isomorphism for any $u \in I$.

Let $L_{u_0} \subset \dot{T}^*N$ be a closed conic subset. We set $L_u = \phi_u(L_0)$, for all $u \in I$. We let $i_u\colon N \to N \times I$ be the inclusion $x \mapsto (x, u)$. Then there exists a unique conic subset

$$L \subset \dot{T}^*(N \times I) \tag{10.1}$$

such that L is non-characteristic for i_u and $L_u = (i_u)_d((i_u)_\pi^{-1}(L))$, for any $u \in I$. The bound for the microsupport of the inverse image implies that $\underline{i_u^{-1}}$ gives a functor

$$\underline{i_u^{-1}}\colon \mathbb{I}_L(\mathbf{k}_{N\times I}) \to \mathbb{I}_{L_u}(\mathbf{k}_N). \tag{10.2}$$

Proposition 10.1. (Prop. 3.12 of [6]) *For any $u \in I$ the functor* (10.2) *is a quasi-equivalence.*

Remark 10.2. Proposition 10.1 has the following consequence (which is also stated in [6] as corollary of the main theorem): the categories $\mathsf{D}^{\mathrm{lb}}_{L_{u_0}}(\mathbf{k}_N)$ and $\mathsf{D}^{\mathrm{lb}}_{L_u}(\mathbf{k}_N)$ are equivalent. In the same way, the categories $\mathsf{D}^{\mathrm{lb}}_{(L_{u_0})}(\mathbf{k}_N)$ and $\mathsf{D}^{\mathrm{lb}}_{(L_u)}(\mathbf{k}_N)$ are equivalent, as well as $\mu\mathrm{Sh}(\mathbf{k}_{L_{u_0}})$ and $\mu\mathrm{Sh}(\mathbf{k}_{L_u})$.

If L_{u_0} is smooth Lagrangian, then the equivalence preserves the simple objects (since the simpleness can be tested at any point of the connected components of L). Hence $\mu\mathrm{Sh}^s(\mathbf{k}_{L_{u_0}})$ and $\mu\mathrm{Sh}^s(\mathbf{k}_{L_u})$ are also equivalent.

Remark 10.3. Proposition 10.1 applies in particular to $N \times N$ and to any ϕ' defined on $\dot{T}^*(N \times N)$ which coincides with $\mathrm{id}_{T^*N} \times \phi$ on $L_{u_0} = T^*_{\Delta_N}(N \times N)$. Then L_u is the graph of ϕ_u and we deduce that there exists a unique sheaf K on $N \times N \times I$ such that $\dot{\mathrm{SS}}(K) = L$ and $i_{u_0}^{-1}(K) \simeq \mathbf{k}_{\Delta_N}$, which is the main result of [6]. In particular $\dot{\mathrm{SS}}(i_u^{-1}(K))$ is the graph of ϕ_u. Conversely, an inverse of (10.2) is given by $F \mapsto K \circ L$, for $F \in \mathsf{D}^{\mathrm{lb}}_{(L_0)}(\mathbf{k}_N)$, where $\circ$ denotes the convolution product $K \circ L = \mathrm{R}q_{2!}(K \overset{\mathrm{L}}{\otimes} q_1^{-1}L)$ with $q_1 \colon N_1 \times N_2 \times I \to N_1$ and $q_2 \colon N_1 \times N_2 \times I \to N_2 \times I$ the projections.

We come back to our Lagrangian submanifold $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$ as in Section 9. Using the notations (8.1) we define

$$\Lambda^+ = \Lambda_p \cup \Lambda_r, \qquad \Lambda_u = \Lambda \cup T'_u(\Lambda). \tag{10.3}$$

Using (9.1) it is not difficult to prove that there exists a Hamiltonian isotopy which keeps Λ fixed and moves $T'_u(\Lambda)$ to $T'_{u+s}(\Lambda)$ for times $s \geq 0$. Hence we can apply Proposition 10.1 with $L = \Lambda^+$ and we obtain the following result.

Corollary 10.4. *The inverse image functor induces an equivalence of categories*

$$i_u^{-1} \colon \mathsf{D}^{\mathrm{lb}}_{\Lambda^+}(\mathbf{k}_{M\times\mathbb{R}\times\mathbb{R}_{>0}}) \to \mathsf{D}^{\mathrm{lb}}_{\Lambda_u}(\mathbf{k}_{M\times\mathbb{R}}), \tag{10.4}$$

for any $u > 0$. In particular, for all $F, G \in \mathsf{D}^{\mathrm{lb}}_{\Lambda^+}(\mathbf{k}_{M\times\mathbb{R}\times\mathbb{R}_{>0}})$ we have

$$\mathrm{RHom}(F, G) \xrightarrow{\sim} \mathrm{RHom}(F|_{M\times\mathbb{R}\times\{u\}}, G|_{M\times\mathbb{R}\times\{u\}}). \tag{10.5}$$

Corollary 10.5. *For any $F_1, F_2 \in \mathsf{D}^{\mathrm{lb}}_{\Lambda}(\mathbf{k}_{M\times\mathbb{R}})$ we have isomorphisms*

$$\mathrm{RHom}(F_1, F_2) \xrightarrow{\sim} \mathrm{RHom}(q^{-1}F_1, r^{-1}F_2) \xrightarrow{\sim} \mathrm{RHom}(F_1, T_{u*}F_2),$$

for any $u \geq 0$.

Proof. Since r is a submersion with fibers diffeomorphic to $\mathbb{R}_{>0}$ we have $\mathrm{RHom}(F_1, F_2) \xrightarrow{\sim} \mathrm{RHom}(r^{-1}F_1, r^{-1}F_2)$. By hypothesis we have $\Lambda \subset \{\tau > 0\}$ and we can consider the distinguished triangle (7.10) (for $U = M \times \mathbb{R}$). Applying $\mathrm{RHom}(\cdot, r^{-1}F_2)$ to (7.10) we obtain the triangle

$$\begin{aligned} &\mathrm{RHom}(r^{-1}F_1, r^{-1}F_2) \to \mathrm{RHom}(q^{-1}F_1, r^{-1}F_2) \\ &\qquad \to \mathrm{RHom}(\Psi_{M\times\mathbb{R}}(F_1), r^{-1}F_2) \xrightarrow{+1} . \end{aligned} \tag{10.6}$$

Using the base change formula we can check that $\mathrm{R}r_!(\Psi_{M\times\mathbb{R}}(F_1)) \simeq 0$. Hence the third term in (10.6) vanishes and we obtain the first isomorphism of the corollary. The second one follows from (10.5) applied with $F = q^{-1}F_1$ and $G = r^{-1}F_2$. ☐

