Interacting vortex pairs in inviscid and viscous planar flows

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Abstract

The aim of this contribution is to make a connection between two recent results concerning the dynamics of vortices in incompressible planar flows. The first one is an asymptotic expansion, in the vanishing viscosity limit, of the solution of the two-dimensional Navier-Stokes equation with point vortices as initial data. In such a situation, it is known [5] that the solution behaves to leading order like a linear superposition of Oseen vortices whose centers evolve according to the point vortex system, but higher order corrections can also be computed which describe the deformation of the vortex cores due to mutual interactions. The second result is the construction by D. Smets and J. van Schaftingen of "desingularized" solutions of the two-dimensional Euler equation [22]. These solutions are stationary in a uniformly rotating or translating frame, and converge either to a single vortex or to a vortex pair as the size parameter ϵ goes to zero. We consider here the particular case of a pair of identical vortices, and we show that the solution of the weakly viscous Navier-Stokes equation is accurately described at time t by an approximate steady state of the rotating Euler equation which is a desingularized solution in the sense of [22] with Gaussian profile and size $\epsilon = \sqrt{\nu t}$.

1 Introduction

Numerical simulations of freely decaying turbulence show that vortex interactions play a crucial role in the dynamics of two-dimensional viscous flows [12, 13]. In particular, vortex mergers are responsible for the appearance of larger and larger structures in the flow, a process which is directly related to the celebrated "inverse energy cascade" [3]. Although nonperturbative interactions such as vortex mergers are extremely complex and desperately hard to analyze from a mathematical point of view [11, 20], rigorous results can be obtained in the perturbative regime where the distances between the vortex centers are large compared to the typical size of the vortex cores.

As an example of this situation, consider the case where the initial flow is a superposition of N point vortices. This means that the initial vorticity ω_0 satisfies

$$\omega_0 \,=\, \sum_{i=1}^{N} \alpha_i \, \delta(\cdot - x_i) \;, \tag{1.1}$$

where $x_1, \ldots, x_N \in \mathbb{R}^2$ are the initial positions and $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ the circulations of the

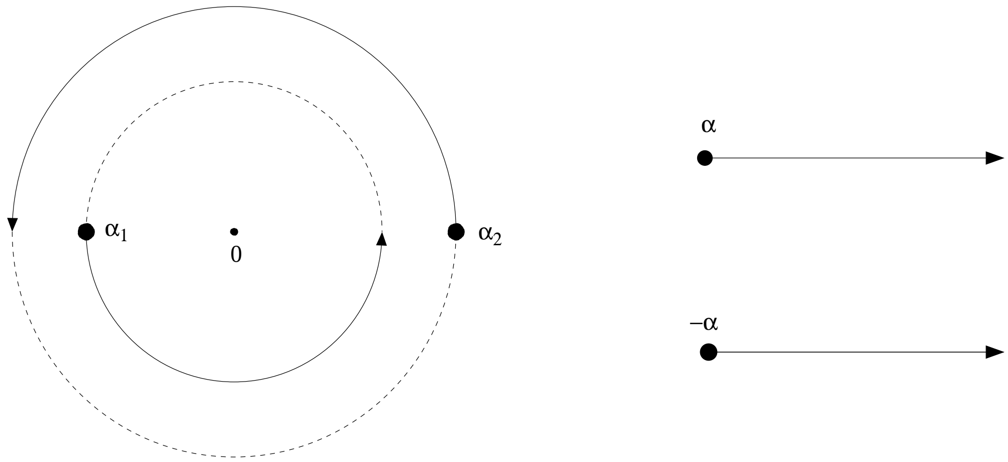

Figure 1: The motion of two point vortices whith circulations $\alpha_1 > \alpha_2 > 0$ (left) and $\alpha_1 + \alpha_2 = 0$ (right).

vortices. Let $\omega(x,t)$ be the solution of the two-dimensional vorticity equation

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega \;=\; \nu \Delta \omega \;, \qquad x \in \mathbb{R}^2 \;, \quad t > 0 \;, \tag{1.2}$$

with initial data ω_0, where $u(x,t)$ is the velocity field defined by the Biot-Savart law

$$u(x,t) \;=\; \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2}\, \omega(y,t)\, \mathrm{d}y \;, \qquad x \in \mathbb{R}^2 \;, \quad t > 0 \;. \tag{1.3}$$

Solutions of (1.2), (1.3) with singular initial data of the form (1.1) were first constructed by Benfatto, Esposito, and Pulvirenti [2]. More generally, if $\omega_0 \in \mathcal{M}(\mathbb{R}^2)$ is any finite measure, Giga, Miyakawa, and Osada [8] have shown that the vorticity equation (1.2) has a global solution with initial data ω_0, which moreover is unique if the total variation norm of atomic part of ω_0 is small compared to the kinematic viscosity ν. This last restriction has been removed recently by I. Gallagher and the author [4], so we know that (1.2) has a unique global solution $\omega \in C^0((0,\infty), L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2))$ with initial data (1.1), no matter how small the viscosity coefficient is. This solution is uniformly bounded in $L^1(\mathbb{R}^2)$, and the total circulation $\int_{\mathbb{R}^2} \omega(x,t)\,\mathrm{d}x$ is conserved.

In the vanishing viscosity limit, the motion of point vortices in the plane is described by a system of ordinary differential equations introduced by Helmholtz [9] and Kirchhoff [10]. If $z_1(t), \ldots, z_N(t) \in \mathbb{R}^2$ denote the positions of the vortices, the system reads

$$\frac{\mathrm{d}}{\mathrm{d}t} z_i(t) \;=\; \frac{1}{2\pi} \sum_{j \neq i} \alpha_j \, \frac{(z_i(t) - z_j(t))^\perp}{|z_i(t) - z_j(t)|^2} \;, \qquad i = 1, \ldots, N \;, \tag{1.4}$$

and the initial conditions $z_i(0) = x_i$ for $i = 1, \ldots, N$ are determined by (1.1). A lot is known about the dynamics of the *point vortex system* (1.4), see e.g. [21] for a recent monograph devoted to this problem. Most remarkably, (1.4) is a Hamiltonian system with N degrees of freedhom, which always possesses three independent involutive first integrals. In particular, system (1.4) is *integrable* if $N \le 3$, whatever the vortex circulations $\alpha_1, \ldots, \alpha_N$ may be. In the simple situation where $N = 2$, both vortices rotate with constant angular speeed around the common vorticity center, see Fig. 1 (left). In the exceptional case where $\alpha_1 + \alpha_2 = 0$, there is no center of vorticity and the vortices move with constant speed along parallel straight lines, see Fig. 1 (right).

It should be remarked that system (1.4) is not always globally well-posed: if $N \geq 3$ and if the circulations $\alpha_1, \ldots, \alpha_N$ are not all of the same sign, vortex collisions may occur in finite time for exceptional initial configurations [19, 21]. In what follows, we always assume that system (1.4) is well-posed on some time interval $[0, T]$, and we denote by d the *minimal distance* between any two vortices on this interval:

$$d = \min_{t \in [0,T]} \min_{i \neq j} |z_i(t) - z_j(t)| > 0 . \tag{1.5}$$

We also introduce the *turnover time*

$$T_0 = \frac{d^2}{|\alpha|} , \quad \text{where} \quad |\alpha| = |\alpha_1| + \cdots + |\alpha_N| , \tag{1.6}$$

which is a natural time scale for the inviscid dynamics described by (1.4). For instance, for a pair of vortices with circulations of the same sign, one can check that the rotation period of each vortex around the center is $4\pi^2 T_0$.

When the viscosity ν is nonzero, the point vortices in the initial data (1.1) are smoothed out by diffusion, and the solution $\omega(x,t)$ of (1.2) is no longer described by the point vortex system. In the particular case where $N = 1$, the unique solution of (1.2) with initial data $\omega_0 = \alpha\delta_0$ is the *Lamb-Oseen vortex:*

$$\omega(x,t) = \frac{\alpha}{\nu t} G\Big(\frac{x}{\sqrt{\nu t}}\Big) , \qquad u(x,t) = \frac{\alpha}{\sqrt{\nu t}} v^G\Big(\frac{x}{\sqrt{\nu t}}\Big) , \tag{1.7}$$

where the vorticity profile G and the velocity profile v^G have the following explicit expressions:

$$G(\xi) = \frac{1}{4\pi} e^{-|\xi|^2/4} , \qquad v^G(\xi) = \frac{1}{2\pi} \frac{\xi^\perp}{|\xi|^2} \Big(1 - e^{-|\xi|^2/4}\Big) . \tag{1.8}$$

As was shown by C.E. Wayne and the author, Oseen vortices describe the long-time asymptotics of all solutions of the two-dimensional Navier-Stokes equation for which the vorticity distribution is integrable, and are also the only self-similar solutions of this equation with integrable vorticity profile [7].

When $N \geq 2$, the solution of (1.2) with initial data (1.1) is not explicit, but in some parameter regimes it can be approximated by a linear superposition of Oseen vortices whose centers evolve according to the point vortex system (1.4). More precisely, we have the following result:

Theorem 1.1 [5] *Given pairwise distinct initial positions $x_1, \ldots, x_N \in \mathbb{R}^2$ and nonzero circulations $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$, fix $T > 0$ such that the point vortex system* (1.4) *is well-posed on the time interval* $[0, T]$*. Let $d > 0$ be the minimal distance* (1.5) *and T_0 the turnover time* (1.6)*. Then the (unique) solution of the two-dimensional vorticity equation* (1.2) *with initial data* (1.1) *satisfies*

$$\frac{1}{|\alpha|} \int_{\mathbb{R}^2} \Big| \omega(x,t) - \sum_{i=1}^N \frac{\alpha_i}{\nu t} G\Big(\frac{x - z_i(t)}{\sqrt{\nu t}}\Big)\Big| \,\mathrm{d}x \leq K \frac{\nu t}{d^2} , \qquad t \in (0, T] , \tag{1.9}$$

where $z(t) = \{z_1(t), \ldots, z_N(t)\}$ is the solution of (1.4) *and K is a (dimensionless) constant depending only on the ratio T/T_0.*

Theorem 1.1 gives nontrivial information on the solution of (1.2) in the *weak interaction regime*, where the size $\mathcal{O}(\sqrt{\nu t})$ of the vortex cores is much smaller than the distance d between the centers. If the initial data are fixed, a convenient way to achieve this is to assume that the viscosity ν is small compared to the total circulation $|\alpha|$, and that the observation time T is smaller than or comparable to the turnover time T_0. Indeed, using definition (1.6), we see that

$$\frac{\nu t}{d^2} = \frac{\nu}{|\alpha|} \frac{t}{T_0} .$$

In particular, in the vanishing viscosity limit, we obtain the following corollary.

Corollary 1.2 **[5]** *Under the assumptions of Theorem 1.1, the solution $\omega^\nu(x,t)$ of the viscous vorticity equation* (1.2) *with initial data* (1.1) *satisfies*

$$\omega^\nu(\cdot,t) \xrightarrow[\nu\to 0]{} \sum_{i=1}^N \alpha_i\, \delta(\cdot - z_i(t)) , \quad \textit{for all } t \in [0,T] , \tag{1.10}$$

where $z(t) = \{z_1(t), \dots, z_N(t)\}$ is the solution of (1.4).

In other words, the solution of the vorticity equation (1.2) with initial data (1.1) converges weakly, in the vanishing viscosity limit, to a superposition of point vortices which evolve according to the point vortex dynamics (1.4). In particular, Corollary 1.2 provides a natural and rigorous derivation of the point vortex system itself from the Navier-Stokes equation. As is well-known, system (1.4) can also be rigorously derived from Euler's equation [15, 18], but in the latter approach it is necessary to regularize the initial vorticity because we do not know how to solve Euler's equation with singular data such as (1.1). In this respect, it is important to keep in mind that the right-hand side of (1.10) is *not* a weak solution of the inviscid vorticity equation $\partial_t\omega + u\cdot\nabla\omega = 0$. For completeness, we also mention that the vanishing viscosity limit for solutions of (1.2) with concentrated vorticity has been studied by Marchioro [16], who obtained the analog of Corollary 1.2 in that context.

The conclusion of Theorem 1.1 can be interpreted in the following simple and somewhat naive way: If we solve the two-dimensional vorticity equation (1.2) with point vortices as initial data, the diffusion term $\nu\Delta\omega$ in (1.2) smooths out the point vortices into Oseen vortices, and the advection term $u\cdot\nabla\omega$ translates the vortex centers according to the point vortex dynamics (1.4). While correct, this interpretation ignores the important fact that the advection of a smooth vortex by the inhomogeneous velocity field created by the other vortices results non only in a translation of the vortex center, but also in a *deformation* of the vortex core. In our case, this deformation is of order $\mathcal{O}(\nu t/d^2)$ in L^1 norm, and therefore does not appear in (1.9) because it is included in the error term. However, as is explained in [5], such a small deformation of the vortex profile creates a *self-interaction* effect of size $\mathcal{O}(1)$, which basically counterbalances the influence of the external velocity field, except for a rigid translation.

As a matter of fact, self-interactions play a crucial role in the proof of Theorem 1.1, and we even believe that is it not possible to establish (1.9) without computing a higher order approximation of the solution. A systematic asymptotic expansion is carried out in [5], but if we only keep the leading order nonradial corrections to the Oseen vortices we obtain the following approximate solution of (1.2):

$$\omega_{\text{app}}(x,t) = \sum_{i=1}^N \frac{\alpha_i}{\nu t}\left\{G\Big(\frac{x-z_i(t)}{\sqrt{\nu t}}\Big) + \frac{\nu t}{d^2}\, F_i\Big(\frac{x-z_i(t)}{\sqrt{\nu t}}\,,t\Big)\right\} , \tag{1.11}$$

where

$$F_i(\xi,t) = a(|\xi|)\sum_{j\neq i}\frac{\alpha_j}{\alpha_i}\,\frac{d^2}{|z_i(t)-z_j(t)|^2}\Big(2\frac{|\xi\cdot(z_i(t)-z_j(t))|^2}{|\xi|^2|z_i(t)-z_j(t)|^2} - 1\Big) + \mathcal{O}\Big(\frac{\nu}{|\alpha|}\Big) . \tag{1.12}$$

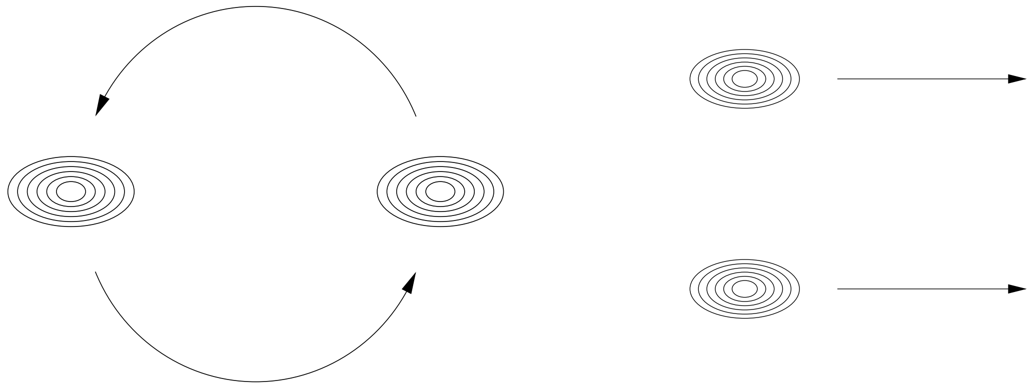

Figure 2: The level lines of the vorticity distribution for a pair of vortices with equal (left) or opposite (right) circulations.

Here $a : (0,\infty) \to \mathbb{R}$ is a smooth, positive function satisfying $a(r) \approx C_1 r^2$ as $r \to 0$ and $a(r) \approx C_2 r^4 e^{-r^2/4}$ as $r \to \infty$ for some $C_1, C_2 > 0$. In the regime where the viscosity ν is much smaller than the circulations α_i of the vortices, we can use formulas (1.11), (1.12) to compute, to leading order in our expansion parameter $\nu t/d^2$, the deformation of the vortex cores due to mutual interactions. For any fixed $t \in (0,T]$, this first order correction depends only on the relative positions of the vortex centers, which are determined by (1.4). In the particular case of a pair of vortices with equal or opposite circulations, the level lines of the vorticity distribution $\omega_{\mathrm{app}}(x,t)$ are represented in Fig. 2.

The discussion above shows that the dynamics of weakly interacting viscous vortices is essentially driven by two different mechanisms: *diffusion*, which is responsible for the continuous growth of the vortex cores, and *advection*, which creates the motion of the vortex centers and the deformation of the vortex profiles. The latter effect persists in the vanishing viscosity limit, and it is therefore reasonable to expect that, if we can find solutions of Euler's equation describing widely separated Gaussian vortices, these inviscid solutions will provide an accurate approximation of the viscous N-vortex solution considered in Theorem 1.1, if ν is sufficiently small. The aim of this contribution is to explore this idea in the particular case of a single *vortex pair*. In this simple situation, we only need to consider solutions of Euler's equation that are stationary in a uniformly rotating or translating frame.

Assume thus that $N = 2$ and, for definiteness, that both circulations α_1, α_2 are positive. Given $r_1, r_2 > 0$ such that $\alpha_1 r_1 = \alpha_2 r_2$, let $d = r_1 + r_2$ and $\Omega = (\alpha_1 + \alpha_2)/(2\pi d^2)$. We consider the inviscid vorticity equation in a rotating frame with angular speed Ω:

$$\partial_t \omega + (u - \Omega x^\perp)\cdot\nabla\omega \,=\, 0\;. \tag{1.13}$$

Formally, the vorticity distribution

$$\omega_0 \,=\, \alpha_1\delta(\cdot - x_1) + \alpha_2\delta(\cdot - x_2)\;, \qquad \text{where} \quad x_1 = \begin{pmatrix} r_1 \\ 0 \end{pmatrix}\;, \quad x_2 = \begin{pmatrix} -r_2 \\ 0 \end{pmatrix}\;, \tag{1.14}$$

is a stationary solution of (1.13). In the laboratory frame, this corresponds to a time periodic solution of (1.4) where both vortices rotate around the origin with angular velocity Ω. Now, let $w_* \in \mathcal{S}(\mathbb{R}^2)$ be a nonnegative vorticity profile which is radially symmetric, decreasing along rays, and normalized in the sense that $\int_{\mathbb{R}^2} w_*(x)\,\mathrm{d}x = 1$. Given $\epsilon > 0$, we look for a stationary solution of (1.13) of the form

$$\omega_\epsilon(x) \,=\, \frac{\alpha_1}{\epsilon^2}\, w_{1,\epsilon}\Big(\frac{x - x_{1,\epsilon}}{\epsilon}\Big) + \frac{\alpha_2}{\epsilon^2}\, w_{2,\epsilon}\Big(\frac{x - x_{2,\epsilon}}{\epsilon}\Big)\;, \tag{1.15}$$

where $x_{i,\epsilon} \to x_i$ and $w_{i,\epsilon} \to w_*$ as $\epsilon \to 0$. Existence of such "desingularized" solutions of Euler's equation has been investigated in a recent work by D. Smets and J. Van Schaftingen [22]. In fact, the authors of [22] do not consider rotating vortex pairs of the form (1.15), but they treat a variety of other interesting cases, including a single stationary vortex in a bounded or unbounded domain, a rotating vortex in a disk, and a translating vortex pair in the plane. They use the "stream function method", which consists in constructing (by variational methods) a nontrivial solution to an elliptic equation of the form $-\Delta\psi = f_\epsilon(\psi + \frac{1}{2}\Omega|x|^2)$, where f_ϵ is a power-like nonlinearity which depends on ϵ in an appropriate way. The vorticity $\omega = -\Delta\psi$ is then a stationary solution of (1.13). For technical reasons, the desingularized vorticity profiles obtained in [22] are always compactly supported, but it is reasonable to expect that similar results can be obtained with Gaussian profiles too. We hope to clarify this issue and to extend the results of [22] to pairs of vortices of the same sign in a future work.