11. Quantization - II

Theorem 11.1. *For any globally defined object $F \in \mu\mathrm{Sh}(\mathbf{k}_\Lambda)(\Lambda)$ there exists $G \in \mathbb{I}_{\Lambda\cup T^*_{M\times\mathbb{R}}(M\times\mathbb{R})}(\mathbf{k}_{M\times\mathbb{R}})$ and an isomorphism $\mathfrak{m}_\Lambda(G) \simeq F$ in the homotopy category of $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)(\Lambda)$.*

Proof. We let $F' \in \mathbb{I}_{\Lambda\cup T'_\varepsilon(\Lambda)}(\mathbf{k}_{M\times\mathbb{R}})$ be the object given by Proposition 9.1. By the equivalence of categories (10.4) there exists $F'' \in \mathsf{D}^{\mathrm{lb}}_{\Lambda^+}(\mathbf{k}_{M\times\mathbb{R}\times\mathbb{R}_{>0}})$ such that, setting $F_u = F''|_{M\times\mathbb{R}\times\{u\}}$, we have $F_\varepsilon \simeq F'$. By Lemma 8.1 we have $\mathfrak{m}_\Lambda(F_u) \simeq \mathfrak{m}_\Lambda(F_\varepsilon) \simeq F$ for all $u > 0$.

There exist A, B such that $\Lambda \subset T^*(M \times]A, B[)$. For $u > B - A + 1$ the sheaf $G' = F_u|_{M\times]-\infty,B+1[}$ satisfies $\dot{\mathrm{SS}}(G') = \Lambda$ and $\mathfrak{m}_\Lambda(G') \simeq F$. We choose a diffeomorphism $f\colon \mathbb{R} \xrightarrow{\sim}]-\infty, B+1[$ which is the identity on $]-\infty, B[$. Then $G = \underline{f}^{-1}(G')$ satisfies the conclusion of the theorem. □

12. Triangulated orbit categories

We will use a very special case of the triangulated hull of an orbit category as described by Keller in [12]. More precisely Definition 12.1 below is inspired by §7 of [12] that we apply to the simple case where we quotient $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ by the autoequivalence $F \mapsto F[1]$ (in [12] much more general equivalences are considered). We will use abusively the name orbit category for "triangulated hull of the orbit category".

In this section we set $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{K} = \mathbf{k}[X]/\langle X^2\rangle$. We let ε be the image of X in $\mathbb{K}$. Hence $\mathbb{K} = \mathbf{k}[\varepsilon]$ with $\varepsilon^2 = 0$. Let M be a manifold. The natural ring morphisms $\mathbf{k} \to \mathbb{K}$ and $\mathbb{K} \to \mathbf{k}$ induce two pairs of adjoint functors (e_M, r_M) and (E_M, R_M), where e_M, E_M are scalar extensions and r_M, R_M restrictions of scalars:

$$\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M) \underset{r_M}{\overset{e_M}{\rightleftarrows}} \mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M), \qquad e_M(F) = \mathbb{K}_M \otimes_{\mathbf{k}_M} F, \quad r_M(G) = G,$$

$$\mathsf{D}^{-}(\mathbb{K}_M) \underset{R_M}{\overset{E_M}{\rightleftarrows}} \mathsf{D}^{-}(\mathbf{k}_M), \qquad E_M(G) = \mathbf{k}_M \overset{\mathrm{L}}{\otimes}_{\mathbb{K}_M} G, \;\; R_M(F) = F.$$

Definition 12.1. We let $\mathsf{perf}(\mathbb{K}_M)$ be the full triangulated subcategory of $\mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M)$ generated by the image of e_M, that is, by the objects of the form $\mathbb{K}_M \otimes_{\mathbf{k}_M} F$ with $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$.

We denote by $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ the quotient $\mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M)/\mathsf{perf}(\mathbb{K}_M)$. We let $Q_M\colon \mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M) \to \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ be the quotient functor and we set $\iota_M = Q_M \circ R_M\colon \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M) \to \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$.

We define in the same way the dg enhancement $\mathbb{I}_{/[1]}(\mathbf{k}_M)$ of $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ (see [20] Cor. 8.7 or [21] §4.3 for the quotient) and we also denote by Q_M, ι_M the morphisms in $Ho(dgCat)$ similar to the above defined Q_M, ι_M.

The exact sequence of $\mathbb{K}$-modules $0 \to \mathbf{k} \to \mathbb{K} \to \mathbf{k} \to 0$ induces a morphism

$$s_M\colon \mathbf{k}_M \to \mathbf{k}_M[1] \qquad \text{in } \mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M) \tag{12.1}$$

and a distinguished triangle, for any $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$,

$$R_M(F) \to e_M(F) \to R_M(F) \xrightarrow{s_M \otimes \mathrm{id}_F} R_M(F)[1]. \tag{12.2}$$

We thus obtain an isomorphism $s_M \otimes \mathrm{id}_F \colon R_M(F) \xrightarrow{\sim} R_M(F)[1]$ in $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$, for any $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$.

Proposition 12.2. *For any $F, G \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ we have*

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)}(\iota_M(F), \iota_M(G)) \simeq \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)}(F[-n], G).$$

We can check that the six Grothendieck operations for sheaves induce similar functors on the orbit categories. In particular we can define a μhom functor from $\mathbb{I}_{/[1]}(\mathbf{k}_M)^{op} \times \mathbb{I}_{/[1]}(\mathbf{k}_M)$ to $\mathbb{I}_{/[1]}(\mathbf{k}_{T^*M})$.

We define a microsupport for $F \in \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ by $\mathrm{SS}(F) = \bigcap_{F'} \mathrm{SS}(F')$ where F' runs over the objects of $\mathsf{D}^{\mathrm{lb}}(\mathbb{K}_M)$ such that $F' \simeq F$ in $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$. We can prove that this microsupport behaves like the usual microsupport of sheaves with respect to the Grothendieck operations.

Then we define the categories $\mathsf{D}^{\mathrm{b}}_{/[1]_{(S)}}(\mathbf{k}_M)$ and $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M; S)$ by replacing D^{lb} by $\mathsf{D}^{\mathrm{b}}_{/[1]}$ in the definition of $\mathsf{D}^{\mathrm{lb}}_{(S)}(\mathbf{k}_M)$ and $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M; S)$. We obtain the analog of the Kashiwara-Schapira stack in this framework, denoted $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$. It comes with a functor $\mathfrak{m}_{/[1],\Lambda} \colon \mathsf{D}^{\mathrm{b}}_{/[1],(\Lambda)}(\mathbf{k}_M) \to \mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$.