In Section 2 below, we study in detail the case of two identical vortices ($\alpha_1 = \alpha_2$). For a large class of radially symmetric profiles w_*, we prove the existence of approximate stationary solutions of (1.13) of the form (1.15). In other words, we construct an asymptotic expansion in powers of ϵ of the vorticity distribution (1.15) as a steady state of (1.13). Under natural symmetry assumptions, we show that this expansion can be performed to arbitrarily high order. Then, in Section 3, we prove that the solution $\omega^\nu(x,t)$ of the rotating viscous vorticity equation $\partial_t\omega + (u - \Omega x^\perp)\cdot\nabla\omega = \nu\Delta\omega$ with initial data (1.14) is very close to the inviscid stationary solution ω_ϵ with asymptotic profile $w_* = G$, if $\epsilon = \sqrt{\nu t}$ is sufficiently small. This means that, when the viscosity is small, the solution of (1.2) with initial data (1.14) slowly travels through a family of uniformly rotating solutions of Euler's equation, whose vorticity profiles are approximately Gaussian and evolve diffusively. We expect a similar picture to be relevant in the general situation considered in Theorem 1.1, although the corresponding inviscid solutions may be more difficult to identify in that case.

Acknowledgements. I am grateful to Felix Otto for suggesting the point of view adopted in this paper, and to Didier Smets for many stimulating discussions.

2 Approximate steady states of Euler's equation

The aim of this section is to construct an asymptotic expansion for a family of stationary solutions of the inviscid vorticity equation, which correspond to weakly interacting vortex pairs. For simplicity, we only consider the particular case where both vortices have the same circulation $\alpha > 0$. As is explained in the introduction, weak interaction means that the distance d between the vortex centers is large compared to the size of the vortex cores. If the vorticity distribution is given by (1.15), this condition is satisfied if $\epsilon > 0$ is sufficiently small. Here, we find it more convenient to fix the size of the vortex cores, and to assume that the distance d between the centers is large. This alternative point of view is of course equivalent to the previous one, up to a rescaling. Note, however, that the rotation speed Ω will now depend on d and behave like $\alpha/(\pi d^2)$ as $d \to \infty$.

From now on, we fix $\alpha > 0$, $d \gg 1$, and we look for a stationary solution ω of (1.13) describing a pair of identical vortices with circulation α. We make the ansatz

$$\omega(x) = \alpha w(x - x_d) + \alpha w(-x - x_d) , \quad u(x) = \alpha v(x - x_d) - \alpha v(-x - x_d) , \tag{2.1}$$

where $x_d = (d/2, 0)$, w is a localized vorticity profile to be determined, and $v = K[w]$ is the velocity field obtained from w via the Biot-Savart law (1.3). We assume that w belongs to the

Schwartz class $\mathcal{S}(\mathbb{R}^2)$, is nonnegative, and satisfies the normalization condition

$$\int_{\mathbb{R}^2} w(x)\,\mathrm{d}x \;=\; 1 \;. \tag{2.2}$$

We also impose the following symmetry

$$w(x_1,-x_2) \;=\; w(x_1,x_2)\;, \quad v_1(x_1,-x_2) \;=\; -v_1(x_1,x_2)\;, \quad v_2(x_1,-x_2) \;=\; v_2(x_1,x_2)\;, \tag{2.3}$$

which implies that $\omega(-x_1,x_2) = \omega(x_1,-x_2) = \omega(x_1,x_2)$ for all $x = (x_1,x_2) \in \mathbb{R}^2$. Finally, without loss of generality, we assume that

$$\int_{\mathbb{R}^2} x_1 w(x)\,\mathrm{d}x \;=\; 0 \;. \tag{2.4}$$

This means that the vorticity distribution ω defined in (2.1) is indeed a superposition of two localized vortices centered at the points $\pm x_d$.

The distribution ω will be a stationary solution of the rotating vorticity equation (1.13) if the profile w satisfies

$$\Bigl(\alpha v(x-x_d) - \alpha v(-x-x_d) - \Omega x^\perp\Bigr)\cdot\nabla w(x-x_d) \;=\; 0\;, \qquad x\in\mathbb{R}^2\;.$$

Replacing x by $x+x_d$ and denoting $\tilde\Omega = \Omega/\alpha$, we obtain the equivalent equation

$$\Bigl(v(x) - v(-x-2x_d) - \tilde\Omega(x+x_d)^\perp\Bigr)\cdot\nabla w(x) \;=\; 0\;, \qquad x\in\mathbb{R}^2\;. \tag{2.5}$$

Note that the rotating term $\tilde\Omega(x+x_d)^\perp$ behaves like the velocity field $v(x)$ when x_2 is changed into $-x_2$; this implies that the symmetry (2.3) is indeed compatible with Eq. (2.5). To determine the rotation speed Ω, we multiply both members of (2.5) by x_2 and we integrate by parts over $\mathbb{R}^2$. We easily obtain

$$\tilde\Omega\int_{\mathbb{R}^2}(x_1+d/2)w(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2}\Bigl(v_2(x) - v_2(-x-2x_d)\Bigr)w(x)\,\mathrm{d}x\;.$$

This identity can be simplified if we use (2.2), (2.4), and the fact that $\int_{\mathbb{R}^2} v_2 w\,\mathrm{d}x = 0$. We thus arrive at the following relation,

$$\tilde\Omega \;=\; \tilde\Omega[w] \;:=\; -\frac{2}{d}\int_{\mathbb{R}^2} v_2(-x-2x_d)w(x)\,\mathrm{d}x\;, \tag{2.6}$$

which determines $\tilde\Omega$ as a function of w.

As we shall see, since the vorticity profile w is normalized by (2.2), the corresponding velocity field $v = K[w]$ satisfies

$$-v_2(-x-2x_d) \;\sim\; \frac{1}{2\pi d}\;, \qquad \text{as} \quad d\to\infty\;,$$

for any fixed $x\in\mathbb{R}^2$. In view of (2.6), this means that $\tilde\Omega \sim (\pi d^2)^{-1}$ as $d\to\infty$. In particular, if we take formally the limit $d\to\infty$ in (2.5), we obtain the limiting equation $v\cdot\nabla w = 0$, which simply means that w (or v) is a stationary solution of Euler's equation. In what follows, we assume that the limiting profile w_* is *radially symmetric* and *stable* in the sense of Arnold [1, 19]. Roughly speaking, this means that $w_*(x)$ is a strictly decreasing function of $|x|$. Given any such profile, we shall construct perturbatively a family of approximate solutions of (2.5), indexed by the parameter $d \gg 1$, which converge to w_* as $d \to \infty$. Under natural assumptions, these approximate solutions are uniquely determined by the asymptotic profile w_*.

An important question that we leave open here is whether we can actually construct *exact* solutions of (2.5) which converge to w_* as $d \to \infty$, in which case our approximate solutions could be recovered by truncating the asymptotic expansion of the exact solutions. As was mentioned in the introduction, it should be possible to prove the existence of such solutions, at least for a particular class of compactly supported profiles w_*, by adapting the variational techniques of D. Smets and J. van Schaftingen [22]. It might also be possible to construct exact solutions of (2.5) for more general profiles using a fixed point argument of Nash-Moser type. We hope to clarify these issues in a future work.

2.1 Asymptotic profile and functional setting

Let $w_* \in \mathcal{S}(\mathbb{R}^2)$ be a radially symmetric, nonnegative function satisfying the normalization condition (2.2), and let $v_* = K[w_*]$ be the velocity field obtained from w_* via the Biot-Savart law (1.3). We can thus write

$$w_*(x) \,=\, \frac{1}{\pi}\, q(|x|^2)\;, \qquad v_*(x) \,=\, \frac{1}{2\pi}\frac{x^\perp}{|x|^2}\, Q(|x|^2)\;, \qquad x \in \mathbb{R}^2\;, \tag{2.7}$$

where $q : [0,\infty) \to \mathbb{R}_+$ is a smooth, rapidly decreasing function and $Q(r) = \int_0^r q(s)\,\mathrm{d}s$. Note that $Q(r) \to 1$ as $r \to \infty$. We assume that w_* satisfies the following *strong stability conditions:*

$$q'(r) \,<\, 0 \text{ for all } r \ge 0\;, \qquad \text{and} \qquad \sup_{r>0} \frac{-r^2 q'(r)}{Q(r)} < 1\;. \tag{2.8}$$

The first condition in (2.8) implies of course that $q(r) > 0$ for all $r \ge 0$, so that $Q(r) > 0$ for all $r > 0$. In particular, compactly supported asymptotic profiles w_* are excluded. This condition also implies that w_* is a stable solution of the two-dimensional inviscid vorticity equation, with respect to perturbations in $L^1 \cap L^\infty$ [17]. The second assumption in (2.8) is more technical in nature, and can probably be relaxed. It is satisfied, for instance, if $q(r) = \gamma e^{-\gamma r}$ for some $\gamma > 0$. Note that we always have

$$\sup_{r>0} \frac{-r^2 q'(r)}{Q(r)} > \frac{1}{4}\;.$$

Indeed, if we assume on the contrary that $Q(r) + 4r^2 q'(r) \ge 0$ for all $r > 0$, then the function $h(r) = Q''(r) + Q(r)/(4r^2)$ is nonnegative and satisfies $h(r) \sim q(0)/(4r)$ as $r \to 0$, and $h(r) \sim 1/(4r^2)$ as $r \to \infty$. Thus $\sqrt{r}h \in L^1((0,\infty))$, but if we integrate by parts we obtain

$$\int_0^\infty \sqrt{r} h(r)\,\mathrm{d}r \,=\, \int_0^\infty \sqrt{r}\Big(Q''(r) + \frac{1}{4r^2}Q(r)\Big)\,\mathrm{d}r \,=\, 0\;,$$

which yields a contradiction. Finally, in addition to (2.8), we also assume that q^2/q' decays rapidly at infinity:

$$\sup_{r>0} \frac{r^k q(r)^2}{|q'(r)|} \,<\, \infty\;, \qquad \text{for all } k \in \mathbb{N}\;. \tag{2.9}$$

As was already observed, the asymptotic profile w_* is already an approximate solution of (2.5), (2.6) in the sense that, if we substitute (w_*, v_*) for (w, v) in (2.5), the left-hand side converges to zero as $d \to \infty$. Our goal is to construct here more accurate approximations, which take into account the interaction of the vortices. We look for solutions of the form

$$w \,=\, w_* + \omega\;, \qquad v = v_* + u\;, \tag{2.10}$$

where $u = K[\omega]$ is the velocity field obtained from ω via the Biot-Savart law (1.3). The symmetry (2.3) implies that

$$\omega(x_1, -x_2) \;=\; \omega(x_1, x_2)\;, \quad u_1(x_1, -x_2) \;=\; -u_1(x_1, x_2)\;, \quad u_2(x_1, -x_2) \;=\; u_2(x_1, x_2)\;, \tag{2.11}$$

and in agreement with (2.2), (2.4) we impose

$$\int_{\mathbb{R}^2} \omega(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_1\omega(x)\,\mathrm{d}x \;=\; 0\;. \tag{2.12}$$

Finally, we assume without loss of generality that ω has no radially symmetric component, namely

$$\int_0^{2\pi} \omega(r\cos\theta, r\sin\theta)\,\mathrm{d}\theta \;=\; 0 \qquad \text{for all} \quad r \ge 0\;. \tag{2.13}$$

We can always realize (2.13) by including, if necessary, the radially symmetric part of ω into the asymptotic profile w_*. In this respect, it is important to note that both conditions in (2.8) are open.

Inserting (2.10) into (2.5), we obtain for ω the following equation

$$\Lambda\omega + u\cdot\nabla\omega + R_d[\omega] \;=\; 0\;, \tag{2.14}$$

where Λ is the linearized operator defined by

$$\Lambda\omega \;=\; v_*\cdot\nabla\omega + u\cdot\nabla w_*\;, \tag{2.15}$$

and $R_d[\omega]$ is a remainder term which depends on the distance d between the vortex centers

$$R_d[\omega](x) \;=\; \Big(v_*(x+2x_d) - u(-x-2x_d) - \tilde{\Omega}[w_*+\omega](x+x_d)^\perp\Big)\cdot\nabla\big(w_*(x)+\omega(x)\big)\;. \tag{2.16}$$

In (2.14)–(2.16), it is understood that $u = K[\omega]$ is the velocity field associated to ω.

We look for solutions ω of (2.14) in the Hilbert space

$$X \;=\; \Big\{\omega\in L^2(\mathbb{R}^2)\;\Big|\; \int_{\mathbb{R}^2} |\omega(x)|^2 p(|x|^2)\,\mathrm{d}x < \infty\Big\}\;, \tag{2.17}$$

equipped with the scalar product

$$\langle\omega_1,\omega_2\rangle \;=\; \int_{\mathbb{R}^2} \omega_1(x)\omega_2(x)\,p(|x|^2)\,\mathrm{d}x\;, \qquad \omega_1,\omega_2\in X\;,$$

and with the associated norm $\|\omega\| \;=\; \langle\omega,\omega\rangle^{1/2}$. Here the weight $p : [0,\infty) \to \mathbb{R}_+$ is defined by

$$p(r) \;=\; \frac{-1}{q'(r)}\;, \qquad r \ge 0\;. \tag{2.18}$$

The reason for this particular choice is that the linear operator Λ has nice properties in the space X, see Section 2.2 below. In view of (2.9), the asymptotic profile w_* and all its moments belongs to X : for any $k\in\mathbb{N}$, we have

$$\||x|^{2k}w_*\|^2 \;=\; \frac{1}{\pi^2}\int_{\mathbb{R}^2} |x|^{4k}q(|x|^2)^2 p(|x|^2)\,\mathrm{d}x \;=\; \frac{1}{\pi}\int_0^\infty r^{2k}q(r)^2p(r)\,\mathrm{d}r \;<\; \infty\;.$$

It is easy to verify that the operator Λ commutes with the rotations about the origin in $\mathbb{R}^2$, see Lemma 2.2 below. It is thus natural to use polar coordinates (r, θ) in the plane, and to decompose our space X as a direct sum

$$X \;=\; \bigoplus_{n=0}^{\infty} X_n \;=\; \bigoplus_{n=0}^{\infty} P_n X \;, \tag{2.19}$$

where P_n is the orthogonal projection in X defined by the formula

$$(P_n\omega)(r\cos\theta, r\sin\theta) \;=\; \frac{2-\delta_{n,0}}{2\pi}\int_0^{2\pi} \omega(r\cos\theta', r\sin\theta')\cos(n(\theta-\theta'))\,\mathrm{d}\theta' \;, \quad n \in \mathbb{N}\;.$$

In particular, $X_0 = P_0 X$ is the subspace of all radially symmetric functions, and for $n \ge 1$ the subspace $X_n = P_n X$ contains functions of the form $\omega(r\cos\theta, r\sin\theta) = a_1(r)\cos(n\theta) + a_2(r)\sin(n\theta)$. With this notation, condition (2.13) means that $P_0\omega = 0$.

2.2 The linearized operator and its right-inverse

We now discuss the main properties of the linearized operator Λ defined in (2.15). In the particular case where w_* is the profile G of Oseen's vortex (1.8), the operator Λ was studied in detail in [7, 14], and we shall obtain here analogous results in a more general situation.

From (2.15) we know that $\Lambda = \Lambda_1 + \Lambda_2$, where $\Lambda_1\omega = v_* \cdot \nabla\omega$ and $\Lambda_2\omega = K[\omega]\cdot\nabla w_*$. As is easily verified, Λ_2 is compact in X, while Λ_1 is unbounded. The maximal domain of Λ is therefore

$$D(\Lambda) \;=\; D(\Lambda_1) \;=\; \{\omega \in X \,|\, v_*\cdot\nabla\omega \in X\}\;. \tag{2.20}$$

The most remarkable property of this operator is that it is *skew-symmetric* in X.

Lemma 2.1 *For all $\omega_1, \omega_2 \in D(\Lambda)$, we have $\langle\Lambda\omega_1, \omega_2\rangle + \langle\omega_1, \Lambda\omega_2\rangle = 0$.*

Proof. We shall prove in fact that both operators Λ_1, Λ_2 are skew-symmetric. First, since the weight $p(|x|^2)$ is radially symmetric, we have

$$\langle v_*\cdot\nabla\omega_1, \omega_2\rangle + \langle\omega_1, v_*\cdot\nabla\omega_2\rangle \;=\; \int_{\mathbb{R}^2} p(|x|^2)\, v_*\cdot\nabla(\omega_1\omega_2)\,\mathrm{d}x \;=\; 0\;,$$

because the velocity field $p(|x|^2)v_*(x)$ is divergence-free. Next, since $\nabla w_*(x) = (2/\pi)xq'(|x|^2)$, we have

$$\langle u_1\cdot\nabla w_*, \omega_2\rangle + \langle\omega_1, u_2\cdot\nabla w_*\rangle \;=\; -\frac{2}{\pi}\int_{\mathbb{R}^2}\Big((x\cdot u_1)\omega_2 + (x\cdot u_2)\omega_1\Big)\,\mathrm{d}x \;=\; 0\;,$$

see e.g. [7, Lemma 4.8]. Combining both equalities, we obtain the desired result. □

As in the Gaussian case [7], the operator Λ is invariant under rotations about the origin in the plane $\mathbb{R}^2$. It is thus natural to work in polar coordinates (r,θ), and to develop the vorticity $\omega(r\cos\theta, r\sin\theta)$ in Fourier series with respect to the angular variable θ. In these variables, the action of Λ can be described fairly explicitly. Let

$$\phi(r) \;=\; \frac{Q(r^2)}{2\pi r^2}\;, \qquad g(r) \;=\; -\frac{2q'(r^2)}{\pi}\;, \qquad r > 0\;. \tag{2.21}$$

Then we have the following result :

Lemma 2.2 *Fix $n \in \mathbb{N}$. If $\omega = a_n(r)\sin(n\theta)$, then $\Lambda\omega = n[\phi(r)a_n(r) - g(r)A_n(r)]\cos(n\theta)$, where A_n is the regular solution of the differential equation*

$$-A_n''(r) - \frac{1}{r}A_n'(r) + \frac{n^2}{r^2}A_n(r) \;=\; a_n(r)\;, \qquad r > 0\;. \tag{2.22}$$

Similarly, if $\omega = -a_n(r)\cos(n\theta)$, then $\Lambda\omega = n[\phi(r)a_n(r) - g(r)A_n(r)]\sin(n\theta)$.