We say that $F \in \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ is simple along Λ if $\mathrm{SS}(F) \cap \dot{T}^*M \subset \Lambda$ and, for any $p \in \Lambda$, there exist a neighborhood Λ_0 of p in Λ and $F' \in \mathsf{D}^{\mathrm{lb}}_{(\Lambda_0)}(\mathbf{k}_M)$ such that F' is simple along Λ_0 and $\iota(F') \simeq F$ in $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M; \Lambda_0)$.

Since any object F of $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$ is isomorphic to its shifts $F[n]$, the Maslov class, interpreted in Section 5 as a shift in cohomological degree, is a priori no longer an obstruction for the existence of a global object in $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$. Indeed we can prove the following result.

Proposition 12.3. *For any smooth closed conic Lagrangian submanifold Λ of $\dot{T}^*M$, there exists a global simple object in $\mu\mathrm{Sh}_{/[1]}((\mathbb{Z}/2\mathbb{Z})_\Lambda)$.*

Moreover we also have the analog of Proposition 4.4, Theorem 11.1 and Theorem 13.3 below for $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ and $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$.

Part 3. Topological consequences

In this part we recover results of [4], [16], and [2] which say that the projection $\Lambda \to M$ is a homotopy equivalence, assuming the vanishing of the Maslov class of Λ, and also [13] which says that, indeed, the Maslov class of Λ vanishes. (A more precise result is proved in [3]: the projection $\Lambda \to M$ is a simple homotopy equivalence). We also see that the class $c^{\mathbf{k}}_\Lambda \in H^2(\Lambda; \mathbf{k}^\times)$ introduced in (5.1) vanishes.

13. Restriction at infinity

Since $\Lambda/\mathbb{R}_{>0}$ is compact we can choose $A > 0$ such that

$$\Lambda \subset T^*_{\tau>0}(M \times \,]-A, A[). \tag{13.1}$$

Then, for $F \in \mathsf{D}^{\mathrm{lb}}_\Lambda(\mathbf{k}_{M\times\mathbb{R}})$, the restrictions $F|_{M\times]-\infty,-A[}$ and $F|_{M\times]A,+\infty[}$ have locally constant cohomology sheaves.

Definition 13.1. For $F \in \mathsf{D}^{\mathrm{lb}}_\Lambda(\mathbf{k}_{M\times\mathbb{R}})$ we define $F_-, F_+ \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ by $F_- = F|_{M\times\{-t\}}$, $F_+ = F|_{M\times\{t\}}$, for any $t \in [A, +\infty[$. We let $\mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ be the full subcategory of $\mathsf{D}^{\mathrm{lb}}_\Lambda(\mathbf{k}_{M\times\mathbb{R}})$ consisting of the F such that $F_- \simeq 0$.

For $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ we have by definition

$$F|_{M\times]A,+\infty[} \simeq F_+ \boxtimes \mathbf{k}_{]A,+\infty[}, \qquad F|_{M\times]-\infty,-A[} \simeq 0. \tag{13.2}$$

Lemma 13.2. *Let $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}})$. We assume that there exists $B > 0$ such that $\mathrm{supp}(F) \subset M \times [-B, B]$. We also assume either $\mathrm{SS}(F) \subset T^*_{\tau\geq 0}(M\times\mathbb{R})$ or $\mathrm{SS}(F) \subset T^*_{\tau\leq 0}(M\times\mathbb{R})$. Let $p_M \colon M\times\mathbb{R} \to M$ be the projection. Then $\mathrm{R}p_{M!}(F) \simeq \mathrm{R}p_{M*}(F) \simeq 0$.*

Proof. By base change we may assume that M is a point. Then the result follows from the "Morse lemma" (Cor. 5.4.19 of [10]). ☐

Theorem 13.3. *Let $F, F' \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$. Let $F_+, F'_+ \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ be as in Definition 13.1. Then*

$$\mathrm{RHom}(F, F') \xrightarrow{\sim} \mathrm{RHom}(F_+, F'_+). \tag{13.3}$$

In particular the functor $\mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}}) \to \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ given by $F \mapsto F_+$ is fully faithful and we have: $F \simeq F'$ if and only if $F_+ \simeq F'_+$.

Proof. Let $p_M \colon M\times\mathbb{R} \to M$ be the projection. Let us choose $u > 2A$. Hence $\mathrm{supp}(T_{u*}F') \subset M\times]A, +\infty[$ and we obtain by Corollary 10.5

$$\begin{aligned} \mathrm{RHom}(F, F') &\simeq \mathrm{RHom}(F, T_{u*}F') \\ &\simeq \mathrm{RHom}(p_M^{-1}(F_+), T_{u*}F') \\ &\simeq \mathrm{RHom}(F_+, \mathrm{R}p_{M*}T_{u*}F'). \end{aligned} \tag{13.4}$$

Let us set $G = (T_{u*}F') \otimes \mathbf{k}_{M\times]-\infty,A+u[}$. By (13.2) we know that $T_{u*}F'$ has locally constant cohomology sheaves in a neighborhood of $M\times\{A+u\}$. We deduce that $\mathrm{SS}(G) \subset T^*_{\tau\geq 0}(M\times\mathbb{R})$. By Lemma 13.2 we obtain $\mathrm{R}p_{M*}(G) \simeq 0$. By (13.2) again we have the distinguished triangle $G \to T_{u*}F' \to F'_+ \boxtimes \mathbf{k}_{[A+u,+\infty[} \xrightarrow{+1}$. Hence we obtain $\mathrm{R}p_{M*}T_{u*}F' \simeq \mathrm{R}p_{M*}(F'_+ \boxtimes \mathbf{k}_{[A+u,+\infty[}) \simeq F'_+$ and (13.4) translates into (13.3). ☐

Theorem 13.4. *Let $F, F' \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$. Then we have an isomorphism*

$$\mathrm{RHom}(F,F') \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \mu hom(F,F')). \tag{13.5}$$

Its composition with (13.3) *gives a canonical isomorphism*

$$\mathrm{RHom}(F_+,F'_+) \simeq \mathrm{R}\Gamma(\Lambda; \mu hom(F,F')). \tag{13.6}$$