Proof. If $n = 0$, namely if ω is radially symmetric, it is straightforward to verify that $\Lambda\omega = 0$. Thus we assume that $n \ge 1$, and that $\omega = a_n(r)\sin(n\theta)$. If ψ denotes the stream function defined by $-\Delta\psi = \omega$, we have $\psi = A_n(r)\sin(n\theta)$, where A_n is the regular solution of (2.22), namely

$$A_n(r) \;=\; \frac{1}{2n}\left(\int_0^r \Bigl(\frac{s}{r}\Bigr)^n s a_n(s)\,\mathrm{d}s + \int_r^\infty \Bigl(\frac{r}{s}\Bigr)^n s a_n(s)\,\mathrm{d}s\right)\;, \qquad r > 0\;.$$

The velocity field $u = K[\omega] = -\nabla^\perp\psi$ is thus

$$u \;=\; \frac{n}{r}A_n(r)\cos(n\theta)\mathbf{e}_r - A_n'(r)\sin(n\theta)\mathbf{e}_\theta\;, \qquad \text{where} \quad \mathbf{e}_r \;=\; \frac{x}{|x|}\;, \quad \mathbf{e}_\theta \;=\; \frac{x^\perp}{|x|}\;.$$

Since $\Lambda\omega = v_* \cdot \nabla\omega + u \cdot \nabla w_*$, we conclude that

$$\Lambda\omega \;=\; \frac{Q(r^2)}{2\pi r^2}\, n a_n(r)\cos(n\theta) + \frac{n}{r}A_n(r)\cos(n\theta)\,\frac{2r}{\pi}q'(r^2)\;,$$

which, in view of (2.21), is the desired result. The case where $\omega = -a_n(r)\cos(n\theta)$ is similar. □

As an application of Lemma 2.2, we can characterize the kernel of the operator Λ. We already know that $\Lambda\omega = 0$ if ω is radially symmetric. Moreover, differentiating the identity $v_* \cdot \nabla w_* = 0$ with respect to x_1 and x_2, we obtain $\Lambda(\partial_1 w_*) = \Lambda(\partial_2 w_*) = 0$. As in the Gaussian case [14], we conclude:

Lemma 2.3 $\mathrm{Ker}(\Lambda) \;=\; X_0 \oplus \{\alpha_1\partial_1 w_* + \alpha_2\partial_2 w_* \,|\, \alpha_1, \alpha_2 \in \mathbb{R}\}$.

Proof. Since the decomposition (2.19) is invariant under the action of Λ, it is sufficient to characterize the kernel in each subspace X_n. The case $n = 0$ is trivial, because $X_0 \subset \mathrm{Ker}(\Lambda)$, hence we assume from now on that $n \ge 1$. If $\omega = a_n(r)\sin(n\theta)$ satisfies $\Lambda\omega = 0$, we know from Lemma 2.2 that $\phi a_n - gA_n = 0$. In view of (2.22), this can be written in the equivalent form

$$-A_n''(r) - \frac{1}{r}A_n'(r) + \Bigl(\frac{n^2}{r^2} - \frac{g(r)}{\phi(r)}\Bigr)A_n(r) \;=\; 0\;, \qquad r > 0\;. \tag{2.23}$$

Now, the second assumption in (2.8) means that

$$\sup_{r>0}\frac{r^2 g(r)}{\phi(r)} \;=\; \sup_{r>0}\frac{4r^2|q'(r)|}{Q(r)} \;<\; 4\;.$$

Thus, if $n \ge 2$, the "potential" term $(n^2/r^2 - g/\phi)$ in (2.23) is positive, and since $A_n(r) \to 0$ as $r \to 0$ and $r \to \infty$, the maximum principle implies that $A_n = 0$, hence also $a_n = 0$. Thus $\mathrm{Ker}(\Lambda) \cap X_n = \{0\}$ if $n \ge 2$.

In the particular case $n = 1$, it is easy to verify that $A_1(r) = r\phi(r)$ is the regular solution of (2.23). Using (2.22) we find $a_1(r) = rg(r)$, so that $\omega = a_1(r)\sin(\theta) = -\partial_2 w_*$. Similarly, $a_1(r)\cos(\theta) = -\partial_1 w_*$, hence the kernel of Λ in X_1 is spanned by the functions $\{\partial_1 w_*, \partial_2 w_*\}$. □

Using the same arguments as in [14], one can show that the operator Λ is not only skew-symmetric, but also *skew-adjoint* in X. This implies that $\mathrm{Ker}(\Lambda) = \mathrm{Ran}(\Lambda)^\perp$, hence

$$\overline{\mathrm{Ran}(\Lambda)} \;=\; \mathrm{Ker}(\Lambda)^\perp\ .$$

Let

$$Y \;=\; \{\omega \in X\,|\,|x|^2\omega \in X\}\ . \tag{2.24}$$

We now show that $\mathrm{Ker}(\Lambda)^\perp \cap Y \subset \mathrm{Ran}(\Lambda)$, and we establish a semi-explicit formula for the inverse of Λ on that subspace.

Proposition 2.4 *If $f \in X_n \cap Y$ for some $n \ge 2$, there exists a unique $\omega \in X_n \cap D(\Lambda)$ such that $\Lambda\omega = f$. Specifically, if $f = b_n(r)\cos(n\theta)$, then $\omega = a_n(r)\sin(n\theta)$, where*

$$a_n(r) \;=\; \frac{g(r)}{\phi(r)}A_n(r) + \frac{b_n(r)}{n\phi(r)}\ , \tag{2.25}$$

and A_n is the regular solution of the differential equation

$$-A_n''(r) - \frac{1}{r}A_n'(r) + \Bigl(\frac{n^2}{r^2} - \frac{g(r)}{\phi(r)}\Bigr)A_n(r) \;=\; \frac{b_n(r)}{n\phi(r)}\ , \qquad r > 0\ . \tag{2.26}$$

Similarly, if $f = b_n(r)\sin(n\theta)$, then $\omega = -a_n(r)\cos(n\theta)$.

Proof. If $\omega = a_n(r)\sin(n\theta)$, then $\Lambda\omega = n[\phi(r)a_n(r) - g(r)A_n(r)]\cos(n\theta)$ by Lemma 2.2, where A_n satisfies (2.22). The equation we have to solve is therefore $n(\phi a_n - gA_n) = b_n$, which gives (2.25). Moreover, combining (2.25) and (2.22), we obtain (2.26).

Proceeding as in [5, Lemma 3.4], we now show that (2.26) has a unique regular solution, and we establish a representation formula. As we observed in the proof of Lemma 2.3, the "potential" term $(n^2/r^2 - g/\phi)$ in (2.23) is positive if $n \ge 2$. Let ψ_+, ψ_- be the (unique) solutions of the homogeneous equation (2.23) such that

$$\psi_-(r) \;\sim\; r^n \quad \text{as } r \to 0\ , \quad \text{and} \quad \psi_+(r) \;\sim\; r^{-n} \quad \text{as } r \to \infty\ . \tag{2.27}$$

By the maximum principle, the functions ψ_+, ψ_- are strictly monotone and linearly independent. The Wronskian determinant $W = \psi_+\psi_-' - \psi_-\psi_+'$ satisfies $W' + W/r = 0$, hence $W(r) = 2n\kappa/r$ for some $\kappa > 0$, and we also have

$$\psi_-(r) \;\sim\; \kappa r^n \quad \text{as } r \to \infty\ , \quad \text{and} \quad \psi_+(r) \;\sim\; \kappa r^{-n} \quad \text{as } r \to 0\ .$$

With these notations, the unique regular solution of (2.26) has the following expression:

$$A_n(r) \;=\; \psi_+(r)\int_0^r \frac{\psi_-(s)}{W(s)}\,\frac{b_n(s)}{n\phi(s)}\,\mathrm{d}s + \psi_-(r)\int_r^\infty \frac{\psi_+(s)}{W(s)}\,\frac{b_n(s)}{n\phi(s)}\,\mathrm{d}s\ , \quad r > 0\ . \tag{2.28}$$

If $f = b_n(r)\cos(n\theta) \in X_n$, it is straightforward to verify that the function A_n defined by (2.28) is continuous and vanishes at the origin and at infinity. Moreover, we know from (2.21) that $\phi(r) \sim 1/(2\pi r^2)$ as $r \to \infty$. Thus, if we assume that $f \in X_n \cap Y$, we see that the function a_n defined by (2.25) satisfies $\int_0^\infty a_n(r)^2 p(r^2) r\,\mathrm{d}r < \infty$. As a consequence, if $\omega = a_n(r)\sin(n\theta)$, we conclude that $\omega \in X_n \cap D(\Lambda)$, and $\Lambda\omega = f$ by construction. $\square$

Remark 2.5 *If $n = 1$, the conclusion of Proposition 2.4 fails because $\partial_j w_* \in X_1 \cap \mathrm{Ker}(\Lambda)$ for $j = 1, 2$. However, if $f \in X_1 \cap Y$ satisfies $\langle f, \partial_j w_*\rangle = 0$ for $j = 1, 2$, one can show that there exists a unique $\omega \in X_1 \cap D(\Lambda) \cap \mathrm{Ker}(\Lambda)^\perp$ such that $\Lambda\omega = f$.*

2.3 The perturbation expansion

Equipped with the technical results of the previous section, we now go back to equation (2.14), which we want to solve perturbatively for large d. This equation can be written as $\Lambda\omega + N_d[\omega] = 0$, where $N_d[\omega] = u \cdot \nabla\omega + R_d[\omega]$. Before starting the calculations, we briefly explain why we expect to find a unique solution, under our symmetry assumptions.