Proof. Here is a sketch of proof assuming F is constructible. In view of (4.3) it is enough to prove $\mathrm{R}\Gamma(M\times\mathbb{R}; \mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} F') \simeq 0$. We define $q_2\colon M\times\mathbb{R}^2 \to \mathbb{R}$ by $q_2(x,t,u) = u$ and $q, r\colon M\times\mathbb{R}^2 \to M\times\mathbb{R}$ as in (7.1). We set $G = \mathrm{R}q_{2*}(q^{-1}\mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} r^{-1}F')$. For $u\in\mathbb{R}$ we then have $G_u = \mathrm{R}\Gamma(M\times\mathbb{R}; \mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} T_{u*}F')$. By microsupport estimates we can check that $G_0 \simeq G_u$ for any $u\leq 0$. As in the proof of Theorem 13.3 we have $G_u \simeq \mathrm{R}\Gamma(M\times\mathbb{R}; \mathrm{D}'(F) \overset{\mathrm{L}}{\otimes} p_M^{-1}(F'_+))$ for $u \ll 0$ and, using Lemma 13.2, we can see that it vanishes. Hence $G_0 \simeq 0$ as required.

This proves the first isomorphism and the second one then follows from (13.3). ☐

Remark 13.5. For $F, G, H \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$ we have a composition morphism (see [10, Cor. 4.4.10])

$$\mu hom(F,G) \overset{\mathrm{L}}{\otimes} \mu hom(G,H) \to \mu hom(F,H) \tag{13.7}$$

which is compatible with the composition morphism for $\mathrm{R}\mathcal{H}om$ and the isomorphism $\mathrm{R}\pi_{M*}(\mu hom(\cdot,\cdot)) \simeq \mathrm{R}\mathcal{H}om(\cdot,\cdot)$. In particular the isomorphism (13.5) is compatible with the composition morphisms on both sides. This is also true for (13.3), hence also for (13.6).

14. Poincaré groups

We let $\pi_\Lambda\colon \Lambda \to M$ be the projection to the base and we denote by $\pi_1(\pi_\Lambda)\colon \pi_1(\Lambda) \to \pi_1(M)$ the induced morphism of Poincaré groups.

Proposition 14.1. *The morphism $\pi_1(\pi_\Lambda)\colon \pi_1(\Lambda) \to \pi_1(M)$ is injective.*

Proof. (i) We set $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ and $G = \pi_1(\Lambda)$. We let $\rho\colon G \to GL(\mathbf{k}[G])$ be the regular representation of G. This means that $\mathbf{k}[G]$ is the vector space with basis $\{e_g\}_{g\in G}$ and the action of G is given by $g \cdot e_h = e_{gh}$, for all $g, h \in G$. We let $\mathcal{L}_\rho$ be the local system on Λ with stalks $\mathbf{k}[G]$ corresponding to this representation ρ.

(ii) By Proposition 12.3 we have a simple object $\mathcal{F}_0 \in \mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$. Then the functor $\mu hom(F_0, \cdot)$ induces an equivalence $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda) \xrightarrow{\sim} \mathsf{Loc}(\mathbf{k}_\Lambda)$. We let $\mathcal{F}_\rho \in \mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda)$ be the object associated with $\mathcal{L}_\rho$ by this equivalence. By the analog of Theorem 11.1 for $\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$, there exist $F_0, F_\rho \in \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_{M\times\mathbb{R}})$ such that $\mathfrak{m}_{/[1],\Lambda}(F_0) \simeq \mathcal{F}_0$ and $\mathfrak{m}_{/[1],\Lambda}(F_\rho) \simeq \mathcal{F}_\rho$. We then have $\mu hom(F_0, F_\rho)|_\Lambda \simeq \mathcal{L}_\rho$. We define $L_0, L_1 \in \mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_M)$ by $L_0 = (F_0)_+$ and $L_1 = (F_1)_+$. We let $p\colon M \times \mathbb{R} \to M$ be the projection and we set $F = F_0 \otimes p^{-1}L_1$ and $F' = F_\rho \otimes p^{-1}L_0$. Then $F_+ \simeq L_0 \otimes L_1 \simeq F'_+$ and Theorem 13.3 implies

$$F_0 \otimes p^{-1}L_1 \simeq F_\rho \otimes p^{-1}L_0. \tag{14.1}$$

Applying $\mathfrak{m}_{/[1],\Lambda}$ to (14.1) and the equivalence $\mu\mathrm{Sh}_{/[1]}(\mathbf{k}_\Lambda) \xrightarrow{\sim} \mathsf{Loc}(\mathbf{k}_\Lambda)$, we find

$$\pi_\Lambda^{-1}L_1 \simeq \mathcal{L}_\rho \otimes \pi_\Lambda^{-1}L_0. \tag{14.2}$$

(iii) We let L'_i be the sheaf on M associated with the presheaf $U \mapsto \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}_{/[1]}(\mathbf{k}_U)}(\mathbf{k}_U, L_i)$, for $i = 0, 1$. Then L'_0 and L'_1 are local systems on M and correspond to representations of $\pi_1(M)$, say ρ'_0 and ρ'_1. They induce representations of $G = \pi_1(\Lambda)$, say ρ''_0 and ρ''_1, through the morphism $\pi_1(\pi_\Lambda)$. Then (14.2) gives the isomorphism of representations of G, $\rho''_1 \simeq \rho \otimes \rho''_0$. We restrict these representations to the subgroup $K = \ker(\pi_1(\pi_\Lambda))$ of G. Then $\rho''_0|_K$ and $\rho''_1|_K$ are trivial representations and we deduce that $\rho|_K$ also is trivial. Since ρ is a faithful representation of G, this gives $K = \{1\}$, as required. □

Let $r\colon M' \to M$ be a covering. The derivative of r induces a covering $r'\colon T^*M' \to T^*M$. We let Λ'_0 be a connected component of $r'^{-1}(\Lambda')$. Then $\Lambda'_0 \to \Lambda$ is a covering and $\pi_1(\Lambda'_0)$ is a subgroup of $\pi_1(\Lambda)$. We have the commutative diagram

$$\begin{array}{ccc} \pi_1(\Lambda'_0) & \hookrightarrow & \pi_1(\Lambda) \\ \downarrow & & \downarrow{\scriptstyle \pi_1(\pi_\Lambda)} \\ \pi_1(M') & \longrightarrow & \pi_1(M), \end{array} \tag{14.3}$$

where $\pi_1(\pi_\Lambda)$ is injective by Proposition 14.1. This implies that the morphism $\pi_1(\Lambda'_0) \to \pi_1(M')$ is injective. In particular, if M' is the universal cover of M, then $\pi_1(\Lambda_0')$ vanishes, that is, Λ_0' is the universal cover of Λ.

We denote by m_Λ the Maslov class of Λ, which is a group morphism $m_\Lambda \colon \pi_1(\Lambda) \to \mathbb{Z}$.

Corollary 14.2. *We assume that $m_\Lambda \neq 0$. Then there exist covering maps $f\colon M_0 \to M_1$, $g\colon M_1 \to M$ where f is a cyclic cover of group $\mathbb{Z}$ and closed conic connected Lagrangian submanifolds $\Lambda_i \subset \dot{T}^*(M_i \times \mathbb{R})$ for $i = 0, 1$, such that the derivatives of f and g induce a cyclic cover of group $\mathbb{Z}$, $\Lambda_0 \to \Lambda_1$, and an isomorphism $\Lambda_1 \xrightarrow{\sim} \Lambda$:*

$$
\begin{array}{ccccc}
\Lambda_0 & \xrightarrow{\mathbb{Z}} & \Lambda_1 & \xrightarrow{\sim} & \Lambda \\
\downarrow & & \downarrow & & \downarrow \\
M_0 & \xrightarrow[f]{\mathbb{Z}} & M_1 & \xrightarrow[g]{} & M.
\end{array}
\tag{14.4}
$$

Moreover the isomorphism $\Lambda_1 \xrightarrow{\sim} \Lambda$ identifies the Maslov classes of Λ_1 and Λ and the Maslov class of Λ_0 is zero.

Proof. We set $K = \ker(m_\Lambda)$. Since $m_\Lambda \neq 0$ we have $\pi_1(\Lambda)/K \simeq \mathbb{Z}$. We let M' be the universal cover of M and we define Λ_0' as in the diagram (14.3). Hence Λ_0' is the universal cover of Λ. Since K and $\pi_1(\Lambda)$ are subgroups of $\pi_1(M)$ they act freely on M'. These actions commute with their actions on Λ_0' through the map $\Lambda_0' \to M$. We obtain the diagram of quotient manifolds

$$
\begin{array}{ccccc}
\Lambda_0' & \longrightarrow & \Lambda_0'/K & \xrightarrow{f'} & \Lambda_0'/\pi_1(\Lambda) \\
\downarrow & & \downarrow & & \downarrow \\
M' & \longrightarrow & M'/K & \xrightarrow{f} & M'/\pi_1(\Lambda),
\end{array}
$$

where f' and f are covering maps with group $\pi_1(\Lambda)/K \simeq \mathbb{Z}$. We set $\Lambda_0 = \Lambda_0'/K$, $M_0 = M'/K$, $\Lambda_1 = \Lambda_0'/\pi_1(\Lambda)$ and $M_1 = M'/\pi_1(\Lambda)$. Then Λ_1 is identified with Λ since it is the quotient of the universal cover of Λ by its Poincaré group. This gives the diagram (14.4). The claim on the Maslov classes follows easily. □

15. Vanishing of the Maslov class

In [13] Kragh and Abouzaid (using a result of [1]) prove that the Maslov class of any compact exact Lagrangian submanifold of a cotangent bundle vanishes. Now we can give a new proof of this result.

Theorem 15.1. *We have $m_\Lambda = 0$.*

Proof. (i) We set $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$. We apply Corollary 14.2 and we replace M by M_1. Hence we have cyclic covers of group $\mathbb{Z}$

$$\begin{array}{ccc} \Lambda_0 & \xrightarrow{\mathbb{Z}} & \Lambda \\ \downarrow & & \downarrow{\scriptstyle \pi_\Lambda} \\ M_0 & \xrightarrow[f]{\mathbb{Z}} & M \end{array} \tag{15.1}$$

such that $m_{\Lambda_0} = 0$ and an exact sequence $\pi_1(\Lambda_0) \to \pi_1(\Lambda) \xrightarrow{m_\Lambda} \mathbb{Z}$. The diagram (15.1) induces the isomorphisms $\pi_1(\Lambda)/\pi_1(\Lambda_0) \simeq \mathbb{Z} \simeq \pi_1(M)/\pi_1(M_0)$ and we deduce that m_Λ factorizes through a morphism $m\colon \pi_1(M) \to \mathbb{Z}$ such that $\ker(m) \simeq \pi_1(M_0)$. Since $m_{\Lambda_0} = 0$, the stack $\mu\mathrm{Sh}(\mathbf{k}_{\Lambda_0})$ has a global object, say F_0.

(ii) We can adapt the proof of Theorem 11.1 to the case of the cyclic cover M_0. More precisely, let $\psi\colon M_0 \to M_0$ be the deck transformation generating the action of $\mathbb{Z}$ on M_0. Let $d \in \mathbb{Z}$ be the generator of $\mathrm{im}(m_\Lambda\colon \pi_1(M) \to \mathbb{Z})$. Then we can deduce from F_0 an object $G \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M_0\times\mathbb{R}})$ which is simple along Λ_0 and such that we have an isomorphism $\psi^{-1}(G) \simeq G[d]$.

We define $G_+ = G|_{M_0\times\{t_0\}}$ for $t_0 \gg 0$ as in Definition 13.1. Theorem 13.3 gives $\mathrm{RHom}(G, G) \xrightarrow{\sim} \mathrm{RHom}(G_+, G_+)$. Since $G \not\simeq 0$ it follows that $G_+ \not\simeq 0$. We also know that G_+ has locally constant cohomology sheaves and is locally bounded (since G is). Since M_0 is connected we deduce that G_+ is bounded.

However $\psi^{-1}(G) \simeq G[d]$ implies $(\psi^n)^{-1}(G_+) \simeq G_+[nd]$ for all $n \in \mathbb{Z}$. Since G_+ is bounded and non zero, we obtain $d = 0$ but this contradicts the hypothesis $m_\Lambda \neq 0$. ☐

16. Behaviour at infinity

Lemma 16.1. *Let N be a connected manifold, $\Lambda \subset \dot{T}^*N$ a smooth conic closed Lagrangian submanifold and $\mathbf{k}$ a ring. Let $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_N)$ be such that $\dot{\mathrm{SS}}(F) \subset \Lambda$ and F is simple along Λ. We set $U = N \setminus \dot\pi_N(\Lambda)$. We assume that there exists $x_0 \in U$ such that $H^iF_{x_0}$ is of finite rank over $\mathbf{k}$, for all $i \in \mathbb{Z}$. Then H^iF_x is of finite rank over $\mathbf{k}$, for all $x \in U$ and all $i \in \mathbb{Z}$.*

Proof. We let $Z \subset \dot\pi_M(\Lambda)$ be the subset of points z such that $\dot\pi_M(\Lambda)$ is a smooth hypersurface in some neighborhood of z.