First, if $\omega \in X$ satisfies (2.11)–(2.13), then $\omega \in \mathrm{Ker}(\Lambda)^\perp$, hence ω is uniquely determined by $\Lambda\omega$. Indeed, as was already observed, (2.13) means that $P_0\omega = 0$. Moreover, it follows from (2.11), (2.12) that

$$\langle \partial_j w_*, \omega\rangle \;=\; -\frac{2}{\pi}\int_{\mathbb{R}^2} x_j \omega \,\mathrm{d}x \;=\; 0 \;, \qquad j = 1, 2 \;,$$

hence $\omega \in \mathrm{Ker}(\Lambda)^\perp$ by Lemma 2.3. Next, if ω and $u = K[\omega]$ have the symmetries (2.11), it is straightforward to verify that the nonlinearity in (2.14) satisfies $N_d[\omega](x_1, -x_2) = -N_d[\omega](x_1, x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. This implies that $P_0 N_d[\omega] = 0$ and $\langle \partial_1 w_*, N_d[\omega]\rangle = 0$. Moreover, we have $\langle \partial_2 w_*, N_d[\omega]\rangle = 0$ by construction, because this is the relation we imposed to determine the angular speed $\tilde{\Omega}$ in (2.6). Thus, we see that $N_d[\omega] \in \mathrm{Ker}(\Lambda)^\perp$, and if we can prove in addition that $|x|^2 N_d[\omega] \in X$, then Proposition 2.4 (and Remark 2.5) will imply that $N_d[\omega] \in \mathrm{Ran}(\Lambda)$. We can therefore hope to find a unique $\omega \in \mathrm{Ker}(\Lambda)^\perp \cap D(\Lambda)$ such that $\Lambda\omega + N_d[\omega] = 0$.

To begin our perturbative approach, we compute the remainder term (2.16) for $\omega = 0$, namely

$$R_d(x) \;\equiv\; R_d[0](x) \;=\; \Big(v_*(x + 2x_d) - \tilde{\Omega}[w_*](x + x_d)^\perp\Big) \cdot \nabla w_*(x) \;, \qquad x \in \mathbb{R}^2 \;. \tag{2.29}$$

From (2.7), we know that

$$v_*(x) \;=\; \frac{1}{2\pi}\frac{x^\perp}{|x|^2}(1 - \tilde{Q}(|x|^2)) \;, \qquad \text{where} \quad \tilde{Q}(r) = \int_r^\infty q(s)\,\mathrm{d}s \;.$$

By assumption, the term $\tilde{Q}(|x|^2)$ decays faster than any inverse power of $|x|$ as $|x| \to \infty$, hence we can neglect its contribution in our calculations. For any fixed $x \in \mathbb{R}^2$ we thus have

$$v_*(x + 2x_d) \;=\; \frac{1}{2\pi}\frac{(2x_d)^\perp}{|2x_d|^2} + \frac{1}{2\pi}V(x, 2x_d) + \mathcal{O}\Big(\frac{1}{d^\infty}\Big) \;, \qquad \text{as} \quad d \to \infty \;, \tag{2.30}$$

where

$$V(x, y) \;=\; \frac{(x+y)^\perp}{|x+y|^2} - \frac{y^\perp}{|y|^2} \;.$$

Setting $x = (r\cos\theta, r\sin\theta)$, $y = 2x_d = (d, 0)$, and proceeding as in [5, Lemma 3.2], we find

$$V_1(x, y) \;=\; \frac{1}{d}\sum_{n=1}^\infty (-1)^n \frac{r^n}{d^n}\sin(n\theta) \;, \qquad V_2(x, y) \;=\; \frac{1}{d}\sum_{n=1}^\infty (-1)^n \frac{r^n}{d^n}\cos(n\theta) \;. \tag{2.31}$$

In particular, returning to (2.30) and using definition (2.6), we obtain

$$\tilde{\Omega}[w_*] \;=\; \frac{2}{d}\int_{\mathbb{R}^2} (v_*)_2(x + 2x_d) w_*(x)\,\mathrm{d}x \;=\; \frac{1}{\pi d^2} + \mathcal{O}\Big(\frac{1}{d^\infty}\Big) \;, \qquad \text{as} \quad d \to \infty \;. \tag{2.32}$$

Note that the term $V(x, 2x_d)$ in (2.30) gives no contribution to the angular velocity $\tilde{\Omega}[w_*]$. On the other hand, inserting (2.30), (2.32) into (2.29) and using the expansion (2.31) together with the relation $\nabla w_* = -xg(|x|)$, where g is defined in (2.21), we find for $x = (r\cos\theta, r\sin\theta)$:

$$R_d(x) \;=\; \frac{g(r)}{2\pi}\sum_{n=2}^\infty (-1)^n \frac{r^n}{d^n}\sin(n\theta) + \mathcal{O}\Big(\frac{1}{d^\infty}\Big) \;, \qquad \text{as} \quad d \to \infty \;. \tag{2.33}$$

Motivated by this result, we now construct inductively an approximate solution of (2.14) of the form

$$\omega(x) \;=\; \sum_{n=2}^{\ell} \frac{1}{d^n}\,\omega^{(n)}(x)\;, \qquad u(x) \;=\; \sum_{n=2}^{\ell} \frac{1}{d^n}\,u^{(n)}(x)\;, \tag{2.34}$$

where each velocity profile $u^{(n)}$ is obtained from $\omega^{(n)}$ via the Biot-Savart law (1.3). The order ℓ of the approximation is in principle arbitrary, but the complexity of the calculations increases rapidly with ℓ, and we shall restrict ourselves to $\ell = 4$ for simplicity. Of course, we assume that the symmetry and normalization conditions (2.11)–(2.13) hold at each order of the approximation. In particular, we have

$$\int_{\mathbb{R}^2} \omega^{(n)}(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_1\omega^{(n)}(x)\,\mathrm{d}x \;=\; \int_{\mathbb{R}^2} x_2\omega^{(n)}(x)\,\mathrm{d}x \;=\; 0\;, \tag{2.35}$$

for all $n \in \{2,\dots,\ell\}$. In view of [6, Appendix B], this implies that the velocity field $u^{(n)}(x)$ decays at least as fast as $|x|^{-3}$ when $|x| \to \infty$. It follows that the term $u(-x-2x_d)$ in (2.16) is $\mathcal{O}(d^{-5})$ as $d \to \infty$, and will therefore not contribute to $\omega^{(n)}$ for $n \le 4$. For the same reason,

$$\tilde\Omega[w_* + \omega] \;=\; \frac{2}{d}\int_{\mathbb{R}^2}\Big((v_*)_2(x+2x_d) - u_2(-x-2x_d)\Big)\Big(w_*(x)+\omega(x)\Big)\,\mathrm{d}x \;=\; \frac{1}{\pi d^2} + \mathcal{O}\Big(\frac{1}{d^6}\Big)\;,$$

as $d \to \infty$. Indeed, the leading term $\tilde\Omega[w_*]$ was computed in (2.32), and we know that the contribution of $u_2(-x-2x_d)$ is negligible. Moreover, using (2.30), (2.31), and (2.35), it is easy to verify that $\int (v_*)_2(x+2x_d)\omega(x)\,\mathrm{d}x = \mathcal{O}(d^{-5})$, as $d \to \infty$. Summarizing, we have shown that

$$\begin{aligned} R_d[\omega](x) \;&=\; \Big(v_*(x+2x_d) - \tilde\Omega[w_*](x+x_d)^\perp\Big)\cdot\nabla(w_*(x)+\omega(x)) + \mathcal{O}\Big(\frac{1}{d^5}\Big) \\ &=\; R_d(x) + \Big(\frac{1}{2\pi}V(x,2x_d) - \frac{x^\perp}{\pi d^2}\Big)\cdot\nabla\omega(x) + \mathcal{O}\Big(\frac{1}{d^5}\Big)\;, \end{aligned} \tag{2.36}$$

as $d \to \infty$. Similarly, the quadratic term $u\cdot\nabla\omega$ in (2.14) satisfies

$$u\cdot\nabla\omega \;=\; \frac{1}{d^4}u^{(2)}\cdot\nabla\omega^{(2)} + \mathcal{O}\Big(\frac{1}{d^5}\Big)\;, \qquad \text{as}\quad d\to\infty\;. \tag{2.37}$$