Let $x \in U$ and let I be an open interval containing 0 and 1. We can choose a C^∞ path $\gamma\colon I \to N$ such that $\gamma(0) = x_0$, $\gamma(1) = x$ and $\gamma([0, 1])$ meets $\dot\pi_N(\Lambda)$ at finitely many points, all contained in Z and with a transversal intersection. Then $\gamma^{-1}(F)$ is a sheaf on I which is simple along finitely many half-lines and the result follows from (2.4). □

Proposition 16.2. *We assume that* $\mathbf{k} = \mathbb{Z}$ *or* $\mathbf{k}$ *is a finite field. Let* $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$*. We assume that* F *is simple along* Λ*. Then* F_+ *is concentrated in one degree, say* i*, and* H^iF_+ *is a local system with stalks isomorphic to* $\mathbf{k}$*.*

Proof. (i) We first assume that $\mathbf{k}$ is a finite field. Let us prove that F_+ is concentrated in one degree. Let $a \leq b$ be respectively the minimal and maximal integers i such that $H^iF_+ \not\simeq 0$. By Lemma 16.1 the local systems H^iF_+ are of finite rank. Since $\mathbf{k}$ is finite we can find a finite cover $r\colon M' \to M$ such that $r^{-1}(H^iF_+)$ are trivial, for $i = a, b$. We set $F' = (r\times\mathrm{id}_{\mathbb{R}})^{-1}F$ and $\Lambda' = d(r\times\mathrm{id}_{\mathbb{R}})^{-1}(\Lambda)$. Then $r^{-1}(H^iF_+) \simeq H^iF'_+$, F' is simple along Λ' and we have $\mu hom(F', F') \simeq \mathbf{k}_{\Lambda'}$. Since $\Lambda'/\mathbb{R}_{>0}$ is compact, Theorem 13.4 gives

$$\mathrm{RHom}(F'_+, F'_+) \simeq \mathrm{R}\Gamma(\Lambda'; \mathbf{k}_{\Lambda'}). \tag{16.1}$$

On the other hand the complex $G = \mathrm{R}\mathcal{H}om(F'_+, F'_+)$ is concentrated in degrees greater than $a-b$ and $H^{a-b}G \simeq \mathcal{H}om(H^aF'_+, H^bF'_+)$ is a non zero constant sheaf. Hence $H^{a-b}\mathrm{RHom}(F'_+, F'_+)$ is non zero. By (16.1) we deduce that $H^{a-b}\mathrm{R}\Gamma(\Lambda'; \mathbf{k}_{\Lambda'})$ also is non zero, which implies $a-b \geq 0$. Hence $a = b$ and F_+ is concentrated in a single degree.

(ii) Now we prove that H^aF_+ is of rank one. There exists $d \geq 1$ such that $H^aF'_+ \simeq \mathbf{k}^d_{M'}$. The isomorphism (16.1) gives in degree 0:

$$\mathrm{Hom}(\mathbf{k}^d, \mathbf{k}^d) \simeq H^0(\Lambda'; \mathbf{k}_{\Lambda'}). \tag{16.2}$$

By Remark 13.5 this isomorphism is compatible with the algebra structures of both terms. Let I be the set of connected components of Λ'. We obtain $|I| = d^2$. The natural decomposition $H^0(\Lambda'; \mathbf{k}_{\Lambda'}) \simeq \bigoplus_{i\in I} H^0(\Lambda'_i; \mathbf{k}_{\Lambda'_i})$ gives an expression of the unit as a sum of orthogonal idempotents, $1 = \sum_{i\in I} e_i$, where e_i is the projection

$$e_i\colon H^0(\Lambda'; \mathbf{k}_{\Lambda'}) \to H^0(\Lambda'_i; \mathbf{k}_{\Lambda'_i}), \qquad i \in I.$$

We let $m_i \in \mathrm{Hom}(\mathbf{k}^d, \mathbf{k}^d)$ be the image of e_i by (16.2). The relation $1 = \sum_{i\in I} e_i$ gives a decomposition of the identity matrix $I_d = \sum_{i\in I} m_i$ as a sum of $|I|$ non-zero orthogonal projections, that is, $m_i^2 = m_i$ and $m_im_j = 0$, for $i \neq j$. We deduce that $|I| \leq d$, that is, $d^2 \leq d$. Hence $d = 1$, as claimed.

(iii) Now we assume that $\mathbf{k} = \mathbb{Z}$. By Lemma 16.1, for each $i \in \mathbb{Z}$, the stalks of the local system H^iF_+ are of finite type. Let us assume that one H^iF_+ has p-torsion for some prime p. Then $G = F \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$

is simple along Λ and $G_+ \simeq F_+ \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is not concentrated in one degree, which contradicts (i).

Hence the $H^i F_+$ are free of finite type. Then the result follows from (i) and (ii) applied to $F \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$, for an arbitrary prime p. ☐

Corollary 16.3. *We assume that* $\mathbf{k} = \mathbb{Z}$ *or* $\mathbf{k}$ *is a finite field and that* $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ *has a global simple object. Then the projection* $\Lambda \to M$ *induces an isomorphism* $\mathrm{R}\Gamma(M; \mathbf{k}_M) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \mathbf{k}_\Lambda)$.

Proof. We choose a simple object $\mathcal{F} \in \mu\mathrm{Sh}(\mathbf{k}_\Lambda)$. By Theorem 11.1 there exists $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ such that $\mathfrak{m}_\Lambda(F) \simeq \mathcal{F}$. By Proposition 16.2 F_+ is concentrated in one degree, say i, and $H^i F_+$ is a local system with stalks isomorphic to $\mathbf{k}$. Hence $\mathrm{R}\mathcal{H}om(F_+, F_+) \simeq \mathbf{k}_M$ and $\mathrm{RHom}(F_+, F_+) \simeq \mathrm{R}\Gamma(M; \mathbf{k}_M)$. Since F is simple we also have $\mu hom(F,F)|_\Lambda \simeq \mathbf{k}_\Lambda$. By Theorem 13.4 we deduce an isomorphism

$$\mathrm{R}\Gamma(M; \mathbf{k}_M) \simeq \mathrm{R}\Gamma(\Lambda; \mathbf{k}_\Lambda). \tag{16.3}$$

By construction (16.3) is given by taking the global sections in the bottom morphism of the commutative diagram:

$$\begin{array}{ccccc}
\mathbf{k}_{M\times\mathbb{R}} & & \xrightarrow{\quad a \quad} & & \mathrm{R}(\dot\pi_{M\times\mathbb{R}})_*(\mathbf{k}_\Lambda) \\
\downarrow{\scriptstyle b} & & & & \wr\downarrow{\scriptstyle c} \\
\mathrm{R}\mathcal{H}om(F,F) & \xrightarrow{\sim} & \mathrm{R}(\pi_{M\times\mathbb{R}})_*\mu hom(F,F) & \longrightarrow & \mathrm{R}(\dot\pi_{M\times\mathbb{R}})_*(\mu hom(F,F)|_\Lambda),
\end{array}$$

where b and c map the sections 1 to the identity morphisms. When taking global sections, b and c induce isomorphisms and a induces the natural morphism $\mathrm{R}\Gamma(M; \mathbf{k}_M) \to \mathrm{R}\Gamma(\Lambda; \mathbf{k}_\Lambda)$ given by the projection of Λ to the base M. The bottom horizontal arrow induces (16.3). This shows that (16.3) is indeed induced by the projection to the base. ☐

Remark 16.4. We have seen in Section 5 that $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ has a global simple object if the Maslov class of Λ and the image of its relative Stiefel-Whitney class in $H^2(\Lambda; \mathbf{k}^\times)$ is zero. By Theorem 15.1 the Maslov class vanishes in our case. Hence, when $\mathbf{k} = \mathbb{Z}/2\mathbb{Z}$ the stack $\mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ has a global simple object and Corollary 16.3 gives: the projection $\Lambda \to M$ induces an isomorphism

$$\mathrm{R}\Gamma(M; \mathbb{Z}/2\mathbb{Z}_M) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \mathbb{Z}/2\mathbb{Z}_\Lambda).$$

17. Vanishing of the Stiefel-Whitney class

Here we prove that the class $c^{\mathbb{Z}}_\Lambda \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z})$ of (5.1) vanishes. For this we will use Theorem 11.1 in the framework of twisted sheaves. Let $c \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ be given and let $\check{c} = \{c_{ijk}\}$, $i, j, k \in$, be a Čech cocycle representing c with respect to a finite covering $\{U_i\}_{i\in I}$ of M.

Definition 17.1. A $\check{c}$-twisted sheaf F on M is the data of sheaves $F_i \in \mathrm{Mod}(\mathbf{k}_{U_i})$ and isomorphisms $\varphi_{ij} \colon F_j|_{U_{ij}} \xrightarrow{\sim} F_i|_{U_{ij}}$ satisfying

$$\varphi_{ij} \circ \varphi_{jk} = c_{ijk}\, \varphi_{ik}. \tag{17.1}$$

The $\check{c}$-twisted sheaves form an abelian category that we denote by $\mathrm{Mod}(\mathbf{k}^{\check{c}}_M)$. We denote by $\mathsf{D}^{\mathrm{lb}}(\mathbf{k}^{\check{c}}_M)$ its derived category.

The prestack $U \mapsto \mathrm{Mod}(\mathbf{k}^{\check{c}|_U}_U)$ is a stack which is locally equivalent to the stack of sheaves. The usual operations on sheaves extend to twisted sheaves. We can define a Kashiwara-Schapira stack $\mu\mathrm{Sh}(\mathbf{k}^{\check{c}}_\Lambda)$ and formulate a version of Theorem 11.1 in this framework: for $\mathcal{F} \in \mu\mathrm{Sh}(\mathbf{k}^{\check{c}}_\Lambda)$ there exists $F \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}^{\check{c}}_{M\times\mathbb{R}})$ such that $\dot{\mathrm{SS}}(F) = \Lambda$, $F|_{M\times\{t\}} \simeq 0$ for $t \ll 0$ and $\mathfrak{m}^{\check{c}}_\Lambda(F) \simeq \mathcal{F}$.

Lemma 17.2. *We assume that there exists $L \in \mathrm{Mod}(\mathbb{Z}^{\check{c}}_M)$ which is locally constant and non zero. Then $c = 0$.*

Proof. The cocycle $\check{c}$ is associated with a covering $\{U_i\}_{i\in I}$ of M and L is given by sheaves $L_i \in \mathrm{Mod}(\mathbb{Z}_{U_i})$ and isomorphisms $\varphi_{ij} \colon L_j|_{U_{ij}} \xrightarrow{\sim} L_i|_{U_{ij}}$ as in Definition 17.1. Up to refining the covering we can assume that U_i is contractible and that U_{ij} is connected for any $i, j \in I$. Since L is locally constant, we can choose an isomorphism $\varphi_i \colon L|_{U_i} \simeq \mathbb{Z}_{U_i}$ for each $i \in I$. Then the composition $b_{ij} = \varphi_i \varphi_{ij} \varphi_j^{-1}$ is an isomorphism $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$, that is, $b_{ij} = \pm 1$, and we see that $\check{c}$ is the coboundary of $\{b_{ij}\}_{i,j\in I}$. ☐

Proposition 17.3. *The class $c^{\mathbf{k}}_\Lambda \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}_\Lambda)$ is zero.*

Proof. By Corollary 16.3 and Remark 16.4 we have $H^2(M; \mathbb{Z}/2\mathbb{Z}_M) \xrightarrow{\sim} H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}_\Lambda)$. We let $c \in H^2(M; \mathbb{Z}/2\mathbb{Z}_M)$ be the inverse image of $c^{\mathbf{k}}_\Lambda$ by this isomorphism and we choose a Čech cocycle $\check{c}$ representing c. Then the twisted Kashiwara-Schapira stack $\mu\mathrm{Sh}(\mathbb{Z}^{\check{c}}_\Lambda)$ has a simple global object and the twisted version of Theorem 11.1 gives $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbb{Z}^{\check{c}}_{M\times\mathbb{R}})$ which is simple along Λ. By Proposition 16.2 we have $F_+ \simeq L[d]$ where $L \in \mathrm{Mod}(\mathbb{Z}^{\check{c}}_M)$ is a twisted locally constant sheaf with stalks isomorphic to $\mathbb{Z}$ and d is some integer. Now the result follows from Lemma 17.2. ☐

18. A canonical quantization

Summing up the results of this part we obtain a canonical quantization for Λ. We recover a previous result of Abouzaid and Kragh that the projection $\Lambda \to M$ is a homotopy equivalence. (Since then they proved that this projection is a simple homotopy equivalence – see [3]).