It is now a straightforward task to determine the first vorticity profiles in the expansion (2.34). From (2.36), (2.37), we know that the nonlinearity $N_d[\omega] = u\cdot\nabla\omega + R_d[\omega]$ in (2.14) satisfies, for $x = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$,

$$N_d[\omega](x) \;=\; \frac{g(r)}{2\pi}\left(\frac{r^2}{d^2}\sin(2\theta) - \frac{r^3}{d^3}\sin(3\theta)\right) + \mathcal{O}\Big(\frac{1}{d^4}\Big)\;, \qquad \text{as}\quad d\to\infty\;. \tag{2.38}$$

Thus, to ensure that $\Lambda\omega + N_d[\omega] = \mathcal{O}(d^{-4})$, we must impose

$$\Lambda\omega^{(n)} + \frac{g(r)}{2\pi}(-1)^n r^n\sin(n\theta) \;=\; 0\;, \qquad \text{for}\quad n = 2,3\;. \tag{2.39}$$

By Proposition 2.4, Eq. (2.39) has a unique solution $\omega^{(n)} \in X_n \cap D(\Lambda)$ of the form

$$\omega^{(n)}(x) \;=\; a_n(r)\cos(n\theta)\;, \qquad u^{(n)}(x) \;=\; -\frac{n}{r}A_n(r)\sin(n\theta)\mathbf{e}_r - A_n'(r)\cos(n\theta)\mathbf{e}_\theta\;, \tag{2.40}$$

where $a_n(r), A_n(r)$ are given by (2.25), (2.26) with $b_n(r) = (-1)^n r^n g(r)/(2\pi)$. As is easily verified, the symmetry conditions (2.11)–(2.13), are satisfied by the profiles $\omega^{(n)}, u^{(n)}$ for $n = 2,3$, and the velocity $|u^{(n)}(x)|$ decays like $|x|^{-n-1}$ as $|x| \to \infty$.

Computing the profiles $\omega^{(4)}, u^{(4)}$ is more cumbersome, but also more representative of what happens in the general case. First of all, the quadratic term (2.37) is no longer negligible, and using (2.40) for $n = 2$ we find

$$u^{(2)}(x) \cdot \nabla\omega^{(2)}(x) \;=\; B_1(r)\sin(4\theta)\;, \qquad \text{where} \quad B_1(r) \;=\; \frac{1}{r}\Big(A_2'(r)a_2(r) - A_2(r)a_2'(r)\Big)\;.$$

Note that, to ensure that $u^{(2)} \cdot \nabla\omega^{(2)} \in X$, we need an assumption on the second derivative of the function q appearing in (2.7). For instance, in analogy with (2.9), one can impose

$$\sup_{r>0} \frac{r^k q''(r)^2}{|q'(r)|} \;<\; \infty\;, \qquad \text{for all } k \in \mathbb{N}\;. \tag{2.41}$$

Next, we must compute the contribution of $\omega^{(2)}$ to the right-hand side of (2.36). Using (2.31), (2.40), we obtain

$$\Big(\frac{d^2}{2\pi}V(x, 2x_d) - \frac{x^\perp}{\pi}\Big) \cdot \nabla\omega^{(2)}(x) \;=\; B_2(r)\sin(4\theta) + C(r)\sin(2\theta)\;,$$

where

$$B_2(r) \;=\; \frac{1}{4\pi}(2a_2(r) - ra_2'(r))\;, \qquad C(r) \;=\; \frac{2}{\pi}a_2(r)\;.$$

Combining these results, we find instead of (2.38) :

$$N_d[\omega](x) \;=\; \frac{g(r)}{2\pi}\sum_{n=2}^{3}(-1)^n\frac{r^n}{d^n}\sin(n\theta) + \frac{B(r)}{d^4}\sin(4\theta) + \frac{C(r)}{d^4}\sin(2\theta) + \mathcal{O}\Big(\frac{1}{d^5}\Big)\;,$$

as $d \to \infty$, where $B(r) = B_1(r) + B_2(r) + r^4 g(r)/(2\pi)$. Therefore, in addition to (2.39), we must impose

$$\Lambda\omega^{(4)} + B(r)\sin(4\theta) + C(r)\sin(2\theta) \;=\; 0\;. \tag{2.42}$$

Using again Proposition 2.4, we see that (2.42) has a unique solution $\omega^{(4)} \in (X_4 + X_2) \cap D(\Lambda)$ of the form $\omega^{(4)}(x) \;=\; a_4(r)\cos(4\theta) + \tilde{a}_2(r)\cos(2\theta)$, where $a_4(r)$ is given by (2.25), (2.26) with $n = 4$ and $b_4(r) = B(r)$, while $\tilde{a}_2(r)$ is given by the same relations with $n = 2$ and $b_2(r) = C(r)$. An explicit expression of the velocity profile $u^{(4)}$ can also be obtained, as in (2.40).

Summarizing, we have shown :

Proposition 2.6 *Let w_* be a radially symmetric vorticity profile of the form* (2.7), *where the function q satisfies* (2.8), (2.9), (2.41), *and let*

$$w(x) \;=\; w_*(x) + \sum_{n=2}^{4}\frac{1}{d^n}\omega^{(n)}(x)\;, \qquad v(x) \;=\; v_*(x) + \sum_{n=2}^{4}\frac{1}{d^n}u^{(n)}(x)\;, \tag{2.43}$$

where the vorticity profiles $\omega^{(n)} \in X \cap D(\Lambda) \cap \mathrm{Ker}(\Lambda)^\perp$ satisfy (2.39), (2.42), *and the velocity profiles $u^{(n)}$ are obtained by the Biot-Savart law* (1.3). *Then w is an asymptotic solution of Eqs.* (2.5), (2.6) *in the sense that*

$$\Big(v(x) - v(-x - 2x_d) - \tilde{\Omega}[w](x + x_d)^\perp\Big) \cdot \nabla w(x) \;=\; \mathcal{O}\Big(\frac{1}{d^5}\Big)\;, \tag{2.44}$$

in the topology of X and uniformly on $\mathbb{R}^2$, as $d \to \infty$.

The asymptotic expansion (2.34) is very natural, and it is clear that it can be performed to any finite order $\ell \in \mathbb{N}$ if we make appropriate assumptions on the derivatives of the profile q, as in (2.41). As was already mentioned, we also conjecture that there exists an *exact* solution of (2.5) for $d \gg 1$ which coincides with (2.43) up to corrections of order $\mathcal{O}(d^{-5})$. It is important to notice that, under the symmetry and normalization conditions (2.2)–(2.4), the exact solution (if it exists) and the asymptotic expansion (2.43) are uniquely determined by the limiting profile w_*.

Remark 2.7 *Rather strong assumptions on the limiting profile w_* were made in this section to ensure that the asymptotic expansion* (2.43) *holds in the function space X defined in* (2.17)*, which we believe is naturally associated to the problem. This does not restrict the scope of our results here, because these conditions are automatically fulfilled by the Gaussian profiles created by the Navier-Stokes evolution. However, within the framework of Euler's equation, it is certainly interesting to construct interacting vortex pairs with more general profiles, including compactly supported ones. If we do not insist on controlling our expansion in the space X, the calculations presented in this section show that the assumptions on the function q can be considerably relaxed. The most important point is that Eq.* (2.26) *should have a unique solution for $n \ge 2$, and for $n = 1$ if the right-hand side satisfies some orthogonality conditions. This is definitely the case if the second inequality in* (2.8) *holds, but that condition does not imply that q is strictly decreasing and can well be satisfied if q is compactly supported.*

To conclude this section, we indicate how approximate solutions of (1.13) of the form (1.15) can be obtained from Proposition 2.6 by a simple rescaling. Given $\alpha, d > 0$, we consider the situation described in (1.14) with $\alpha_1 = \alpha_2 = \alpha$ and $r_1 = r_2 = d/2$. If w_* is a radially symmetric vorticity profile satisfying the assumptions of Proposition 2.6, we define, for all sufficiently small $\epsilon > 0$,

$$w_\epsilon(x) \;=\; w_*(x) + \sum_{n=2}^{4} \frac{\epsilon^n}{d^n}\,\omega^{(n)}(x) \;, \qquad v_\epsilon(x) \;=\; v_*(x) + \sum_{n=2}^{4} \frac{\epsilon^n}{d^n}\,u^{(n)}(x) \;, \tag{2.45}$$

where $\omega^{(n)}, u^{(n)}$ are as in (2.43). Then, by construction, the vorticity distribution

$$\omega_\epsilon(x) \;=\; \frac{\alpha}{\epsilon^2}\,w_\epsilon\Big(\frac{x - x_d}{\epsilon}\Big) + \frac{\alpha}{\epsilon^2}\,w_\epsilon\Big(\frac{-x - x_d}{\epsilon}\Big) \;, \tag{2.46}$$

where $x_d = (d/2, 0)$, is an approximate solution of Eq. (1.13) with $\Omega = \alpha/(\pi d^2)$. More precisely, it follows from (2.44) that

$$\partial_t \omega_\epsilon + (u_\epsilon - \Omega x^\perp)\cdot\nabla\omega_\epsilon \;=\; \mathcal{O}(\epsilon) \;, \qquad \text{as} \quad \epsilon \to 0 \;, \tag{2.47}$$

where u_ϵ is obtained from ω_ϵ via the Biot-Savart law (1.3).

3 Inviscid approximation of viscous vortex pairs

In this final section, we describe in some detail the result of [5] in the particular case of a vortex pair, and we interprete it using the approximate solutions of Euler's equation constructed in Section 2. Given $\alpha, d > 0$, we set $\Omega = \alpha/(\pi d^2)$ and we denote by ω_0 the vorticity distribution (1.14) where $\alpha_1 = \alpha_2 = \alpha$ and $r_1 = r_2 = d/2$. For any $\nu > 0$, we consider the (unique) solution $\omega^\nu(x, t)$ of the rotating viscous vorticity equation

$$\partial_t \omega + (u - \Omega x^\perp)\cdot\nabla\omega \;=\; \nu\Delta\omega \;, \qquad x \in \mathbb{R}^2 \;, \quad t > 0 \;, \tag{3.1}$$

with initial data ω_0. Up to a rotation of angle Ωt, the vorticity distribution $\omega^\nu(x,t)$ coincides with the solution of the nonrotating equation (1.2) with the same initial data, which is studied in [5]. The advantage of using a rotating frame is that the vortex centers remain fixed, instead of evolving according to the point vortex dynamics (1.4). As a matter of fact, Theorem 2.1 in [5] establishes that $\omega^\nu(\cdot,t) \rightharpoonup \omega_0$ as $\nu \to 0$, for any $t > 0$.

To obtain a more precise convergence result, we decompose the solution of (3.1) into a sum of viscous vortices :

$$\omega^\nu(x,t) \;=\; \frac{\alpha}{\nu t}\, w_1^\nu\Bigl(\frac{x-x_1}{\sqrt{\nu t}},t\Bigr) + \frac{\alpha}{\nu t}\, w_2^\nu\Bigl(\frac{x-x_2}{\sqrt{\nu t}},t\Bigr)\;, \tag{3.2}$$

where $x_1 = -x_2 = (d/2,0)$. As is shown in [5], both vorticity profiles $w_1^\nu(\xi,t), w_2^\nu(\xi,t)$ can be approximated by the same function

$$w_{\text{app}}^\nu(\xi,t) \;=\; G(\xi) + \Bigl(\frac{\nu t}{d^2}\Bigr) F_\nu(\xi)\;, \qquad \xi \in \mathbb{R}^2\;, \quad t > 0\;, \tag{3.3}$$

where G is the Gaussian profile (1.8) and the first order correction F_ν is constructed as follows. Let $\mathcal{L}$ be the Fokker-Planck operator

$$\mathcal{L} \;=\; \Delta_\xi + \frac{1}{2}\xi\cdot\nabla_\xi + 1\;, \qquad \xi \in \mathbb{R}^2,$$

and Λ be the linearized operator (2.15) with $w_* = G$, namely

$$\Lambda w \;=\; v^G\cdot\nabla w + K[w]\cdot\nabla G\;.$$

Here we use the functional setting of Section 2 in the particular case where the asymptotic profile w_* is the Oseen vortex G. This means that $q(r) = \frac{1}{4}e^{-r/4}$ in (2.7), and the assumptions (2.8), (2.9), (2.41) are clearly satisfied. With these notations, the profile F_ν is the unique solution of the linear equation

$$\frac{\nu}{\alpha}(1-\mathcal{L})F_\nu + \Lambda F_\nu + A \;=\; 0\;, \tag{3.4}$$

where $A(\xi) = \frac{1}{2\pi}\xi_1\xi_2 G(\xi)$, see [5, Section 3.3]. In polar coordinates $\xi = (r\cos\theta, r\sin\theta)$, we thus have

$$A(\xi) \;=\; \frac{1}{16\pi^2}\, r^2 e^{-r^2/4}\sin(2\theta) \;=\; \frac{1}{2\pi}\, r^2 g(r)\sin(2\theta)\;,$$

where g is defined in (2.21). In particular, if we use the angular decomposition (2.19) of the function space (2.17), we see that $A \in X_2$, and it follows that $F_\nu \in X_2$ too. Now, setting $\nu = 0$ in (3.4), we obtain the simple equation $\Lambda F_0 + A = 0$, which coincides with (2.39) for $n = 2$. Since $X_2 \cap \mathrm{Ker}(\Lambda) = \{0\}$, we conclude that $F_0 = \omega^{(2)}$. It is clear from (3.4) that the actual profile F_ν is close to F_0 if the viscosity ν is small compared to the circulation α of the vortices. As a matter of fact, it is shown in [5, Lemma 3.5] that

$$\|F_\nu - F_0\|_X \;\le\; C\frac{\nu}{\nu+\alpha}\;. \tag{3.5}$$

To formulate our main approximation result, we introduce a function space with a weaker norm than X. Given any $\beta > 0$, we denote by Z_β the space

$$Z_\beta \;=\; \Bigl\{ w \in L^2(\mathbb{R}^2) \,\Big|\, \|w\|_\beta < \infty \Bigr\}\;, \qquad \text{where} \quad \|w\|_\beta^2 \;=\; \int_{\mathbb{R}^2} |w(\xi)|^2 e^{\beta|\xi|}\,\mathrm{d}\xi\;.$$

Applying Theorem 2.5 of [5] to the particular situation considered here, we obtain

Proposition 3.1 *Fix $T > 0$, and let $\omega^\nu(x,t)$ be the solution of the rotating viscous vorticity equation* (3.1) *with initial data ω_0. There exist positive constants K, β, depending only on the product ΩT, such that, if $\omega^\nu(x,t)$ is decomposed as in* (3.2)*, then the vorticity profiles $w_i^\nu(\xi,t)$ satisfy*

$$\max_{i=1,2} \|w_i^\nu(\cdot,t) - w_{\text{app}}^\nu(\cdot,t)\|_\beta \le K\Big(\frac{\nu t}{d^2}\Big)^{3/2} , \tag{3.6}$$

for all $t \in (0,T]$, where $w_{\text{app}}^\nu(\xi,t)$ is given by (3.3).

This result can be reformulated in a slightly different way, using the approximate solutions of Euler's equation constructed in Section 2. Indeed, if we set $\epsilon = \sqrt{\nu t}$, and if we remember that $w_* = G$ in the present case, we see that the inviscid profile w_ϵ defined in (2.45) satisfies

$$w_{\sqrt{\nu t}}(\xi) = w_{\text{app}}^\nu(\xi,t) + \Big(\frac{\nu t}{d^2}\Big)\Big(F_0(\xi) - F_\nu(\xi)\Big) + \mathcal{O}\Big(\Big(\frac{\nu t}{d^2}\Big)^{3/2}\Big) .$$

Thus, combining (3.5) and (3.6), we obtain:

Corollary 3.2 *Under the assumptions of Proposition 3.1, we have*

$$\max_{i=1,2} \|w_i^\nu(\cdot,t) - w_{\sqrt{\nu t}}\|_\beta \le K\Big(\frac{\nu t}{d^2}\Big)^{3/2} + C\,\frac{\nu t}{d^2}\,\frac{\nu}{\nu+\alpha} , \tag{3.7}$$

for all $t \in (0,T]$, where w_ϵ is the inviscid profile defined by (2.45) *with $w_* = G$.*

Finally, using the continuous inclusion $Z_\beta \hookrightarrow L^1(\mathbb{R}^2)$, we can formulate an approximation result for the original function $\omega^\nu(x,t)$. If we compare (2.46), (3.2) and use the fact that $G(\xi)$ and $F_0(\xi)$ are even functions of ξ, we arrive at:

Corollary 3.3 *Under the assumptions of Proposition 3.1, we have*

$$\frac{1}{\alpha}\int_{\mathbb{R}^2} |\omega^\nu(x,t) - \omega_{\sqrt{\nu t}}(x)|\,\mathrm{d}x \le K\Big(\frac{\nu t}{d^2}\Big)^{3/2} + C\,\frac{\nu t}{d^2}\,\frac{\nu}{\nu+\alpha} , \tag{3.8}$$

for all $t \in (0,T]$, where ω_ϵ is the approximate steady state of the rotating Euler equation defined in (2.46) *with $w_* = G$.*

Comparing this last result with Theorem 1.1, we see that replacing a linear superposition of Oseen vortices by a more accurate solution of Euler's equation, which takes into account the deformation of the vortex cores due to mutual interaction, results in a better approximation. If we believe that there exists an *exact* solution of Euler's equation which is close to ω_ϵ for ϵ sufficiently small, then (3.8) shows that the solution $\omega^\nu(x,t)$ of the rotating viscous vorticity equation (3.1) slowly travels through a family of steady states of the inviscid equation (1.13) indexed by the length parameter ϵ, which evolves diffusively according to $\epsilon = \sqrt{\nu t}$.

Remark 3.4 *The right-hand side of* (3.8) *suggests that both dimensionless quantities ν/α and $\nu t/d^2$ play an important role in the evolution of viscous vortices. It is tempting to eliminate one of these quantities by considering, for instance, the limit $\nu \to 0$ while $\epsilon = \sqrt{\nu t}$ is kept fixed. Under the assumptions of Proposition 3.1, one may conjecture that the solution $\omega^\nu(x,t)$ of* (3.1) *satisfies, if $\epsilon > 0$ is sufficiently small,*

$$\omega^\nu(x,\epsilon^2/\nu) \xrightarrow[\nu\to 0]{} \omega_\epsilon(x) , \qquad \text{for all } x \in \mathbb{R}^2 ,$$

where $\omega_\epsilon(x)$ is an exact stationary solution of (1.13) *of the form* (2.46)*. Unfortunately, the results of [5] do not provide any control on $\omega^\nu(x,t)$ in the limit where $\Omega t \to \infty$.*

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