Theorem 18.1. *Let $\mathbf{k}$ be a ring.*

(i) *There exists* $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ *such that* $F_- \simeq 0$ *and* $F_+ \simeq \mathbf{k}_M$.

(ii) *The object* F *in* (i) *is unique up to a unique isomorphism: for another such* $F' \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ *we have a canonical isomorphism* $\mathrm{Hom}(F, F') \simeq \mathrm{Hom}(F_+, F'_+) \simeq \mathbf{k}$.

(iii) *The projection* $\Lambda \to M$ *yields an isomorphism* $\mathrm{R}\Gamma(M; \mathbf{k}_M) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \mathbf{k}_\Lambda)$.

Proof. We first assume that $\mathbf{k} = \mathbb{Z}$. By Theorem 15.1 and Proposition 17.3 we know that $m_\Lambda = 0$ and $c^{\mathbf{k}}_\Lambda = 0$. By Theorem 11.1 there exists $F^0 \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ which is simple along Λ. By Proposition 16.2 we have $F^0_+ \simeq L[d]$ where $L \in \mathrm{Mod}(\mathbf{k}_M)$ is locally constant with stalks isomorphic to $\mathbb{Z}$ and d is some integer. Let $p\colon M \times \mathbb{R} \to M$ be the projection. Then $F^1 = F^0 \otimes p^{-1}L^{\otimes -1}[-d]$ satisfies the required properties. For a general ring $\mathbf{k}$ we set $F = F^1 \overset{\mathrm{L}}{\otimes}_{\mathbb{Z}_{M\times\mathbb{R}}} \mathbf{k}_{M\times\mathbb{R}}$.

Then (ii) is given by Theorem 13.3 and (iii) by Corollary 16.3. □

In [2] Abouzaid gives a result more precise than Theorem 18.1: the projection $\pi_\Lambda\colon \Lambda \to M$ induces an isomorphism of the fundamental groups. Since we already have an isomorphism between the cohomology groups, it is enough to show that $\pi_1(\Lambda) \to \pi_1(M)$ is an isomorphism and that $\mathrm{R}\Gamma(M; L) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \pi_\Lambda^{-1}L)$ for all local systems L on M. The assertion on π_1 is equivalent to: the inverse image by π_Λ induces an equivalence of categories $\mathsf{Loc}(\mathbf{k}_M) \xrightarrow{\sim} \mathsf{Loc}(\mathbf{k}_\Lambda)$, for some field $\mathbf{k}$.

Proposition 18.2. *Let* $\mathbf{k}$ *be a field. Let* $\pi_\Lambda\colon \Lambda \to M$ *be the projection. Then the inverse image functor* $\pi_\Lambda^{-1}\colon \mathsf{Loc}(\mathbf{k}_M) \to \mathsf{Loc}(\mathbf{k}_\Lambda)$ *is an equivalence of categories.*

Proof. (i) We first prove that π_Λ^{-1} is fully faithful. Let $F \in \mathsf{D}^{\mathrm{lb}}_{\Lambda,+}(\mathbf{k}_{M\times\mathbb{R}})$ be the simple object given by Theorem 18.1. Since F is simple we have $\mu hom(F, F)|_\Lambda \simeq \mathbf{k}_\Lambda$ and we deduce, for $L, L' \in \mathsf{Loc}(\mathbf{k}_M)$,

$$\mu hom(F \otimes p^{-1}L, F \otimes p^{-1}L') \simeq \mathcal{H}om(\pi_\Lambda^{-1}L, \pi_\Lambda^{-1}L'), \tag{18.1}$$

where $p\colon M \times \mathbb{R} \to M$ is the projection. We have $(F \otimes p^{-1}L)_+ \simeq L$ and (13.6) together with (18.1) imply

$$\begin{aligned}\mathrm{Hom}(L, L') &\simeq H^0(\Lambda; \mu hom(F \otimes p^{-1}L, F \otimes p^{-1}L')) \\ &\simeq \mathrm{Hom}(\pi_\Lambda^{-1}L, \pi_\Lambda^{-1}L'),\end{aligned}$$

which means that π_Λ^{-1} is fully faithful.

(ii) We prove that π_Λ^{-1} is essentially surjective. Let $L_1 \in \mathsf{Loc}(\mathbf{k}_\Lambda)$ be given. We recall that the functor $\mu hom(F, \cdot)$ induces an equivalence $\mu\mathrm{Sh}(\mathbf{k}_\Lambda) \xrightarrow{\sim} \mathsf{Loc}(\mathbf{k}_\Lambda)$ (see Proposition 4.4, where the induced functor is denoted $\overline{\mu hom}(F, \cdot)$). Hence there exists $\mathcal{L}_1 \in \mu\mathrm{Sh}(\mathbf{k}_\Lambda)$ such that

$\overline{\mu hom}(F, \mathcal{L}_1) \simeq \mathcal{L}$. By Theorem 11.1 there exists $F_1 \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_{M\times\mathbb{R}})$ such that $\mathfrak{m}_\Lambda(F_1) \simeq L_1$. Then we have $\mu hom(F, F_1)|_\Lambda \simeq L_1$.

We set $L = (F_1)_+ \in \mathsf{D}^{\mathrm{lb}}(\mathbf{k}_M)$. Then $\dot{\mathrm{SS}}(L) = \emptyset$ and, since $F_+ \simeq \mathbf{k}_M$, we also have $L \simeq (F \otimes p^{-1}L)_+$. Hence $(F_1)_+ \simeq (F \otimes p^{-1}L)_+$ and Theorem 13.3 gives $F_1 \simeq F \otimes p^{-1}L$. We deduce

$$\mu hom(F, F_1)|_\Lambda \simeq \mu hom(F, F \otimes p^{-1}L)|_\Lambda \simeq \pi_\Lambda^{-1}L.$$

Hence $L_1 \simeq \pi_\Lambda^{-1}L$ as required. □

Corollary 18.3. *The projection $\Lambda \to M$ is a homotopy equivalence.*

Proof. In view of Proposition 18.2 it only remains to prove that, for any local system L on M, we have $\mathrm{R}\Gamma(M; L) \xrightarrow{\sim} \mathrm{R}\Gamma(\Lambda; \pi_\Lambda^{-1}L)$. This follows from (13.6) applied with the F of Theorem 18.1 and $F' = F \otimes p^{-1}L$, where $p\colon M \times \mathbb{R} \to M$ is the projection. □

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